1) Introduce derivation of $\{F\} = [K]\{D\}$ $F = kx$

2) Study bar elements for 1-D FEA

3) Study truss elements for 2-D truss structures

4) (Thursday) Study beam elements for bending FEA

The fundamental FEA equation

$$\{F\} = [K]\{D\}$$

Forces

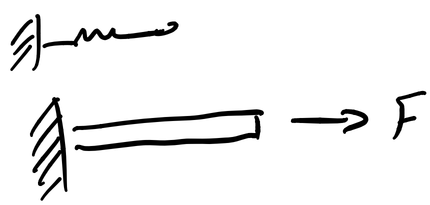
Global Stiffness matrix

Displacements

$\{ \; \}$ → column vector

We want Displacements of our system, given some boundary conditions and given applied forces

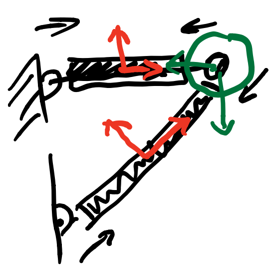
Bar



Uniaxial tension / comp.

$\sigma_x = E\epsilon_x$

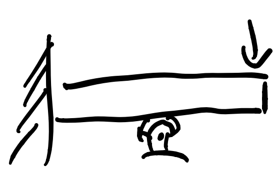
Truss



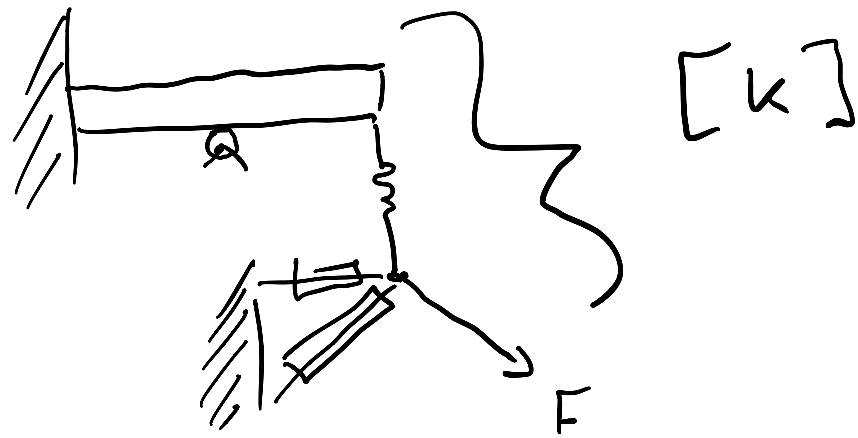
Tension & compression but arranged in 2D/3D

$\sigma_x = E\epsilon_x$

Beam



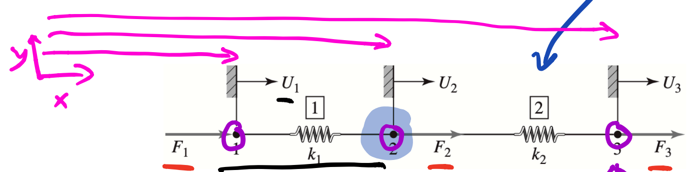
Bending moment but no tension/compression



[K]
F

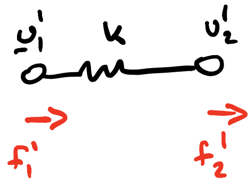
$$\{F\} = [K]\{U\}$$

Elements



Forces are applied at nodes only

Nodes



$$f_1' = k u_1 - k u_2$$

$$f_2' = k u_2 - k u_1$$

$$\boxed{k'\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix}}$$

Element stiffness matrix

Element 1 →
$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

Element 2 →
$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Relative elemental position

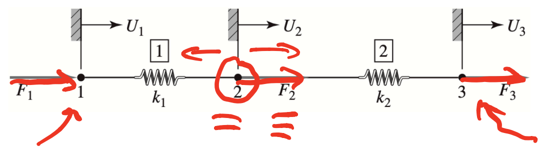
Absolute position

$$u_1^{(1)} = U_1 \qquad u_2^{(1)} = U_2 \qquad u_1^{(2)} = U_2 \qquad u_2^{(2)} = U_3$$

#1
$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

#2
$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$



#1

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \end{Bmatrix}$$

#2

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

$$f_1^{(1)} = F_1 \qquad f_2^{(1)} + f_2^{(2)} = F_2 \qquad f_3^{(2)} = F_3$$

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$[k]\underline{\underline{\{U\}}} = \{F\}$$

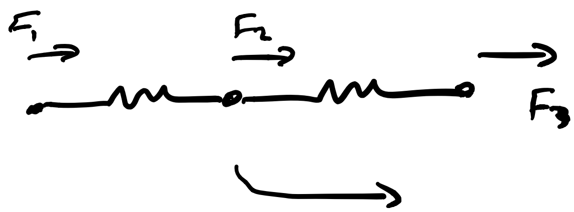
$$\underline{\underline{[K]\{U\} = \{F\}}} \qquad [K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

Our goal is to solve for displacements with applied forces {F} at equilibrium

$$\{\underline{U}\} = [K]^{-1}\{F\} \qquad ??$$

known variables

$[K]^{-1}$ as written above is singular, no inverse

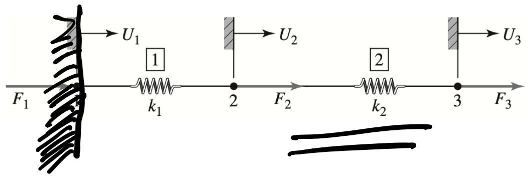


Singular matrix ⇒ unsolvable, and in this case signifies the rigid body motion of the springs

To solve $\{F\} = [K]\{U\}$

we need boundary conditions

Node 1 is attached to a fixed support, yielding the displacement constraint $U_1 = 0$.
$k_1 = 50$ lb./in., $k_2 = 75$ lb./in., $F_2 = F_3 = 75$ lb.



$\{U\}$

$$\begin{bmatrix} \cancel{50} & \cancel{-50} & \cancel{0} \\ \cancel{-50} & 125 & -75 \\ \cancel{0} & -75 & 75 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 75 \\ 75 \end{Bmatrix}$$

$[K]\{U\} = \{F\}$

$$\begin{bmatrix} 125 & -75 \\ -75 & 75 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 75 \\ 75 \end{Bmatrix}$$

$$\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = [K]^{-1} \begin{Bmatrix} 75 \\ 75 \end{Bmatrix}$$

$U_2 = 3$ in. and $U_3 = 4$ in

$$\begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix} \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} -150 \\ 150 \end{Bmatrix} \text{lb.} \quad \text{for element 1}$$

$$\begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} -75 \\ 75 \end{Bmatrix} \text{lb.} \quad \text{for element 2}$$

1) Construct element stiffness equations
2) Combined into a global stiffness matrix using compatibility conditions at adjacent nodes
3) Apply boundary conditions
4) Solve reduced system for U
5) Solve for unknown forces

3 spring elements in series

spring : $k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

#1 $[k^{(1)}] = \begin{bmatrix}3k & -3k\\ -3k & 3k\end{bmatrix}$

#2 $[k^{(2)}] = \begin{bmatrix}2k & -2k\\ -2k & 2k\end{bmatrix}$

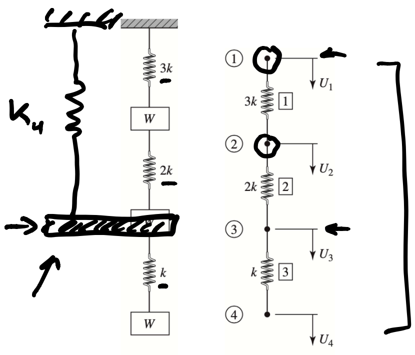
#3 $[k^{(3)}] = \begin{bmatrix}k & -k\\ -k & k\end{bmatrix}$

$\rightarrow$ $u_1^{(1)} = U_1$ $\quad u_2^{(1)} = u_1^{(2)} = U_2$

$u_2^{(2)} = u_1^{(3)} = U_3$ $\quad u_2^{(3)} = U_4$

#1 $3k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$

$2k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_2\\ U_3\end{Bmatrix} = \begin{Bmatrix}F_2\\ F_3\end{Bmatrix}$



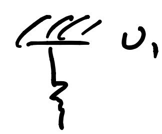
$k_4\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ F_3\end{Bmatrix}$

#4 $\begin{bmatrix}k_4 & 0 & -k_4 & 0\\ 0 & 0 & 0 & 0\\ -k_4 & 0 & k_4 & \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}$

$$\begin{bmatrix}3k & -3k & 0 & 0\\ -3k & 3k & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix} = \begin{Bmatrix}f_1^{(1)}\\ f_2^{(1)}\\ 0\\ 0\end{Bmatrix}$$ #1

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 2k & -2k & 0\\ 0 & -2k & 2k & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix} = \begin{Bmatrix}0\\ f_1^{(2)}\\ f_2^{(2)}\\ 0\end{Bmatrix}$$ #2

$$\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & k & -k\\ 0 & 0 & -k & k\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix} = \begin{Bmatrix}0\\ 0\\ f_1^{(3)}\\ f_2^{(3)}\end{Bmatrix}$$ #3



$U_1 = 0$

$$k\begin{bmatrix}\cancel{3} & \cancel{-3} & \cancel{0} & \cancel{0}\\ -3 & 5 & -2 & 0\\ 0 & -2 & 3 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix} = \begin{Bmatrix}F_1\\ W\\ W\\ W\end{Bmatrix}$$

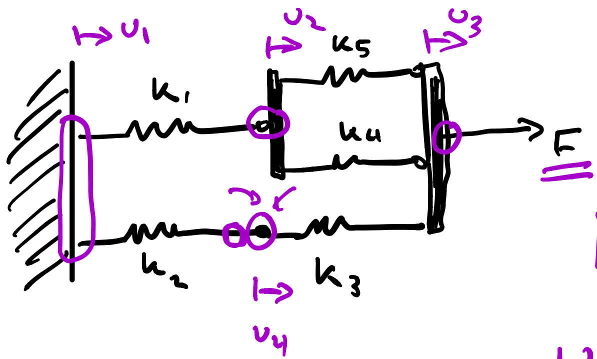
$$k\begin{bmatrix}5 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_2\\ U_3\\ U_4\end{Bmatrix} = \begin{Bmatrix}W\\ W\\ W\end{Bmatrix}$$

Imposed forces

$\{U\} = [K]^{-1}\{F\}$

$$U_2 = \frac{W}{k} \qquad U_3 = \frac{2W}{k} \qquad U_4 = \frac{3W}{k}$$

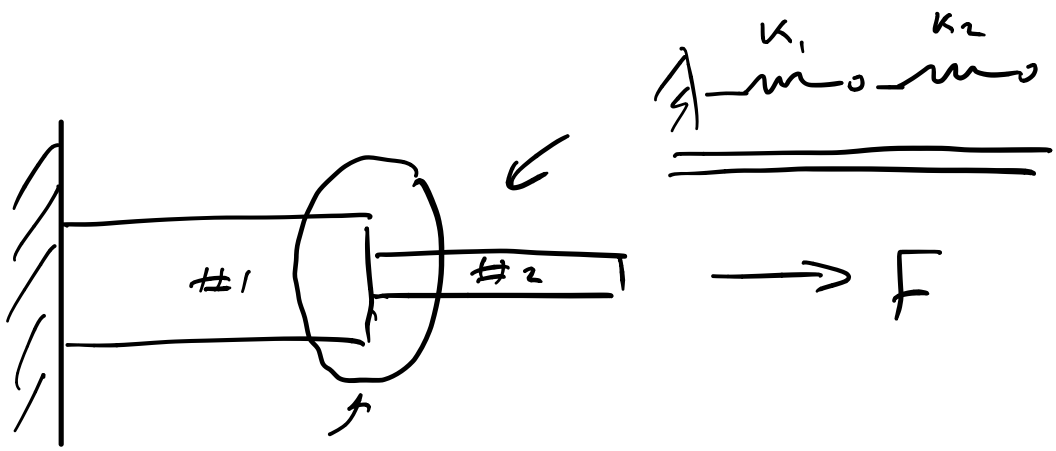
Spring elements have two nodes, each described by 1-variable of displacement

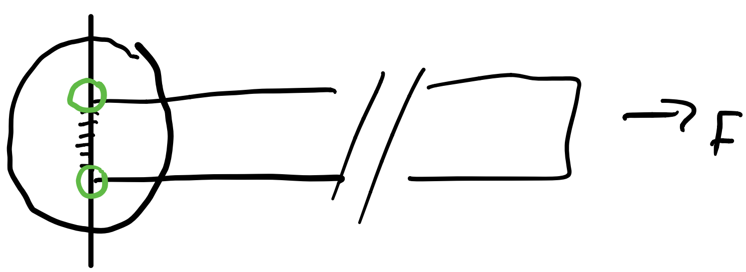


What is size of global stiffness [k]?

[k] is 4x4

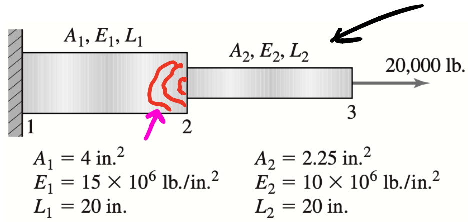
$$U = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

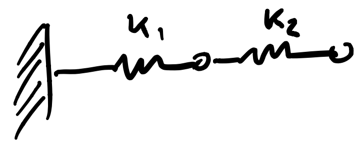

K1
K2
#1
#2
F


F

Bar element

Area, Youngs modulus, Length





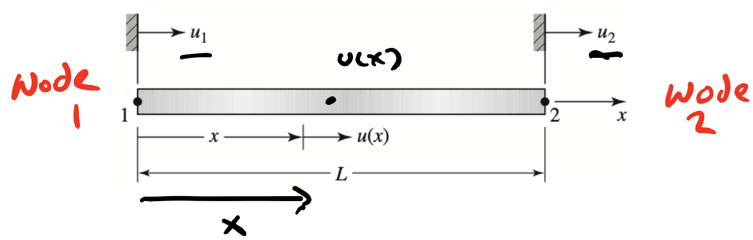
$$k_1 = \frac{A_1 E_1}{L_1} \qquad k_2 = \frac{A_2 E_2}{L_2}$$

#1 $k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

#2 $k_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Goal in FEA is to have a continuous representation of stress and strain

$$u(x) = N_1(x)u_1 + N_2(x)u_2$$



B.C. $u(x = 0) = u_1 \qquad u(x = L) = u_2$

$$N_1(0) = 1 \qquad N_2(0) = 0$$
$$N_1(L) = 0 \qquad N_2(L) = 1$$

$$N_1(x) = a_0 + a_1 x$$
$$N_2(x) = b_0 + b_1 x$$

$$N_1(x) = 1 - x/L$$
$$N_2(x) = x/L$$

$$u(x) = (1 - x/L)u_1 + (x/L)u_2$$

Continuous displacement field of 1-D Bar

$$u(x) = [N_1(x) \quad N_2(x)]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{u\}$$

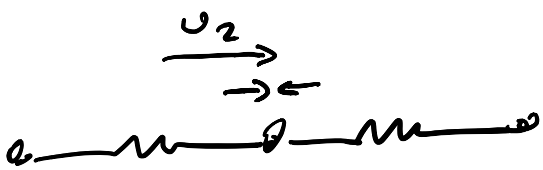
$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \{U\}$$

$$u(x) = [N]\{U\}$$

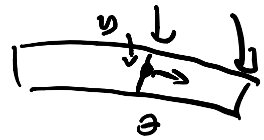
shape functions

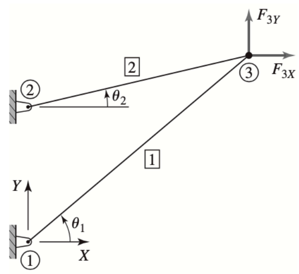
Interpolation functions

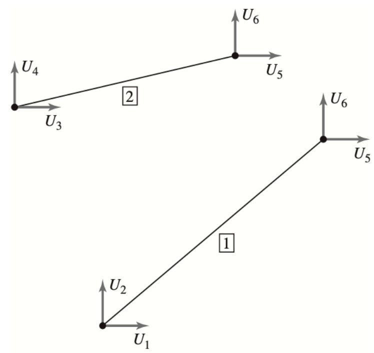
Typically polynomials

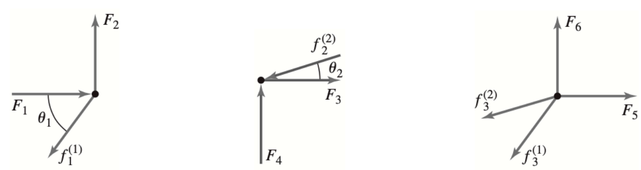


compatibility

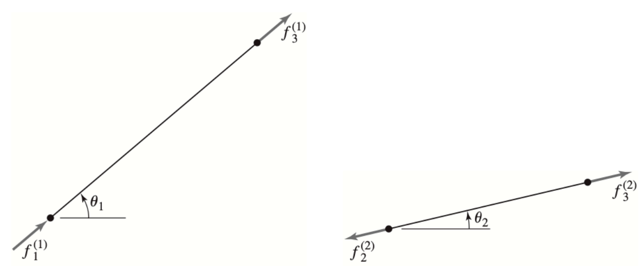








(a) (b) (c)

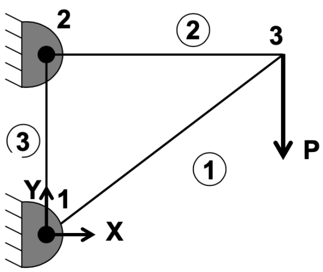


$$F_1 - f_1^{(1)} \cos\theta_1 = 0 \qquad F_3 - f_2^{(2)} \cos\theta_2 = 0 \qquad F_5 - f_3^{(1)} \cos\theta_1 - f_3^{(2)} \cos\theta_2 = 0$$

$$F_2 - f_1^{(1)} \sin\theta_1 = 0 \qquad F_4 - f_2^{(2)} \sin\theta_2 = 0 \qquad F_6 - f_3^{(1)} \sin\theta_1 - f_3^{(2)} \sin\theta_2 = 0$$

$$\begin{bmatrix} k^{(1)}c^2\theta_1 & k^{(1)}s\theta_1 c\theta_1 & 0 & 0 & -k^{(1)}c^2\theta_1 & -k^{(1)}s\theta_1 c\theta_1 \\ k^{(1)}s\theta_1 c\theta_1 & k^{(1)}s^2\theta_1 & 0 & 0 & -k^{(1)}s\theta_1 c\theta_1 & -k^{(1)}s^2\theta_1 \\ 0 & 0 & k^{(2)}c^2\theta_2 & k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}c^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 \\ 0 & 0 & k^{(2)}s\theta_2 c\theta_2 & k^{(2)}s^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}s^2\theta_2 \\ -k^{(1)}c^2\theta_{12} & -k_1 s\theta_1 c\theta_1 & -k^{(2)}c^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 & \begin{matrix} k^{(1)}c^2\theta_1 + \\ k^{(2)}c^2\theta_2 \end{matrix} & \begin{matrix} k^{(1)}s\theta_1 c\theta_1 + \\ k^{(2)}s\theta_2 c\theta_2 \end{matrix} \\ -k_1 s\theta_1 c\theta_1 & -k^{(1)}s^2\theta_1 & -k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}s^2\theta_2 & \begin{matrix} k^{(1)}s\theta_1 c\theta_1 + \\ k^{(2)}s\theta_2 c\theta_2 \end{matrix} & \begin{matrix} k^{(1)}s^2\theta_1 + \\ k^{(2)}s^2\theta_2 \end{matrix} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

$$[K^e] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$



$$[K^e] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} & -\frac{k_1}{2} & -\frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & \frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix},$$

$$K_2 = \begin{bmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix},$$

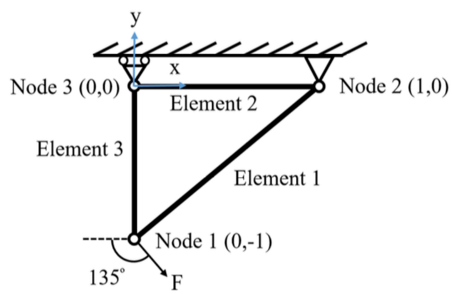
$$\begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ 0 \\ 0 \\ -P \end{Bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix},$$

$$\begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} + k_3 & 0 & -k_3 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & -k_3 & 0 & k_3 & 0 & 0 \\ -\frac{k_1}{2} & -\frac{k_1}{2} & -k_2 & 0 & k_2 + \frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ -P \end{Bmatrix}.$$

$$\begin{bmatrix} k_2 + \frac{k_1}{2} & \frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \end{Bmatrix}$$


y
x
Node 3 (0,0)
Node 2 (1,0)
Element 2
Element 3
Element 1
Node 1 (0,-1)
135°
F

10.15 Two dimensional FEM analysis

10.15.1 Triangular elements

Figure 10.12 shows a plane constant-strain triangle. The triangle has six degrees of freedom. The shape functions (interpolation functions)

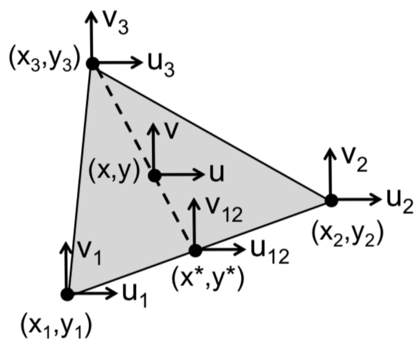


Figure 10.12: Plane, constant strain triangle.

defined at the nodes allows calculating the displacement at each point within the element:

$$u_{12} = \frac{1}{2}\left(\frac{x^* - x_2}{x_1 - x_2} + \frac{y^* - y_2}{y_1 - y_2}\right)u_1 + \frac{1}{2}\left(\frac{x^* - x_1}{x_2 - x_1} + \frac{y^* - y_1}{y_2 - y_1}\right)u_2. \tag{10.164}$$

Along the line (x^*, y^*) to (x_3, y_3) one can define

$$u = \frac{1}{2}\left(\frac{x - x_3}{x^* - x_3} + \frac{y - y_3}{y^* - y_3}\right)u_{12} + \frac{1}{2}\left(\frac{x - x^*}{x_3 - x^*} + \frac{y - y^*}{y_3 - y^*}\right)u_3. \tag{10.165}$$

Slope condition:

$$\frac{y^* - y_1}{x^* - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \tag{10.166}$$

and

$$\frac{y - y_3}{x - x_3} = \frac{y^* - y_3}{x^* - x_3}. \tag{10.167}$$

Solving eqns. (10.165), (10.166), and (10.167), one obtains

$$u = \frac{p_1(x,y)\,u_1 + p_2(x,y)\,u_2 + p_3(x,y)\,u_3}{\Delta}, \tag{10.168}$$

where

$$p_1(x,y) = x_2 y_3 - x_3 y_2 + x(y_2 - y_3) + y(x_3 - x_2), \tag{10.169}$$

$$p_2(x,y) = x_3 y_1 - x_1 y_3 + x(y_3 - y_1) + y(x_1 - x_3), \tag{10.170}$$

$$p_3(x,y) = x_1 y_2 - x_2 y_1 + x(y_1 - y_2) + y(x_2 - x_1), \tag{10.171}$$

and

$$\Delta = x_2 y_3 - x_1 y_3 - x_3 y_2 + x_1 y_2 + x_3 y_1 - x_2 y_1. \tag{10.172}$$

A similar expression can be derived for v in terms of v_1, v_2 and v_3. Thus the displacement of an arbitrary point within an element can be written in terms of the displacement at the nodes as

$$\boldsymbol{d} = \left\{ \begin{array}{c} u \\ v \end{array} \right\} = \frac{1}{\Delta} \begin{bmatrix} p_1(x,y) & p_2(x,y) & p_3(x,y) & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1(x,y) & p_2(x,y) & p_3(x,y) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{10.173}$$

or

$$\boldsymbol{d} = [N]\,\boldsymbol{d^e}. \tag{10.174}$$

Expressing the strains ε as a function of the nodal displacements $\boldsymbol{d^e}$, one finds

$$\boldsymbol{\varepsilon} = \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{array} \right\} = [B]\,\boldsymbol{d^e}. \tag{10.175}$$

Since $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$ and $\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{ \begin{array}{c} u \\ v \end{array} \right\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N]\,\boldsymbol{d^e}. \tag{10.176}$$

Thus,

$$[B] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N] \tag{10.177}$$

or

$$[B] = \frac{1}{\Delta} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix}. \tag{10.178}$$

Note that $[B]$ is not a function of x or y. The stiffness matrix

$$[\boldsymbol{K}] = \int_V [B]^t\,[D]\,[B]\,dV \tag{10.179}$$