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COS302-F23

Assignment #11

Due: 6:00pm Friday 15 December 2023
(Dean's Date, so no extensions possible)

Discussants: N/A

Upload at: https://www.gradescope.com/courses/606160/assignments/3745386

Problem 1 (20pts)
Solve the following problem using Lagrange multipliers:

$$\begin{aligned} \text{minimize:} \quad & x^2 + y^2 \\ \text{subject to:} \quad & x - y = 3 \end{aligned}$$

Start with $L(x, y, \lambda) = x^2 + y^2 + \lambda(x - y - 3)$.

We then consider the stationary conditions:

$\frac{\partial L}{\partial x} = 2x + \lambda = 0$

$\frac{\partial L}{\partial y} = 2y - \lambda = 0$

$x - y = 3$

From the first two conditions, we observe that $\lambda = -2x$ and $\lambda = 2y \implies y = -x$.

Consider the constraint: $x - y = x - (-x) = 3 \implies 2x = 3 \implies x = \frac{3}{2}$ and $y = -\frac{3}{2}$.

We minimize $x^2 + y^2 = (\frac{3}{2})^2 + (-\frac{3}{2})^2 = \frac{9}{2}$.

This provides a solution: $(x, y) = (\frac{3}{2}, -\frac{3}{2})$ with minimum value of $\frac{9}{2}$.

Problem 2 (25pts)

An investment firm wants to allocate a budget of \$100,000 across four different investment options: stocks, bonds, real estate, and a startup venture. The expected annual return rates for these investments are 5% for stocks, 3% for bonds, 4% for real estate, and 10% for the startup venture. However, each investment comes with different risk factors and the firm wants to manage its risk exposure. The firm has established the following risk limits:

- No more than 40% of the total investment should be in high-risk options (stocks and startup venture).
- At least 30% of the total investment should be in low-risk options (bonds).
- The investment in real estate should not exceed 35% of the total investment.

Formulate a linear programming model to maximize the firm's expected annual return. Define the decision variables for the amounts to invest in each option, the objective function for maximizing return, and the constraints based on both the risk limits and the required properties for the percentages to make sense.

We can define variables s, b, r, v for stocks, bonds, real estate, and venture capital respectively. These variables are subject to the constraints:

$s + b + r + v = 100{,}000$
$s + v \leq 40{,}000$
$b \geq 30{,}000$
$r \leq 35{,}000$
Additionally, all of s, b, r, v should be non-negative.

We want to maximize the expected return: $0.05s + 0.03b + 0.04r + 0.1v$.

$$\max\ c^T x = \begin{bmatrix} 0.05 \\ 0.03 \\ 0.04 \\ 0.10 \end{bmatrix}^T \begin{bmatrix} s \\ b \\ r \\ v \end{bmatrix}$$

Subject to $Ax \leq b,\ Ex = d,\ x \geq 0$, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ b = \begin{bmatrix} 40{,}000 \\ -30{,}000 \\ 35{,}000 \end{bmatrix},\ E = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix},\ d = \begin{bmatrix} 100{,}000 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ b \\ r \\ v \end{bmatrix} \leq \begin{bmatrix} 40{,}000 \\ -30{,}000 \\ 35{,}000 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ b \\ r \\ v \end{bmatrix} = \begin{bmatrix} 100{,}000 \end{bmatrix}$$

Problem 3 (25pts)
Determine whether the following functions are convex and explain your reasoning.

(A) $f(x) = e^x$

(B) $f(x) = \tanh(x)$

(C) $f(x) = |x|$

(D) $f(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{a})^T \boldsymbol{B}\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^d$, constant $\boldsymbol{a}$, and constant symmetric positive definite $\boldsymbol{B}$.

We can use the second derivative (or Hessian) to test if a function is convex. The second derivative should always be non-negative ($f''(x) \geq 0$ for all x).

(A) $f''(x) = e^x$

Because $f''(x)$ is positive across the entire domain, $f(x) = e^x$ is convex.

(B) $f''(x) = -2\tanh(x)\,\mathrm{sech}^2(x)$

Because $f''(x)$ can be negative, $f(x) = \tanh(x)$ is not convex.

(C) $f''(x) = 0$

Because $f''(x)$ is non-negative across the entire domain, $f(x) = |x|$ is convex.

(D) $f(x) = (x - a)^T Bx = x^T Bx - a^T Bx$

$\nabla f(x) = (B + B^T)x - B^T a$

$\nabla^2 f(x) = B + B^T = 2B$ (B is symmetric)

Because the Hessian matrix $\nabla^2 f(x)$ is positive and definite, the function $f(x) = (x - a)^T Bx$ is convex.

Problem 4 (28pts)

The Rosenbrock function

$$f(x, y) = 100(y - x^2)^2 + (1 - x)^2 \,,$$

is a classic test function for optimization routines. It has a global minimum of 0 at $(1, 1)$.

(A) Use `meshgrid` and `contourf` to make a contour plot of the function on the range $x \in [-2, 2]$, $y \in [-1, 3]$. You'll want to write code that looks kind of like this.

(B) Write a function that computes the gradient of the Rosenbrock function.

(C) Starting from $(-1, 2)$, perform gradient descent to minimize the function. You'll need to set the learning rate (probably to a very small number) and it may require quite a few steps (like thousands). Plot the path of your optimization on the contour plot from part (A).

(D) Now, minimize the problem using an off-the-shelf optimization tool. One powerful and widely-used method is the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm. Fortunately, `scipy.optimize.minimize` implements it (and many other methods). Check out the documentation, but you'll want to use it doing something along the lines of:

```
from scipy.optimize import minimize

steps = []
def cb(x):
  global steps
  steps.append(x)

result = minimize(func, x0, jac=grad_func, method='BFGS', callback=cb)
```

There are a few things going on here. `func` is the function you want to minimize. The argument `x0` is the initialization. The keyword argument `jac` takes the gradient function. The keyword argument `callback` is letting us keep track of the steps the algorithm takes, putting them into the variable `steps`. The structure `result` contains various things telling us how the optimization went and what minimum was found, if any.

As in (C), plot the path of the optimization (in `steps`). Explain what differences you see with the path from part (C).

Problem 5 (2pts)
Approximately how many hours did this assignment take you to complete?

My notebook URL: `https://colab.research.google.com/drive/114_3jjy0VPHXtgwwEDS3v2jovbIT_yli?usp=sharing`

Changelog

- 29 November 2023 – Initial version.