TA: GINA OH/ gina.oh@wisc.edu / 1275C MSC
Office Hours: MW 2:30 PM - 3:45 PM / 1219 MSC

1 Inference for one population mean

1.1 Assumptions

1. The first sample $Y_1, \ldots, Y_n$ is a random sample of size n from $N(\mu, \sigma^2)$.
2. σ^2 is unknown

1.2 Hypothesis testing

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_A : \mu_1 \neq \mu_0 \text{ (two-tailed test)} \tag{1}$$
$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_A : \mu_1 > \mu_0 \text{ (one-tailed test)} \tag{2}$$

where μ_0 is a hypothesized value.

1.3 Test statistic

1. $\bar{Y}$
   (a) $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$
2. T test statistic
   (a) If $Z \sim N(0, 1)$, $V \sim \chi^2_v$, Z and V are independent, then
   $$\frac{Z}{\sqrt{V/v}} \sim T_v.$$
   (b) Thus,
   $$T = \frac{\bar{Y} - \mu}{\sqrt{S^2(n^{-1})}} \sim T_{n-1}.$$
   (c) Under $H_0 : \mu = \mu_0$,
   $$T = \frac{\bar{Y} - \mu_0}{\sqrt{S^2(n^{-1})}} \sim T_{n-1}.$$
3. Critical values at the significance level α

(a) $t_{n-1,\alpha}$ is the value such that $\mathbb{P}(T \geq t_{n-1,\alpha}) = \alpha$.

(b) In two-sided test, $H_0 : \mu_1 - \mu_2 = 0$ vs $H_A : \mu_1 - \mu_2 \neq 0$
Reject H_0 if $t > t_{n_1+n_2-2,\alpha/2}$ or $t < t_{n_1+n_2-2,\alpha/2}$.

(c) One-tailed test, $H_0 : \mu_1 - \mu_2 = 0$ vs $H_A : \mu_1 - \mu_2 > 0$
Reject H_0 if $t > t_{n_1+n_2-2,\alpha}$.

1.4 $(1-\alpha)$ Confidence Interval for $\mu_1 - \mu_2$

1. $\bar{Y}_1 - \bar{Y}_2$ and S_p^2 are random variables.

$$\mathbb{P}\left(-t_{n_1+n_2-2,\alpha/2} \leq \frac{\bar{Y}_1-\bar{Y}_2-(\mu_1-\mu_2)}{\sqrt{S_p^2(n_1^{-1}+n_2^{-1})}} \leq t_{n_1+n_2-2,\alpha/2}\right) = 1-\alpha$$

$$\implies \mathbb{P}\Big(\bar{Y}_1 - \bar{Y}_2 - t_{n_1+n_2-2,\alpha/2} \times \sqrt{S_p^2(n_1^{-1}+n_2^{-1})} \leq \mu_1 - \mu_2 \leq \bar{Y}_1 - \bar{Y}_2 + t_{n_1+n_2-2,\alpha/2} \times \sqrt{S_p^2(n_1^{-1}+n_2^{-1})}\Big) = 1-\alpha$$

2. Thus, a $(1-\alpha)$ CI for $\mu_1 - \mu_2$ is

$$\bar{y_1}-\bar{y_2}-t_{n_1+n_2-2,\alpha/2}\times\sqrt{s_p^2(n_1^{-1}+n_2^{-1})} \leq \mu_1-\mu_2 \leq \bar{y_1}-\bar{y_2}+t_{n_1+n_2-2,\alpha/2}\times\sqrt{s_p^2(n_1^{-1}+n_2^{-1})}.$$

1.5 Remarks on one-sample t test

The t procedures are exactly correct when the population is distributed exactly Normally. However, most real data are not exactly Normal. The t procedures are robust to small deviations from Normality—the results will not be affected too much. Factors that strongly matter:

- Random sampling. The sample must be an SRS from the population.
- Outliers and skewness. They strongly influence the mean and therefore the t procedures. However, their impact diminishes as the sample size gets larger because of the central limit theorem

Specifically:

- When n < 15, the data must be close to Normal and without outliers.
- When 15 < n < 40, mild skewness is acceptable but not outliers.
- When n < 40, the t statistic will be valid even with strong skewness.

2 Example 1 : Paired T test

An experiment was conducted to compare the effectiveness of mosquito repellent lotions from Company 1 and Company 2. In the experiment, a total of 8 persons took lotion 1 on their left arms and took lotion 2 on their right arms. The data, the number of mosquito bites during 2 hours experiment, are as below:

Person	1	2	3	4	5	6	7	8
Left arm	37	40	11	29	99	71	6	31
Right arm	41	44	21	105	108	170	15	29

Paired T test is appropriate for these data.

```
left  <- c(37, 40, 11,  29,  99,  71,  6, 31)
right <- c(41, 44, 21, 105, 108, 170, 15, 29)

stem(left-right)
hist(left-right)
boxplot(left-right)

t.test(left, right, paired=T)
t.test(left, right, paired=T, alternative = "two.sided")
```

3 Example 2 : Drinking Water

Trace metals in drinking water affect the flavor and an unusually high concentration can pose a health hazard. Ten pairs of data were taken measuring zinc concentration in bottom water and surface water. The data is on the file (zincconc.txt).

(a) State an appropriate H_0 and H_A. Describe in words the parameter(s) that appear in your hypothesis.

(b) Make a graphical check for outliers or strong skewness in the data that you will use in your statistical test, and report your conclusions on the validity of the test.

(c) Carry out a test. Draw your conclusion in the context of the study.

(d) Give a 80% confidence interval for the mean difference.

```r
zinc = read.csv("zinc.csv")

bottom = zinc[,2]
surface = zinc[,3]
t.test(bottom, surface, paired=T)
t.test(bottom, surface, paired=T, alternative = "greater")
t.test(bottom, surface, paired=T,conf.level = 0.8)
```

4 Example 3

Do piano lessons improve the spatial-temporal reasoning of preschool children? Neurobiological arguments suggest that this may be true. A study designed to test this hypothesis measured the spatial-temporal reasoning of 34 preschool children before and after six months of piano lessons. The changes in the reasoning scores are saved in the file 'piano.csv'.
(a) Display the data and summarize the distribution.
(b) Find the mean, the standard deviation, and the standard error of the mean.
(c) Give a 95% confidence interval for the mean improvement in reasoning scores.
(d) Test the null hypothesis that there is no improvement versus the alternative suggested by the neurobiological arguments. State the hypotheses, and give the test statistic with degrees of freedom and the P-value. What do you conclude? From your answer to part (c) of the previous exercise, what can be concluded from this significance test?

Answers: a) The distribution is left skewed

```
The decimal point is at the |

-2 | 000
-0 | 0
0 | 000
2 | 00000000
4 | 000000000
6 | 00000000
8 | 00
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b) $\bar{x} = 3.618$, s =3.055, se is 0.524. c) $(2.55, 4.68)$.

d) The hypotheses are $H_0 : \mu = 0$ and $H_a : \mu > 0$, where μ represents the average improvement in scores over six months for preschool children. t = 6.90 with 33 df and P-value < 0.0005. There is extremely strong evidence that the scores improved over six months, which is in agreement with the confidence interval (2.55, 4.68) obtained in Exercise 7.34.

R code:

```
rm(list=ls(all=TRUE))
data <- read.csv('piano.csv', header=T); View(data)
attach(data)
stem(Reasoning)
mean(Reasoning)
sd(Reasoning)
sd(Reasoning)/sqrt(length(Reasoning))
t.test(Reasoning)

> t.test(Reasoning, alternative='greater')

One Sample t-test

data:  Reasoning
t = 6.9044, df = 33, p-value = 3.46e-08
alternative hypothesis: true mean is greater than 0
95 percent confidence interval:
2.730915      Inf
sample estimates:
mean of x
3.617647
```