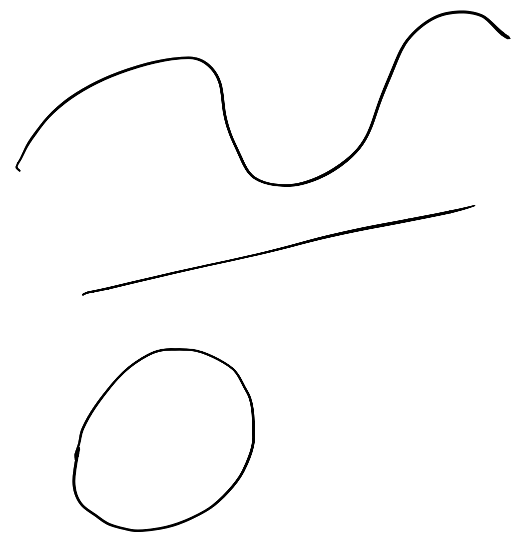
Today: Curves

Representations of curves



a. explicit

b. implicit

c. Parametric

Explicit curves: $y = f(x)$

$y = ax + b$

$y = ax^2 + bx^2 + c$

$y = \log(x)$

implicit curves: $f(x, y) = 0$

$y - ax - b = 0$ line

$y - ax^2 + bx + c = 0$ parabola

$x^2 + y^2 + R^2 = 0$ circle

$\frac{x^2}{A} + \frac{y^2}{B} + R^2 = 0$ ellipse

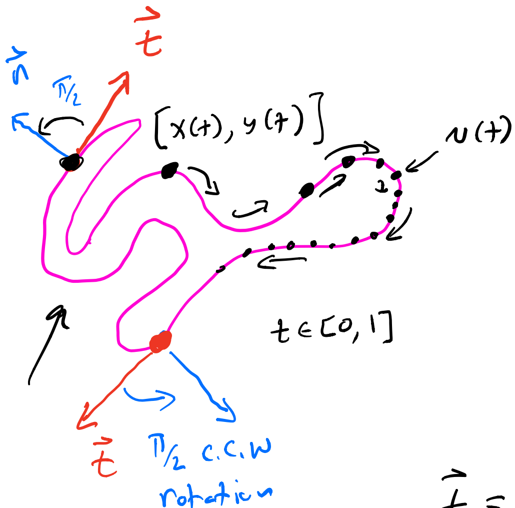
Parametric curves: $(x(t), y(t))$ $t \in [a, b]$

(t, t^2) parabolic

$(\cos(t), \sin(t))$ circle $t \in [0, 2\pi]$

Parametric curves!

$$v(t) = \sqrt{x'(t)^2 + y'(t)^2}$$ speed



$$L = \int_0^1 v(t)\, dt$$ Total length of curve

$$L_{ab} = \int_a^b v(t)\, dt$$ Arc length between a, b

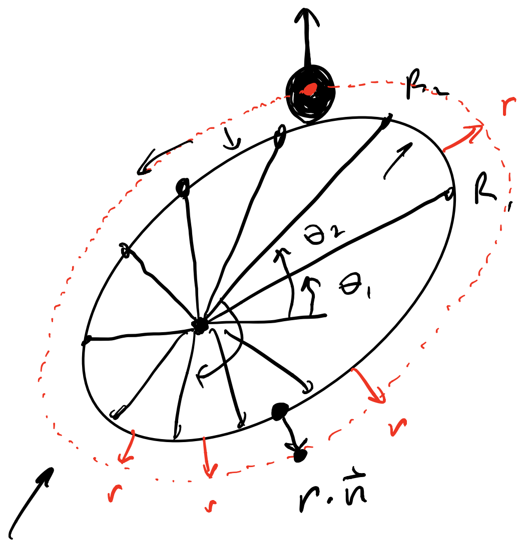
$$\vec{t} = \left(\frac{x'(t)}{v(t)}, \frac{y'(t)}{v(t)} \right)$$ tangent vector

$$\vec{n} = \underline{\underline{R(\pi/2)}} \begin{bmatrix} x'(t)/v(t) \\ y'(t)/v(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x'(t)/v(t) \\ y'(t)/v(t) \end{bmatrix}$$

$$\vec{n} = \left(\frac{-y'(t)}{v(t)}, \frac{x'(t)}{v(t)} \right)$$ Normal Vector

Example:



$t \in [0, 2\pi]$

Cam $(x(t), y(t))$

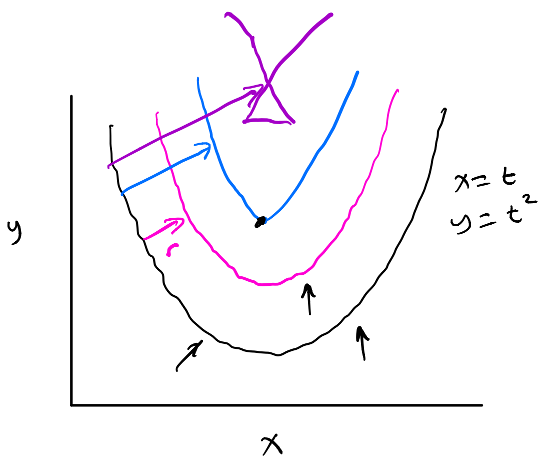
$C(t)$

Offset of a para... curve as

$$\underline{\underline{O_r(t)}} = \underline{\underline{\vec{C}(t)}} + \underline{\underline{r \cdot \vec{n}(t)}}$$

$$\underline{\underline{O_r(t)}} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + r \begin{bmatrix} -\frac{y'(t)}{\nu(t)} \\ \frac{x'(t)}{\nu(t)} \end{bmatrix}$$

$$O_r(t) = \begin{bmatrix} x(t) - \frac{r}{\nu(t)} y'(t) \\ y(t) + \frac{r}{\nu(t)} x'(t) \end{bmatrix}$$

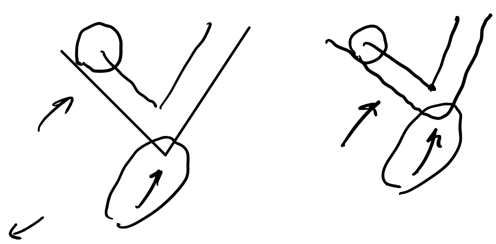


$$C(t) = (t, t^2)$$

$$\nu = \sqrt{x'(t)^2 + y'(t)^2}$$

$$= \sqrt{1 + 4t^2}$$

$$\underline{\underline{O_r(t) = \begin{bmatrix} t - \frac{r}{\sqrt{1+4t^2}} 2t \\ t^2 + \frac{r}{\sqrt{1+4t^2}} 1 \end{bmatrix}}}$$



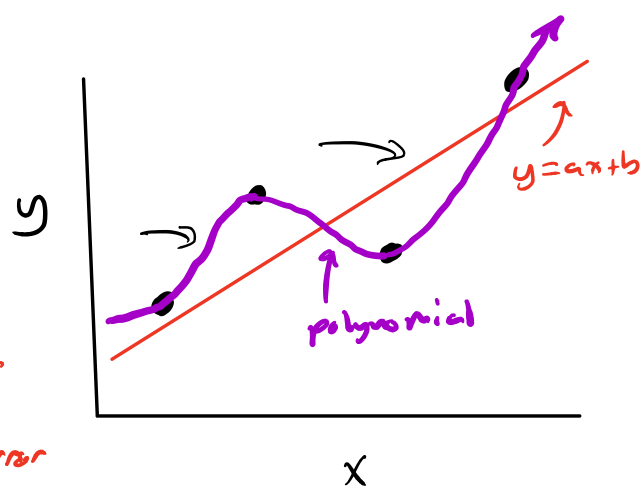
How to generate curve shapes?

Two distinct cases:

- Curves defined by data
  "Fitting"

- Curves designed by anchor and feature points

Fitting Curves:



1) fit data such that curve doesn't pass through all points, but minimizes some quantity. Typ: error

2) Curve that passes through every data point and subject to conditions on $y'(x)$ and $y''(x)$ etc...

1) Least squares fitting

$(x_i, y_i) \quad i \leq N$

linear

$ax_i + b = y_i$

find best a, b

$$f(x_i) = y_i + \epsilon_i$$

$$\epsilon_i = y_i - f(x_i)$$

$$\underline{\underline{\text{error}}} = \sum_i^N \left[y_i - f(x_i)\right]^2$$

for linear fit

$$\text{min} \rightarrow \begin{cases} \dfrac{\partial e}{\partial a} = 0 \\ \dfrac{\partial e}{\partial b} = 0 \end{cases}$$

minimize error by varying parameter in fit function

Rewrite LLS as matrix eqn

$$a x_1 + b = y_0$$
$$a x_2 + b = y_1$$
$$\vdots$$
$$a x_N + b = y_N$$

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

$$\underline{\underline{X}} \cdot \beta = Y$$

$$\boxed{a x_1^3 + b x^2 + c x + d = y_1}$$

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ \vdots & \vdots & & \vdots \\ x_N^3 & x_N^2 & x_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

$$X \cdot \beta = Y$$

$$y = \underline{\underline{a}} \cdot (1 - \underline{\underline{b}} x)$$

$$X\beta = Y$$

Design matrix — coefficents — observation

$$\text{error} = Y - X\beta \longrightarrow \min_{\beta}\left(e^T e\right) = \text{Sum of squares error}$$

$$e^T e = (Y^T - \beta^T X^T)(Y - X\beta)$$

$$= Y^T Y - \underline{\underline{Y^T X\beta}} - \underline{\underline{\beta^T X^T Y}} + \beta^T X^T X\beta$$

$$e^T e = \cancel{Y^T Y} - 2\beta^T X^T Y + \beta^T X^T X\beta$$

$$\frac{\partial(e^T e)}{\partial\beta} = -2X^T Y + \underline{\underline{X^T X\beta}} + \underline{\underline{\beta^T X^T X}} = 0$$

$$= -2X^T Y + 2X^T X\beta = 0$$

$$X^T X\beta = X^T Y$$

$$\boxed{\beta = (X^T X)^{-1} X^T Y}$$

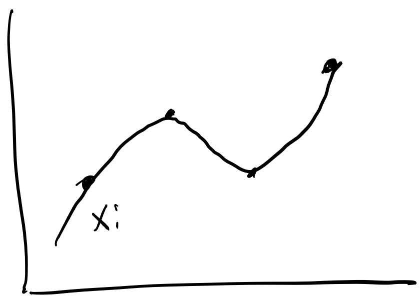
Normal equation

$$y = Ae^x + B$$

$$\begin{bmatrix} e^{x_1} & 1 \\ e^{x_2} & \vdots \\ \vdots & \\ e^{x_N} & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

Define a polynomial passing through all points...

- OLS routine can be problematic
- Lagrange polynomials



$$f(x_i) = y_i$$

N points

$$y(x) = \sum_{k=1}^{N} \ell_k(x) y_k$$

3 #'s 2nd order poly

(0,0), (1,1), (2,4)

$$y(x) = \ell_1(x) y_1 + \ell_2(x) y_2 + \ell_3(x) y_3$$

$$\ell_k = \prod_{\substack{0 \le m \le k \\ m \ne j}} \frac{(x - x_m)}{(x_j - x_m)}$$

$$\ell_k(x) = \begin{cases} 1 & x = x_k \\ 0 & x \ne x_k \end{cases}$$

(0,0)
(1,1)
(2,4)

$$\ell_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x-1)(x-2)}{(-1)(-2)}$$

$$\ell_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x-0)\cdot(x-2)}{(1)\cdot(-1)}$$

$$\ell_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(x-0)(x-1)}{2 \cdot 1}$$

$$y(x) = \cancel{l_1(x)\, y_1} + l_2(x)\, y_2 + l_3(x)\, y_3$$

$$= \frac{x(x-2)}{-1}\, 1 + \frac{x \cdot (x-1)}{2}\, 4$$

$$= -x^2 + \cancel{2x} + 2x^2 - \cancel{2x}$$

$$\boxed{y(x) = x^2}$$

Lagrange polynomial