Exeter Math Club Competition
January 28, 2023

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Organizing Acknowledgments

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- **Contest Editors** Chelsea Drescher, Jeffrey Ibbotson, Kevin Cong
- **Publicity** Minseo Kim
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Jane Street

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Contest Day Acknowledgments

- **Tournament Directors** Alan Bu, Daria Ivanova, Minseo Kim, Jack Kugler, Anish Mudide, Max Xu
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- **Exeter Panelists** Chelsea Drescher, Jeffrey Ibbotson, Daria Ivanova, Anish Mudide, Harini Venkatesh, Ava Zhao

Chapter 1

EMC^2 2023 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Evaluate the following expression, giving your answer as a decimal: $\frac{20\times 2\times 3}{20+2+3}$.

2. Given real numbers x and y, we have that $2x + 3y = 20$ and $3x + 4y = 12$. Find the value of $x + y$.

3. Alan, Daria, and Max want to sit in a row of three airplane seats. If Alan cannot sit in the middle, in how many ways can they sit down?

4. Jack thinks of two distinct positive integers a and b. He notices that neither a nor b is a perfect square, but ab is a perfect square. What is the smallest possible value of $a + b$?

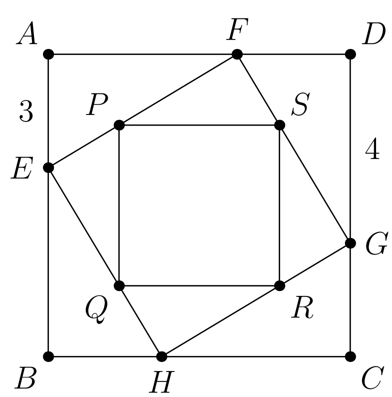
5. What is the smallest integer greater than 2023 whose digits sum to 4?

6. Triangle ABC has $AB = AC$ and $\angle B = 60^\circ$. The altitude drawn from C intersects AB at X, where $BX = 4$. What is the area of ABC?

7. Archyuta writes a program to create words with at least one letter. The probability of having n letters in the word for each positive integer n is $\frac{1}{2^n}$. Each letter of the word is chosen randomly and independently from the uppercase English alphabet. The probability of Archyuta's program outputting "EMCC" can be written as $\frac{1}{k}$ for some positive integer k. What is the greatest nonnegative integer a such that 2^a divides k?

8. What is the greatest whole number less than 1000 that can be expressed as the sum of seven consecutive whole numbers, as the sum of five consecutive whole numbers, and as the sum of three consecutive whole numbers?

9. Given a square $ABCD$ with side length 7, square $EFGH$ is inscribed in $ABCD$ such that E is on side AB and G is on side CD such that $EA = 3$ and $GD = 4$. If square $PQRS$ inscribed in $EFGH$ such that $PQ \parallel AB$, find the side length of $PQRS$.



10. Michael wants to do some exercise by going up and down a moving escalator. He first runs up the escalator, taking 30 seconds to reach the top. Tired, he then walks at one-third of his running speed back down the escalator, taking 30 seconds to reach the bottom. Assuming his running speed and the escalator's speed are constant, what is the ratio of his running speed to the escalator's speed?

11. Bob the architect has 4 bricks shaped like rectangular prisms each of dimension 1 foot by 1 foot by 2 feet which he stores inside a 2 feet by 2 feet by 2 feet hollow box. In how many ways can he fit his bricks into the box? (Rotations and reflections of a configuration are considered distinct.)

12. P is a point lying inside rectangle $ABCD$. If $\angle PAB = 40^\circ$, $\angle PBC = 50^\circ$ and $\angle PCD = 60^\circ$, find $\angle PDA$ in degrees.

13. Let N be a positive integer. If 4 of N's divisors are prime and 346 of N's divisors are composite, how many of N's divisors are perfect squares?

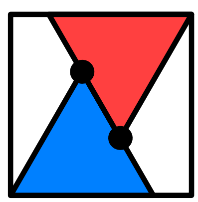
14. Two positive integers have a product of 2^{23}. Let S be the sum of all distinct possible values of their absolute difference. Find the remainder when S is divided by 1000.

15. A rectangle has area 216. The internal angle bisectors of each of its four vertices are drawn, bounding a square region with area 18. Find the perimeter of the rectangle.

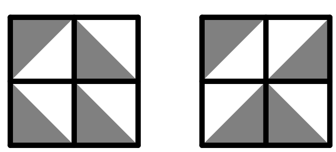
16. Let $\triangle ABC$ be a right triangle, with a right angle at B. The perpendicular bisector of hypotenuse $\overline{AC}$ splits the triangle into a smaller triangle and a quadrilateral. If the triangle has an area of 5 and the quadrilateral has an area of 13, find the length of $\overline{AC}$.

17. Let N be the sum of the 2023 smallest positive perfect squares minus the sum of the 2023 smallest positive odd numbers. What is the largest prime factor of N?

18. Anna designs a logo, shown below, consisting of a large square with side length 12 and two congruent equilateral triangles placed inside the square, one in each corner and with one overlapping side. What is the distance between the marked vertices?



19. How many ways are there to place an isosceles right triangle with legs of length 1 in each unit square of a two-by-two grid, such that no two isosceles triangles share an edge? One valid construction is shown on the left, followed by an invalid construction on the right.



20. Mr. Ibbotson and Dr. Drescher are playing a game where they write numbers on the blackboard. On the first turn, Mr. Ibbotson begins by writing 1, followed by Dr. Drescher writing another 1 on the second turn. Each turn afterwards, they take the two newest numbers on the board and concatenate them, writing the resulting number of the board. For instance, the first few numbers on the board are $1, 1, 11, 111, 11111, \ldots$ How many turns does it take for them to write a number which is divisible by 63?

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. Minseo writes all of the divisors of 1,000,000 on the whiteboard. She then erases all of the numbers which have the digit 0 in their decimal representation. How many numbers are left?

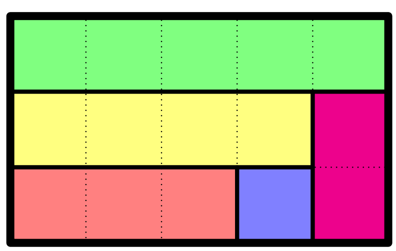
2. $n < 100$ is an odd integer and can be expressed as $3k - 2$ and $5m + 1$ for positive integers k and m. Find the sum of all possible values of n.

3. Mr. Pascal is a math teacher who has the license plate SQUARE. However, at night, a naughty student scrambles Mr. Pascal's license plate to UQRSEA. The math teacher luckily has an unscrambler that is able to move license plate letters. The unscrambler swaps the positions of any two adjacent letters. What is the minimum number of times Mr. Pascal must use the unscrambler to restore his original license plate?

4. Find the number of distinct real numbers x which satisfy $x^2 + 4\lfloor x \rfloor + 4 = 0$.

5. All four faces of tetrahedron $ABCD$ are acute. The distances from point D to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are all 7, and the distance from point D to face ABC is 5. Given that the volume of tetrahedron $ABCD$ is 60, find the surface area of tetrahedron $ABCD$.

6. Forrest has a rectangular piece of paper with a width of 5 inches and a height of 3 inches. He wants to cut the paper into five rectangular pieces, each of which has a width of 1 inch and a distinct integer height between 1 and 5 inches, inclusive. How many ways can he do so? (One possible way is shown below.)



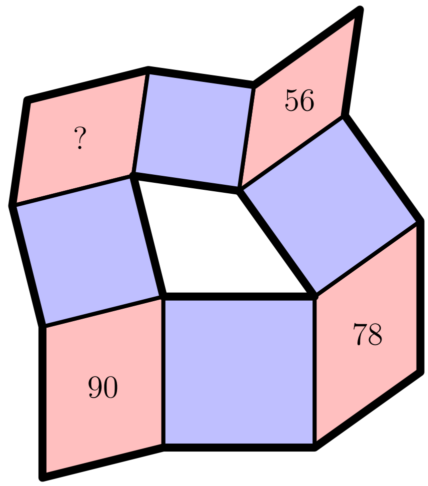
7. In convex quadrilateral $ABCD$, $AB = CD = 5$, $BC = 4$ and $AD = 8$. If diagonal $\overline{AC}$ bisects $\angle DAB$, find the area of quadrilateral $ABCD$.

8. Let x and y be real numbers such that

$$x + y = x^3 + y^3 + \frac{3}{4} = \frac{1}{8xy}.$$

   Find the value of $x + y$.

9. Four blue squares and four red parallelograms are joined edge-to-edge alternately to form a ring of quadrilateral as shown. The areas of three of the red parallelograms are shown. Find the area of the fourth red parallelogram.



10. Define $f(x,n) = \sum_{d|n} \frac{x^n-1}{x^d-1}$. For how many integers n between 1 and 2023 inclusive is $f(3,n)$ an odd integer?

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 60 minutes.

1. We define $a \oplus b = \frac{ab}{a+b}$. Compute $(3 \oplus 5) \oplus (5 \oplus 4)$.

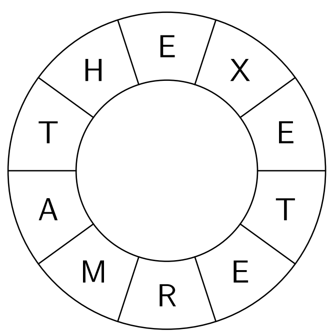
2. Let $ABCD$ be a quadrilateral with $\angle A = 45^\circ$ and $\angle B = 45^\circ$. If $BC = 5\sqrt{2}$, $AD = 6\sqrt{2}$, and $AB = 18$, find the length of side CD.

3. A positive real number x satisfies the equation $x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 10$. Find the sum of all possible values of $x + 1 + \frac{1}{x}$.

4. David writes 6 positive integers on the board (not necessarily distinct) from least to greatest. The mean of the first three numbers is 3, the median of the first four numbers is 4, the unique mode of the first five numbers is 5, and the range of all 6 numbers is 6. Find the maximum possible value of the product of David's 6 integers.

5. Let $ABCD$ be a convex quadrilateral such that $\angle A = \angle B = 120^\circ$ and $\angle C = \angle D = 60^\circ$. There exists a circle with center I which is tangent to all four sides of $ABCD$. If $IA \cdot IB \cdot IC \cdot ID = 240$, find the area of quadrilateral $ABCD$.

6. The letters EXETERMATH are placed into cells on an annulus as shown below. How many ways are there to color each cell of the annulus with red, blue, green, or yellow such that each letter is always colored the same color and adjacent cells are always colored differently?



7. Let $ABCD$ be a square, and let ω be a quarter circle centered at A passing through points B and D. Points E and F lie on sides BC and CD respectively. Line EF intersects ω at two points, G and H. Given that $EG = 2$, $GH = 16$ and $HF = 9$, find the length of side AB.

8. Let x be equal to
$$\frac{2022! + 2021!}{2020! + 2019! + 2018!}.$$
Find the closest integer to $2\sqrt{x}$.

9. For how many ordered pairs of positive integers (m, n) is the absolute difference between $\operatorname{lcm}(m, n)$ and $\gcd(m, n)$ equal to 2023?

10. There are 2023 distinguishable frogs sitting on a number line with one frog sitting on i for all integers i between -1011 and 1011, inclusive. Each minute, every frog randomly jumps either one unit left or one unit right with equal probability. After 1011 minutes, over all possible arrangements of the frogs, what is the average number of frogs sitting on the number 0?

11. Albert has a calculator initially displaying 0 with two buttons: the first button increases the number on the display by one, and the second button returns the square root of the number on the display. Each second, he presses one of the two buttons at random with equal probability. What is the probability that Albert's calculator will display the number 6 at some point?

12. For a positive integer $k \geq 2$, let $f(k)$ be the number of positive integers n such that n divides $(n-1)!+k$. Find $f(2) + f(3) + f(4) + f(5) + ... + f(100)$.

13. Mr. Atf has nine towers shaped like rectangular prisms. Each tower has a 1 by 1 base. The first tower has height 1, the next has height 2, up until the ninth tower, which has height 9. Mr. Atf randomly arranges these 9 towers on his table so that their square bases form a 3 by 3 square on the surface of his table. Over all possible solids Mr. Atf could make, what is the average surface area of the solid?

14. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular. Let E be the intersection of AC and BD, and let the feet of the altitudes from E to the sides AB, BC, CD, DA be W, X, Y, Z respectively. Given that $EW = 2EY$ and $EW \cdot EX \cdot EY \cdot EZ = 36$, find the minimum possible value of $\frac{1}{[EAB]} + \frac{1}{[EBC]} + \frac{1}{[ECD]} + \frac{1}{[EDA]}$. The notation $[XYZ]$ denotes the area of triangle XYZ.

15. Given that $x^2 - xy + y^2 = (x+y)^3$, $y^2 - yz + z^2 = (y+z)^3$, and $z^2 - zx + x^2 = (z+x)^3$ for complex numbers x, y, z, find the product of all distinct possible nonzero values of $x + y + z$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [6] What is the sum of the digits in the binary representation of 2023?

2. [6] Jack is buying fruits at the EMCCmart. Three apples and two bananas cost \$11.00. Five apples and four bananas cost \$19.00. In cents, how much more does an apple cost than a banana?

3. [6] Define $a \sim b$ as $a! - ab$. What is $(4 \sim 5) \sim (5 \sim (3 \sim 1))$?

1.4.2 Round 2

4. [7] Alan has 24 socks in his drawer. Of these socks, 4 are red, 8 are blue, and 12 are green. Alan takes out socks one at a time from his drawer at random. What is the minimum number of socks he must pull out to guarantee that the number of green socks is at least twice the number of red socks?

5. [7] What is the remainder when the square of the 24th smallest prime number is divided by 24?

6. [7] A cube and a sphere have the same volume. If k is the ratio of the length of the longest diagonal of the cube to the diameter of the sphere, find k^6.

1.4.3 Round 3

7. [9] Equilateral triangle ABC has side length $3\sqrt{3}$. Point D is drawn such that BD is tangent to the circumcircle of triangle ABC and $BD = 4$. Find the distance from the circumcenter of triangle ABC to D.

8. [9] If $\frac{2023!}{2^k}$ is an odd integer for an integer k, what is the value of k?

9. [9] Let S be a set of 6 distinct positive integers. If the sum of the three smallest elements of S is 8, and the sum of the three largest elements of S is 19, find the product of the elements in S.

1.4.4 Round 4

10. [11] For some integers b, the number $1 + 2b + 3b^2 + 4b^3 + 5b^4$ is divisible by $b + 1$. Find the largest possible value of b.

11. [11] Let a, b, c be the roots of cubic equation $x^3 + 7x^2 + 8x + 1$. Find $a^2 + b^2 + c^2 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

12. [11] Let $\mathcal{C}$ be the set of real numbers c such that there are exactly two integers n satisfying $2c < n < 3c$. Find the expected value of a number chosen uniformly at random from $\mathcal{C}$.

1.4.5 Round 5

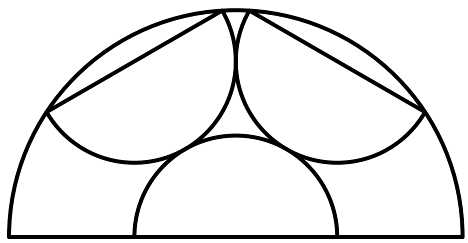
13. [13] For a square pyramid whose base has side length 9, a square is formed by connecting the centroids of the four triangular faces. What is the area of the square formed by the centroids?

14. [13] Farley picks a real number p uniformly at random in the range $\left(\frac{1}{3}, \frac{2}{3}\right)$. She then creates a special coin that lands on heads with probability p and tails with probability $1 - p$. She flips this coin, and it lands on heads. What is the probability that $p > \frac{1}{2}$?

15. [13] Let $ABCD$ be a quadrilateral with $\angle A = \angle C = 90^\circ$. Extend AB and CD to meet at point P. Given that $PB = 3, BA = 21$, and $PC = 1$, find BD^2.

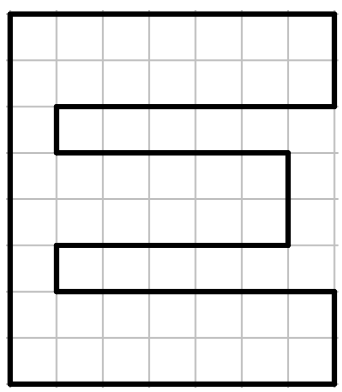
1.4.6 Round 6

16. [15] Three congruent, mutually tangent semicircles are inscribed in a larger semicircle, as shown in the diagram below. If the larger semicircle has a radius of 30 units, what is the radius of one of the smaller semicircles?



17. [15] In isosceles trapezoid $ABCD$ with $BC \parallel AD$, the distances from A and B to line CD are 3 and 9, respectively. If the distance between the two bases of trapezoid $ABCD$ is 5, find the area of quadrilateral $ABCD$.

18. [15] How many ways are there to tile the "E" shape below with dominos? A domino covers two adjacent squares.



1.4.7 Round 7

19. [18] In isoceles triangle ABC, $AC = BC$ and $\angle ACB = 20^\circ$. Let Ω be the circumcircle of triangle ABC with center O, and let M be the midpoint of segment BC. Ray $\overrightarrow{\mathrm{OM}}$ intersects Ω at D. Let ω be the circle with diameter OD. AD intersects ω again at a point X not equal to D. Given $OD = 2$, find the area of triangle OXD.

20. [18] Find the smallest odd prime factor of $2023^{2029} + 2026^{2029} - 1$.

21. [18] Achyuta, Alan, Andrew, Anish, and Ava are playing in the EMCC games. Each person starts with a paper with their name taped on their back. A person is eliminated from the game when anybody rips their paper off of their back. The game ends when one person remains. The remaining person then rips their paper off of their own back. At the end of the game, each person collects the papers that they ripped off. How many distinct ways can the papers be distributed at the end of the game?

1.4.8 Round 8

22. [21] Anthony has three random number generators, labelled A, B and C.

    - Generator A returns a random number from the set $\{12, 24, 36, 48, 60\}$.
    - Generator B returns a random number from the set $\{15, 30, 45, 60\}$.
    - Generator C returns a random number from the set $\{20, 40, 60\}$.

    He uses generator A, B, and then C in succession, and then repeats this process indefinitely. Anthony keeps a running total of the sum of all previously generated numbers, writing down the new total every time he uses a generator. After he uses each machine 10 times, what is the average number of multiples of 60 that Anthony will have written down?

23. [21] A laser is shot from one of the corners of a perfectly reflective room shaped like an equilateral triangle. The laser is reflected 2497 times without shining into a corner of the room, but after the 2497th reflection, it shines directly into the corner it started from. How many different angles could the laser have been initially pointed?

24. [21] We call a k-digit number blissful if the number of positive integers n such that n^n ends in that k-digit number happens to be nonzero and finite. What is the smallest value of k such that there exists a blissful k-digit number?

Chapter 2

EMC^2 2023 Solutions

2.1 Speed Test Solutions

1. Evaluate the following expression, giving your answer as a decimal: $\frac{20\times2\times3}{20+2+3}$.

   Solution. The answer is $\boxed{4.8}$.

   Computing gives the value as $\frac{120}{25}=\frac{24}{5}$, or 4.8.

2. Given real numbers x and y, we have that $2x+3y=20$ and $3x+4y=12$. Find the value of $x+y$.

   Solution. The answer is $\boxed{-8}$.

   Subtracting the two equations, we get that $x+y=(3x+4y)-(2x+3y)=12-20=-8$.

3. Alan, Daria, and Max want to sit in a row of three airplane seats. If Alan cannot sit in the middle, in how many ways can they sit down?

   Solution. The answer is $\boxed{4}$.

   There are $3!=6$ ways to arrange the three among the airplane seats. Among these 6 configurations, Alan is in the middle in 2 of them, so there are 4 ways for them to sit.

4. Jack thinks of two distinct positive integers a and b. He notices that neither a nor b is a perfect square, but ab is a perfect square. What is the smallest possible value of $a+b$?

   Solution. The answer is $\boxed{10}$.

   Let $a=k_1x^2$ and $b=k_2y^2$, where k_1 and k_2 are both squarefree positive integers, and x and y are positive integers. Then, in order for ab to be a square and a and b not squares, we must have $k_1=k_2\neq1$. Thus, in order for a and b to be distinct, we must have $x\neq y$. Now, $a+b=k_1\left(x^2+y^2\right)$, so to minimize this expression, we take $k_1=2$, and $(x,y)=(1,2)$ for a sum of 10.

5. What is the smallest integer greater than 2023 whose digits sum to 4?

   Solution. The answer is $\boxed{2101}$.

   It is clear that any integer in the range $[2030,2099]$ has digit sum greater than 4, since the thousands and tens digits sum to more than 4. For integers in the range $[2024,2029]$, we also have a digit sum of greater than 4. Thus, the smallest possible candidate is 2100, which doesn't work, however, 2101 works, and is thus the smallest.

6. Triangle ABC has $AB=AC$ and $\angle B=60^\circ$. The altitude drawn from C intersects AB at X, where $BX=4$. What is the area of ABC?

   Solution. The answer is $\boxed{16\sqrt{3}}$.

   From the conditions, we must have that ABC is equilateral, and since $BX=4$, we have the side length of ABC is 8. Furthermore, from 30-60-90 triangles, $CX=4\sqrt{3}$, so the area of ABC is $\frac{1}{2}\cdot8\cdot4\sqrt{3}=16\sqrt{3}$.

7. Archyuta writes a program to create words with at least one letter. The probability of having n letters in the word for each positive integer n is $\frac{1}{2^n}$. Each letter of the word is chosen randomly and independently from the uppercase English alphabet. The probability of Archyuta's program outputting "EMCC" can be written as $\frac{1}{k}$ for some positive integer k. What is the greatest nonnegative integer a such that 2^a divides k?

   Solution. The answer is $\boxed{8}$.

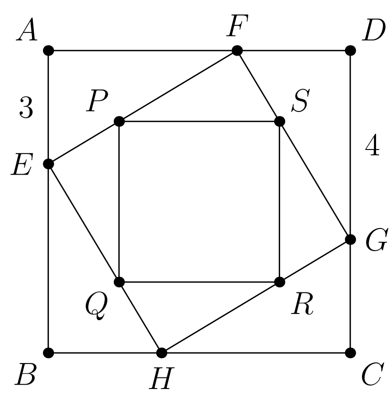
   The probability that Archyuta's program outputs a 4-letter word is $\frac{1}{16}$, and the probability of each letter being the corresponding letter in "EMCC" is $\frac{1}{26}$. Thus, the probability of Archyuta's program outputting "EMCC" is $\frac{1}{16} \cdot \left(\frac{1}{26}\right)^4 = \frac{1}{16 \cdot 26^4}$. Since $2 \mid 26$ and $4 \nmid 26$, we have that $2^8 \mid 16 \cdot 26^4$ and $2^9 \nmid 16 \cdot 26^4$, so the largest possible a is 8.

8. What is the greatest whole number less than 1000 that can be expressed as the sum of seven consecutive whole numbers, as the sum of five consecutive whole numbers, and as the sum of three consecutive whole numbers?

   Solution. The answer is $\boxed{945}$.

   Suppose the 7 consecutive whole numbers are $x - 3, x - 2, \ldots, x + 3$. The sum of these is $7x$, so we have that the whole number is divisible by 7. Similarly, we must have that the whole number is divisible by 5 and 3 as well, so in all, it must be divisible by $\text{lcm}(7, 5, 3) = 105$. The largest multiple of 105 less than 1000 is 945.

9. Given a square $ABCD$ with side length 7, square $EFGH$ is inscribed in $ABCD$ such that E is on side AB and G is on side CD such that $EA = 3$ and $GD = 4$. If square $PQRS$ inscribed in $EFGH$ such that $PQ \parallel AB$, find the side length of $PQRS$.



   Solution. The answer is $\boxed{\frac{25}{7}}$.

   Since $EFGH$ is a square, $EF = FG$, and $\angle AFE = 90 - \angle GFD = \angle FGD$, so by AAS, $\triangle AFE \cong \triangle DGF$. Thus, $AF = 4$ and $FD = 3$, and as a result, $EF = 5$. Now, since $AB||PQ$, we have $\angle AEF = \angle EPQ$, so in fact we have $ABCD\&EFGH \sim EFGH\&PQRS$. Thus, $\frac{AB}{EF} = \frac{EF}{PQ}$, or $PQ = \frac{EF^2}{AB} = \frac{25}{7}$.

10. Michael wants to do some exercise by going up and down a moving escalator. He first runs up the escalator, taking 30 seconds to reach the top. Tired, he then walks at one-third of his running speed

back down the escalator, taking 30 seconds to reach the bottom. Assuming his running speed and the escalator's speed are constant, what is the ratio of his running speed to the escalator's speed?

Solution. The answer is $\boxed{3}$.

It is clear the escalator is moving down. Let Michael's speed be m, and the escalator's x. Since the up and down trips take the same amount of time, and cover the same amount of distance, the speed at which Michael traveled must be the same for both trips, or $m - x = \frac{1}{3}m + x$. Thus, we get $x = \frac{1}{3}m$, so the ratio of Michael's speed to the escalator's speed is 3.

11. Bob the architect has 4 bricks shaped like rectangular prisms each of dimension 1 foot by 1 foot by 2 feet which he stores inside a 2 feet by 2 feet by 2 feet hollow box. In how many ways can he fit his bricks into the box? (Rotations and reflections of a configuration are considered distinct.)

    Solution. The answer is $\boxed{9}$.

    There are two cases. If all the bricks are aligned along the same axis, there are 3 ways to put the bricks in the box. Now, if not all the bricks are aligned along the same axis, it is clear there must two in one direction and two in another direction. Further, these must be arranged as 2 by 2 by 1 blocks, and then put together to form the 2 by 2 by 2 box. Therefore, there are 3 choices for which direction the 2 by 2 by 1 blocks face, and 2 choices for which direction to put on the front. Thus, there are 6 arrangements in this case, for a total of 9.

12. P is a point lying inside rectangle $ABCD$. If $\angle PAB = 40^\circ$, $\angle PBC = 50^\circ$ and $\angle PCD = 60^\circ$, find $\angle PDA$ in degrees.

    Solution. The answer is $\boxed{30}$.

    We have that $\angle PBA = 90 - \angle PBC = 40^\circ$, so P lies on the perpendicular bisector of AB. As such, $\angle PDA = 90 - \angle PDC = 90 - \angle PCD = 90 - 60 = 30^\circ$.

13. Let N be a positive integer. If 4 of N's divisors are prime and 346 of N's divisors are composite, how many of N's divisors are perfect squares?

    Solution. The answer is $\boxed{56}$.

    Since all the divisors of N are either prime, composite, or 1, N has a total of 351 divisors. Thus, since $351 = 3^3 \cdot 13$, N must be of the form $p_1^2 \cdot p_2^2 \cdot p_3^2 \cdot p_4^{12}$, where all p_i are distinct primes. Thus, N has a total of $2^3 \cdot 7 = 56$ square divisors of N.

14. Two positive integers have a product of 2^{23}. Let S be the sum of all distinct possible values of their absolute difference. Find the remainder when S is divided by 1000.

    Solution. The answer is $\boxed{25}$.

    The possible differences are $2^{23} - 1$, $2^{22} - 2, 2^{21} - 2^2, \ldots, 2^{12} - 2^{11}$. Summing these all gives

$$\begin{aligned} 2^{23} + 2^{22} + 2^{21} + \ldots + 2^{12} - 2^{11} - 2^{10} - \ldots - 1 &= 2^{23} + 2^{22} + \ldots + 1 - 2\left(2^{11} + 2^{10} + \ldots + 1\right) \\ &= 2^{24} - 1 - 2\left(2^{12} - 1\right) = 2^{24} - 2^{13} + 1. \end{aligned}$$

We have $2^{13} \equiv 192 \pmod{1000}$, and $2^{24} \equiv \left(2^{12}\right)^2 \equiv 96^2 \equiv 216 \pmod{1000}$, so $2^{24} - 2^{13} + 1 \equiv 216 - 192 + 1 = 25 \pmod{1000}$.

15. A rectangle has area 216. The internal angle bisectors of each of its four vertices are drawn, bounding a square region with area 18. Find the perimeter of the rectangle.

    Solution. The answer is $\boxed{60}$.

    It is fairly straightforward to see that if the difference of the sides of the rectangle is d, then the square formed has side length $\frac{d}{\sqrt{2}}$. Thus, if our rectangle has sides $x \geq y$, we have $x - y = 6$ and $xy = 216$. Solving for x and y (in this case, some simple trial and error works) gives sides of length 18 and 12, for a perimeter of 60.

16. Let $\triangle ABC$ be a right triangle, with a right angle at B. The perpendicular bisector of hypotenuse $\overline{AC}$ splits the triangle into a smaller triangle and a quadrilateral. If the triangle has an area of 5 and the quadrilateral has an area of 13, find the length of $\overline{AC}$.

    Solution. The answer is $\boxed{2\sqrt{30}}$.

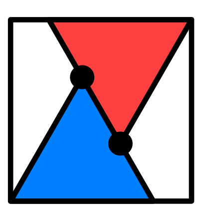
    WLOG $AB > AC$. Let the midpoint of AC be M, and let the perpendicular bisector of AC hit AB at D. Then, we have that $\triangle ABC \sim \triangle AMD$. Since the area ratio of ABC to AMD is 18 to 5, we know the similarity ratio between the two is $\frac{3\sqrt{2}}{\sqrt{5}}$. Thus, if AC is x, we have that AM is $\frac{x}{2}$ and AD is $\frac{\sqrt{5}}{3\sqrt{2}}x$. Then, $\frac{AD}{AM} = \frac{\sqrt{10}}{3}$, so $\frac{MD}{AM} = \frac{1}{3}$. Since the area of AMD is 5, we have that $AM \cdot MD = 10$, so $AM = \sqrt{30}$ and $MD = \frac{\sqrt{10}}{\sqrt{3}}$, so $AC = 2AM = 2\sqrt{30}$.

17. Let N be the sum of the 2023 smallest positive perfect squares minus the sum of the 2023 smallest positive odd numbers. What is the largest prime factor of N?

    Solution. The answer is $\boxed{809}$.

    The sum of the 2023 smallest positive odd numbers is 2023^2, so N is the sum of the first 2022 positive perfect squares, or $\frac{2022 \cdot 2023 \cdot 4045}{6}$. To factor N, we just factor each of 2022, 2023, and 4045, which factor as $2 \cdot 3 \cdot 337$, $7 \cdot 17^2$, and $5 \cdot 809$. The 6 in the denominator removes the 2 and 3, so the largest prime factor of N is 809.

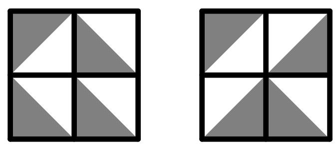
18. Anna designs a logo, shown below, consisting of a large square with side length 12 and two congruent equilateral triangles placed inside the square, one in each corner and with one overlapping side. What is the distance between the marked vertices?

    Solution. The answer is $\boxed{12 - 4\sqrt{3}}$.

Let the square be $ABCD$, where A is also a vertex of the blue triangle, and B is in the top left of the square. Then, let the marked vertex of the blue triangle be E and the marked vertex of the red triangle be F. Then, extend AE to meet BC at G. Since ABG is a 30-60-90 triangle, we have BG is $4\sqrt{3}$. Then, since $AE||CF$, and $\angle EFC = \angle BCF$, $CFEG$ is an isosceles trapezoid, so $EF = CG = 12 - 4\sqrt{3}$.

19. How many ways are there to place an isosceles right triangle with legs of length 1 in each unit square of a two-by-two grid, such that no two isosceles triangles share an edge? One valid construction is shown on the left, followed by an invalid construction on the right.



Solution. The answer is $\boxed{81}$.

Note that choosing an isosceles triangle is equivalent to choosing a vertical side and a horizontal side of the unit square. Then, we look at the left column of the two-by-two grid. We need to choose two distinct horizontal edges from the three horizontal edges: one for the top square and one for the bottom square. Note that any way we choose two edges, there will be one for the top square and one for the bottom square, so there are 3 choices there. Similarly, for the right column, there are 3 choices for horizontal edges, and for the top and bottom rows, there are 3 choices each for the vertical edges. Thus, in total there are 81 possible placements.

20. Mr. Ibbotson and Dr. Drescher are playing a game where they write numbers on the blackboard. On the first turn, Mr. Ibbotson begins by writing 1, followed by Dr. Drescher writing another 1 on the second turn. Each turn afterwards, they take the two newest numbers on the board and concatenate them, writing the resulting number of the board. For instance, the first few numbers on the board are $1, 1, 11, 111, 11111, \ldots$ How many turns does it take for them to write a number which is divisible by 63?

Solution. The answer is $\boxed{12}$.

Factoring 63, we get that it is $3^2 \cdot 7$, so we need the number to be divisible by 7 and 9. Now, we note that $11\ldots1$ with n ones is $\frac{10^n-1}{9}$. Thus, we need this expression to be divisible by 7 and 9. Therefore, we need $10^n - 1$ to be divisible by 7 and $9 \cdot 9 = 81$. In order for $10^n - 1$ to be divisible by 7, we must have that $3^n \equiv 10^n \equiv 1 \pmod 7$. The order of 3 mod 7 is 6, so we must have $6|n$. In order for $10^n - 1$ to be divisible by 81, we need $v_3(10^n - 1) \geq 4$. Using the lifting the exponent lemma, we have that $v_3(10^n - 1) = v_3(9) + v_3(n) = v_3(n) + 2$, so we need $9|n$. Thus, in all, we need $18|n$ in order to have 63 divide $\frac{10^n-1}{9}$. Listing out the n for the terms we write down, it is the Fibonacci sequence. The first number in the Fibonacci sequence divisible by 18 is 144, which corresponds to the number written on turn 12.

2.2 Accuracy Test Solutions

1. Minseo writes all of the divisors of 1,000,000 on the whiteboard. She then erases all of the numbers which have the digit 0 in their decimal representation. How many numbers are left?

   Solution. The answer is $\boxed{13}$.

   The prime factorization of 1,000,000 is is $2^6 \cdot 5^6$. It is clear that if a divisor has both a 2 and 5 in its prime factorization, then it is a multiple of 10 and thus has a zero in its decimal representation. The powers of 2^n and 5^n for $1 \le x \le 6$ all do not have zero in their decimal representations. In addition, 1 must be counted as a divisor. Thus, Minseo is left with $6 + 6 + 1 = 13$ divisors.

2. $n < 100$ is an odd integer and can be expressed as $3k - 2$ and $5m + 1$ for positive integers k and m. Find the sum of all possible values of n.

   Solution. The answer is $\boxed{183}$.

   First, note that m must be even, otherwise $5m + 1$ yields an even integer. Therefore, $1 \le m \le 18$. Also note that k must be odd in order for n to be odd. Next, $k = \frac{5m}{3} + 1$, so m must be a multiple of 6 for k to be an odd integer. Therefore, possible values of m are $6, 12$, and 18 corresponding to n equalling $31, 61$, and 91 respectively. Summing these values yields 183.

3. Mr. Pascal is a math teacher who has the license plate SQUARE. However, at night, a naughty student scrambles Mr. Pascal's license plate to UQRSEA. The math teacher luckily has an unscrambler that is able to move license plate letters. The unscrambler swaps the positions of any two adjacent letters. What is the minimum number of times Mr. Pascal must use the unscrambler to restore his original license plate?

   Solution. The answer is $\boxed{6}$.

   We call a pair of letters that appears in the wrong order when reading from left to right an inverted pair. There are 6 inverted pairs total in UQRSEA. Notice that each time the unscrambler is used, at most one inverted pair is corrected, which means it needs to be used at least 6 times to fix all the inverted pairs. One way to restore the license plate in six uses is as follows: UQRSEA $\to$ QURSEA $\to$ QUSREA $\to$ QSUREA $\to$ SQUREA $\to$ SQURAE $\to$ SQUARE.

4. Find the number of distinct real numbers x which satisfy $x^2 + 4\lfloor x \rfloor + 4 = 0$.

   Solution. The answer is $\boxed{3}$.

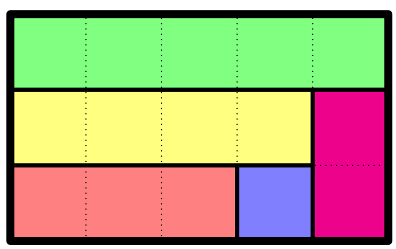
   Notice that $x - 1 < \lfloor x \rfloor$. Thus, we have the inequality $0 > x^2 + 4(x-1) + 4 \iff 2^2 > (x+2)^2 \iff -4 < x < 0$. Thus, we can divide into four cases depending on the value of $\lfloor x \rfloor$. If it is -4, then $x = -\sqrt{12}$ satisfies $-4 \le -\sqrt{12} < -3$. If it is -3, then $x = -\sqrt{8}$ satisfies $-3 \le -\sqrt{8} < -2$. If it is -2, then $x = -\sqrt{4} = -2$ satisfies $-2 \le -2 < -1$. If it is -1, then $x = 0$. However, $\lfloor 0 \rfloor = 0 \neq -1$, so this case is impossible. Therefore, there are three possible solutions: $x = -2\sqrt{3}, -2\sqrt{2}, -2$.

5. All four faces of tetrahedron $ABCD$ are acute. The distances from point D to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are all 7, and the distance from point D to face ABC is 5. Given that the volume of tetrahedron $ABCD$ is 60, find the surface area of tetrahedron $ABCD$.

Solution. The answer is $\boxed{21\sqrt{6}+36}$.

Let D' be the foot of the altitude from D to face ABC. Then by the pythagorean theorem, the distance from D' to each of the sides of triangle ABC is the same and equal to $\sqrt{7^2-5^2} = 2\sqrt{6}$. Notice that this means a circle centered at D' with radius $2\sqrt{6}$ would be tangent to all three edges of triangle ABC. Now, since all the faces are acute, D' is inside ABC, so D' is the incenter of triangle ABC. Notice that the volume of tetrahedron $ABCD$ satisfies $V = \frac{bh}{3}$ so the area of base ABC is equal to 36. Since $A = rs$ where r is the inradius and $s = \frac{AB+BC+CA}{2}$ is the semiperimeter of triangle ABC, it follows that $AB + BC + CA = 6\sqrt{6}$. The surface area of the tetrahedron is the area of the base plus the area of the other three faces, each of which has an altitude from D of 7, so the surface area is $36 + \frac{AB\cdot 7}{2} + \frac{BC\cdot 7}{2} + \frac{CA\cdot 7}{2} = 36 + \frac{6\sqrt{6}\cdot 7}{2} = 21\sqrt{6} + 36$.

6. Forrest has a rectangular piece of paper with a width of 5 inches and a height of 3 inches. He wants to cut the paper into five rectangular pieces, each of which has a width of 1 inch and a distinct integer height between 1 and 5 inches, inclusive. How many ways can he do so? (One possible way is shown below.)



Solution. The answer is $\boxed{40}$.

Perform casework on the position of the 1×5 rectangle. If we do this, we see two possibilities, either the 1×5 is on the end of the original paper, or is in the middle. If it is in the center, the whole shape is determined, as the resulting left over segments are only tillable by a 1×1 and a 1×4, or a 1×2 and a 1×3. This yields $2\cdot 2\cdot 2 = 8$ ways. For the second case, we are left tiling a 2×5 grid with a 1×1, 1×2, 1×3, and 1×4 piece respectively. There are four ways to place the 1×4 block, and then three ways to place the 1×3 block. These three placements always create $2, 1, 1$ ways to place the 1×2 and 1×1 blocks respectively, so in total there are $4\times(2+1+1) = 16$ placements in this case. Thus in total, we obtain $8 + 16 + 16 = 40$ solutions.

7. In convex quadrilateral $ABCD$, $AB = CD = 5$, $BC = 4$ and $AD = 8$. If diagonal $\overline{AC}$ bisects $\angle DAB$, find the area of quadrilateral $ABCD$.

Solution. The answer is $\boxed{26}$.

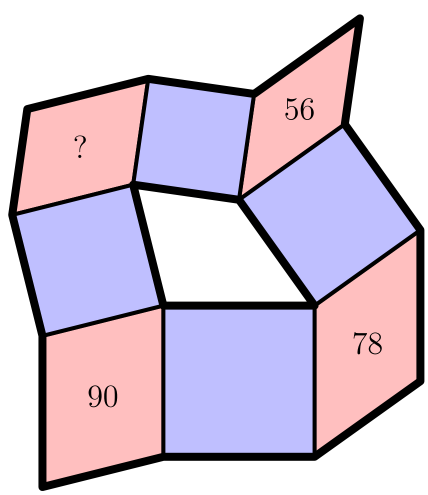
Let B' be the reflection of B across line AC, then B' lies on segment AD and satisfies $AB' = 5$. Thus $B'D = 3$ and $B'C = 4$, so $B'CD$ is a right triangle with sides $3, 4, 5$ by the Pythagorean theorem so $\angle CB'D = 90°$. Therefore the area of triangle ACD is $\frac{bh}{2} = \frac{8\cdot 4}{2} = 16$ and the area of triangle ABC equals the area of triangle $AB'C$ which is $\frac{bh}{2} = \frac{5\cdot 4}{2} = 10$, so the area of quadrilateral $ABCD$ is $16 + 10 = 26$.

8. Let x and y be real numbers such that
$$x + y = x^3 + y^3 + \frac{3}{4} = \frac{1}{8xy}.$$
Find the value of $x + y$.

Solution. The answer is $\boxed{\frac{-1-\sqrt{13}}{4}}$.

Let $p = x + y$ and $q = xy$. Then we have $p = p^3 - 3pq + \frac{3}{4} = \frac{1}{8q}$. Since $q = \frac{1}{8p}$, we know that $p = p^3 + \frac{3}{8}$. This can be rearranged as $(p - \frac{1}{2})(p^2 + \frac{1}{2}p - \frac{3}{4}) = 0$, which has roots using the quadratic formula of $p = \frac{1}{2}, \frac{-1\pm\sqrt{13}}{4}$. If p is positive, then $q = \frac{1}{8p}$ must be positive, so x, y must have the same sign, so since $x + y$ is positive, it follows that both x, y would need to be positive. This means that by $AM - GM$, $p \geq 2\sqrt{q}$ so $p^2 \geq 4q = \frac{1}{2p}$ so $p^3 \geq \frac{1}{2}$. However, this fails for both $p = \frac{1}{2}$ and $p = \frac{\sqrt{13}-1}{4}$ since $\left(\frac{\sqrt{13}-1}{4}\right)^3 = \frac{2\sqrt{13}-5}{8} < \frac{1}{2}$ as $2\sqrt{13} < 9$ as $52 < 81$, so p must be negative and equal to $\frac{-1-\sqrt{13}}{4}$. Notice that the graph of $xy = \frac{1}{8\cdot\frac{-1-\sqrt{13}}{4}}$ and $x + y = c$ for any constant c must intersect since along the line $x + y = c$, xy approaches negative infinity as x goes to infinity and y goes to negative infinite, and xy is equal to 0 when $x = 0$, so by the intermediate value theorem, xy must achieve every negative value as a point varies along the line $x + y = c$ and thus $x + y = \frac{-1-\sqrt{13}}{4}$ must intersect the graph of $xy = \frac{1}{8\cdot\frac{-1-\sqrt{13}}{4}}$ so there exists real numbers x, y such that the equation is satisfied and such that $x + y = \frac{-1-\sqrt{13}}{4}$. Thus, the only possible value is $\frac{-1-\sqrt{13}}{4}$ and it is achievable.

9. Four blue squares and four red parallelograms are joined edge-to-edge alternately to form a ring of quadrilateral as shown. The areas of three of the red parallelograms are shown. Find the area of the fourth red parallelogram.



Solution. The answer is $\boxed{68}$.

Notice that the diagonals of the center empty quadrilateral split it into pairs of triangles. Each of these triangles has half the area its corresponding red parallelogram. Thus, the sum of the areas of opposite red parallelograms is the same. So the answer is $56 + 90 - 78 = 68$.

10. Define $f(x,n) = \sum_{d|n} \frac{x^n-1}{x^d-1}$. For how many integers n between 1 and 2023 inclusive is $f(3,n)$ an odd integer?

    Solution. The answer is $\boxed{75}$.

    Notice that $f(3,n) \equiv \sum_{d|n} \frac{3^n-1}{3^d-1} \equiv \sum\limits_{d|n} \sum\limits_{i=0}^{n/d-1} 3^{di} \equiv \sum\limits_{d|n} \sum\limits_{i=0}^{n/d-1} 1 \equiv \sum\limits_{d|n} \frac{n}{d} \equiv \sum\limits_{d|n} d \pmod 2$. Thus $f(3,n)$ is an odd integer if and only if the sum of the divisors of n is an odd integer. Notice even divisors do not affect the parity of this sum, so this is equivalent to the number of odd divisors of n being odd. We can divide out any factors of 2 from n, since multiplying by 2 does not add or remove any odd factors, so we look only at the case where n is odd. Then, since n is odd, every divisor of n is also odd. Thus n has an odd number of odd divisors if and only if n is a perfect square, since each divisor d other than $\sqrt{n}$ can be paired with a different divisor $\frac{n}{d}$, giving an even number of divisors if n is not a perfect square and an odd number of divisors if n is a perfect square. Hence, we are looking for n which can be represented as $2^k \cdot m^2$ for odd integers m. This is equivalent to $n = s^2$ or $n = 2s^2$ for some integer s, so the total number of distinct n with $f(3,n)$ odd is $\lfloor \sqrt{2023} \rfloor + \left\lfloor \sqrt{\frac{2023}{2}} \right\rfloor = 44 + 31 = 75$.

2.3 Team Test Solutions

1. We define $a \oplus b = \frac{ab}{a+b}$. Compute $(3 \oplus 5) \oplus (5 \oplus 4)$.

   Solution. The answer is $\boxed{\frac{60}{59}}$.

   Note that $a \oplus b = \frac{1}{\frac{1}{a}+\frac{1}{b}}$. So

$$\begin{aligned}(3 \oplus 5) \oplus (5 \oplus 4) &= \frac{1}{\frac{1}{3\oplus 5} + \frac{1}{5\oplus 4}} \\ &= \frac{1}{\frac{1}{\frac{1}{1/3+1/5}} + \frac{1}{\frac{1}{1/5+1/4}}} \\ &= \frac{1}{\frac{1}{3}+\frac{1}{5}+\frac{1}{5}+\frac{1}{4}} \\ &= \frac{60}{59}.\end{aligned}$$

2. Let $ABCD$ be a quadrilateral with $\angle A = 45^\circ$ and $\angle B = 45^\circ$. If $BC = 5\sqrt{2}$, $AD = 6\sqrt{2}$, and $AB = 18$, find the length of side CD.

   Solution. The answer is $\boxed{5\sqrt{2}}$.

   Let E and F be the feet of the perpendiculars from C and D onto AB, respectively. Then $\triangle AFD$ and $\triangle BEC$ are isosceles right triangles, so $AF = FD = 6$ and $BE = EC = 5$. Now create point G by dropping a perpendicular from C onto DF. Note that $AB = AF + FE + EB = AF + GC + EB$, so $GC = 18 - 6 - 5 = 7$. Finally, since $GD = FD - EC = 6 - 5 = 1$, from the Pythagorean Theorem we get

$$CD = \sqrt{GC^2 + GD^2} = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}.$$

3. A positive real number x satisfies the equation $x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 10$. Find the sum of all possible values of $x + 1 + \frac{1}{x}$.

   Solution. The answer is $\boxed{\frac{1+3\sqrt{5}}{2}}$.

   Let $A = x + \frac{1}{x}$. Note that $A^2 = x^2 + 2 + \frac{1}{x^2}$. So the equation can be rewritten as

$$A^2 + A - 11 = 0,$$

   which has solutions $A = \frac{-1\pm 3\sqrt{5}}{2}$. But since x is positive, A must be at least 2 by AM-GM, so the only possible value for A is $\frac{-1+3\sqrt{5}}{2}$. Then our answer is $A + 1 = \frac{1+3\sqrt{5}}{2}$.

4. David writes 6 positive integers on the board (not necessarily distinct) from least to greatest. The mean of the first three numbers is 3, the median of the first four numbers is 4, the unique mode of the first five numbers is 5, and the range of all 6 numbers is 6. Find the maximum possible value of the product of David's 6 integers.

Solution. The answer is $\boxed{3675}$.

Since the mean of the first three numbers is 3, the first three numbers must sum to 9. But the median of the first four numbers is 4, so the second and third numbers must sum to 8. Therefore the first number is 1.

The range of all six numbers is 6, so the sixth number must be 7.

The unique mode of the first five numbers is 5, so the third number cannot be greater than 5, or else there will be at most one 5 among the six positive integers. Therefore the second and third numbers are either 4 and 4 or 3 and 5. But if the second and third numbers are 4 and 4, there can be at most two 5's. Then, there is either no unique mode or the unique mode is 4. So the first four numbers have to be 1, 3, 5, 5 to create a unique mode of 5 among the first five numbers.

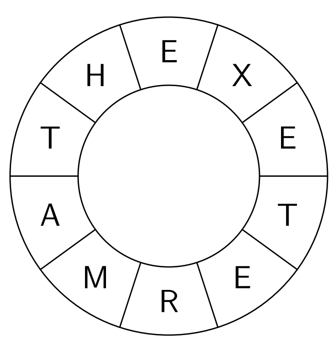
Since the sixth number is 7, to maximize the product we let the fifth number also be 7. So our final set of six integers is 1, 3, 5, 5, 7, 7, which has a product of 3675.

5. Let $ABCD$ be a convex quadrilateral such that $\angle A = \angle B = 120^\circ$ and $\angle C = \angle D = 60^\circ$. There exists a circle with center I which is tangent to all four sides of $ABCD$. If $IA \cdot IB \cdot IC \cdot ID = 240$, find the area of quadrilateral $ABCD$.

   Solution. The answer is $\boxed{8\sqrt{15}}$.

   Note that $ABCD$ is an isosceles trapezoid with bases AB and CD. Furthermore, from the angle conditions, we also know that $CD - AB = BC = AD$. Let this value be x Since $ABCD$ has an incircle, the sum of opposite sides must be equal, so we must have that $CD + AB = BC + AD = 2x$. Thus, $AB = \frac{x}{2}$ and $CD = \frac{3}{2}x$. Now, we have that since I is the incenter of $ABCD$, it lies on the angle bisector of $\angle BAD$. Thus, $\angle IAB = 60^\circ$, and furthermore $IA = IB$. Thus, IAB is equilateral and $IA = IB = AB = \frac{x}{2}$. Similarly, $\angle ICD = 30^\circ$, and $IC = ID$, so we have that $IC = ID = \frac{CD}{\sqrt{3}} = \frac{\sqrt{3}}{2}x$. Therefore, we have that $\frac{3x^4}{16} = 240$. The height of the trapezoid is $\frac{\sqrt{3}}{2}x$, and the sum of the bases is $2x$, so the area of the trapezoid is $\frac{\sqrt{3}}{2}x^2$. Now, $x^4 = 1280$, so $x^2 = 16\sqrt{5}$, so the area is $\frac{\sqrt{3}}{2} \cdot 8\sqrt{5} = 8\sqrt{15}$.

6. The letters EXETERMATH are placed into cells on an annulus as shown below. How many ways are there to color each cell of the annulus with red, blue, green, or yellow such that each letter is always colored the same color and adjacent cells are always colored differently?



Solution. The answer is $\boxed{1440}$.

There are 4 choices for the letter E, and since T is adjacent to E, there are now 3 choices for what T can be. Now, X is only adjacent to E, so there are again 3 choices for X. H is adjacent to both E and T, so there are 2 choices for H. Now, we are left selecting colors for the A, M, and R in the bottom left. Note that E and T are different colors, so WLOG they are red and blue, respectively. Then, if A is blue, there are 3 choices for M and 2 choices for R. If A is not blue, there are 2 other choices for A. Now, if M is blue, there are 3 choices for R, and if M is not blue, there are 2 choices for M, and 2 choices for R. In total, we have $4 \cdot 3 \cdot 3 \cdot 2 \cdot (3 \cdot 2 + 2 \cdot (3 + 2 \cdot 2)) = 1440$ colorings.

7. Let $ABCD$ be a square, and let ω be a quarter circle centered at A passing through points B and D. Points E and F lie on sides BC and CD respectively. Line EF intersects ω at two points, G and H. Given that $EG = 2$, $GH = 16$ and $HF = 9$, find the length of side AB.

Solution. The answer is $\boxed{\frac{21+9\sqrt{17}}{2}}$.

Note that EB and FD are tangent to the quarter circle, so we must have by power of a point that $EB^2 = EG \cdot EH = 2 \cdot 18 = 36$ and $FD^2 = FG \cdot FH = 25 \cdot 9 = 225$. Thus, $EB = 6$, and $FD = 15$. If $AB = x$, then we have that $EC = x - 6$ and $FC = x - 15$, so since we have $EC^2 + FC^2 = EF^2 = 27^2$, we have $(x - 6)^2 + (x - 15)^2 = 27^2$. Solving for x yields that it is $\frac{21 \pm 9\sqrt{17}}{2}$, but we need $x > 15$, so x must be $\frac{21+9\sqrt{17}}{2}$.

8. Let x be equal to
$$\frac{2022! + 2021!}{2020! + 2019! + 2018!}.$$
Find the closest integer to $2\sqrt{x}$.

Solution. The answer is $\boxed{4043}$.

Factoring the top and bottom, we have the following:

$$\begin{aligned}
\frac{2022! + 2021!}{2020! + 2019! + 2018!} &= \frac{2021!(1 + 2022)}{2018!(1 + 2019 + 2019 \cdot 2020)} \\
&= \frac{2021! \cdot 2023}{2018!(1 + 2019 \cdot 2021)} = \frac{2021! \cdot 2023}{2018! \cdot 2020^2} \\
&= \frac{2019 \cdot 2021 \cdot 2023}{2020}.
\end{aligned}$$

Then, if $2020 = a$, the expression is equivalent to the following:

$$\frac{(a-1)(a+1)(a+3)}{a} = \frac{(a^2-1)(a+3)}{a} = \frac{a^3 + 3a^2 - a - 3}{a} = a^2 + 3a - 1 - \frac{3}{a}.$$

Now, $(a + \frac{3}{2})^2 = a^2 + 3a + \frac{9}{4}$, so our expression only differs from this by $\frac{13}{4} + \frac{3}{2020}$. This is a very small ϵ when we are dealing with $\sqrt{2021.5^2 \pm \epsilon}$, so we have that $2\sqrt{x}$ is nearest to $2 \cdot 2021.5$, or 4043.

9. For how many ordered pairs of positive integers (m, n) is the absolute difference between $\text{lcm}(m, n)$ and $\gcd(m, n)$ equal to 2023?

Solution. The answer is $\boxed{32}$.

Let $d = \gcd(m, n)$ and $a = \frac{m}{d}, b = \frac{n}{d}$ then we have $\gcd(a, b) = 1$ and want to find positive integers a, b, d such that $2023 = abd - d = d(ab - 1)$. We perform casework on the value of d. Since $2023 = 7 \cdot 17^2$, $d = 1, 7, 17, 119, 289, 2023$. If $d = 1$ then $ab = 2024 = 2^3 \cdot 11 \cdot 23$. Notice it suffices to distribute the distinct prime factors between a and b since they cannot share the same prime factor, so since there are 3 distinct prime factors, there are 2^3 solutions in this case. Similarly $d = 7$ gives $ab = 290 = 2 \cdot 5 \cdot 29$ and 2^3 solutions, $d = 17$ gives $ab = 120 = 2^3 \cdot 3 \cdot 5$ and 2^3 solutions, $d = 119$ gives $ab = 18 = 2 \cdot 3^2$ and 2^2 solutions, $d = 289$ gives $ab = 8 = 2^3$ and 2^1 solutions, and $d = 2023$ gives $ab = 2$ and 2^1 solutions. Thus, we have a total of $8 + 8 + 8 + 4 + 2 + 2 = 32$ solutions.

10. There are 2023 distinguishable frogs sitting on a number line with one frog sitting on i for all integers i between -1011 and 1011, inclusive. Each minute, every frog randomly jumps either one unit left or one unit right with equal probability. After 1011 minutes, over all possible arrangements of the frogs, what is the average number of frogs sitting on the number 0?

Solution. The answer is $\boxed{1}$.

Note that since 1011 is odd, only frogs standing on odd numbers initially can end up at the number 0 after 1011 minutes. By symmetry, we consider the frogs standing on positive numbers, then multiply by 2 at the end. Consider the frog standing on the number $2i + 1$. Then it needs to have $506 + i$ left jumps and $505 - i$ right jumps. There are $\binom{1011}{505-i}$ ways to choose the sequence of jumps, and each jump has an equal probability of being left or right. Thus this frog has a $\frac{\binom{1011}{505-i}}{2^{1011}}$ probability of landing on the number 0 after 1011 minutes. Summing over all frogs standing on positive numbers, and then multiplying by 2 for the negative numbers, our final value is

$$2\sum_{i=0}^{505} \frac{\binom{1011}{505-i}}{2^{1011}} = \frac{1}{2^{1011}} \sum_{i=0}^{505} 2\binom{1011}{505-i}.$$

So we need to calculate $\sum_{i=0}^{505} 2\binom{1011}{505-i}$. Using the fact that $\binom{1011}{505-i} = \binom{1011}{506+i}$, we have

$$\begin{aligned}\sum_{i=0}^{505} 2\binom{1011}{505-i} &= \sum_{i=0}^{505} \binom{1011}{505-i} + \binom{1011}{506+i} \\ &= \sum_{i=0}^{1011} \binom{1011}{i} \\ &= 2^{1011}.\end{aligned}$$

Thus our final answer is $\frac{1}{2^{1011}} \cdot 2^{1011} = 1$.

11. Albert has a calculator initially displaying 0 with two buttons: the first button increases the number on the display by one, and the second button returns the square root of the number on the display. Each second, he presses one of the two buttons at random with equal probability. What is the probability that Albert's calculator will display the number 6 at some point?

Solution. The answer is $\boxed{\frac{1}{14}}$.

Note that Albert will always eventually reach 2, as the square root button does nothing when he is at 0 or 1. Also note that if Albert ever gets to an irrational number, he can never get back to a rational number. Thus, for $2 \leq k \leq 6$, let the probability of Albert reaching 6 at some point starting from k be p_k. Then, $p_6 = 1$, and since square rooting 5 gives an irrational, $p_5 = \frac{1}{2}$. Now, from 4, we can either reach 5 or 2, so $p_4 = \frac{1}{2}p_5 + \frac{1}{2}p_2 = \frac{1}{4} + \frac{1}{2}p_2$. Further, from 3, we can only reach 4, so $p_3 = \frac{1}{2}p_4$. Similarly, $p_2 = \frac{1}{2}p_3$. Solving this, we do the following:

$$p_4 = \frac{1}{4} + \frac{1}{2}p_2 = \frac{1}{4} + \frac{1}{4}p_3 = \frac{1}{4} + \frac{1}{8}p_4.$$

This gives $p_4 = \frac{2}{7}$, and so $p_2 = \frac{1}{4}p_4 = \frac{1}{14}$.

12. For a positive integer $k \geq 2$, let $f(k)$ be the number of positive integers n such that n divides $(n-1)!+k$. Find $f(2) + f(3) + f(4) + f(5) + ... + f(100)$.

Solution. The answer is $\boxed{479}$.

To start, we note that if n is prime, then from Wilson's Theorem, $(n-1)! \equiv -1 \pmod{n}$, so n divides $(n-1)!+k$ if and only if n divides $k-1$. If n is 4, then n divides $(n-1)!+k$ if and only if $k \equiv 2 \pmod{4}$. If n is anything else, it is clear that n divides $(n-1)!$, so n divides $(n-1)!+k$ if and only if n divides k. Thus, aside from $n = 4$, which is special, $f(k)$ counts the number of nonprimes that divide k and the number of primes that divide $k-1$. Let $g(k)$ be the number of nonprimes and non-4 divisors of k, and let $h(k)$ be the number of prime divisors of k. Then, $f(k) = g(k) + h(k-1) + \epsilon_k$, where ϵ_k is 1 for $k \equiv 2 \pmod{4}$ and 0 otherwise. Summing over all the terms and noting that $h(1) = 0$, we get that the expression is equivalent to $g(2)+h(2)+g(3)+h(3)+\cdots+g(99)+h(99)+g(100)+25$. Since 100 has only 2 prime divisors and there are 25 terms between 2 and 100 divisible by 4, so this is also equivalent to $d(1) + d(2) + d(3) + \cdots + d(100) - 3$, where $d(k)$ is the number of divisors of k.

To sum this, we double count, counting by the divisor. There are 100 numbers between 1 and 100 with 1 as a divisor, 50 with 2 as a divisor, and so on (the full table is below). Summing them all, we get 482, so our final answer is 479.

13. Mr. Atf has nine towers shaped like rectangular prisms. Each tower has a 1 by 1 base. The first tower has height 1, the next has height 2, up until the ninth tower, which has height 9. Mr. Atf randomly arranges these 9 towers on his table so that their square bases form a 3 by 3 square on the surface of his table. Over all possible solids Mr. Atf could make, what is the average surface area of the solid?

Solution. The answer is $\boxed{118}$.

To start, the bottom always has area 9 and the tops of the solids also always have area 9. From there, we use linearity of expectation. There are 12 vertical faces on the "outside" of the solid around the sides, and the area of these faces corresponds to the height of the tower it is attached to. Since the expected value of the height is 5, linearity of expectation tells us the average area of the "outside" is $12 \cdot 5 = 60$. Now, there are also 12 vertical faces in the "middle" of the solid, with areas corresponding to the positive difference of the heights of the two adjacent towers. Now, we need to find the expected value of the difference between any two tower heights. There is 1 way to achieve a difference of 8 (namely between height 1 and height 9), 2 ways to achieve a difference of 7, and so on, until 8 ways to achieve a difference of 1. Averaging all these gives the average difference as $\frac{10}{3}$. Thus, the expected area for these vertical faces is $\frac{10}{3} \cdot 12 = 40$. Adding all these up, we get the expected surface area is $9 + 9 + 60 + 40$, or 118.

Divisor	# of Times Counted
1	100
2	50
3	33
4	25
5	20
6	16
7	14
8	12
9	11
10	10
11	9
12	8
13–14	7
15–16	6
17–20	5
21–25	4
26–33	3
34–50	2
51–100	1

14. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular. Let E be the intersection of AC and BD, and let the feet of the altitudes from E to the sides AB, BC, CD, DA be W, X, Y, Z respectively. Given that $EW = 2EY$ and $EW \cdot EX \cdot EY \cdot EZ = 36$, find the minimum possible value of $\frac{1}{[EAB]} + \frac{1}{[EBC]} + \frac{1}{[ECD]} + \frac{1}{[EDA]}$. The notation $[XYZ]$ denotes the area of triangle XYZ.

Solution. The answer is $\boxed{\frac{2}{3}}$.

First, we note that $\triangle EAB \sim \triangle EDC$, so we have that $\frac{EA}{ED} = \frac{EB}{EC} = \frac{EW}{EY}$. Furthermore, $\triangle EAD \sim \triangle EBC$, so $\frac{[EAD]}{EA^2} = \frac{[EBC]}{EB^2}$, or $\frac{\frac{1}{EA^2}}{\frac{1}{[EAD]}} = \frac{\frac{1}{EB^2}}{\frac{1}{[EBC]}}$. Since $EA \perp EB$, we have that $\frac{1}{EA^2} + \frac{1}{EB^2} = \frac{1}{EW^2}$, so in fact we also have the following:

$$\frac{\frac{1}{EA^2}}{\frac{1}{[EAD]}} = \frac{\frac{1}{EB^2}}{\frac{1}{[EBC]}} = \frac{\frac{1}{EW^2}}{\frac{1}{[EAD]} + \frac{1}{[EBC]}}.$$

Now, these previous expressions are in fact equal to $\frac{ED}{2EA}$, which is also equal to $\frac{EY}{2EW}$. Setting the last expression and this equal gives the following:

$$\frac{\frac{1}{EW^2}}{\frac{1}{[EAD]} + \frac{1}{[EBC]}} = \frac{EY}{2EW}$$
$$\frac{1}{[EAD]} + \frac{1}{[EBC]} = \frac{2}{EW \cdot EY}.$$

Similarly, we have that $\frac{1}{[ECD]} + \frac{1}{[EAB]} = \frac{2}{EX \cdot EZ}$. Thus, we have that:

$$\frac{1}{[EAB]} + \frac{1}{[EBC]} + \frac{1}{[ECD]} + \frac{1}{[EDA]} = \frac{2}{EW \cdot EY} + \frac{2}{EX \cdot EZ} \geq 2\sqrt{\frac{4}{EW \cdot EX \cdot EY \cdot EZ}} = \frac{2}{3}.$$

The equality case can be achieved when $ABCD$ is a kite with $AB = AD = 5\sqrt{3}$, $BC = CD = \frac{5\sqrt{3}}{2}$, and $AC = \frac{5\sqrt{15}}{2}$.

Author's Note: The condition $EW = 2EY$ is actually irrelevant for the purposes of the final answer. However, it provides a semiunique equality case (up to some permutations of the vertices). Additionally, it serves as a deterrent from simply looking at the easy case when $ABCD$ is a square.

15. Given that $x^2 - xy + y^2 = (x + y)^3$, $y^2 - yz + z^2 = (y + z)^3$, and $z^2 - zx + x^2 = (z + x)^3$ for complex numbers x, y, z, find the product of all distinct possible nonzero values of $x + y + z$.

Solution. The answer is $\boxed{-\frac{57}{512}}$.

First, consider the case where x, y, z are all distinct.

Expand to get:

$$\begin{aligned}
x^3 + 3x^2y + 3xy^2 + y^3 - x^2 + xy - y^2 &= 0 \\
y^3 + 3y^2z + 3yz^2 + z^3 - y^2 + yz - z^2 &= 0 \\
z^3 + 3z^2x + 3zx^2 + x^3 - z^2 + zx - x^2 &= 0
\end{aligned}$$

Now take differences between pairs of equations to get:

$$\begin{aligned}
(x - z)(x^2 + xz + z^2 + 3xy + 3yz + 3y^2 - x - z + y) &= 0 \\
(y - x)(y^2 + yx + x^2 + 3yz + 3zx + 3z^2 - y - x + z) &= 0 \\
(z - y)(z^2 + zy + y^2 + 3zx + 3xy + 3x^2 - z - y + x) &= 0
\end{aligned}$$

Divide out $x - z, y - x, z - y$ respectively and then take differences again to get:

$$\begin{aligned}
2(y - z)(x + y + z + 1) &= 0 \\
2(z - x)(x + y + z + 1) &= 0 \\
2(x - y)(x + y + z + 1) &= 0
\end{aligned}$$

So it follows that $x + y + z = -1$. To show that this value is achievable, we can reverse engineer this cyclic difference process. If we look at the second set of equations, divide out $x - z, y - x, z - y$ respectively and then take a sum instead, we get:

$$\begin{aligned}
0 &= 5x^2 + 5y^2 + 5z^2 + 7xy + 7yz + 7zx - x - y - z \\
&= 5(x + y + z)^2 - 3(xy + yz + zx) - (x + y + z) \\
&= 6 - 3(xy + yz + zx)
\end{aligned}$$

So $xy+yz+zx=2$. This condition suffices for the second set of equations to be true given $x+y+z=-1$ since knowing the sum of the expressions is 0 and that each pairwise difference is 0 implies they are all equal to 0.

Finally, if we look at the initial set of equations, we can take a sum instead to get:

$$\begin{aligned}0&=2x^3+2y^3+2z^3+3x^2y+3xy^2+3y^2z+3yz^2+3z^2x+3zx^2-2x^2-2y^2-2z^2+xy+yz+zx\\&=2(x+y+z)^3-3(x+y+z)(xy+yz+zx)-3xyz-2(x+y+z)^2+5(xy+yz+zx)\\&=12-3xyz\end{aligned}$$

So $xyz=4$. Similarly, this condition suffices for the original set of equations to be true given $xy+yz+zx=2$ and $x+y+z=-1$.

Thus, if we simply let x,y,z be the three roots of the cubic equation x^3+x^2+2x-4, the equations will be satisfied by Vieta's formulas and we will get $x+y+z=-1$. Specifically, notice $0=x^3+x^2+2x-4=(x-1)(x^2+2x+4)$, which has roots $1,-1-i\sqrt{3},-1+i\sqrt{3}$. These values achieve $x+y+z=-1$.

Next, consider the case where some two of x,y,z are equal. Then WLOG $x=y$. From the initial equations, we get that: $x^2=8x^3$ so $x=0$ or $x=\frac{1}{8}$.

If $x=0$ then we get that $z^2=z^3$ so $z=0,1$ giving us $x+y+z=0,1$ respectively.

If $x=\frac{1}{8}$ then we get that $z^2-\frac{1}{8}z+\frac{1}{64}=z^3+\frac{3}{8}z^2+\frac{3}{64}z+\frac{1}{512}$. Letting $8z=t$, we get that $0=t^3-5t^2+11t-7=(t-1)(t^2-4t+7)$. Letting $t=1$ gives $x+y+z=\frac{3}{8}$. Otherwise, $t=2+i\sqrt{3},2-i\sqrt{3}$. These give $x+y+z=\frac{4+i\sqrt{3}}{8},\frac{4-i\sqrt{3}}{8}$ respectively. Thus, all cases have been completed.

Now, the product of all nonzero values of $x+y+z$ is $1\cdot(-1)\cdot\frac{3}{8}\cdot\frac{4+i\sqrt{3}}{8}\cdot\frac{4-i\sqrt{3}}{8}=-\frac{57}{512}$.