Team Round

EMCC

February 2021

1. Suppose that Yunseo wants to order a pizza that is cut into 4 identical slices. For each slice, there are 2 toppings to choose from: pineapples and apples. Each slice must have exactly one topping. How many distinct pizzas can Yunseo order? Pizzas that can be obtained by rotating one pizza are considered the same.

2. How many triples of distinct positive integers (E, M, C) are there such that $E = MC^2$ and $E \leq 50$?

3. Given that the cubic polynomial $p(x)$ has leading coefficient 1 and satisfies $p(0) = 0$, $p(1) = 1$, and $p(2) = 2$. Find $p(3)$.

4. Olaf asks Anna to guess a two-digit number and tells her that it's a multiple of 7 with two distinct digits. Anna makes her first guess. Olaf says one digit is right but in the wrong place. Anna adjusts her guess based on Olaf's comment, but Olaf answers with the same comment again. Anna now knows what the number is. What is the sum of all the numbers that Olaf could have picked?

5. Vincent the Bug draws all the diagonals of a regular hexagon with area 720, splitting it into many pieces. Compute the area of the smallest piece.

6. Given that $y - \frac{1}{y} = 7 + \frac{1}{7}$, compute the least integer greater than $y^4 + \frac{1}{y^4}$.

7. At 9:00 A.M., Joe sees three clouds in the sky. Each hour afterwards, a new cloud appears in the sky, while each old cloud has a 40% chance of disappearing. Given that the expected number of clouds that Joe will see right after 1:00 P.M. can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers, what is $p + q$?

8. Compute the unique three-digit integer with the largest number of divisors.

9. Jo has a collection of 101 books, which she reads one each evening for 101 evenings in a predetermined order. In the morning of each day that Jo reads a book, Amy chooses a random book from Jo's collection and burns one page in it. What is the expected number of pages that Jo misses?

10. Given that x, y, z are positive real numbers satisfying $2x + y = 14 - xy$, $3y + 2z = 30 - yz$, and $z + 3x = 69 - zx$, the expression $x + y + z$ can be written as $p\sqrt{q} - r$, where p, q, r are positive integers and q is not divisible by the square of any prime. Compute $p + q + r$.

11. In rectangle $TRIG$, points A and L lie on sides TG and TR respectively such that $TA = AG$ and $TL = 2LR$. Diagonal GR intersects segments IL and IA at B and E respectively. Suppose that the area of the convex pentagon with vertices $TABLE$ is equal to 21. What is the area of $TRIG$?

12. Call a number *nice* if it can be written in the form $2^m \cdot 3^n$, where m and n are nonnegative integers. Vincent the Bug fills in a 3 by 3 grid with distinct nice numbers, such that the product of the numbers in each row and each column are the same. What is the smallest possible value of the largest number Vincent wrote?

13. Let $s(n)$ denote the sum of digits of positive integer n and define $f(n) = s(202n) - s(22n)$. Given that M is the greatest possible value of $f(n)$ for $0 < n < 350$ and N is the least value such that $f(N) = M$, compute $M + N$.

14. In triangle ABC, let M be the midpoint of BC and let E, F be points on AB, AC, respectively, such that $\angle MEF = 30°$ and $\angle MFE = 60°$. Given that $\angle A = 60°$, $AE = 10$, and $EB = 6$, compute $AB + AC$.

15. A unit cube moves on top of a 6×6 checkerboard whose squares are unit squares. Beginning in the bottom left corner, the cube is allowed to roll up or right, rolling about its bottom edges to travel from square to square, until it reaches the top right corner. Given that the side of the cube facing upwards in the beginning is also facing upwards after the cube reaches the top right corner, how many total paths are possible?