Exeter Math Club Competition
January 27, 2024

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Organizing Acknowledgments

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Jane Street

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Contest Day Acknowledgments

- **Tournament Directors** Alan Bu, Bryan Chen, Daria Ivanova, Michael Lu, Ava Zhao
- **Food Czars** Daria Ivanova, Yecheng (Ava) Zhao
- **Runners** Sofiya Goncharova, Ari Lee, Crane Lee
- **Proctors** Claire Chetwynd, Derrick Chu, Isabel Evans, Ella Fang, Ashley Gong, Emily Huang, Robert Joo, Emily Kim, Chayyah Lewis, Ronald Qiao, Sophia Qiu, Yash Shah, Tiffany Sun, Shaoshao Tang, Jiayu Wang, Oron Wang, Catherine Yan, Martin Yau, Chengyue Zhang, Albert Zhu
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- **Exeter Panelists** Bryan Chen, Daria Ivanova, Harini Venkatesh, Benny Wang

Chapter 1

EMC^2 2024 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Compute
$$\frac{2024}{2+0\times 2-4}.$$

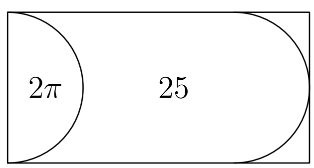
2. Find the smallest integer that can be written as the product of three distinct positive odd integers.

3. Bryan's physics test score is a two-digit number. When Bryan reverses its digits and adds the tens digit of his test score, he once again obtains his test score. Determine Bryan's physics test score.

4. Grant took four classes today. He spent 70 minutes in math class. Had his math class been 40 minutes instead, he would have spent 15% less total time in class today. Find how many minutes he spent in his other classes combined.

5. Albert's favorite number is a nonnegative integer. The square of Albert's favorite number has 9 digits. Find the number of digits in Albert's favorite number.

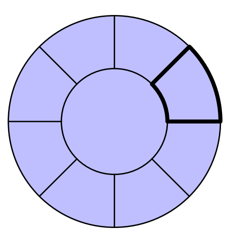
6. Two semicircular arcs are drawn in a rectangle, splitting it into four regions as shown below. Given the areas of two of the regions, find the area of the entire rectangle.



7. Daria is buying a tomato and a banana. She has a 20%-off coupon which she may use on one of the two items. If she uses it on the tomato, she will spend \$1.21 total, and if she uses it on the banana, she will spend \$1.31 total. In cents, find the absolute difference between the price of a tomato and the price of a banana.

8. Celine takes an 8×8 checkerboard of alternating black and white unit squares and cuts it along a line, creating two rectangles with integer side lengths, each of which contains at least 9 black squares. Find the number of ways Celine can do this. (Rotations and reflections of the cut are considered distinct.)

9. Each of the nine panes of glass in the circular window shown below has an area of π, eight of which are congruent. Find the perimeter of one of the non-circular panes.

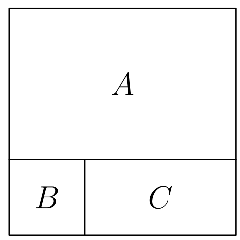


10. In Alan's favorite book, pages are numbered with consecutive integers starting with 1. The average of the page numbers in Chapter Five is 95 and the average of the page numbers in Chapter Six is 114. Find the number of pages in Chapters Five and Six combined.

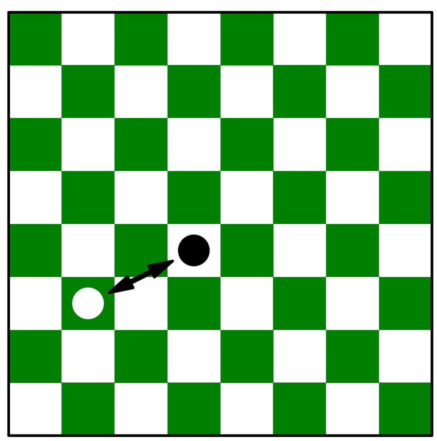
11. Find the number of ordered pairs (a, b) of positive integers such that $a + b = 2024$ and

$$\frac{a}{b} > \frac{1000}{1025}.$$

12. A square is split into three smaller rectangles A, B, and C. The area of A is 80, B is a square, and the area of C is 30. Compute the area of B.



13. A knight on a chessboard moves two spaces horizontally and one space vertically, or two spaces vertically and one space horizontally. Two knights attack each other if each knight can move onto the other knight's square. Find the number of ways to place a white knight and a black knight on an 8×8 chessboard so that the two knights attack each other. One such possible configuration is shown below.



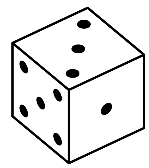
14. Find the sum of all positive integers N for which the median of the positive divisors of N is 9.

15. Let x, y, and z be nonzero real numbers such that

$$\begin{cases} 20x + 24y = yz \\ 20y + 24x = xz. \end{cases}$$

Find the sum of all possible values of z.

16. Ava glues together 9 standard six-sided dice in a 3×3 grid so that any two touching faces have the same number of dots. Find the number of dots visible on the surface of the resulting shape. (On a standard six-sided die, opposite faces sum to 7.)



17. Harini has a regular octahedron of volume 1. She cuts off its 6 vertices, turning the triangular faces into regular hexagons. Find the volume of the resulting solid.

18. Each second, Oron types either **O** or **P** with equal probability, forming a growing sequence of letters. Find the probability he types out **POP** before **OOP**.

19. For an integer $n \geq 10$, define $f(n)$ to be the number formed after removing the first digit from n (and removing any leading zeros) and define $g(n)$ to be the number formed after removing the last digit from n. Find the sum of the solutions to the equation $f(n) + g(n) = 2024$.

20. In convex trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $AD = BC$, let M be the midpoint of $\overline{BC}$. If $\angle AMB = 24^\circ$ and $\angle CMD = 66^\circ$, find $\angle ABC$, in degrees.

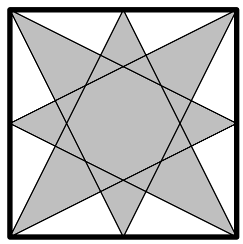
1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. Find the smallest positive multiple of 9 whose digits are all even.

2. Anika writes down a positive real number x in decimal form. When Nat erases everything to the left of the decimal point, the remaining value is one-fifth of x. Find the sum of all possible values of x.

3. A star-like shape is formed by joining up the midpoints and vertices of a unit square, as shown in the diagram below. Compute the area of this shape.

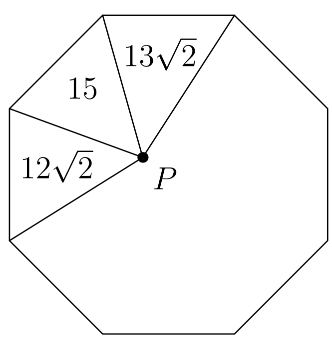


4. Benny and Daria are running a 200 meter foot race, each at a different constant speed. When Daria finishes the race, she is 14 meters ahead of Benny. The next time they race, Daria starts 14 meters behind Benny, who starts at the starting line. Both runners run at the same constant speed as in the first race. When Daria reaches the finish line, compute, *in centimeters*, how far she is ahead of Benny.

5. In one semester, Ronald takes ten biology quizzes, earning a distinct integer score from 91 to 100 on each quiz. He notices that after the first three quizzes, the average of his three most recent scores always increased. Compute the number of ways Ronald could have earned the ten quiz scores.

6. Ant and Ben are playing a game with stones. They begin with Z stones on the ground. Ant and Ben take turns removing a prime number of stones. Ant moves first. The player who is first unable to make a valid move loses. Find the sum of all positive integers $Z \leq 30$ such that Ben can guarantee a win with perfect play.

7. Let P be a point in a regular octagon as shown in the diagram below. The areas of three triangles are shown. Find the area of the octagon.



8. Find the number of ordered triples (a, b, c) of nonnegative integers with $a \leq b \leq c$ for which

$$5a + 4b + 6c = 1200.$$

9. Define
$$f(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \le x \le 1. \end{cases}$$
Michael picks a real number $0 \le x \le 1$. Michael applies f repeatedly to x until he reaches x again. Find the number of real numbers x for which Michael applies f exactly 12 times.

10. In $\triangle ABC$, let point H be the intersection of its altitudes and let M be the midpoint of side $\overline{BC}$. Given that $BC = 4$, $MA = 3$, and $\angle HMA = 60°$, find the circumradius of $\triangle ABC$.

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 60 minutes.

1. Warren interrogates the 25 members of his cabinet, each of whom always lies or always tells the truth. He asks them all, "How many of you always lie?" He receives every integer answer from 1 to 25 exactly once. Find the actual number of liars in his cabinet.

2. Abraham thinks of distinct nonzero digits E, M, and C such that

$$E + M = \overline{CC}.$$

   Help him evaluate the sum of the two digit numbers $\overline{EC}$ and $\overline{MC}$. (Note that $\overline{CC}$, $\overline{EC}$, and $\overline{MC}$ are read as two-digit numbers.)

3. Let ω, Ω, Γ be concentric circles such that Γ is inside Ω and Ω is inside ω. Points A, B, C on ω and D, E on Ω are chosen such that line AB is tangent to Ω, line AC is tangent to Γ, and line DE is tangent to Γ. If $AB = 21$ and $AC = 29$, find DE.

4. Let a, b, and c be three prime numbers such that $a + b = c$. If the average of two of the three primes is four less than four times the fourth power of the last, find the second-largest of the three primes.

5. At Stillwells Ice Cream, customers must choose one type of scoop and two different types of toppings. There are currently 630 different combinations a customer could order. If another topping is added to the menu, there would be 840 different combinations. If, *instead*, another type of scoop were added to the menu, compute the number of different combinations there would be.

6. Eleanor the ant takes a path from $(0, 0)$ to $(20, 24)$, traveling either one unit right or one unit up each second. She records every lattice point she passes through, including the starting and ending point. If the sum of all the x-coordinates she records is 271, compute the sum of all the y-coordinates. (A lattice point is a point with integer coordinates.)

7. Teddy owns a square patch of desert. He builds a dam in a straight line across the square, splitting the square into two trapezoids. The perimeters of the trapezoids are 64 miles and 76 miles, and their areas differ by 135 square miles. Find, in miles, the length of the segment that divides them.

8. Michelle is playing Spot-It with a magical deck of 10 cards. Each card has 10 distinct symbols on it, and every pair of cards shares exactly 1 symbol. Find the minimum number of distinct symbols on all of the cards in total.

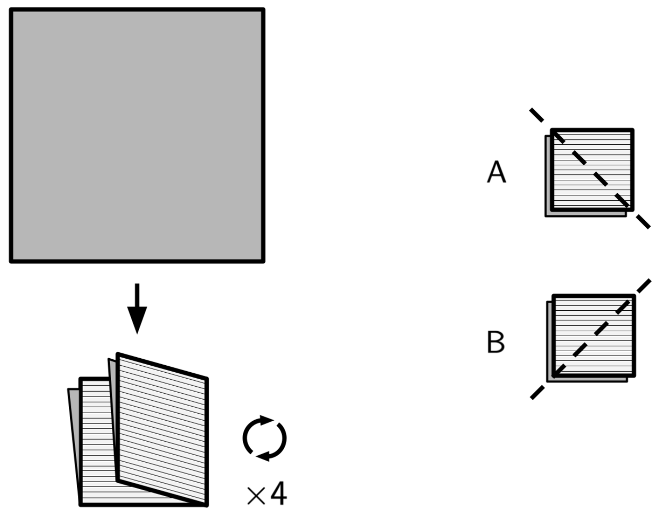
9. Define the function $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$ for integers $n \geq 2$. Find

$$f(2) + f(4) + f(6) + \cdots .$$

10. There are 9 indistinguishable ants standing on a 3×3 square grid. Each ant is standing on exactly one square. Compute the number of different ways the ants can stand so that no column or row contains more than 3 ants.

11. Let $s(N)$ denote the sum of the digits of N. Compute the sum of all two-digit positive integers N for which $s(N^2) = s(N)^2$.

12. Martha has two square sheets of paper, A and B. With each sheet, she repeats the following process four times: fold bottom side to top side, fold right side to left side. With sheet A, she then makes a cut from the top left corner to the bottom right. With sheet B, she makes a cut from the bottom left corner to the top right. Find the total number of pieces of paper yielded from sheets A and sheets B.



13. Let x and y be positive integers such that

$$\gcd(x^y, y^x) = 2^{28}.$$

Find the sum of all possible values of $\min(x, y)$.

14. Convex hexagon *TRUMAN* has opposite sides parallel. If each side has length 3 and the area of this hexagon is 5, compute

$$TU \cdot RM \cdot UA \cdot MN \cdot AT \cdot NR.$$

15. Let x, y, and z be positive real numbers satisfying the system

$$\begin{cases} x^2 + xy + y^2 = 25 \\ y^2 + yz + z^2 = 36 \\ z^2 + zx + x^2 = 49. \end{cases}$$

Compute $x^2 + y^2 + z^2$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [6] When Shiqiao sells a bale of kale, he makes x dollars, where
$$x = \frac{1+2+3+4+5+6+7+8}{3+4+5+6}.$$
Find x.

2. [6] The fraction of Shiqiao's kale that has gone rotten is equal to
$$\sqrt{\frac{100^2}{99^2} - \frac{100}{99}}.$$
Find the fraction of Shiqiao's kale that has gone rotten.

3. [6] Shiqiao is growing kale. Each day the number of kale plants doubles, but 4 of his kale plants die afterwards. He starts with 6 kale plants. Find the number of kale plants Shiqiao has after five days.

1.4.2 Round 2

4. [7] Today the high is 68 degrees Fahrenheit. If C is the temperature in Celsius, the temperature in Fahrenheit is equal to $1.8C + 32$. Find the high today in Celsius.

5. [7] The internal angles in Evan's triangle are all at most 68 degrees. Find the minimum number of degrees an angle of Evan's triangle could measure.

6. [7] Evan's room is at 68 degrees Fahrenheit. His thermostat has two buttons, one to increase the temperature by one degree, and one to decrease the temperature by one degree. Find the number of combinations of 10 button presses Evan can make so that the temperature of his room never drops below 67 degrees or rises above 69 degrees.

1.4.3 Round 3

7. [9] In a digital version of the SAT, there are four spaces provided for either a digit (0-9), a fraction sign (/), or a decimal point (.). The answer must be in simplest form and at most one space can be a non-digit character. Determine the largest fraction which, when expressed in its simplest form, fits within this space, but whose exact decimal representation does not.

8. [9] Rounding Rox picks a real number x. When she rounds x to the nearest hundred, its value increases by 2.71828. If she had instead rounded x to the nearest hundredth, its value would have decreased by y. Find y.

9. [9] Let a and b be real numbers satisfying the system of equations
$$\begin{cases} a + \lfloor b \rfloor = 2.14 \\ \lfloor a \rfloor + b = 2.72. \end{cases}$$
Determine $a + b$.

1.4.4 Round 4

10. [11] Carol and Lily are playing a game with two unfair coins, both of which have a $1/4$ chance of landing on heads. They flip both coins. If they both land on heads, Lily loses the game, and if they both land on tails, Carol loses the game. If they land on different sides, Carol and Lily flip the coins again. They repeat this until someone loses the game. Find the probability that Lily loses the game.

11. [11] Dongchen is carving a circular coin design. He carves a regular pentagon of side length 1 such that all five vertices of the pentagon are on the rim of the coin. He then carves a circle inside the pentagon so that the circle is tangent to all five sides of the pentagon. Find the area of the region between the smaller circle and the rim of the coin.

12. [11] Anthony flips a fair coin six times. Find the probability that at some point he flips 2 heads in a row.

1.4.5 Round 5

13. [13] Mandy is baking cookies. Her recipe calls for N grams of flour, where N is the number of perfect square divisors of $20! + 24!$. Find N.

14. [13] Consider a circular table with center R. Beef-loving Bryan places a steak at point I on the circumference of the table. Then he places a bowl of rice at points C and E on the circumference of the table such that $CE \parallel IR$ and $\angle ICE = 25°$. Find $\angle CIE$.

15. [13] Enya writes the 4-letter words LEEK, BEAN, SOUP, PEAS, HAMS, and TACO on the board. She then thinks of one of these words and gives Daria, Ava, Harini, and Tiffany a slip of paper containing exactly one letter from that word such that if they ordered the letters on their slips correctly, they would form the word.

    Each person announces at the same time whether they know the word or not. Ava, Harini, and Tiffany all say they do not know the word, while Daria says she knows the word. After hearing this, Ava, Harini, and Tiffany all know the word. Assuming all four girls are perfect logicians and they all thought of the same correct word, determine Daria's letter.

1.4.6 Round 6

16. [15] Michael receives a cheese cube and a chocolate octahedron for his 5th birthday. On every day after, he slices off each corner of his cheese and chocolate with a knife. Each slice cuts off exactly one corner. He then eats each corner sliced off. Find the difference between the total number of cheese and chocolate pieces he has eaten by the end of his 6th birthday. (Michael's 5th and 6th birthdays do not occur on leap years.)

17. [15] Let D be the average of all positive integers n satisfying

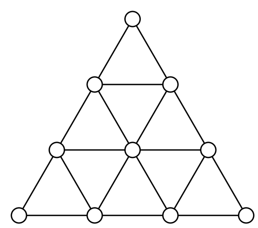
    $$\operatorname{lcm}(\gcd(n, 2000), \gcd(n, 24)) = \gcd(\operatorname{lcm}(n, 2000), \operatorname{lcm}(n, 24)).$$

    Find $3D$.

18. [15] The base $\triangle ABC$ of the triangular pyramid $PABC$ is an equilateral triangle with a side length of 3. Given that $PA = 3$, $PB = 4$, and $PC = 5$, find the circumradius of $PABC$.

1.4.7 Round 7

19. [18] 2049300 points are arranged in an equilateral triangle point grid, a smaller version of which is shown below, such that the sides contain 2024 points each. Peter starts at the topmost point of the grid. At 9:00 am each day, he moves to an adjacent point in the row below him. Derrick wants to prevent Peter from reaching the bottom row, so at 12:00 pm each day, he selects a point on the bottom row and places a rock at that point. Peter stops moving as soon as he is guaranteed to end up at a point with a rock on it. At least how many moves will Peter complete, no matter how Derrick places the rocks?



20. [18] There are N stones in a pile, where N is a positive integer. Ava and Anika take turns playing a game, with Ava moving first. If there are n stones in the pile, a move consists of removing x stones, where $1 < \gcd(x, n) \leq x < n$. Whoever first has no possible moves on their turn wins. Both Ava and Anika play optimally. Find the 2024th smallest value of N for which Ava wins.

21. [18] Alan is bored and alone, so he plays a fun game with himself. He writes down all quadratic polynomials with leading coefficient 1 whose coefficients are integers between -10 and 10, inclusive, on a blackboard. He then erases all polynomials which have a non-integer root. Alan defines the size of a polynomial $P(x)$ to be $P(1)$ and spends an hour adding up the sizes of all the polynomials remaining on the blackboard. Assuming Alan does computation perfectly, find the sum Alan obtains.

1.4.8 Round 8

22. [21] A prime number is a positive integer with exactly two distinct divisors. You must submit a prime number for this problem. If you do not submit a prime number, you gain 0 points, and your submission will not be considered valid. The median of all valid submitted numbers is M (duplicates are counted). Estimate $2M$. If your team's absolute difference between $2M$ and your submission is the ith smallest absolute difference among all teams, you gain $\max(23 - 2i, 0)$ points. All teams who did not submit any number gain 0 points. (In the case of a tie, all teams that tied gain the same amount of points.)

23. [21] Ribbotson the Frog is at the point $(0, 0)$ and wants to reach the point $(18, 18)$ in 36 steps. Each step, he either moves one unit in the $+x$ direction or one unit in the $+y$ direction. However, Ribbotson hates turning, so he must make at least two steps in any direction before switching directions. If m is the number of different paths Ribbotson the Frog can make, estimate m. If N is your team's submitted number, your team earns points equal to the closest integer to $21(1 - |\log_{10} \frac{N}{m}|^2)$.

24. [21] Let $M = \pi^{\pi^{\pi^{\pi}}}$. Estimate k, where $M = 10^{10^k}$. If N is your team's submitted number, your team earns points equal to the closest integer to $21 \cdot 1.01^{(-|N-k|^3)}$.

Chapter 2

EMC^2 2024 Solutions

2.1 Speed Test Solutions

1. Compute

$$\frac{2024}{2+0\times 2-4}.$$

Solution. The answer is $\boxed{-1012}$.

We have

$$\frac{2024}{2+0\times 2-4}=\frac{2024}{2+0-4}=\frac{2024}{-2}=-1012.$$

2. Find the smallest integer that can be written as the product of three distinct positive odd integers.

Solution. The answer is $\boxed{15}$.

The three smallest odd positive integers are 1, 3, and 5, so the answer is

$$1\cdot 3\cdot 5=15.$$

3. Bryan's physics test score is a two-digit number. When Bryan reverses its digits and adds the tens digit of his test score, he once again obtains his test score. Determine Bryan's physics test score.

Solution. The answer is $\boxed{98}$.

Letting Bryan's score equal $10a+b$ for digits a and b, we have the equation

$$\begin{aligned} 10a+b&=10b+a+a\\ 8a&=9b. \end{aligned}$$

Since a and b are digits and cannot both equal 0, we must have $a=9$ and $b=8$, making Bryan's score 98.

4. Grant took four classes today. He spent 70 minutes in math class. Had his math class been 40 minutes instead, he would have spent 15% less total time in class today. Find how many minutes he spent in his other classes combined.

Solution. The answer is $\boxed{130}$.

If cutting 30 minutes from Grant's day reduced it by 15%, the original 100% must have been

$$30\cdot\frac{100\%}{15\%}=200$$

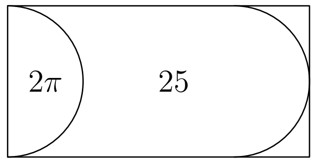
minutes. Subtracting off the 70 minutes of math class, his other classes must have taken 130 minutes.

5. Albert's favorite number is a nonnegative integer. The square of Albert's favorite number has 9 digits. Find the number of digits in Albert's favorite number.

Solution. The answer is $\boxed{5}$.

A number with 9 digits is at least 10^8 and less than 10^9, so its square root must be at least 10^4 (which has five digits) and at most $10^{4.5}$. In fact, $10^{4.5}$ is strictly less than 10^5-1 which has five digits, so $10^{4.5}$ must also have five digits. Thus, Albert's number must have exactly 5 digits.

6. Two semicircular arcs are drawn in a rectangle, splitting it into four regions as shown below. Given the areas of two of the regions, find the area of the entire rectangle.



Solution. The answer is $\boxed{33}$.

Letting the radius of the semicircle equal r, we have the equation

$$\frac{r^2\pi}{2} = 2\pi,$$

so $r = 2$. Thus, the height of the rectangle is 4. Now imagine sliding the left semicircle so that it perfectly fits inside the right semicircular arc. There will be two rectangles, each with height 4. The left rectangle has area 25, since sliding the semicircle does not change the area of the middle region. The right rectangle has width $r = 2$, so its area is $4 \cdot 2 = 8$. Summing these two areas up gives us a total area of $25 + 8 = 33$.

7. Daria is buying a tomato and a banana. She has a 20%-off coupon which she may use on one of the two items. If she uses it on the tomato, she will spend \$1.21 total, and if she uses it on the banana, she will spend \$1.31 total. In cents, find the absolute difference between the price of a tomato and the price of a banana.

Solution. The answer is $\boxed{50}$.

Letting the price of a tomato be t and a banana be b, we have the equations

$$\begin{cases} 0.8 \cdot t + b = 121 \\ t + 0.8 \cdot b = 131. \end{cases}$$

Subtracting the first equation from the second one gives

$$0.2 \cdot t - 0.2 \cdot b = 10,$$

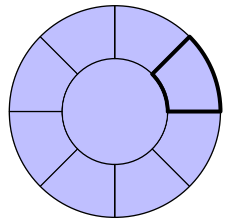
so $t - b = 50$.

8. Celine takes an 8×8 checkerboard of alternating black and white unit squares and cuts it along a line, creating two rectangles with integer side lengths, each of which contains at least 9 black squares. Find the number of ways Celine can do this. (Rotations and reflections of the cut are considered distinct.)

Solution. The answer is $\boxed{6}$.

A cut will split the board along rows or columns, so both rectangles will have one dimension of 8. To contain at least 9 black squares, the other dimension must be at least 3, because a 2×8 rectangle only has 8 black squares, while a 3×8 rectangle has 12 black squares. Thus, for vertical cuts there will be 3 possibilities, which come from splitting the width of 8 into $3 + 5$, $4 + 4$, or $5 + 3$. Similarly, horizontal cuts also give 3 possibilities. Therefore, the answer is $3 + 3 = 6$.

9. Each of the nine panes of glass in the circular window shown below has an area of π, eight of which are congruent. Find the perimeter of one of the non-circular panes.



Solution. The answer is $\boxed{4+\pi}$.

The central pane is a circle with area π, so it has radius 1. The whole window is a circle made of all 9 panes, which have a total area of 9π, so the window has radius 3. Thus, the straight sides of the side-panes have lengths equal to $3-1=2$. The two circular arcs of each side-pane make up $\frac{1}{8}$ of their respective circles, so they have lengths $\frac{6\pi}{8}=\frac{3}{4}\pi$ and $\frac{2\pi}{8}=\frac{1}{4}\pi$. Adding up these lengths gives $2+2+\frac{3}{4}\pi+\frac{1}{4}\pi=4+\pi$.

10. In Alan's favorite book, pages are numbered with consecutive integers starting with 1. The average of the page numbers in Chapter Five is 95 and the average of the page numbers in Chapter Six is 114. Find the number of pages in Chapters Five and Six combined.

Solution. The answer is $\boxed{38}$.

The 18 pages strictly between pages 95 and 114 form the latter half of Chapter Five and the former half of Chapter Six. (The latter/former halves do not contain the exact halfway point in a chapter.) Since the former and latter halves of each chapter have the same number of pages, the total number of pages in the former half of Chapter Five and the latter half of Chapter Six is also 18. Adding pages 95 and 114 gives a total of

$$18 \cdot 2 + 2 = 38.$$

11. Find the number of ordered pairs (a, b) of positive integers such that $a+b=2024$ and

$$\frac{a}{b} > \frac{1000}{1025}.$$

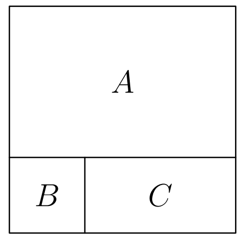
Solution. The answer is $\boxed{1024}$.

The fraction $\frac{999}{1025}$ fails the property since its numerator is too small, whereas the fraction $\frac{1000}{1024}$ satisfies the property since its denominator is small enough. All fractions onwards work as well, since each fraction is larger than the previous; our solution set is

$$\frac{1000}{1024}, \frac{1001}{1023}, \ldots, \frac{2023}{1},$$

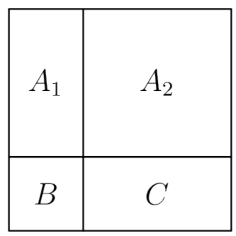
for a total of 1024 solutions.

12. A square is split into three smaller rectangles A, B, and C. The area of A is 80, B is a square, and the area of C is 30. Compute the area of B.



Solution. The answer is $\boxed{18}$.

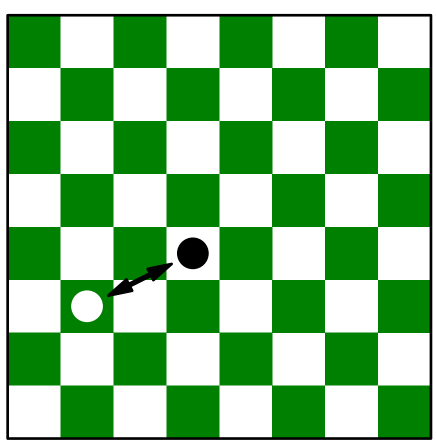
Extend the line separating rectangles B and C as shown in the diagram below.



Since B is a square and the outer rectangle is a square as well, it follows that A_1 and C are congruent, so they both have an area of 30. It also follows that A_2 is a square with area $80 - 30 = 50$.

Thus, the side length of A_2 is $\sqrt{50} = 5\sqrt{2}$, so the height of A_1 is also $5\sqrt{2}$; then, the width of A_1 is $\frac{30}{5\sqrt{2}} = 3\sqrt{2}$, so the width of B is $3\sqrt{2}$. Squaring this gives the area of B, which is $\left(3\sqrt{2}\right)^2 = 18$.

13. A knight on a chessboard moves two spaces horizontally and one space vertically, or two spaces vertically and one space horizontally. Two knights attack each other if each knight can move onto the other knight's square. Find the number of ways to place a white knight and a black knight on an 8×8 chessboard so that the two knights attack each other. One such possible configuration is shown below.



Solution. The answer is $\boxed{336}$.

First we count the number of 2 by 3 rectangles in the chessboard. There are 2 possible orientations for this rectangle, along with $7 \cdot 6 = 42$ possible positions for the rectangle, so there are

$$2 \cdot 42 = 84$$

rectangles that we could choose. For each rectangle, we can pick one of the four corners to be the black knight – then, it immediately follows that the opposite corner of the rectangle contains the white knight, giving us a working configuration.

Our final answer is $84 \cdot 4 = 336$.

14. Find the sum of all positive integers N for which the median of the positive divisors of N is 9.

Solution. The answer is $\boxed{320}$.

Divisors pair up (possibly with themselves) to multiply to N, so the pair which is closest together must have a median of 9. We consider each case:

- We could have $N = 9 \cdot 9 = 81$.
- We could have $N = 8 \cdot 10 = 80$.
- We could have $N = 7 \cdot 11 = 77$.
- We could *not* have $N = 6 \cdot 12 = 72$, since $(6, 12)$ is not the closest pair of divisors; that award goes to $(8, 9)$.
- We could have $N = 5 \cdot 13 = 65$.
- We could *not* have $N = 4 \cdot 14 = 56$, since $(4, 14)$ is not the closest pair of divisors; that award goes to $(7, 8)$.
- We could *not* have $N = 3 \cdot 15 = 45$, since $(3, 15)$ is not the closest pair of divisors; that award goes to $(5, 9)$.
- We could *not* have $N = 2 \cdot 16 = 32$, since $(2, 16)$ is not the closest pair of divisors; that award goes to $(4, 4)$.
- We could have $N = 1 \cdot 17 = 17$.

Summing together the working possibilities gives us a total of

$$81 + 80 + 77 + 65 + 17 = 320.$$

15. Let x, y, and z be nonzero real numbers such that

$$\begin{cases} 20x + 24y = yz \\ 20y + 24x = xz. \end{cases}$$

Find the sum of all possible values of z.

Solution. The answer is $\boxed{48}$.

Summing the two equations and factoring gives us

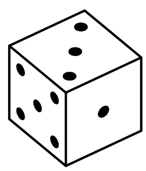
$$\begin{aligned} 44x + 44y &= yz + xz \\ 44(x + y) &= z(x + y) \\ (44 - z)(x + y) &= 0. \end{aligned}$$

So, we either have $z = 44$ (attainable by setting $x = y$) or $x + y = 0$. The latter is equivalent to $x = -y$; plugging this back into the first equation in our original system gives us

$$\begin{aligned} -20y + 24y &= yz \\ 4y &= yz \\ 4 &= z. \end{aligned}$$

(It's okay to divide by y at the last step since we're given that its nonzero.) We can check that $z = 4$ is obtainable whenever $x = -y$. Summing these two possible values gives us a total of $4 + 44 = 48$.

16. Ava glues together 9 standard six-sided dice in a 3×3 grid so that any two touching faces have the same number of dots. Find the number of dots visible on the surface of the resulting shape. (On a standard six-sided die, opposite faces sum to 7.)



Solution. The answer is $\boxed{105}$.

Opposite faces sum to 7. Thus the number of dots on the two faces with nine die showing must be $9 \cdot 7 = 63$. This is because they are on opposite sides. To find the number of dots on the side, for each face of a die with number of dots n, the face on the opposite side will show the number of dots $7 - n$. This can be shown using the fact that two touching faces have the same number of dots. Thus the number of dots on the side will be $6 \cdot 7 = 42$ from the 6 pairs of opposite die. The total number of visible dots is $63 + 42 = 105$.

17. Harini has a regular octahedron of volume 1. She cuts off its 6 vertices, turning the triangular faces into regular hexagons. Find the volume of the resulting solid.

Solution. The answer is $\boxed{\frac{8}{9}}$.

To turn the triangular faces into regular hexagons, for octahedron side length l, a square pyramid with all side lengths equal to $\frac{l}{3}$ must be cut from each vertex so that the side lengths of the resulting hexagonal faces will be $\frac{l}{3}$. These square pyramids will be similar to the square pyramids formed from half the octahedron. Since the length ratio is $\frac{1}{3}$, the volume ratio will be $\frac{1}{27}$, and thus the volume of each cut square pyramid will be $\frac{1}{54}$ of the volume of the octagon. Since there are 6 cut pieces, the resulting solid will have volume $1 - \frac{6}{54} = \frac{8}{9}$.

18. Each second, Oron types either **O** or **P** with equal probability, forming a growing sequence of letters. Find the probability he types out **POP** before **OOP**.

Solution. The answer is $\boxed{\frac{3}{8}}$.

Note that once Oron types two **O**'s in a row, he is doomed to write "**OOP**" the moment he types his next **P**.

We split into two cases:

- In the $\frac{1}{2}$ chance that Oron's first letter is an **O**, then he must write a **P** next, which happens with probability $\frac{1}{2}$; then, after Oron types his second **O**, he must type a **P**, which happens with probability $\frac{1}{2}$. Now, Oron has successfully typed **POP**. The probability that this case occurs and succeeds is
$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$
- In the $\frac{1}{2}$ chance that Oron's first letter is a **P**: after Oron types his first **O**, he must type a **P**, which happens with probability $\frac{1}{2}$. Now, Oron has, again, successfully typed **POP**. This case occurs and succeeds with probability
$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Summing the two cases gives us a total of $\frac{1}{2} + \frac{1}{4} = \frac{3}{8}$.

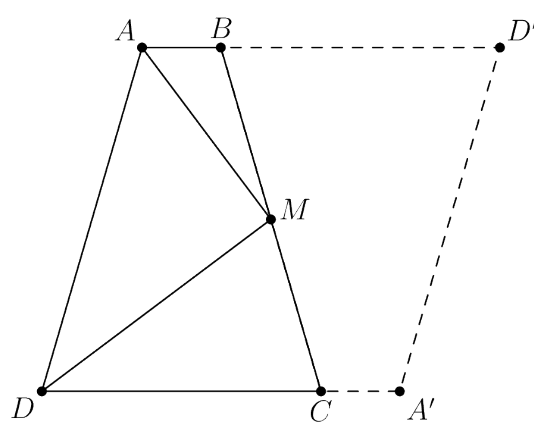
19. For an integer $n \geq 10$, define $f(n)$ to be the number formed after removing the first digit from n (and removing any leading zeros) and define $g(n)$ to be the number formed after removing the last digit from n. Find the sum of the solutions to the equation $f(n) + g(n) = 2024$.

    Solution. The answer is $\boxed{30953}$.

    First, note that n must have exactly 5 digits. If n has more than 5 digits, then $g(n)$ will be too large, while if n has fewer than 5 digits, both $f(n)$ and $g(n)$ will be less than 1000. Then we can choose either 1 or 2 for the first digit, as any other selection will fail. Then we can work left to right on the digits to determine that the final answer is $10931 + 20022 = 30953$.

20. In convex trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $AD = BC$, let M be the midpoint of $\overline{BC}$. If $\angle AMB = 24°$ and $\angle CMD = 66°$, find $\angle ABC$, in degrees.

    Solution. The answer is $\boxed{104}$.



Rotate the trapezoid and glue it along edge $\overline{BC}$, forming parallelogram $AD'A'D$ as shown in the diagram above. Since $\angle A'MC = \angle AMB$, it follows that M lies on $\overline{AA'}$; similarly, M lies on $\overline{DD'}$. So, the diagonals $\overline{DD'}$ and $\overline{AA'}$ intersect at point M.

Note that $\angle AMD = 180^\circ - \angle AMB - \angle CMD = 90^\circ$, meaning the diagonals of parallelogram $AD'A'D$ are perpendicular – in fact, $AD'A'D$ is a rhombus. Due to properties of rhombuses, $\overline{AM}$ bisects $\angle DAB$. Letting $\angle ABM = x$, it follows that $\angle BAM = \frac{x}{2}$. Since the angles in $\triangle ABM$ sum to 180°, we have

$$x + \frac{x}{2} + 24^\circ = 180^\circ \implies x = 104^\circ.$$

EXETER MATH CLUB COMPETITION

2.2 Accuracy Test Solutions

1. Find the smallest positive multiple of 9 whose digits are all even.

   Solution. The answer is $\boxed{288}$.

   The divisibility of 9 rule forces the sum of the digits of this number to be a multiple of 9. Since each digit is even, it follows that the sum of the digits of this number is at least 18.

   The only such two-digit number is 99 which has odd digits, so we search for the smallest possible three-digit number. Its first digit must be even so it must be at least 2. The second and third digits are both even and sum to 16, so they must both be 8. This gives us 288, which can be checked to work.

   On the other hand, if the sum of the digits of our number is 36, it must have at least four digits which is certainly bigger than 288.

2. Anika writes down a positive real number x in decimal form. When Nat erases everything to the left of the decimal point, the remaining value is one-fifth of x. Find the sum of all possible values of x.

   Solution. The answer is $\boxed{\frac{15}{2}}$.

   We can express x as $N + a$ for a nonnegative integer N and $0 \le a < 1$. The condition in the problem tells us that

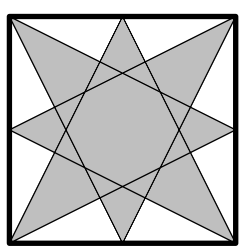
$$\begin{aligned} \frac{N+a}{5} &= a \\ N &= 4a. \end{aligned}$$

   Since N is a positive integer, the only possible values for a are $\frac{1}{4}$, $\frac{2}{4}$ and $\frac{3}{4}$. (In particular, $a = 0$ would result in $N = 0$, which is not allowed.) These correspond to

$$x = 1\tfrac{1}{4},\ 2\tfrac{2}{4},\ 3\tfrac{3}{4},$$

   which sum to $\frac{15}{2}$.

3. A star-like shape is formed by joining up the midpoints and vertices of a unit square, as shown in the diagram below. Compute the area of this shape.



   Solution. The answer is $\boxed{\frac{3}{5}}$.

Each of the eight empty triangles has side length ratio $1 : 2 : \sqrt{5}$ with hypotenuse $\frac{1}{2}$, so its area is

$$\frac{1}{2}(1)(2) \cdot \left(\frac{\frac{1}{2}}{\sqrt{5}}\right)^2 = \frac{1}{20}.$$

Hence, the area of the shaded shape is

$$1^2 - 8 \cdot \frac{1}{20} = \frac{3}{5}.$$

4. Benny and Daria are running a 200 meter foot race, each at a different constant speed. When Daria finishes the race, she is 14 meters ahead of Benny. The next time they race, Daria starts 14 meters behind Benny, who starts at the starting line. Both runners run at the same constant speed as in the first race. When Daria reaches the finish line, compute, *in centimeters*, how far she is ahead of Benny.

Solution. The answer is $\boxed{98}$.

Let v_{B} and v_{D} be the speeds of Benny and Daria respectively. Since

$$\frac{200-14}{v_{\text{B}}} = \frac{200}{v_{\text{D}}}$$

in the first race, we have

$$\frac{v_{\text{B}}}{v_{\text{D}}} = \frac{200-14}{200} = \frac{93}{100}.$$

The distance that Benny runs in the second race is

$$\frac{200+14}{v_{\text{D}}} \cdot v_{\text{B}} = (200+14) \cdot \frac{93}{100} = \frac{9951}{50},$$

so when Daria reaches the finish line, she is ahead of Benny by

$$200 - \frac{9951}{50} = \frac{49}{50}$$

meters, or 98 centimeters.

5. In one semester, Ronald takes ten biology quizzes, earning a distinct integer score from 91 to 100 on each quiz. He notices that after the first three quizzes, the average of his three most recent scores always increased. Compute the number of ways Ronald could have earned the ten quiz scores.

Solution. The answer is $\boxed{4200}$.

Define Ronald's "recent average" to be the average of his three recent scores. For $i > 3$, Ronald's ith quiz replaces his $(i-3)$th quiz in his recent average, so this average increases if and only if his ith quiz score is greater than his $(i-3)$th quiz score.

It follows that the 1st, 4th, 7th, and 10th quiz scores form an increasing sequence; the 2nd, 5th, and 8th quiz scores form an increasing sequence; and the 3rd, 6th, and 9th quiz scores form an increasing sequence.

Thus, it suffices to partition $\{91, 92, \ldots, 100\}$ into one set of size 4 and two sets of size 3; we then allocate the first set to be the scores of tests numbers $\{1, 4, 7, 10\}$, the second set to be the scores of tests numbers $\{2, 5, 8\}$ and the third set to be the scores of tests numbers $\{3, 6, 9\}$.

We now compute the number of ways to partition a set of 10 elements into one set of size 4 and two sets of size 3:

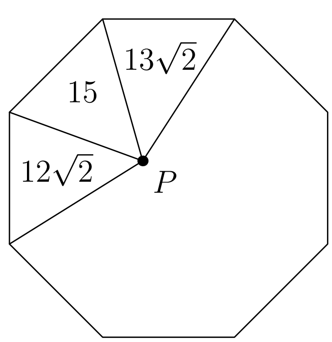
$$\binom{10}{4,3,3} = \frac{10!}{4!3!3!} = 4200.$$

6. Ant and Ben are playing a game with stones. They begin with Z stones on the ground. Ant and Ben take turns removing a prime number of stones. Ant moves first. The player who is first unable to make a valid move loses. Find the sum of all positive integers $Z \le 30$ such that Ben can guarantee a win with perfect play.

Solution. The answer is $\boxed{45}$.

We say a position is winning for the first player if the player who moves first from that position can force a win, and a position is winning for the second player if the player who moves second from that position can force a win. $Z = 0$ and $Z = 1$ are winning for the second player since the first player is unable to make a valid move. $Z = 2$ is winning for the first player since the first player must remove 2 stones, leaving 0 stones for the second player, which is losing for the second player. Similarly, $Z = 3$ is winning for the first player since the first player removes either 2 or 3 stones, which results in 1 or 0 stones for the second player, which are both losing for the second player. $Z = 4$ is winning for the first player, since the first player can remove 3 stones, leaving 1 stones for the second player. From our prior logic, we know this position is winning for the new second player, who is the original first player. In general, if Ant can take away some stones that results in a position winning for the second player, they win, since from that position Ant would be considered the second player. If all of Ant's possible first moves can only result in a position winning for the first player, then Ben wins, since in any of those positions Ben would be considered the first player. We can notice that for $Z > 4$ primes and numbers one above primes will be winning for the first player by subtracting that prime and leaving 0 or 1 for the next term. Continue our inductive logic after $Z = 4$ and checking non-prime and non-one-above-prime numbers, we find that $1, 9, 10, 25$ are winning for the second player, and all other positive integers in our bound is winning for the first player. Thus the sum of positive integers Z from which Ben wins is $1 + 9 + 10 + 25 = 45$.

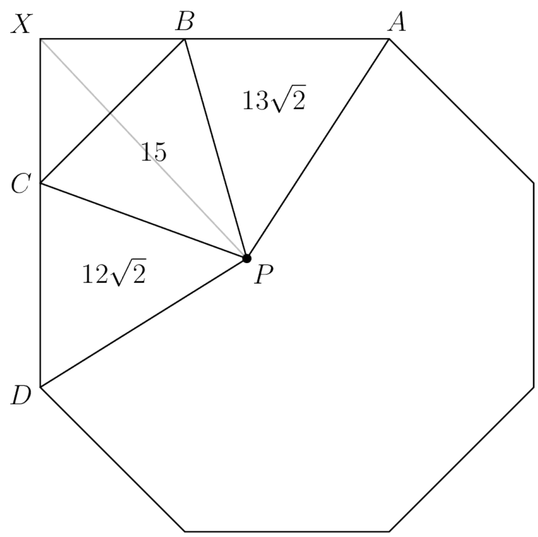
7. Let P be a point in a regular octagon as shown in the diagram below. The areas of three triangles are shown. Find the area of the octagon.



Solution. The answer is $\boxed{80 + 80\sqrt{2}}$.

Throughout this solution we use square brackets to denote the area of polygons.

Label points A, B, C and D as shown below, and let X be the intersection of lines AB and CD.



By right isosceles triangle ratios, we have that

$$AB : BX = DC : CX = \sqrt{2} : 1,$$

so we have

$$[XBP] = \frac{[ABP]}{\sqrt{2}} = 13, \qquad [XCP] = \frac{[DCP]}{\sqrt{2}} = 12.$$

Thus, $[XCPB] = 13 + 12 = 25$, so

$$[XCB] = [XCPB] - [CBP] = 10.$$

Since $\triangle XCB$ is a right isosceles triangle, we can compute its hypotenuse, BC, to have a length of $2\sqrt{10}$. This is the side length of the octagon, which we can now compute to have an area of $80 + 80\sqrt{2}$.

8. Find the number of ordered triples (a, b, c) of nonnegative integers with $a \leq b \leq c$ for which

$$5a + 4b + 6c = 1200.$$

Solution. The answer is $\boxed{861}$.

Define $x = c - b$, $y = b - a$ and $z = a$. Then, the condition becomes

$$6x + 10y + 15z = 1200,$$

where x, y and z can be any nonnegative integers. Taking mod 2, 3, and 5, it follows that x is a multiple of 5, y is a multiple of 3, and z is a multiple of 2. Letting $x = 5p$, $y = 3q$, and $z = 2r$, our equation becomes

$$30p + 30q + 30r = 1200$$

where p, q and r are nonnegative integers. Hence $p + q + r = 40$, which has

$$\binom{42}{2} = 861$$

solutions by stars and bars.

9. Define
$$f(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2}, \\ 2-2x & \frac{1}{2} \le x \le 1. \end{cases}$$
Michael picks a real number $0 \le x \le 1$. Michael applies f repeatedly to x until he reaches x again. Find the number of real numbers x for which Michael applies f exactly 12 times.

Solution. The answer is $\boxed{4020}$.

Let $f^i(x)$ denote the function f applied on x exactly i times.

Claim: For positive integers i, the equation
$$x = f^i(x)$$
has 2^i solutions.

Proof: Let $g(x) = 2x$ and $h(x) = 2 - 2x$. In fact, there is a one-to-one correspondence between a solution to $f^i(x) = x$ and a list of i binary choices of whether to replace $x \to g(x)$ or $x \to h(x)$ at each of the i steps. We prove this in three parts:

(a) Choosing a sequence of i choices leaves at most one possible value for x. After choosing these i steps, we will have an equation of the form
$$x = ax + b,$$
where $ax + b$ is equal to the function that results from our i choices. a and b are most certainly even integers – thus, there is at most one value of x after we make these i choices, which is the unique solution to the linear equation above.

(b) The solution x to the aforementioned $x = ax + b$ will always work. Consider the sequence S of $i+1$ values that results from our i choices, starting and ending with x. If some element in S is not in the range $[0, 1]$, any following values in S will also not be in the range $[0, 1]$, which contradicts that the starting and ending value of S are both equal. Thus, each element in S is in the range $[0, 1]$.
For any real $x \in [0, 1]$, there is exactly one element in the set $\{g(x), h(x)\}$ that is in the range $[0, 1]$; this element is equal to $f(x)$. This must mean that our sequence S is actually just
$$x, f(x), f^2(x), \ldots, f^i(x),$$
proving that the aforementioned x does indeed give us a solution to $x = f^i(x)$.

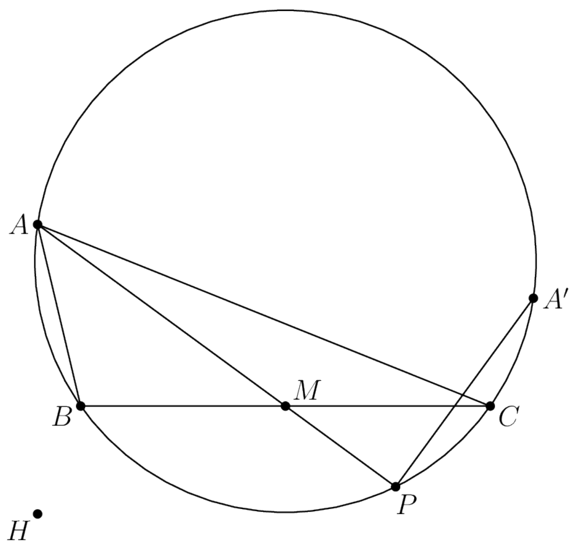
(c) No two different sequences of i choices will begin with the same x. For a certain x, we know that our i choices of either $g(x)$ or $h(x)$ at each iteration will produce the sequence $x, f(x), f^2(x), \ldots, f^i(x)$. This uniquely determines whether we choose $g(x)$ or $h(x)$ at each step, *except* for if two consecutive elements in this sequence are $\frac{1}{2} \to 1$; then, both $g(\frac{1}{2})$ and $h(\frac{1}{2}) = 1$. However, this situation will not arise because no element in our sequence can equal 1, since all following elements would equal 0.

Our proof is complete. $\square$

We now extract the final answer. The answer is not just 2^{12} because, reading carefully, the problem tells us that for $0 < i < 12$ we have $f^i(x) \neq x$. So, among the 2^{12} values of x satisfying $f^{12}(x) = x$, we must subtract away the 2^6 values of x satisfying $f^6(x) = x$ and the 2^4 values of x satisfying $f^4(x) = x$. But we subtract the values of x satisfying $f^2(x) = x$ twice instead of once, so we must add back 2^2. Our final answer is
$$2^{12} - 2^6 - 2^4 + 2^2 = 4020.$$

10. In $\triangle ABC$, let point H be the intersection of its altitudes and let M be the midpoint of side $\overline{BC}$. Given that $BC = 4$, $MA = 3$, and $\angle HMA = 60^\circ$, find the circumradius of $\triangle ABC$.

Solution. The answer is $\boxed{\frac{\sqrt{217}}{6}}$.



Let A' be the point on the circumcircle of $\triangle ABC$ which is diametrically opposite of A, and let P be the second intersection of line AM with the circumcircle of $\triangle ABC$. By power of a point,

$$MP = \frac{BM \cdot MC}{MA} = \frac{4}{3}.$$

It's well known that M is the midpoint of $\overline{HA'}$, so $\angle A'MP = \angle AMH = 60^\circ$. Since $\angle APA' = 90^\circ$, it follows that $\triangle MPA'$ is a 30-60-90 triangle, so $PA' = MP \cdot \sqrt{3} = \frac{4\sqrt{3}}{3}$.

Thus, we compute the circumradius of the triangle to be

$$\frac{AA'}{2} = \frac{\sqrt{AP^2 + PA'^2}}{2} = \frac{\sqrt{217}}{6}.$$

2.3 Team Test Solutions

1. Warren interrogates the 25 members of his cabinet, each of whom always lies or always tells the truth. He asks them all, "How many of you always lie?" He receives every integer answer from 1 to 25 exactly once. Find the actual number of liars in his cabinet.

   Solution. The answer is $\boxed{24}$.

   Since Warren receives different answers from each person, at least 24 people are liars. If everybody was a liar, the person that reported 25 liars would've actually been telling the truth, so there must be exactly 24 liars. (The person who said that there are 24 liars was the unique person telling the truth.)

2. Abraham thinks of distinct nonzero digits E, M, and C such that

   $$E + M = \overline{CC}.$$

   Help him evaluate the sum of the two digit numbers $\overline{EC}$ and $\overline{MC}$. (Note that $\overline{CC}$, $\overline{EC}$, and $\overline{MC}$ are read as two-digit numbers.)

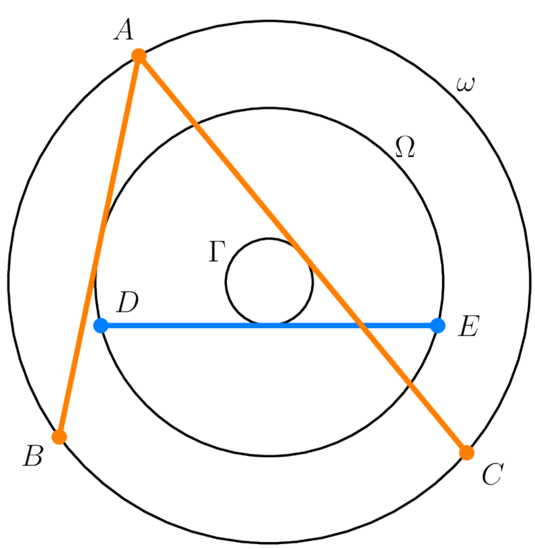
   Solution. The answer is $\boxed{112}$.

   Because $E + M$ is at most 17, it follows that $E + M = \overline{CC} = 11$. Then,

   $$\overline{EC} + \overline{MC} = 10E + 10M + 2 = 10(E + M) + 2 = 10 \cdot 11 + 2 = \boxed{112}.$$

3. Let ω, Ω, Γ be concentric circles such that Γ is inside Ω and Ω is inside ω. Points A, B, C on ω and D, E on Ω are chosen such that line AB is tangent to Ω, line AC is tangent to Γ, and line DE is tangent to Γ. If $AB = 21$ and $AC = 29$, find DE.

   Solution. The answer is $\boxed{20}$.



Let the radii of ω, Ω, Γ be $r_\omega, r_\Omega, r_\Gamma$ respectively.

By the Pythagorean theorem, we have that

$$r_\omega^2 - r_\Gamma^2 = \left(\frac{AC}{2}\right)^2 = \left(\frac{29}{2}\right)^2$$

and

$$r_\omega^2 - r_\Omega^2 = \left(\frac{AB}{2}\right)^2 = \left(\frac{21}{2}\right)^2,$$

so subtracting these two equations gives

$$\left(\frac{DE}{2}\right)^2 = r_\Omega^2 - r_\Gamma^2 = \left(\frac{29}{2}\right)^2 - \left(\frac{21}{2}\right)^2 = \left(\frac{20}{2}\right)^2.$$

Therefore, $DE = 20$.

4. Let a, b, and c be three prime numbers such that $a + b = c$. If the average of two of the three primes is four less than four times the fourth power of the last, find the second-largest of the three primes.

   Solution. The answer is $\boxed{59}$.

   We are given

   $$\frac{x+y}{2} = 4z^4 - 4,$$

   where x, y, z is some reordering of a, b, c. Since c is larger than a and b, it must be an odd prime, so exactly one of a or b is equal to 2. Therefore, exactly one of x, y, or z is equal to 2. Since $x + y = 2(4z^4 - 4)$ is even, we must have $z = 2$, so $x + y = 2(4(2)^4 - 4) = 120$. Moreover, we have $|x - y| = z = 2$, so the three prime numbers are $2, 59$, and 61.

5. At Stillwells Ice Cream, customers must choose one type of scoop and two different types of toppings. There are currently 630 different combinations a customer could order. If another topping is added to the menu, there would be 840 different combinations. If, *instead*, another type of scoop were added to the menu, compute the number of different combinations there would be.

   Solution. The answer is $\boxed{651}$.

   Let a be the number of scoops and b be the number of toppings. Then, we have the system of equations

   $$\begin{cases} a \cdot \binom{b}{2} & = 630 \\ a \cdot \binom{b+1}{2} & = 840. \end{cases}$$

   Dividing these two equations, we get that

   $$\frac{b+1}{b-1} = \frac{4}{3},$$

   so $b = 7$. The problem asks for

   $$(a+1) \cdot \binom{b}{2} = a \cdot \binom{b}{2} + \binom{b}{2} = 630 + 21 = 651.$$

6. Eleanor the ant takes a path from $(0,0)$ to $(20,24)$, traveling either one unit right or one unit up each second. She records every lattice point she passes through, including the starting and ending point. If the sum of all the x-coordinates she records is 271, compute the sum of all the y-coordinates. (A lattice point is a point with integer coordinates.)

   Solution. The answer is $\boxed{719}$.

   The sum of the coordinates of the first point Eleanor writes down is 0 (since it is the origin). Then, after she takes one step, either the x-coordinate or y-coordinate increases by 1; therefore, the sum of the coordinates of the second point Eleanor writes down is 1. Repeating this reasoning, the sum of the coordinates of the third point she writes down is 2, the sum of the coordinates of the fourth point she writes down is 3, and in general, the sum of the coordinates of the nth point Eleanor writes down is $n-1$, as n varies from 1 to 45.

   So, the sum of the coordinates of all points that Eleanor writes down is $0+1+2+\cdots+44 = \frac{44\cdot 45}{2} = 990$; if the x-coordinates sum to 271, the y-coordinates sum to $990-271 = 719$.

7. Teddy owns a square patch of desert. He builds a dam in a straight line across the square, splitting the square into two trapezoids. The perimeters of the trapezoids are 64 miles and 76 miles, and their areas differ by 135 square miles. Find, in miles, the length of the segment that divides them.

   Solution. The answer is $\boxed{25}$.

   Give the two trapezoids with perimeters 64 and 76 the names $\mathcal{A}$ and $\mathcal{B}$ respectively. The difference in the perimeters of $\mathcal{A}$ and $\mathcal{B}$ is 12; this difference is caused by the sum of the bases of $\mathcal{B}$ being 12 greater than the sum of the bases of $\mathcal{A}$. Thus, the average of the bases of $\mathcal{B}$ is 6 greater than the average of the bases of $\mathcal{B}$.

   The heights of $\mathcal{A}$ and $\mathcal{B}$ are the same; let h be this common height. Then, the area of $\mathcal{B}$ would be $6h$ greater than the area of $\mathcal{A}$. Hence, $6h = 135$, so $h = 22.5$. Note that h is the side length of the square, so the perimeter of the square is $4 \cdot 22.5 = 90$.

   If we sum the perimeters of $\mathcal{A}$ and $\mathcal{B}$, we would get the perimeter of the square plus double the length of the segment that divides $\mathcal{A}$ and $\mathcal{B}$; thus, letting the desired answer be x, we have

   $$64 + 76 = 90 + 2x,$$

   so $x = 25$.

8. Michelle is playing Spot-It with a magical deck of 10 cards. Each card has 10 distinct symbols on it, and every pair of cards shares exactly 1 symbol. Find the minimum number of distinct symbols on all of the cards in total.

   Solution. The answer is $\boxed{55}$.

   We attempt to fill in the 100 symbols on the 10 cards by filling in the 10 symbols on each card before moving on to the next card. Each symbol is either a new symbol or a reused symbol that has appeared before in the filling order. The number of different symbols on the 10 cards is equal to the number of new symbols. Since every pair of cards has exactly one symbol in common, the nth card has at most $n-1$ reused symbols, so there are at least

   $$0 + 1 + 2 + 3 + \cdots + 9 = 45$$

reused symbols and thus at most

$$100 - 45 = 55$$

new symbols. An example configuration is shown below, where new symbols are shown in black and reused symbols are shown in gray.

Card 1:	1	2	3	4	5	6	7	8	9	10
Card 2:	2	11	12	13	14	15	16	17	18	19
Card 3:	3	12	20	21	22	23	24	25	26	27
Card 4:	4	13	21	28	29	30	31	32	33	34
Card 5:	5	14	22	29	35	36	37	38	39	40
Card 6:	6	15	23	30	36	41	42	43	44	45
Card 7:	7	16	24	31	37	42	46	47	48	49
Card 8:	8	17	25	32	38	43	47	50	51	52
Card 9:	9	18	26	33	39	44	48	51	53	54
Card 10:	10	19	27	34	40	45	49	52	54	55

9. Define the function $f(n) = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$ for integers $n \geq 2$. Find

$$f(2) + f(4) + f(6) + \cdots.$$

Solution. The answer is $\boxed{\frac{3}{4}}$.

We can regroup terms as shown below to use the infinite geometric series formula, and then manipulate using partial fraction decomposition to get

$$\begin{aligned}
&\left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \dots\right) + \left(\frac{1}{4^2} + \frac{1}{4^4} + \frac{1}{4^6} + \dots\right) + \dots \\
= \;& \frac{\frac{1}{2^2}}{1 - \frac{1}{2^2}} + \frac{\frac{1}{3^2}}{1 - \frac{1}{3^2}} + \frac{\frac{1}{4^2}}{1 - \frac{1}{4^2}} + \dots \\
= \;& \frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \frac{1}{4^2 - 1} + \dots \\
= \;& \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots \\
= \;& \frac{1}{2} \cdot \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots\right) \\
= \;& \frac{1}{2} \cdot \left(\frac{1}{1} + \frac{1}{2}\right) \\
= \;& \frac{3}{4}.
\end{aligned}$$

10. There are 9 indistinguishable ants standing on a 3×3 square grid. Each ant is standing on exactly one square. Compute the number of different ways the ants can stand so that no column or row contains more than 3 ants.

Solution. The answer is $\boxed{55}$.

Clearly, no row/column may contain less than 3 ants – each row/column must contain exactly 3 ants. Thus, there can only be 0, 1, 2 or 3 ants in each cell. We now split into cases, based on the first row of the grid:

(a) If one cell of the first row contains 3 ants and the rest contain 0, there are 3 ways to place ants in the first row (by choosing which cell contains all 3 ants). WLOG assume we pick the top left cell to contain 3 ants:

3	0	0
0		
0		

We then must consider all possibilities for the 2×2 square at the bottom right corner of the grid. There are only 4 possibilities:

3	0	0
0	0	3
0	3	0

3	0	0
0	3	0
0	0	3

3	0	0
0	1	2
0	2	1

3	0	0
0	2	1
0	1	2

So, this case yields $4 \cdot 3 = 12$ possibilities.

(b) If one cell of the first row contains 2 ants and another cell of the first row contains 1 ant, there are $3 \cdot 2 = 6$ ways to place ants in the first row. WLOG assume we place 0, 1 and 2 ants in the first row from left to right. Then we have 6 possibilities for the bottom two rows:

0	1	2
1	1	1
2	1	0

0	1	2
2	1	0
1	1	1

0	1	2
0	2	1
3	0	0

0	1	2
3	0	0
0	2	1

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

So, this case yields $6 \cdot 6 = 36$ possibilities.

(c) Finally, if the first row is 1,1,1, then the remaining rows have 7 possibilities:

1	1	1
1	1	1
1	1	1

1	1	1
2	0	1
0	2	1

1	1	1
0	2	1
2	0	1

1	1	1
1	0	2
1	2	0

1	1	1
1	2	0
1	0	2

1	1	1
0	1	2
2	1	0

1	1	1
2	1	0
0	1	2

This case yields 7 possibilities.

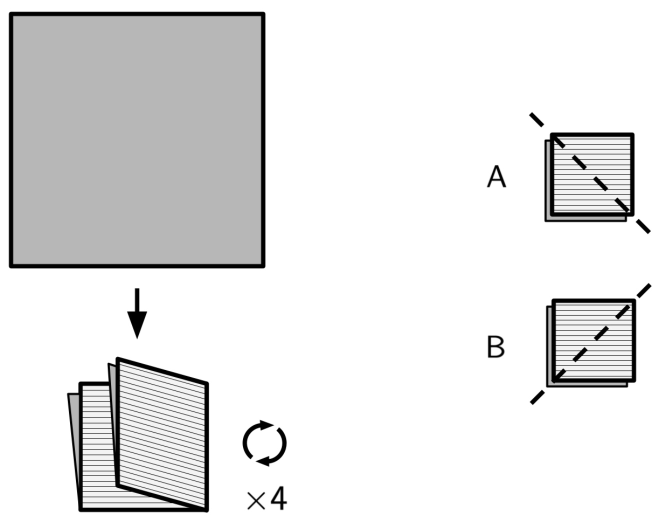
In total, there are $12 + 36 + 7 = 55$ possible arrangements of ants on our grid.

11. Let $s(N)$ denote the sum of the digits of N. Compute the sum of all two-digit positive integers N for which $s(N^2) = s(N)^2$.

Solution. The answer is $\boxed{170}$.

Let $N = \overline{ab} = 10a + b$, so $N^2 = 100a^2 + 20ab + b^2$. We have $s(N^2) < s(N)^2$ if the evaluation of N^2 involves carrying (since each carry decreases the sum of the digits by 9) and $s(N^2) = s(N)^2$ otherwise. Hence, a^2, $2ab$, and b^2 are all at most 9, and the set of all such N is $\{10, 11, 12, 13, 20, 21, 22, 30, 31\}$. The sum of these numbers is 170.

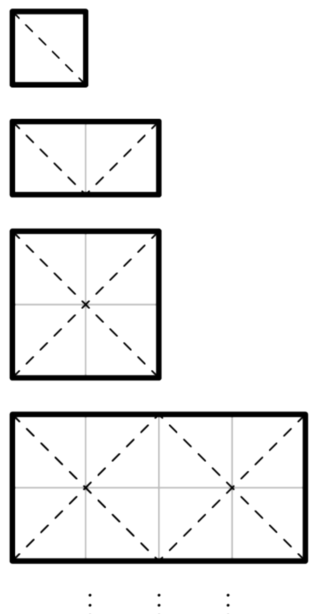
12. Martha has two square sheets of paper, A and B. With each sheet, she repeats the following process four times: fold bottom side to top side, fold right side to left side. With sheet A, she then makes a cut from the top left corner to the bottom right. With sheet B, she makes a cut from the bottom left corner to the top right. Find the total number of pieces of paper yielded from sheets A and sheets B.



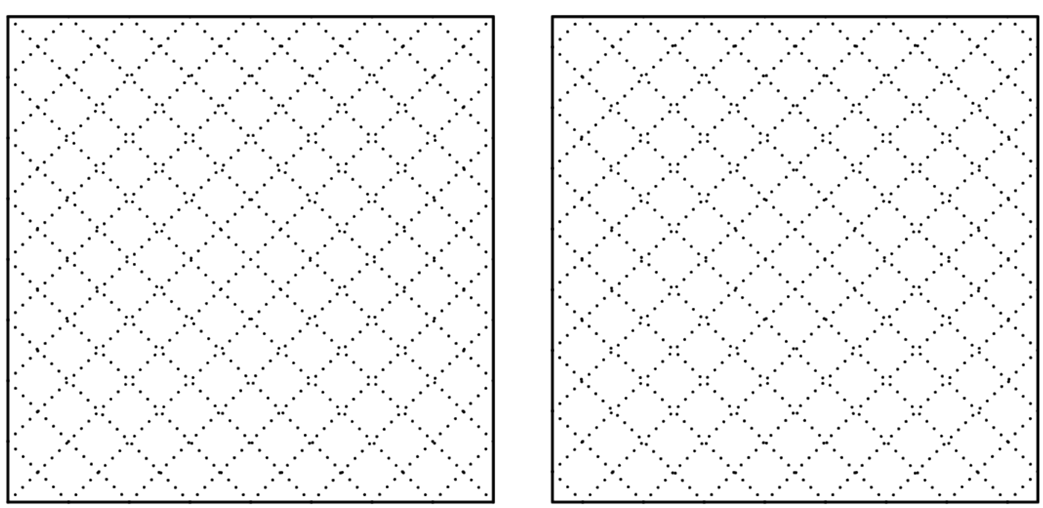
Solution. The answer is $\boxed{289}$.

Instead of making incisions, we will imagine drawing a dotted line on both sides of sheets A and B where there would be an incision.

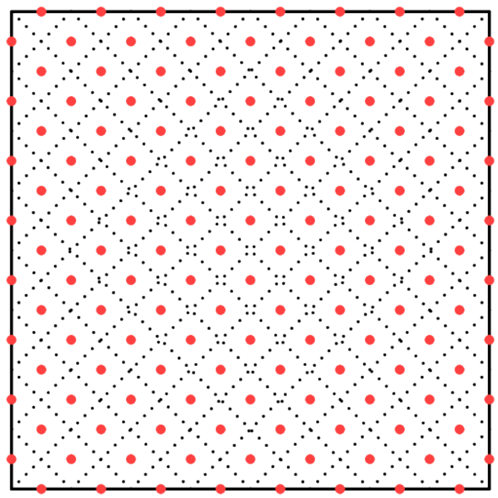
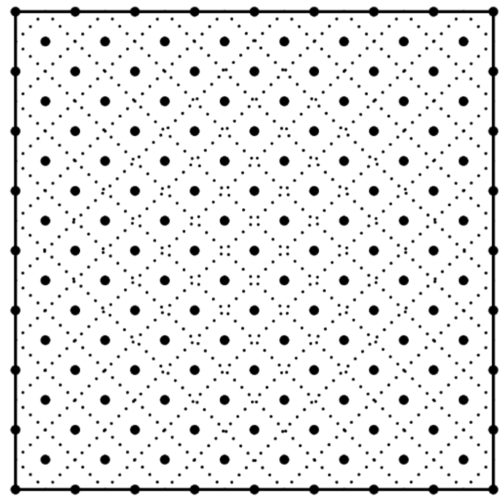
Once we fold up sheet A, we will unfold it, adding in the dotted lines as we unfold. Each "unfold" is a reflection of the existing rectangle across one of its sides, so the incisions are also reflected across the side.



Once we unfold squares A and B completely, the dotted lines we draw will form the patterns shown below:



Now we count the total number of pieces formed when we cut along the dotted lines. To do so, consider drawing a red dot in the "center" of each piece formed from sheet A, and a black dot in the "center" of each piece formed from sheet B, as shown in the diagram below.

It suffices to count the total number of red dots and black dots. But overlaying the two diagrams gives a 17 by 17 grid of dots, so there are $17^2 = 289$ dots in total.

Remark: By mere coincidence, the number of pieces obtained from sheet A is a perfect square, $12^2 = 144$. This is because of the special fact that $2 \cdot 12^2 + 1 = 17^2$.

13. Let x and y be positive integers such that

$$\gcd(x^y, y^x) = 2^{28}.$$

Find the sum of all possible values of $\min(x, y)$.

Solution. The answer is $\boxed{114}$.

Let $v_p(n)$ denote the largest nonnegative integer k for which p^k divides n. Then the given equation implies

$$\min(v_2(x^y), v_2(y^x)) = 28.$$

Without loss of generality assume that $v_2(x^y) = 28$ and $v_2(y^x) \geq 28$.

Since $v_2(x^y) = y \cdot v_2(x)$, it follows that $y \mid 28$. It's also clear that y must be even so that 2^{28} divides y^x. We split into cases depending on the value of y:

- If $y = 2$, we have $v_2(x) = 14$, so x is at least 2^{14}, so $\min(x, y) = 2$. The pair $\left(2^{14}, 2\right)$ is a working example of this case.
- If $y = 4$, we have $v_2(x) = 7$, so x is at least 2^7, so $\min(x, y) = 4$. The pair $\left(2^7, 4\right)$ is a working example of this case.
- If $y = 14$, we have $v_2(x) = 2$. Furthermore, x must be at least 14 so that $v_2(y^x) \geq 28$. Thus, $\min(x, y) = 14$. The pair $(20, 14)$ is a working example of this case.
- If $y = 28$, we have $v_2(x) = 1$. In order to satisfy $v_2(y^x) \geq 28$, we must have $x \geq 14$. We can't have $x = 14$, since then $\gcd(x^y, y^x)$ is a multiple of 7. However, pairs of the form $(2k, 28)$ with $k > 7$ and $\gcd(k, 14) = 1$ all work, so $\min(x, y)$ can equal 18, 22, 26 or 28.

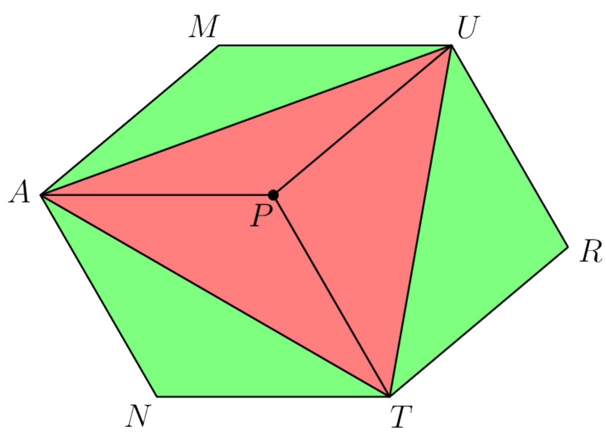
Summing all values from all cases gives us a total of

$$2 + 4 + 14 + 18 + 22 + 26 + 28 = 114.$$

14. Convex hexagon $TRUMAN$ has opposite sides parallel. If each side has length 3 and the area of this hexagon is 5, compute

$$TU \cdot RM \cdot UA \cdot MN \cdot AT \cdot NR.$$

Solution. The answer is $\boxed{900}$.



Let P be the point inside $TRUMAN$ for which $ANTP$ is a rhombus. It is easy to see that $TRUP$ and $UMAP$ are rhombii as well.

Then, $PA = PU = PT = 3$, hence P is the circumcenter of $\triangle AUT$ and its circumradius is 3. Furthermore, the area of $\triangle AUT$ is 2.5, half of the whole hexagon, since for each of the three green triangles there is a corresponding congruent red triangle.

Thus, $AU \cdot UT \cdot AT = 4 \cdot PA \cdot [AUT] = 30$, so

$$TU \cdot RM \cdot UA \cdot MN \cdot AT \cdot NR = 30^2 = 900.$$

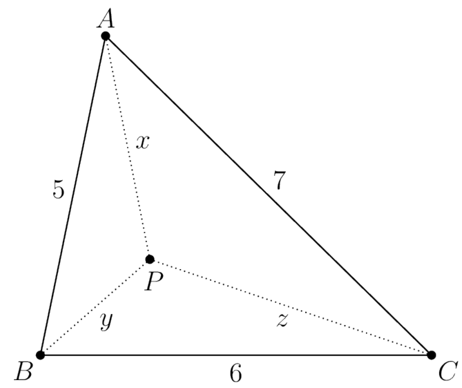
15. Let x, y, and z be positive real numbers satisfying the system

$$\begin{cases} x^2 + xy + y^2 = 25 \\ y^2 + yz + z^2 = 36 \\ z^2 + zx + x^2 = 49. \end{cases}$$

Compute $x^2 + y^2 + z^2$.

Solution. The answer is $\boxed{55 - 12\sqrt{2}}$.

Method I.



Define $\triangle ABC$ with side lengths $AB = 5$, $BC = 6$, and $CA = 7$, and let P be a point inside the triangle such that $PA = x$, $PB = y$, $PC = z$, and $\angle APB = \angle BPC = \angle CPA = 120°$. These conditions are equivalent to the given equations according to the Law of Cosines. On one hand, the area of $\triangle ABC$ is

$$\begin{aligned} & [PAB] + [PBC] + [PCA] \\ = \ & \frac{1}{2}\sin 120°(xy + yz + zx) \\ = \ & \frac{\sqrt{3}}{4}(xy + yz + zx). \end{aligned}$$

On the other hand, the area of $\triangle ABC$ is also equal to

$$\sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$$

by Heron's formula. Equating these two gives us

$$\begin{aligned} \frac{\sqrt{3}}{4}(xy + yz + zx) &= 6\sqrt{6} \\ xy + yz + xz &= 24\sqrt{2}. \end{aligned}$$

To finish, adding up the three given equations gives

$$\begin{aligned} 2(x^2 + y^2 + z^2) + (xy + yz + zx) &= 110 \\ x^2 + y^2 + z^2 &= \frac{110 - 24\sqrt{2}}{2} = 55 - 12\sqrt{2}. \end{aligned}$$

Method II. Subtracting the first equation from the second equation results in

$$z^2 - x^2 + y(z - x) = 11,$$

so

$$(x + y + z)(z - x) = 11.$$

Let $s = x + y + z$, so

$$z - x = \frac{11}{s}.$$

Hence,

$$z = x + \frac{11}{s}.$$

Similar calculation with the second and third equations shows

$$y = x - \frac{13}{s}.$$

Thus,

$$x + \left(x - \frac{13}{s}\right) + \left(x + \frac{11}{s}\right) = s,$$

so

$$3x = s + \frac{2}{s} = \frac{s^2+2}{s}.$$

Hence,

$$x = \frac{s^2+2}{3s},$$

so

$$y = \frac{s^2-37}{3s},$$

and

$$z = \frac{s^2+35}{3s}.$$

Substituting these values into the first equation results in

$$(s^2+2)^2 + (s^2+2)(s^2-37) + (s^2-37)^2 = 225s^2,$$

which simplifies to

$$s^4 - 110s^2 + 433 = 0.$$

Hence,

$$s^2 = \frac{110 \pm \sqrt{110^2 - 4 \times 433}}{2} = 55 \pm 36\sqrt{2}.$$

Since $y \geq 0$, we have $s^2 \geq 37$, so $s^2 = 55 + 36\sqrt{2}$. Therefore,

$$\begin{aligned}
x^2 + y^2 + z^2 &= \frac{1}{9s^2}\big((s^2+2)^2 + (s^2-37)^2 + (s^2+35)^2\big) \\
&= \frac{1}{3s^2}(s^4 + 866) \\
&= \frac{1}{3s^2}\big(s^4 - 2(s^4 - 110s^2)\big) \\
&= \frac{220 - s^2}{3} \\
&= \frac{220 - (55 + 36\sqrt{2})}{3} \\
&= 55 - 12\sqrt{2}.
\end{aligned}$$

2.4 Guts Test Solutions

2.4.1 Round 1

1. [6] When Shiqiao sells a bale of kale, he makes x dollars, where

$$x = \frac{1+2+3+4+5+6+7+8}{3+4+5+6}.$$

Find x.

Solution. The answer is $\boxed{2}$.

2. [6] The fraction of Shiqiao's kale that has gone rotten is equal to

$$\sqrt{\frac{100^2}{99^2} - \frac{100}{99}}.$$

Find the fraction of Shiqiao's kale that has gone rotten.

Solution. The answer is $\boxed{\frac{10}{99}}$.

3. [6] Shiqiao is growing kale. Each day the number of kale plants doubles, but 4 of his kale plants die afterwards. He starts with 6 kale plants. Find the number of kale plants Shiqiao has after five days.

Solution. The answer is $\boxed{68}$.

2.4.2 Round 2

4. [7] Today the high is 68 degrees Fahrenheit. If C is the temperature in Celsius, the temperature in Fahrenheit is equal to $1.8C + 32$. Find the high today in Celsius.

Solution. The answer is $\boxed{20}$.

5. [7] The internal angles in Evan's triangle are all at most 68 degrees. Find the minimum number of degrees an angle of Evan's triangle could measure.

Solution. The answer is $\boxed{44}$.

6. [7] Evan's room is at 68 degrees Fahrenheit. His thermostat has two buttons, one to increase the temperature by one degree, and one to decrease the temperature by one degree. Find the number of combinations of 10 button presses Evan can make so that the temperature of his room never drops below 67 degrees or rises above 69 degrees.

   Solution. The answer is $\boxed{32}$.

2.4.3 Round 3

7. [9] In a digital version of the SAT, there are four spaces provided for either a digit (0-9), a fraction sign (/), or a decimal point (.). The answer must be in simplest form and at most one space can be a non-digit character. Determine the largest fraction which, when expressed in its simplest form, fits within this space, but whose exact decimal representation does not.

   Solution. The answer is $\boxed{\frac{98}{3}}$.

8. [9] Rounding Rox picks a real number x. When she rounds x to the nearest hundred, its value increases by 2.71828. If she had instead rounded x to the nearest hundredth, its value would have decreased by y. Find y.

   Solution. The answer is $\boxed{0.00172}$.

9. [9] Let a and b be real numbers satisfying the system of equations

$$\begin{cases} a + \lfloor b \rfloor = 2.14 \\ \lfloor a \rfloor + b = 2.72. \end{cases}$$

   Determine $a + b$.

   Solution. The answer is $\boxed{2.86}$.

2.4.4 Round 4

10. [11] Carol and Lily are playing a game with two unfair coins, both of which have a 1/4 chance of landing on heads. They flip both coins. If they both land on heads, Lily loses the game, and if they both land on tails, Carol loses the game. If they land on different sides, Carol and Lily flip the coins again. They repeat this until someone loses the game. Find the probability that Lily loses the game.

Solution. The answer is $\boxed{\frac{1}{10}}$.

11. [11] Dongchen is carving a circular coin design. He carves a regular pentagon of side length 1 such that all five vertices of the pentagon are on the rim of the coin. He then carves a circle inside the pentagon so that the circle is tangent to all five sides of the pentagon. Find the area of the region between the smaller circle and the rim of the coin.

    Solution. The answer is $\boxed{\frac{\pi}{4}}$.

12. [11] Anthony flips a fair coin six times. Find the probability that at some point he flips 2 heads in a row.

    Solution. The answer is $\boxed{\frac{43}{64}}$.

2.4.5 Round 5

13. [13] Mandy is baking cookies. Her recipe calls for N grams of flour, where N is the number of perfect square divisors of $20! + 24!$. Find N.

    Solution. The answer is $\boxed{800}$.

14. [13] Consider a circular table with center R. Beef-loving Bryan places a steak at point I on the circumference of the table. Then he places a bowl of rice at points C and E on the circumference of the table such that $CE \parallel IR$ and $\angle ICE = 25^\circ$. Find $\angle CIE$.

    Solution. The answer is $\boxed{40}$.

15. [13] Enya writes the 4-letter words LEEK, BEAN, SOUP, PEAS, HAMS, and TACO on the board. She then thinks of one of these words and gives Daria, Ava, Harini, and Tiffany a slip of paper containing exactly one letter from that word such that if they ordered the letters on their slips correctly, they would form the word.

    Each person announces at the same time whether they know the word or not. Ava, Harini, and Tiffany all say they do not know the word, while Daria says she knows the word. After hearing this, Ava, Harini, and Tiffany all know the word. Assuming all four girls are perfect logicians and they all thought of the same correct word, determine Daria's letter.

    Solution. The answer is $\boxed{\text{U}}$.

2.4.6 Round 6

16. [15] Michael receives a cheese cube and a chocolate octahedron for his 5th birthday. On every day after, he slices off each corner of his cheese and chocolate with a knife. Each slice cuts off exactly one corner. He then eats each corner sliced off. Find the difference between the total number of cheese and chocolate pieces he has eaten by the end of his 6th birthday. (Michael's 5th and 6th birthdays do not occur on leap years.)

    Solution. The answer is $\boxed{2}$.

17. [15] Let D be the average of all positive integers n satisfying

    $$\operatorname{lcm}(\gcd(n, 2000), \gcd(n, 24)) = \gcd(\operatorname{lcm}(n, 2000), \operatorname{lcm}(n, 24)).$$

    Find $3D$.

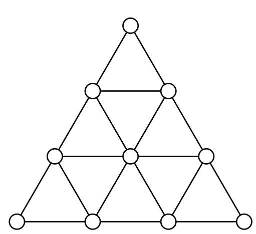
    Solution. The answer is $\boxed{2808}$.

18. [15] The base $\triangle ABC$ of the triangular pyramid $PABC$ is an equilateral triangle with a side length of 3. Given that $PA = 3$, $PB = 4$, and $PC = 5$, find the circumradius of $PABC$.

    Solution. The answer is $\boxed{\frac{9\sqrt{11}}{11}}$.

2.4.7 Round 7

19. [18] 2049300 points are arranged in an equilateral triangle point grid, a smaller version of which is shown below, such that the sides contain 2024 points each. Peter starts at the topmost point of the grid. At 9:00 am each day, he moves to an adjacent point in the row below him. Derrick wants to prevent Peter from reaching the bottom row, so at 12:00 pm each day, he selects a point on the bottom row and places a rock at that point. Peter stops moving as soon as he is guaranteed to end up at a point with a rock on it. At least how many moves will Peter complete, no matter how Derrick places the rocks?



    Solution. The answer is $\boxed{1012}$.

20. [18] There are N stones in a pile, where N is a positive integer. Ava and Anika take turns playing a game, with Ava moving first. If there are n stones in the pile, a move consists of removing x stones, where $1 < \gcd(x, n) \leq x < n$. Whoever first has no possible moves on their turn wins. Both Ava and Anika play optimally. Find the 2024th smallest value of N for which Ava wins.

    Solution. The answer is $\boxed{2038}$.

21. [18] Alan is bored and alone, so he plays a fun game with himself. He writes down all quadratic polynomials with leading coefficient 1 whose coefficients are integers between -10 and 10, inclusive, on a blackboard. He then erases all polynomials which have a non-integer root. Alan defines the size of a polynomial $P(x)$ to be $P(1)$ and spends an hour adding up the sizes of all the polynomials remaining on the blackboard. Assuming Alan does computation perfectly, find the sum Alan obtains.

    Solution. The answer is $\boxed{70}$.

2.4.8 Round 8

22. [21] A prime number is a positive integer with exactly two distinct divisors. You must submit a prime number for this problem. If you do not submit a prime number, you gain 0 points, and your submission will not be considered valid. The median of all valid submitted numbers is M (duplicates are counted). Estimate $2M$. If your team's absolute difference between $2M$ and your submission is the ith smallest absolute difference among all teams, you gain $\max(23 - 2i, 0)$ points. All teams who did not submit any number gain 0 points. (In the case of a tie, all teams that tied gain the same amount of points.)

    Solution. The answer is $\boxed{152}$.

    We received the 14 valid responses:

    $$11, 23, 47, 53, 53, 71, 73, 79, 97, 191, 197, 307, 331, 467.$$

    Many groups accidentally submitted a composite number, including responses of 52, 2024 and 1434. Unfortunately, all of these responses had to be invalidated.

23. [21] Ribbotson the Frog is at the point $(0, 0)$ and wants to reach the point $(18, 18)$ in 36 steps. Each step, he either moves one unit in the $+x$ direction or one unit in the $+y$ direction. However, Ribbotson hates turning, so he must make at least two steps in any direction before switching directions. If m is the number of different paths Ribbotson the Frog can make, estimate m. If N is your team's submitted number, your team earns points equal to the closest integer to $21(1 - |\log_{10} \frac{N}{m}|^2)$.

    Solution. The answer is $\boxed{2192902}$.

24. [21] Let $M = \pi^{\pi^{\pi^{\pi}}}$. Estimate k, where $M = 10^{10^k}$. If N is your team's submitted number, your team earns points equal to the closest integer to $21 \cdot 1.01^{(-|N-k|^3)}$.

Solution. The answer is $\boxed{\text{approximately } 17.823645339416962939967754375012845583956903321517077327864654505}$.