Convex optimization problems

Mathématiques Informatique et Statistique Appliquées (MISA)
Université d'Antananarivo

Optimization problem in standard form

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{s.t } & f_i(x) \leq 0, \ i = 1, ..., m \\ & g_i(x) = 0, \ i = 1, ..., p \end{aligned}$$

- $x \in \mathbb{R}^n$ is the optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the **objective or cost function**
- $f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, ..., m$ are the **inequality constraint** functions
- $h_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, ..., p$ are the **equality constraint** functions

Feasibility

- The **domain** (also called **implicit constraint**) is the set of points for which the objective and all constraint functions are defined

$$\mathcal{D} = \bigcap_{i=0}^{m} \textbf{dom}\ f_i \cap \bigcap_{i=1}^{p} \textbf{dom}\ h_i$$

- A problem is **unconstrained** if it has no explicit contraints.
- A point $x \in \mathcal{D}$ is **feasible** if it satisfies the constraints.
- The set of all feasible points is called the **feasible set** or the **constraint set**.
- The problem is said to be **feasible** if there exists at least one feasible point, and **infeasible** otherwise.

Optimal value and points

- The **optimal value** $p^\star$ of the problem is

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, ..., m,\ g_i(x) = 0,\ i = 1, ..., p\}$$

  - $p^\star = \infty$ if the problem is infeasible
  - $p^\star = -\infty$ if the problem is unbounded below
- $x^\star$ is an **optimal point** (or solves the problem) if $x^\star$ is feasible and $f_0(x^\star) = p^\star$
- The set of optimal points is the **optimal set**

$$X_{opt} = \inf\{x \mid f_i(x) \leq 0,\ i = 1, ..., m,\ g_i(x) = 0,\ i = 1, ..., p,\ f_0(x^\star) = p^\star\}$$

- If there exists an optimal point for the problem , we say the optimal value is **attained** or **achieved**, and the problem is **solvable**.
- A point x is **locally optimal** if it is feasible and there is an $R > 0$ such that

$$f(x) = \inf\{f_0(z) \mid f_i(z) \leq 0,\ i = 1, ..., m,\ g_i(z) = 0,\ i = 1, ..., p,\ ||z - x||_2 < R)\}$$

Convex optimization problems

A **convex optimization problem** is of the form

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{s.t } & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & a_i^\top x = b_i, \; i = 1, \ldots, p \text{ (or } Ax = b) \end{aligned}$$

where $f_i, \; i = 0, \ldots, m$ are convex functions. Compared to standard problems, the convex problem has three more requirements:

- the objective function must be convex,
- the inequality constraint functions must be convex,
- the equality constraint functions must be affine

The feasible set of a convex optimization problem is convex.

Local and global optima

Any locally optimal point of a convex problem is also (globally) optimal.

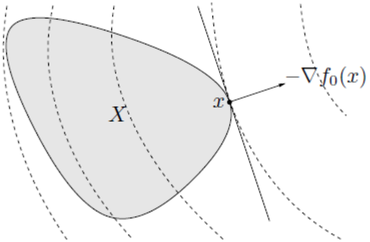
Optimality condition for differentiable f_0

x is optimal iff it is feasible and

$$\nabla f_0(x)^\top (y - x) \geq 0 \text{ for all feasible } y$$

For an unconstrained problem, it reduces to

$$\nabla f_0(x) = 0$$



If $\nabla f_0 \neq 0$, $-\nabla f_0$ defines a supporting hyperplane to the feasible set at x.

Equivalent convex problems

We call two problems **equivalent** if from a solution of one, a solution of the other is readily found, and vice versa.
There are general transformations that yield equivalent problems including:

- change of variables
- transformation of objective and constraint functions
- slack variables
- eliminating or introducing equality constraints
- epigraph problem form

Linear program (LP)

$$\begin{aligned} \text{minimize } & c^\top x + d \\ \text{s.t } & Gx \preceq 0 \\ & Ax = b \end{aligned}$$

where $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$

- It is a convex problem with affine objective and constraint functions.
- The feasible set is a polyhedron.

Quadratic program (QP)

$$
\begin{aligned}
\text{minimize } & (1/2)x^\top Px + q^\top x + r \\
\text{s.t } & Gx \preceq 0 \\
& Ax = b
\end{aligned}
$$

where $P \in S_+^n$ (positive semi-definite), $G \in \mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{p \times n}$

- we minimize a convex quadratic function over a polyhedron

Quadratically constrained quadratic program (QCQP)

$$\begin{aligned} \text{minimize } & (1/2)x^\top P_0 x + q_0^\top x + r_0 \\ \text{s.t } & (1/2)x^\top P_i x + q_i^\top x + r_i \leq 0, \ i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

where $P_i \in S_+^n$

- the objective and the inequality constraint functions are (convex) quadratic
- if $P_i \succ 0$, we minimize a convex quadratic function over a feasible region that is the intersection of ellipsoids

Second-order cone programming (SOCP)

$$\begin{aligned} \text{minimize } & f^\top x \\ \text{s.t } & ||A_i x + b_i||_2 \leq c_i^\top x + d_i, \; i = 1, ..., m \\ & Fx = g \end{aligned}$$

where $A_i \in \mathbb{R}^{n_i \times n}$ and $F \in \mathbb{R}^{p \times n}$

- The inequalities are called **second-order cone constraint** since $(A_i x + b_i, c_i^\top x + d_i) \in$ second-order cone in $\mathbb{R}^{n_i+1}$
- It generalizes LP$(n_i = 0)$ and QCQP$(c_i = 0)$.