Non-commutative Minimal $\overset{\text{rational}}{\wedge}$ Surfaces (Susan Sierra)

A Sklyanin algebra can be written as

$$S_{abc} = \frac{k\langle X,Y,Z\rangle}{\begin{pmatrix} aXY+bYX+cZ^2 \\ aYZ+bZY+cX^2 \\ aZX+bXZ+cY^2 \end{pmatrix}}$$

for $a,b,c \in k = \bar{k}$ generic.

<u>Question</u>: What are the connected graded Noetherian rings R birational to S

(i.e. what are the noncommutative rational surfaces qgr(R)?)

Again, for technical reasons consider $T = S^{(3)}$ and look for $R \subset D_{gr}(T)$ with $D_{gr}(R) = D_{gr}(T)$

The case $R \subset T$ was already covered by [RSS], see page 3 of these notes.

"<u>Cheap</u>" <u>example</u>: $R = k\langle T_2 \cdot g^{-1}\rangle \supseteq k\langle T_1 g \cdot g^{-1}\rangle = T$ and $R \subset D_{gr}(T)$.

However this is no true example as

$$R \cong T^{(2)}$$
$$x \in r_n \longmapsto x g^n$$

$\Rightarrow$ Consider only $R \subseteq T_{(g)} := T\langle h^{-1} \mid h \text{ homogeneous}, h \notin (g)\rangle$

① <u>THM [RSS]</u>: Let $T \subseteq R \subseteq T_{(g)}$ with R f.g. Noetherian. Then $R = T$.

(Recall: blowing down gives you an overring, hence a "minimal surface" should have no overring)

Def: A ring $X \subseteq T_{(g)}$ s.t. "$X \subseteq R \subseteq T_{(g)}$ with R cg Noetherian $\Rightarrow X = R$" is called a minimal model for $D_{gr}(T)$.

Idea of proof THM (1)

Let $T \subsetneq R \subsetneq T_{(g)}$ with R c.g. Noetherian.

Take $K \subset R$ such that K/T is 2-GK-critical and g-torsion free submodule of R/T.

As T has no line modules (This is a nontrivial fact!)

$$K/T \otimes_T T/gT = \underbrace{\oplus \text{ point modules}}_{\text{at least } 2}$$

$\Downarrow$ Lemma

$\exists K' \supsetneq K$ such that $(K')^\circ / T^\circ \cong K^\circ / T^\circ$

(where $(-)^\circ : T\text{-mod} \longrightarrow (T_{(g)})_0 = (D_{gr}(T))_0\text{-mod}$ Function field

similarly $(\hat{-}) \longleftarrow$ and $(\hat{M^\circ}) \supset M$.

As T° is Hereditary by [ATV]

$\Downarrow$

Some explanation I did not write down

$\Downarrow$

$K' \subset \hat{R^\circ}$

End by contradiction via "finiteness of hats" - Lemma.

Now let Q be a NC quadric surface (in their definition) $\dfrac{\text{generic 4D Sklyanin algebra}}{\text{central deg 2 element}}$

These are closely related to the second Veronese of noncommutative quadrics as defined by Van den Bergh

$\Rightarrow$ Q has a central element in degree 2

$\Rightarrow$ consider $Q^{(2)}$. This is an elliptic algebra.

$Q^{(2)}$ has similar properties as T, however

$$\operatorname{gldim}((Q^{(2)})^{\circ} = \begin{cases} 1 \\ 2 \\ \infty \end{cases}$$

THM 2 $\Rightarrow$ proof only generalizes for $\operatorname{gldim}(Q^{(2)})^{\circ} = 1$.

The key observation for tackling the gldim $\geq \infty$ cases is that $(Q^{(2)})^{\circ}$ is only mildly singular.

$$D_{sing}(Q^{(2)}) = \frac{D^b((Q^{(2)})^{\circ})}{\mathrm{Perf}} \simeq \mathrm{Vect} \quad (A_1\text{-sing.})$$

$\Rightarrow$ some version of Goodearl localization will still work.

For GLDim 2 one uses a result by [VdB] that these are related to singular quadrics.

THM 3: Let U be either $S^{(3)}$ or $Q^{(2)}$. Then for any R c.g. Noetherian $U \subseteq R \subseteq D_{gr}(U)$ we have

R is cofinite with some "cheap" overring $k\langle U_n g^{1-n} \rangle$

I.e. equal in qgr(U)

[RSS8]

Remark: Although T has almost no Noetherian overrings, $(T_{\mathfrak{g}})^0$ has a lot of noetherian subrings

(in fact any subring is noetherian [RSS])

Open questions:

- Find all minimal models in $T_{(\mathfrak{g})}$
- Is any subring ~~compact~~ contained in a minimal model?