APPENDIX A

THE KKT TRANSFORMATION AND THE LINEARIZATION PROCESS FOR POWER FLOW CONSTRAINT

The lagrangian function for the original lower-level problem (32)-(37) is:

$$\begin{aligned}
L = & \sum_{n\in V_T} \Delta Pd_n - \sum_{l\in L} \mu_l \left(p_l - v_l \cdot \frac{1}{x_l} \cdot \sum_{n\in V_T} A_{nl} \cdot \delta_n \right) \\
& - \sum_{n\in V_T} \lambda_n \left(\sum_{g\in G_n} p_g - \sum_{l\in L} A_{nl} \cdot p_l + \Delta Pd_n - Pd_n \right) \\
& - \sum_{l\in L} \underline{\omega}_l \left(p_l + \overline{p}_l \right) - \sum_{l\in L} \overline{\omega}_l \left(\overline{p}_l - p_l \right) \\
& - \sum_{j\in G} \underline{\theta}_j \left(g_j - \underline{g}_j \right) - \sum_{j\in G} \overline{\theta}_j \left(\overline{g}_j - g_j \right) \\
& - \sum_{n\in V_T} \underline{\alpha}_n \left(\Delta pd_n \right) - \sum_{n\in V_T} \overline{\alpha}_n \left(Pd_n - \Delta Pd_n \right)
\end{aligned} \tag{A1}$$

Where μ_l, λ_n, $\underline{\omega}_l$, $\overline{\omega}_l$, $\underline{\theta}_j$, $\overline{\theta}_j$, $\underline{\alpha}_n$, $\overline{\alpha}_n$ are the Lagrangian multipliers associated with the DC power flow constraints (33)-(37), respectively, in addition to the original feasibility constraints (33)-(37), the optimality conditions for the original problem KKT are:

$$\frac{\partial L}{\partial \delta_n} = \sum_{l\in L} \frac{1}{x_l} A_{nl} \mu_l v_l = 0, \qquad n \in V_T \tag{A2}$$

$$\frac{\partial L}{\partial g_j} = -\lambda_n - \underline{\theta}_j + \overline{\theta}_j = 0, \qquad j \in G \tag{A3}$$

$$\frac{\partial L}{\partial p_l} = \sum_{n\in V_T} A_{nl} \lambda_n - \mu_l - \underline{\omega}_l + \overline{\omega}_l = 0, \quad l \in L \tag{A4}$$

$$\frac{\partial L}{\partial \Delta Pd_n} = 1 - \lambda_n - \underline{\alpha}_n + \overline{\alpha}_n = 0, \qquad n \in V_T \tag{A5}$$

$$\underline{\omega}_l \ge 0, \qquad l \in L \tag{A6}$$

$$\overline{\omega}_l \ge 0, \qquad l \in L \tag{A7}$$

$$\underline{\theta}_j \ge 0, \qquad j \in G \tag{A8}$$

$$\overline{\theta}_j \ge 0, \qquad j \in G \tag{A9}$$

$$\underline{\alpha}_n \ge 0, \qquad n \in V_T \tag{A10}$$

$$\overline{\alpha}_n \ge 0, \qquad n \in V_T \tag{A11}$$

$$\underline{\omega}_l \left(p_l + \overline{p}_l \right) = 0, \qquad l \in L \tag{A12}$$

$$\overline{\omega}_l \left(\overline{p}_l - p_l \right) = 0, \qquad l \in L \tag{A13}$$

$$\underline{\theta}_j \left(g_j - \underline{g}_j \right) = 0, \qquad j \in G \tag{A14}$$

$$\overline{\theta}_j \left(\overline{g}_j - g_j \right) = 0, \qquad j \in G \tag{A15}$$

$$\underline{\alpha}_n \left(\Delta Pd_n \right) = 0, \qquad n \in V_T \tag{A16}$$

$$\overline{\alpha}_n \left(Pd_n - \Delta Pd_n \right) = 0, \qquad n \in V_T \tag{A17}$$

where (A2)- (A11) denotes the original lower-level problem dual constraints and (A12)- (A17) denotes the complementary slackness constraints.

It can be seen that the original lower-level problem line flow formulation (33), dual constraints (A2) and complementary slackness constraints (A12)- (A17) are all non-linear terms. The above constraints are linearized by the following method:

(1) For the linearization of formulation ((33)

In formulation (33), there are two non-linear terms-- v_l multiplied by the phase angle δ_l^f at the beginning of the line and v_l multiplied by the phase angle δ_l^t at the end of the line, so continuous variables s_l^f and s_l^t are introduced to represent $v_l\delta_l^f$ and $v_l\delta_l^t$ respectively. Further intermediate variables z_l^f and z_l^t are introduced to equate the non-linear term (33) with the following linear terms:

$$p_l = \frac{1}{x_l} \cdot \left(z_l^f - z_l^t \right), \qquad l \in L \tag{A18}$$

$$z_l^f = \delta_l^f - s_l^f, \qquad l \in L \tag{A19}$$

$$z_l^t = \delta_l^t - s_l^t, \qquad l \in L \tag{A20}$$

$$\underline{\delta} \cdot v_l \le z_l^f \le \overline{\delta} \cdot v_l, \qquad l \in L \tag{A21}$$

$$\underline{\delta} \cdot v_l \le z_l^t \le \overline{\delta} \cdot v_l, \qquad l \in L \tag{A22}$$

$$\underline{\delta} \cdot \left(1 - v_l\right) \le s_l^f \le \overline{\delta} \cdot \left(1 - v_l\right), \qquad l \in L \tag{A23}$$

$$\underline{\delta} \cdot \left(1 - v_l\right) \le s_l^t \le \overline{\delta} \cdot \left(1 - v_l\right), \qquad l \in L \tag{A24}$$

The formulations(A18)-(A24) represents the linearized expression for the calculation of the line DC power flow, if line l is destroyed ($v_l = 0$), then according to the formulations (A21), (A22) can be obtained: z_l^f=0, z_l^t=0, so the line power flow p_l=0, $s_l^f = \delta_l^f$, $s_l^t = \delta_l^t$; if l is not destroyed (v_l=1), then according to the formulations (A23), (A24) can be obtained: s_l^f =0, s_l^t =0, so $z_l^f = \delta_l^f$, $z_l^t = \delta_l^t$, and the line plow flow p_l is determined by the phase angle difference between the two ends of the line.

(2) Linearization for the dual constraint (A2)

Similarly, by introducing the continuous variables t_l and h_l, the nonlinear dual constraint (A2) is equivalently represented by the following set of linearization constraints:

$$\sum_{l\in L} \frac{1}{x_l} \cdot A_{nl} \cdot t_l = 0, \qquad n \in V_T \tag{A25}$$

$$t_l = u_l - h_l, \qquad l \in L \tag{A26}$$

$$\underline{u}_l \cdot v_l \le t_l \le v_l \cdot \overline{u}_l, \qquad l \in L \tag{A27}$$

$$\underline{u}_l \cdot \left(1 - v_l\right) \le h_l \le \overline{u}_l \cdot \left(1 - v_l\right), \quad l \in L \tag{A28}$$

(3) Linearization for the complementary slackness constraints (A12)- (A17)

The nonlinear complementary slackness constraints (A12)-(A17) are equivalently represented by the following set of linearization constraints by introducing the 0-1 variables $\omega_l^{\underline{\omega}}$, $\omega_l^{\overline{\omega}}$, $\omega_g^{\underline{\theta}}$, $\omega_g^{\overline{\theta}}$, $\omega_n^{\underline{\alpha}}$, $\omega_n^{\overline{\alpha}}$:

$$\underline{\omega}_l \le M \cdot \omega_l^{\underline{\omega}}, \qquad l \in L \tag{A29}$$

$$p_l + \overline{p}_l \le M \cdot \left(1-\omega_l^{\omega}\right), \qquad l \in \boldsymbol{L} \qquad \text{(A30)}$$

$$\overline{\omega}_l \le M \cdot \omega_l^{\overline{\omega}}, \qquad l \in \boldsymbol{L} \qquad \text{(A31)}$$

$$\overline{p}_l - p_l \le M \cdot \left(1-\omega_l^{\overline{\omega}}\right), \qquad l \in \boldsymbol{L} \qquad \text{(A32)}$$

$$\underline{\theta}_j \le M \cdot \omega_j^{\theta}, \qquad j \in \boldsymbol{G} \qquad \text{(A33)}$$

$$g_j - \underline{g}_j \le M \cdot \left(1-\omega_j^{\theta}\right), \qquad j \in \boldsymbol{G} \qquad \text{(A34)}$$

$$\overline{\theta}_j \le M \cdot \omega_j^{\overline{\theta}}, \qquad j \in \boldsymbol{G} \qquad \text{(A35)}$$

$$\overline{g}_j - g_j \le M \cdot \left(1-\omega_j^{\overline{\theta}}\right), \qquad j \in \boldsymbol{G} \qquad \text{(A36)}$$

$$\underline{\alpha}_n \le M \cdot \omega_n^{\alpha}, \qquad n \in \boldsymbol{V_T} \qquad \text{(A37)}$$

$$\Delta p_n^d \le M \cdot \left(1-\omega_n^{\alpha}\right), \qquad n \in \boldsymbol{V_T} \qquad \text{(A38)}$$

$$\overline{\alpha}_n \le M \cdot \omega_n^{\overline{\alpha}}, \qquad n \in \boldsymbol{V_T} \qquad \text{(A39)}$$

$$p_n^d - \Delta p_n^d \le M \cdot \left(1-\omega_n^{\overline{\alpha}}\right), \qquad n \in \boldsymbol{V_T} \qquad \text{(A40)}$$

$$\omega_l^{\omega} + \omega_l^{\overline{\omega}} \le 1, \qquad l \in \boldsymbol{L} \qquad \text{(A41)}$$

$$\omega_j^{\theta} + \omega_j^{\overline{\theta}} \le 1, \qquad j \in \boldsymbol{G} \qquad \text{(A42)}$$

$$\omega_n^{\alpha} + \omega_n^{\overline{\alpha}} \le 1, \qquad n \in \boldsymbol{V_T} \qquad \text{(A43)}$$

where (A29)- (A30), (A31)- (A32), (A33)- (A34), (A35)- (A36), (A37)-(A38) and (A39)-(A40) are linearised equivalent representations of the constraints (A12), (A13), (A14), (A15), (A16) and (A17) respectively. The intermediate variables introduced satisfies formulations (A41)-A43).

In summary, the original lower-level problem after transformation by KKT is expressed as formulations (A18)-(A24), formulations (33)- (37), (A25)- (A28), (A3)- (A11) and (A29)-(A43). Therefore, the transformed vulnerability identification problem is as follows:

$$\max_{a} \sum_{T \in \Omega_T} \pi_T \left(\sum_{i \in \boldsymbol{V}} \Delta Pd_{i*}^T \right) \qquad \text{(A44)}$$

s.t. (26)-(29), (33)-(37), (A18)-(A43).

Appendix B

Table VI contains information such as the road lengths, the number of cables on the roads, and E_{ij} for each road in the case of Figure B1.

TABLE VI

ROAD NETWORK CONNECTION RELATION AND PARAMETER

l_{ij}	d_{ij}(km)	mra_{ij}	mrb_{ij}	mrc_{ij}	E_{ij}
1-2	1.8	0	0	0	0
1-3	1.6	0	0	0	0
2-51	1.7	0	0	0	0
2-52	1.7	0	0	0	0
3-4	1.3	0	0	0	0
3-12	1.9	0	0	0	0
4-5	1.7	0	0	0	0
4-12	1.7	0	0	1	0.43
5-6	1.8	0	0	0	0
6-9	1.5	3	0	0	0.93
6-10	1.5	2	0	0	0.95
6-13	1.5	2	0	0	0.78
7-8	1.3	2	0	0	0.87
7-11	1.3	4	0	0	0.95
8-10	1.32	3	0	0	0.97
8-30	1.2	3	0	0	0.92
9-13	1.5	3	0	0	0.83
9-16	1.5	2	0	0	0.84
9-49	1.6	2	0	0	0.86
10-16	1.6	3	0	0	0.97
11-17	1.3	0	0	0	0
11-30	1.1	3	0	0	0.96
12-13	1.3	0	1	0	0.6
12-52	1.8	0	0	0	0
13-15	1.5	2	0	0	0.75
14-32	1	0	0	0	0
14-50	1	0	0	0	0
15-49	1.3	3	0	0	0.97
15-54	1.27	3	0	0	0.98
16-31	1.2	2	0	0	0.9
16-50	1.12	1	0	0	0.97
17-18	0.9	4	0	0	0.36
18-19	1	0	2	0	0
18-21	1.28	3	0	0	0.55
18-30	1.22	2	0	0	0
19-20	1.12	3	0	0	0
19-22	1.15	1	0	0	0.57
20-23	1.3	4	0	0	0
20-24	1.35	3	0	0	0
21-22	1.4	0	0	1	0.63
21-31	1.39	0	0	0	0.92
21-33	1.29	0	2	0	0.92
22-29	1.23	0	0	0	0
23-25	2.1	0	0	0	0
24-25	1.4	0	0	2	0
24-26	1.6	0	0	0	0
25-27	1.5	0	0	0	0
26-27	1.57	0	0	2	0.76
26-29	1.05	4	0	0	0.75
26-37	4	3	0	0	0
27-28	1.9	0	0	0	0
28-37	3	0	0	0	0
28-38	3.5	0	0	0	0
29-35	2	0	0	0	0.93
30-31	2.3	3	0	0	0.95
31-32	1.8	3	0	0	0.93
32-33	2	5	0	0	0.93

32-45	2.2	0	0	0	0
33-34	1.7	3	0	0	0.89
34-35	1.82	4	0	0	0.89
34-44	2	2	0	0	0.68
35-36	1.5	2	0	0	0.89
36-37	1.4	3	0	0	0.9
36-41	2.7	0	0	0	0
36-44	1.7	2	0	0	0.87
37-40	3.7	0	0	0	0
38-39	1.9	0	0	0	0
39-40	1.7	1	0	0	0.56
39-85	2.1	0	0	0	0
40-41	1.82	1	0	0	0.6
40-65	1.95	0	0	0	0
41-42	1.67	3	0	0	0.96
41-43	1.32	2	0	0	0.87
42-61	1.65	4	0	0	0.95
42-65	0.89	0	0	0	0
43-44	1.32	3	0	0	0.93
44-46		2	0	0	0.89
45-46	1.4	0	0	0	0
45-47	1.6	3	0	0	0.79
46-61	1.62	4	0	0	0.95
47-48	1.45	3	0	0	0.9
47-50	1.7	0	0	0	0
47-60	1.89	0	0	0	0
48-49	1.43	4	0	0	0.97
48-58	1.05	3	0	0	0.89
49-50	0.89	2	0	0	0.88
51-55	1.97	0	0	0	0
52-53	1.9	2	0	0	0.89
52-54	1.9	2	0	0	0.93
53-55	1.82	0	0	0	0
53-56	1.89	4	0	0	0.94
54-56	2.2	2	0	0	0.9
54-57	0.93	4	0	0	0.92
55-66	1.1	0	0	1	0.5
56-66	1.25	0	0	0	0
56-70	1.19	0	0	0	0
57-58	0.65	2	0	0	0.78
57-70	1.8	0	0	0	0
58-59	1.1	0	1	0	0.78
58-72	1.4	0	2	0	0.57
59-60	1.4	0	0	0	0
59-79	1.62	0	0	3	0.78
60-61	1.63	0	0	0	0
60-62	1.4	0	0	0	0
62-63	1.3	0	0	0	0
62-80	1.6	0	2	0	0.76
63-64	1	0	0	0	0
64-65	0.9	0	0	0	0
64-82	1.2	2	0	0	0.9
64-87	1.3	0	0	0	0
65-84	1.7	0	0	0	0
66-67	1.02	0	0	0	0
66-69	1.02	0	1	0	0.7
67-68	1.9	0	0	0	0
68-76	1.79	0	0	0	0
69-70	1.76	0	0	2	0.67
69-71	1.69	0	0	0	0
69-75	1.95	4	0	0	0.89
70-72	1.05	0	1	0	0.78
71-73	2.3	0	0	0	0
72-73	1.21	0	2	0	0.74
73-74	1.25	0	0	1	0.32
74-75	2.52	0	0	0	0
74-77	2.1	0	0	2	0.56
74-80	2.9	0	0	3	0.67
75-77	1.92	4	0	0	0.97
76-77	1.68	0	0	0	0
77-78	2.6	4	0	0	0.96
78-81	2	2	0	0	0.92
79-80	1.97	0	2	0	0.59
80-81	2.01	4	0	0	0.95
81-82	2	4	0	0	0.94
81-83	2.32	4	0	0	0.96
82-83	1.87	3	0	0	0.89
83-87	1.62	0	0	0	0
84-85	1.82	0	0	0	0
84-86	1.3	0	0	0	0
86-87	2.05	0	0	0	0

TABLE VII

Node PARAMETERS OF UTN

Node Number	Node Types	Load/MW	Active Output/MW
1	pq	0	0
2	pq	0	0
3	pq	342	0
4	pq	385	0
5	pq	0	0
6	pq	0	0
7	pq	233.8	0
8	pq	268.5	0
9	pq	0	0
10	pq	0	0
11	pq	0	0
12	pq	287.5	0
13	pq	0	0
14	pq	0	0

15	pq	220	0
16	pq	329	0
17	pq	0	0
18	pq	258	0
19	pq	0	0
20	pq	228	0
21	pq	274	0
22	pq	385	0
23	pq	247.5	0
24	pq	368.6	0
25	pq	324	0
26	pq	339	0
27	pq	261.5	0
28	pq	306	0
29	pq	583.5	0
30	pq	0	600
31	vθ	0	0
32	pv	9.5	750
33	pv	0	850
34	pv	0	508
35	pv	0	650
36	pv	0	250
37	pv	0	540
38	pv	0	830
39	pv	0	868

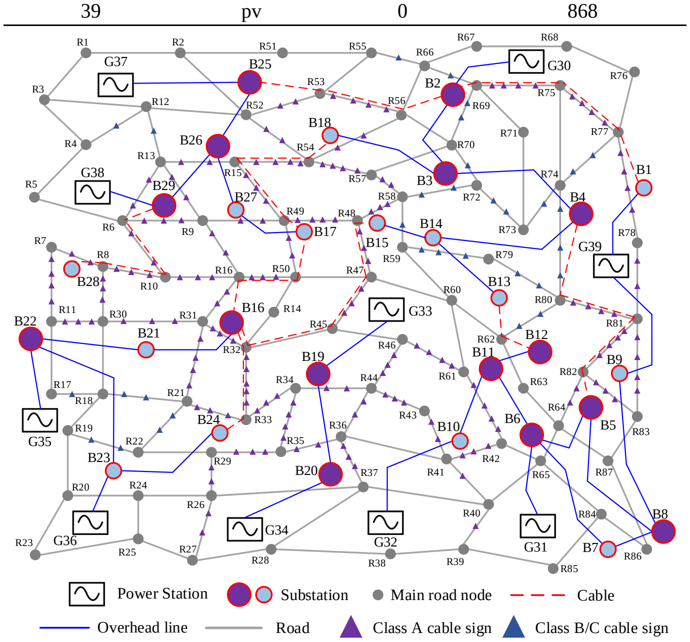

Fig. B1 The coupling network between incomplete UTN and the road network of the case