1) Generalized definition of elemental stiffness and global stiffness

2) Energy derivation of stiffness eqn

3) Truss structures

4) ~~Beams~~

5) Fusion 360 Demo

$$u(x) = (1 - x/L)u_1 + (x/L)u_2$$

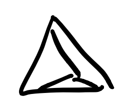
$$u(x) = [N_1(x) \quad N_2(x)]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{u\}$$

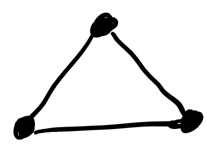
We express the continuous displacement across each element as

$[N]\{u\}$

shape or interpolation functions

nodal coordinates





$[N]\{u\}$ allows us to estimate the true deflection within all elements

Let's derive the stiffness equation from the shape functions this time:

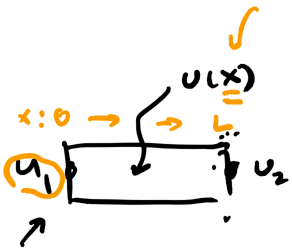
We want to solve equilibrium equations

thus: want nodal displacements vs forces

Displacement → strain

strain → stress

stress → loading



$$\varepsilon_x = \frac{\partial u(x)}{\partial x}$$

$$u(x) = [N_1(x) \quad N_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$

$u(0) = u_1$

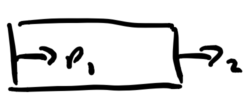
$u(L) = u_2$

$$\varepsilon_x = -\frac{u_1}{L} + \frac{u_2}{L}$$

$$= \frac{u_2 - u_1}{L}$$

$$\sigma_x = E\,\varepsilon_x = E\left(\frac{u_2 - u_1}{L}\right)$$

$$P = AE\left(\frac{u_2 - u_1}{L}\right)$$



$$P_1 = -\frac{AE}{L}(u_2 - u_1)$$

$$P_2 = \frac{AE}{L}(u_2 - u_1)$$

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

We derive elemental stiffness matrices by taking appropriate derivatives of $u(x)$ and considering the elastic behavior of material

Minimum potential energy formulation of equilibrium equations:

1D

$$[L]_1 = \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x} u(x)$$

$$\underline{\underline{\varepsilon}} = [L]\, \underline{u(x)}$$

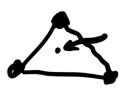
shape fcn

$$u(x) = [N(x)]\{U\}$$

2D

$$[L] = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}$$

2D derivative



(x, y)

$$\varepsilon = [L][N(x)]\{u\}$$

derivative operator

Shape function

Nodal displacements

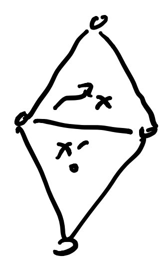
1D

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Elasticity matrix

$$\sigma = [\underline{\underline{E}}]\, \underline{\underline{\varepsilon}}$$

$$\sigma = [E]\underbrace{[L][N(x)]}_{[B]}\{u\}$$



$$\sigma = [E][B]\{u\}$$

$$\Pi_e = \underbrace{\frac{1}{2}\int_{V_e} \underline{\sigma}^T \underline{\varepsilon}\, dV}_{\text{Strain energy}} - \int_{V_e} \underbrace{\{u\}^T p}_{\text{Body forces}}\, dV - \int_{S_e} \underbrace{\{u\}^T q}_{\text{Surface forces}}\, ds$$

$$\Pi_e = \frac{1}{2}\int_{V_e} \{\underline{\underline{u}}\}^T [B][E][B]\{u\}\,dV$$

$$- \int_V \{\underline{\underline{u}}\}^T p\,dV - \int_{S_e} \{u\}^T \underline{q}\,dS$$

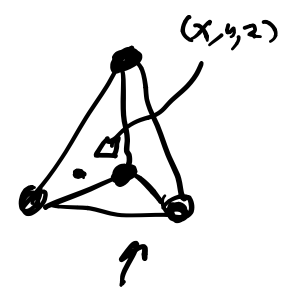
$$\frac{\partial \Pi_e}{\partial u_e} = 0$$

$$\Rightarrow \left(\underbrace{\int_{V_e} [B]^T[E][B]\,dV}\right)\{u\} - F_e = 0$$

$$[K]\{u\} = \{F\}$$

$$[B] = [L][N]$$

$$[K] = \int_{V_e} [N][L][E][L][N]\,dV$$

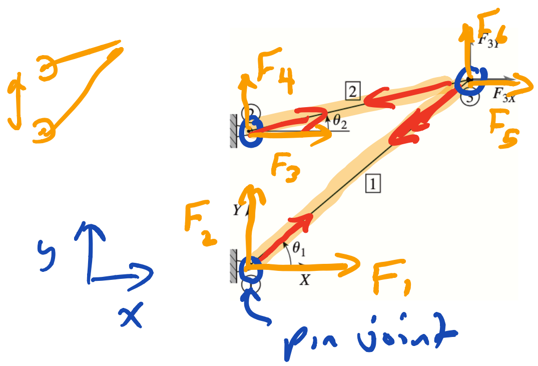


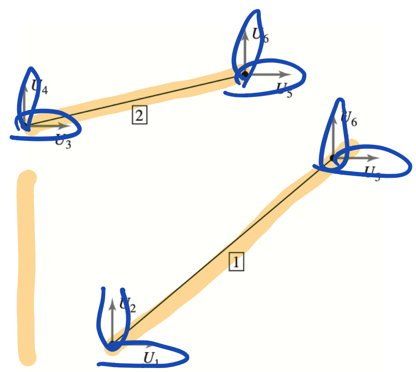
Shape functions

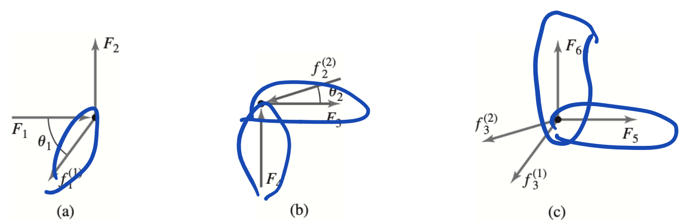
$$[k_e] = \left(\int_{V_e} [B]^T[E][B]\,dV\right)$$

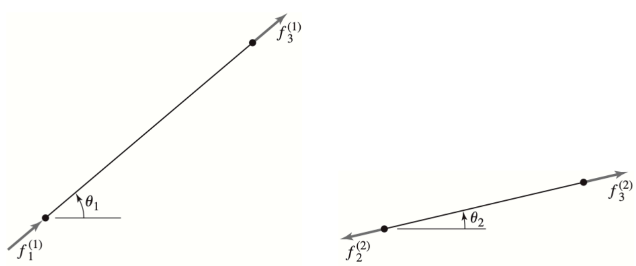
Move on from 1D bars → 2D configurations of bars

Truss problems



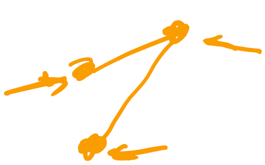






$$F_1 - f_1^{(1)}\cos\theta_1 = 0 \qquad F_3 - f_2^{(2)}\cos\theta_2 = 0 \qquad F_5 - f_3^{(1)}\cos\theta_1 - f_3^{(2)}\cos\theta_2 = 0$$

$$F_2 - f_1^{(1)}\sin\theta_1 = 0 \qquad F_4 - f_2^{(2)}\sin\theta_2 = 0 \qquad F_6 - f_3^{(1)}\sin\theta_1 - f_3^{(2)}\sin\theta_2 = 0$$



$U_1^{(e)}$ = element node 1 displacement in the global X direction
$U_2^{(e)}$ = element node 1 displacement in the global Y direction
$U_3^{(e)}$ = element node 2 displacement in the global X direction
$U_4^{(e)}$ = element node 2 displacement in the global Y direction

$$u_1^{(e)} = U_1^{(e)}\cos\theta + U_2^{(e)}\sin\theta$$
$$v_1^{(e)} = -U_1^{(e)}\sin\theta + U_2^{(e)}\cos\theta$$
$$u_2^{(e)} = U_3^{(e)}\cos\theta + U_4^{(e)}\sin\theta$$
$$v_1^{(e)} = -U_3^{(e)}\sin\theta + U_4^{(e)}\cos\theta$$

$$-k^{(1)}[(U_5 - U_1)\cos\theta_1 + (U_6 - U_2)\sin\theta_1]\cos\theta_1 = F_1$$
$$-k^{(1)}[(U_5 - U_1)\cos\theta_1 + (U_6 - U_2)\sin\theta_1]\sin\theta_1 = F_2$$
$$-k^{(2)}[(U_5 - U_3)\cos\theta_2 + (U_6 - U_4)\sin\theta_2]\cos\theta_2 = F_3$$
$$-k^{(2)}[(U_5 - U_3)\cos\theta_2 + (U_6 - U_4)\sin\theta_2]\sin\theta_2 = F_4$$
$$k^{(2)}[(U_5 - U_3)\cos\theta_2 + (U_6 - U_4)\sin\theta_2]\cos\theta_2 + k^{(1)}[(U_5 - U_3)\cos\theta_1 + (U_6 - U_4)\sin\theta_1]\cos\theta_1 = F_5$$
$$k^{(2)}[(U_5 - U_3)\cos\theta_2 + (U_6 - U_4)\sin\theta_2]\sin\theta_2 + k^{(1)}[(U_5 - U_1)\cos\theta_1 + (U_6 - U_2)\sin\theta_1]\sin\theta_1 = F_6$$

$c_{\theta_1} \to \cos\theta_1$ $\quad$ $s_{\theta_1} \to \sin\theta_1$

6 x 6

$$\begin{bmatrix} k^{(1)}c^2\theta_1 & k^{(1)}s\theta_1 c\theta_1 & 0 & 0 & -k^{(1)}c^2\theta_1 & -k^{(1)}s\theta_1 c\theta_1 \\ k^{(1)}s\theta_1 c\theta_1 & k^{(1)}s^2\theta_1 & 0 & 0 & -k^{(1)}s\theta_1 c\theta_1 & -k^{(1)}s^2\theta_1 \\ 0 & 0 & k^{(2)}c^2\theta_2 & k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}c^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 \\ 0 & 0 & k^{(2)}s\theta_2 c\theta_2 & k^{(2)}s^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}s^2\theta_2 \\ -k^{(1)}c^2\theta_{12} & -k_1 s\theta_1 c\theta_1 & -k^{(2)}c^2\theta_2 & -k^{(2)}s\theta_2 c\theta_2 & k^{(1)}c^2\theta_1 + k^{(2)}c^2\theta_2 & k^{(1)}s\theta_1 c\theta_1 + k^{(2)}s\theta_2 c\theta_2 \\ -k_1 s\theta_1 c\theta_1 & -k^{(1)}s^2\theta_1 & -k^{(2)}s\theta_2 c\theta_2 & -k^{(2)}s^2\theta_2 & k^{(1)}s\theta_1 c\theta_1 + k^{(2)}s\theta_2 c\theta_2 & k^{(1)}s^2\theta_1 + k^{(2)}s^2\theta_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

6x1 $\quad$ 6x1

$[K]\{u\} = \{F\}$

2-node

$k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[K^e] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

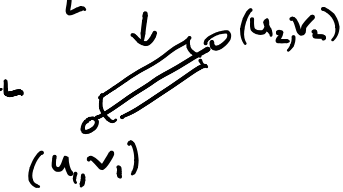
Elemental stiffness matrix for a truss

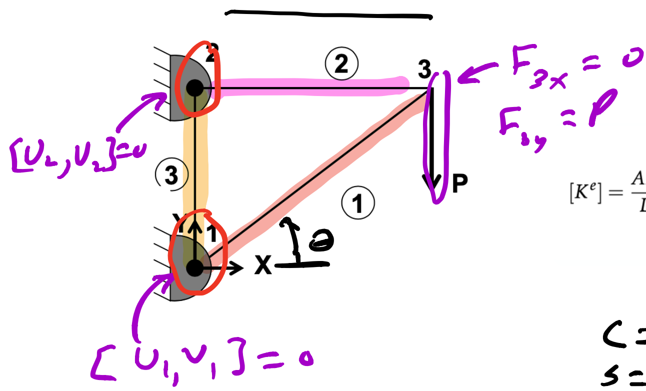
A = Area

E = Young's mod.

L = Length

$\frac{AE}{L}$





$$[K^e] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$c = \cos\theta$
$s = \sin\theta$

$$K_1 = \begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} & -\frac{k_1}{2} & -\frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & \frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix},$$

$$K_2 = \begin{bmatrix} k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix},$$

$$[K_1] = \begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ 0 \\ 0 \\ -P \end{Bmatrix}, \quad +$$

$$[K_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix}, \quad +$$

$$[K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix}, \quad +$$

$[K]\{u\} = \{F\}$

Not complete

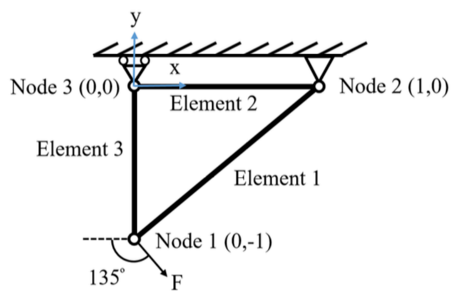
$$\begin{bmatrix} \frac{k_1}{2} & \frac{k_1}{2} & 0 & 0 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2}+k_3 & 0 & -k_3 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & -k_3 & 0 & k_3 & 0 & 0 \\ -\frac{k_1}{2} & -\frac{k_1}{2} & -k_2 & 0 & k_2+\frac{k_1}{2} & \frac{k_1}{2} \\ -\frac{k_1}{2} & -\frac{k_1}{2} & 0 & 0 & \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix}\begin{Bmatrix} \cancel{u_1}\ 0 \\ \cancel{v_1}\ 0 \\ \cancel{u_2}\ 0 \\ \cancel{v_2}\ 0 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ -P \end{Bmatrix}.$$

$$\begin{bmatrix} k_2+\frac{k_1}{2} & \frac{k_1}{2} \\ \frac{k_1}{2} & \frac{k_1}{2} \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \end{Bmatrix}$$

$[K]\{u\} = \{F\}$
$\{u\} = [K]^{-1}\{F\}$

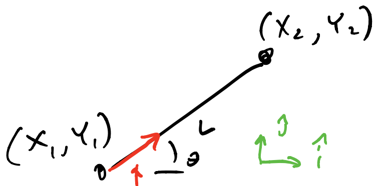
known disp = unknown force

known force = unknown disp


y
x
Node 3 (0,0)
Node 2 (1,0)
Element 2
Element 3
Element 1
Node 1 (0,-1)
135°
F

Direction cosines:

We don't actually need the angles, just the nodal coords



$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\hat{r} = \frac{1}{L}\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

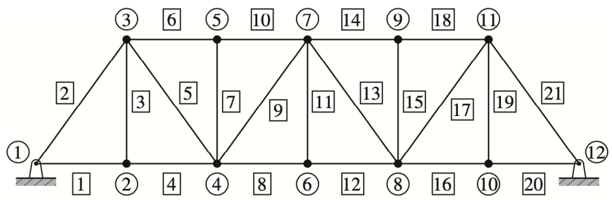
$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

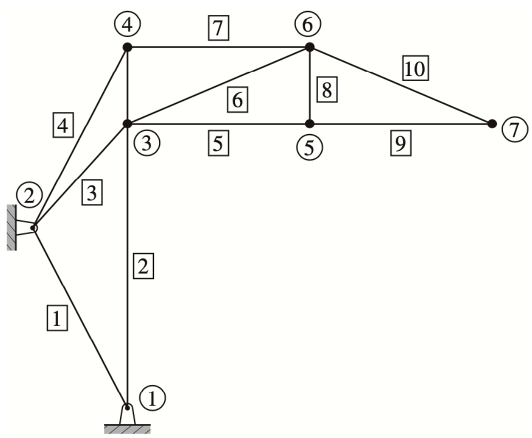
$$\hat{r} \cdot \hat{i} = \cos\theta$$

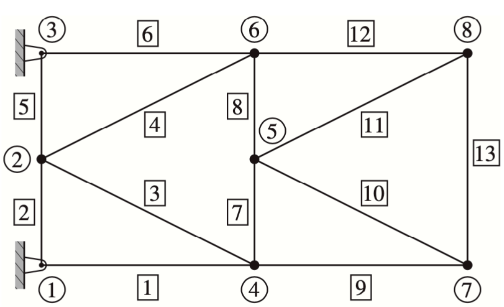
$$\hat{r} \cdot \hat{j} = \sin\theta$$

$$\cos\theta = \frac{x_2 - x_1}{L} \qquad \sin\theta = \frac{y_2 - y_1}{L}$$

These are the direction <u>cosines</u>


3 6 5 10 7 14 9 18 11
2 3 5 7 9 11 13 15 17 19 21
1 12
1 2 4 4 8 6 12 8 16 10 20
(a)


4 7 6
10
6 8
4 7
3 5 5 9
2 3
2
1
1


3 6 6 12 8
5 8
4 5 11
2 13
3 10
2 7
1 1 4 9 7

What about