第四节 多元复合函数的求导法则

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

微分法则 $$\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$$

推广

本节内容:

一、多元复合函数求导的链式法则

二、多元复合函数的全微分

一、多元复合函数求导链式法则

根据复合的不同情形，分三种情况

(一) 复合函数的中间变量均为一元函数

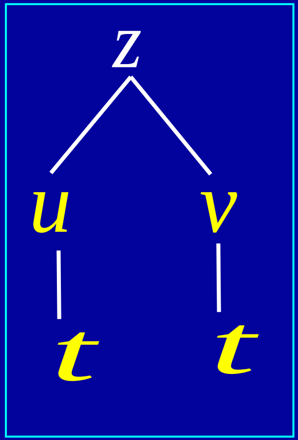
定理: (1) 函数 $u=u(t),\ v=v(t)$ 在点 t 可导,

(2) 函数 $z=f(u,v)$ 在对应点 (u,v) 具有连续偏导数;

则复合函数 $z=f(u(t),v(t))$ 在点 t 可导

且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



证明: 设 t 获得增量 Δt，相应中间变量 u, v 有增量 Δu，Δv，

函数 $z = f(u, v)$ 相应获得增量 Δz，

$$\Delta z = \frac{\partial z}{\partial u}\Delta u + \frac{\partial z}{\partial v}\Delta v + o(\rho) \qquad \left(\rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t} + \frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t} + \frac{o(\rho)}{\Delta t} \longrightarrow (*)$$

令 $\Delta t \to 0$，则有 $\Delta u \to 0$，$\Delta v \to 0$，

$$\frac{\Delta u}{\Delta t} \to \frac{du}{dt}, \qquad \frac{\Delta v}{\Delta t} \to \frac{dv}{dt}$$

$$\frac{o(\rho)}{\Delta t} = \frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2 + \left(\frac{\Delta v}{\Delta t}\right)^2} \to 0$$

($\Delta t < 0$ 时,根式前加“–”号)

对(*)式，取极限 $\Delta t\to 0$, 从而有

$$\boxed{\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}}$$

（全导数公式）

$\dfrac{dz}{dt}$ 称为全导数.

说明: 1^0. 关键搞清中间变量与自变量. 复合后是关于t的一元函数,求的是导数.

$$\begin{tikzcd}
 & u \arrow[dr] & \\
z \arrow[ur] \arrow[dr] & & t \\
 & v \arrow[ur] &
\end{tikzcd}$$

中　　自

2^0. 从$z \to t$有两条线路，故有两项.

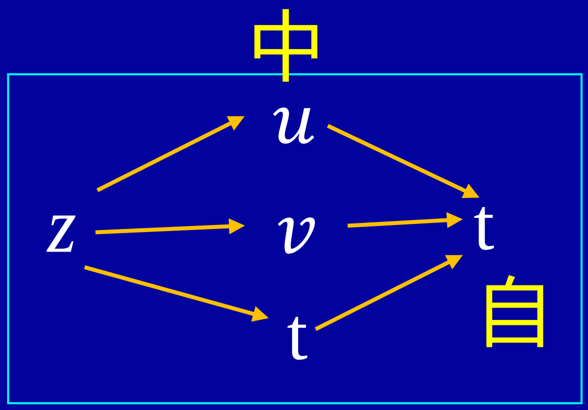
3^0. 推广三个及以上中间变量情形.

$z=f(u,v,w)$,$u=u(t),v=v(t),w=w(t)$满足定理，则

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$

例1. 设 $z = uv + sint$, 而 $u = e^t, v = \cos t$

求全导数 $\frac{dz}{dt}$.



解: $\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial t}$

$= ve^t + u(-\sin t) + \cos t$

$= e^t\,(\cot - \sin t) + \cos t$

t 中间变量本身又是自变量

说明: 若定理中 $f(u,v)$ 在点 (u,v) **偏导数连续**减弱为**偏导数存在**, 则定理结论不一定成立.

例如: $$z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$$

$$u = t, \quad v = t$$

易知: $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0$, $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$

但复合函数 $z = f(t,t) = \dfrac{t}{2}$

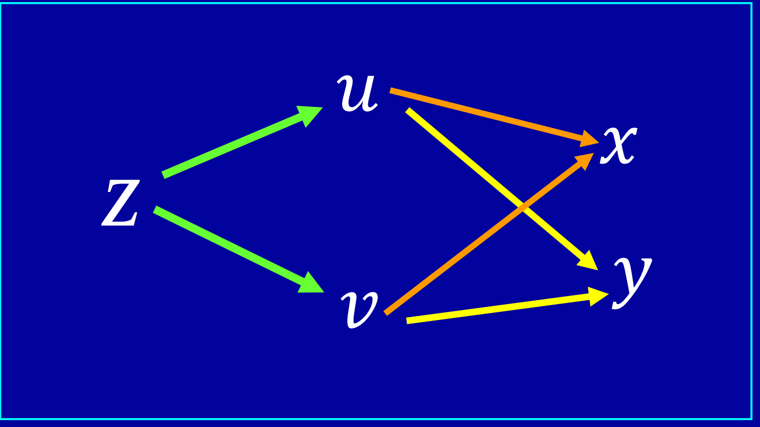
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

(二) 复合函数的中间变量均为多元函数情形

定理: 若函数$u=u(x,y),v=v(x,y)$都在点(x,y)处具有对x及对y的偏导数, 函数$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数 $z=f(u(x,y),v(x,y))$ 在点(x,y)的两个偏导数都存在，且有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$



例2. 设 $z = u^2 \ln v$,而 $u = \frac{x}{y}$, $v = 3x - 2y$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$= 2u \ln v \cdot \frac{1}{y} + \frac{3u^2}{v}$$
$$= \frac{2x}{y^2}\ln(3x - 2y) + \frac{3x^2}{y^2(3x-2y)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2u \ln v \left(-\frac{x}{y^2}\right) - \frac{2u^2}{v}$$

$$= -\frac{2x^2}{y^3}\ln(3x - 2y) - \frac{2x^2}{(3x - 2y)y^2}$$

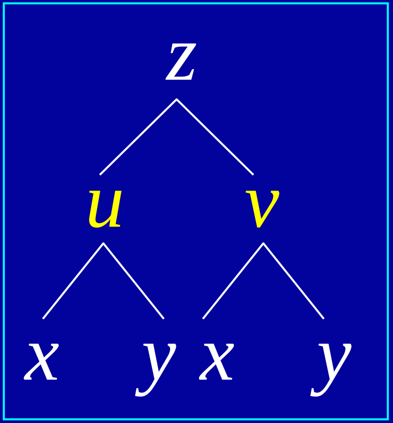
例3. 设 $z = e^u \sin v$, $u = xy$, $v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^{xy}[y \cdot \sin(x + y) + \cos(x + y)]$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy}[x \cdot \sin(x + y) + \cos(x + y)]$$

(三) 中间变量既有一元函数，又有多元函数情形

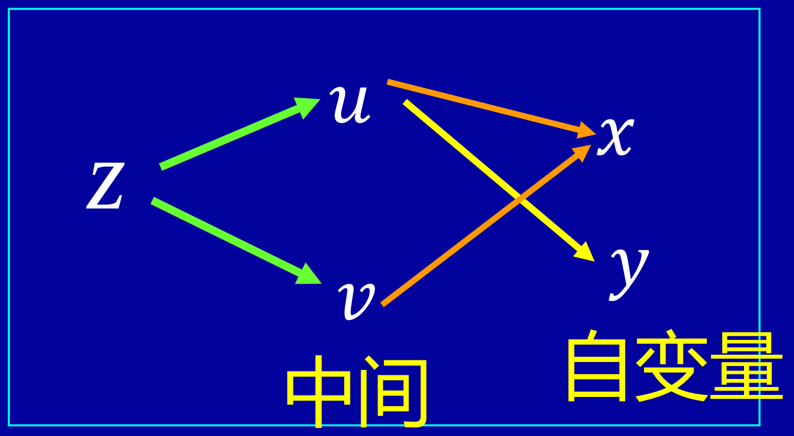
定理: (1) 函数$u = u(x, y)$在点(x, y)处具有对x及对y的偏导数; 函数$v = v(x)$在点x 可导;

(2) 函数$z = f(u, v)$在对应点(u, v)具有连续偏导数;

则复合函数$z = f(u(x, y), v(x))$在点(x, y)的两个偏导存在，且

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{d v}{d x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y}$$



例4. 设 $z=(u^2+v^3)\sin v^2,\ u=xy,\ v=2y.$ 求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}.$

解: $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=2u\sin v^2\cdot y=2xy^2\sin(4y^2)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{dv}{dy}$$

$$=2u\sin v^2\cdot x+[3v^2\sin v^2+v^3\cos v^2\cdot 2v$$

$$+u^2\cos v^2\cdot 2v]\cdot 2$$

$$=2x^2y\sin(4y^2)+2[12y^2\sin(4y^2)+32y^4\cos 4y^2$$

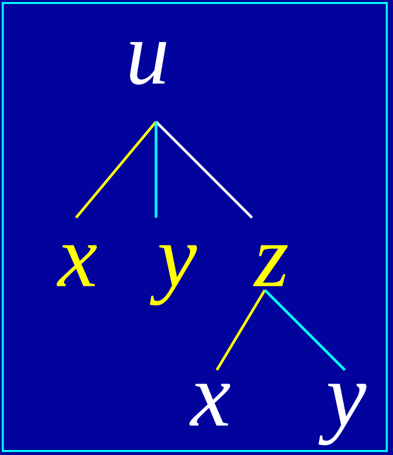
$$+4x^2y^3\cos(4y^2)]$$

例5. $u=f(x,y,z)=e^{x^2+y^2+z^2}$, $z=x^2\sin y$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

解: $\boxed{\frac{\partial u}{\partial x}}=\boxed{\frac{\partial f}{\partial x}}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$

$$=2xe^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$



$\boxed{\frac{\partial u}{\partial y}}=\boxed{\frac{\partial f}{\partial y}}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$

$$=2ye^{x^2+y^2+z^2}+2ze^{x^2+y^2+z^2}\cdot x^2\cos y$$

$$=2(y+x^4\sin y\cos y)e^{x^2+y^2+x^4\sin^2 y}$$

x, y既是中间变量，又是自变量

注意: 这里 $\dfrac{\partial u}{\partial x}$ 与 $\dfrac{\partial f}{\partial x}$ 不同,

$\dfrac{\partial u}{\partial x}$ 是把复合函数$u = f(x, y, z(x, y))$中的y看做不变.

即固定 y 对 x 求偏导;

$\dfrac{\partial f}{\partial x}$ 是把 $f(x, y, z)$看成三元函数，将y, z看做不变,

对x求偏导数.

$\dfrac{\partial u}{\partial y}$ 与 $\dfrac{\partial f}{\partial y}$ 也有类似的区别.

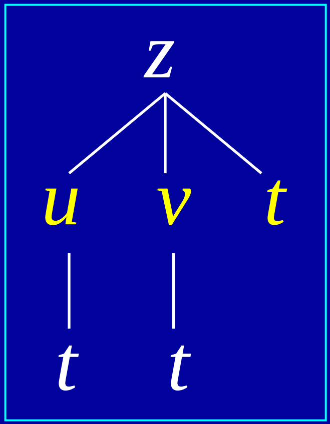
例6. 设 $z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

$$= v e^t - u\sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t$$



注意：多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到.

二、抽象函数求偏导

例1. 设$z=f(x,y,u), y=g(x), u=\varphi(x)$, 求$\frac{dz}{dx}$.

解: $$\boxed{\frac{dz}{dx}}=\boxed{\frac{\partial z}{\partial x}}\cdot\frac{dx}{dx}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial z}{\partial u}\frac{du}{dx}$$

$$=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot g'(x)+\frac{\partial z}{\partial u}\varphi'(x)$$

$$=f_x'+f_y'g'(x)+f_u'\varphi'(x)$$

说明: $\frac{dz}{dx}$是$z=f(x,g(x),\varphi(x))$作为x的函数, 对自变量x求导. $\frac{\partial z}{\partial x}$是外层函数$z=f(x,y,u)$作为$x,y,u$的三元函数对$x$求偏导，即$z$对中间变量$x$求偏导.

例2. 设$z=f(x,y,u),u=\varphi(x,y)$, 求$\frac{\partial z}{\partial x}$

解: $$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}=f'_x+f'_u\varphi'_x.$$

说明: x既是自变量又是中间变量, $\frac{\partial z}{\partial x}$是$z$对自变量$x$的偏导, $\frac{\partial f}{\partial x}$是$z$对中间变量$x$的偏导.

例3. 设$z=f(u),\ u=\varphi(x,y)$, 求$\frac{\partial z}{\partial x}$

解: $$\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=f'(u)\varphi'_x(x,y)$$

说明: 一元函数必须写“d”，二元写偏导符号.

例4. 设$\omega = f(x+y+z, xyz)$, f 具有二阶连续偏导数,

求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.

解: (可引入中间变量，将函数进行分解)

令 $u = x+y+z, v = xyz$, 则 $\omega = f(u, v)$.

为方便引入记号:

$\dfrac{\partial f}{\partial u} = f_1'$, (下标1表示对$f$的第一个变量$u$,求偏导)

$\dfrac{\partial f}{\partial v} = f_2'$, 则 $\dfrac{\partial w}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}$

$$= f_1' + f_2' yz$$

(应注意:f_1'，f_2'仍为抽象函数，它们仍是复合函数,与f具有相同的复合结构，即中间变量是u, v,自变量是x, y, z.)

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial z}(f_1' + f_2' yz) = \frac{\partial f_1'}{\partial z} + y f_2' + yz\frac{\partial f_2'}{\partial z}$$

而 $$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v}\frac{\partial v}{\partial z} = f_{11}'' + f_{12}'' \cdot xy$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v}\frac{\partial v}{\partial z} = f_{21}'' + xy f_{22}''$$

$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + (xy + yz) f_{12}'' + y f_2' + xy^2 z f_{22}''$$

说明:f_{11}''表示对f_1'的第一个变量u求偏导. 类似f_{12}'', f_{21}'',f_{22}''.

由于f 具有二阶连续偏导数 $f_{12}'' = f_{21}''$.

例5. 设f具有二阶连续偏导数，求函数$u = f\left(x, \frac{x}{y}\right)$的混合二阶偏导数.

解： 由条件两个混合二阶偏导数相等，只需求一个.

$$\frac{\partial u}{\partial x} = f_1' + f_2' \cdot \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = f_{12}''\left(-\frac{x}{y^2}\right) - \frac{1}{y^2} f_2' + \frac{1}{y} f_{22}''\left(-\frac{x}{y^2}\right)$$

$$= -\frac{x}{y^2} f_{12}'' - \frac{x}{y^3} f_{22}'' - \frac{1}{y^2} f_2'$$

例6. 设 $z = xf\left(\frac{y}{x}\right) + yf\left(\frac{x}{y}\right)$, 求 $\frac{\partial^2 z}{\partial x \partial y}$

解: $\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) + yf'\left(\frac{x}{y}\right)\cdot\frac{1}{y}$

$$= f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right) + f'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = f'\left(\frac{y}{x}\right)\frac{1}{x} - \frac{1}{x}f'\left(\frac{y}{x}\right) - \frac{y}{x}f''\left(\frac{y}{x}\right)\cdot\frac{1}{x} + f''\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$= -\frac{y}{x^2}f''\left(\frac{y}{x}\right) - \frac{x}{y^2}f''\left(\frac{x}{y}\right)$$

例7. 设$z = f\left(xy, \frac{y}{x}, y+x\right)$，求$\frac{\partial z}{\partial y}$.

解：$$\frac{\partial z}{\partial y} = xf_1' + f_2' \cdot \frac{1}{x} + f_3'$$

说明：设$u = xy, v = \frac{y}{x}, w = y + x$, 比较麻烦，故用下标 1,2,3来简记. 不允许下标既出现数字，又出现字母.

例8. 设 $u = f(x, y)$ 二阶偏导数连续,求下列表达式在极坐标系下的形式

$$(1)\ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2, \quad (2)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

解析: 由直角坐标与极坐标关系：$x = r\cos\theta$, $y = r\sin\theta$

可把 $u = f(x, y)$ 换成 r, θ 的函数,

$$u = f(r\cos\theta, r\sin\theta) = F(r, \theta)$$

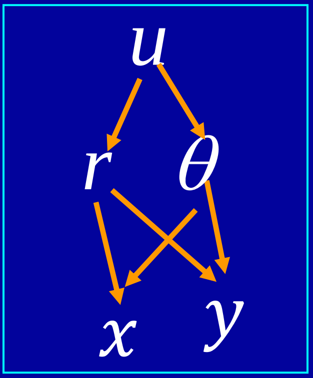
$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}$$

r, θ分别是关于 x, y的函数. 把u 看成是以r, θ 为中间变量， x, y 为自变量的函数

(1) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

关于θ的说明见课本P83说明

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial \theta}{\partial x} = \frac{\frac{-y}{x^2}}{1 + (\frac{y}{x})^2} = \frac{-y}{x^2 + y^2}$$



$$= \frac{\partial u}{\partial r}\frac{x}{r} - \frac{\partial u}{\partial \theta}\frac{y}{r^2} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}$$

$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

已知 $\frac{\partial u}{\partial x}=\boxed{\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}}$

$$\frac{\partial u}{\partial x}$$

$$r\qquad\theta$$

$$x\quad y\quad x\quad y$$

注意利用已有公式

(2) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$= \frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right]$$

例9. 用多元复合函数微分法求下列一元函数的导数.

(1) $y = x^{\cos x}$;　(2) $y = \dfrac{(1+x^2)\ln x}{\sin x+\cos x}$

解：（通常用“对数求导法”求此一元函数的导数，现在可用多元复合函数链式法则，全导数公式来求.)

(1) 令 $y = u^v$，$u = x$, $v = cosx$，则

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\cdot\frac{du}{dx} + \frac{\partial y}{\partial v}\cdot\frac{dv}{dx}$$

$$= vu^{v-1} + u^v \ln u\cdot(-\sin x)$$

$$= x^{\cos x-1}\cos x - \sin x\cdot\ln x\cdot x^{\cos x}$$

(2) 令 $y = \frac{vw}{u}$, $u = sinx + cosx, v = 1 + x^2, w = lnx$

则 $$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} + \frac{\partial y}{\partial \omega}\frac{dw}{dx}$$

$$= -\frac{vw}{u^2}(\cos x - \sin x) + \frac{w}{u} \cdot 2x + \frac{v}{u} \cdot \frac{1}{x}$$

$$= -\frac{(1 + x^2)\ln x}{(\sin x + \cos x)^2}(\cos x - \sin x) +$$

$$\frac{2x \ln x}{\sin x + \cos x} + \frac{1 + x^2}{x(\sin x + \cos x)}$$

二、多元复合函数的全微分

情形一：设$z=f(u,v)$可微，求 dz

这里u,v是自变量, 则

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

情形二：设函数 $z=f(u,v),u=\varphi(x,y),v=\psi(x,y)$

都可微,则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$\boxed{\mathrm{d}z} = \boxed{\frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y}$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$= \boxed{\frac{\partial z}{\partial u}\mathrm{d}u + \frac{\partial z}{\partial v}\mathrm{d}v}$$

说明：当z是u, v的函数时，无论 u , v 是自变量，还是中间变量, 其全微分表达形式都一样,这性质叫做**全微分形式不变性**. 即

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

四则运算的全微分法则:

$$d(u \pm v) = du \pm dv$$

$$d(uv) = u\,dv + v\,du$$

$$d(Cu) = C\,du \qquad (C\text{为常数})$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \quad (v \neq 0)$$

说明：利用多元函数的全微分形式不变性，可来求全微分及一阶偏导数.

例 10. 利用全微分形式不变性再解例1.

解:

$$dz = d(e^u \sin v)$$

$$= e^u \sin v\, du + e^u \cos v\, dv$$

$$= e^{xy}[\sin(x+y)\, d(xy) + \cos(x+y)\, d(x+y)]$$

$$= e^{xy}[\sin(x+y)(y\,dx + x\,dy) + \cos(x+y)(dx+dy)]$$

$$= e^{xy}[y\sin(x+y) + \cos(x+y)]dx$$

$$+ e^{xy}[x\sin(x+y) + \cos(x+y)]dy$$

所以
$$\frac{\partial z}{\partial x} = e^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$$

$$\frac{\partial z}{\partial y} = e^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$$

例 11. 设 $z = f(u, v)$，$u = x\sin y$, $v = g(x, y)$

求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解：$dz = f_u' \, du + f_v' \, dv$

$$= f_u' \, d(x \sin y) + f_v' \, d\big(g(x, y)\big)$$

$$= f_u'\big(\sin y \, dx + x \, d(\sin y)\big) + f_v'\big(g_x' \, dx + g_y' \, dy\big)$$

$$= f_u' \sin y \, dx + x \cos y \cdot f_u' \, dy + f_v' g_x' \, dx + f_v' g_y' \, dy$$

$$= (\underbrace{f_u' \sin y + f_v' g_x'}_{\frac{\partial z}{\partial x}}) \, dx + \big(\underbrace{x \cos y \cdot f_u' + f_v' g_y'}_{\frac{\partial z}{\partial y}}\big) \, dy$$

备用题

1. 已知 $f(x,y)\big|_{y=x^2}=1$, $f_1'(x,y)\big|_{y=x^2}=2x$, 求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

2. 设函数 $z = f(x, y)$ 在点$(1,1)$处可微，且

$$f(1,1)=1, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 3,$$

$\varphi(x) = f(x, f(x,x))$，求 $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. **(2001考研)**

解: 由题设 $\varphi(1) = f(1, f(1,1)) = f(1,1) = 1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1} = 3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$= 3\Big[f_1'(x, f(x,x)) + f_2'(x, f(x,x))\big(f_1'(x,x) + f_2'(x,x) \big) \Big]\Big|_{x=1}$$

$$= 3\cdot\big[2 + 3\cdot(2+3)\big] = 51$$

3. 设 $z=f(u)$, 方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定 u 是 x , y 的函数 , 其中 $f(u),\varphi(u)$ 可微 , $p(t),\varphi'(u)$ 连续, 且 $\varphi'(u)\neq 1$, 求 $p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}$.

解: $\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x}$, $\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[\,p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\,\right]=0$$

作业

P85　4， 5， 6， 7,

8: (1),(3), 9, 11