① (again) (ugh) Assignment 2

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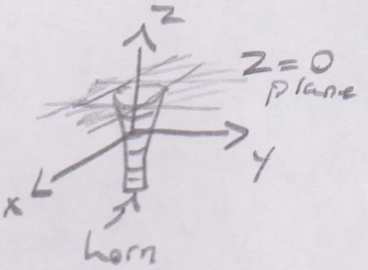
yes this is my work

Assume this →



$$\tilde{E}_{F\theta} = C_e(\theta)\sin\phi$$

$$\tilde{E}_{F\phi} = C_h(\theta)\cos\phi$$

a) $C_e = \cos^{q_e}\theta$ for $\theta \in [0, \pi/2]$

$C_h = \cos^{q_h}\theta$ 0 otherwise

Steps
I) find U
II) find $\langle U \rangle$
III) find D
IV) max(D)

Find $D(\theta,\phi)$, $\underset{\theta,\phi}{\arg\max}\{D(\theta,\phi)\} = D_{max} = D(\theta_0,\phi_0)$

$$\bar{E}_F = \begin{bmatrix} 0 \\ C_e\sin\phi \\ C_h\cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$U = \frac{\bar{E}_F \bar{E}_F^*}{2\eta} = \frac{\begin{bmatrix} 0 \\ C_e\sin\phi \\ C_h\cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ C_e^*\sin\phi \\ C_h^*\cos\phi \end{bmatrix}}{2\eta} = \frac{|C_e|^2\sin^2\phi + |C_h|^2\cos^2\phi}{2\eta}$$

$$\langle U \rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} U\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{8\pi\eta}\left(\int_0^{2\pi}\int_0^{\pi/2} C_e^2\sin^2\phi \sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi/2} C_h^2\cos^2\phi \sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{1}{8\pi\eta}\left(\int_0^{2\pi}\sin^2\phi\, d\phi \int_0^{\pi/2} C_e^2\sin\theta\, d\theta + \int_0^{2\pi}\cos^2\phi\, d\phi \int_0^{\pi/2} C_h^2\sin\theta\, d\theta\right)$$

$$c = \cos^n\theta \quad \text{for } \theta \in [0, \pi/2]$$

$$c^2 = \cos^{2n}\theta \quad \text{"} \qquad \text{"}$$

$$\int_0^{\pi} c^2 \sin\theta \, d\theta = \int_0^{\pi/2} c^2 \sin\theta \, d\theta = \int_0^{\pi/2} \cos^{2n}\theta \sin\theta \, d\theta$$

$$\int \cos^{2n}\theta \sin\theta \, d\theta = -\frac{\cos^{2n+1}(\theta)}{2n+1}$$

$$\int_0^{\pi/2} \cos^{2n}\theta \sin\theta \, d\theta = + \left(\cancelto{0}{\frac{\cos^{2n+1}(\pi/2)}{2n+1}} + \frac{\cancelto{1}{\cos^{2n+1}(0)}}{2n+1} \right)$$

$$= \frac{1}{2n+1}$$

$$\langle u \rangle = \frac{1}{8\pi\eta} \left(\cancel{\pi} \cdot \frac{1}{2q_e+1} + \cancel{\pi} \cdot \frac{1}{2q_h+1} \right)$$

$$= \frac{1}{8\eta} \left(\frac{2(q_e+q_h)+2}{(2q_e+1)(2q_h+1)} \right)$$

$$= \frac{1}{4\eta} \left(\frac{q_e+q_h}{(2q_e+1)(2q_h+1)} \right)$$

$$D(\theta,\phi;q_e;q_h) = \frac{u}{\langle u \rangle}$$

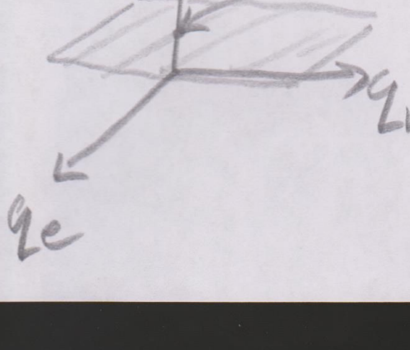
$$\doteq \frac{\frac{1}{2\cancel{\pi}}}{\frac{1}{4\cancel{\pi}}} \frac{\dfrac{\cos^{2q_e}\theta\sin^2\phi + \cos^{2q_h}\theta\cos^2\phi}{q_e+q_h}}{(2q_e+1)(2q_h+1)}$$

$$= \frac{2(2q_e+1)(2q_h+1)(\cos^{2q_e}\theta\sin^2\phi + \cos^{2q_h}\theta\cos^2\phi)}{q_e+q_h}$$

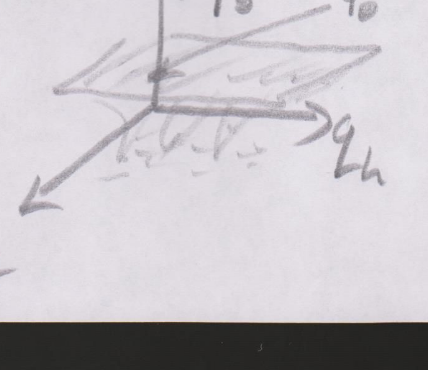
→ the dir of max{D} is the same dir of max{u}

$$u_0 = u(\theta_0,\phi_0) \Rightarrow D_0 = D(\theta_0,\phi_0)$$

$$u = \frac{\cos^{2q_e}\theta\sin^2\phi + \cos^{2q_h}\theta\cos^2\phi}{}$$



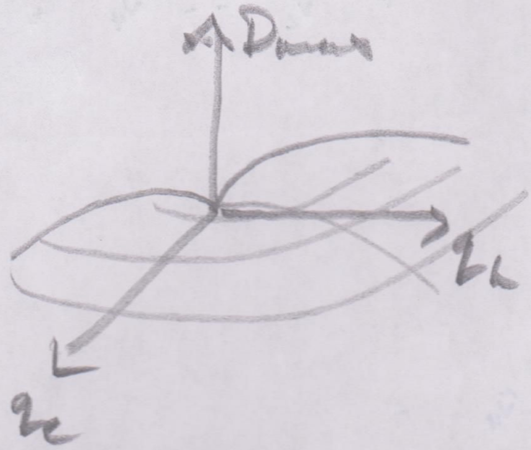
$\theta_0 = 0°$

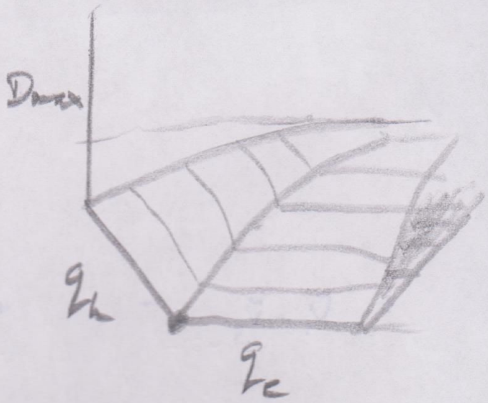


$\phi_0 = 12°$ ← probably an artifact when $\theta = 0$, ϕ = undefined/doesn't ~~matter~~

→ max D is $\theta_0 = 0$, ϕ_0 = doesn't matter
so where the horn points

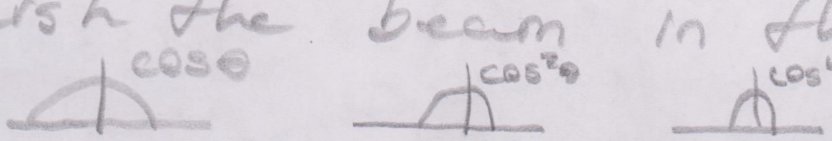
$$D_{max}(q_e, q_h) = \frac{2(2q_e+1)(2q_h+1)}{q_e+q_h}$$





→ no info about beam width can be obtained from expression of D_{max} (unless there is an equivalent to Gain Bandwidth Product for antennas)

→ for U, q_e and q_h do squish the beam in the θ direction



b) $\hat{u}_{co} = \sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$

$$= \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$\hat{u}_{cr} = \cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} = \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

for orthogonality $\hat{u}_{co} \cdot \hat{u}_{cr} = 0$

$$\begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix} = 0 + \sin\phi\cos\phi - \cos\phi\sin\phi$$

$$= 0 \checkmark$$

c) $\bar{\bar{E}}_F = C_e \sin\phi\,\hat{\theta} + C_h \cos\phi\,\hat{\phi} = \begin{bmatrix} 0 \\ C_e \sin\phi \\ C_h \cos\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$

$$= \tilde{\bar{E}}_{Fco}\,\hat{u}_{co} + \tilde{\bar{E}}_{Fcr}\,\hat{u}_{cr}$$

$$= \tilde{\bar{E}}_{Fco}\left(\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}\right)$$
$$+ \tilde{\bar{E}}_{Fcr}\left(\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}\right)$$

$$= \begin{bmatrix} 0 \\ \tilde{\bar{E}}_{Fco}\sin\phi + \tilde{\bar{E}}_{Fcr}\cos\phi \\ \tilde{\bar{E}}_{Fco}\cos\phi - \tilde{\bar{E}}_{Fcr}\sin\phi \end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}}$$

$$\begin{bmatrix} C_e \sin\phi \\ C_h \cos\phi \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \tilde{E}_{Fco} \\ \tilde{E}_{Fcr} \end{bmatrix} = \begin{bmatrix} \tilde{E}_{F\theta} \\ \tilde{E}_{F\phi} \end{bmatrix}$$

$$\begin{vmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{vmatrix} = -\sin^2\phi - \cos^2\phi$$

$$= -1$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|A|}$$

$$\begin{bmatrix} \tilde{E}_{Fco} \\ \tilde{E}_{Fcr} \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \tilde{E}_{F\theta} \\ \tilde{E}_{F\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} C_e \sin\phi \\ C_h \cos\phi \end{bmatrix}$$

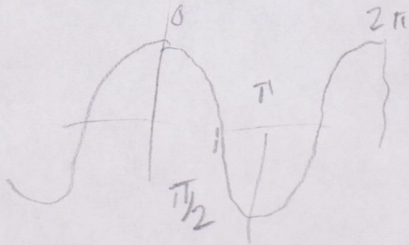
$$\tilde{E}_{F_{co}} = C_e \sin^2\phi + C_h \cos^2\phi$$

$$\tilde{E}_{Fcr} = C_e \sin\phi\cos\phi - C_h \sin\phi\cos\phi$$

$$= \frac{(C_e - C_h)}{2} \sin 2\phi$$

d) $C_e = \cos^{q_e}\theta$

$C_h = \cos^{q_h}\theta$



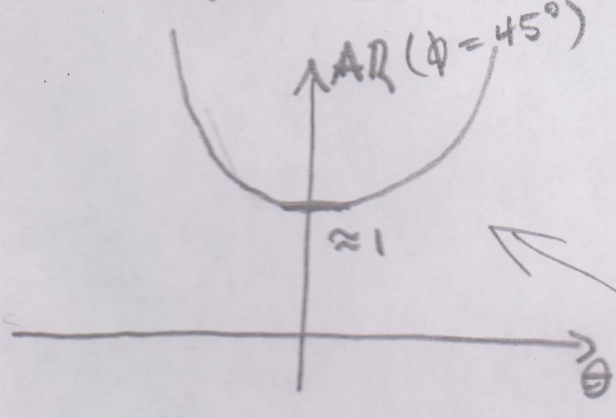
$$\tilde{E}_{Fco} = \cos^{q_e}\theta \sin^2\phi + \cos^{q_h}\theta \cos^2\phi$$

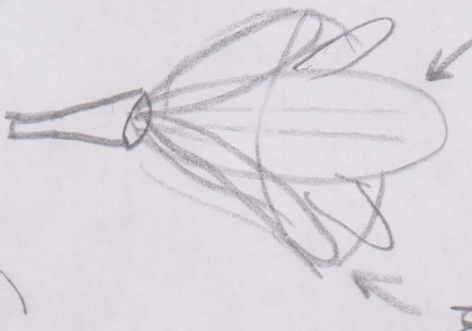
$$\tilde{E}_{Fcr} = \frac{\cos^{q_e}\theta - \cos^{q_h}\theta}{2} \sin 2\phi$$

Plot: $\dfrac{\tilde{E}_{Fco}(\theta, \phi_0)}{|\tilde{E}_{Fco}(\theta, \phi_0)|}$ and $\dfrac{\tilde{E}_{Fcr}(\theta, \phi_0)}{|\tilde{E}_{Fcr}(\theta, \phi_0)|}$

for $\phi_0 = [0°, 45°, 90°]$

$q_e = 3.2$, $q_h = 2.16$





CP here

4 lobes ~~cone~~ of LP here

my guess (to lazy to actually plot it)

you should have asked for this!

e) $C_L = C_R = C$

$$\tilde{E}_{Fco} = C \sin^2\phi + C\cos^2\phi$$
$$= C(\sin^2\phi + \cos^2\phi)$$
$$= C$$

$$\tilde{E}_{Fcr} = \frac{C - C}{2} \sin 2\phi = 0$$

$$\bar{E}_F = C\hat{u}_{co} = C \begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi \end{bmatrix}_{\{\hat{r}, \hat{\theta}, \hat{\phi}\}}$$

→ for this case antenna is perfectly circularly polarized in one sense and the other sense (cross pol) is zero

polarization of any antenna = some const · LHCP + some other const · RHCP

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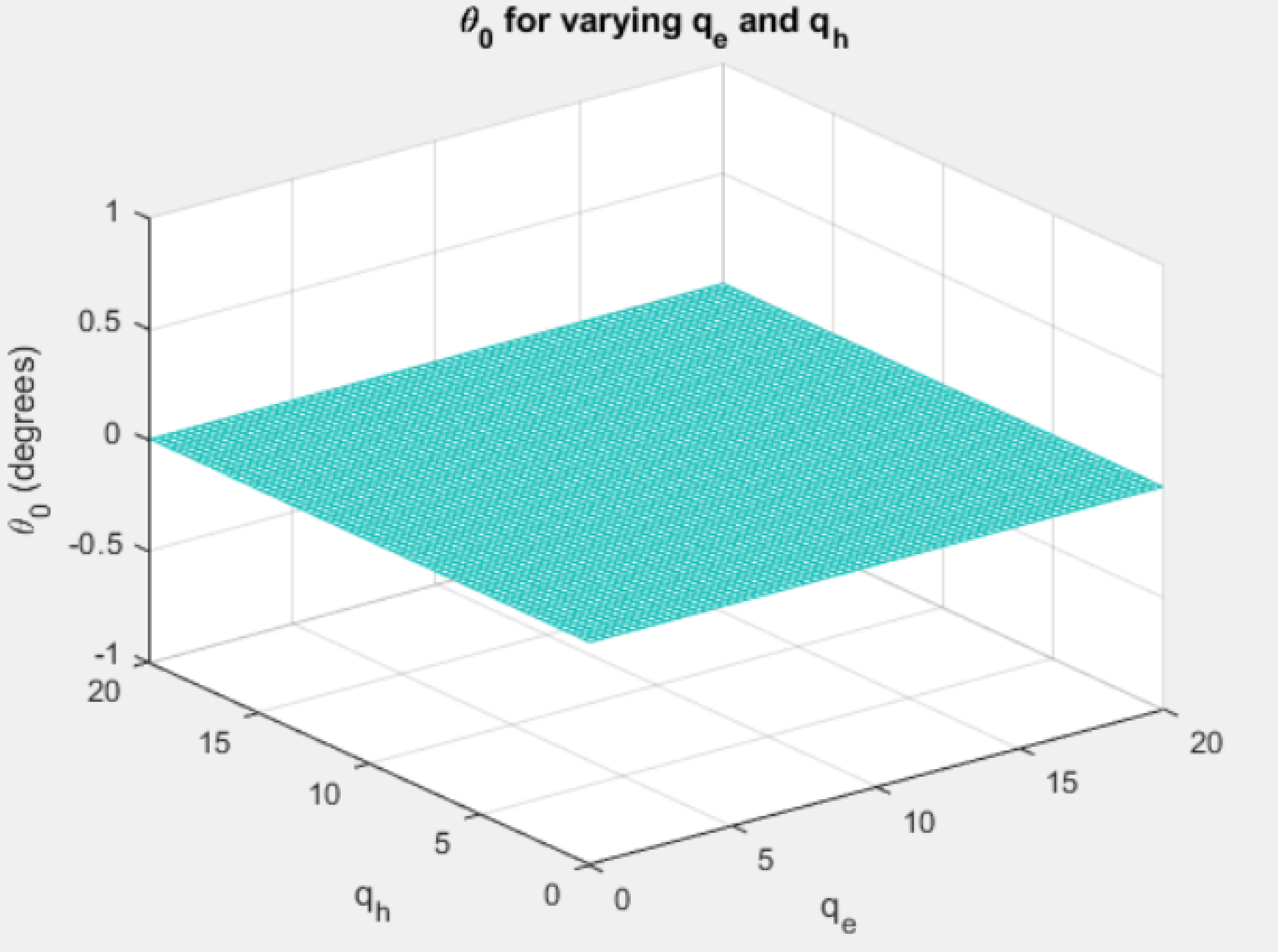
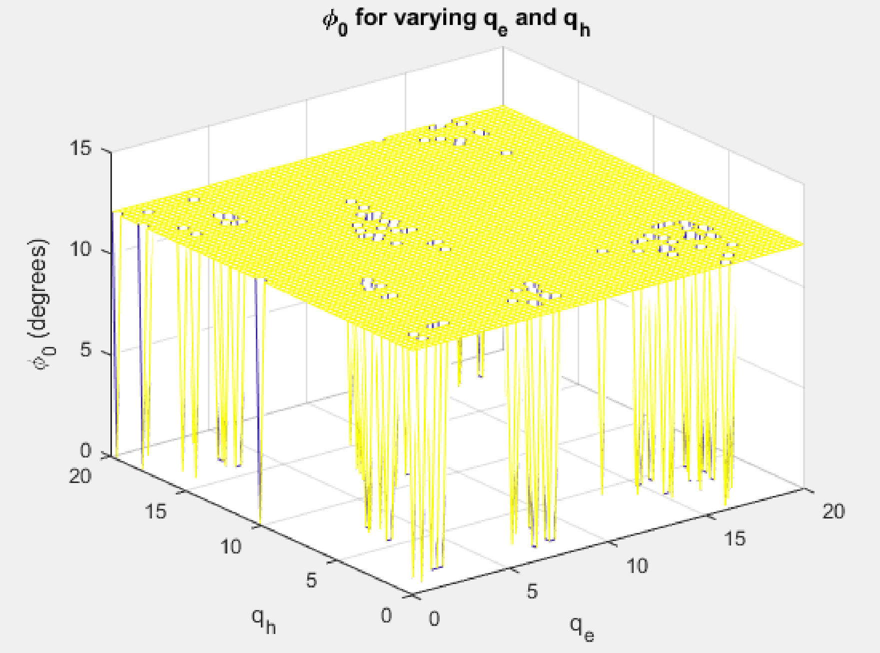
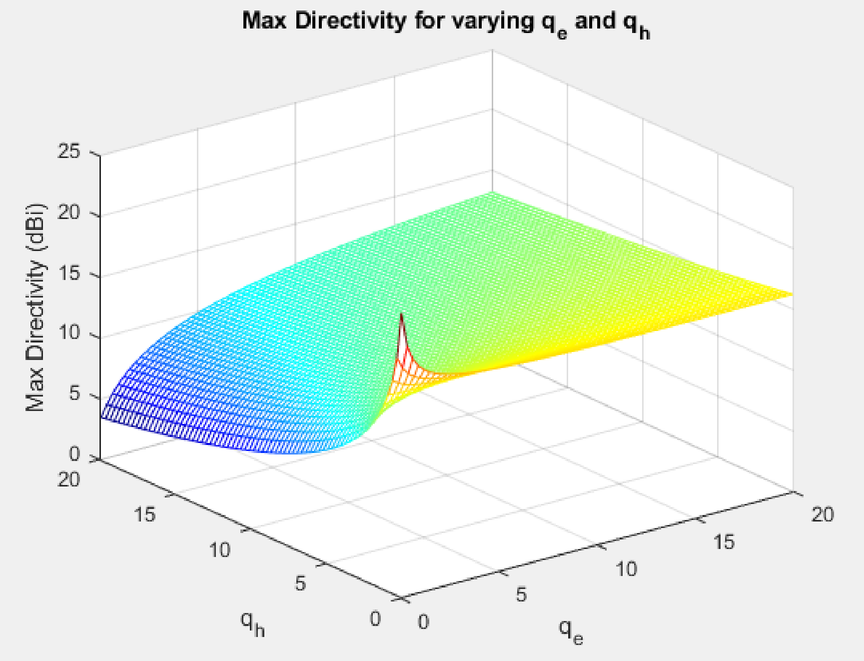
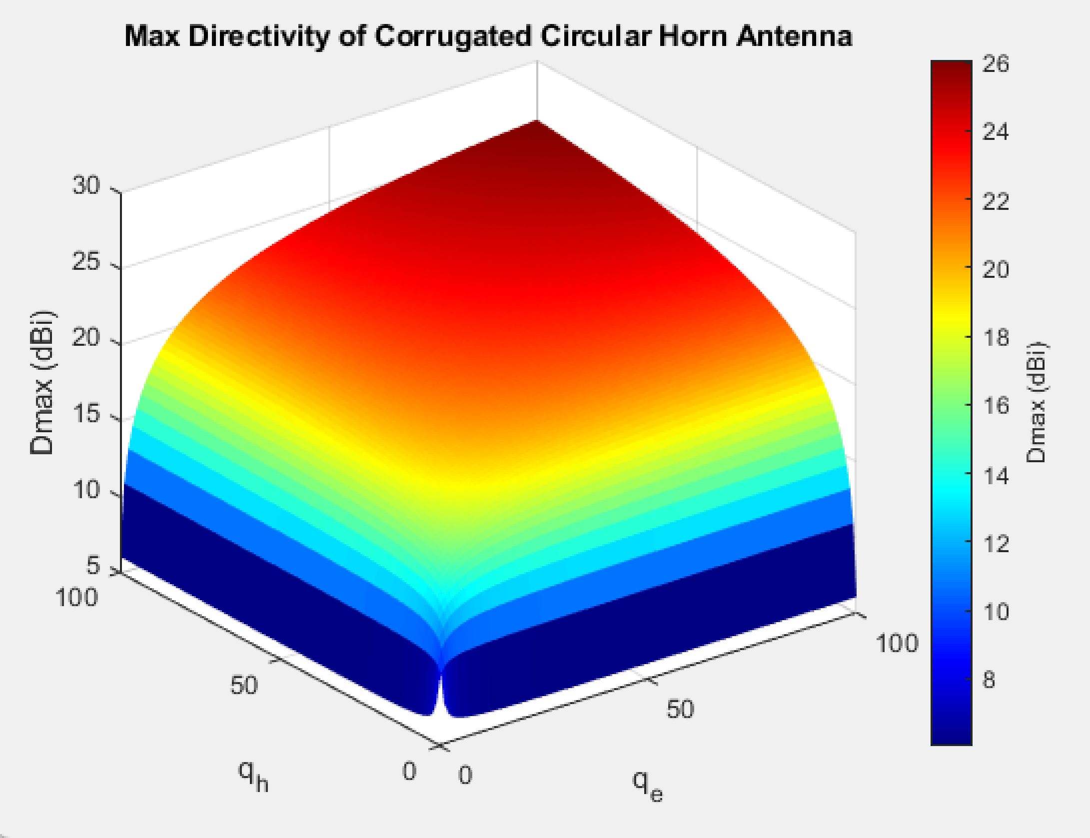

θ_0 for varying q_e and q_h
θ_0 (degrees)
1
0.5
0
-0.5
-1
20
15
10
5
0
q_h
0
5
10
15
20
q_e



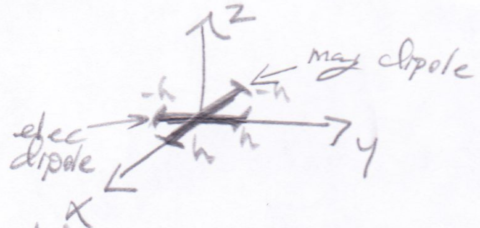
ϕ_0 for varying q_e and q_h
ϕ_0 (degrees)
15
10
5
0
20
15
10
5
0
q_h
0
5
10
15
20
q_e


Max Directivity for varying q_e and q_h
Max Directivity (dBi)
25
20
15
10
5
0
20
15
10
5
0
q_h
0
5
10
15
20
q_e



Max Directivity of Corrugated Circular Horn Antenna
Dmax (dBi)
30
25
20
15
10
5
100
50
0
q_h
0
50
100
q_e
26
24
22
20
18
16
14
12
10
8
Dmax (dBi)

② $\frac{C_m}{C_e} = \eta$, Find $\bar{E}$, $C_m = 1$



$\bar{E} = \bar{E}_F \frac{e^{-jkr}}{r}$ * Assume far field

Method I: radiation integrals + duality

Method II: $\bar{A}$ then $\bar{E}$ + duality

Method III: $\bar{E}$ + change of basis + duality

Method I steps

i) elec dipole on y axis find $\bar{J}$

ii) find f_x, f_y, f_z

iii) find $\tilde{E}_\theta$, $\tilde{E}_\phi$

iv) elec dipole on x axis find J

v) find f_x, f_y, f_z

vi) find $\tilde{E}_\theta$, $\tilde{E}_\phi$

vii) duality

viii) use maxwell curl to get back to $\bar{E}$

ix) sum for overall $\bar{E}$

Step i

$$\bar{J} = C_e\, \delta(x)\,\delta(y)\,\delta(z)\,\hat{y}$$

Step ii

$$J_x = J_z = 0 \Rightarrow f_x = f_z = 0$$

$$f_y = \lim_{h\to 0} \int_{-h}^{h} C_e \delta(y') e^{jk\beta \sin k y'} dy'$$

$$= C_e$$

Step iii

$$\tilde{E}_{F\theta} = C_0(0 - \cos\theta \sin\phi f_y)$$

$$= -C_e C_0 \cos\theta \sin\phi$$

$$= -\frac{j\omega\mu_0 C_e}{4\pi} \cos\theta \sin\phi$$

$$= -\frac{jk\eta C_e}{4\pi} \cos\theta \sin\phi$$

$$\tilde{E}_{F\phi} = C_0(0 - \cos\phi f_y)$$

$$= -C_0 C_e \cos\phi = -\frac{j\omega\mu_0}{4\pi} C_e \cos\phi$$

$$= -\frac{jk\eta C_e}{4\pi} \cos\phi$$

$$\frac{C_m}{C_e} = \eta$$

$$C_0 = \frac{j\omega\mu_0}{4\pi}$$

$$K = \frac{2\pi}{\lambda}$$

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad \lambda f = c$$

$$\omega = 2\pi f \qquad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$\omega\mu_0 = 2\pi f \mu_0$$

$$= \frac{2\pi c}{\lambda} \mu_0$$

$$= \frac{2\pi}{\lambda} \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \mu_0$$

$$= \frac{2\pi}{\lambda} \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= K\eta$$

5
4

1

Step iv

$$\bar{J} = C_e\,\delta(x)\,\delta(y)\,\delta(z)\,\hat{x}$$

Step v

$$f_y = f_z = 0$$

$$f_x = \lim_{k \to 0} \int_{-h}^{h} C_e\,\delta(x')\,e^{jk\xi \sin k x'}\,dx'$$

$$= C_e$$

Step vi

$$\tilde{E}_{F\theta} = C_0(-\cos\theta\cos\phi f_x - 0 + 0)$$

$$= -C_0 C_e \cos\theta\cos\phi$$

$$= \frac{-jk\eta C_e \cos\theta\cos\phi}{4\pi}$$

$$\tilde{E}_{F\phi} = C_0(\sin\phi f_x - 0)$$

$$= C_0 C_e \sin\phi$$

$$= \frac{jk\eta C_e}{4\pi}\sin\phi$$

3

$$\bar{E}_{F_2} = \frac{jK\eta C_e}{4\pi}\begin{bmatrix}0\\ -\cos\theta\cos\phi\\ \sin\phi\end{bmatrix}_{\{\hat{r},\hat{\theta},\hat{\phi}\}} \cdot \frac{e^{+jkr}}{r}$$

<u>Step vii</u>

Duality: $C_e \to C_m$ $\quad \bar{H}_m = \frac{e^{-jkr}}{r}\frac{jKC_m}{4\pi\eta}\begin{bmatrix}0\\ -\cos\theta\cos\phi\\ \sin\phi\end{bmatrix}$

$\eta \to \eta^{-1}$

$\mu_0 \to \varepsilon_0$

$\bar{E} \to \bar{H}$

<u>Step viii</u>

$\nabla \times \bar{H} = j\omega\varepsilon_0 \bar{E}$ (far field/free space only)

$$= \frac{1}{r}\begin{bmatrix}\csc\theta\left(\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right)\\ -\frac{\partial}{\partial r}(rH_\phi)\\ \frac{\partial}{\partial r}(rH_\theta)\end{bmatrix}$$

$$= \frac{jKC_m}{4\pi\eta r}\begin{bmatrix}\csc\theta\left(\frac{\partial}{\partial\theta}\left(\frac{e^{-jkr}}{r}\sin\phi\sin\theta\right) + \frac{\partial}{\partial\phi}\left(\frac{e^{-jkr}}{r}\cos\theta \cos\phi\right)\right)\\ -\frac{\partial}{\partial r}\left(\cancel{r}\frac{e^{-jkr}}{\cancel{r}}\sin\phi\right)\\ -\frac{\partial}{\partial r}\left(r\frac{e^{-jkr}}{r}\cos\theta\cos\phi\right)\end{bmatrix}$$

3

$$\nabla \times \bar{H} = \frac{jkC_m}{4\pi\eta r}\begin{bmatrix} \csc\theta\left(\cancel{\frac{e^{-jkr}}{r}\sin\phi\cos\theta} + \cancel{\frac{e^{-jkr}}{r}\cos\theta\sin\phi}\right) \\ jk\,e^{-jkr}\sin\phi \\ jk\,e^{-jkr}\cos\theta\cos\phi \end{bmatrix}$$

$$= \frac{j^2 k^2 C_m}{4\pi\eta}\cdot\frac{e^{-jkr}}{r}\begin{bmatrix} 0 \\ \sin\phi \\ \cos\theta\cos\phi \end{bmatrix}$$

$$= \frac{-k^2 C_m}{4\pi\eta}\cdot\frac{e^{-jkr}}{r}\begin{bmatrix} 0 \\ \sin\phi \\ \cos\theta\cos\phi \end{bmatrix} \quad \{\hat{r}, \hat{\theta}, \hat{\phi}\}$$

$$= j\omega\varepsilon_0 \bar{E}$$

$$\bar{E} = \frac{-k^2 C_m}{4\pi\eta\cdot j\omega\varepsilon_0}\cdot\frac{e^{-jkr}}{r}\begin{bmatrix} 0 \\ \sin\phi \\ \cos\phi\cos\theta \end{bmatrix}$$

$$= \frac{-k^2 C_m}{j4\pi\omega\varepsilon_0\eta} = \frac{-k^2 C_m}{j4\pi\cancel{\eta}\cdot k/\cancel{\eta}}$$

$$\varepsilon_0 \omega = 2\pi f \varepsilon_0$$
$$= \frac{2\pi c}{\lambda} \varepsilon_0 = K \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \varepsilon_0 = K \sqrt{\frac{\varepsilon_0}{\mu_0}} = K/\eta$$

$$\bar{E}_m = \frac{jKC_m}{4\pi} \cdot \frac{e^{-jKr}}{r} \begin{bmatrix} 0 \\ \sin\phi \\ \cos\theta \cos\phi \end{bmatrix}_{\{\hat{r}, \hat{\theta}, \hat{\phi}\}}$$

<u>Step ix</u>

$$\bar{E}_e = -\frac{jK\eta C_e}{4\pi} \cdot \frac{e^{-jKr}}{r} \begin{bmatrix} 0 \\ \cos\theta \sin\phi \\ \cos\phi \end{bmatrix}_{\{r, \theta, \phi\}}$$

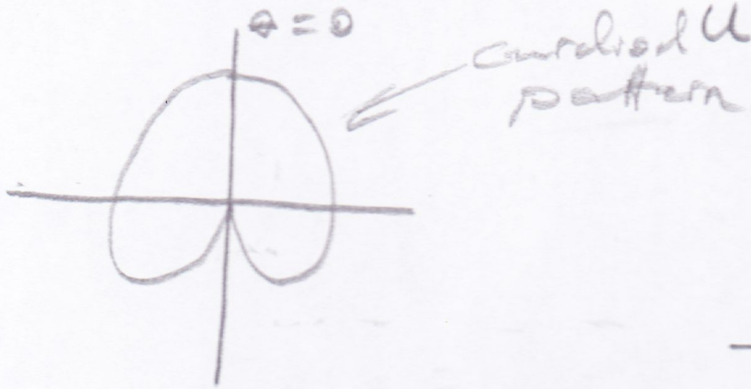
$$\bar{E} = \bar{E}_e + \bar{E}_m$$
$$= -\frac{jK\eta C_e}{4\pi} \cdot \frac{e^{-jKr}}{r} \begin{bmatrix} 0 \\ \cos\theta \sin\phi \\ \cos\phi \end{bmatrix}$$
$$+ \frac{jKC_m}{4\pi} \cdot \frac{e^{-jKr}}{r} \begin{bmatrix} 0 \\ \sin\phi \\ \cos\theta \cos\phi \end{bmatrix}$$

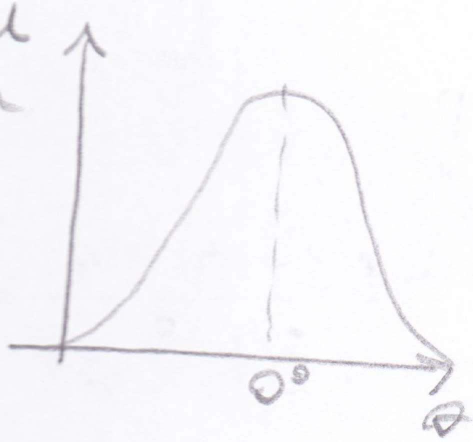
$$\frac{C_m}{C_e} = \eta \Rightarrow C_e \eta = C_m = 1$$

$$\bar{E} = \frac{jK}{4\pi} \cdot \frac{e^{-jkr}}{r} \begin{bmatrix} 0 \\ \sin\phi - \cos\theta\sin\phi \\ \cos\theta\cos\phi - \cos\phi \end{bmatrix} \{\hat{r}, \hat{\theta}, \hat{\phi}\}$$

$$U = \frac{|\bar{E}_F|^2}{2\eta}$$

also has high pass response $K \propto f$

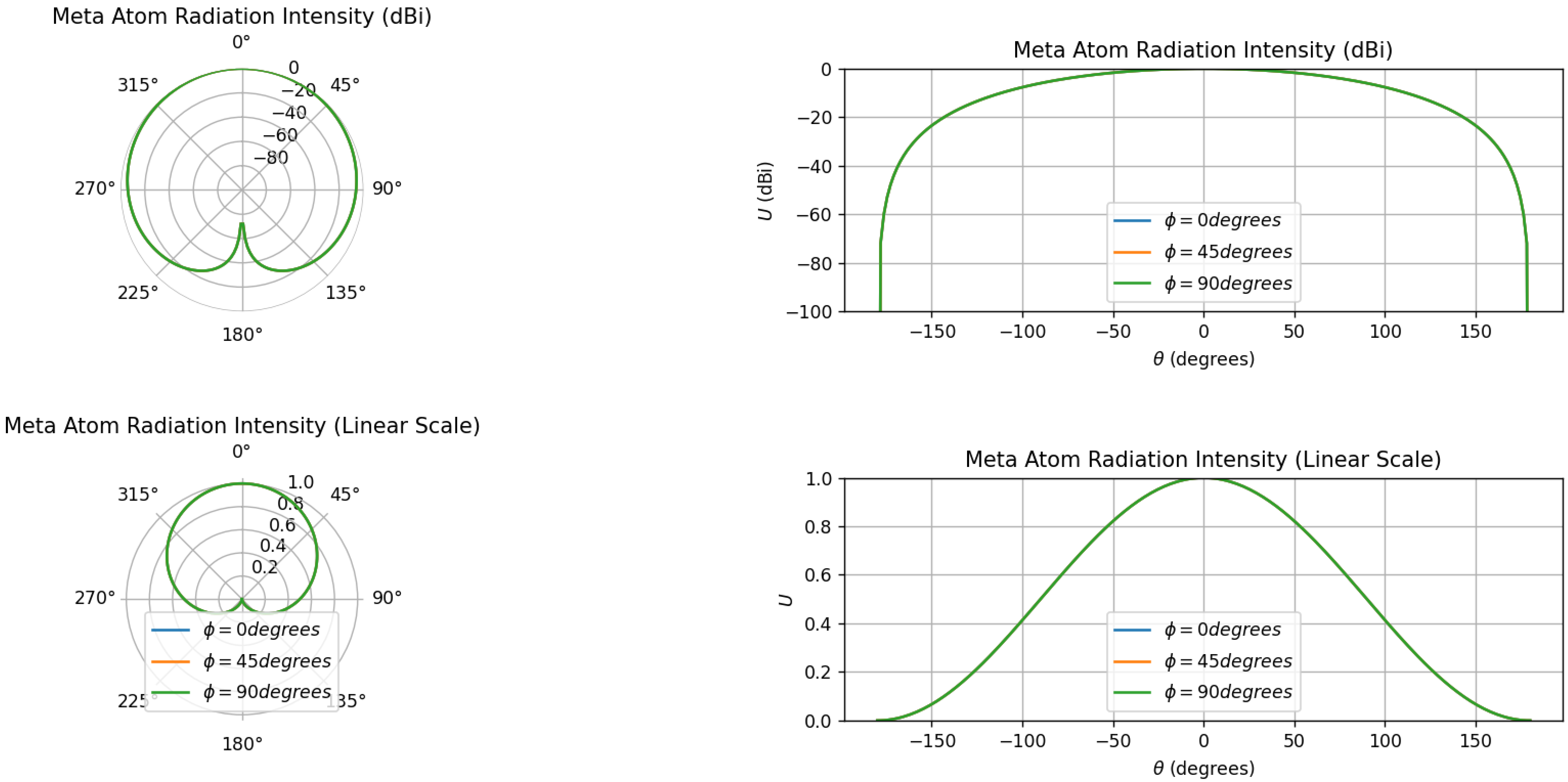




→ no backlobe

→ plot actually looks like this

but I manually flipped
not sure if that
is a bug somewhere



Meta Atom Radiation Intensity (dBi)
0°
45°
90°
135°
180°
225°
270°
315°
0
−20
−40
−60
−80
Meta Atom Radiation Intensity (dBi)
U (dBi)
θ (degrees)
ϕ = 0degrees
ϕ = 45degrees
ϕ = 90degrees
Meta Atom Radiation Intensity (Linear Scale)
1.0
0.8
0.6
0.4
0.2
ϕ = 0degrees
ϕ = 45degrees
ϕ = 90degrees
Meta Atom Radiation Intensity (Linear Scale)
U
θ (degrees)
ϕ = 0degrees
ϕ = 45degrees
ϕ = 90degrees