Homework 6

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1

a. True.

- Let $M_1 = (S_1, I_1, R_1, L_1), H := \{(s, s) \mid s \in S_1\}$.
- H is a simulation relation:
  Let $(s, s') \in H$. by construction $s = s'$, thus they satisfy matching atomic propositions. Furthermore - let f be successor to s, thus f is also a successor to s' and $(f, f) \in H$.

- Let $s \in I_1$, thus $(s, s) \in H$. So $R_1 \preceq R_1$.

b. False.

- Assume $AP = \{F_1, F_2\}$ and take the structures:

  $$M := (S, I, R, L)$$

  $$S := \{i_1, f_1, i_2, f_2\}, I := \{i_1, i_2\},$$

  $R := \{(i_1, f_1), (i_2, f_2), (f_1, f_1), (f_2, f_2)\}, L := \{(i_j, \emptyset), (f_j, \{F_j\}) \mid j \in \{1, 2\}\}$

  $M' := (S' = \{i', f'\}, I' = \{i'\}, R' = \{(i', f'), (f', f')\}, L' = \{(i', \emptyset), (f', \{F_1\})\})$

- Let $H := \{(i', i_1), (f', f_1)\}$. H is a simulation relation $H \subseteq S' \times S$:
  Let $(a, b) \in H$. it is easy to see that the atomic propositions match. Also, the only possible successor to a is f' (wether $a = i'$ or $a = f'$), and the only successor to b is also f_1, so since $(f', f_1) \in H$ we have that H is indeed a simutation relation.
- Let $s \in I = \{i'\}$, meaning $s = i'$, we have that $(i', i_1) \in H$ and $i_1 \in I_1$, Hence $M' \preceq M$.
- Assume towards contrediction that exists H' - a simulation relation $\subseteq S \times S'$ and which gives $M \preceq M'$. Then either $\exists s' : (s', i_2) \in H'$ or not.

- If $\exists s' : (s', i_2) \in H'$:
  Consider f_2 - it cannot be that $(x, f_2) \in H'$ since no $x \in S'$ has the atomic proposition F_2 (which f_2 does). Hence $\forall y : (i_2, y) \notin H'$, but this directly contredicts the definition of $M \preceq M'$.
- So the assumption is false and $M \not\preceq M'$.
- Hence the relation is not symmetric.

c. False.

- Consider:

  $$M := (S = \{i_1, i_2\}, I = \{i_1, i_2\}, R = \{(i_1, i_1), (i_2, i_2)\}, L = \{(x, \emptyset) \mid x \in S\})$$

  $$M' := (S' = \{i'\}, I' = \{i'\}, R' = \{(i', i')\}, L' = \{(x, \emptyset) \mid x \in S\})$$

  Let $H_e := \{(i', i_1), (i', i_2)\}, H_c := \{(i_1, i'), (i_2, i')\}$.
- Clearly $M \neq M'$.
- H_e is a simulation relation:
  The atomic propositions always match since they are always $\emptyset$.
  Let $(i', b) \in H_e$. In any case the only successor to b is b, and since i' is also a successor to itself, we have that for any successor to b, there is a matching successor to i' (itself) s.t. $(i', b) \in H_e$.
- $M' \preceq M$:
  We have a simulation relation H_e, and for all initial states $i' \in I'$, we have that $(i', i_2) \in H_e$.
- H_c is a simulation relation:
  Atomic propositions like before. Let $(a, i') \in H_c$. a is a successor to itself, and i' is the only successor to itself and $(a, i') \in H_c$.
- $M \preceq M'$:
  We have a simulation relation and for each initial state $a \in I$, we have $(a, i') \in H_c$.
- We have seen $M \preceq M' \wedge M' \preceq M \wedge M \neq M'$.

d. True.

- Let:

  $$M_1 = (S_1, I_1, R_1, L_1), M_2 = (S_2, I_2, R_2, L_2), M_3 = (S_3, I_3, R_3, L_3)$$

  And assume $M_1 \preceq M_2 \wedge M_2 \preceq M_3$ denote the simulating relations as $H_{1,2}, H_{2,3}$ repsectively.
- Let $H_{1,3} := \{(a, c) \mid \exists b : (a, b) \in H_{1,2} \wedge (b, c) \in H_{2,3}\}$.

- $H_{1,3}$ is a simulation relation:
  Let $(a,c) \in H_{1,3}$. The atomic propositions match by: $AP(a) = AP(b) = AP(c)$.
  Let a' be a successor to a, then $\exists b'$ a successor of b s.t. $(a',b') \in H_{1,2}$, thus $\exists c'$ a successor of c s.t. $(b',c') \in H_{2,3}$, and by definition: $(a',c') \in H_{1,3}$. To sum up the last conclusion: if a' is a successor of a, then exists some c' a successor of c s.t. $(a',c') \in H_{1,3}$.
  So we have that $H_{1,3}$ is a simulation relation.

- Let $i_1 \in I_1$, thus $\exists i_2 \in I_2$ s.t. $(i_1,i_2) \in H_{1,2}$. Also $i_2 \in I_2 \Rightarrow \exists i_3 \in I_3$ s.t. $(i_2,i_3) \in H_{2,3}$. So by definition of $H_{1,3}$ we have that $(i_1,i_3) \in H_{1,3}$, and $i_3 \in I_3$.
  Thus $M_1 \preceq M_3$.