Bayesian Analysis

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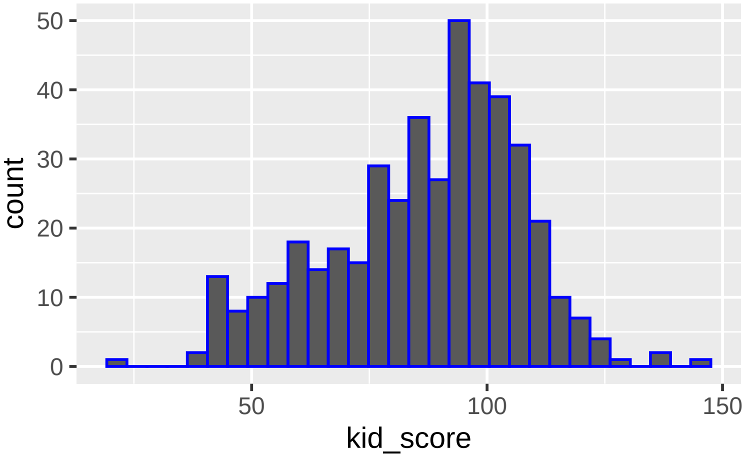
Single Parameter Normal Model

Example: Cognitive Test Scores

Data (y) are available on the cognitive test scores of three- and four-year-old children in the USA. The sample contains 434 observations.



Normal distribution with known variance

We will assume a normal model for the data where $y_i|\mu, \sigma^2 \sim N(\mu, \sigma^2)$. Assume σ^2 is known where $\sigma = 20.4$

- Specify the likelihood for μ

$p(y|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(- \frac{1}{2\sigma^2}(y_i - \mu)^2 \right)$

- Specify a prior for μ

$\mu \sim N(\mu_0, \sigma_0^2)$

- Use Bayes' rule to obtain the posterior distribution

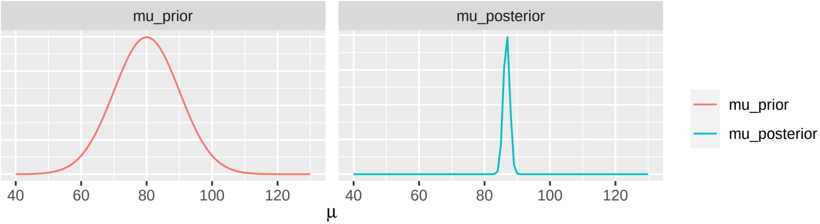
$p(\mu|y) \propto p(y|\mu)p(\mu)$

- As it turns out, the posterior is also a normal distribution

$$\mu|y \sim N\left(\frac{n\bar{y}/\sigma^2 + \mu_0/\sigma_0^2}{n/\sigma^2 + 1/\sigma_0^2}, \frac{1}{n/\sigma^2 + 1/\sigma_0^2} \right)$$

Prior vs Posterior for μ

- Choose prior mean and variance, e.g., $\mu_0 = 80$, $\sigma_0 = 10$



- Result: $\hat{\mu} = 86.73$, 95% CI: (84.82, 88.64)

Normal distribution with known mean

- Assume μ is known where $\mu = 86.79$
- Usually work with precision i.e., $\tau = 1/\sigma^2$
- Specify a prior for τ
  - Popular prior for the precision of a normal distribution is a gamma prior e.g., $\tau \sim Gamma(a, b)$ where $E[\tau] = \frac{a}{b}$ and $Var[\tau] = \frac{a}{b^2}$
  - $p(\tau|a, b) = \frac{b^a}{\Gamma(a)}\tau^{a-1}e^{-b\tau}$ for $\tau > 0$ and $a, b > 0$
- Use Bayes' rule to obtain the posterior distribution

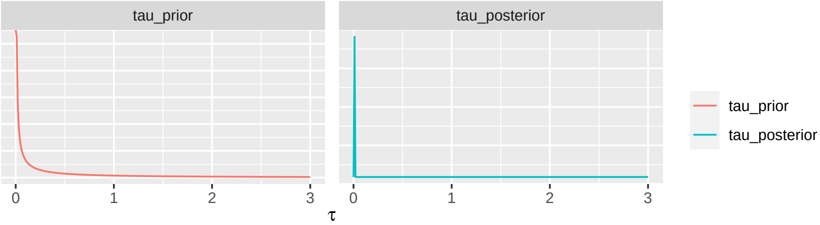
$$p(\tau|y) \propto p(y|\tau)p(\tau)$$

- The posterior will also be a gamma distribution

$$\tau|y \sim Gamma\left(a + n/2, b + 1/2\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Prior vs Posterior for τ

- Choose parameter values for the prior distribution for τ, e.g., $a = 0.1$, $b = 0.1$



- Result: Through simulation from the posterior for σ we find $\hat{\sigma} = 20.23$, 95% CI: (18.96, 21.63)