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# ELG 6367 – Fundamentals of Antenna Engineering

## Assignment#3 Winter 2024

**Hand-Out Date** : 28th February 2024
**Due Date** : 20th March 2024

Your assignment report must be uploaded on Brightspace as a single PDF document before or on 20th March 2024.

If you wish to meet with me to discuss this assignment you should make an appointment with me, but only after you have attempted it yourself. Bring preliminary results with you to such a meeting.

Please attach a copy of this page to your solution, with your details filled in, and the declaration signed and dated.

Lastname: ……………………………………..

First Name : …………………………………...

Student Number : ……………………………..

*I hereby confirm that this is entirely* ***my own work****. I did not discuss my solution with anyone else other than the ELG 6367 instructor, and I did not copy it from someone else's solution.*

Signature : ………………………………….

Date : ……………………………………….

# Problem 3.1 [50]

"The helical beam antenna is one of the most unique and widely used antennas ever invented. I discovered it only because I tried what some might have called a foolish experiment."
*John D. Kraus, "Big Ear"*
(Cygnus-Quasar Books, 1976)

### *A. Introduction to Helix Antennas*

■ The helix antenna[1] shown in Fig.3.1-1 was invented by Kraus in 1947.

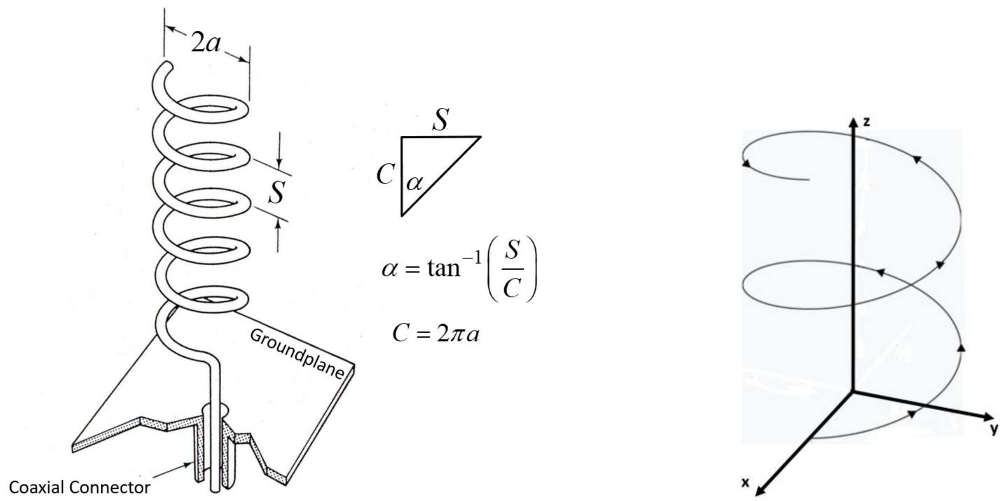



**Fig.3.1-1 : Helix antenna geometry (left) and defining curve (right).**

■ The helical curve lies on a cylinder of radius $a$, and points on the curve are given by coordinates

$$(x, y, z) = \left( a\cos\tau, a\sin\tau, \frac{S}{2\pi}\tau \right) \tag{3.1-1}$$

where $\tau$ is a parameter "along" the curve. Each time $\tau$ increases by $2\pi$, the $x$ and $y$ coordinates return to their original values, whereas $z$ has increased by $S$, the spacing between each turn. The circumference of the helix is $C = 2\pi a$. The pitch angle is $\alpha = \tan^{-1}(S/C)$; this is the angle formed by a line tangent to the helix curve and a plane perpendicular to the helix axis (in this case the xy-plane). If we let $N_{turn}$ be the number of turns used (with $N_{turn}$ not necessarily an integer), then the length of the helix is $L = N_{turn}S$. The helix is not simply an arbitrary tightly wound wire coil; it is the very specific curve define above.

■ The helix antenna has two useful modes of operation, the *axial mode* and the *normal*[2] *mode*.

[1] J.D.Kraus, *Antennas* (McGraw-Hill, 1950)
[2] The word *normal* is used here in the sense of *perpendicular* rather than *usual*.

■ The word "axial" in axial mode is used to signify that the radiation pattern maximum is in the direction of the axis of the helix (in this case it is coincident with the z-axis). In order to excite this axial mode the circumference $C$ and spacing between turns $S$ must be an appreciable fraction of a wavelength ($\lambda$). In order to obtain circular polarization in the main lobe the circumference must be in the range

$$\frac{3}{4} < \frac{C}{\lambda} < \frac{4}{3} \tag{3.1-2}$$

with $C/\lambda = 1$ and $S/\lambda \approx 0.25$ being near optimum. If $f_u$ is the upper frequency of operation and $f_\ell$ the lower frequency, over this band, then the above constraint can be written as

$$\frac{f_u}{f_\ell} = \frac{\lambda_\ell}{\lambda_u} = \frac{4/3}{3/4} = 1.78 \tag{3.1-3}$$

The pitch angle for axial mode circularly-polarized operation should be in the range $12^\circ < \alpha < 18^\circ$, with $14^\circ$ near optimum. The input impedance of a helix operating in the axial mode is found to be nearly resistive with values between 100Ω and 200Ω. In the case of an axial mode helix the directivity increases with $N_{turn}$. In the words of its inventor, the axial mode helix "is noncritical to an unprecedented degree"[3]. It represents the first antenna which was of its very nature circularly polarised. If the helix is wound in a left-hand (right-hand) sense, then in the axial mode its polarization in the direction of the pattern maximum will be of a left-hand (right-hand) sense. In the axial mode the current distribution along the helix wire has the form of a travelling wave, with a propagation constant different from the free space wavenumber $k$, and is classified as "endfire" radiation.

■ The word "normal" is used to signify that the radiation pattern maximum is normal (perpendicular) to the axis of the helix; this is classified as broadside radiation. The normal mode prevails when the helix dimensions are relatively small in terms of the wavelength ($C/\lambda << 1$ and $N_{turn}S << 1$) and it radiates as if it were a straight wire monopole with continuous inductive loading (and so can be made electrically shorter than an ordinary monopole).

### *B. Assignment Task*

A particular axial mode helix antenna, operating at 3.5 GHz, is shown in Fig.3.1-2, along with 3D colour plots of its normalized radiation intensity. The *attached spreadsheet* gives the complex fields $F_\theta(\theta,\phi)$ and $F_\phi(\theta,\phi)$ of this helix antenna as a function of $\theta$ in the pattern cut $\phi = 0^\circ$. Use this data to do the following:

**(a).** Give a rectangular plot of the normalized radiation intensity (in dB) in this pattern cut over the angular range $-180^\circ \le \theta \le 180^\circ$. **[10]**

**(b).** Give a rectangular plot of the signed AR (not in dB) versus $\theta$ in this pattern cut over the angular range $-180^\circ \le \theta \le 180^\circ$. Comment on what it tells you about the antenna's state of polarization. **[10]**

**(c).** Give a rectangular plot of $|AR|$ (in dB) versus $\theta$ in this pattern cut over the angular range $-180^\circ \le \theta \le 180^\circ$. **[5]**

[3] J.D.Kraus, "Antennas since Hertz and Marconi", IEEE Trans. Antennas & Propagation, Vol.AP-33, No.2, pp.131-137, Feb.1985.

**(d).** Are you able to determine the directivity from this single pattern cut? Explain. **[2]**

**(e).** If we assume that the pattern is rotationally symmetric (same for all $\phi$), determine a fairly accurate estimate of the maximum directivity of the helix antenna (in dBi). [You will need to do, and show, some analytical work before being able to calculate this using the data in the spreadsheet. **Do not use** the expressions derived in the examples following Section 2.8.2 of Chapter 2.] **[20]**

**(f).** Are you able to determine the radiation efficiency? Explain. **[3]**

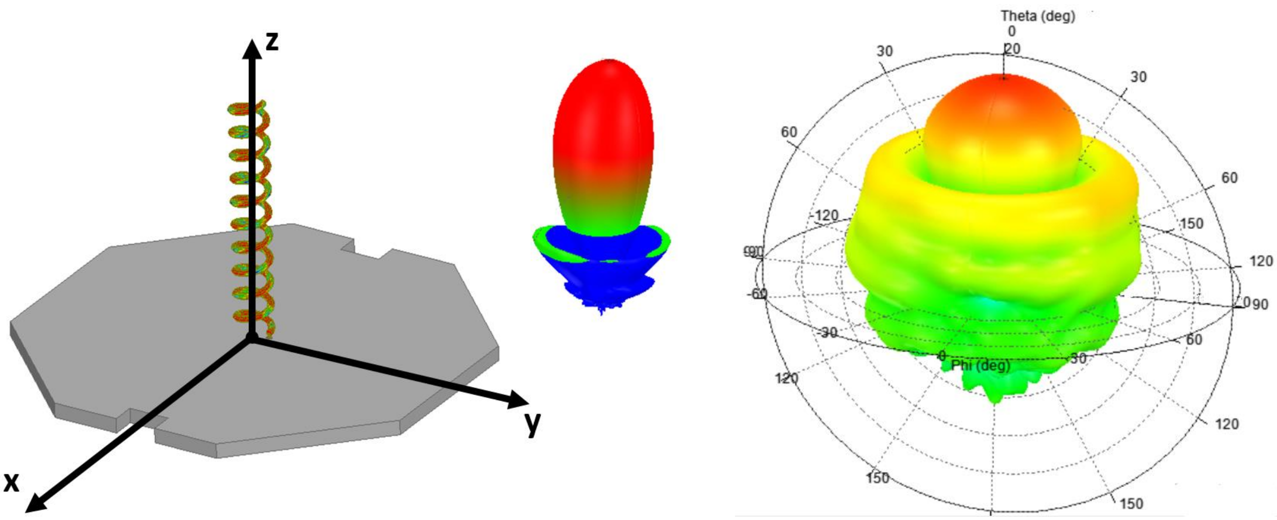



**Fig.3.1-2 : Specific axial mode helix antenna geometry (left) and its normalized radiation intensity at 3.5 GHz as non-dB values (centre) and in dB (right).**

## Problem 3.2 [60]

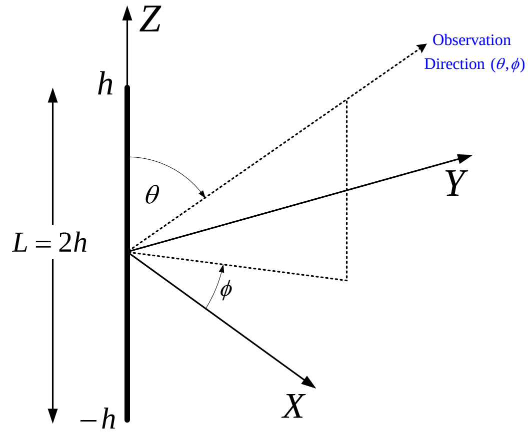



**Fig.3.2-1 : Straight filamentary dipole**

Assume that the straight filamentary antenna shown in Fig.3.2-1 has a current distribution[4]

$$I(z) = I_o e^{jbz} \quad -h \le z \le h \tag{3.2-1}$$

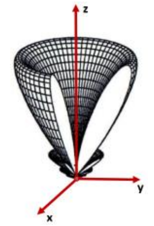
with $I_o$ some complex constant, and real number $b$ the phase constant (in radians/metre) of the wave travelling along the filament. The phase velocity of the wave on the filament is $v_p = \omega / b$, whereas that of a wave in free space is $c = \omega / b$, and so

$$v_p = \frac{k}{b} c \tag{3.2-2}$$

If $k/b > 1$ we have $v_p > c$, and it is said to be a "fast wave". Otherwise it is a "slow wave". The helix structure in Problem 3.1 acts as a slow-wave structure. If $b = 0$ the filament simply has a uniform amplitude distribution.

**(a).** Show that for the filamentary distribution (3.2-1) we have **[10]**

$$F_\theta(\theta,\phi) = \frac{j\eta k L I_o}{4\pi} \sin\theta \frac{\sin\left[kh\left(\cos\theta - \frac{b}{k}\right)\right]}{kh\left(\cos\theta - \frac{b}{k}\right)} \qquad F_\phi(\theta,\phi) = 0 \tag{3.2-3}$$




[4] It would not be possible in practice to achieve a current distribution like this on a simple straight filament. However, what we are doing here is simply using the straight filament as a vehicle to recognize some antenna principles.

**(b).** Determine an expression for $D(\theta,\phi)$, valid for any value of $b$, given that for the $F_\theta(\theta,\phi)$ and $F_\phi(\theta,\phi)$ in expression (3.2-3), it is possible to show that[5] **[8]**

$$P_{rad} = \frac{1}{2\eta}\int_0^{2\pi}\int_0^{\pi}\left|\bar{F}(\theta,\phi)\right|^2 \sin\theta \; d\theta \, d\phi = \frac{\eta}{4\pi}\left|I_o\right|^2 \left\{1.415 + \ln\left(\frac{2kh}{\pi}\right) - Ci(4kh) + \frac{\sin(4kh)}{4kh}\right\} \tag{3.2-4}$$

■ Set $b = 0$

**(c).** The radiation intensity in this case always has a maximum in the $\theta = 90^\circ$ direction, irrespective of the value of $L/\lambda$. Use your just-derived expression to plot the maximum directivity $D_{\max}$ (in dBi) versus $L/\lambda$ over at least the range $0 \le L/\lambda \le 20$. Comment on your results. **[10]**

**(d).** Plot (in dB) the normalized radiation intensity $U_{norm}(\theta,0)$ versus $\theta$ (on a single rectangular plot) over the range $0 \le \theta \le 180^\circ$ for the following four $L/\lambda$ values : $L/\lambda = 5$, $L/\lambda = 10$, $L/\lambda = 20$, $L/\lambda = 40$. Comment on the trends in the beamwidths and the highest side lobe level. **[10]**

■ Consider $b \neq 0$

**(e).** Give an expression for the magnitude, and then the phase, of $I(z)$ as a function of $z$. **[2]**

**(f).** The pattern maximum will no longer occur in the $\theta = 90^\circ$ direction. Find a way to determine the direction $\theta = \theta_s$ of the pattern maximum (that is, of the radiation intensity) in terms of the value of $b$. You must discuss the procedure. Then plot $\theta_s$ (in degrees) versus $b$ for $0 \le \theta_s \le 90^\circ$. Also plot $D(\theta_s,0)$ in dBi versus $\theta_s$ (in degrees). Do this for at least two filament lengths in the range $6 \le L/\lambda \le 20$. **[20]**

[5] The function $Ci(x)$ has been identified in Chapter 3.

# Problem 3.3 [50]

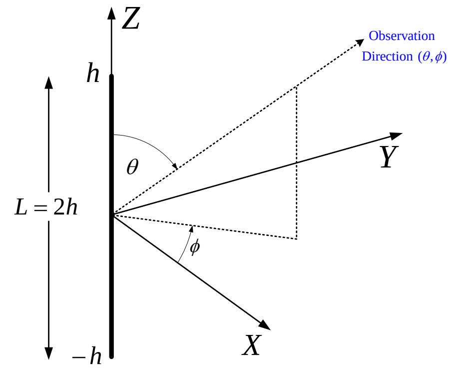



**Fig.3.3-1 : Straight filamentary dipole**

Now assume that the straight filamentary antenna shown in Fig.3.3-1 has a current distribution

$$I(z) = I_o e^{-jbz^2} = I_o e^{-j\Phi(z)} \quad -h \le z \le h \tag{3.3-1}$$

with

$$\Phi(z) = bz^2 \qquad -h \le z \le h \tag{3.3-2}$$

being the (quadratic) phase distribution along the filament. Clearly $\Phi(0) = 0$, and if its value at $z = \pm h$ is denoted by $\Phi_o$ then $b = \Phi_o / h^2$. We thus have

$$f_z(\theta,\phi) = \int_{-h}^{h} I_o e^{-jbz'^2} e^{jkz'\cos\theta} dz' = \int_{-h}^{h} I_o e^{-j\left[b\,z'^2 - k_z z'\right]} dz' \tag{3.3-3}$$

with $k_z = k\cos\theta$. Completing the square of the term in the exponent gives

$$bz'^2 - k_z z' = b\left(z' - \frac{k_z}{2b}\right)^2 + \frac{k_z^2}{4b} \tag{3.3-4}$$

Substitution of this into (3.3-3) then gives

$$f_z(\theta,\phi) = \int_{-h}^{h} I_o\, e^{-j\left[b\left(z' - \frac{k_z}{2b}\right)^2 + \frac{k_z^2}{4b}\right]} dz' = I_o\, e^{-j\frac{k_z^2}{4b}} \int_{-h}^{h} e^{-jb\left(z' - \frac{k_z}{2b}\right)^2} dz' \tag{3.3-5}$$

The integral in (3.3-5) cannot be found in closed form. It can be found numerically using some quadrature algorithm, but in this case it can also be manipulated so that it can be expressed in terms of known special functions. With this end in mind we let

$$b\left(z' - \frac{k_z}{2b}\right)^2 = \frac{\pi}{2}\tau^2 \qquad \Rightarrow \qquad \tau = \sqrt{\frac{2b}{\pi}}\left(z' - \frac{k_z}{2b}\right) \tag{3.3-6}$$

When $z' = -h$ we are at

$$\tau_1 = -\sqrt{\frac{2b}{\pi}}\left(h + \frac{k_z}{2b}\right) \tag{3.3-7}$$

whereas when $z' = h$ we are at

$$\tau_2 = \sqrt{\frac{2b}{\pi}}\left(h - \frac{k_z}{2b}\right) \tag{3.3-8}$$

Furthermore,

$$dz' = \sqrt{\frac{\pi}{2b}}d\tau \tag{3.3-9}$$

and so we can write

$$f_z(\theta,\phi) = I_o\sqrt{\frac{\pi}{2b}}e^{-j\frac{k_z^2}{4b}}\int_{\tau_1}^{\tau_2} e^{-j\frac{\pi}{2}\tau^2}d\tau \tag{3.3-10}$$

There are two special functions[6] known as the cosine and sine Fresnel integrals, defined by

$$C(\tau) = \int_0^{\tau}\cos\left(\frac{\pi}{2}\tau^2\right)d\tau \tag{3.3-11}$$

and

$$S(\tau) = \int_0^{\tau}\sin\left(\frac{\pi}{2}\tau^2\right)d\tau \tag{3.3-12}$$

It then follows that we have

$$f_z(\theta,\phi) = I_o\sqrt{\frac{\pi}{2b}}e^{-j\frac{k_z^2}{4b}}\left\{\left[C(\tau_2)-C(\tau_1)\right] - j\left[S(\tau_2)-S(\tau_1)\right]\right\} \tag{3.3-13}$$

►***Check the above mathematical manipulations and inform everyone of any errors***◄

We want to use the above analysis to demonstrate (using plots and associated descriptions) to someone what the quadratic phase behaviour is, what it does to the radiation patterns, **and** what it does to the maximum directivity[7]. Do this for a range of $\Phi_o$, up to at least a value where it starts distorting the pattern to a point where the main lobe starts being deformed. Derive any additional expressions, and select the plots, that you need to do this in a thorough manner.

[6] Available in Matlab, amongst other places.
[7] In this case of a current distribution with a quadratic phase behaviour there does not appear to be any convenient expression for $P_{rad}$ in terms of the easily accessible special functions that we saw in Problem 3.2. You can use numerical integration (in which case you **must** use the approach described in the addendum) or material from Section 2.10 of Chapter 2.

## Appendix : Numerical Integration

Suppose we have a smooth function $g(z)$ of a real variable $z$, and wish to determine the value of its integral between the limits z = *a* and z = *b*, namely

$$\text{Int} = \int_a^b g(z)dz \tag{1}$$

In engineering the situation often arises where it is not possible to do this in closed-form. Then we have to resort to numerical integration. This can be done using algorithms referred to as “quadrature rules”. The first step is to transform $g(z)$ into the new function

$$f(\xi) = \frac{b-a}{2} g\left\{\frac{b-a}{2}\xi + \frac{b+a}{2}\right\} \tag{2}$$

and then rewrite (1) in the form

$$\text{Int} = \int_{-1}^{1} f(\xi)d\xi \tag{3}$$

This integral is then approximated by the sum

$$\text{Int} \;=\; \int_{-1}^{1} f(\xi)d\xi \;\approx\; \sum_{i=1}^{N} w_i f(\xi_i) \tag{4}$$

where the distinct $\xi_i$ are symmetrically located points in the interval $-1 \le \xi \le 1$. The $w_i$ are referred to as the *weights*, and the $\xi_i$ as the *nodes*[8] where the value of the function is to be evaluated. The $w_i$ and $\xi_i$ are independent of $f(\xi)$. The expression (4) is called a *numerical quadrature rule or formula*, or *numerical integration formula*. Integer $N$ is the order of the quadrature algorithm. What they in effect do (at least in classical quadrature) is to approximate the integrand by a polynomial and then integrate the polynomial. We do not actually perform these steps each time we use a quadrature formula of course; it is already "built in".

[8] The words *node*, *sampling point*, *station*, *quadrature point* and *abscissa* are used synonymously.

Example : Suppose we wish to evaluate the integral $\text{Int} = \int_{z_\ell}^{z_u} z\, e^{-jkz} dz$ . In this case $g(z) = z\, e^{-jkz}$ . The integral transforms to $\text{Int} = \int_{-1}^{1} f(\xi)\, d\xi$ where, using (2) with $a = z_\ell$ and $b = z_u$ , we have

$$f(\xi) = \frac{z_u - z_\ell}{2}\, g\left\{\left(\frac{z_u - z_\ell}{2}\right)\xi + \left(\frac{z_u + z_\ell}{2}\right)\right\}$$

This can be rewritten

$$f(\xi) = \frac{z_u - z_\ell}{2}\left[\left(\frac{z_u - z_\ell}{2}\right)\xi + \left(\frac{z_u + z_\ell}{2}\right)\right] e^{-jk\left\{\left(\frac{z_u - z_\ell}{2}\right)\xi + \left(\frac{z_u + z_\ell}{2}\right)\right\}}$$

Then we use (4), as per the tabulated information below. These cannot be used blindly. One has to examine how rapidly the real and imaginary parts of the integrand vary over the interval of integration and then determine what order (value of N) of integration one needs to use. Better still, divide the interval of integration up into several sub-intervals, apply the quadrature rule over each sub-interval, and then sum the results for each sub-interval. $\blacksquare$

**Nodes and weights for N-point Gauss-Legendre quadrature for integration interval [-1,1].**

| Points | Weighting Factors | Nodes (Points at Which Function Value is Sampled) |
|---|---|---|
| 2 | $w_1 = 1.000000000$ | $\xi_1 = -0.577350269$ |
| | $w_2 = 1.000000000$ | $\xi_2 = 0.577350269$ |
| 3 | $w_1 = 0.555555556$ | $\xi_1 = -0.774596669$ |
| | $w_2 = 0.888888889$ | $\xi_2 = 0.000000000$ |
| | $w_3 = 0.555555556$ | $\xi_3 = 0.774596669$ |
| 4 | $w_1 = 0.347854845$ | $\xi_1 = -0.861136312$ |
| | $w_2 = 0.652145155$ | $\xi_2 = -0.339981044$ |
| | $w_3 = 0.652145155$ | $\xi_3 = 0.339981044$ |
| | $w_4 = 0.347854845$ | $\xi_4 = 0.861136312$ |

**Nodes and weights for N-point Gauss-Legendre quadrature for integration interval [-1,1].**

| | | |
|---|---|---|
| 5 | $w_1 = 0.236926885$<br>$w_2 = 0.478628670$<br>$w_3 = 0.568888889$<br>$w_4 = 0.478628670$<br>$w_5 = 0.236926885$ | $\xi_1 = -0.906179846$<br>$\xi_2 = -0.538469310$<br>$\xi_3 = \ \ 0.000000000$<br>$\xi_4 = \ \ 0.538469310$<br>$\xi_5 = \ \ 0.906179846$ |
| 6 | $w_1 = 0.171324492$<br>$w_2 = 0.360761573$<br>$w_3 = 0.467913935$<br>$w_4 = 0.467913935$<br>$w_5 = 0.360761573$<br>$w_6 = 0.171324492$ | $\xi_1 = -0.932469514$<br>$\xi_2 = -0.661209386$<br>$\xi_3 = -0.238619186$<br>$\xi_4 = \ \ 0.238619186$<br>$\xi_5 = \ \ 0.661209386$<br>$\xi_6 = \ \ 0.932469514$ |
| 7 | $w_1 = 0.12948\ 49661\ 68870$<br>$w_2 = 0.27970\ 53914\ 89277$<br>$w_3 = 0.38183\ 00505\ 05119$<br>$w_4 = 0.41795\ 91836\ 73469$<br>$w_5 = 0.38183\ 00505\ 05119$<br>$w_6 = 0.27970\ 53914\ 89277$<br>$w_7 = 0.12948\ 49661\ 68870$ | $\xi_1 = -0.94910\ 79123\ 42759$<br>$\xi_2 = -0.74153\ 11855\ 99394$<br>$\xi_3 = -0.40584\ 51513\ 77397$<br>$\xi_4 = \ \ 0.00000\ 00000\ 00000$<br>$\xi_5 = \ \ 0.40584\ 51513\ 77397$<br>$\xi_6 = \ \ 0.74153\ 11855\ 99394$<br>$\xi_1 = 0.94910\ 79123\ 42759$ |

**Some Possibly Useful Mathematical Handbook Type Information**

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \sin^3 ax\,dx = -\frac{\cos ax}{a} + \frac{\cos^3 ax}{3a}$$

$$\int \cos^2 ax \sin^2 ax\; dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \qquad \int \cos^n ax \sin ax\; dx = -\frac{\cos^{n+1} ax}{(n+1)a} \qquad n \neq -1$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \qquad \int \sin ax\,dx = -\frac{\cos ax}{a} \qquad \int_{-b}^{b} e^{ja\xi} d\xi = 2b\ \mathrm{sinc}(ab)$$