Guts Round

EMCC

February 2021

1. What is the remainder when 2021 is divided by 102?
2. Brian has 2 left shoes and 2 right shoes. Given that he randomly picks 2 of the 4 shoes, the probability he will get a left shoe and a right shoe is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.
3. In how many ways can 59 be written as a sum of two perfect squares? (The order of the two perfect squares does not matter.)

4. Two positive integers have a sum of 60. Their least common multiple is 273. What is the positive difference between the two numbers?

5. How many ways are there to distribute 13 identical apples among 4 identical boxes so that no two boxes receive the same number of apples? A box may receive zero apples.

6. In square $ABCD$ with side length 5, P lies on segment AB so that $AP = 3$ and Q lies on segment AD so that $AQ = 4$. Given that the area of triangle CPQ is x, compute $2x$.

7. Find the number of ordered triples (a, b, c) of nonnegative integers such that $2a+3b+5c = 15$.

8. What is the greatest integer $n \leq 15$ such that $n + 1$ and $n^2 + 3$ are both prime?

9. For positive integers a, b, and c, suppose that $\gcd(a, b) = 21$, $\gcd(a, c) = 10$, and $\gcd(b, c) = 11$. Find $\frac{abc}{\operatorname{lcm}(a,b,c)}$. (Note: gcd is the greatest common divisor function and lcm is the least common multiple function.)

10. The vertices of a square in the coordinate plane are at $(0,0)$, $(0,6)$, $(6,0)$, and $(6,6)$. Line ℓ intersects the square at exactly two lattice points (that is, points with integer coordinates). How many such lines ℓ are there that divide the square into two regions, one of them having an area of 12?

11. Let $f(n)$ be defined as follows for positive integers n: $f(1) = 0$, $f(n) = 1$ if n is prime, and $f(n) = f(n-1) + 1$ otherwise. What is the maximum value of $f(n)$ for $n \leq 120$?

12. The graph of the equation $y = x^3 + ax^2 + bx + c$ passes through the points $(2,4)$, $(3,9)$, and $(4,16)$. What is b?

13. Vincent the Bug is at the vertex A of square $ABCD$. Each second, he moves to an adjacent vertex with equal probability. The probability that Vincent is again on vertex A after 4 seconds is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.

14. Let ABC be a triangle with $AB = 2$, $AC = 3$, and $\angle BAC = 60^\circ$. Let P be a point inside the triangle such that $BP = 1$ and $CP = \sqrt{3}$, let x equal the area of APC. Compute $16x^2$.

15. Let n be the number of multiples of 3 between 2^{2020} and 2^{2021}. When n is written in base two, how many digits in this representation are 1?

16. Let $f(n)$ be the least positive integer with exactly n positive integer divisors. Find $\frac{f(200)}{f(50)}$.

17. The five points A, B, C, D, and E lie in a plane. Vincent the Bug starts at point A and, each minute, chooses a different point uniformly at random and crawls to it. Then the probability that Vincent is back at A after 5 minutes can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.

18. A circle is divided in the following way. First, four evenly spaced points A, B, C, D are marked on its perimeter. Point P is chosen inside the circle and the circle is cut along the rays PA, PB, PC, PD into four pieces. The piece bounded by PA, PB, and minor arc AB of the circle has area equal to one fifth of the area of the circle, and the piece bounded by PB, PC, and minor arc BC has area equal to one third of the area of the circle. Suppose that the ratio between the area of the second largest piece and the area of the circle is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.

19. There exists an integer n such that $|2^n - 5^{50}|$ is minimized. Compute n.

20. For nonnegative integers $a = \overline{a_n a_{n-1} \cdots a_2 a_1}$, $b = \overline{b_m b_{m-1} \cdots b_2 b_1}$, define their *distance* to be

$$d(a, b) = \overline{|a_{\max(m,n)} - b_{\max(m,n)}||a_{\max(m,n)-1} - b_{\max(m,n)-1}| \cdots |a_1 - b_1|},$$

where $a_k = 0$ if $k > n$, $b_k = 0$ if $k > m$. For example, $d(12321, 5067) = 13346$. For how many nonnegative integers n is $d(2021, n) + d(12345, n)$ minimized?

21. Let $ABCDE$ be a regular pentagon and let P be a point outside the pentagon such that $\angle PEA = \angle 6^\circ$ and $\angle PDC = 78^\circ$. Find the degree-measure of $\angle PBD$.

22. What is the least positive integer n such that $\sqrt{n+3} - \sqrt{n} < 0.02$?

23. What is the greatest prime divisor of $20^4 + 21 \cdot 23 - 6$?

24. Let $ABCD$ be a parallelogram and let M be the midpoint of AC. Suppose the circumcircle of triangle ABM intersects BC again at E. Given that $AB = 5\sqrt{2}$, $AM = 5$, $\angle BAC$ is acute, and the area of $ABCD$ is 70, what is the length of DE?