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COS302-F23

Assignment #7

Due: 6:00pm Wednesday 8 November 2023

Discussants: N/A

Upload at: https://www.gradescope.com/courses/606160/assignments/3593250

Problem 1 (20pts)
Consider the following cumulative distribution function for a random variable X that takes values in $\mathbb{R}$:

$$F(x) = P(X \leq x) = \frac{1}{1+e^{-x}}$$

(A) What is the probability density function for this random variable?

(B) Find the inverse distribution (quantile) function $F^{-1}(u)$ that maps from $(0, 1)$ to $\mathbb{R}$.

(C) In a Colab notebook, implement inversion sampling and draw 1000 samples from this distribution. Make a histogram of your results.

(A) We can find the PDF by taking the derivative of the CDF:

$$\frac{d}{dx}\frac{1}{1+e^{-x}} = -\frac{\frac{d}{dx}(1+e^{-x})}{(1+e^{-x})^2} = -\frac{-e^{x}}{(1+e^{-x})^2} = \frac{e^{x}}{(1+e^{-x})^2} = PDF(x)$$

(B)

$$F(x) = \frac{1}{1+e^{-x}} = u \text{ (Solve for x)}$$

$$\frac{1}{u} = 1 + e^{-x}$$

$$\frac{1-u}{u} = e^{-x}$$

$$x = -ln(\frac{1-u}{u}) = ln(\frac{u}{1-u}) = F^{-1}(u)$$

(C) See the associated Colab.

Problem 2 (15pts)
Consider two random variables X and Y with a joint probability density function $p(x, y)$. Show that

$$\mathbb{E}_X[x] = \mathbb{E}_Y[\mathbb{E}_X[x \mid Y = y]]$$

where the notation $\mathbb{E}_X[x \mid Y = y]$ denotes the expectation of X under the conditional distribution $P(X \mid Y = y)$.

The joint PDF $p(x, y)$ has marginal distributions $p_X(x) = \int p(x, y)\, dy$ and $p_Y(y) = \int p(x, y)\, dx$.

From this, $\mathbb{E}_X[x] = \int x * p_X(x)\, dx = \int \int x * p(x, y)\, dy\, dx$.

Now we can approach the other side of the equation. $\mathbb{E}_X[x \mid Y = y] = \int x * p(x|y)\, dx$

We define the conditional PDF $p(x|y)$ as $p(x|y) = \frac{p(x,y)}{p_Y(y)}$.

With substitution, $\mathbb{E}_Y[\mathbb{E}_X[x \mid Y = y]] = \mathbb{E}_Y[\int x * \frac{p(x,y)}{p_Y(y)}\, dx] = \int [\int x * \frac{p(x,y)}{p_Y(y)}\, dx]\, p_Y(y)\, dy = \int \int x * p(x, y)\, dx\, dy$.

If the integrals can be exchanged, then $\int \int x * p(x, y)\, dx\, dy = \int \int x * p(x, y)\, dy\, dx = \mathbb{E}_X[x]$.

Problem 3 (20pts)

The covariance between two random variables X and Y can be computed as:

$$\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

If X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ which implies that $\text{cov}(X, Y) = 0$. However, the converse is not true: if the covariance between two random variables is zero, this *does not* imply that these variables are independent. Let's construct a counterexample to show this is the case in a Colab notebook. Be sure to append your PDF and insert your link as usual.

(A) Let X be a random variable that can take on only the values -1 and 1 and $P(X = -1) = 0.5 = P(X = 1)$. Generate 1,000,000 samples of X.

(B) Let Y be random variable such that $Y = 0$ if $X = -1$, and Y is randomly either -1 or 1 with probability 0.5 if $X = 1$. Construct 1,000,000 samples of Y based on the samples of X you generated in part (A).

(C) Use numpy to numerically compute the covariance between X and Y.

Problem 4 (23pts)

You're playing a game at a carnival in which there are three cups face down and if you choose the one with a ball under it, you win a prize. This is sometimes called a *shell game*. At this carnival, there is a twist to the game: after you pick a cup, but before you're shown what is beneath it, the game operator reveals to you that one of the other cups (one of the two you did not choose) is empty. The operator now gives you the opportunity to switch your selection to the other unrevealed cup.

To clarify with an example: imagine there are cups A, B, and C. It is equally probable that the ball is beneath any of the three. You choose B. Before you see what is under B, the operator lifts A and shows you there is nothing under it. You are now presented with the option to keep your selection of B, or switch to the still-unrevealed cup C.

(A) Is it better, worse, or the same to switch to the other cup? Explain your reasoning in terms of probabilities.

(B) In a Colab notebook, simulate this game. Run 1,000 games with the *stay* strategy and 1,000 games with the *switch* strategy. Report the win rate of each strategy and explain which one empirically seems better.

(C) Now imagine that there are $N > 3$ cups face down and one has a ball under it. As a function of N, what are the win probabilities for the *stay* and *switch* strategies? You can assume that under the *switch* strategy you choose uniformly from the other cups.

(D) Verify your answers empirically by simulating these strategies for $N = 5$, $N = 10$, and $N = 25$ as in part (B) above.

(A) Let's consider the two strategies.

Stay

If you stay, your probability of winning is simply the probability that your randomly selected cup/door/option is the winner, i.e. $1/3$.

Switch

If you play the switch strategy, there are two possibilities after you make your initial selection.

- $1/3$ of times, if you are already on correct cup. You will then switch to a losing cup.
- $2/3$ of times, you are initially on a losing cup. The operator then reveals the other losing cup. Because there are only three cups in total, the other unrevealed cup is the winning cup. You switch to the winning cup.

With 3 cups, the switch strategy wins $2/3$ of the time in expectation.

For **(B - D)**, see the associated Google Colab.

Problem 5 (20pts)

Recall that two random variables X and Y are independent if and only if $P(X, Y) = P(X)P(Y)$. Two random variables X and Y are **conditionally** independent given a third event Z if $P(X, Y|Z) = P(X|Z)P(Y|Z)$. The following problem explores the relationship between independence and conditional independence, specifically whether independence implies conditional independence or vice versa.

(A) Imagine that you have two coins: one regular fair coin ($P(\text{heads}) = 0.5$) and one fake two-headed coin ($P(\text{heads}) = 1$). Consider the following experiment: choose a coin at random and toss it twice. Define the following events:

- A is the event that the first coin toss results in a heads.
- B is the event that the second coin toss results in a heads.
- C is the event that the regular fair coin has been selected.

Are the events A and B independent? Give a qualitative answer, i.e., either yes because... or no because... Are they conditionally independent given C? Show your work, i.e., explicitly show the definition holds as given in the problem statement.

(B) Roll a single six-sided die and consider the following events:

- X is that you roll 1 or 2, i.e., $X = \{1, 2\}$.
- Y is that you roll an even number, i.e., $Y = \{2, 4, 6\}$.
- Z is that you roll 1 or 4, i.e., $Z = \{1, 4\}$.

Are the events X and Y independent? Are they conditionally independent given Z? For both of these questions show the definitions hold as in the problem statement.

(A) No, A and B are not independent. Consider a situation where we observe that the first coin toss was tails ($A = 0$, or A is not true). We know then that the fair coin was selected, and that we can expect B to be true ($B = 1$, second toss is heads) 50% of the time. If we observe that the first coin toss was heads (A was true), we don't know which coin was selected and have a higher expected value for B, because the coin could be fake.

Events A and B are conditionally independent given C. To satisfy conditional independence, $P(A, B|C) = P(A|C)P(B|C)$.

With a regular fair coin (C), the probability of heads on any flip is $\frac{1}{2}$: $P(A|C) = 0.5$, $P(B|C) = 0.5$

With a regular fair coin (C), the probability of getting two heads in a row is $\frac{1}{4}$: $P(A, B|C) = 0.25$.

Conditional independence is satisfied: $P(A|C)P(B|C) = 0.5 * 0.5 = 0.25 = P(A, B|C)$.

(B) The events X and Y are independent:

$P(X) = \frac{1}{3}$, $P(Y) = \frac{1}{2}$

The only outcome that satisfies both X and Y is 2, which occurs with probability $\frac{1}{6}$. This equals the product of the probability of the two events: $P(X, Y) = \frac{1}{6} = P(X)P(Y)$, so they are definitionally independent.

The events X and Y are not conditionally independent given Z:

$P(X|Z) = \frac{1}{2}$, $P(Y|Z) = \frac{1}{2}$

However, $P(X, Y|Z) = 0$. There is no outcome in Z that simultaneously satisfies X and Y. As a result, $P(X, Y|Z) \neq P(X|Z)P(Y|Z)$ and the events are not independent conditioned on Z.

Problem 6 (2pts)

Approximately how many hours did this assignment take you to complete?

My notebook URL: `https://colab.research.google.com/drive/1PL8xTez-ADT6Hl--paSk7VAMo41TaG_m?usp=sharing`

Changelog

- 26 Oct 2023 – Initial F23 version