Bayesian Analysis

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Beta Prior Distribution

Quick Recap

Recall:

$$\underset{\text{posterior}}{P(\theta|Y)} \propto \underset{\text{likelihood}}{P(Y|\theta)}\underset{\text{prior}}{P(\theta)}$$

For the Happiness example:

- Data: n = 20 women, y = 14 women reported being happy
- $y \sim Binomial(n = 20, \theta)$

$$p(y|\theta) = c\theta^y(1-\theta)^{n-y} \text{ with } c = \binom{n}{y}$$

- We want to find the posterior distribution for θ

Now we will consider defining the prior, $p(\theta)$, with a known probability distribution, such that

$$\theta \sim Beta(a, b)$$

The Beta Prior

A Beta distribution is defined on the interval [0,1] and has two parameters, a and b. The density function is defined as

$$p(\theta|a,b) = \frac{1}{B(a,b)}\theta^{a-1}(1-\theta)^{b-1}$$

where $B(a,b)$ is a normalising constant that insures a valid probability density function.

If $\theta \sim Be(a,b)$ then $E(\theta) = \frac{a}{a+b}$ and $Var(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$

Note $B(a,b)$ is not a function of θ, so we can write

$$p(\theta|a,b) \propto \theta^{a-1}(1-\theta)^{b-1}$$

This will become useful later.

The Beta Prior with a=1 and b=1

Let's use Bayes' theorem now to find the form of the posterior distribution for θ assuming $\theta \sim Beta(a = 1, b = 1)$

This means we've assumed a prior mean and variance for θ of $\frac{1}{2}$ and $\frac{1}{12}$ respectively.

So the posterior is

$$\underset{\text{posterior}}{p(\theta|y)} \propto \underset{\text{likelihood}}{\theta^y(1-\theta)^{n-y}} \underset{\text{prior}}{\theta^{a-1}(1-\theta)^{b-1}}$$

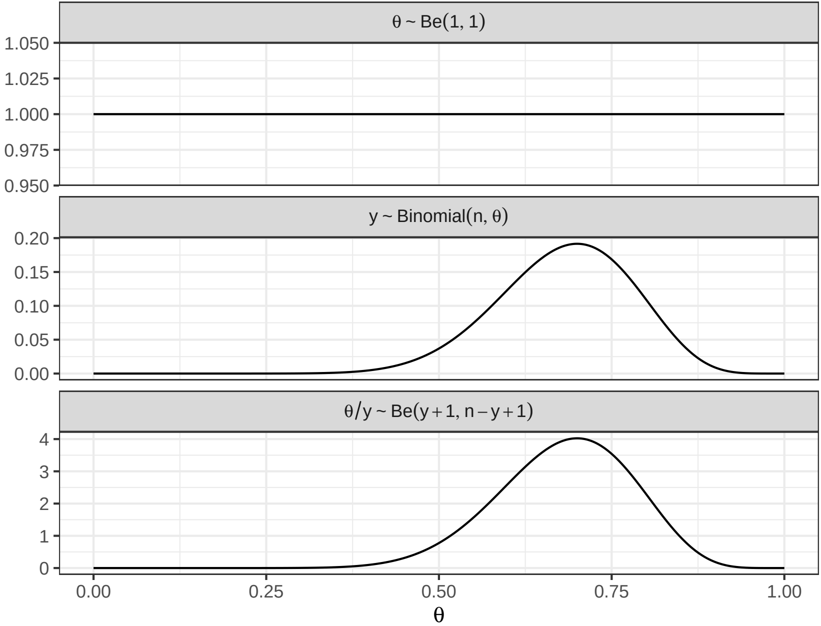
and given a $=$ 1 and b $=$ 1

$$\underset{\text{posterior}}{p(\theta|y)} \propto \theta^y(1-\theta)^{n-y}$$

This posterior actually takes the form of another Beta distribution with parameters $y + 1$ and $n - y + 1$. So,

$$\theta|y \sim Beta(y + 1, n - y + 1)$$

What does this look like for the Happiness example?



More on the Binomial Likelihood and the Beta Prior

It turns out anytime you use a Binomial likelihood and a Beta prior, such that

$\theta \sim Be(a, b)$

$y \sim Binomal(n, \theta)$

then you get a posterior distribution which is also Beta, where

$$\theta | y \sim Beta(y + a, n - y + b)$$

When the posterior is the same form as the prior, the prior is said to be a **conjugate prior**. The Beta prior is a conjugate prior for the Binomial likelihood.

The Posterior is a Compromise of Prior and Likelihood

The posterior distribution is always a compromise between the prior distribution and the likelihood function.

We can illustrate this easily with the Beta-Binomial example.

We've seen that for a $Be(a, b)$ prior and a $Binomial(n, \theta)$ likelihood that the posterior will be of the form

$$\theta|y \sim Beta(y + a, n - y + b)$$

and so the posterior mean is $E(\theta|y) = \frac{y+a}{n+a+b}$.

This can be written as a weighted sum of the prior mean $(\frac{a}{a+b})$ and the data proportion $(\frac{y}{n})$, as follows:

$$E(\theta|y) = \underbrace{\frac{y}{n}}_{data} \underbrace{\frac{n}{n+a+b}}_{weight} + \underbrace{\frac{a}{a+b}}_{prior} \underbrace{\frac{a+b}{n+a+b}}_{weight}$$

Expressing prior knowledge as a Beta distribution

Suppose for the Happiness example, you want to express your underlying belief about θ - the probability a woman age 65+ is happy.

Your beliefs may be based on previous studies or perhaps expert opinion.

So for example, supppose you want your prior to reflect beliefs that the proportion is 0.6 ± 0.1.

How do we express this belief in the Beta distribution?

We will use something called **moment matching**.

Moment Matching

Recall if $\theta \sim Be(a, b)$ then $E(\theta) = \frac{a}{a+b}$ and $Var(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$

Based on our prior beliefs we want:

$E(\theta) = \frac{a}{a+b} = 0.6$

$Var(\theta) = \frac{ab}{(a+b)^2(a+b+1)} = 0.1^2$

We can use these equations to solve for a and b, the parameters of the Beta prior.