1) Loads not at nodes

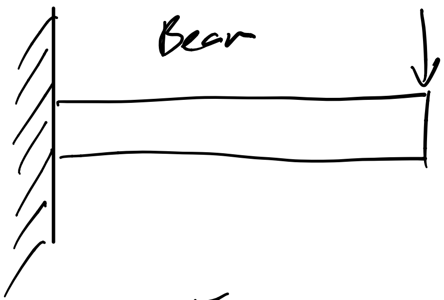
2) 3-D elements

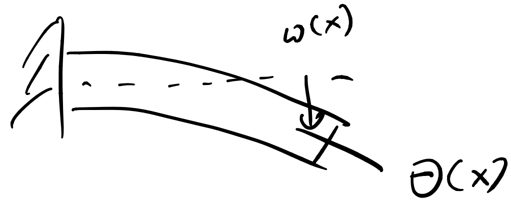
- plane stress (thin elements)
- plane strain (thick elements)
- General 3D element

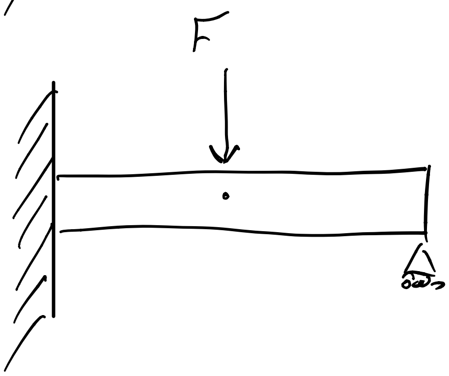
3) modal analysis

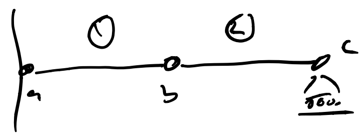
4) Fusion 360

- modal analysis
- Shape optimization
- Constraint peculiarities

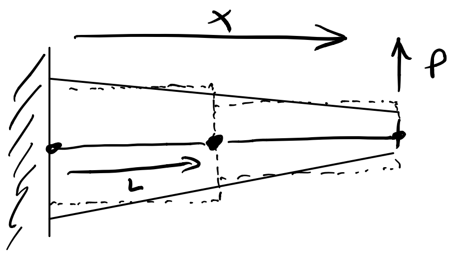








How do we deal with distributed loads?

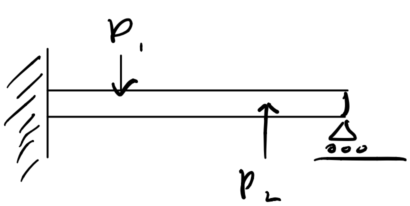


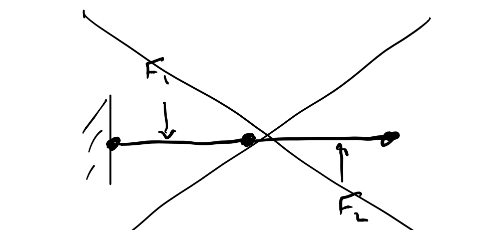
$$I(x)$$

$$I_{ave} = \frac{1}{L}\int_{x_{start}}^{x_{end}} I(x)\,dx$$

Loads also inform node locations

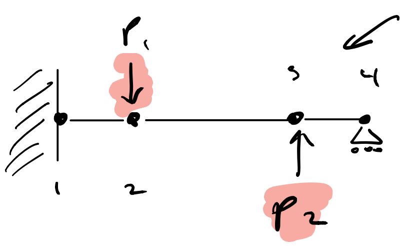
"Loads at nodes"

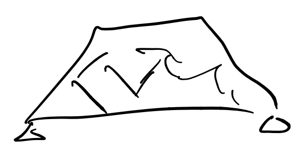




forces not at nodes

not convenient to solve





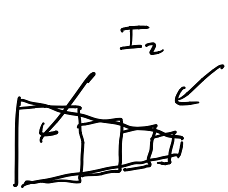
Constraints: $w_1 = 0$ $\quad \theta_1 = 0$ $\quad w_4 = 0$

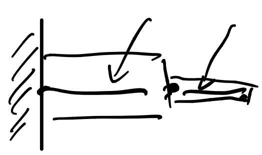
$F_2 = P_1 \quad F_3 = P_2 \quad F_4 = ?$

$M_2 = 0 \quad M_3 = 0 \quad M_4 = 0$

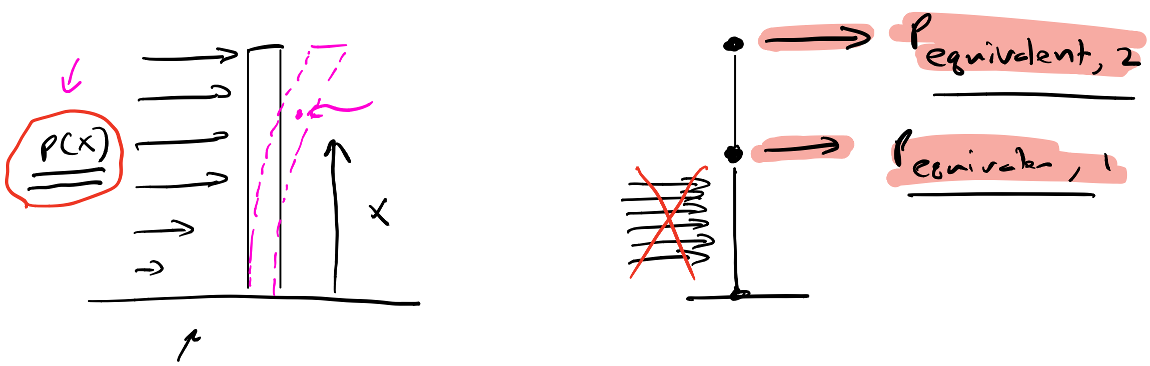
$F_1 = ?$

$M_1 = ?$



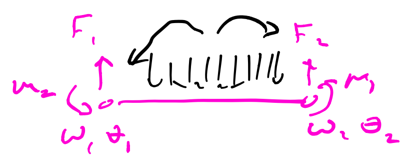


→ **Node locations informed by geometry or load locations**



[When we have continuous pressure on beams, need to apply equivalent forces (moments) at nodes

How to get equivalent loads?



Work equivalence

$$W = \int_0^L p(x)\, w(x)\, dx = F_1 w_1 + M_1 \theta_1 + F_2 w_2 + M_2 \theta_2$$

work done on beam

pressure

deflection

$$w(x) = [N_1(x) w_1 + N_2(x) \theta_1 + N_3(x) w_2 + N_4(x) \theta_2]$$

$$W = \int_0^L p(x)\left[N_1(x)\,w_1 + N_2(x)\,\theta_1 + N_3(x)\,w_2 + N_4\,\theta_2\right] = F_1 w_1 + M_1\theta_1 + F_2 w_2 + M_2\theta_2$$

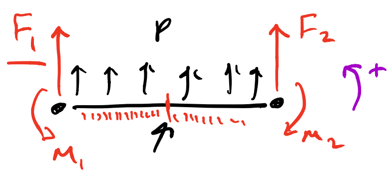
$$w_1 \int_0^L p(x) N_1(x) \;\cdots\; = w_1 F_1 \;\cdots$$

$$F_1 = \int_0^L p(x) N_1(x)\,dx \qquad M_1 = \int_0 p(x) N_2(x)\,dx$$

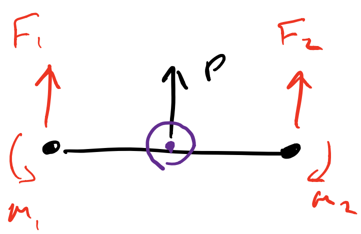
$$F_2 = \int_0^L p(x) N_3(x)\,dx \qquad M_2 = \int_0^L p(x) N_4(x)\,dx$$

$p(x) = p_0$

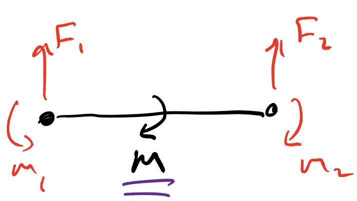
Equivalent forces & moments solved from above 4 equations



$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix}$$

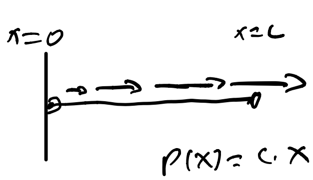


$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} P/2 \\ PL/8 \\ P/2 \\ -PL/8 \end{Bmatrix}$$



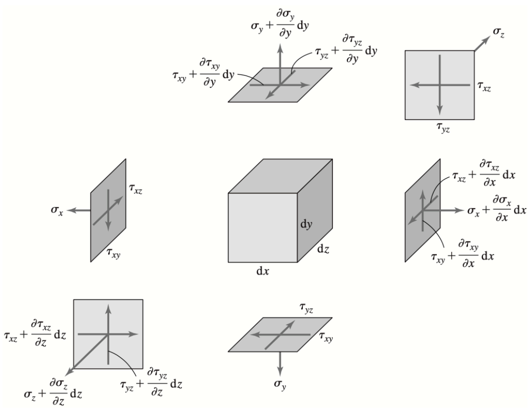
$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 3M/2L \\ M/4 \\ -3M/2L \\ M/4 \end{Bmatrix}$$

Bar or spring



$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \int p(x)u(x) = F_1 u_1 + F_2 u_2$$

Full 3D structures & 3D elements



$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad [L] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix}$$

$$\{\varepsilon\} = [L]\{\delta\}$$

$\{F\} = [K]\{\delta\}$ displacements

$= [K]\{u\}$

x, y, z

$\delta = [u, v, w]$

$u(x,y,z)$

$\delta = [u(x,y,z),\ v(x,y,z),\ w(x,y,z)]$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

6×6

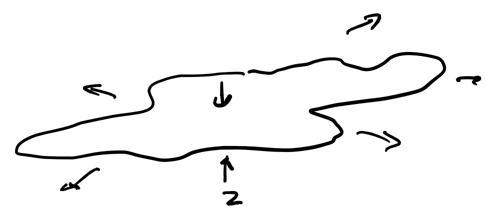
$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = [D]\{\varepsilon\} = [D][L]\{\delta\}$$

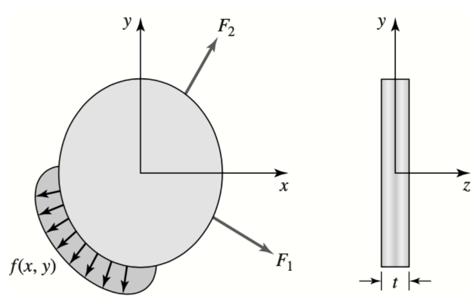
$$E = \frac{\sigma_x}{\varepsilon_x} \qquad \nu = -\frac{\text{unit lateral contraction}}{\text{unit axial elongation}}$$

Continuous representation

$$\Sigma F = 0 \qquad \begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + B_x = 0 \\[2ex] \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + B_y = 0 \\[2ex] \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + B_z = 0 \end{cases}$$

Plane stress condition





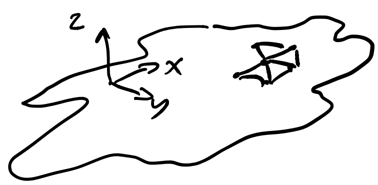
1. The body is small in one coordinate direction (the z direction by convention) in comparison to the other dimensions; the dimension in the z direction (hereafter, the thickness) is either uniform or symmetric about the xy plane; thickness t, if in general, is less than one-tenth of the smallest dimension in the xy plane, would qualify for "small."
2. The body is subjected to loading only in the xy plane.
3. The material of the body is linearly elastic, isotropic, and homogeneous.

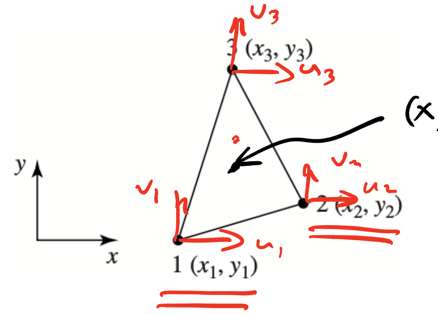
$$\underline{\underline{\sigma_z = \tau_{xz} = \tau_{yz} = 0}}$$

$$\overbrace{\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}}^{3\times 1} = \underbrace{\frac{E}{1-\nu^2} \overbrace{\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}}^{3\times 3}}_{D_{\text{plane-strain}}} \overbrace{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}}$$

Plane stress

triangular element





u(x, y)
v(x, y)
displacements

Shape function

$$N_1(x, y) = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2(x, y) = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$N_3(x, y) = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$u(x, y) = N_1(x, y)u_1 + N_2(x, y)u_2 + N_3(x, y)u_3 = [N]\{u\}$$
$$v(x, y) = N_1(x, y)v_1 + N_2(x, y)v_2 + N_3(x, y)v_3 = [N]\{v\}$$

$$\{\delta^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \qquad \{f\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \\ f_{1y} \\ f_{2y} \\ f_{3y} \end{Bmatrix}$$

$\varepsilon = [L]\{\delta\}$

$= [L][N]\{\delta^e\}$

$[B] = [L][N]$

$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [B]\{\delta^{(e)}\}$$

$$[k] = V^e [B]^T [D][B]$$

$$[k] = \frac{Et}{4A(1-\nu^2)} \begin{bmatrix} \beta_1^2 + C\gamma_1^2 & \beta_1\beta_2 + C\gamma_1\gamma_2 & \beta_1\beta_3 + C\gamma_1\gamma_3 & \frac{1+\nu}{2}\beta_1\gamma_1 & \nu\beta_1\gamma_2 + C\beta_2\gamma_1 & \nu\beta_1\gamma_3 + C\beta_3\gamma_1 \\ & \beta_2^2 + C\gamma_2^2 & \beta_2\beta_3 + C\gamma_2\gamma_3 & \nu\beta_2\gamma_1 + C\beta_1\gamma_2 & \frac{1+\nu}{2}\beta_2\gamma_2 & \nu\beta_2\gamma_3 + C\beta_3\gamma_2 \\ & & \beta_3^2 + C\gamma_3^2 & \nu\beta_3\gamma_1 + C\beta_1\gamma_3 & \nu\beta_3\gamma_2 + C\beta_2\gamma_3 & \frac{1+\nu}{2}\beta_3\gamma_3 \\ & SYM & & \gamma_1^2 + C\beta_1^2 & \gamma_1\gamma_2 + C\beta_1\beta_2 & \gamma_1\gamma_3 + C\beta_1\beta_3 \\ & & & & \gamma_2^2 + C\beta_2^2 & \gamma_2\gamma_3 + C\beta_2\beta_3 \\ & & & & & \gamma_3^2 + C\beta_3^2 \end{bmatrix}$$

$$C = (1-\nu)/2$$

$$N_1(x, y) = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$
$$= \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y)$$
$$N_2(x, y) = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$
$$= \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y)$$
$$N_3(x, y) = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$
$$= \frac{1}{2A}(\alpha_3 + \beta 2x + \gamma_3 y)$$

$\rightarrow$ $\alpha_1 = x_2y_3 - x_3y_2$

$\rightarrow$ $\beta_1 = y_2 - y_3$

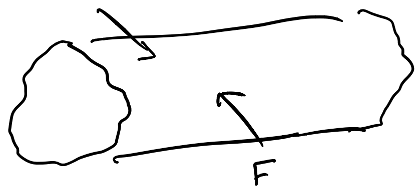
$\rightarrow$ $\gamma_1 = x_3 - x_2$

$$\{F\} = [k]\{\delta^e\}$$

$\rightarrow$ $\{F\} = [k]\{u\}$

solve for $\{u\}$

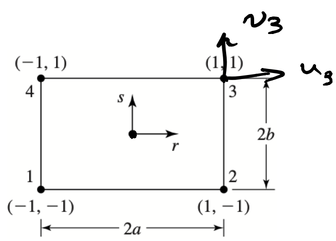
Plane strain condition:



$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Let's look at a rectangular element



$$N_1(r,s) = \frac{1}{4}(1-r)(1-s)$$
$$N_2(r,s) = \frac{1}{4}(1+r)(1-s)$$
$$N_3(r,s) = \frac{1}{4}(1+r)(1+s)$$
$$N_4(r,s) = \frac{1}{4}(1-r)(1+s)$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

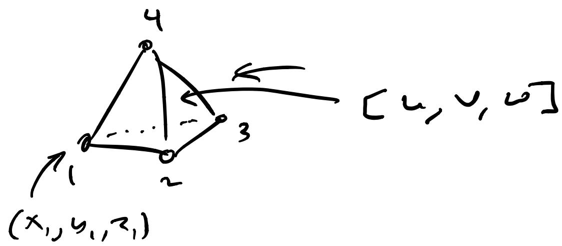
$$[B] = [L][N]$$

$$\{\varepsilon\} = [B]\{\delta\}$$

nodal displacements

$$\left[k^{(e)}\right] = \iiint_{V^{(e)}} [B^T][D][B]\, \mathrm{d}V^{(e)}$$

A full 3D general element



$$\{\varepsilon\} = \overbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix}}^{L} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [L] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

shape function

nodal coords

$$\left[\begin{aligned} u(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) u_i \\ v(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) v_i \\ w(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) w_i \end{aligned}\right.$$

$$\{\delta\} = [N]\{\delta^e\}$$

$$\{\delta\} = [\underbrace{u_1 \quad u_2 \quad \cdots \quad u_M}_{x\text{-displace}} \quad \underbrace{v_1 \quad v_2 \quad \cdots \quad v_M}_{y} \quad \underbrace{w_1 \quad w_2 \quad \cdots \quad w_M}_{z}]^T$$

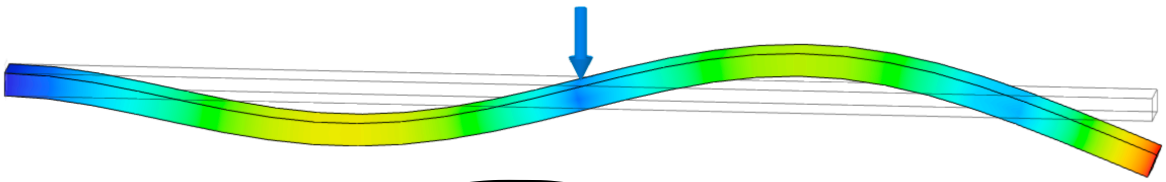
$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix} \{\delta\} = [N_3]\{\delta\}$$

$$[N] = [N_1 \quad N_2 \quad \cdots \quad N_M]$$

$$[B] = [L][N_3] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix}$$

$$[k] = \iiint_V [B]^T [D][B] \, \mathrm{d}V$$

$$[K]\{\Delta\} = \{F\}$$



$\{F\} = [K]\{x\} \rightarrow \{F\} = [K]\{u\} \rightarrow \{F\} = [K]\{\delta\}$

The static equilibrium problem

$\dot{x} = 0$
$\ddot{x} = 0$

Vibrations

acceleration

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0$$

$[M]$ = mass matrix

$$x = A\sin\omega t.$$

$2\pi f$

$$\rightarrow \dot{x} = A\omega\cos\omega t$$

$$\rightarrow \ddot{x} = -A\omega^2\sin\omega t = -\omega^2 x.$$

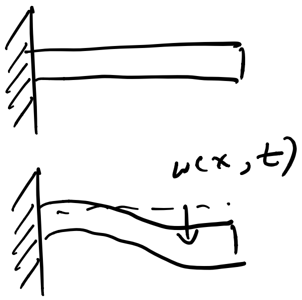
$$\rightarrow \left(-\omega^2[M] + [K]\right)\boldsymbol{x} = 0$$

$$\left\{[M]^{-1}[K] - \omega^2[I]\right\}\boldsymbol{x} = 0.$$

$$\left|[M]^{-1}[K] - \omega^2[I]\right| = 0$$

solve for ω and then x

Need mass matrix



$$w(x,t) = \sum_{i=1}^{4} \psi_i(x) w_i(t)$$

B. C.

$$\psi_1(0) = 1; \psi_1'(0) = \psi_1(L) = \psi_1'(L) = 0$$
$$\psi_2'(0) = 1; \psi_2(0) = \psi_2(L) = \psi_2'(L) = 0$$
$$\psi_3(L) = 1; \psi_3(0) = \psi_3'(0) = \psi_3'(L) = 0$$
$$\psi_4'(L) = 1; \psi_4(0) = \psi_4'(0) = \psi_4(L) = 0.$$

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = \rho A \frac{\partial^2 w}{\partial t^2} = 0$$

$$\left| [M]^{-1} [K] - \omega^2 [I] \right| = 0$$

$$K_{ij} = \int_0^L EI \psi_i''(x) \psi_j''(x)\, dx.$$

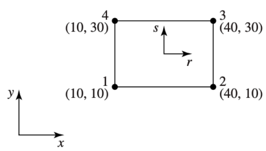
$$M_{ij} = \int_0^L \rho A \psi_i(x) \psi_j(x)\, dx,$$

Same K from equilib.

$$[M] = \left(\frac{\rho A L}{420}\right) \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

$$[K] = \left(\frac{EI}{L^3}\right) \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Formulate the mass matrix for the two-dimensional rectangular element depicted in Figure 10.12. The element has uniform thickness 5 mm and density $\rho = 7.83 \times 10^{-6}$ kg/mm^3.



$$[m^{(e)}] = \begin{bmatrix} 2.6 & 1.3 & 0.7 & 1.3 & 0 & 0 & 0 & 0 \\ 1.3 & 2.6 & 1.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0.7 & 1.3 & 2.6 & 1.3 & 0 & 0 & 0 & 0 \\ 1.3 & 0.7 & 1.3 & 2.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.6 & 1.3 & 0.7 & 1.3 \\ 0 & 0 & 0 & 0 & 1.3 & 2.6 & 1.3 & 0.7 \\ 0 & 0 & 0 & 0 & 0.7 & 1.3 & 2.6 & 1.3 \\ 0 & 0 & 0 & 0 & 1.3 & 0.7 & 1.3 & 2.6 \end{bmatrix} (10)^{-3} \text{ kg}$$

shape functions

constant density

$$[m^{(e)}] = \iiint_{V^{(e)}} \begin{bmatrix} [N]^T[N] & 0 & 0 \\ 0 & [N]^T[N] & 0 \\ 0 & 0 & [N]^T[N] \end{bmatrix} \rho \, dV^{(e)}$$