Exeter Math Club Competition
January 25, 2020

Table of Contents

Organizing Acknowledgments

- **Tournament Directors** Benjamin Wright, Sanath Govindarajan
- **Tournament Supervisor** Zuming Feng
- **System Administrator and Webmaster** Sanath Govindarajan
- **Problem Committee** Adam Bertelli, Benjamin Wright, Kevin Cong, Thomas Guo
- **Problem Committee Assistants** William Park, James Wang
- **Contest Editors** Chris Jeuell
- **Solution Writers** Adam Bertelli, Benjamin Wright, Thomas Guo, Yuan "Yannick" Yao
- **Solution Reviewers** Zhuoqun "Alex" Song, Yuan "Yannick" Yao,
- **Problem Contributors** Adam Bertelli, Benjamin Wright, Victor Luo, Thomas Guo, Kevin Cong, Ivan Borsenco, Brian Liu, Zuming Feng, Jinpyo Hong, James Wang, Sanath Govindarajan, Eric Yang, Neil Chowdhury
- **Treasurer** Sanath Govindarajan
- **Publicity** Narmana Vale, Yunseo Choi
- **Facilities Coordinator** Connie Simmons
- **Primary Tournament Sponsor** We would like to thank Jane Street Capital for their generous support of this competition.

Jane Street

- **Tournament Sponsors** We would also like to thank the Phillips Exeter Academy Math, History, and Religion Departments for their classroom space.

Contest Day Acknowledgments

- **Tournament Directors** Victor Luo, Adam Bertelli
- **Food Czar** Benjamin Wright
- **Registration** Jiaying “Lucy” Cai, Zac Feng, Maxwell Wang, Sanath Govindarajan, Thomas Guo
- **Setup Crew** Yunseo Choi, Kevin Cong, Jacob David, Eric Yang, Subin “Rachael” Kim
- **Tour Guides** Honglin “Lin” Zhu, Yi “Alex” Liang, Zheheng “Tony” Xiao, Maxine Park, Jocelyn Sides, Evan Chandran, Jasmine Xi
- **Proctors** Subin “Rachael” Kim, Kevin Cong, Honglin “Lin” Zhu, Jasmine Xi, Nathan Sun, Jocelyn Sides, Cyril Jazra, Sunyu “Gordon” Chi, Sophia Cho, Narmana Vale, Logan Valenti, Angela Liu, Maxine Park, Eric Yang, Yi “Alex” Liang, Manuel “Manny” Paez, Celine Tan, Evan Chandran, Yeonjae “Jeannie” Eom, Stella Shattuck, Taehoon Lee, Amy Lum, Richard Huang, Ian Rider, Jenny Yang, Ayush Noori, Alta Magruder, Emily Jetton, Valentina “Tina” Fernandez, Alexander Masoudi, Sophie Fernandez, Jason Huang, Jacob David, Thomas Yun, Isa Matsubayashi, Josh Frost
- **Runners** Zheheng “Tony” Xiao, Yunseo Choi, Jiaying “Lucy” Cai
- **Head Graders** Adam Bertelli, Ivan Borsenco
- **Grading System Managers** Sanath Govindarajan, Zhuoqun “Alex” Song, James Lin
- **Graders** Adam Bertelli, Ivan Borsenco, James Lin, Zhuoqun “Alex” Song, Daniel Whatley, Kristy Chang, Benjamin Wright, Yuan “Yannick” Yao, Thomas Guo, Maxwell Wang, William Park, Arun Wongprommoon, Brian Liu, Zac Feng, Sanath Govindarajan
- **Guts Round Graders** Kristy Chang, Daniel Whatley, Benjamin Wright, Maxwell Wang, Zac Feng, Yuan “Yannick” Yao, Brian Liu, William Park, Kevin Cong, Subin “Rachael” Kim, Yeonjae “Jeannie” Eom, Evan Chandran, Arun Wongprommoon, Yi “Alex” Liang, Eric Yang
- **Cleanup Crew** Victor Luo, Thomas Guo, Sanath Govindarajan

Chapter 1

EMC^2 2020 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. What is $20 \div 2 - 0 \times 1 + 2 \times 5$?

2. Today is Saturday, January 25, 2020. Exactly four hundred years from today, January 25, 2420, is again a Saturday. How many weekend days (Saturdays and Sundays) are in February, 2420? (January has 31 days and in year 2040, February has 29 days.)

3. Given that there are four people sitting around a circular table, and two of them stand up, what is the probability that the two of them were originally sitting next to each other?

4. What is the area of a triangle with side lengths 5, 5, and 6?

5. Six people go to OBA Noodles on Main Street. Each person has 1/2 probability to order Duck Noodle Soup, 1/3 probability to order OBA Ramen, and 1/6 probability to order Kimchi Udon Soup. What is the probability that three people get Duck Noodle Soup, two people get OBA Ramen, and one person gets Kimchi Udon Soup?

6. Among all positive integers a and b that satisfy $a^b = 64$, what is the minimum possible value of $a + b$?

7. A positive integer n is called *trivial* if its tens digit divides n. How many two-digit trivial numbers are there?

8. Triangle ABC has $AB = 5$, $BC = 13$, and $AC = 12$. Square $BCDE$ is constructed outside of the triangle. The perpendicular line from A to side DE cuts the square into two parts. What is the positive difference in their areas?

9. In an increasing arithmetic sequence, the first, third, and ninth terms form an increasing geometric sequence (in that order). Given that the first term is 5, find the sum of the first nine terms of the arithmetic sequence.

10. Square $ABCD$ has side length 1. Let points C' and D' be the reflections of points C and D over lines AB and BC, respectively. Let P be the center of square $ABCD$. What is the area of the concave quadrilateral $PD'BC'$?

11. How many four-digit palindromes are multiples of 7? (A palindrome is a number which reads the same forwards and backwards.)

12. Let A and B be positive integers such that the absolute value of the difference between the sum of the digits of A and the sum of the digits of $(A + B)$ is 14. What is the minimum possible value for B?

13. Clark writes the following set of congruences: $x \equiv a \pmod 6$, $x \equiv b \pmod{10}$, $x \equiv c \pmod{15}$, and he picks a, b, and c to be three randomly chosen integers. What is the probability that a solution for x exists?

14. Vincent the bug is crawling on the real number line starting from 2020. Each second, he may crawl from x to $x - 1$, or teleport from x to $\frac{x}{3}$. What is the least number of seconds needed for Vincent to get to 0?

15. How many positive divisors of 2020 do not also divide 1010?

16. A *bishop* is a piece in the game of chess that can move in any direction along a diagonal on which it stands. Two bishops *attack* each other if the two bishops lie on the same diagonal of a chessboard. Find the maximum number of bishops that can be placed on an 8×8 chessboard such that no two bishops attack each other.

17. Let ABC be a right triangle with hypotenuse 20 and perimeter 41. What is the area of ABC?

18. What is the remainder when $x^{19} + 2x^{18} + 3x^{17} + \cdots + 20$ is divided by $x^2 + 1$?

19. Ben splits the integers from 1 to 1000 into 50 groups of 20 consecutive integers each, starting with $\{1, 2, \ldots, 20\}$. How many of these groups contain at least one perfect square?

20. Trapezoid $ABCD$ with AB parallel to CD has $AB = 10$, $BC = 20$, $CD = 35$, and $AD = 15$. Let AD and BC intersect at P and let AC and BD intersect at Q. Line PQ intersects AB at R. What is the length of AR?

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. What is $(2+4+\cdots+20)-(1+3+\cdots+19)$?

2. Two ants start on opposite vertices of a dodecagon (12-gon). Each second, they randomly move to an adjacent vertex. What is the probability they meet after four moves?

3. How many distinct 8-letter strings can be made using 8 of the 9 letters from the words FORK and KNIFE (e.g., FORKNIFE)?

4. Let ABC be an equilateral triangle with side length 8 and let D be a point on segment BC such that $BD = 2$. Given that E is the midpoint of AD, what is the value of $CE^2 - BE^2$?

5. You have two fair six-sided dice, one labeled 1 to 6, and for the other one, each face is labeled 1, 2, 3, or 4 (not necessarily all numbers are used). Let p be the probability that when the two dice are rolled, the number on the special die is smaller than the number on the normal die. Given that $p = 1/2$, how many distinct combinations of 1, 2, 3, 4 can appear on the special die? The arrangement of the numbers on the die does not matter.

6. Let ω_1 and ω_2 be two circles with centers A and B and radii 3 and 13, respectively. Suppose $AB = 10$ and that C is the midpoint of AB. Let ℓ be a line that passes through C and is tangent to ω_1 at P. Given that ℓ intersects ω_2 at X and Y such that $XP < YP$, what is XP?

7. Let $f(x)$ be a cubic polynomial. Given that $f(1) = 13$, $f(4) = 19$, $f(7) = 7$, and $f(10) = 13$, find $f(13)$.

8. For all integers $0 \le n \le 202$ not divisible by seven, define $f(n) = \{\sqrt{7n}\}$. For what value n does $f(n)$ take its minimum value? (Note: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

9. Let ABC be a triangle with $AB = 14$ and $AC = 25$. Let the incenter of ABC be I. Let line AI intersect the circumcircle of BIC at D (different from I). Given that line DC is tangent to the circumcircle of ABC, find the area of triangle BCD.

10. Evaluate the infinite sum
$$\frac{4^2+3}{1\cdot 3\cdot 5\cdot 7} + \frac{6^2+3}{3\cdot 5\cdot 7\cdot 9} + \frac{8^2+3}{5\cdot 7\cdot 9\cdot 11} + \cdots .$$

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 60 minutes.

1. The number 2020 is very special: the sum of its digits is equal to the product of its nonzero digits. How many such four digit numbers are there? (Numbers with only one nonzero digit, like 3000, also count)

2. A locker has a combination which is a sequence of three integers between 0 and 49, inclusive. It is known that all of the numbers in the combination are even. Let the *total* of a lock combination be the sum of the three numbers. Given that the product of the numbers in the combination is 12160, what is the sum of all possible totals of the locker combination?

3. Given points $A = (0, 0)$ and $B = (0, 1)$ in the plane, the set of all points P in the plane such that triangle ABP is isosceles partitions the plane into k regions. The sum of the areas of those regions that are bounded is s. Find ks.

4. Three families sit down around a circular table, each person choosing their seat at random. One family has two members, while the other two families have three members. What is the probability that every person sits next to at least one person from a different family?

5. Jacob and Alexander are walking up an escalator in the airport. Jacob walks twice as fast as Alexander, who takes 18 steps to arrive at the top. Jacob, however, takes 27 steps to arrive at the top. How many of the upward moving escalator steps are visible at any point in time?

6. Points A, B, C, D, E lie in that order on a circle such that $AB = BC = 5$, $CD = DE = 8$, and $\angle BCD = 150^\circ$. Let AD and BE intersect at P. Find the area of quadrilateral $PBCD$.

7. Ivan has a triangle of integers with one number in the first row, two numbers in the second row, and continues up to eight numbers in the eighth row. He starts with the first 8 primes, 2 through 19, in the bottom row. Each subsequent row is filled in by writing the least common multiple of two adjacent numbers in the row directly below. For example, the second last row starts with 6, 15, 35, etc. Let P be the product of all the numbers in this triangle. Suppose that P is a multiple of a^b, where a and b are positive integers and $a > 1$. Given that b is maximized, and for this value of b, a is also maximized, find $a + b$.

8. Let $ABCD$ be a cyclic quadrilateral. Given that triangle ABD is equilateral, $\angle CBD = 15^\circ$, and $AC = 1$, what is the area of $ABCD$?

9. Let S be the set of all integers greater than 1. The function f is defined on S and each value of f is in S. Given that f is nondecreasing and $f(f(x)) = 2x$ for all x in S, find $f(100)$.

10. An origin-symmetric parallelogram P (that is, if (x, y) is in P, then so is $(-x, -y)$) lies in the coordinate plane. It is given that P has two horizontal sides, with a distance of 2020 between them, and that there is no point with integer coordinates except the origin inside P. Also, P has the maximum possible area satisfying the above conditions. The coordinates of the four vertices of P are $(a, 1010)$, $(b, 1010)$, $(-a, -1010)$, $(-b, -1010)$, where a, b are positive real numbers with $a < b$. What is b?

11. What is the remainder when $5^{200} + 5^{50} + 2$ is divided by $(5 + 1)(5^2 + 1)(5^4 + 1)$?

12. Let $f(n) = n^2 - 4096n - 2045$. What is the remainder when $f(f(f(\cdots f(2046)\cdots)))$ is divided by 2047, where the function f is applied 47 times?

13. What is the largest possible area of a triangle that lies completely within a 97-dimensional hypercube of side length 1, where its vertices are three of the vertices of the hypercube?

14. Let $N = \left\lfloor \frac{1}{61} \right\rfloor + \left\lfloor \frac{3}{61} \right\rfloor + \left\lfloor \frac{3^2}{61} \right\rfloor + \cdots + \left\lfloor \frac{3^{2019}}{61} \right\rfloor$. Given that $122N$ can be expressed as $3^a - b$, where a, b are positive integers and a is as large as possible, find $a + b$. (Note: $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x.)

15. Among all ordered triples of integers (x, y, z) that satisfy $x + y + z = 8$ and $x^3 + y^3 + z^3 = 134$, what is the maximum possible value of $|x| + |y| + |z|$?

1.4 Guts Test

Chapter 2

EMC^2 2020 Solutions

2.1 Speed Test Solutions

1. What is $20 \div 2 - 0 \times 1 + 2 \times 5$?

   Solution. The answer is $\boxed{20}$.

   $20 \div 2 - 0 \times 1 + 2 \times 5 = 10 - 0 + 10 = 20.$

2. Today is Saturday, January 25, 2020. Exactly four hundred years from today, January 25, 2420, is again a Saturday. How many weekend days (Saturdays and Sundays) are in February, 2420? (January has 31 days and in year 2040, February has 29 days.)

   Solution. The answer is $\boxed{9}$.

   Since January 25 is a Saturday, February 1, 8, 15, 22, and 29 are also Saturdays. Thus, February 2, 9, 16, and 23 are Sundays. The total number of weekend days is $5 + 4 = 9$.

3. Given that there are four people sitting around a circular table, and two of them stand up, what is the probability that the two of them were originally sitting next to each other?

   Solution. The answer is $\boxed{\frac{2}{3}}$.

   Say the people, in clockwise order, are A,B,C, and D. Then, if the pair of people who stand up were sitting next to each other, it must be (A,B), (B,C), (C,D), or (D,A). Since there are $\binom{4}{2} = 6$ ways to choose two people to stand up, the answer is $\frac{4}{6} = \frac{2}{3}$.

4. What is the area of a triangle with side lengths 5, 5, and 6?

   Solution. The answer is $\boxed{12}$.

   Let the triangle be ABC, with $AB = AC = 5, BC = 6$. If M is the midpoint of BC, then $AM \perp BC$. So, $BM = 3$ and $AM = \sqrt{5^2 - 3^2} = 4 =$. Finally, $[ABC] = \frac{6 \cdot 4}{2} = 12$.

5. Six people go to OBA Noodles on Main Street. Each person has 1/2 probability to order Duck Noodle Soup, 1/3 probability to order OBA Ramen, and 1/6 probability to order Kimchi Udon Soup. What is the probability that three people get Duck Noodle Soup, two people get OBA Ramen, and one person gets Kimchi Udon Soup?

   Solution. The answer is $\boxed{\frac{5}{36}}$.

   There are $\binom{6}{3}$ ways to choose 3 people of the 6 to get Duck Noodle Soup and $\binom{3}{2}$ ways to choose 2 people of the remaining 3 to get OBA Ramen. The last person must get Kimchi Udon Soup. This yields $20 \cdot 3 = 60$ ways to choose which people order which soup. Then, for each one of the 60 arrangements, the probability it occurs is $(\frac{1}{2})^3(\frac{1}{3})^2(\frac{1}{6})^1 = \frac{1}{432}$. Thus, the desired probability is $60 \cdot \frac{1}{432} = \frac{5}{36}$.

6. Among all positive integers a and b that satisfy $a^b = 64$, what is the minimum possible value of $a + b$?

   Solution. The answer is $\boxed{7}$.

   Since $a^b = 64$, a must be a power of 2. Since b must also be an integer, we see that (a, b) can be $(2, 6)$, $(4, 3)$, $(8, 2)$, or $(64, 1)$. The minimum possible value of $a + b$ is $4 + 3 = 7$.

7. A positive integer n is called *trivial* if its tens digit divides n. How many two-digit trivial numbers are there?

   Solution. The answer is $\boxed{32}$.

   If our number is $10A+B$, where A is the tens digit and B is the ones digit, then A divides $10A+B$ if and only if A divides B. Then, using casework on the value of A, the answer is $10+5+4+3+2+2+2+2+2 = 32$.

8. Triangle ABC has $AB = 5$, $BC = 13$, and $AC = 12$. Square $BCDE$ is constructed outside of the triangle. The perpendicular line from A to side DE cuts the square into two parts. What is the positive difference in their areas?

   Solution. The answer is $\boxed{119}$.

   Let the perpendicular line from A to side DE cut BC at K and DE at L. Then, since $\angle ABC = \angle KBA = 90^\circ - \angle BAK = \angle KAC$ and $\angle BAC = \angle BKA = \angle AKC = 90^\circ$, by AA-similarity, $\triangle BKA \sim \triangle AKC \sim \triangle BAC$. Therefore, $\frac{BK}{BA} = \frac{BA}{BC}$, so $BK = \frac{25}{13}$. Similarly, we find that $CK = \frac{144}{13}$. Since $BELK$ and $CDLK$ are rectangles, the difference in their areas is $13 \cdot \frac{144}{13} - 13 \cdot \frac{25}{13} = 119$.

9. In an increasing arithmetic sequence, the first, third, and ninth terms form an increasing geometric sequence (in that order). Given that the first term is 5, find the sum of the first nine terms of the arithmetic sequence.

   Solution. The answer is $\boxed{225}$.

   Let the nth term of the sequence be a_n, and say the common difference is d. Then, $a_3 = 5 + 2d$ and $a_9 = 5 + 8d$. Since a_1, a_3, a_9 is a geometric sequence, $(5)(5 + 8d) = (5 + 2d)^2$, or $4d^2 = 20d$. Since d is positive, $d = 5$. Thus, $a_n = 5n$, and the answer is $5(1 + 2 + \ldots + 9) = 225$.

   Remark: Since $1, 3, 9$ is a geometric sequence, one can realize that the arithmetic sequence should be $a_n = kn$ where k is constant. Then $k = 5$ because $a_1 = 5$.

10. Square $ABCD$ has side length 1. Let points C' and D' be the reflections of points C and D over lines AB and BC, respectively. Let P be the center of square $ABCD$. What is the area of the concave quadrilateral $PD'BC'$?

    Solution. The answer is $\boxed{\frac{3}{4}}$.

    Notice that $PB = PC$, $BC' = CD' = 1$, and $\angle PBC' = 135^\circ = \angle PCD'$. Therefore, $\triangle PBC' \cong \triangle PCD'$. So,

    $$[PD'BC'] = [PBC'] + [PBD'] = [PCD'] + [PBD'] = [PBCD'] = [PBC] + [BCD'],$$

    where $[F]$ is the area of figure F. But

    $$[PBC] + [BCD'] = \frac{[ABCD]}{4} + \frac{1 \cdot 1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4},$$

    so our desired answer is $\frac{3}{4}$.

11. How many four-digit palindromes are multiples of 7? (A palindrome is a number which reads the same forwards and backwards.)

Solution. The answer is $\boxed{18}$.

Call a four-digit palindrome $ABBA = 1001A + 110B$ *good* if it is divisible by 7. Then, because 7 divides 1001, a palindrome is *good* if an only if $110B$ is divisible by 7. This happens if and only if $B = 0$ or $B = 7$. So, since A ranges from 1 to 9, the number of *good* palindromes is $9 \cdot 2 = 18$.

12. Let A and B be positive integers such that the absolute value of the difference between the sum of the digits of A and the sum of the digits of $(A + B)$ is 14. What is the minimum possible value for B?

Solution. The answer is $\boxed{4}$.

Observe that the sum of digits of a number is congruent to that number modulo 9. Indeed,

$$\overline{a_n a_n - 1 \ldots a_0} = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_0 \cdot 10^0 \cong a_n + a_{n-1} + \ldots + a_0 \pmod 9,$$

where the last step follows from

$$10^k \cong 1^k \cong 1 \pmod 9.$$

So, since $14 \cong 5 \pmod 9$, we know that either $B \cong 5 \pmod 9$ or $-B \cong 5 \pmod 9$. Therefore, $B \cong 4 \pmod 9$ or $B \cong 5 \pmod 9$. So, $B \geq 4$.

Also, if $A = 99$ and $B = 4$, the sum of digits of A is 18, while the sum of digits of $A + B = 103$ is 4. Since $18 - 4 = 14$, the minimum possible value of B is 4.

13. Clark writes the following set of congruences: $x \equiv a \pmod 6$, $x \equiv b \pmod{10}$, $x \equiv c \pmod{15}$, and he picks a, b, and c to be three randomly chosen integers. What is the probability that a solution for x exists?

Solution. The answer is $\boxed{\frac{1}{30}}$.

Notice that the system of congruences

$$x \equiv a \pmod 6,$$
$$x \equiv b \pmod{10},$$
$$x \equiv c \pmod{15}$$

is equivalent to the system of congruences

$$x \equiv a_1 \pmod 2,$$
$$x \equiv a_2 \pmod 3;$$
$$x \equiv b_1 \pmod 2,$$
$$x \equiv b_2 \pmod 5;$$
$$x \equiv c_1 \pmod 3,$$
$$x \equiv c_2 \pmod 5;$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are randomly chosen integers. This is because we simply choose a_1, a_2 so that $a \equiv a_1 \pmod 2$ and $a \equiv a_2 \pmod 3$, and similar for b_1, b_2, c_1, c_2.

Then, by the Chinese Remainder Theorem, there is a solution x if and only if $a_1 = b_1$, $a_2 = c_1$, and $b_2 = c_2$. Thus, our answer - the probability that this occurs - is $(\frac{1}{2})(\frac{1}{3})(\frac{1}{5}) = \frac{1}{30}$.

14. Vincent the bug is crawling on the real number line starting from 2020. Each second, he may crawl from x to $x-1$, or teleport from x to $\frac{x}{3}$. What is the least number of seconds needed for Vincent to get to 0?

Solution. The answer is $\boxed{16}$.

Let $f(n)$ be the least number of seconds it takes Vincent to get to 0 if he starts at n. Define $s(n)$ to be the sum of digits of n in base three and $t(n)$ to be the number of digits of n in base three.

Observing that Vincent wants to get to 0, we want to have Vincent teleport whenever he can (when $\frac{n}{3}$ is an integer). This leads to:

Claim: $f(n) = s(n) + t(n) - 1$.

We prove this by induction.

Base Case: $n = 1$. Then we want to show that $f(1) = 1 + 1 - 1 = 1$, which is clear because Vincent takes at least one second, and he can use only one second by crawling from 1 to 0.

Inductive Step: Suppose the statement is true for $n = 1, 2, \ldots, k$. We claim it is true for $n = k + 1$.

If n is not a multiple of three, then Vincent cannot teleport to $\frac{n}{3}$, since then he would never again arrive at an integer. So, he must crawl to $n-1$, and by the inductive hypothesis and the facts $s(n) = s(n-1) + 1$, $t(n) = t(n-1)$, we get that $f(n) = s(n-1) + t(n-1) + 1 - 1 = s(n) + t(n) - 1$.

If n is a multiple of three, say $n = 3k$, then Vincent can either teleport or crawl.

If he teleports, he takes $s(k) + t(k)$ steps, and if he crawls, he takes $s(3k-1) + t(3k-1)$ steps.

But if

$$k = \overline{a_m a_{m-1} \ldots a_1 00 \ldots 0}$$

in base three, where $a_1 \neq 0$ and there are $t \geq 0$ zeroes after a_1,

$$s(k) = a_m + a_{m-1} + \ldots + a_1$$

while

$$s(3k-1) = a_m + a_{m-1} + \ldots + (a_1 - 1) + 1 + 1 + \ldots + 1$$

where there are t ones. So,

$$s(3k_1) - s(k) = t - 1 \geq 1 - 1 \geq 0.$$

Since $t(3k-1) \geq t(k)$, we have that $t(3k-1) + s(3k-1) \geq t(k) = s(k)$.

So, $f(n) = s(k) + t(k) = s(n) + t(3k) - 1 = s(n) + t(n)$.

This completes the induction and the proof of the claim.

Finally, 2020 is 2202211 in base three, so the answer is $10 + 7 - 1 = 16$.

Remark: Although the rigorous solution is very long, the intuition is fairly simple - whenever Vincent can teleport, he does so, because it covers a longer distance. This heuristic leads to the answer.

15. How many positive divisors of 2020 do not also divide 1010?

Solution. The answer is $\boxed{4}$.

Observe that $2020 = 2^2 \cdot 5 \cdot 101$. So, any divisor of 2020 is of the form $2^a \cdot 5^b \cdot 101^c$, where $a = 0, 1, 2$, $b = 0, 1$, $c = 0, 1$. This does not divide $1010 = 2 \cdot 5 \cdot 101$ if and only if $a = 2$. Thus, there are a total of $1 \cdot 2 \cdot 2 = 4$ divisors.

16. A *bishop* is a piece in the game of chess that can move in any direction along a diagonal on which it stands. Two bishops *attack* each other if the two bishops lie on the same diagonal of a chessboard. Find the maximum number of bishops that can be placed on an 8×8 chessboard such that no two bishops attack each other.

    Solution. The answer is $\boxed{14}$.

    Consider the chessboard as the 8×8 grid of points (x, y) such that both x and y are between 0 and 7, inclusive. Then, partition the chessboard into 15 diagonals, $x + y = k$ for $k = 0, 1, ...14$. Each diagonal can have at most 1 bishops on it, so we have at most 15 bishops.

    However, if we have 15 bishops, then the diagonals $x + y = 0$ and $x + y = 14$ must both have bishops on them, so there are bishops at $(0, 0)$ and $(7, 7)$. But these bishops are both on the diagonal $y = x$, so they attack each other.

    Therefore, there are at most 14 bishops. Furthermore, if we put bishops on $(0, 0), (0, 1), \dots (0, 6)$ and $(7, 0), (7, 1), \dots (7, 6)$, it is clear that no two of these bishops lie on a diagonal $x + y = k$ $(0 \leq k \leq 14)$ or $x - y = k$ $(-7 \leq k \leq 6)$. Thus, the answer is 14 bishops.

17. Let ABC be a right triangle with hypotenuse 20 and perimeter 41. What is the area of ABC?

    Solution. The answer is $\boxed{\frac{41}{4}}$.

    Suppose that the legs of the triangle have side lengths a and b. Then, $a + b + \sqrt{a^2 + b^2} = 41$ and $\sqrt{a^2 + b^2} = 20$. Subtracting the equations gives $a + b = 21$. So, $\frac{ab}{2} = \frac{1}{2}\left(\frac{(a+b)^2 - (a^2+b^2)}{2}\right) = \frac{21^2 - 20^2}{4} = \frac{41}{4}$

18. What is the remainder when $x^{19} + 2x^{18} + 3x^{17} + \cdots + 20$ is divided by $x^2 + 1$?

    Solution. The answer is $\boxed{10x + 10}$.

    Note that $(x^2 + 1)(x^2 - 1) = x^4 - 1$, so $x^n - x^{n-4}$ is divisible by $x^2 + 1$. Therefore, the remainder when

    $$x^{19} + 2x^{18} + 3x^{17} + \cdots + 20$$

    is divided by $x^2 + 1$ is equivalent to the remainder when

    $$(1 + 5 + 9 + 13 + 17)x^3 + (2 + 6 + 10 + 14 + 18)x^2 + (3 + 7 + 11 + 15 + 19)x + (4 + 8 + 12 + 16 + 20)$$

    is divided by $x^2 + 1$. Noting that $x^2 + 1$ and $x^3 + x$ are both multiples of $x^2 + 1$, we can rewrite this expression as

    $$(1 + 5 + 9 + 13 + 17)(x^3 + x) + 2 \cdot 5x + (2 + 6 + 10 + 14 + 18)(x^2 + 1) + 2 \cdot 5,$$

    which has remainder $10x + 10$ when divided by $x^2 + 1$.

19. Ben splits the integers from 1 to 1000 into 50 groups of 20 consecutive integers each, starting with $\{1, 2, \dots, 20\}$. How many of these groups contain at least one perfect square?

    Solution. The answer is $\boxed{26}$.

    Label the group $\{20k - 19, 20k - 18, \dots, 20k\}$ as group G_k.

Then, observe that $(k+1)^2 - k^2 \leq 20$ for $k \leq 9$, so each group G_1, G_2, G_3, G_4, G_5 contain a perfect square (otherwise some consecutive perfect squares would differ by more than 20.

Further, $(k+1)^2 - k^2 > 20$ for $k \geq 10$, and all squares at least 11^2 are in group G_k for some $k \geq 6$. Therefore, all of the squares $11^2, 12^2, \ldots 31^2$ are in different groups G_i, $i \geq 6$. This gives 21 more groups containing at least one perfect square.

So, our answer is $5 + 21 = 26$.

20. Trapezoid $ABCD$ with AB parallel to CD has $AB = 10$, $BC = 20$, $CD = 35$, and $AD = 15$. Let AD and BC intersect at P and let AC and BD intersect at Q. Line PQ intersects AB at R. What is the length of AR?

Solution. The answer is $\boxed{5}$.

Suppose that line PQ intersects CD at S. Then, since $AB \parallel CD$, there is a dilation centered at P taking DC to AB, so $\triangle PAB \sim \triangle PDC$. Furthermore, by Ceva's Theorem,

$$\frac{DS}{SC} \cdot \frac{CB}{BP} \cdot \frac{PA}{AD} = 1$$

.

Therefore,

$$\begin{aligned} \frac{PC}{PB} &= \frac{PD}{PA} \implies \\ \frac{PC - PB}{PB} &= \frac{PD - PA}{PA} \implies \\ \frac{CB}{BP} &= \frac{AD}{AP} \implies \\ \frac{DS}{SC} &= 1 \end{aligned}$$

.

Furthermore, since P, R, and S are collinear and $R \in AB, S \in DC$, the dilation centered at P taking DC to AB also takes S to R. Thus, because S is the midpoint of DC, R is the midpoint of AB. Therefore $AR = \frac{AB}{2} = 5$.

2.2 Accuracy Test Solutions

1. What is $(2+4+\cdots+20)-(1+3+\cdots+19)$?

   Solution. The answer is $\boxed{10}$.

   $(2+4+\cdots+20)-(1+3+\cdots+19)=(2-1)+(4-3)+\cdots+(20-19)=1\cdot 10=10$

2. Two ants start on opposite vertices of a dodecagon (12-gon). Each second, they randomly move to an adjacent vertex. What is the probability they meet after four moves?

   Solution. The answer is $\boxed{\frac{1}{16} \text{ or } \frac{3}{64}}$.

   Each move, the two ants can either move closer to each other by two edges (units), away from each other by two units, or stay at the same distance from each other. Since they begin with a distance of six units apart, there must be three moves of getting closer to each other and one move of staying. When the ants are six units apart, there is a $\frac{1}{2}$ probability of moving closer, and a $\frac{1}{2}$ probability of staying at the same distance. When they are fewer than six units apart, there is a $\frac{1}{4}$ probability of moving closer, and a $\frac{1}{2}$ probability of staying at the same distance. Since the ants can stay during any one of the four moves, the total probability would be $4\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$.

   Remark. This problem was ambiguously worded that one could interpret the condition to be that the ants must meet, for the first time, after the forth move. In this interpretation, the last move cannot be a stay. Thus, the probability would be $3\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{64}$.

   We accepted both answers and apologize for the poor wording.

3. How many distinct 8-letter strings can be made using 8 of the 9 letters from the words FORK and KNIFE (e.g., FORKNIFE)?

   Solution. The answer is $\boxed{90720}$.

   There are two cases. First, if only one of F and K is used twice and every other letter is used once, there are $2\cdot\binom{8}{2}\cdot 6! = 8!$ strings. Second, if both F and K are used twice, then one of the remaining letters is unused, giving $5\cdot\binom{8}{2}\cdot\binom{6}{2}\cdot 4! = \frac{5}{4}\cdot 8!$ strings. This gives a total of $\frac{9}{4}\cdot 40320 = 90720$ distinct strings.

4. Let ABC be an equilateral triangle with side length 8 and let D be a point on segment BC such that $BD=2$. Given that E is the midpoint of AD, what is the value of CE^2-BE^2?

   Solution. The answer is $\boxed{16}$.

   Let M and N be the feet of perpendiculars from A and E to segment BC, respectively. Then, $DM=2$. Since E is the midpoint of AD, N is the midpoint of DM. Thus, $CN=5$ and $BN=3$. Therefore, $CE^2-BE^2=(CE^2-EN^2)-(BE^2-EN^2)=CN^2-BN^2=16$.

5. You have two fair six-sided dice, one labeled 1 to 6, and for the other one, each face is labeled 1, 2, 3, or 4 (not necessarily all numbers are used). Let p be the probability that when the two dice are rolled, the number on the special die is smaller than the number on the normal die. Given that $p=1/2$, how many distinct combinations of 1, 2, 3, 4 can appear on the special die? The arrangement of the numbers on the die does not matter.

Solution. The answer is $\boxed{7}$.

Let a_1, a_2, a_3, a_4 be the number of 1's, 2's, 3's, and 4's on the special die, respectively. Then, $a_1 + a_2 + a_3 + a_4 = 6$. The condition that $p = 1/2$ yields the equation $\frac{1}{2} = \frac{a_1}{6} \cdot \frac{5}{6} + \frac{a_2}{6} \cdot \frac{4}{6} + \frac{a_3}{6} \cdot \frac{3}{6} + \frac{a_4}{6} \cdot \frac{2}{6}$, which is $5a_1 + 4a_2 + 3a_3 + 2a_4 = 18$. Subtracting this by two times the first equation, we get that $3a_1 + 2a_2 + a_3 = 6$. By solving for non-negative integer solutions to this and plugging back into the first equation to get a_4, we get that all possible sets of (a_1, a_2, a_3, a_4) are $(2, 0, 0, 4), (1, 1, 1, 3), (1, 0, 3, 2), (0, 3, 0, 3), (0, 2, 2, 2), (0, 1, 4, 1), (0, 0, 6, 0)$, giving us 7 arrangements.

6. Let ω_1 and ω_2 be two circles with centers A and B and radii 3 and 13, respectively. Suppose $AB = 10$ and that C is the midpoint of AB. Let ℓ be a line that passes through C and is tangent to ω_1 at P. Given that ℓ intersects ω_2 at X and Y such that $XP < YP$, what is XP?

   Solution. The answer is $\boxed{4\sqrt{10} - 8}$.

   First notice that APC is a 3-4-5 triangle. Let M be the midpoint of XY. Since B is the center of ω_2, BM is perpendicular to XY. Since $BC = 5$, $BM = AP = 3$. Since XB is a radius, we can find that $XM = \sqrt{13^2 - 3^2} = 4\sqrt{10}$. Thus, $XP = XM - MP = 4\sqrt{10} - 2CP = 4\sqrt{10} - 8$.

7. Let $f(x)$ be a cubic polynomial. Given that $f(1) = 13$, $f(4) = 19$, $f(7) = 7$, and $f(10) = 13$, find $f(13)$.

   Solution. The answer is $\boxed{73}$.

   Consider the cubic polynomial $g(x) = f(x+7) = ax^3 + bx^2 + cx + d$, and let $h(x) = g(x) + g(-x)$. We see that $h(x) = 2bx^2 + 2d$, and we have

$$\begin{aligned} h(0) &= 2d = 2f(7) = 14 \\ h(3) &= 18b + 2d = f(4) + f(10) = 32 \\ h(6) &= 72b + 2d = f(1) + f(13). \end{aligned}$$

   From the first two equations, it can be easily found that $b = 1$ and $d = 7$. Substituting into the third equation, we get that $f(13) = 73$.

8. For all integers $0 \leq n \leq 202$ not divisible by seven, define $f(n) = \{\sqrt{7n}\}$. For what value n does $f(n)$ take its minimum value? (Note: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

   Solution. The answer is $\boxed{156}$.

   Suppose $\lfloor 7n \rfloor = k$. Let $a = 7n - k^2$. Since n is not divisible by 7, it is not a perfect square, and $a > 0$. Then,

$$\{7n\} = \sqrt{k^2 + a} - k = \frac{a}{\sqrt{k^2 + a} + k},$$

   and

$$\frac{a}{2k+1} < \frac{a}{\sqrt{k^2 + a} + k} < \frac{a}{2k}.$$

   To minimize $f(n) = \{\sqrt{7n}\}$, we wish to minimize a and maximize k. Since $-k^2$ can only be $3, 5, 6$ mod 7, $a \geq 3$. When $a = 3$, the largest k where n is not divisible by 7 is 33 and $n = 156$. Then, $\{7n\} < \frac{1}{22}$.

If $a \geq 5$,

$$\{x\} > \frac{5}{2 \cdot 37 + 1} > \frac{1}{22}.$$

Thus, $n = 156$ gives the minimum $f(n)$.

9. Let ABC be a triangle with $AB = 14$ and $AC = 25$. Let the incenter of ABC be I. Let line AI intersect the circumcircle of BIC at D (different from I). Given that line DC is tangent to the circumcircle of ABC, find the area of triangle BCD.

   Solution. The answer is $\boxed{300}$.

   Let E be the intersection of line DC and the circumcircle of ABC. By the tangent condition, we have that $\angle DCE = \angle CAE$. Since AI bisects $\angle BAC$, $\angle CAE = \angle BAE = \angle BCE$. Then, $\angle BCD = 2\angle DCE = \angle BID = \angle BAE + \angle ABI$. Thus, $\angle ABI = \angle DCE = \angle CBI = \angle IDC$. Therefore, we have that $CA = CB = CD = 25$ ($\angle BAC = \angle ABC = 2\angle CAD = 2\angle CDA$), and since $\angle BAD = \angle ADC$, CD is parallel to AB. Thus, the height of triangle ABC from C to side AB is equal to the height of triangle BCD from B to side CD. Also, notice that triangle ABC is two 7-24-25 triangles put together. Therefore,

$$[BCD] = \frac{25}{14}[ABC] = \frac{25}{14} \cdot \frac{1}{2} \cdot 14 \cdot 24 = 300.$$

10. Evaluate the infinite sum

$$\frac{4^2+3}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{6^2+3}{3 \cdot 5 \cdot 7 \cdot 9} + \frac{8^2+3}{5 \cdot 7 \cdot 9 \cdot 11} + \cdots.$$

   Solution. The answer is $\boxed{\frac{3}{10}}$.

   We wish to use partial fractions to express the term $\frac{(2a)^2+3}{(2a-3)(2a-1)(2a+1)(2a+3)}$. Suppose

$$\frac{4a^2+3}{(2a-3)(2a-1)(2a+1)(2a+3)} = \frac{x}{2a-3} + \frac{y}{2a-1} + \frac{z}{2a+1} + \frac{w}{2a+3}.$$

   Then,

$$x(2a+3)(4a^2-1) + w(2a-3)(4a^2-1) + y(2a+1)(4a^2-9) + z(2a-1)(4a^2-9) = 4a^2+3.$$

   Since there are no cubic terms, we can let $w = -x$, $z = -y$. Then,

$$6x(4a^2-1) + 2y(4a^2-9) = 4a^2+3.$$

   We find that $x = \frac{1}{4}, y = -\frac{1}{4}$. Thus, the infinite sum now becomes

$$\begin{aligned}
&\sum_{a=2}^{\infty} \frac{1}{4}\left(\frac{1}{2a-3} + \frac{-1}{2a-1} + \frac{1}{2a+1} + \frac{-1}{2a+3}\right) \\
&= \frac{1}{4}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} \cdots\right) \\
&= \frac{1}{4}\left(\frac{1}{1} + \frac{1}{5}\right) \\
&= \frac{3}{10}.
\end{aligned}$$

2.3 Team Test Solutions

1. The number 2020 is very special: the sum of its digits is equal to the product of its nonzero digits. How many such four digit numbers are there? (Numbers with only one nonzero digit, like 3000, also count)

   Solution. The answer is $\boxed{42}$.

2. A locker has a combination which is a sequence of three integers between 0 and 49, inclusive. It is known that all of the numbers in the combination are even. Let the *total* of a lock combination be the sum of the three numbers. Given that the product of the numbers in the combination is 12160, what is the sum of all possible totals of the locker combination?

   Solution. The answer is $\boxed{240}$.

3. Given points $A = (0, 0)$ and $B = (0, 1)$ in the plane, the set of all points P in the plane such that triangle ABP is isosceles partitions the plane into k regions. The sum of the areas of those regions that are bounded is s. Find ks.

   Solution. The answer is $\boxed{8\pi + 3\sqrt{3}}$.

4. Three families sit down around a circular table, each person choosing their seat at random. One family has two members, while the other two families have three members. What is the probability that every person sits next to at least one person from a different family?

   Solution. The answer is $\boxed{\frac{53}{70}}$.

5. Jacob and Alexander are walking up an escalator in the airport. Jacob walks twice as fast as Alexander, who takes 18 steps to arrive at the top. Jacob, however, takes 27 steps to arrive at the top. How many of the upward moving escalator steps are visible at any point in time?

   Solution. The answer is $\boxed{54}$.

6. Points A, B, C, D, E lie in that order on a circle such that $AB = BC = 5$, $CD = DE = 8$, and $\angle BCD = 150^\circ$. Let AD and BE intersect at P. Find the area of quadrilateral $PBCD$.

   Solution. The answer is $\boxed{20}$.

7. Ivan has a triangle of integers with one number in the first row, two numbers in the second row, and continues up to eight numbers in the eighth row. He starts with the first 8 primes, 2 through 19, in the bottom row. Each subsequent row is filled in by writing the least common multiple of two adjacent numbers in the row directly below. For example, the second last row starts with 6, 15, 35, etc. Let P be the product of all the numbers in this triangle. Suppose that P is a multiple of a^b, where a and b are positive integers and $a > 1$. Given that b is maximized, and for this value of b, a is also maximized, find $a + b$.

   Solution. The answer is $\boxed{97}$.

8. Let $ABCD$ be a cyclic quadrilateral. Given that triangle ABD is equilateral, $\angle CBD = 15°$, and $AC = 1$, what is the area of $ABCD$?

   Solution. The answer is $\boxed{\frac{\sqrt{3}}{4}}$.

9. Let S be the set of all integers greater than 1. The function f is defined on S and each value of f is in S. Given that f is nondecreasing and $f(f(x)) = 2x$ for all x in S, find $f(100)$.

   Solution. The answer is $\boxed{136}$.

10. An origin-symmetric parallelogram P (that is, if (x, y) is in P, then so is $(-x, -y)$) lies in the coordinate plane. It is given that P has two horizontal sides, with a distance of 2020 between them, and that there is no point with integer coordinates except the origin inside P. Also, P has the maximum possible area satisfying the above conditions. The coordinates of the four vertices of P are $(a, 1010)$, $(b, 1010)$, $(-a, -1010)$, $(-b, -1010)$, where a, b are positive real numbers with $a < b$. What is b?

    Solution. The answer is $\boxed{\frac{1011}{1010}}$.

11. What is the remainder when $5^{200} + 5^{50} + 2$ is divided by $(5 + 1)(5^2 + 1)(5^4 + 1)$?

    Solution. The answer is $\boxed{28}$.

12. Let $f(n) = n^2 - 4096n - 2045$. What is the remainder when $f(f(f(\cdots f(2046) \cdots)))$ is divided by 2047, where the function f is applied 47 times?

    Solution. The answer is $\boxed{129}$.

13. What is the largest possible area of a triangle that lies completely within a 97-dimensional hypercube of side length 1, where its vertices are three of the vertices of the hypercube?

Solution. The answer is $\boxed{28}$.

14. Let $N = \left\lfloor \frac{1}{61} \right\rfloor + \left\lfloor \frac{3}{61} \right\rfloor + \left\lfloor \frac{3^2}{61} \right\rfloor + \cdots + \left\lfloor \frac{3^{2019}}{61} \right\rfloor$. Given that $122N$ can be expressed as $3^a - b$, where a, b are positive integers and a is as large as possible, find $a + b$. (Note: $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x.)

    Solution. The answer is $\boxed{125241}$.

15. Among all ordered triples of integers (x, y, z) that satisfy $x + y + z = 8$ and $x^3 + y^3 + z^3 = 134$, what is the maximum possible value of $|x| + |y| + |z|$?

    Solution. The answer is $\boxed{34}$.