Bayesian Data Analysis

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Inferring a Binomial Probability

Recall: Bayes' rule

Given a set of observed data points y and a set of parameters θ, we write Bayes' rule as

$$\underbrace{P(\theta|y)}_{\text{posterior}} = \frac{\underbrace{P(y|\theta)}_{\text{likelihood}}\underbrace{P(\theta)}_{\text{prior}}}{\underbrace{P(y)}_{\text{marginal likelihood}}}$$

and as a proportional statement

$$\underbrace{P(\theta|Y)}_{\text{posterior}} \propto \underbrace{P(Y|\theta)}_{\text{likelihood}}\underbrace{P(\theta)}_{\text{prior}}$$

We will now consider an example that will build some intuition for how prior distributions and data interact to produce posterior distributions.

The Happiness example

Suppose females, aged 65+ in a general social survey were asked about being happy. If this is a representative sample of the population of women, what is the probability that a 65+ woman is happy?

What is our goal? To estimate the probability that a 65+ woman is happy. This is an unknown parameter which we'll call θ.

What data do we have? Data: n = 20 women, y = 14 women reported being happy

How do we do Bayesian inference for θ?

- Decide on a descriptive model for the data (i.e., the likelihood) with meaningful parameter(s), θ (e.g., the probability a 65+ woman is happy)
- Information about θ will be summarized in a prior probability distribution
- and updated using the data, via the likelihood, to obtain the posterior distribution for the parameters using Bayes' rule.

Likelihood function - $p(y|\theta)$

Data: n = 20 women, y = 14 women reported being happy

We will assume that y is Binomial(θ, n) such that

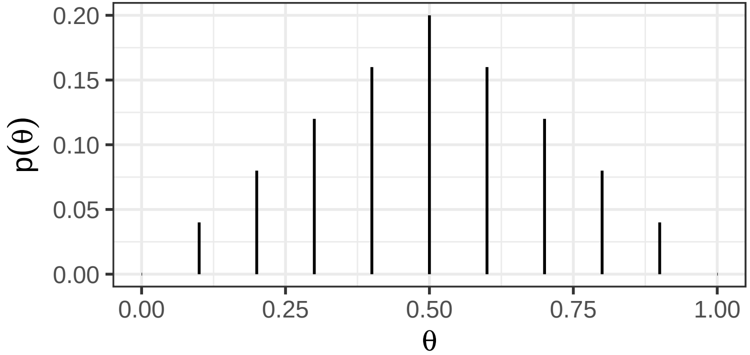
$$p(y|\theta) = c\theta^{y}(1-\theta)^{n-y} \text{ with } c = \binom{n}{y}$$

We'll refer to $y|\theta \sim Bin(\theta, n)$ as the data model (or the likelihood). It tells us how the data are related to the parameter(s) we want to estimate.

Prior distribution - $p(\theta)$

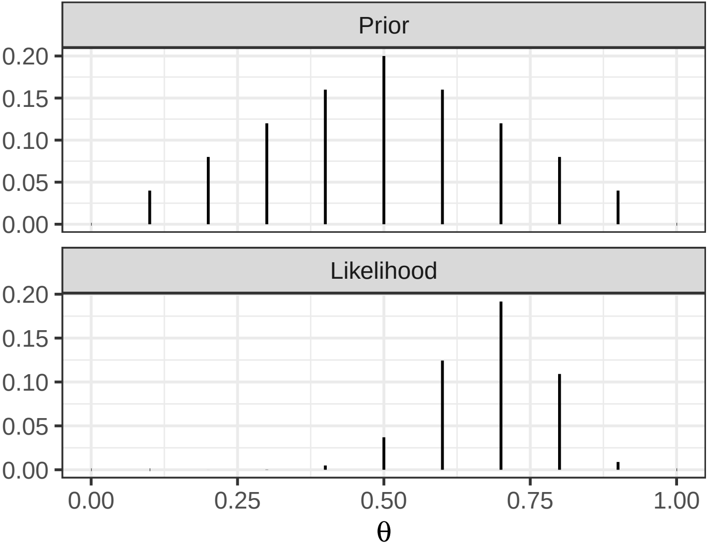
Now that we've defined the data model, the next step is to establish a prior distribution over the parameter values.

- Let's start simple and assume θ can only take on values k $= 0, 0.1, 0.2, \ldots, 1$.
- Suppose that we believe that θ is most likely to be 0.5 and we assign lower weight to θ values far above or below 0.5.
- A prior distribution incorporating these beliefs might look like

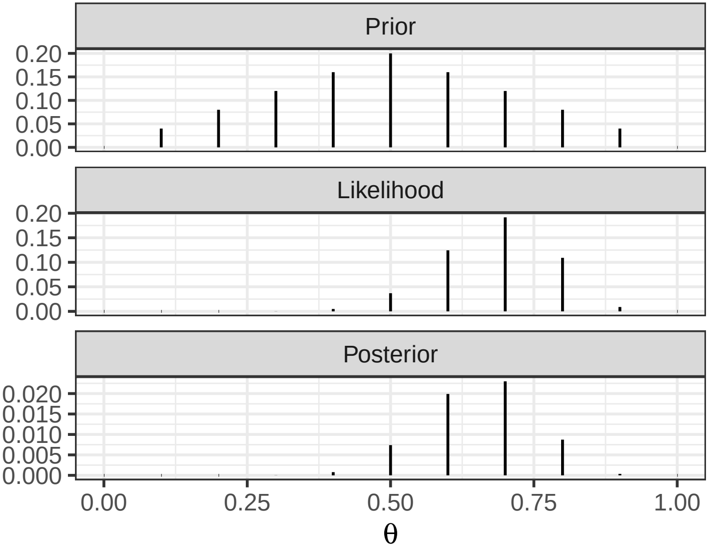


Likelihood & Prior

Given that y = 14 and n = 20 with $\frac{y}{n} = 0.7$, which θ out of $0, 0.1, 0.2, \ldots, 1$ do you expect to have the largest value of the likelihood function?

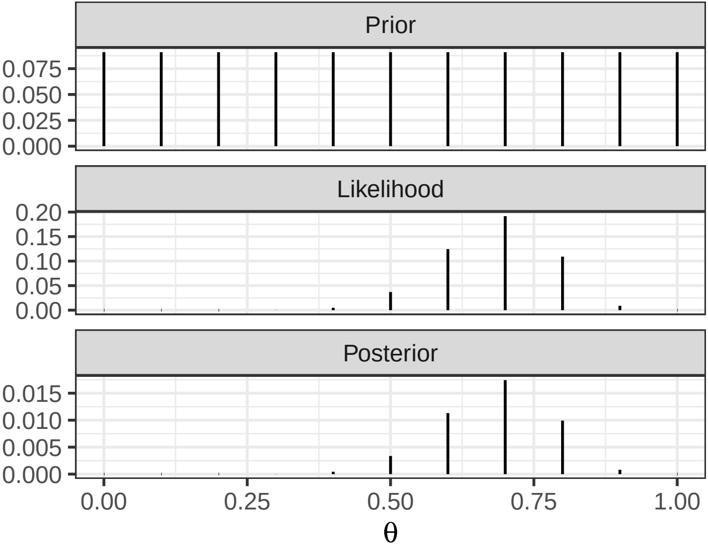


Posterior distribution - $\underbrace{P(\theta|Y)}_{\text{posterior}} \propto \underbrace{P(Y|\theta)}_{\text{likelihood}}\underbrace{P(\theta)}_{\text{prior}}$



Changing prior assumptions

Instead of the "triangular" prior let's make a more uniform assumption. So for $k = 0, 0.1, 0.2, \ldots, 1$, $Pr(\theta = k) = 1/11$ (i.e., all are equally likely).



Marginal likelihood - $p(y)$

Recall:

$$\underset{\text{posterior}}{P(\theta|y)} = \frac{\underset{\text{likelihood}}{P(y|\theta)}\underset{\text{prior}}{P(\theta)}}{\underset{\text{marginal likelihood}}{P(y)}}$$

What is $P(y)$?

$$P(y) = \sum_{\theta *} P(y|\theta^*)P(\theta^*)$$

So for $k = 0, 0.1, 0.2, \ldots, 1$, $Pr(\theta = k) = 1/11$ (i.e., all are equally likely)

$P(y) = p(y|\theta = 0)Pr(\theta = 0) + P(y|\theta = 0.1)Pr(\theta = 0.1) + \ldots = 0.04$

To do this in R:

```
n_grid = 11
theta <- seq(0,1,length = n_grid)
p_y <- (1/n_grid)*(sum(dbinom(14, 20, prob = theta)))
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