Team Round Solutions

EMCC

February 2021

1. Suppose that Yunseo wants to order a pizza that is cut into 4 identical slices. For each slice, there are 2 toppings to choose from: pineapples and apples. Each slice must have exactly one topping. How many distinct pizzas can Yunseo order? Pizzas that can be obtained by rotating one pizza are considered the same.

   Solution. The answer is $\boxed{6}$.

   We can begin by considering how many slices have pineapples. If either none or all slices have pineapples, then there is a unique pizza, giving us 2 possibilities. If there is one or three slices with pineapples, there is also a unique pizza if we account for rotations, giving us another 2 possibilities. Lastly, if there are two slices with pineapples and two slices with apples, there are another 2 possibilities: either the two slices with pineapples are adjacent or they are opposite. Thus, in total, we have $2 + 2 + 2 = 6$ distinct pizzas.

2. How many triples of distinct positive integers (E, M, C) are there such that $E = MC^2$ and $E \leq 50$?

   Solution. The answer is $\boxed{22}$.

   We can easily list all possibilities of C. When $C = 2$, $M = 1, 3, 4, 5, \ldots, 12$. When $C = 3$, $M = 1, 2, 4, 5$. When $C = 4$, $M = 1, 2, 3$. When $C = 5$, $M = 1, 2$. When $C = 6, 7$, $M = 1$. Thus, we have $11 + 4 + 3 + 2 + 2 = 22$ such triples.

3. Given that the cubic polynomial $p(x)$ has leading coefficient 1 and satisfies $p(0) = 0$, $p(1) = 1$, and $p(2) = 2$. Find $p(3)$.

   Solution. The answer is $\boxed{9}$.

   Consider polynomial $q(x) = p(x) - x$, observe that q has leading coefficient 1 and has roots $0, 1, 2$. Hence, $q(x) = x(x-1)(x-2)$ and so $p(x) = x(x-1)(x-2)+x$. Thus, $p(3) = 6+3 = 9$.

4. Olaf asks Anna to guess a two-digit number and tells her that it's a multiple of 7 with two distinct digits. Anna makes her first guess. Olaf says one digit is right but in the wrong place. Anna adjusts her guess based on Olaf's comment, but Olaf answers with the same comment again. Anna now knows what the number is. What is the sum of all the numbers that Olaf could have picked?

   Solution. The answer is $\boxed{581}$.

Olaf's answer based on the initial description could be $14, 21, 28, 35, 42, 49, 56, 63, 70, 84, 91, 98$. Each of them except 70 works. The chart shown below gives feasible ways for them to happen. Note that 70 doesn't work since no other multiples of 7 contains 7 or 0.

	Answer	1st guess	2nd guess
1	14	21	42
2	21	14	42
3	28	84	42
4	35	56	63
5	42	21	14
6	49	91	14
7	56	63	35
8	63	35	56
9	84	42	14
10	91	14	49
11	98	49	84

By summing them, we know the answer is 581.

5. Vincent the Bug draws all the diagonals of a regular hexagon with area 720, splitting it into many pieces. Compute the area of the smallest piece.

Solution. The answer is $\boxed{20}$.

It is obvious that $\triangle OST$ has the smallest area. It is also a $30-60-90$ triangle. Thus $\frac{ST}{LT} = \frac{1}{3}$. Because $[LTO] = 1/12$ of the whole hexagon, $[OST] = \frac{720}{3 \cdot 12} = 20$.

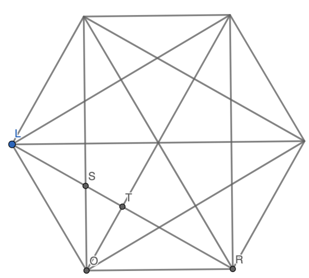

Figure 1: The hexagon of Vincent the bug

6. Given that $y - \frac{1}{y} = 7 + \frac{1}{7}$, compute the least integer greater than $y^4 + \frac{1}{y^4}$.

Solution. The answer is $\boxed{2810}$.

Squaring the given relation, we see that $y^2 + \frac{1}{y^2} - 2 = 7^2 + \frac{1}{7^2} + 2$, hence $y^2 + \frac{1}{y^2} = 7^2 + 4 + \frac{1}{7^2}$. Squaring again yields $y^4 + \frac{1}{y^4} + 2 = 7^4 + 16 + \frac{1}{7^4} + 2 \cdot 4 \cdot 7^2 + 2 + 2 \cdot 4 \cdot \frac{1}{7^2}$. Hence, $y^4 + \frac{1}{y^4} = 2401 + 16 + 392 + \frac{8}{49} + \frac{1}{2401}$. The final two terms add up to less than 1, so the answer is $2401 + 16 + 392 + 1 = 2810$.

7. At 9:00 A.M., Joe sees three clouds in the sky. Each hour afterwards, a new cloud appears in the sky, while each old cloud has a 40% chance of disappearing. Given that the expected number of clouds that Joe will see right after 1:00 P.M. can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers, what is $p+q$?

   Solution. The answer is $\boxed{2228}$.

   By linearity of expectation, the total expected number of clouds in the sky will equal the total of the expected value of each individual cloud being there. For each of the initial three clouds, the expected value each of them being there at 1:00 P.M (functionally equivalent to probability) is $\left(\frac{3}{5}\right)^4$, since to exist at the end, each cloud must not have dissipated for any of the earlier hours. Similarly, the clouds that appeared at 10:00 A.M, 11:00 A.M, 12:00 P.M, and 1:00 P.M each have an expected value of $\left(\frac{3}{5}\right)^3, \left(\frac{3}{5}\right)^2, \left(\frac{3}{5}\right), 1$ respectively. Adding everything up, we compute $3 \cdot \left(\frac{3}{5}\right)^4 + \left(\frac{3}{5}\right)^3 + \left(\frac{3}{5}\right)^2 + \left(\frac{3}{5}\right) + 1 = \frac{1603}{625}$.

8. Compute the unique three-digit integer with the largest number of divisors.

   Solution. The answer is $\boxed{840}$.

   The total number of divisors depends solely on the multiplicities of prime factors rather than their size, so we may assume that our three digits number is minimized with respect to the multiplicities involved in calculating the number of divisors, thus the exponents of prime factors will decrease as the prime factor we are looking at increases. Thus, the only factors of our number are 2, 3, 5, and 7 since if 11 is a factor then our number is at least $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 > 999$. If our number has a factor of 7, then it must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$, and we find that 840 has the most divisors out of these numbers, with a total of 32 divisors. If our number does not have a factor of 7, it only has prime factors 2, 3 and 5. If the exponent of 5 is 2, then we simply check 900, the only case, with only 27 factors. If the exponent of 5 is 0, then we may use $(\frac{m+n+2}{2})^2 \geq (n+1)(m+1) \geq 32$ through AM-GM to find that $2^n \cdot 3^m \geq 2^{n+m} > 1024$, contradiction. Thus, the exponent of 5 is 1. Let the number be $2^{n+1} \cdot 3^{m+1} \cdot 5$, then $2^m \cdot 3^n \leq 33$, while $(m+2)(n+2) \geq 16$. It is easy to test that there are no solutions to both these inequalities, hence 840 is the answer.

9. Jo has a collection of 101 books, which she reads one each evening for 101 evenings in a predetermined order. In the morning of each day that Jo reads a book, Amy chooses a random book from Jo's collection and burns one page in it. What is the expected number of pages that Jo misses?

   Solution. The answer is $\boxed{51}$.

   When Amy burns the first book, Jo will definitely miss that page. When Amy burns the second book, Jo has a $\frac{100}{101}$ probability of missing that page, since there is only a a $\frac{1}{101}$ chance that she has read it. This continues until the last day where the lastly burnt book has a $\frac{1}{101}$ chance of not been read. Thus, our total expected value is $1 + \frac{100}{101} + \frac{99}{101} + \frac{98}{101} + \cdots + \frac{1}{101} = 51$.

10. Given that x, y, z are positive real numbers satisfying $2x + y = 14 - xy$, $3y + 2z = 30 - yz$, and $z + 3x = 69 - zx$, the expression $x + y + z$ can be written as $p\sqrt{q} - r$, where p, q, r are positive integers and q is not divisible by the square of any prime. Compute $p + q + r$.

Solution. The answer is $\boxed{23}$.

$$\begin{aligned}
xy + 2x + y &= 14\\
yz + 3y + 2z &= 30\\
zx + z + 3x &= 69\\
(x+1)(y+2) &= 16\\
(y+2)(z+3) &= 36\\
(z+3)(x+1) &= 72\\
(x+1)(y+2)(z+3) &= \sqrt{16 \cdot 36 \cdot 72} = 144\sqrt{2}\\
x+1 &= \frac{144\sqrt{2}}{36} = 4\sqrt{2}\\
y+2 &= \frac{144\sqrt{2}}{72} = 2\sqrt{2}\\
z+3 &= \frac{144\sqrt{2}}{16} = 9\sqrt{2}\\
x+y+z &= 15\sqrt{2} - 6\\
p+q+r &= 15+2+6 = 23
\end{aligned}$$

11. In rectangle $TRIG$, points A and L lie on sides TG and TR respectively such that $TA = AG$ and $TL = 2LR$. Diagonal GR intersects segments IL and IA at B and E respectively. Suppose that the area of the convex pentagon with vertices $TABLE$ is equal to 21. What is the area of $TRIG$?

    Solution. The answer is $\boxed{56}$.

    Let $[TRIG] = s$. We have $[IRG] = s/2$, $[IGA] = s/4$ and $[IRL] = s/6$. Note that the triangle RLE is similar to triangle GIE with side ratio $LR/IG = LR/TR = 1/3$, from which it follows that $RE/EG = 1/3$ or $RE/RG = 1/4$. Likewise, triangle GAB is similar to triangle RIB with side ratio $1/2$, and so $GB/BR = 1/2$ or $GB/RG = 1/3$. It follows that $GB : BE : ER = 4 : 5 : 3$ and $[IBE] = \frac{5}{12}[IRG] = \frac{5}{24}s$. Consequently, we have $21 = [TABLE] = [TRIG] - [IGA] - [IRL] - [IBE] = \frac{3}{8}s$ and $s = 56$.

12. Call a number *nice* if it can be written in the form $2^m \cdot 3^n$, where m and n are nonnegative integers. Vincent the Bug fills in a 3 by 3 grid with distinct nice numbers, such that the product of the numbers in each row and each column are the same. What is the smallest possible value of the largest number Vincent wrote?

    Solution. The answer is $\boxed{24}$.

    Note that 24 works in the configuration above: There are 10 nice numbers less than 24: $1, 2, 3, 4, 6, 8, 9, 12, 16, 18$. By eliminating one of the numbers, we know the final product is a cube. The only choice is 12. Now we look at the product of threes. We have $0, 0, 1, 0, 1, 0, 2, 0, 2$. We need to arrange them into 3×3 arrays so that column sums and row sums are the same. Because this sum is 2 and the two $1s$ can not appear in the same row and column at the same time, we cannot only use numbers strictly under 24.

18	16	1
8	3	12
2	6	24

13. Let $s(n)$ denote the sum of digits of positive integer n and define $f(n) = s(202n) - s(22n)$. Given that M is the greatest possible value of $f(n)$ for $0 < n < 350$ and N is the least value such that $f(N) = M$, compute $M + N$.

    Solution. The answer is $\boxed{64}$.

    Let $2n$ be a three-digit number $\overline{abc}$ (here, some of a, b, c could be 0). Observe that $s(a+b) = s(a) + s(b) - 9c$, where c is the number of carries when adding a and b. Then, note that $22n = \overline{abc0} + \overline{abc}$, so its sum of digits is at least $a + b + c + 0 + a + b + c - 27$, since $c + 0$ never carries. If equality held, then every other digit must carry. So, $a = 9$, but $2n \leq 700$, contradiction. Hence, $M \leq 2a + 2b + 2c - (2a + 2b + 2c - 18)$, as $s(202n) \leq 2a + 2b + 2c$.

    Equality holds here when $\overline{abc0} + \overline{abc}$ has two carries. If $a \geq 1$, then $n \geq 50$. Otherwise, if $a = 0$, then $\overline{bc0} + \overline{bc}$ has two carries. Thus, $b + c \geq 10$ and $b \geq 9$ (otherwise, either the tens or hundreds addition would not carry). Hence the minimal even value of $\overline{bc}$ in this case is 92, implying that the minimal value of n is 46, which indeed yields $f(46) = s(9292) - s(1012) = 18$. Hence, $M = 18$ and $N = 46$, so $M + N = 64$.

14. In triangle ABC, let M be the midpoint of BC and let E, F be points on AB, AC, respectively, such that $\angle MEF = 30°$ and $\angle MFE = 60°$. Given that $\angle A = 60°$, $AE = 10$, and $EB = 6$, compute $AB + AC$.

    Solution. The answer is $\boxed{42}$.

    Reflect F over M to F'. Then, observe that FEF' is an equilateral triangle. In addition, since M is the midpoint of BC, $BM = CM$ and $FM = F'M$, so $BF'CF$ is a parallelogram. Now, observe that $\angle EBF' = \angle ABF' = 180° - \angle BAC = 120° = 180° - \angle EFF'$ so quadrilateral $EFF'B$ is cyclic. Thus, $\angle BFA = \angle FBF' = \angle FEF' = 60°$, implying that AFB is equilateral.

    By Ptolemy's Theorem, $BF \cdot EF' = EB \cdot FF' + BF' \cdot EF \implies BF = BE + BF'$ as $EF'F$ is equilateral. Hence, $CF = BF' = BF - BE = AB - BE = AE$. Thus, $CF = 10$ and $AF = AB = 16$, and the answer is $16 + 10 + 16 = 42$.

15. A unit cube moves on top of a 6×6 checkerboard whose squares are unit squares. Beginning in the bottom left corner, the cube is allowed to roll up or right, rolling about its bottom edges to travel from square to square, until it reaches the top right corner. Given that the side of the cube facing upwards in the beginning is also facing upwards after the cube reaches the top right corner, how many total paths are possible?

    Solution. The answer is $\boxed{26}$.

One key observation to solving this problem is that since the cube cannot move left or down, before the cube can face up again on another square, it must first face down. By the same reasoning, before the cube can face down on another square, it must first rise to face up. Using this reasoning, we can divide the cube's path into a combination of sequences where it changes from facing up to down, or facing down to up.

Now let us break down what are the requirements for such a sequence. Starting facing up, the first move is either up or right. Without loss of generality, assume that the first move is right, after which the cube will be facing right. Next, the cube will move up k times, where $k \geq 0$. After these k moves, the cube will still be facing to the right. Finally, the cube is only able to move right one more time before it is facing downwards. So the cube moved right twice and up k times for some $k \geq 0$. Similarly, if the cube had moved up first, it would have moved up twice and right k times before facing down.

Represented as a vector, if the cube starts facing up, the first time it faces down relative to its starting location is of the form $[2, k]$ or $[k, 2]$. Similarly, this also applies to starting facing down and ending facing up. Since the cube starts and ends facing up, its path can be expressed as a combination of either two or four of these sequences (since each of these sequences constitutes $2 + k$ moves, six sequences would exceed the ten needed to travel to the opposite corner of the board).

So it suffices to express $[5, 5]$ as the ordered sum of either two or four of these vectors. There are 2 ways to order the vectors $\{[2, 3], [3, 2]\}$ and 24 ways to order the vectors $\{[0, 2], [1, 2], [2, 1], [2, 0]\}$, accounting for $\boxed{26}$ total paths.