1) Constraints in trusses ←

2) Direction cosines ←

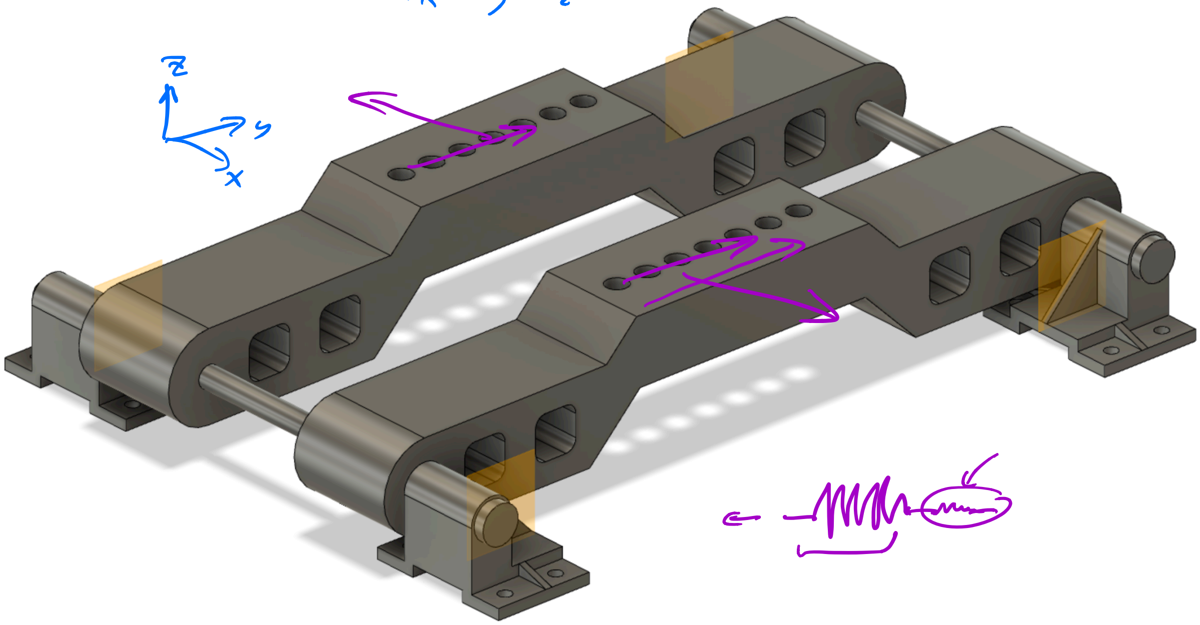
3) Beam elements
   a. Derive shape functions
   b. The stiffness matrix

4) An extended beam problem ] ✓

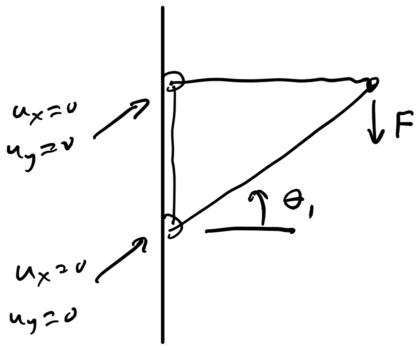
5) Compare with fusion 360

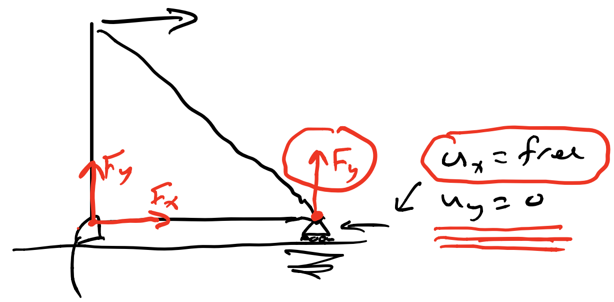
$$\{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_y \\ 0 \end{Bmatrix}$$

$u_x = 0,\ u_z = 0$



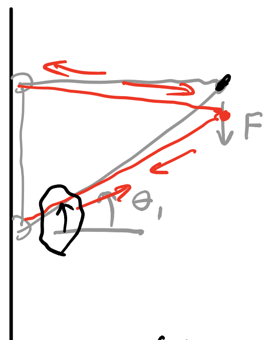
# Truss constraints




$u_x = 0$

$u_y = 0$



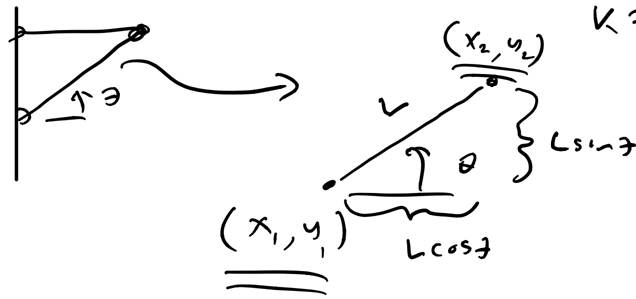

$$u = \begin{Bmatrix} 0 \\ 0 \\ u_x \\ 0 \\ 0 \end{Bmatrix} \quad F = \begin{Bmatrix} F_1 \\ F_2 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$K_{11} = \cos^2 \theta$$

$$\underbrace{[k]}_{\text{constant}}[u] = [F]$$

[k] never changes during structural deformation

# Direction cosines:




$$K = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$c = \cos\theta$

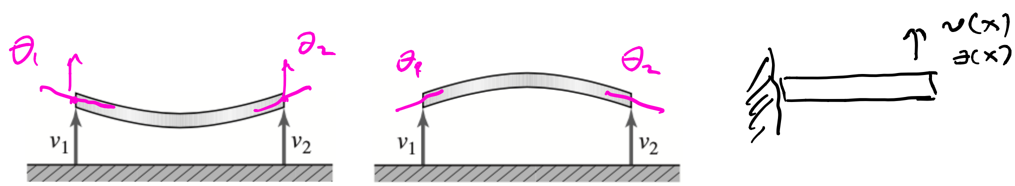
$s = \sin\theta$

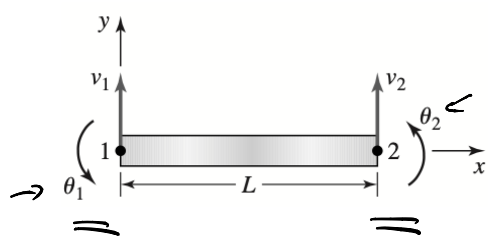
$$\rightarrow \quad \cos\theta = \frac{x_2 - x_1}{L}$$

$$\rightarrow \quad \sin\theta = \frac{y_2 - y_1}{L}$$

# Beam elements

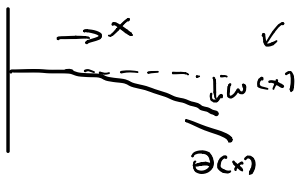
Bar & spring elements can only support load axially. Beams can handle bending. To model bending we need to "track" not only the deflection but also the angle.




$$\left\{ \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \right\} = \{u\}$$

$v(x) = w(x) =$ vertical deflection



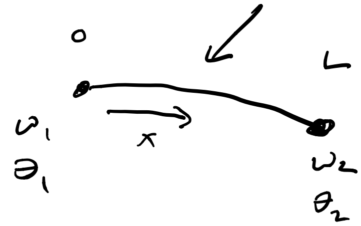

Beam equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w(x)}{dx^2}\right) = 0$$

$\int\int\int\int$

$$w(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4.$$

Element




$$w(0) = w_1, \qquad \left.\frac{dw}{dx}\right|_{x=0} = \theta_1,$$

$$w(L) = w_2, \qquad \left.\frac{dw}{dx}\right|_{x=L} = \theta_2.$$

$$w(0) = w_1 = C_4, \qquad w(L) = C_1 L^3 + C_2 L^2 + C_3 L + C_4 = w_2,$$

$$\left.\frac{dw}{dx}\right|_{x=0} = \theta_1 = C_3, \qquad \left.\frac{dw}{dx}\right|_{x=L} = 3C_1 L^2 + 2C_2 L + C_3 = \theta_2.$$

$$\left.\begin{aligned}
C_1 &= \frac{1}{L^3}\left(2w_1 + L\theta_1 - 2w_2 + L\theta_2\right),\\
C_2 &= \frac{1}{L^2}\left(-3w_1 - 2L\theta_1 + 3w_2 - L\theta_2\right),\\
C_3 &= \theta_1,\\
C_4 &= w_1.
\end{aligned}\right\}$$

$$w(x) = C_1x^3 + C_2x^2 + C_3x + C_4$$

$$w(x) = \left[N_1(x) \;\; N_2(x) \;\; \ldots\right]\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$w(x) = \left[\frac{1}{L^3}\left(2w_1 + L\theta_1 - 2w_2 + L\theta_2\right)\right]x^3 + \left[\frac{1}{L^2}\left(-3w_1 - 2L\theta_1 + 3w_2 - L\theta_2\right)\right]x^2 + \theta_1 x + w_1$$

$$w(x) = \underbrace{\left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right]}_{N_1(x)} w_1 + \underbrace{\left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]}_{N_2(x)} L\theta_1 + \underbrace{\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]}_{N_3(x)} w_2 + \underbrace{\left[-\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right)^3\right]}_{N_4(x)} L\theta_2$$

$$\underline{w(x)} = \left[N_1(x) \;\; N_2(x) \;\; N_3(x) \;\; N_4(x)\right]\underline{\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}}$$

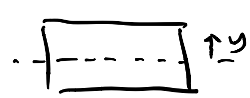
What about $k$?

let's write out potential energy

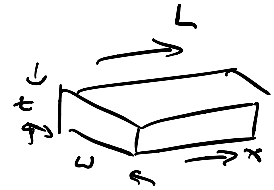
$$U_e = \frac{1}{2}\int_V \sigma_x \, \varepsilon_x \, \partial V$$

$$\underline{\varepsilon_x} = -\underline{y}\,\frac{\partial^2 w}{\partial x^2}$$

$$\underline{\sigma_x} = E\,\varepsilon_x$$



$$U_e = \frac{E}{2}\int_V y^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \partial V$$



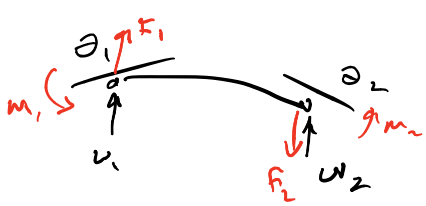

$$U_e = \frac{E}{2}\int \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \underbrace{\left(\int_A y^2 \partial A\right)}_{I_z} \partial x$$

$$\partial V = \partial A \cdot \partial x$$

$$U_e = \frac{EI_z}{2}\int_0^L \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \partial x$$

Castigliano's first theorem:

force acting along a nodal direction is equal partial derivative of potential energy w.r.t. the nodal direction




$$F_1 = \frac{\partial U}{\partial w_1} \qquad F_2 = \frac{\partial U}{\partial w_2}$$

$$M_1 = \frac{\partial U}{\partial \theta_1} \qquad M_2 = \frac{\partial U}{\partial \theta_2}$$

$$w(x) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\left(\frac{\partial^2}{\partial x^2} w(x)\right)^2 = \left(\begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \end{bmatrix} \{U\}\right)^2$$

$$\frac{\partial}{\partial x} f(x)^2 = 2f(x)\frac{\partial f}{\partial x}$$

$$F_1 = \frac{\partial U}{\partial w_1} = EI \int_0^L \left( \frac{\partial^2 N_1}{\partial x^2} w_1 + \frac{\partial^2 N_2}{\partial x^2} \theta_1 + \frac{\partial^2 N_3}{\partial x^2} w_2 + \frac{\partial^2 N_4}{\partial x^2} \theta_2 \right) \frac{\partial^2 N_1}{\partial x^2} \, \partial x$$

$$M_1 = \frac{\partial U}{\partial \theta_1} = EI \int_0^L \left( \frac{\partial^2 N_1}{\partial x^2} w_1 + \frac{\partial^2 N_2}{\partial x^2} \theta_1 + \frac{\partial^2 N_3}{\partial x^2} w_2 + \frac{\partial^2 N_4}{\partial x^2} \theta_2 \right) \frac{\partial^2 N_2}{\partial x^2} \, \partial x$$

$$F_2 = \frac{\partial U}{\partial w_2} = EI \int_0^L \left( \frac{\partial^2 N_1}{\partial x^2} w_1 + \frac{\partial^2 N_2}{\partial x^2} \theta_1 + \frac{\partial^2 N_3}{\partial x^2} w_2 + \frac{\partial^2 N_4}{\partial x^2} \theta_2 \right) \frac{\partial^2 N_3}{\partial x^2} \, \partial x$$

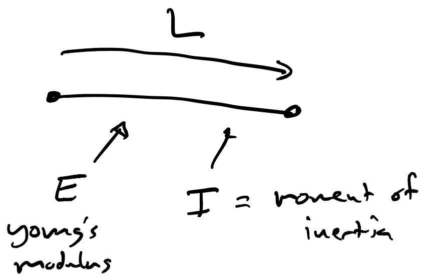
$$M_2 = \frac{\partial U}{\partial \theta_2} = EI \int_0^L \left( \frac{\partial^2 N_1}{\partial x^2} w_1 + \frac{\partial^2 N_2}{\partial x^2} \theta_1 + \frac{\partial^2 N_3}{\partial x^2} w_2 + \frac{\partial^2 N_4}{\partial x^2} \theta_2 \right) \frac{\partial^2 N_4}{\partial x^2} \, \partial x$$

$k_{11}$

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} & & \\ & k & \\ & & \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$K_{11} = EI \int_0^L \frac{\partial^2 N_1}{\partial x^2} \frac{\partial^2 N_1}{\partial x^2} \, \partial x$$

$$K_{mn} = EI \int_0^L \frac{\partial^2 N_m}{\partial x^2} \frac{\partial^2 N_n}{\partial x^2} \, \partial x$$

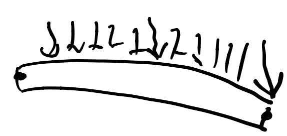


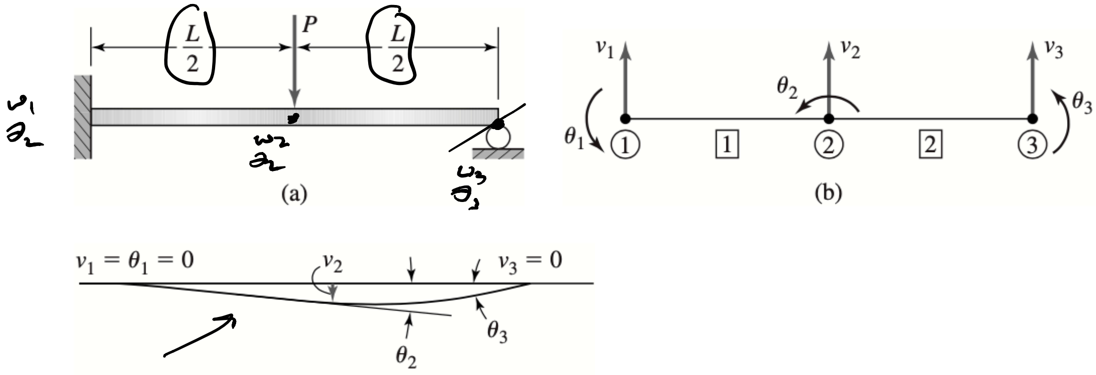

$$\frac{12EI}{L^3}$$

$$k_{11} = EI\int_0^L \frac{\partial^2 N_1}{\partial x^2}\frac{\partial^2 N_1}{\partial x^2}\,\partial x$$

$$[k_e] = \frac{EI_z}{L^3}\begin{array}{c} \left[\begin{array}{cccc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \end{array}\begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array}$$

Loads only act at nodes!!





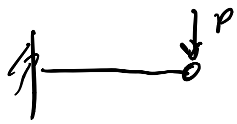

$$[k^{(1)}] = [k^{(2)}] = \frac{EI_z}{(L/2)^3}\begin{bmatrix} 12 & 6L/2 & -12 & 6L/2 \\ 6L/2 & 4L^2/4 & -6L/2 & 2L^2/4 \\ -12 & -6L/2 & 12 & -6L/2 \\ 6L/2 & 2L^2/4 & -6L/2 & 4L^2/4 \end{bmatrix}$$

$$= \frac{8EI_z}{L^3}\begin{bmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & L^2/2 \\ -12 & -3L & 12 & -3L \\ 3L & L^2/2 & -3L & L^2 \end{bmatrix}$$

$$K = 6 \times 6$$

$$K_1 = \frac{8EI}{L^3}\begin{bmatrix} 12 & 3L & -12 & 3L & 0 & 0 \\ 3L & L^2 & -3L & L^2/2 & 0 & 0 \\ -12 & -3L & 12 & -3L^2 & 0 & 0 \\ 3L & L^2/2 & -3L & L^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_2 = \frac{8EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 3L & -12 & 3L \\ 0 & 0 & 3L & L^2 & -3L & L^2/2 \\ 0 & 0 & -12 & -3L & 12 & -3L^2 \\ 0 & 0 & 3L & L^2/2 & -3L & L^2 \end{bmatrix}$$




$w_1 = 0$ $\quad w_2 = \text{free}$

$\theta_1 = 0$ $\quad \theta_2 = \text{free}$


$w_2 = \text{free}$ $\quad w_3 = 0$

$\theta_2 = \text{free}$ $\quad \theta_3 = \text{free}$

$$\frac{EI_z}{L^3}\overbrace{\begin{bmatrix} 96 & 24L & -96 & 24L & 0 & 0 \\ 24L & 8L^2 & -24L & 4L^2 & 0 & 0 \\ -96 & -24L & 192 & 0 & -96 & 24L \\ 24L & 4L^2 & 0 & 16L^2 & -24L & 4L^2 \\ 0 & 0 & -96 & -24L & 96 & 24L \\ 0 & 0 & 24L & 4L^2 & 24L & 8L^2 \end{bmatrix}}^{K}\begin{Bmatrix} \cancel{v_1} \\ \cancel{\theta_1} \\ v_2 \\ \theta_2 \\ \cancel{v_3} \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix}$$

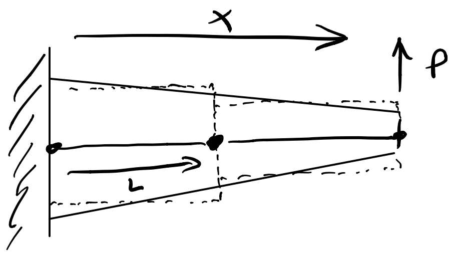
$$\frac{EI_z}{L^3}\begin{bmatrix} 192 & 0 & 24L \\ 0 & 16L^2 & 4L^2 \\ 24L & 4L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix}$$

$$v_2 = \frac{-7PL^3}{768EI_z} \qquad \theta_2 = \frac{-PL^2}{128EI_z} \qquad \theta_3 = \frac{PL^2}{32EI_z}$$

$$F_1 = \frac{EI_z}{L^3}(-96v_2 + 24L\theta_2) = \frac{11P}{16}$$

$$F_3 = \frac{EI_z}{L^3}(-96v_2 - 24L\theta_2 - 24L\theta_3) = \frac{5P}{16}$$

$$M_1 = \frac{EI_z}{L^3}(-24Lv_2 + 4L^2\theta_2) = \frac{3PL}{16}$$

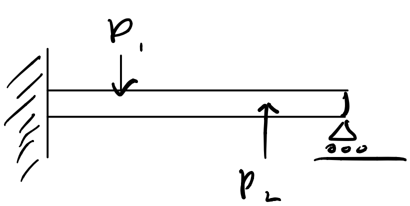



$$I(x)$$

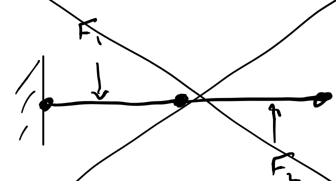
$$I_{ave} = \frac{1}{L}\int_{x_{start}}^{x_{end}} I(x)\,dx$$

Loads also inform node locations

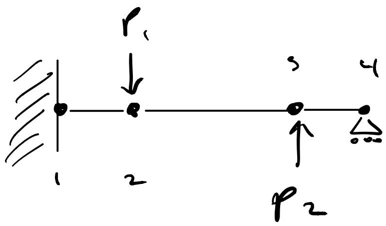
"Loads at nodes"

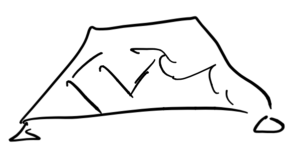



→




forces not at nodes
not convenient
to solve



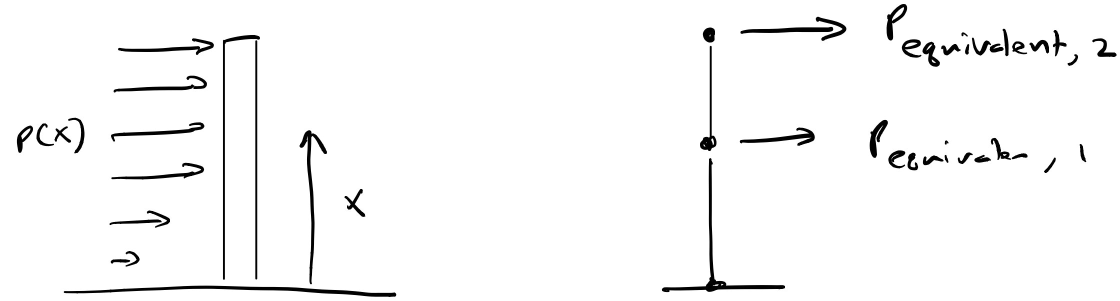
Constraints: $w_1 = 0$ $\quad w_4 = 0$
$\theta_1 = 0$

$F_2 = P_1 \quad F_3 = P_2 \quad F_4 = ?$

$M_2 = 0 \quad M_3 = 0 \quad F_1 = ?$

$M_4 = 0 \quad M_1 = ?$

→ Node locations informed by geometry or load locations




When we have continuous pressure on beams, need to apply equivalent forces / moments at nodes

How to get equivalent loads?

Work equivalence

$$W = \int_0^L p(x)\, w(x)\, dx = F_1 w_1 + M_1 \theta_1 + F_2 w_2 + M_2 \theta_2$$

work done on beam

pressure

deflection

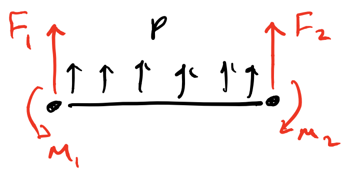
$$w(x) = \left[ N_1(x) w_1 + N_2(x) \theta_1 + N_3(x) w_2 + N_4 \theta_2 \right]$$

$$W = \int_0^L p(x)\left[N_1(x) w_1 + N_2(x)\theta_1 + N_3(x) w_2 + N_4 \theta_2\right] = F_1 w_1 + M_1\theta_1 + F_2 w_2 + M_2\theta_2$$

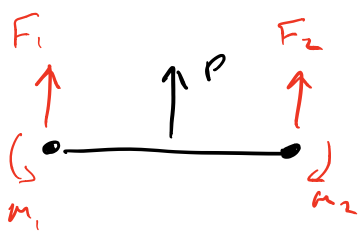
$$F_1 = \int_0^L p(x) N_1(x)\,dx \qquad M_1 = \int_0^L p(x) N_2(x)\,dx$$

$$F_2 = \int_0^L p(x) N_3(x)\,dx \qquad M_2 = \int_0^L p(x) N_4(x)\,dx$$

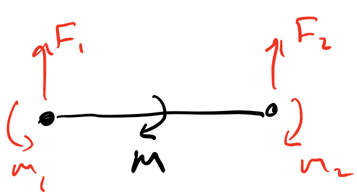
Equivalent forces & moments solved from above 4 equations




$$\longrightarrow \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix}$$




$$\longrightarrow \begin{Bmatrix} P/2 \\ PL/8 \\ P/2 \\ PL/8 \end{Bmatrix}$$




$$\longrightarrow \begin{Bmatrix} 3M/2L \\ M/4 \\ -3M/2L \\ M/4 \end{Bmatrix}$$

## 10.15 *Two dimensional FEM analysis*

### 10.15.1 *Triangular elements*

Figure 10.12 shows a plane constant-strain triangle. The triangle has six degrees of freedom. The shape functions (interpolation functions)

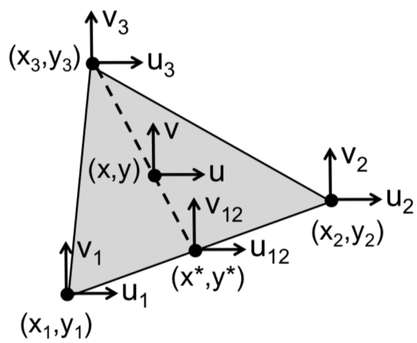



Figure 10.12: Plane, constant strain triangle.

defined at the nodes allows calculating the displacement at each point within the element:

$$u_{12} = \frac{1}{2}\left(\frac{x^*-x_2}{x_1-x_2} + \frac{y^*-y_2}{y_1-y_2}\right)u_1 + \frac{1}{2}\left(\frac{x^*-x_1}{x_2-x_1} + \frac{y^*-y_1}{y_2-y_1}\right)u_2. \tag{10.164}$$

Along the line $(x^*, y^*)$ to $(x_3, y_3)$ one can define

$$u = \frac{1}{2}\left(\frac{x-x_3}{x^*-x_3} + \frac{y-y_3}{y^*-y_3}\right)u_{12} + \frac{1}{2}\left(\frac{x-x^*}{x_3-x^*} + \frac{y-y^*}{y_3-y^*}\right)u_3. \tag{10.165}$$

Slope condition:

$$\frac{y^*-y_1}{x^*-x_1} = \frac{y_2-y_1}{x_2-x_1} \tag{10.166}$$

and

$$\frac{y-y_3}{x-x_3} = \frac{y^*-y_3}{x^*-x_3}. \tag{10.167}$$

Solving eqns. (10.165), (10.166), and (10.167), one obtains

$$u = \frac{p_1(x,y)\,u_1 + p_2(x,y)\,u_2 + p_3(x,y)\,u_3}{\Delta}, \tag{10.168}$$

where

$$p_1(x,y) = x_2y_3 - x_3y_2 + x(y_2-y_3) + y(x_3-x_2), \tag{10.169}$$

$$p_2(x,y) = x_3y_1 - x_1y_3 + x(y_3-y_1) + y(x_1-x_3), \tag{10.170}$$

$$p_3(x,y) = x_1 y_2 - x_2 y_1 + x(y_1 - y_2) + y(x_2 - x_1), \tag{10.171}$$

and

$$\Delta = x_2 y_3 - x_1 y_3 - x_3 y_2 + x_1 y_2 + x_3 y_1 - x_2 y_1. \tag{10.172}$$

A similar expression can be derived for $v$ in terms of $v_1$, $v_2$ and $v_3$. Thus the displacement of an arbitrary point within an element can be written in terms of the displacement at the nodes as

$$\boldsymbol{d} = \left\{ \begin{array}{c} u \\ v \end{array} \right\} = \frac{1}{\Delta} \begin{bmatrix} p_1(x,y) & p_2(x,y) & p_3(x,y) & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1(x,y) & p_2(x,y) & p_3(x,y) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{10.173}$$

or

$$\boldsymbol{d} = [N]\, \boldsymbol{d}^e. \tag{10.174}$$

Expressing the strains $\varepsilon$ as a function of the nodal displacements $\boldsymbol{d}^e$, one finds

$$\boldsymbol{\varepsilon} = \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{array} \right\} = [B]\, \boldsymbol{d}^e. \tag{10.175}$$

Since $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$ and $\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{ \begin{array}{c} u \\ v \end{array} \right\} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N]\, \boldsymbol{d}^e. \tag{10.176}$$

Thus,

$$[B] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N] \tag{10.177}$$

or

$$[B] = \frac{1}{\Delta} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix}. \tag{10.178}$$

Note that $[B]$ is not a function of $x$ or $y$. The stiffness matrix

$$[\boldsymbol{K}] = \int_V [B]^t\, [D]\, [B]\, dV \tag{10.179}$$

where $[D]$ is a matrix that defines the material properties. Assuming plane stress, the matrix is

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \qquad (10.180)$$

but assuming plane strain provides

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}. \qquad (10.181)$$

Equation (10.179) produces the element stiffness matrix. Assembling all individual element stiffness matrices, using the method described in section 10.5, produces the global stiffness matrix.