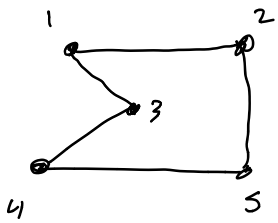
CAD operations $\longrightarrow$ transformations of 2D & 3D points

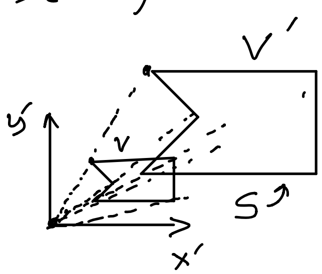


$\leftarrow (x_1, y_1), (x_2, y_2) \ldots (x_5, y_5)$

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} x_5 \\ y_5 \end{bmatrix}$$

$$V = \begin{bmatrix} x_1 & x_2 & \cdots & x_5 \\ y_1 & y_2 & & y_5 \end{bmatrix}$$

a. Scaling



$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$V' = S\,V$$

$$\begin{bmatrix} s_x \cdot x_1 & \cdots & s_x x_5 \\ s_y \cdot y_1 & & s_y y_5 \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_5 \\ y_1 & y_2 & & y_5 \end{bmatrix}$$

b. Translation

$$x_i' = x_i + t_x \qquad \vec{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$y_i' = y_i + t_y$$

$$\underset{V'}{\begin{bmatrix} x_1+t_x & \cdots & x_5+t_x \\ y_1+t_y & & y_5+t_x \end{bmatrix}} = \underbrace{\begin{bmatrix} x_1 & \cdots & x_5 \\ y_1 & & y_5 \end{bmatrix}}_{\text{points}} + \underbrace{\begin{bmatrix} t_x & \cdots & t_x \\ t_y & & t_y \end{bmatrix}}_{\text{translation}}$$

$$V' = S_1 V$$

$$V'' = V' + \underline{\underline{t_1}}$$

$$V''' = S_2 V''$$

$$V'''' = V''' + \underline{\underline{t_2}}$$

$$V'' = SV + t_1$$

$$V'' = S_2 \cdot (S_1 V + t_1)$$

$$V'''' = S_2 \cdot (S_1 V + t_1) + t_2$$

$$= \underline{S_2 \cdot S_1 V} + \underline{S_2 \cdot t_1} + \underline{t_2}$$

Tedious, ungraceful

Better way

Homogeneous coordinates

Define points N-dimensions using $\underline{\underline{N+1}}$ dimensional representation

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad 2D \qquad\qquad 3D \quad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

added row

$$V = \begin{bmatrix} x_1 & \cdots & x_5 \\ y_1 & \cdots & y_5 \\ 1 & & 1 \end{bmatrix} \leftarrow \text{dummy row}$$

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$V' = T \cdot V$$

$$\begin{bmatrix} x_1 + t_x & \cdots & x_5 + t_x \\ y_1 + t_y & \cdots & y_5 + t_y \\ 1 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 0 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_5 \\ y_1 & \cdots & y_5 \\ 1 & & 1 \end{bmatrix}$$

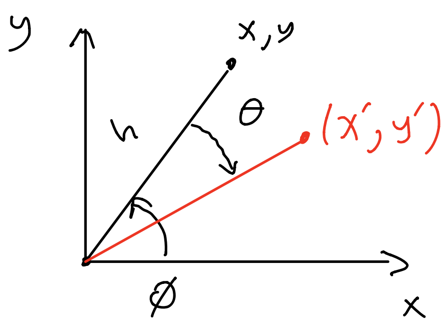
$$V' = T \cdot V$$

$$\underline{\underline{V''''}} = T_2 S_2 T_1 S_1 V$$

$$S_1 = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

a. Rotations



$$x = h\cos\phi$$
$$y = h\sin\phi$$

$$x' = h\cos(\phi - \theta)$$
$$y' = h\sin(\phi - \theta)$$

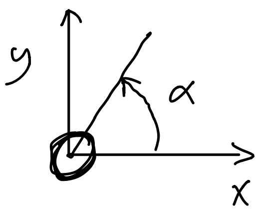
$$\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$$
$$\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$$

$$x' = \underbrace{h[\cos(\phi)}_{x}\cos(\theta) + \underbrace{\sin(\phi)}_{y}\sin(\theta)]$$
$$y' = \underbrace{h[\sin(\phi)}_{y}\cos(\theta) - \underbrace{\cos(\phi)}_{x}\sin(\theta)]$$

$$x' = x\cos(\theta) + y\sin(\theta)$$
$$y' = y\cos(\theta) - x\sin(\theta)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Clockwise θ



Counter-clockwise

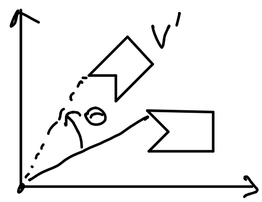
$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

CCW

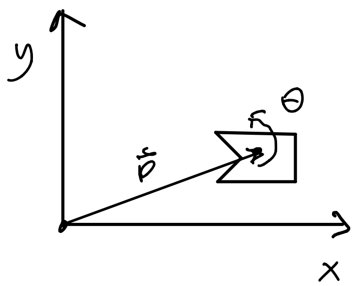
$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta = 0 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

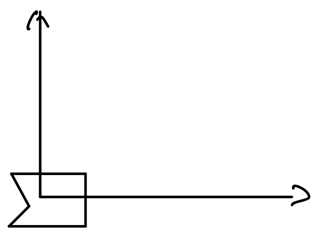
Rotations about an arbitrary point?



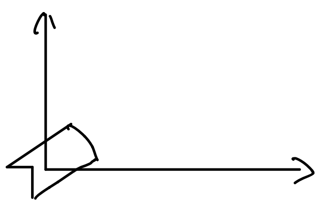
$$V' = \underline{\underline{R(\theta)}}\ V \qquad R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



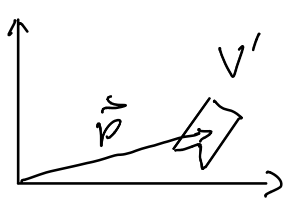
Rotate about $\vec{p}$ by θ CCW



$T(-p_x, -p_y)$ brings $\vec{p}$ to the origin

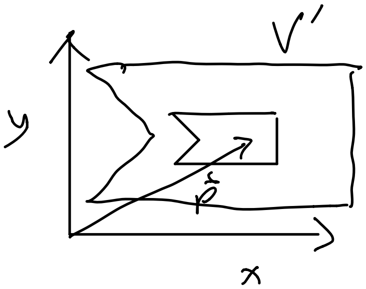


$R(\theta)$



$T(p_x, p_y)$

$$V' = \underline{\underline{T(p_x,p_y)\ R(\theta)}}\ \underline{\underline{T(-p_x,-p_y)}}\ V$$



$$V' = \underbrace{T(p_x, p_y)\, S(s_x, s_y)\, T(-p_x, -p_y)}_{3\times 3} V$$

$$H = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

← Rigid body transformation

Combined rotation and translation

$$S = \begin{bmatrix} S_x & 0 & t_x \\ 0 & S_y & t_y \\ 0 & 0 & 1 \end{bmatrix}$$