① (again) (ugh)

Assignment 2

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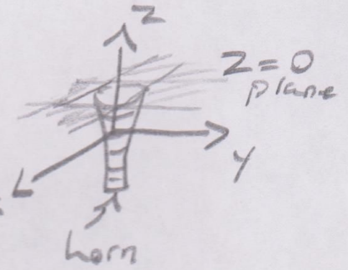
yes this is my work

Assume this →



$$\tilde{E}_{F\theta} = C_e(\theta)\sin\phi$$

$$\tilde{E}_{F\phi} = C_h(\theta)\cos\phi$$

Steps
I) find U
II) find $\langle U \rangle$
III) find D
IV) $\max(D)$

a) $C_e = \cos^{q_e}\theta$ for $\theta \in [0, \pi/2]$

$C_h = \cos^{q_h}\theta$

0 otherwise

Find $D(\theta,\phi)$, $\underset{\theta,\phi}{\arg\max}\{D(\theta,\phi)\} = D_{max} = D(\theta_0,\phi_0)$

$$\bar{E}_F = \begin{bmatrix} 0 \\ C_e\sin\phi \\ C_h\cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$U = \frac{\bar{E}_F \bar{E}_F^*}{2\eta} = \frac{\begin{bmatrix} 0 \\ C_e\sin\phi \\ C_h\cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ C_e^*\sin\phi \\ C_h^*\cos\phi \end{bmatrix}}{2\eta} = \frac{|C_e|^2\sin^2\phi + |C_h|^2\cos^2\phi}{2\eta}$$

$$\langle U \rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} U\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{8\pi\eta}\left(\int_0^{2\pi}\int_0^{\pi/2} C_e^2\sin^2\phi\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi/2} C_h^2\cos^2\phi\sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{1}{8\pi\eta}\left(\int_0^{2\pi}\sin^2\phi\, d\phi\int_0^{\pi/2} C_e^2\sin\theta\, d\theta + \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^{\pi/2} C_h^2\sin\theta\, d\theta\right)$$

$c = \cos^n\theta$ for $\theta \in [0, \pi/2]$

$c^2 = \cos^{2n}\theta$ " "

$$\int_0^{\pi} c^2 \sin\theta \, d\theta = \int_0^{\pi/2} c^2 \sin\theta \, d\theta = \int_0^{\pi/2} \cos^{2n}\theta \sin\theta \, d\theta$$

$$\int \cos^{2n}\theta \sin\theta \, d\theta = -\frac{\cos^{2n+1}(\theta)}{2n+1}$$

$$\int_0^{\pi/2} \cos^{2n}\theta \sin\theta \, d\theta = + \left(\cancel{\frac{\cos^{2n+1}(\pi/2)}{2n+1}}^{\,0} + \frac{\cos^{2n+1}(0)}{2n+1}^{\,1} \right)$$

$$= \frac{1}{2n+1}$$

$$\langle u \rangle = \frac{1}{8\cancel{\pi}\eta} \left(\cancel{\pi} \cdot \frac{1}{2q_e+1} + \cancel{\pi} \cdot \frac{1}{2q_h+1} \right)$$

$$= \frac{1}{8\eta} \left(\frac{2(q_e+q_h)+2}{(2q_e+1)(2q_h+1)} \right)$$

$$= \frac{1}{4\eta} \left(\frac{q_e+q_h}{(2q_e+1)(2q_h+1)} \right)$$

$$D(\theta,\phi;\, q_e;\, q_h) = \frac{u}{\langle u \rangle}$$

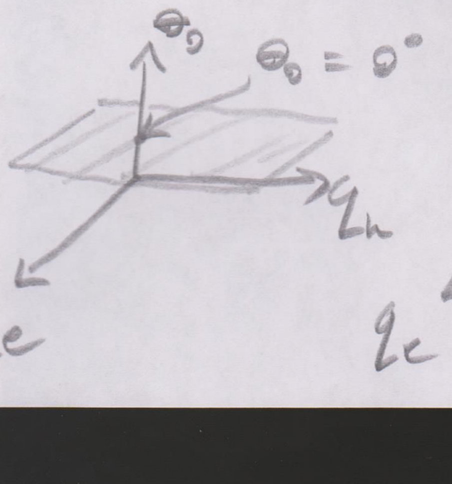
$$\doteq \frac{\frac{1}{2\cancel{\pi}}}{\frac{1}{4\cancel{\pi}}} \; \frac{\cos^{2q_e}\theta \sin^2\phi + \cos^{2q_h}\theta \cos^2\phi}{\frac{q_e + q_h}{(2q_e+1)(2q_h+1)}}$$

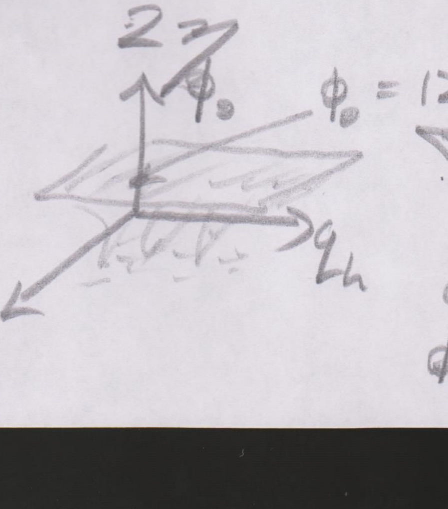
$$= \frac{2(2q_e+1)(2q_h+1)\left(\cos^{2q_e}\theta \sin^2\phi + \cos^{2q_h}\theta \cos^2\phi\right)}{q_e + q_h}$$

→ the dir of $\max\{D\}$ is the same dir of $\max\{u\}$

$$u_0 = u(\theta_0, \phi_0) \implies D_0 = D(\theta_0, \phi_0)$$

$$u = \cos^{2q_e}\theta \sin^2\phi + \cos^{2q_h}\theta \cos^2\phi$$



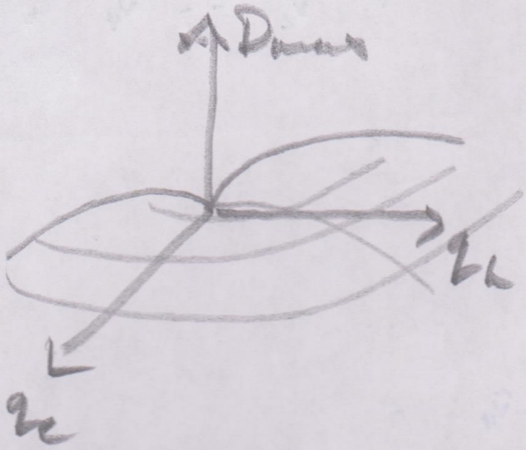


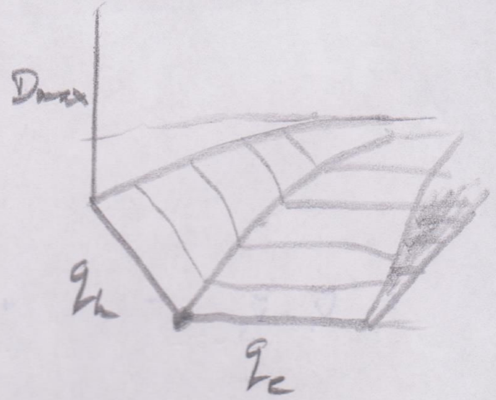
← probably an artifact when $\theta = 0$, ϕ = undefined/doesn't matter

③

→ max D is $\theta_0 = 0$, ϕ_0 = doesn't matter
so where the horn points

$$D_{max}(q_e, q_h) = \frac{2(2q_e+1)(2q_h+1)}{q_e + q_h}$$

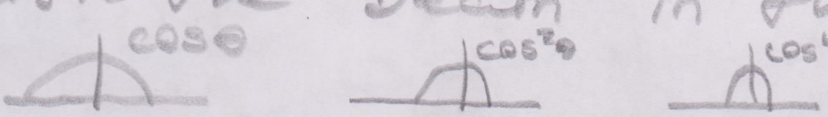




→ no info about beam width can be obtained from expression of D_{max} (unless there is an equivalent to Gain Bandwidth Product for antennas)

→ for U, q_e and q_h do squish the beam in the θ direction

4



b) $\hat{u}_{co} = \sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$

$$= \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$\hat{u}_{cr} = \cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} = \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

for orthogonality $\hat{u}_{co} \cdot \hat{u}_{cr} = 0$

$$\begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix} = 0 + \sin\phi\cos\phi - \cos\phi\sin\phi$$
$$= 0 \checkmark$$

c) $$\bar{E}_F = C_e\sin\phi\,\hat{\theta} + C_h\cos\phi\,\hat{\phi} = \begin{bmatrix} 0 \\ C_e\sin\phi \\ C_h\cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$= \tilde{E}_{Fco}\,\hat{u}_{co} + \tilde{E}_{Fcr}\,\hat{u}_{cr}$$

$$= \tilde{E}_{Fco}\left(\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right)$$
$$+ \tilde{E}_{Fcr}\left(\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right)$$

$$= \begin{bmatrix} 0 \\ \tilde{E}_{Fco}\sin\phi + \tilde{E}_{Fcr}\cos\phi \\ \tilde{E}_{Fco}\cos\phi - \tilde{E}_{Fcr}\sin\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$\begin{bmatrix} C_E \sin\phi \\ C_h \cos\phi \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \tilde{E}_{Fc\theta} \\ \tilde{E}_{Fcr} \end{bmatrix} = \begin{bmatrix} \tilde{E}_{F\theta} \\ \tilde{E}_{F\phi} \end{bmatrix}$$

$$\begin{vmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{vmatrix} = -\sin^2\phi - \cos^2\phi = -1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|A|}$$

$$\begin{bmatrix} \tilde{E}_{Fc\theta} \\ \tilde{E}_{Fcr} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \tilde{E}_{F\theta} \\ \tilde{E}_{F\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} C_E \sin\phi \\ C_h \cos\phi \end{bmatrix}$$

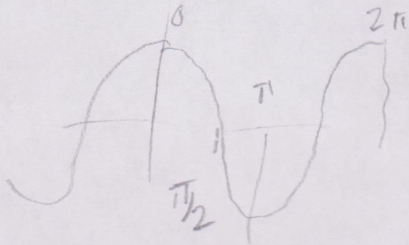
$$\tilde{E}_{Fc\theta} = C_E \sin^2\phi + C_h \cos^2\phi$$

$$\tilde{E}_{Fcr} = C_E \sin\phi\cos\phi - C_h \sin\phi\cos\phi$$

$$= \frac{(C_E - C_h)}{2} \sin 2\phi$$

d) $C_e = \cos^{q_e}\theta$

$C_h = \cos^{q_h}\theta$



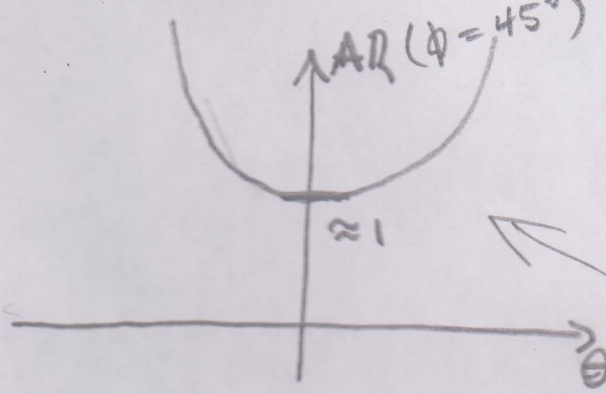
$$\tilde{E}_{Fco} = \cos^{q_e}\theta \sin^2\phi + \cos^{q_h}\theta \cos^2\phi$$

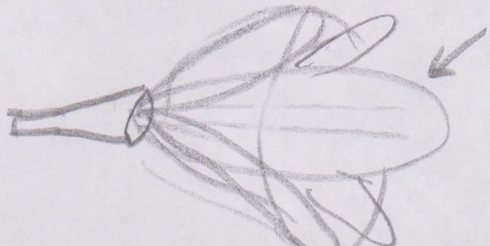
$$\tilde{E}_{Fcr} = \frac{\cos^{q_e}\theta - \cos^{q_h}\theta}{2} \sin 2\phi$$

Plot: $\dfrac{\tilde{E}_{Fco}(\theta,\phi_0)}{|\tilde{E}_{Fco}(\theta,\phi_0)|}$ and $\dfrac{\tilde{E}_{Fcr}(\theta,\phi_0)}{|\tilde{E}_{Fcr}(\theta,\phi_0)|}$

for $\underline{\phi_0} = [0°, 45°, 90°]$

$q_e = 3.2$, $q_h = 2.16$





my guess (to lazy to actually plot it)

you should have asked for this! →

e) $C_L = C_R = C$

$$\tilde{E}_{Fco} = C \sin^2\phi + C \cos^2\phi$$
$$= C(\sin^2\phi + \cos^2\phi)$$
$$= C$$

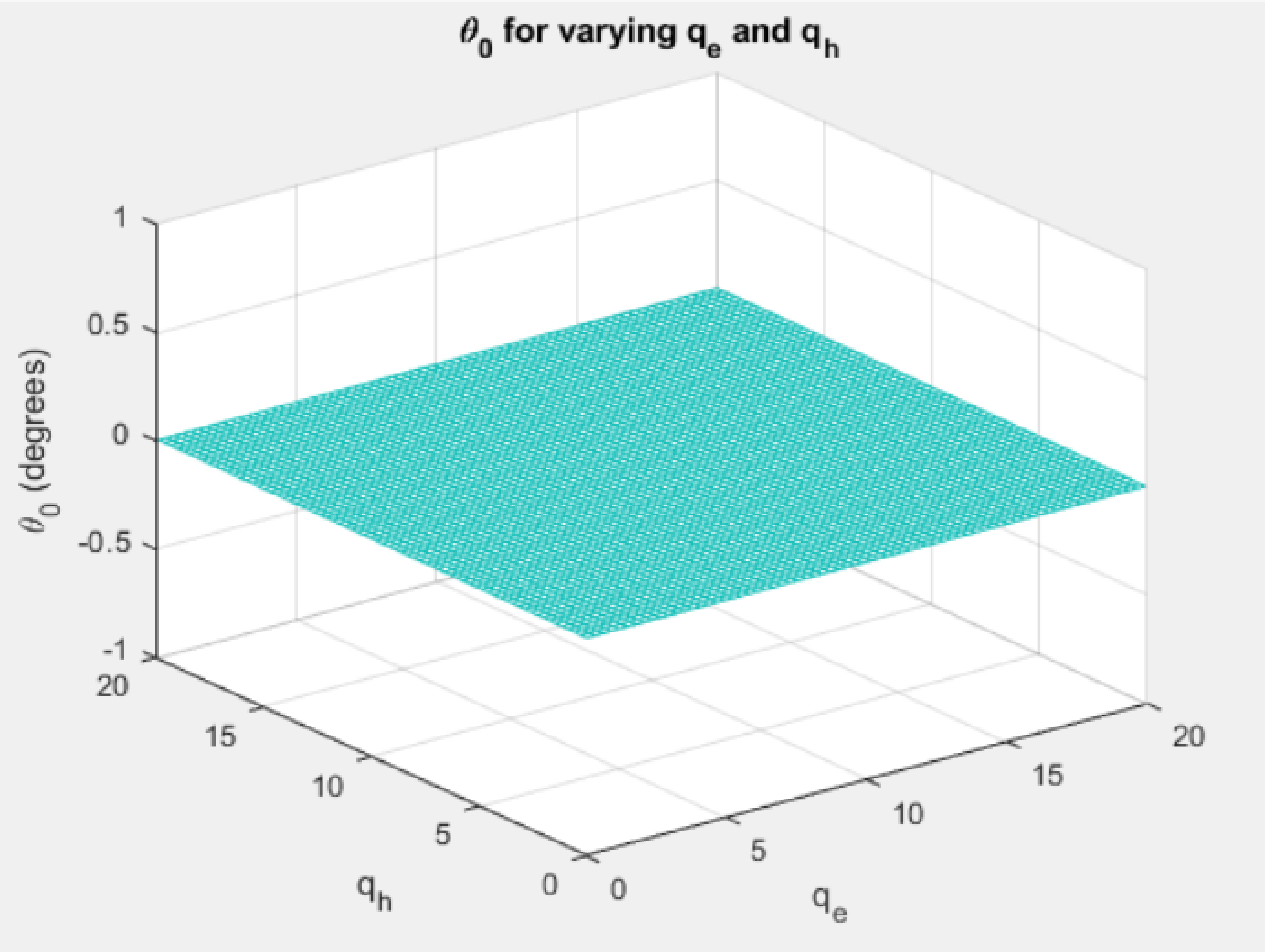
$$\tilde{E}_{Fcr} = \frac{C - C}{2} \sin 2\phi = 0$$

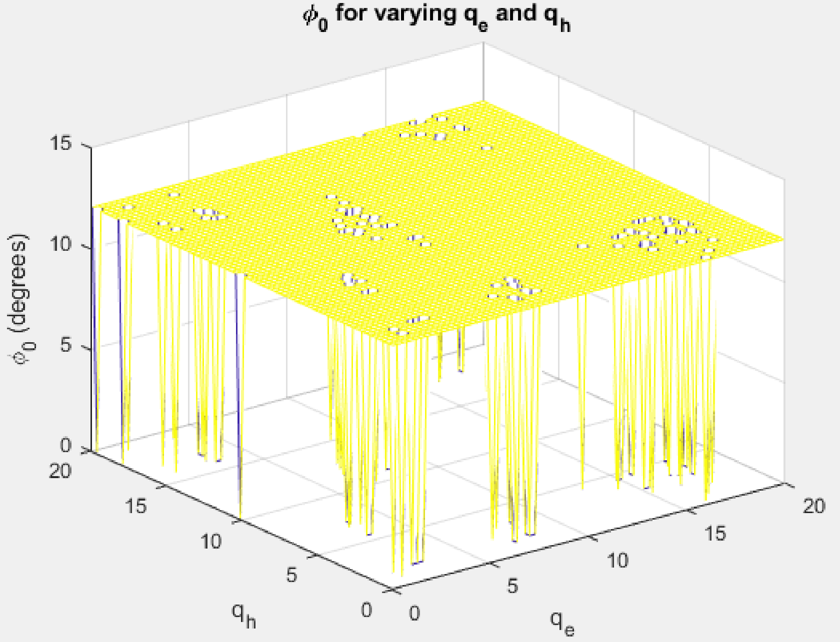
$$\bar{E}_F = C \hat{u}_{co} = C \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix}_{\{\hat{r}, \hat{\theta}, \hat{\phi}\}}$$

→ for this case antenna is perfectly circularly polarized in one sense and the other sense (cross pol) is zero

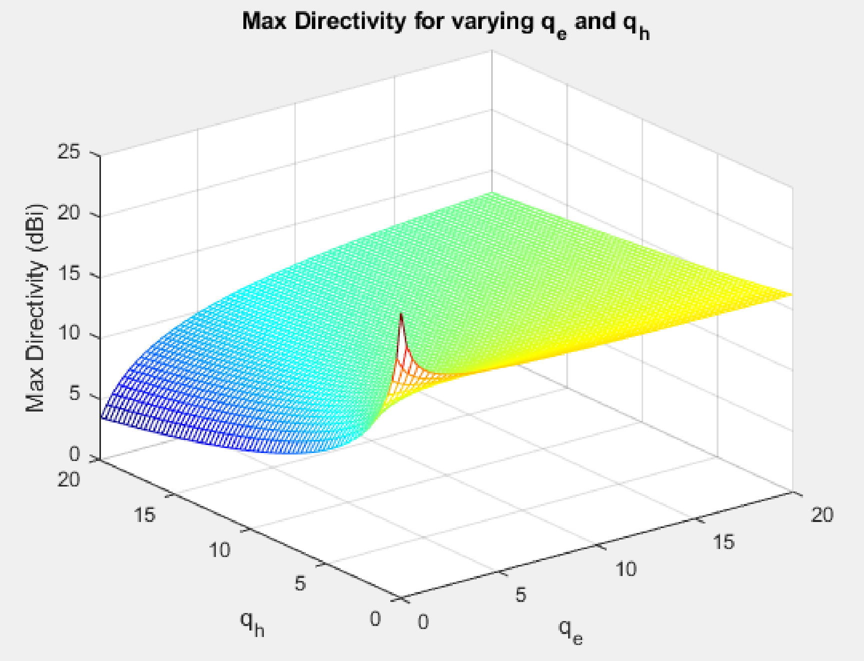
polarization of any antenna = some const · LHCP + some other const · RHCP

8

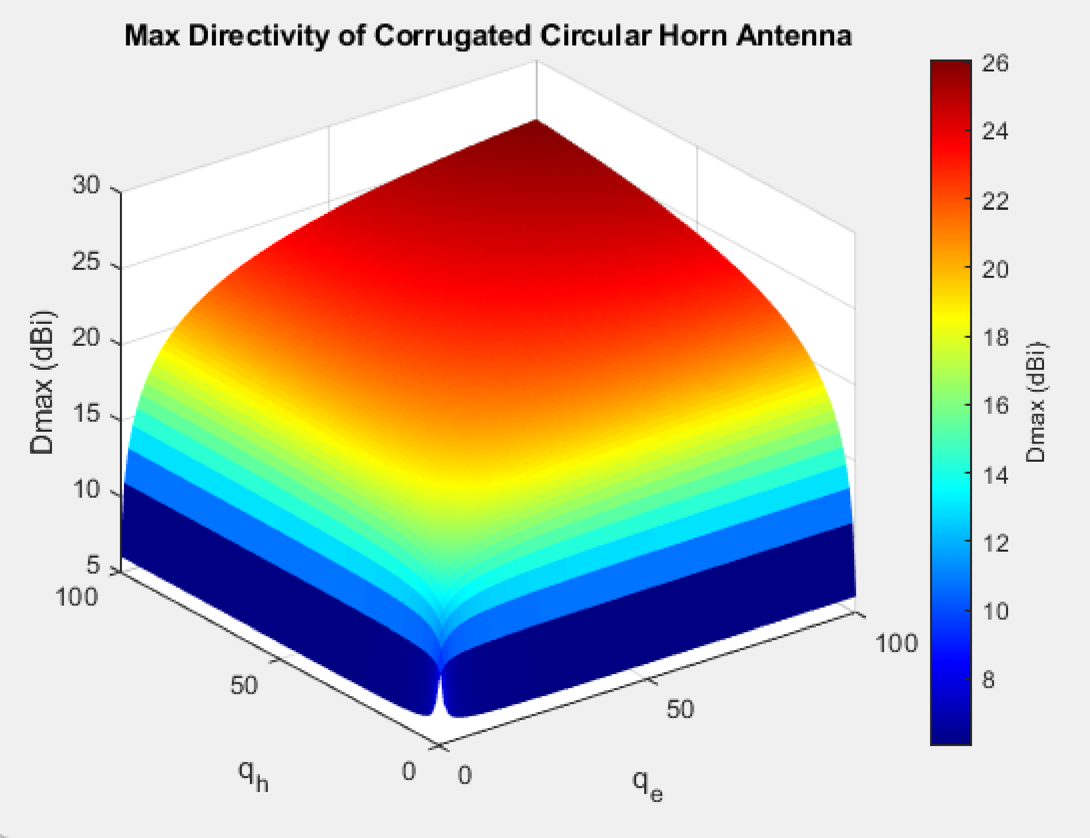

θ_0 for varying q_e and q_h
θ_0 (degrees)
1
0.5
0
-0.5
-1
20
15
10
5
0
q_h
0
5
10
15
20
q_e



ϕ_0 for varying q_e and q_h
15
10
5
0
ϕ_0 (degrees)
20
15
10
5
0
q_h
0
5
10
15
20
q_e

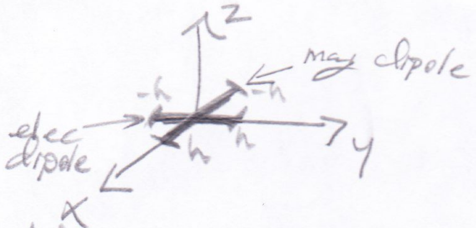


Max Directivity for varying q_e and q_h
Max Directivity (dBi)
0
5
10
15
20
25
q_h
q_e



Max Directivity of Corrugated Circular Horn Antenna
Dmax (dBi)
30
25
20
15
10
5
100
50
0
q_h
0
50
100
q_e
26
24
22
20
18
16
14
12
10
8
Dmax (dBi)

② $\frac{C_m}{C_e} = \eta$, Find $\bar{E}$, $C_m = 1$



$$\bar{E} = \bar{E}_F \frac{e^{-jkr}}{r}$$ *** Assume far field**

Method I : radiation integrals + duality

Method II : $\bar{A}$ then $\bar{E}$ + duality

Method III : $\bar{E}$ + change of basis + duality

Method I steps

i) elec dipole on y axis find $\bar{J}$

ii) find f_x , f_y , f_z

iii) find $\tilde{E}_\theta$, $\tilde{E}_\phi$

iv) elec dipole on x axis find $\bar{J}$

v) find f_x , f_y , f_z

vi) find $\tilde{E}_\theta$, $\tilde{E}_\phi$

vii) duality

viii) use maxwell curl to get back to $\bar{E}$

ix) sum for overall $\bar{E}$

Step i

$$\bar{J} = C_e\, \delta(x)\, \delta(y)\, \delta(z)\, \hat{y}$$

Step ii

$$J_x = J_z = 0 \Rightarrow f_x = \int 0 \, e^{jK\,\text{junk}} \, dx = f_z = 0$$

$$f_y = \lim_{h \to 0} \int_{-h}^{h} C_e \, \delta(y') \, e^{jK \sin\theta \sin\phi y'} \, dy'$$

$$= \frac{C_e}{jK \sin\theta \sin\phi} \cancel{e^{jK\,\text{junk} \cdot 0}}$$

$$C\mu_0 = \frac{\mu_0}{\sqrt{\varepsilon_0 \mu_0}} = \frac{\sqrt{\mu_0}}{\sqrt{\varepsilon_0}} = \eta$$

Step iii

$$\tilde{E}_{F\theta} = C_0 \left(0 - \cos\theta \sin\phi f_y + 0 \right) \qquad C_0 = \frac{j\omega\mu_0}{4\pi}$$

$$= \frac{-\cancel{j}\omega\mu_0}{4\pi} \cdot \frac{C_e}{\cancel{j}K \sin\theta \cancel{\sin\phi}} \cdot \cos\theta \cancel{\sin\phi}$$

$$= \frac{-\omega\mu_0 C_e}{4\pi K} \cot\theta$$

$$E_c = \frac{-\eta C_e}{4\pi} \cdot \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ \cot\theta \\ \csc\theta \cot\phi \end{bmatrix} \{\hat{r}, \hat{\theta}, \hat{\phi}\}$$

$$\tilde{E}_{F\phi} = C_0 \left(0 - \cos\phi f_y \right)$$

$$= \frac{-j\omega\mu_0}{4\pi} \cdot \frac{C_e}{jK \sin\theta \sin\phi} \cdot \cos\phi$$

$$= \frac{-\omega\mu_0 C_e}{4\pi K} \csc\theta \cot\phi$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad \lambda f = c \qquad K = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f \qquad \frac{\omega\mu_0}{K} = \frac{2\pi \frac{c}{\lambda} \mu_0}{\frac{2\pi}{\lambda}} = c\mu_0$$

10

Step iv

$$\bar{J} = C_e\,\delta(x)\delta(y)\delta(z)\hat{x}$$

Step v

$$f_y = f_z = 0$$

$$f_x = \lim_{h\to 0}\int_{-h}^{h} C_e\,\delta(x')\,e^{jK\sin\theta\cos\phi x'}\,dx'$$

$$= \frac{C_e}{jK\sin\theta\cos\phi}$$

Step vi

$$\tilde{E}_{F\theta} = C_o(-\cos\theta\cos\phi f_x - 0 + 0)$$

$$= \frac{-\cancel{j}\omega\mu_o}{4\pi}\cdot\cos\theta\,\cancel{\cos\phi}\cdot\frac{C_e}{\cancel{j}K\sin\theta\,\cancel{\cos\phi}}$$

$$= \frac{-\omega\mu_o C_e}{4\pi K}\cot\theta$$

$$\tilde{E}_{F\phi} = C_o(\sin\phi f_x - 0) = \frac{\cancel{j}\omega\mu_o}{4\pi}\cdot\sin\phi\cdot\frac{C_e}{\cancel{j}K\sin\theta\cos\phi}$$

$$= \frac{\omega\mu_o C_e}{4\pi K}\csc\theta\tan\phi$$

11

$$\bar{E}_2 = \frac{\omega \mu_0 C_e}{4\pi K} \cdot \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ -\cot\theta \\ \csc\theta \tan\phi \end{bmatrix}$$

$$\left\{ \hat{r}, \hat{\theta}, \hat{\phi} \right\}$$

Step vii

Duality: $C_e \to C_m$

$\eta \to \eta^{-1}$

$\mu_0 \to \varepsilon_0$

$\bar{E} \to \bar{H}$

$$\bar{H}_m = \frac{\omega \varepsilon_0 C_m}{4\pi K} \cdot \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ -\cot\theta \\ \csc\theta \tan\phi \end{bmatrix}$$

Step viii

$\nabla \times \bar{H} = j\omega\varepsilon_0 \bar{E}$ (free space/far field)

$$\nabla \times \bar{H} = \frac{1}{r} \begin{bmatrix} \csc\theta \left(\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial\phi} \right) \\ -\frac{\partial}{\partial r}(r H_\phi) \\ \frac{\partial}{\partial r}(r H_\theta) \end{bmatrix}$$

$$\nabla\times\bar{H} = \frac{\omega\varepsilon_0 C_m}{4\pi K r}\begin{bmatrix} \csc\theta\left(\cancelto{0}{\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{e^{-jkr}}{r}\csc\theta\tan\phi\right)} + \cancelto{0}{\frac{\partial}{\partial\phi}\left(\frac{e^{-jkr}}{r}\cot\theta\right)}\right) \\ -\frac{\partial}{\partial r}\left(\cancel{r}\,\frac{e^{-jkr}}{\cancel{r}}\csc\theta\tan\phi\right) \\ -\frac{\partial}{\partial r}\left(\cancel{r}\,\frac{e^{-jkr}}{\cancel{r}}\cot\theta\right) \end{bmatrix}$$

$$= \frac{\omega\varepsilon_0 C_m}{4\pi K r}\begin{bmatrix} 0 \\ jKe^{-jKr}\csc\theta\tan\phi \\ jKe^{-jKr}\cot\theta \end{bmatrix}$$

$$= \frac{j\omega\varepsilon_0 C_m}{4\pi}\cdot\frac{e^{-jKr}}{r}\begin{bmatrix} 0 \\ \csc\theta\tan\phi \\ \cot\theta \end{bmatrix}$$

$$= j\omega\varepsilon_0\bar{E}$$

$$\bar{E}_m = \frac{C_m}{4\pi}\cdot\frac{e^{-jKr}}{r}\begin{bmatrix} 0 \\ \csc\theta\tan\phi \\ \cot\theta \end{bmatrix}$$

Step ix

$$\bar{E} = \bar{E}_e + \bar{E}_m$$

$$= -\frac{\eta C_e}{4\pi} \cdot \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ \cot\theta \\ \csc\theta \cot\phi \end{bmatrix} + \frac{C_m}{4\pi} \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ \csc\theta \tan\phi \\ \cot\theta \end{bmatrix}$$

$$\frac{C_m}{C_e} = \eta \Rightarrow C_e \eta = C_m = 1$$

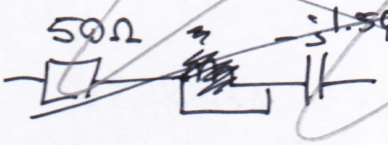
$$\bar{E} = \frac{e^{-jkr}}{4\pi r} \begin{bmatrix} 0 \\ \csc\theta \tan\phi - \cot\theta \\ \cot\theta - \csc\theta \cot\phi \end{bmatrix}_{\{\hat{r}, \hat{\theta}, \hat{\phi}\}}$$

$$\bar{E}_F = \frac{1}{4\pi} \begin{bmatrix} 0 \\ \csc\theta \tan\phi - \cot\theta \\ \cot\theta - \csc\theta \cdot \cot\phi \end{bmatrix}_{\{\hat{r}, \hat{\theta}, \hat{\phi}\}}$$

$$U = \frac{|E_F|^2}{2\eta} = \frac{1}{32\pi^2 \eta}\left((\csc\theta \tan\phi - \cot\theta)^2 + (\cot\theta - \csc\theta \cdot \cot\phi)^2 \right)$$

$$\cancel{= \frac{1}{16\pi^2 \eta}(\csc\theta \tan\phi - \cot\theta)^2}$$

~~Now, let's simplify this~~



$\frac{1(\cos)}{\cos}\frac{}{\sin}$ $\frac{\cos}{\sin}$

$\frac{1}{\sin}$ $\cancel{\frac{\sin}{\cos}}$

$$\cancel{U = \frac{1}{16\pi^2\eta}\left(\csc^2\theta\tan^2\theta - 2\csc\theta\cot\theta\right.}$$

$$U = \frac{1}{32\pi^2\eta}\left(\csc^2\theta\tan^2\phi - 2\csc\theta\tan\phi\cot\theta + \cot^2\theta + \cot^2\theta - 2\cot\theta\csc\theta\cot\phi + \csc^2\theta\cot^2\phi\right)$$

$$= \frac{1}{32\pi^2\eta}\left(\frac{1}{\sin^2\theta}\cdot\frac{\sin^2\phi}{\cos^2\phi} - 2\frac{1}{\sin\theta}\frac{\sin\phi}{\cos\phi}\frac{\cos\theta}{\sin\theta} + 2\frac{\cos^2\theta}{\sin^2\theta} - 2\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\phi}{\sin\phi} + \frac{1}{\sin^2\theta}\frac{\cos^2\phi}{\sin^2\phi}\right)$$

$$\cancel{= \frac{1}{32\pi^2\eta}\left(\frac{}{\sin^2\theta}\right.}$$

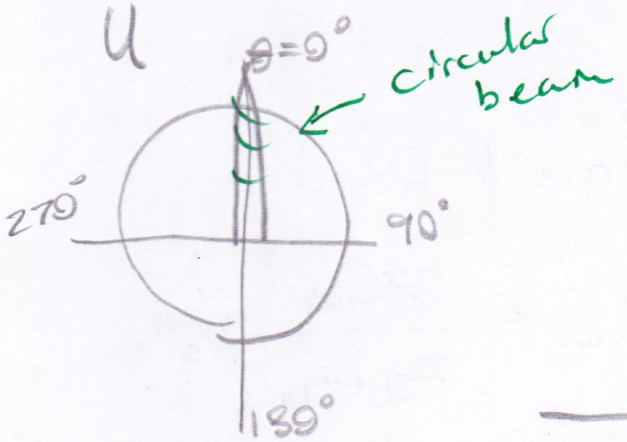
$$= \frac{1}{32\pi^2\eta}\left(2\cot^2\theta - 4\sec\phi\cot^2\theta\tan\phi + \csc^2\theta\left(\tan^2\phi + \cot^2\phi\right)\right)$$

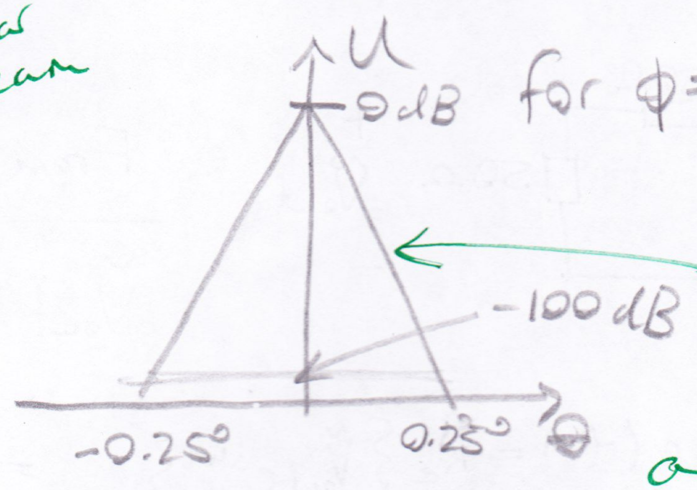
$$= \frac{1}{32\pi^2\eta}\left(2\cot^2\theta - 4\cot^2\theta\sec\theta\tan\phi + (1+\cot^2\theta)(\tan^2\phi + \csc^2\phi - 1)\right)$$

$$= \frac{1}{32\pi^2\eta}\left(\begin{array}{l}\cot^2\theta - 4\cot^2\theta\sec\theta\tan\phi + \tan^2\phi + \csc^2\phi - 1 \\ \quad + \cot^2\theta\tan^2\phi + \cot^2\theta\csc^2\phi - \cancel{\cot^2\theta}\end{array}\right)$$

I was going to check this by using one of the other methods

but I've spent too long on this assignment





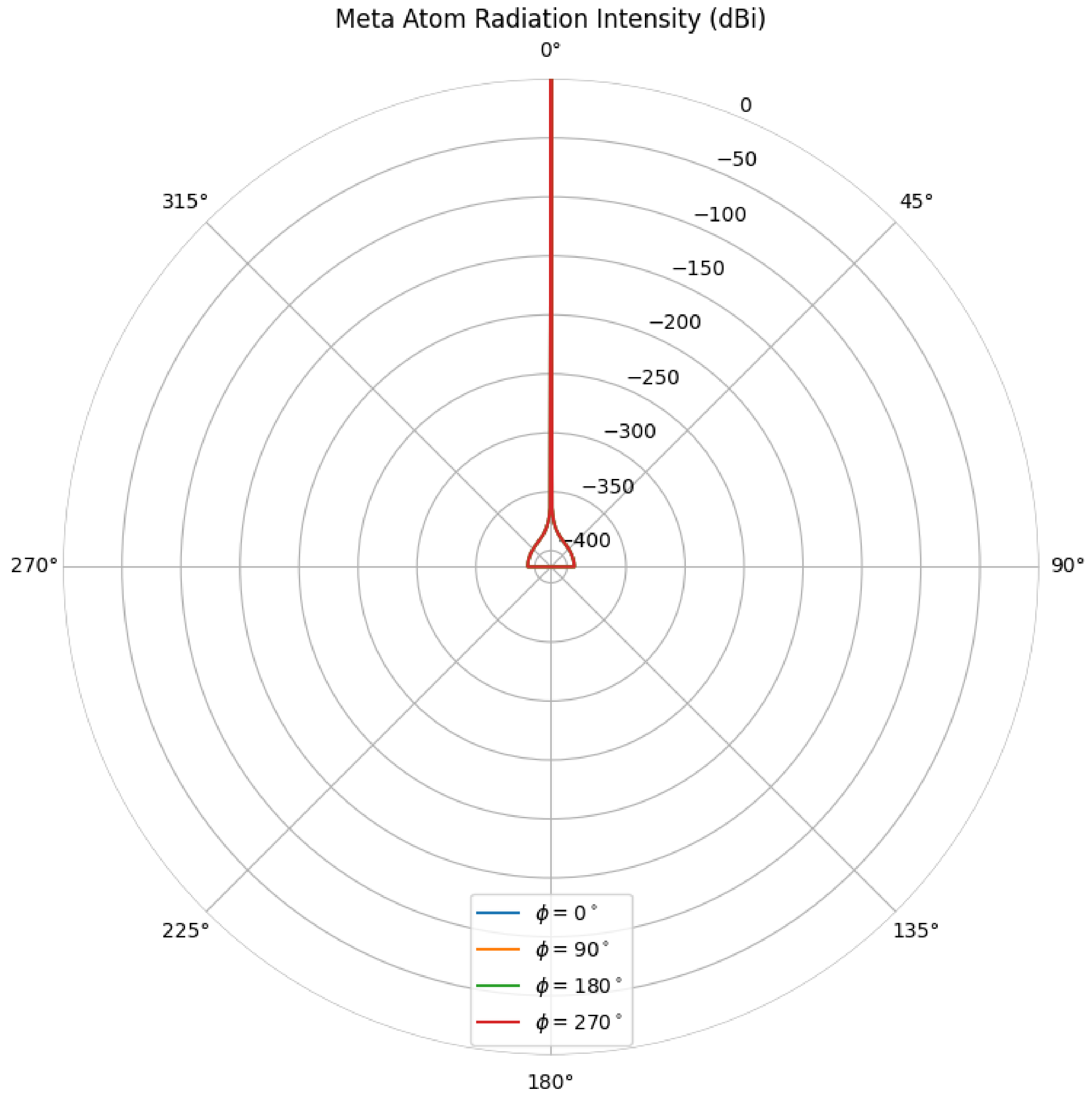
for $\phi = 0°, 90°, 180°, 270°$

~~laser pointer eq for~~ antenna eq to a laser pointer

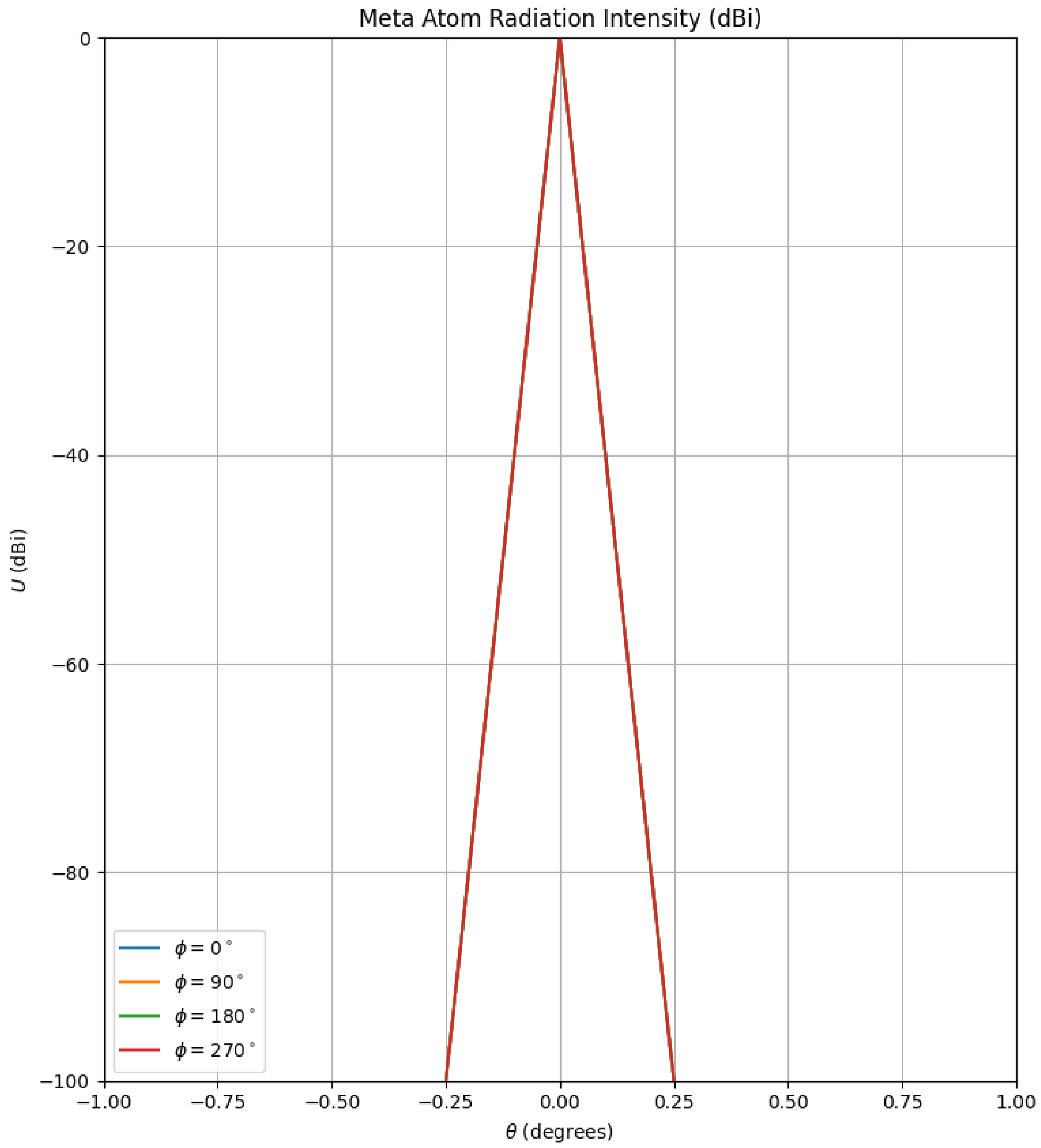
→ would have been nice to have the 1454 Chlavin paper... (See plots)

(no back lobe (= 0) → practically impossible for symmetric structure)

10



Meta Atom Radiation Intensity (dBi)
0°
45°
90°
135°
180°
225°
270°
315°
0
−50
−100
−150
−200
−250
−300
−350
−400
$\phi = 0^\circ$
$\phi = 90^\circ$
$\phi = 180^\circ$
$\phi = 270^\circ$



Meta Atom Radiation Intensity (dBi)
U (dBi)
0
−20
−40
−60
−80
−100
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
θ (degrees)
$\phi = 0^\circ$
$\phi = 90^\circ$
$\phi = 180^\circ$
$\phi = 270^\circ$