Convex functions

Mathématiques Informatique et Statistique Appliquées (MISA)
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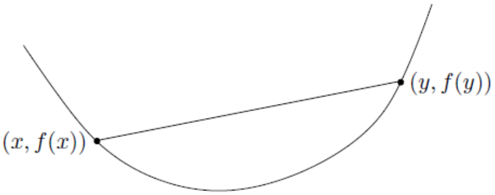
Definition

$f : \mathbb{R}^n \to \mathbb{R}$ is convex if **dom** f is convex and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

for all $x, y \in$ **dom** f, $0 \leq \theta \leq 1$



- f is striclty convex if **dom** f is convex and

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$

  for all $x, y \in$ **dom** f,$x \neq y$, $0 < \theta < 1$
- f is concave if $-f$ is convex
- f is striclty concave if $-f$ is striclty convex

Examples on $\mathbb{R}$

Convex

- affine : $ax + b$ on $\mathbb{R}$, $a, b \in$ on $\mathbb{R}$
- exponential : e^{ax} on $\mathbb{R}$, $a \in \mathbb{R}$
- powers : x^{α} on $\mathbb{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$

Concave

- affine
- powers : x^{α} on on $\mathbb{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm : $\log(x)$ on $\mathbb{R}_{++}$

Examples on $\mathbb{R}^n$

- Affine (linear) functions are both convex and concave

$$f(x) = a^\top x + b$$

- Norms are convex

$$||x||_p = (\sum_{i=1}^{n} |x|^p)^{1/p}$$

First-order condition

f is differentiable if the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$$

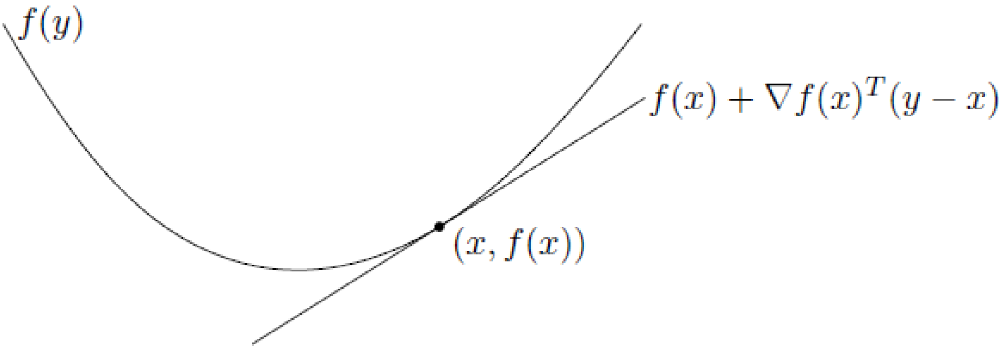
exists at each $x \in \textbf{dom } f$

First-order condition

f is differentiable with a convex domain. f is convex iff

$$f(y) \geq f(x) + \nabla f(x)^\top (x - y)$$

for all $x, y \in \textbf{dom } f$


$f(y)$
$f(x) + \nabla f(x)^T(y - x)$
$(x, f(x))$

Second-order condition

f is twice differentiable if the Hessian $\nabla^2 f(x)$,

$$(\nabla^2 f(x))_{i,j} = \frac{\partial f}{\partial x_i \partial x_j} \quad i, j = 1, ..., n$$

exists at each $x \in$ **dom** f

First-order condition

f is twice differentiable with a convex domain. f is convex iff its Hessian is positive semidefinite.

$$\nabla^2 f(x) \succeq 0, \text{for all} x \in \textbf{dom } f$$

Sublevel sets

The α-sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$C_\alpha = \{x \in \textbf{dom } f | f(x) \leq \alpha\}$$

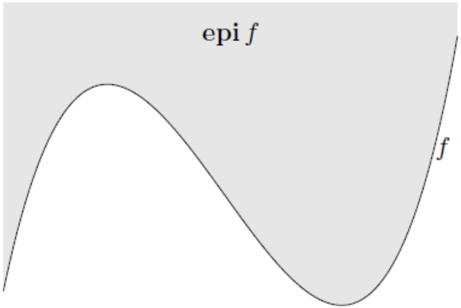
Sublevel sets of convex functions are convex (converse is false)

Epigraph

The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\textbf{epi } f = \{(x,t) \in \mathbb{R}^{n+1} | x \in \textbf{dom } f, f(x) < t\}$$

f is convex iff **epi** f is a convex set.



Convexity in practice

Practical method for establishing convexity of a function :

1. Verify definition
2. For twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
3. Show that f is obtained from simple convex functions by operations that preserve convexity

Operations preserving convexity

- nonnegative scaling : f convex, $\alpha \geq 0 \implies \alpha f$ convex
- sum : f, g convex $\implies f + g$ convex
- affine composition : f convex $\implies f(Ax + b)$ convex
- pointwise maximum : $f_1, ..., f_k$ convex $\implies max_i f_i(x)$ convex
- composition : h convex increasing, f convex $\implies h(f(x))$ convex

similar rules for concave functions (there are more)

The only rule you need to know

$h(f_1(x), \ldots, f_k(x))$ is convex if

- h convex, h increasing in each argument, and f_i are convex, or
- h convex, h decreasing in each argument, and f_i are concave, or
- f_i are affine

similar rules for concave (just swap convex and concave)