Exeter Math Club Competition
January 29, 2022

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Jane Street

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- **The Bug** Vincent

Chapter 1

EMC^2 2022 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Compute $(2+0)(2+2)(2+0)(2+2)$.

2. Given that 25% of x is 120% of 30% of 200, find x.

3. Jacob had taken a nap. Given that he fell asleep at 4:30 PM and woke up at 6:23 PM later that same day, for how many minutes was he asleep?

4. Kevin is painting a cardboard cube with side length 12 meters. Given that he needs exactly one can of paint to cover the surface of a rectangular prism that is 2 meters long, 3 meters wide, and 6 meters tall, how many cans of paint does he need to paint the surface of his cube?

5. How many nonzero digits does $200 \times 25 \times 8 \times 125 \times 3$ have?

6. Given two real numbers x and y, define $x \# y = xy + 7x - y$. Compute the absolute value of $0 \# (1 \# (2 \# (3 \# 4)))$.

7. A 3-by-5 rectangle is partitioned into several squares of integer side length. What is the fewest number of such squares? Squares in this partition must not overlap and must be contained within the rectangle.

8. Points A and B lie in the plane so that $AB = 24$. Given that C is the midpoint of AB, D is the midpoint of BC, E is the midpoint of AD, and F is the midpoint of BD, find the length of segment EF.

9. Vincent the Bug and Achyuta the Anteater are climbing an infinitely tall vertical bamboo stalk. Achyuta begins at the bottom of the stalk and climbs up at a rate of 5 inches per second, while Vincent begins somewhere along the length of the stalk and climbs up at a rate of 3 inches per second. After climbing for t seconds, Achyuta is half as high above the ground as Vincent. Given that Achyuta catches up to Vincent after another 160 seconds, compute t.

10. What is the minimum possible value of $|x - 2022| + |x - 20|$ over all real numbers x?

11. Let $ABCD$ be a rectangle. Lines ℓ_1 and ℓ_2 divide $ABCD$ into four regions such that ℓ_1 is parallel to AB and line ℓ_2 is parallel to AD. Given that three of the regions have area 6, 8, and 12, compute the sum of all possible areas of the fourth region.

12. A *diverse* number is a positive integer that has two or more distinct prime factors. How many diverse numbers are less than 50?

13. Let x, y, and z be real numbers so that $(x+y)(y+z) = 36$ and $(x+z)(x+y) = 4$. Compute $y^2 - x^2$.

14. What is the remainder when $1^{10} + 3^{10} + 7^{10}$ is divided by 58?

15. Let $A = (0, 1)$, $B = (3, 5)$, $C = (1, 4)$, and $D = (3, 4)$ be four points in the plane. Find the minimum possible value of $AP + BP + CP + DP$ over all points P in the plane.

16. In trapezoid $ABCD$, points E and F lie on sides BC and AD, respectively, such that $AB \parallel CD \parallel EF$. Given that $AB = 3$, $EF = 5$, and $CD = 6$, the ratio $\frac{[ABEF]}{[CDFE]}$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$. (Note: $[\mathcal{F}]$ denotes the area of $\mathcal{F}$.)

17. For sets X and Y, let $|X \cap Y|$ denote the number of elements in both X and Y and $|X \cup Y|$ denote the number of elements in at least one of X or Y. How many ordered pairs of subsets (A, B) of $\{1, 2, 3, \ldots, 8\}$ are there such that $|A \cap B| = 2$ and $|A \cup B| = 5$?

18. A *tetromino* is a polygon composed of four unit squares connected orthogonally (that is, sharing a edge). A *tri-tetromino* is a polygon formed by three orthogonally connected tetrominoes. What is the maximum possible perimeter of a tri-tetromino?

19. The numbers from 1 through 2022, inclusive, are written on a whiteboard. Every day, Hermione erases two numbers a and b and replaces them with $ab + a + b$. After some number of days, there is only one number N remaining on the whiteboard. If N has k trailing nines in its decimal representation, what is the maximum possible value of k?

20. Evaluate $5(2^2 + 3^2) + 7(3^2 + 4^2) + 9(4^2 + 5^2) + \cdots + 199(99^2 + 100^2)$.

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. At a certain point in time, 20% of seniors, 30% of juniors, and 50% of sophomores at a school had a cold. If the number of sick students was the same for each grade, the fraction of sick students across all three grades can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

2. The average score on Mr. Feng's recent test is a 63 out of 100. After two students drop out of the class, the average score of the remaining students on that test is now a 72. What is the maximum number of students that could initially have been in Mr. Feng's class? (All of the scores on the test are integers between 0 and 100, inclusive.)

3. Madeline is climbing Celeste Mountain. She starts at $(0, 0)$ on the coordinate plane and wants to reach the summit at $(7, 4)$. Every hour, she moves either 1 unit up or 1 unit to the right. A strawberry is located at each of $(1, 1)$ and $(4, 3)$. How many paths can Madeline take so that she encounters exactly one strawberry?

4. Let E be a point on side AD of rectangle $ABCD$. Given that $AB = 3$, $AE = 4$, and $\angle BEC = \angle CED$, the length of segment CE can be written as $\sqrt{a}$ for some positive integer a. Find a.

5. Lucy has some spare change. If she were to convert it into quarters and pennies, the minimum number of coins she would need is 66. If she were to convert it into dimes and pennies, the minimum number of coins she would need is 147. How much money, in cents, does Lucy have?

6. For how many positive integers x does there exist a triangle with altitudes of length 20, 22, and x?

7. Compute the number of positive integers x for which $\frac{x^{20}}{x+22}$ is an integer.

8. Vincent the Bug is crawling along an octagonal prism. He starts on a fixed vertex A, visits all other vertices exactly once by traveling along the edges, and returns to A. Find the number of paths Vincent could have taken.

9. Point U is chosen inside square $ALEX$ so that $\angle AUL = 90^\circ$. Given that $UL = 56$ and $UE = 65$, what is the sum of all possible values for the area of square $ALEX$?

10. Miranda has prepared 8 outfits, no two of which are the same quality. She asks her intern Andrea to order these outfits for the new runway show. Andrea first randomly orders the outfits in a list. She then starts removing outfits according to the following method: she chooses a random outfit which is both immediately preceded and immediately succeeded by a better outfit and then removes it. Andrea repeats this process until there are no outfits that can be removed. Given that the expected number of outfits in the final routine can be written as $\frac{a}{b}$ for some relatively prime positive integers a and b, find $a+b$.

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 60 minutes.

1. Compute $1+3+6+10+15+21+28+36+45+55$.

2. Given that a, b, and c are positive integers such that $a+b=9$ and $bc=30$, find the minimum possible value of $a+c$.

3. Points X and Y lie outside regular pentagon $ABCDE$ such that ABX and DEY are equilateral triangles. Find the degree measure of $\angle XCY$.

4. Let N be the product of the positive integer divisors of $8!$, including itself. The largest integer power of 2 that divides N is 2^k. Compute k.

5. Let $= A(-20,22)$, $B=(k,0)$, and $C=(202,2)$ be points on the coordinate plane. Given that $\angle ABC = 90^\circ$, find the sum of all possible values of k.

6. Tej is typing a string of Ls and Os that consists of exactly 7 Ls and 4 Os. How many different strings can he type that do not contain the substring 'LOL' anywhere? A *substring* is a sequence of consecutive letters contained within the original string.

7. How many ordered triples of integers (a,b,c) satisfy both $a+b-c=12$ and $a^2+b^2-c^2=24$?

8. For how many three-digit base-7 numbers $\overline{ABC}_7$ does $\overline{ABC}_7$ divide $\overline{ABC}_{10}$? (Note: $\overline{ABC}_D$ refers to the number whose digits in base D are, from left to right, A, B, and C; for example, $\overline{123}_4$ equals 27 in base ten).

9. Natasha is sitting on one of the 35 squares of a 5-by-7 grid of squares. Wanda wants to walk through every square on the board exactly once except the one Natasha is on, starting and ending on any 2 squares she chooses, such that from any square she can only go to an adjacent square (two squares are adjacent if they share an edge). How many squares can Natasha choose to sit on such that Wanda cannot go on her walk?

10. In triangle ABC, $AB=13$, $BC=14$, and $CA=15$. Point P lies inside ABC and points D,E, and F lie on sides BC, CA, and AB, respectively, so that $PD \perp BC$, $PE \perp CA$, and $PF \perp AB$. Given that PD, PE, and PF are all integers, find the sum of all possible distinct values of $PD \cdot PE \cdot PF$.

11. A *palindrome* is a positive integer which is the same when read forwards or backwards. Find the sum of the two smallest palindromes that are multiples of 137.

12. Let $P(x)=x^2+px+q$ be a quadratic polynomial with positive integer coefficients. Compute the least possible value of p such that 220 divides p and the equation $P(x^3)=P(x)$ has at least four distinct integer solutions.

13. Everyone at a math club is either a truth-teller, a liar, or a piggybacker. A truth-teller always tells the truth, a liar always lies, and a piggybacker will answer in the style of the previous person who spoke (i.e., if the person before told the truth, they will tell the truth, and if the person before lied, then they will lie). If a piggybacker is the first one to talk, they will randomly either tell the truth or lie. Four seniors in the math club were interviewed and here was their conversation:

   Neil: There are two liars among us.

   Lucy: Neil is a piggybacker.

Kevin: Excluding me, there are more truth-tellers than liars here.

Neil: Actually, there are more liars than truth-tellers if we exclude Kevin.

Jacob: One plus one equals three.

Define the base-4 number $M = \overline{NLKJ}_4$, where each digit is 1 for a truth-teller, 2 for a piggybacker, and 3 for a liar (N corresponds to Neil, L to Lucy, K corresponds to Kevin, and J corresponds to Jacob). What is the sum of all possible values of M, expressed in base 10?

14. An equilateral triangle of side length 8 is tiled by 64 equilateral triangles of unit side length to form a triangular grid. Initially, each triangular cell is either living or dead. The grid evolves over time under the following rule: every minute, if a dead cell is edge-adjacent to at least two living cells, then that cell becomes living, and any living cell remains living. Given that every cell in the grid eventually evolves to be living, what is the minimum possible number of living cells in the initial grid?

15. In triangle ABC, $AB = 7$, $BC = 11$, and $CA = 13$. Let Γ be the circumcircle of ABC and let M, N, and P be the midpoints of minor arcs $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$ of Γ, respectively. Given that $\mathcal{K}$ denotes the area of ABC and $\mathcal{L}$ denotes the area of the intersection of ABC and MNP, the ratio $\frac{\mathcal{L}}{\mathcal{K}}$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $a + b$.

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [6] Let $ABCDEF$ be a regular hexagon. How many acute triangles have all their vertices among the vertices of $ABCDEF$?

2. [6] A rectangle has a diagonal of length 20. If the width of the rectangle is doubled, the length of the diagonal becomes 22. Given that the width of the original rectangle is w, compute w^2.

3. [6] The number $\overline{2022A20B22}$ is divisible by 99. What is $A+B$?

1.4.2 Round 2

4. [7] How many two-digit positive integers have digits that sum to at least 16?

5. [7] For how many integers k less than 10 do there exist positive integers x and y such that $k = x^2 - xy + y^2$?

6. [7] Isosceles trapezoid $ABCD$ is inscribed in a circle of radius 2 with $AB \parallel CD$, $AB = 2$, and one of the interior angles of the trapezoid equal to 110°. What is the degree measure of minor arc $\overset{\frown}{CD}$?

1.4.3 Round 3

7. [9] In rectangle $ALEX$, point U lies on side EX so that $\angle AUL = 90^\circ$. Suppose that $UE = 2$ and $UX = 12$. Compute the square of the area of $ALEX$.

8. [9] How many digits does 20^{22} have?

9. [9] Compute the units digit of $3 + 3^3 + 3^{3^3} + \ldots + 3^{3^{\cdot^{\cdot^{3}}}}$, where the last term of the series has 2022 3s.

1.4.4 Round 4

10. [11] Given that $\sqrt{x-1} + \sqrt{x} = \sqrt{x+1}$ for some real number x, the number x^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

11. [11] Eric the Chicken Farmer arranges his 9 chickens in a 3-by-3 grid, with each chicken being exactly one meter away from its closest neighbors. At the sound of a whistle, each chicken simultaneously chooses one of its closest neighbors at random and moves $\frac{1}{2}$ of a unit towards it. Given that the expected number of pairs of chickens that meet can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers, compute $p+q$.

12. [11] For a positive integer n, let $s(n)$ denote the sum of the digits of n in base 10. Find the greatest positive integer n less than 2022 such that $s(n) = s(n^2)$.

1.4.5 Round 5

13. [13] Find the number of six-digit positive integers that satisfy all of the following conditions:

    (i) Each digit does not exceed 3.

    (ii) The number 1 cannot appear in two consecutive digits.

    (iii) The number 2 cannot appear in two consecutive digits.

14. [13] Find the sum of all distinct prime factors of 103040301.

15. [13] Let $ABCA'B'C'$ be a triangular prism with height 3 where bases ABC and $A'B'C'$ are equilateral triangles with side length $\sqrt{6}$. Points P and Q lie inside the prism so that $ABCP$ and $A'B'C'Q$ are regular tetrahedra. The volume of the intersection of these two tetrahedra can be expressed in the form $\frac{\sqrt{m}}{n}$, where m and n are positive integers and m is not divisible by the square of any prime. Find $m+n$.

1.4.6 Round 6

16. [15] Let $a_0, a_1, \ldots$ be an infinite sequence such that $a_n^2 - a_{n-1}a_{n+1} = a_n - a_{n-1}$ for all positive integers n. Given that $a_0 = 1$ and $a_1 = 4$, compute the smallest positive integer k such that a_k is an integer multiple of 220.

17. [15] Vincent the Bug is on an infinitely long number line. Every minute, he jumps either 2 units to the right with probability $\frac{2}{3}$ or 3 units to the right with probability $\frac{1}{3}$. The probability that Vincent never lands exactly 15 units from where he started can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p+q$?

18. [15] Battler and Beatrice are playing the "Octopus Game." There are 2022 boxes lined up in a row, and inside one of the boxes is an octopus. Beatrice knows the location of the octopus, but Battler does not. Each turn, Battler guesses one of the boxes, and Beatrice reveals whether or not the octopus is contained in that box at that time. Between turns, the octopus teleports to an adjacent box and secretly communicates to Beatrice where it teleported to. Find the least positive integer B such that Battler has a strategy to guarantee that he chooses the box containing the octopus in at most B guesses.

1.4.7 Round 7

19. [18] Given that $f(x) = x^2 - 2$ the number $f(f(f(f(f(f(f(2.5)))))))$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Find the greatest positive integer n such that 2^n divides $ab + a + b - 1$.

20. [18] In triangle ABC, the shortest distance between a point on the A-excircle ω and a point on the B-excircle Ω is 2. Given that $AB = 5$, the sum of the circumferences of ω and Ω can be written in the form $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. What is $m + n$? (Note: The A-excircle is defined to be the circle outside triangle ABC that is tangent to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and to the side BC. The B-excircle is defined similarly for vertex B.)

21. [18] Let $a_0, a_1, ...$ be an infinite sequence such that $a_0 = 1$, $a_1 = 1$, and there exists two fixed integer constants x and y for which a_{n+2} is the remainder when $xa_{n+1} + ya_n$ is divided by 15 for all nonnegative integers n. Let t be the least positive integer such that $a_t = 1$ and $a_{t+1} = 1$ if such an integer exists, and let $t = 0$ if such an integer does not exist. Find the maximal value of t over all possible ordered pairs (x, y).

1.4.8 Round 8

22. [21] A *mystic square* is a 3 by 3 grid of distinct positive integers such that the least common multiples of the numbers in each row and column are the same. Let M be the least possible maximal element in a mystic square and let N be the number of mystic squares with M as their maximal element. Find $M + N$.

23. [21] In triangle ABC, $AB = 27$, $BC = 23$, and $CA = 34$. Let X and Y be points on sides AB and AC, respectively, such that $BX = 16$ and $CY = 7$. Given that O is the circumcenter of BXY, the value of CO^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

24. [21] Alan rolls ten standard fair six-sided dice, and multiplies together the ten numbers he obtains. Given that the probability that Alan's result is a perfect square is $\frac{a}{b}$, where a and b are relatively prime positive integers, compute a.

Chapter 2

EMC^2 2022 Solutions

2.1 Speed Test Solutions

1. Compute $(2+0)(2+2)(2+0)(2+2)$.

   Solution. The answer is $\boxed{64}$.

   The expression is equal to $2 \times 4 \times 2 \times 4 = 64$.

2. Given that 25% of x is 120% of 30% of 200, find x.

   Solution. The answer is $\boxed{288}$.

   We set up the equation $0.25x = 1.20 \cdot 0.30 \cdot 200$. This gives $x = 4 \cdot 1.20 \cdot 0.3 \cdot 200$. Multiplying it out, we get 288.

3. Jacob had taken a nap. Given that he fell asleep at 4:30 PM and woke up at 6:23 PM later that same day, for how many minutes was he asleep?

   Solution. The answer is $\boxed{113}$.

   Note that 6:23 PM is $6 \cdot 60 + 23$ minutes after noon, while 4:30 PM is $4 \cdot 60 + 30$ minutes after noon. Thus Jacob slept for $(6 \cdot 60 + 23) - (4 \cdot 60 + 30) = 2 \cdot 60 - 7 = 113$ minutes.

4. Kevin is painting a cardboard cube with side length 12 meters. Given that he needs exactly one can of paint to cover the surface of a rectangular prism that is 2 meters long, 3 meters wide, and 6 meters tall, how many cans of paint does he need to paint the surface of his cube?

   Solution. The answer is $\boxed{12}$.

   The surface area of a 2-by-3-by-6 rectangular prism is $2(2 \cdot 3 + 3 \cdot 6 + 6 \cdot 2) = 72$ square meters. The surface area of a cube with side length 12 is $6 \cdot 12^2$. So it takes $\frac{6 \cdot 144}{72} = 12$ cans of paint.

5. How many nonzero digits does $200 \times 25 \times 8 \times 125 \times 3$ have?

   Solution. The answer is $\boxed{2}$.

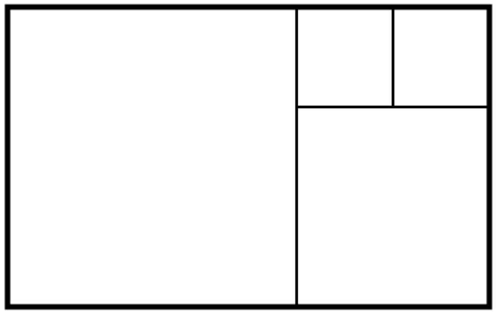
   We can group terms as follows: $200 \times 25 \times 8 \times 125 \times 3 = (2 \times 100) \times 25 \times (1000) \times 3$. Since the powers of ten only contribute trailing zeroes, we can ignore them, and note that $2 \times 25 \times 3 = 150$ only has two nonzero digits.

6. Given two real numbers x and y, define $x \# y = xy + 7x - y$. Compute the absolute value of $0 \# (1 \# (2 \# (3 \# 4)))$.

   Solution. The answer is $\boxed{7}$.

   Note that for all y, $1 \# y = y + 7 - y = 7$. Therefore $1 \# (2 \# (3 \# 4)) = 7$. So our answer is $|0 \# 7| = |-7| = 7$.

7. A 3-by-5 rectangle is partitioned into several squares of integer side length. What is the fewest number of such squares? Squares in this partition must not overlap and must be contained within the rectangle.



Solution. The answer is $\boxed{4}$.

Note that one square of side length 3, one of side length 2, and two of side length 1 will suffice. Now we can show that 3 squares will not work. Note that $3 \times 5 = 15$, so the only squares we can use have area 9, 4, or 1. Without any squares of area 9, the maximum total area we can get with 3 squares is $4+4+4 = 12 < 15$, so we need a square of area 9. Then the remaining 2 squares must have total area 6, but since we only have squares of areas 1 and 4 left, this is impossible. Therefore our answer is 4.

8. Points A and B lie in the plane so that $AB = 24$. Given that C is the midpoint of AB, D is the midpoint of BC, E is the midpoint of AD, and F is the midpoint of BD, find the length of segment EF.

   Solution. The answer is $\boxed{12}$.

   Note that E divides AD into 2 segments both of length x and F divides DB into 2 segments both of length y for some x and y. Then $AB = 2x + 2y = 24$, and $EF = x + y$. Therefore our answer is $EF = \frac{24}{2} = 12$.

9. Vincent the Bug and Achyuta the Anteater are climbing an infinitely tall vertical bamboo stalk. Achyuta begins at the bottom of the stalk and climbs up at a rate of 5 inches per second, while Vincent begins somewhere along the length of the stalk and climbs up at a rate of 3 inches per second. After climbing for t seconds, Achyuta is half as high above the ground as Vincent. Given that Achyuta catches up to Vincent after another 160 seconds, compute t.

   Solution. The answer is $\boxed{64}$.

   Note that in 160 seconds, Achyuta climbs $5 \cdot 160 = 800$ inches, while Vincent climbs $3 \cdot 160 = 480$ inches, so Achyuta gains $800 - 480 = 320$ inches on Vincent. Therefore, at time t, Achyuta should have been 320 inches below Vincent and 320 inches above the ground. Since Achyuta climbs at 5 inches per second, $t = \frac{320}{5} = 64$.

10. What is the minimum possible value of $|x - 2022| + |x - 20|$ over all real numbers x?

    Solution. The answer is $\boxed{2002}$.

    If x is greater than 2022, then both $x - 2022$ and $x - 20$ are positive and the sum becomes $2x - 2042$, and since $x > 2022$ then $2x - 2042 > 2022$. If x is less than 20, then both $x - 2022$ and $x - 20$ are negative and the sum becomes $2042 - 2x$, and since $x < 20$ then $2042 - 2x > 2002$. However, if x is anywhere between 20 and 2002, then $x - 2022$ is negative and $x - 20$ is positive so the sum becomes $2022 - x + x - 20 = 2002$, since the minimum in all three cases is 2002, that is our answer.

11. Let $ABCD$ be a rectangle. Lines ℓ_1 and ℓ_2 divide $ABCD$ into four regions such that ℓ_1 is parallel to AB and line ℓ_2 is parallel to AD. Given that three of the regions have area 6, 8, and 12, compute the sum of all possible areas of the fourth region.

    Solution. The answer is $\boxed{29}$.

    Suppose that AB is divided into two segments of length x_1 and x_2, and AD is divided into two segments of length y_1 and y_2. Then the areas of the four regions are x_1y_1, x_1y_2, x_2y_1, and x_2y_2. Note that $(x_1y_1)(x_2y_2) = (x_1y_2)(x_2y_1)$, and thus the product of the areas of two non-adjacent regions equals the product of the areas of the other two regions. Since we know that three of the regions have area 6, 8, and 12, two of these must correspond to non-adjacent regions, and so our cases for the final area are $\frac{6\cdot 8}{12}$, $\frac{6\cdot 12}{8}$, and $\frac{8\cdot 12}{6}$, which sum to 29.

12. A *diverse* number is a positive integer that has two or more distinct prime factors. How many diverse numbers are less than 50?

    Solution. The answer is $\boxed{25}$.

    We use complementary counting. A non-diverse number must be either 1, a prime, or a prime power. The primes less than 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47, for a total of 15 primes. Furthermore, 2, 3, 5, and 7 yield prime powers: 2 can become 4, 8, 16, and 32. 3 has 9 and 27. 5 and 7 have 25 and 49 respectively. There are 8 prime powers, which means that there are $49 - 1 - 15 - 8 = 25$ diverse numbers.

13. Let x, y, and z be real numbers so that $(x+y)(y+z) = 36$ and $(x+z)(x+y) = 4$. Compute $y^2 - x^2$.

    Solution. The answer is $\boxed{32}$.

    Expanding both equations gives

    $$xy + xz + y^2 + yz = 36$$
    $$x^2 + xy + zx + zy = 4$$

    which when subtracted, gives $y^2 - x^2 = 36 - 4 = 32$.

14. What is the remainder when $1^{10} + 3^{10} + 7^{10}$ is divided by 58?

    Solution. The answer is $\boxed{1}$.

    Remark that $3^{10} + 7^{10}$ is a multiple of $3^2 + 5^2 = 58$ because of the factorization $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$. So, $1^{10} + 3^{10} + 7^{10}$ leaves a remainder of 1 when divided by 58.

    Solution 2:

    We wish to find $1^{10} + 3^{10} + 7^{10} \pmod{58}$. We can perform a couple of manipulations:

    $$1^{10} + 3^{10} + 7^{10} \equiv 1 + 9^5 + 49^5 \pmod{58}$$

$$\equiv 1 + 9^5 + (-9)^5 \equiv 1 \pmod{58}$$

and thus we are done.

15. Let $A = (0, 1)$, $B = (3, 5)$, $C = (1, 4)$, and $D = (3, 4)$ be four points in the plane. Find the minimum possible value of $AP + BP + CP + DP$ over all points P in the plane.

    Solution. The answer is $\boxed{7}$.

    By the triangle inequality, $(AP + BP) + (CP + DP) \geq AB + CD$. Furthermore, this lower bound can be achieved by setting P to be the intersection of segments AB and CD. Thus our answer is $AB + CD = 5 + 2 = 7$.

16. In trapezoid $ABCD$, points E and F lie on sides BC and AD, respectively, such that $AB \parallel CD \parallel EF$. Given that $AB = 3$, $EF = 5$, and $CD = 6$, the ratio $\frac{[ABEF]}{[CDFE]}$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$. (Note: $[\mathcal{F}]$ denotes the area of $\mathcal{F}$.)

    Solution. The answer is $\boxed{27}$.

    Construct point P as the intersection of lines AD and BC. Then $\triangle PAB \sim \triangle PFE \sim \triangle PDC$. Note that the area ratios of these triangles are $3^2 : 5^2 : 6^2 = 9 : 25 : 36$. Since $[ABEF] = [PFE] - [PAB]$ and $[CDFE] = [PDC] - [PFE]$, the desired ratio is $\frac{25-9}{36-25} = \frac{16}{11}$, so the desired sum is $16 + 11 = 27$.

17. For sets X and Y, let $|X \cap Y|$ denote the number of elements in both X and Y and $|X \cup Y|$ denote the number of elements in at least one of X or Y. How many ordered pairs of subsets (A, B) of $\{1, 2, 3, \ldots, 8\}$ are there such that $|A \cap B| = 2$ and $|A \cup B| = 5$?

    Solution. The answer is $\boxed{4480}$.

    There are $\binom{8}{2} = 28$ ways to choose the two elements common to both X and Y. Since $|X \cup Y| = 5$, there are three elements in neither X nor Y and there are $\binom{6}{3} = 20$ ways to choose these three elements from the remaining six. The final three elements can be in either X or Y but not both, for a total of $2^3 = 8$ possibilities. Thus the answer is $28 \cdot 20 \cdot 8 = 4480$.

18. A *tetromino* is a polygon composed of four unit squares connected orthogonally (that is, sharing a edge). A *tri-tetromino* is a polygon formed by three orthogonally connected tetrominoes. What is the maximum possible perimeter of a tri-tetromino?

    Solution. The answer is $\boxed{26}$.

    The total perimeter of twelve disconnected unit squares is clearly $12 \cdot 4 = 48$. When two squares are connected orthogonally, 2 units of perimeter are lost as the overlapping edges become part of the shape's interior. Since it takes a minimum of 11 such connections to connect all twelve squares, we should expect the maximum to be $48 - 2 \cdot 11 = 26$. This can be achieved in multiple ways, including simply gluing three 1-by-4 rectangles together via their 1 unit edges.

19. The numbers from 1 through 2022, inclusive, are written on a whiteboard. Every day, Hermione erases two numbers a and b and replaces them with $ab + a + b$. After some number of days, there is only one number N remaining on the whiteboard. If N has k trailing nines in its decimal representation, what is the maximum possible value of k?

Solution. The answer is $\boxed{503}$.

Consider a new whiteboard that has all the numbers of the original whiteboard, but incremented by one. Then when two numbers a and b are erased on the original whiteboard and replaced by $ab+a+b$, the corresponding $a+1$ and $b+1$ values on the new whiteboard are replaced by $ab+a+b+1 = (a+1)(b+1)$.

Hence no matter which order we erase the numbers in, the final number on the new whiteboard will be $(1+1)(2+1)...(2022+1) = 2023!$ and the final number on our original whiteboard must necessarily be $2023! - 1$. $2023!$ has

$$\left\lfloor \frac{2023}{5} \right\rfloor + \left\lfloor \frac{2023}{25} \right\rfloor + \left\lfloor \frac{2023}{125} \right\rfloor + \left\lfloor \frac{2023}{625} \right\rfloor = 503$$

trailing zeroes, which correspond to 503 trailing nines in N.

20. Evaluate $5(2^2+3^2)+7(3^2+4^2)+9(4^2+5^2)+\cdots+199(99^2+100^2)$.

Solution. The answer is $\boxed{99999984}$.

Note that each term is of the form $(k+[k+1])(k^2+[k+1]^2)$, which we can rewrite as $([k+1]-k)([k+1]+k)([k+1]^2+k^2)$. Using the difference of squares factorization twice, this becomes $[k+1]^4 - k^4$. Therefore, the original expression is equal to

$$(3^4-2^4)+(4^4-3^4)+\cdots+(100^4-99^4)$$

which telescopes into $100^4 - 2^4 = 99999984$.

2.2 Accuracy Test Solutions

1. At a certain point in time, 20% of seniors, 30% of juniors, and 50% of sophomores at a school had a cold. If the number of sick students was the same for each grade, the fraction of sick students across all three grades can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

   Solution. The answer is $\boxed{40}$.

   Let the number of sick people in each grade be x. Then, we have that there are a total of $5x$ seniors, $\frac{10}{3}x$ juniors, and $2x$ sophomores, for a total of $\frac{31}{3}x$ students. Since we have a total of $3x$ sick students, the fraction of total students that are sick is $\frac{9}{31}$, giving us the answer of 40.

2. The average score on Mr. Feng's recent test is a 63 out of 100. After two students drop out of the class, the average score of the remaining students on that test is now a 72. What is the maximum number of students that could initially have been in Mr. Feng's class? (All of the scores on the test are integers between 0 and 100, inclusive.)

   Solution. The answer is $\boxed{16}$.

   If there were originally $x+2$ students in the class, then the original total score of all the students was $63(x+2)$. Afterwards, the total score was $72x$. Since clearly the total score cannot decrease, we have the inequality $63(x+2) \geq 72x$. Solving yields $x \leq 14$, or there were at most 16 students originally. Note that equality can occur if the original two students received a 0.

3. Madeline is climbing Celeste Mountain. She starts at $(0,0)$ on the coordinate plane and wants to reach the summit at $(7,4)$. Every hour, she moves either 1 unit up or 1 unit to the right. A strawberry is located at each of $(1,1)$ and $(4,3)$. How many paths can Madeline take so that she encounters exactly one strawberry?

   Solution. The answer is $\boxed{148}$.

   To go through $(1,1)$ to get to $(7,4)$, we must first make exactly one move up and one right, with $\binom{2}{1}$ possibilities, and then go up 6 times and right 3 times for $\binom{9}{3}$ possibilities. Thus, to go through $(1,1)$, there are $\binom{2}{1}\binom{9}{3} = 168$ paths. Similarly, to go through $(4,3)$, there are $\binom{7}{3}\binom{4}{1} = 140$ paths. Since we've counted the paths that go through both $(1,1)$ and $(4,3)$ twice, we need to subtract these off. There are $\binom{2}{1}\binom{5}{2}\binom{4}{1} = 80$ paths passing through both points, so in total, to go through exactly one strawberry, there are $168 + 140 - 2 \cdot 80 = 148$ paths.

4. Let E be a point on side AD of rectangle $ABCD$. Given that $AB = 3$, $AE = 4$, and $\angle BEC = \angle CED$, the length of segment CE can be written as $\sqrt{a}$ for some positive integer a. Find a.

   Solution. The answer is $\boxed{10}$.

   Doing some angle chasing, we have that $\angle BEC = \angle CED = \angle BCE$. Thus triangle BEC is isosceles and $BE = BC$. Using the Pythagorean Theorem, we have that $BE = BC = 5$. Therefore, $ED = 1$, and using Pythagorean Theorem again, we get that $EC = \sqrt{10}$, so the answer is $a = 10$.

5. Lucy has some spare change. If she were to convert it into quarters and pennies, the minimum number of coins she would need is 66. If she were to convert it into dimes and pennies, the minimum number of coins she would need is 147. How much money, in cents, does Lucy have?

Solution. The answer is $\boxed{1434}$.

Suppose that a minimal conversion into dimes and pennies requires x pennies and $147 - x$ dimes. Then the total amount of change is $x + 10(147 - x) = 1470 - 9x$, and furthermore, $0 \leq x \leq 9$, as having ten pennies is strictly less efficient than if we were to use a single dime instead. Similarly, if a minimal conversion into quarters and pennies requires y pennies, then the total amount of change will be $y+25(66-y) = 1650-24y$, and $0 \leq y \leq 24$. We have that $1470-9x = 1650-24y$, so $24y-9x = 180$ and $8y - 3x = 60$. Taking both sides modulo 4, the equation becomes $-3x \equiv 0 \pmod 4$, so we know that 4 must divide x. Then x is either 0, 4, or 8. Of these, only 4 yields an integer value for y. Thus Lucy has $1470 - 9x = 1470 - 9(4) = 1434$ cents in change.

6. For how many positive integers x does there exist a triangle with altitudes of length 20, 22, and x?

Solution. The answer is $\boxed{209}$.

Let the area of the triangle be A. Then, the sides of the triangle are $\frac{A}{10}$, $\frac{A}{11}$, $\frac{2A}{x}$, and the ratio of the sides is $11x : 10x : 220$. Since we can scale the triangle up and down, we can simply assume that the triangle has sides $11x$, $10x$, and 220. Now, $11x > 10X$, so the longest side is either $11x$ or 220. If the longest side is $11x$, then we have that $11x \geq 220$ and $10x + 220 \geq 11x$ by the triangle inequality, giving us $20 \leq x \leq 220$. Now, if 220 is the longest side, then we have $220 > 11x$ and $10x + 11x > 220$, which gives us $\frac{220}{21} < x < 20$. Combining the two inequalities, we get that $\frac{220}{21} < x < 220$, which is equivalent to $11 \leq x \leq 219$, giving us 209 solutions for x.

7. Compute the number of positive integers x for which $\frac{x^{20}}{x+22}$ is an integer.

Solution. The answer is $\boxed{434}$.

Plug in $x = y - 22$ to get $\frac{(y-22)^{20}}{y}$. Expanding with the Binomial Theorem, we get

$$\frac{\sum_{i=0}^{20} y^i(-22)^{20-i}}{y} = \frac{\sum_{i=1}^{20} y^i(-22)^{20-i}}{y} + \frac{(-22)^{20}}{y}$$

Note that for the numerator of the first fraction on the right hand side, all terms have at least one factor of y in them. Therefore, we only need $\frac{(-22)^{20}}{y}$ to be an integer. We want y to be a factor of 22^{20}, and from the initial statement that x must be a positive integer, y must be an integer greater than 22. There are $21 \cdot 21 = 441$ factors of 22^{20}, and 7 of them, 1, 2, 4, 8, 11, 16, and 22, are all less than or equal to 22. Therefore our answer is $441 - 7 = 434$.

8. Vincent the Bug is crawling along an octagonal prism. He starts on a fixed vertex A, visits all other vertices exactly once by traveling along the edges, and returns to A. Find the number of paths Vincent could have taken.

Solution. The answer is $\boxed{20}$.

Without loss of generality, assume Vincent is on the top octagon of the prism, and let the vertices of the octagon be $ABCDEFGH$. Similarly, let the corresponding vertices of the bottom octagon be $A'B'C'D'E'F'G'H'$. We do casework on where we go and where we end.

Case 1: We first visit A'. Then, without loss of generality, we visit B'. From there, if we visit C', then we cannot ever return to the top octagon until we have finished visiting all vertices on the bottom octagon, so we have 1 path. Otherwise, we visit B next. In this case, we must from there visit C, and then C' (because if we visit D, we cannot ever return to C' as there is only one path in) and D'. Again, if we move to E', then we "cut off" D, and thus we must move to D. This logic continues, forcing the rest of the path, as $EE'F'FGG'H'HA$, thus the initial sequence $AA'B'$ gives 2 paths. Similarly, the initial sequence $AA'H'$ gives 2 paths also.

Case 2: If we end with the sequence $A'A$, then there are also 4 paths here.

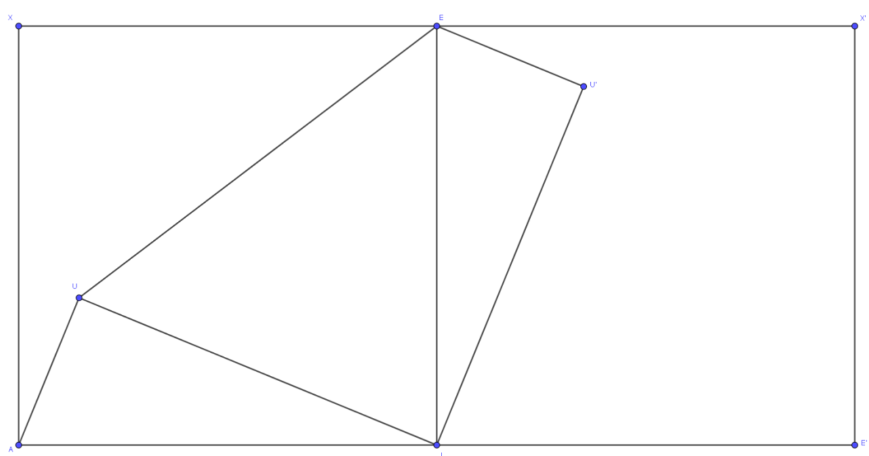
Case 3: If we do not start or end the sequence with A', then we must start with one of B and H and end with the other. Without loss of generality, assume we start with B and end with H. Then, consider when we move down to the lower octagon. If we move down, at say D to D', then we cannot move to E', since then we would cut off all of A' through C' (note that we cannot use A as a neighbor of A' anymore since we do not start or end on A'). Thus, we must move to C', then B', then A', and so on. Working from the end, a similar logical sequence occurs, so for example if we last move up at E' to E, we must have come from F', and G', and H', and so on. Therefore, the vertex that we first move down on and the vertex that we last move up on have to be next to each other. Thus, there are 6 paths here (for the 6 choices of the vertex that we first move down on). Similarly, if we start with H and end with B, there are also 6 paths, for a total of 12 paths in this case.

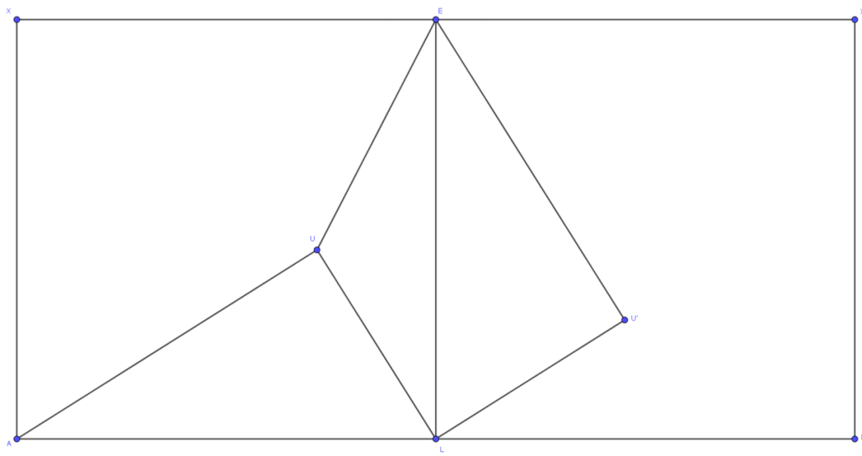
Now, in total there are 20 possible paths.

9. Point U is chosen inside square $ALEX$ so that $\angle AUL = 90°$. Given that $UL = 56$ and $UE = 65$, what is the sum of all possible values for the area of square $ALEX$?

Solution. The answer is $\boxed{14722}$.

Rotate $ALEX$ by 90 degrees about L so that A is sent to E. Let the image of U be U'. Now, we have that $\angle ULU' = 90°$ and $\angle EU'L = \angle AUL = 90°$. Now, $U'L = UL = 56$ and $UE = 65$, so we now have two cases: Either $\angle EUL$ is obtuse, in which case $EU' = UL + \sqrt{65^2 - 56^2} = 89$, or $\angle EUL$ is acute, in which case $EU' = UL - \sqrt{65^2 - 56^2} = 23$. Thus, the two possible cases for EL^2 are $56^2 + 23^2$ and $56^2 + 89^2$. Thus, the sum of these is 14722.





10. Miranda has prepared 8 outfits, no two of which are the same quality. She asks her intern Andrea to order these outfits for the new runway show. Andrea first randomly orders the outfits in a list. She then starts removing outfits according to the following method: she chooses a random outfit which is both immediately preceded and immediately succeeded by a better outfit and then removes it. Andrea repeats this process until there are no outfits that can be removed. Given that the expected number of outfits in the final routine can be written as $\frac{a}{b}$ for some relatively prime positive integers a and b, find $a + b$.

    Solution. The answer is $\boxed{761}$.

    First we will prove that an outfit will be in the final routine if and only if it either comes before all the better outfits, or comes after all the better outfits. It is easy to prove sufficiency: because the relative order of outfits never changes, an outfit that comes before or after all the better outfits will never be surrounded by better outfits.

    To prove necessity, we will show that any outfit that contradicts this condition must be removed. Suppose that Andrea orders the outfits $a_1, a_2, \ldots, a_8$ and suppose further that for some $i < j < k$, we have that $a_i > a_j$ and $a_i < a_k$. We will induct on $|k - i|$. The base case of 2 is trivial, as a_j will be removed immediately.

    For the inductive step, assume that the statement is true if $|k - i| \leq d$ for some d. Suppose that $|k - i| = d + 1$ and observe that if both $a_{j-1} > a_j$ and $a_{j+1} > a_j$ that a_j will be removed immediately. Thus assume that without loss of generality $a_{j-1} < a_j$. Then because $a_{j-1} < a_j < a_i$, and furthermore that $|j - i| < |k - i| = d + 1 \implies |j - i| \leq d$, a_j will be removed by Andrea eventually. At this point, the distance between a_k and a_i decreases, and since a_j is surrounded by better outfits that are at most d apart, induction tells us that a_j will inevitably be removed.

    To compute the expected number of remaining outfits, define the indicator random variable X_i as follows:

    $$X_i = \begin{cases} 1 & \text{if the } i\text{-th best outfit remains} \\ 0 & \text{otherwise} \end{cases}$$

    By linearity,

    $$E\left[\sum_1^8 X_i\right] = \sum_1^8 E[X_i] = \sum_1^8 P(i)$$

where $P(i)$ is the probability that the i-th best outfit remains.

Observe that $P(i) = 1$ for $i = 1$ and $i = 2$, and that $P(i) = \frac{2}{i}$ for larger i. This is because the i best elements are equally likely to be in any of the $i!$ permutations, and that the i-th best element will only remain if it is either at the start or the end, for a total of $2 \cdot (i-1)!$ ways.

Finally, taking the sum yields $1 + 1 + \frac{2}{3} + \frac{2}{4} + \cdots + \frac{2}{8} = \frac{621}{140}$, so the answer is $621 + 140 = 761$.

2.3 Team Test Solutions

1. Compute $1+3+6+10+15+21+28+36+45+55$.

   Solution. The answer is $\boxed{220}$.

   By grouping consecutive terms, we obtain $(1+3+6)+10+(15+21)+(28+36)+(45+55) = 10+10+36+64+100 = 220$.

   Solution 2.

   The sum is equivalent to $\displaystyle\sum_{n=2}^{11}\binom{n}{2}$. By the Hockey-Stick Identity, this equals $\binom{12}{3} = 220$.

2. Given that a, b, and c are positive integers such that $a+b=9$ and $bc=30$, find the minimum possible value of $a+c$.

   Solution. The answer is $\boxed{8}$.

   Observe that $a = 9-b$ and $c = \frac{30}{b}$ are both individually minimized when b is maximized. Therefore, we wish to find the largest $b < 9$ such that $\frac{30}{b}$ is an integer. By inspection, $b = 6$, hence $a+c = 3+5 = 8$.

3. Points X and Y lie outside regular pentagon $ABCDE$ such that ABX and DEY are equilateral triangles. Find the degree measure of $\angle XCY$.

   Solution. The answer is $\boxed{96}$.

   First, note that triangles XBC and YDC are isosceles with $XB = BC = CD = DY$. Also, $\angle XBC = 108^\circ + 60^\circ = 168^\circ$. Therefore, $\angle XCB = \frac{1}{2}(180^\circ - 168^\circ) = 6^\circ$. Similarly, $\angle YCD = 6^\circ$. Therefore, $\angle XCY = \angle BCD - \angle BCX - \angle YCD = 108^\circ - 6^\circ - 6^\circ = 96^\circ$.

4. Let N be the product of the positive integer divisors of $8!$, including itself. The largest integer power of 2 that divides N is 2^k. Compute k.

   Solution. The answer is $\boxed{336}$.

   Observe that $8! = 2^7 \cdot 3^2 \cdot 5 \cdot 7$. Therefore, $8!$ has $8\cdot 3\cdot 2\cdot 2 = 96$ divisors.

   Now, note that for each divisor a, $\frac{8!}{a}$ is also a divisor. In this manner, we can pair all 96 divisors into 48 pairs, each of whose product is $8!$. Thus, the product of all the divisors is $(8!)^{48} = 2^{336}\cdot 3^{96}\cdot 5^{48}\cdot 7^{48}$. Thus, the answer is 336.

5. Let $= A(-20,22)$, $B = (k,0)$, and $C = (202,2)$ be points on the coordinate plane. Given that $\angle ABC = 90^\circ$, find the sum of all possible values of k.

   Solution. The answer is $\boxed{182}$.

   Note that B is on the circle with diameter AC, which has center at the midpoint of AC, $M = (91,12)$. The equation of this circle is thus $(x-91)^2 + (y-12)^2 = 111^2 + 10^2$. Since $(k,0)$ is on the circle, we know $(k-91)^2 = 111^2 + 10^2 - 12^2$. This yields $k = 91 \pm c$, where $c^2 = 111^2 + 10^2 - 12^2$. So, the sum of all possible k is $(91-c)+(91+c) = 182$.

Solution 2.

We make the same observation as in the first solution that B must lie on the circle with diameter AC and center M. This circle intersects the x-axis twice, at two locations which we will call B_1 and B_2. Since $MB_1 = MB_2$, M lies on the perpendicular bisector of B_1B_2. The average x-coordinate of B_1 and B_2 should equal the x-coordinate of M, which is 91. Therefore the sum of all possible k is $2 \cdot 91 = 182$.

6. Tej is typing a string of Ls and Os that consists of exactly 7 Ls and 4 Os. How many different strings can he type that do not contain the substring 'LOL' anywhere? A *substring* is a sequence of consecutive letters contained within the original string.

Solution. The answer is $\boxed{56}$.

The condition is equivalent to having every O be either next to another O or at the start/end of the string. We first place the 7 Ls, and then do casework on where we can insert Os. Note that there are 6 places between two Ls, one place before the first L, and one place after the last L to insert blocks of Os.

Case 1: There is one block of 4 Os. Placing the 7 Ls first, there are 8 spaces ways to place the O block.

Case 2: There are two blocks of 2 Os. There are 8 places to insert a block of Os, but the two blocks cannot be inserted in the same location. There are $\binom{8}{2} = 28$ ways to do this.

Case 3: There is one block of 1 O at the edge and one block of 3 Os. There are 2 places (both endpoints) the singular O block could be, and 7 places for the remaining O block. This yields 14 ways.

Case 4: There are two blocks of 1 O at the edge and one block of 2 Os. Of the 8 locations to insert a block, there must be one O block in both endpoints, so there are 6 places to put the other O block. This yields 6 ways.

Adding everything up, we obtain 56 possible strings.

7. How many ordered triples of integers (a, b, c) satisfy both $a + b - c = 12$ and $a^2 + b^2 - c^2 = 24$?

Solution. The answer is $\boxed{24}$.

First, observe that $b - c = 12 - a$. Rearranging the second equation, $24 - a^2 = b^2 - c^2 = (b+c)(b-c) = (b+c)(12-a)$. Therefore $b + c = \frac{24-a^2}{12-a}$. Solving the system of equations for b and c we get:

$$b = \frac{1}{2}\left(12 - a + \frac{24 - a^2}{12 - a}\right)$$
$$c = \frac{1}{2}\left(\frac{24 - a^2}{12 - a} - (12 - a)\right)$$

Hence, both b and c are integers if and only if $12 - a$ and $\frac{24-a^2}{12-a}$ are integers of the same parity. Thus, it suffices to find the number of integers a for which $12 - a + \frac{24-a^2}{12-a} = 24 - \frac{120}{12-a}$ is an even integer.

Note that for $\frac{120}{12-a}$ to be even, $\frac{60}{12-a}$ must be an integer, so $12 - a$ must divide 60. Using the prime factorization $60 = 2^2 \cdot 3 \cdot 5$, we find $3 \times 2 \times 2 = 12$ positive divisors of 60. Accounting for negative divisors, we have 24 possible values for $12 - a$.

This implies that there are 24 possible values of a, and thus 24 triples.

Solution 2.

Isolating c in both equations, we have that $c = a + b - 12$ and $c^2 = a^2 + b^2 - 24$. Squaring the first equation, we get that $(a+b-12)^2 = c^2 = a^2+b^2-24$, which simplifies to $2ab-24a-24b+144 = -24$, or $ab - 12a - 12b + 84 = 0$. Adding 60 to both sides and factoring, we obtain $(a - 12)(b - 12) = 60$. Since $a-12$ must be a factor (either positive or negative) of 60, we get 24 values for a. Since a uniquely determines the values of b and c, there are 24 triples.

8. For how many three-digit base-7 numbers $\overline{ABC}_7$ does $\overline{ABC}_7$ divide $\overline{ABC}_{10}$? (Note: $\overline{ABC}_D$ refers to the number whose digits in base D are, from left to right, A, B, and C; for example, $\overline{123}_4$ equals 27 in base ten).

   Solution. The answer is $\boxed{11}$.

   Take $\overline{ABC}_{10} = 100A + 10B + C$ and $\overline{ABC}_7 = 49A + 7B + C$. We want that for some integer k,
   $$100A + 10B + C = k(49A + 7B + C)$$

   If $k \geq 3$, $k(49A + 7B + C) \geq 147A + 21B + 3C > 100A + 10B + C$, which is a contradiction. Thus, $k = 1$ or $k = 2$.

   If $k = 1$, $100A + 10B + C = 49A + 7B + C$, so $51A + 3B = 0$, which is a contradiction.

   If $k = 2$, $100A + 10B + C = 98A + 14B + 2C$. Therefore, $2A = 4B + C$, so C is even. Let $C = 2D$, then $A = 2B + D$ and since $\overline{ABC}_7$ is a 3-digit base 7 number, $1 \leq A \leq 6, 0 \leq B \leq 6$, and $0 \leq D \leq 3$. Doing casework on the value of A, we find one solution when $A = 1$, and 2 solutions for each of $A \geq 2$. This gives a total of 11 solutions.

9. Natasha is sitting on one of the 35 squares of a 5-by-7 grid of squares. Wanda wants to walk through every square on the board exactly once except the one Natasha is on, starting and ending on any 2 squares she chooses, such that from any square she can only go to an adjacent square (two squares are adjacent if they share an edge). How many squares can Natasha choose to sit on such that Wanda cannot go on her walk?

   Solution. The answer is $\boxed{17}$.

   Color the grid in a checkerboard pattern, such that there the top left corner is white, yielding a total of 17 black squares and 18 white squares. Every move, Natasha moves from a black-colored square to a white-colored square, or vice versa. Therefore, any path of length 34 that Natasha takes must pass through exactly 17 black squares and 17 white squares. If we remove any of the 17 original black squares, we cannot generate a run, as there will not be enough black squares. For the other 18 squares, we can show that all of these still allow for Natasha to have her run, as constructed below. Note that we only have to test 6 of these, as the rest follow by symmetry.

10. In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Point P lies inside ABC and points D,E, and F lie on sides BC, CA, and AB, respectively, so that $PD \perp BC$, $PE \perp CA$, and $PF \perp AB$. Given that PD, PE, and PF are all integers, find the sum of all possible distinct values of $PD \cdot PE \cdot PF$.

    Solution. The answer is $\boxed{210}$.

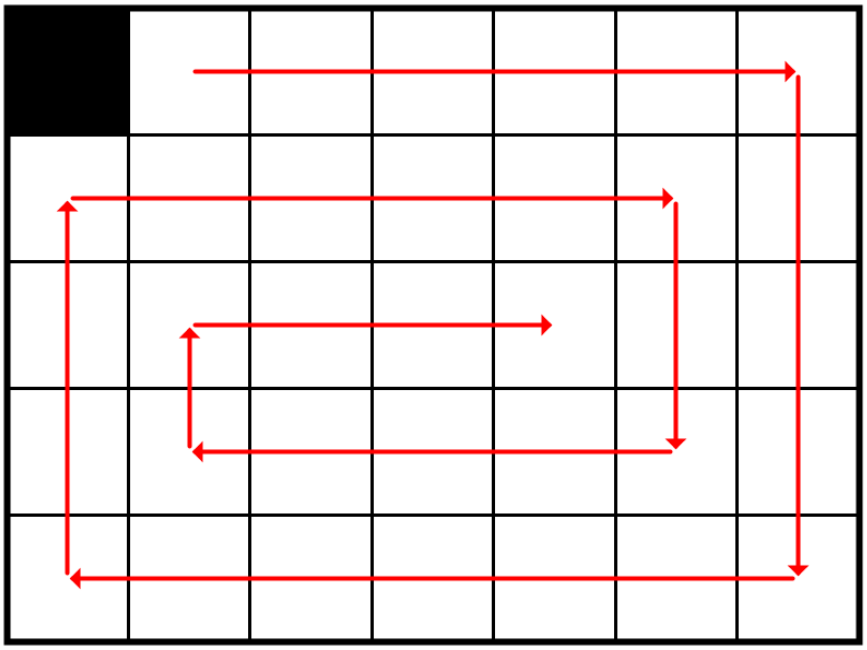


Figure 2.1: First case.

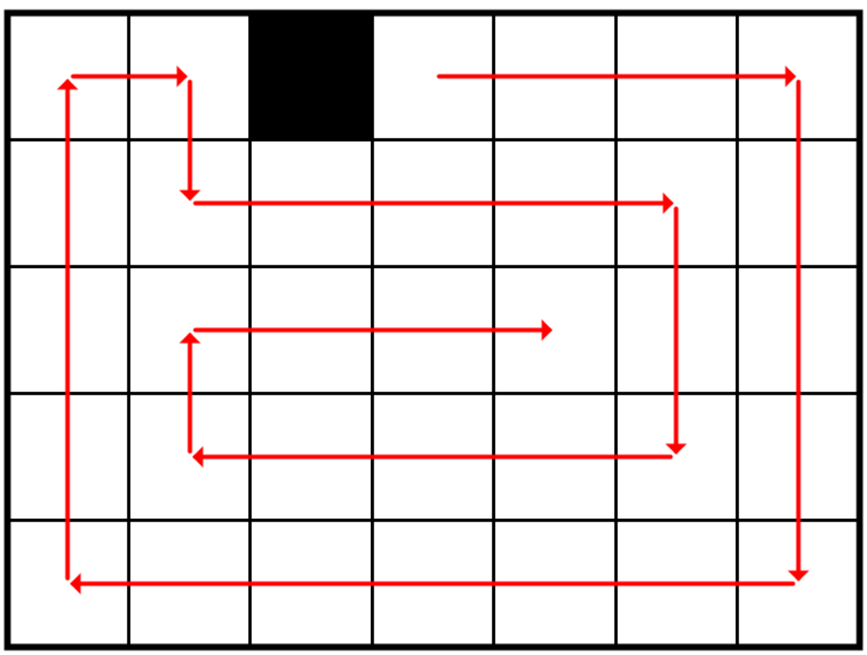


Figure 2.2: Second case.

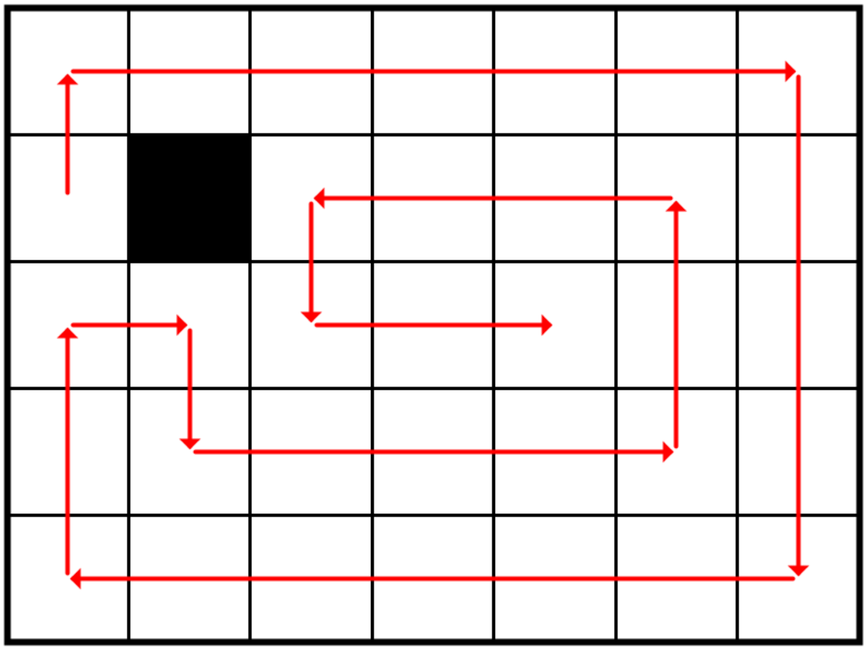


Figure 2.3: Third case.

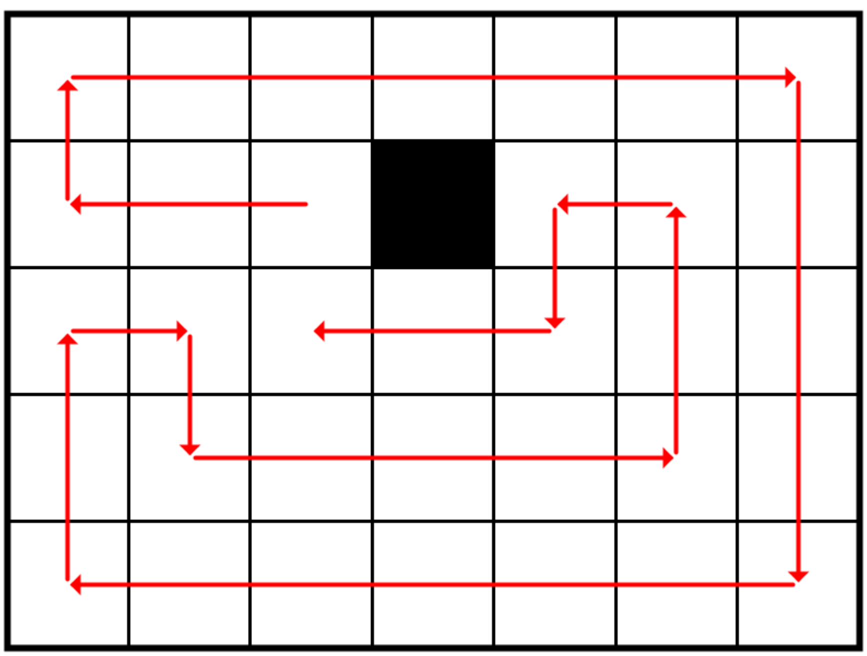


Figure 2.4: Fourth case.

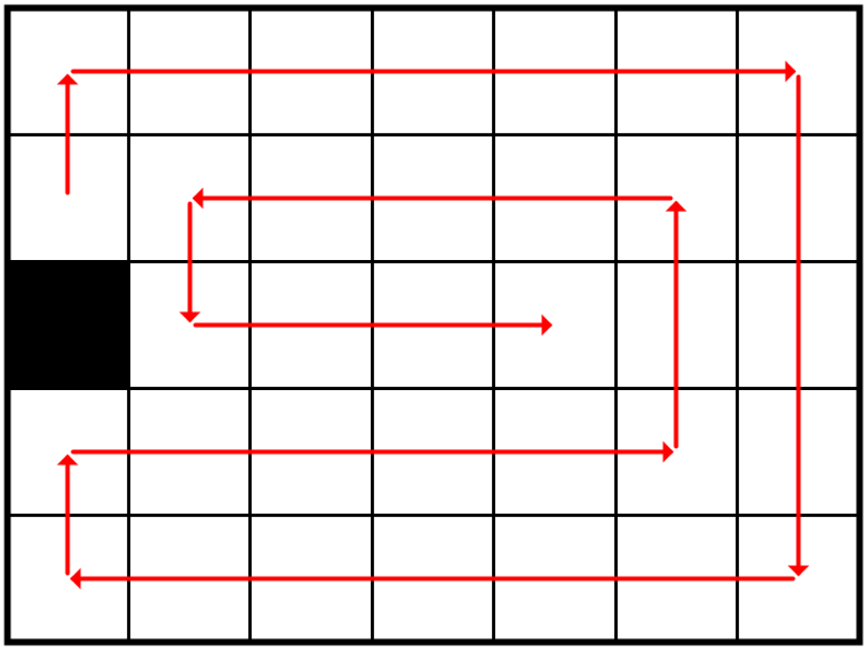


Figure 2.5: Fifth case.

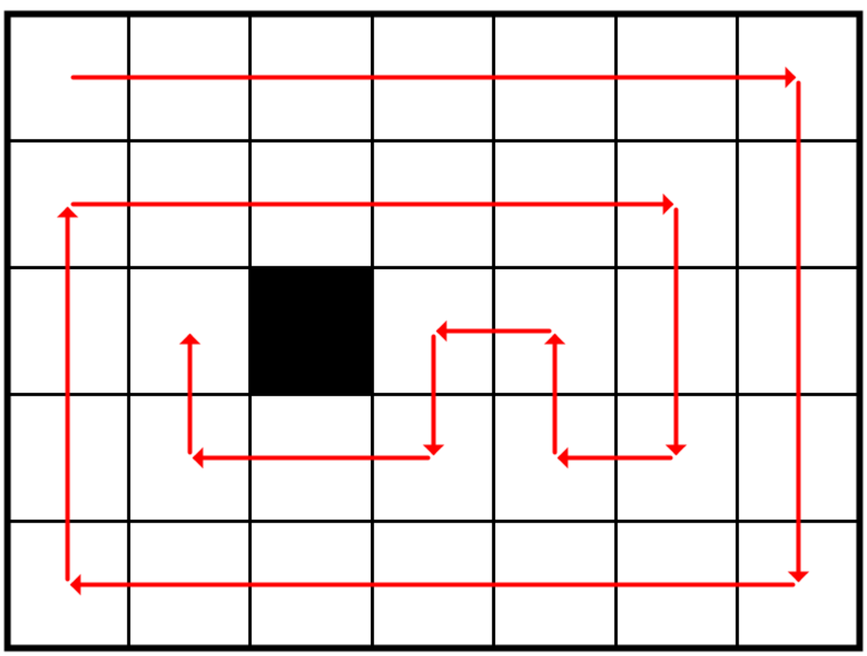


Figure 2.6: Sixth case.

By Heron's Formula, the area of ABC is $[ABC] = 84$. However, we may also write $[ABC] = [ABP] + [BCP] + [CAP] = \frac{1}{2}(13 \cdot FP) + \frac{1}{2}(14 \cdot DP) + \frac{1}{2}(15 \cdot EP)$. Thus, $168 = 13FP + 14DP + 15EP$. Taking this equation modulo 14, we get $0 \equiv -FP + EP \pmod{14}$ so $EP \equiv FP \pmod{14}$. In addition, EP and FP are less than $\frac{168}{15}$ and $\frac{168}{13}$ respectively, so both must be between 1 and 14, inclusive. Therefore, $EP = FP$. Plugging this back in, we get $168 = 28EP + 14DP$ and thus $DP = 12 - 2EP$. So EP must be between 1 and 5, yielding the ordered triples $(DP, EP, FP) = (2, 5, 5), (4, 4, 4), (6, 3, 3), (8, 2, 2)$, or $(10, 1, 1)$. The sum of all possible values of $PD \cdot PE \cdot PF$ is $50 + 64 + 54 + 32 + 10 = 210$.

11. A *palindrome* is a positive integer which is the same when read forwards or backwards. Find the sum of the two smallest palindromes that are multiples of 137.

    Solution. The answer is $\boxed{10960}$.

    We split into cases based on the number of digits of the palindrome. Clearly, since the number is at least 137, it has at least 3 digits.

    Case 1: The palindrome has three digits. By inspection, the three-digit multiples of 137 are 137, 274, 411, 548, 685, 822, and 959. Thus, 959 is the smallest such palindrome.

    Case 2: The palindrome has four digits. Then, it can be written as $\overline{abba} = 1001a + 11b = 11(91a + b)$ so it must be a multiple of 11. As both 11 and 137 are prime, the palindrome is a multiple of $137 \cdot 11 = 1507$. By inspection, the only four-digit multiples of 1507 are $1507, 3014, 4521, 6028, 7535$, and 9042, none of which are palindromes.

    Case 3: The palindrome has five digits. Then, it can be written as $\overline{abcba} = 10001a + 1010b + 100c$. In fact, 10001 itself equals $73 \cdot 137$, so we are already done. Thus, our answer is $959 + 10001 = 10960$.

12. Let $P(x) = x^2 + px + q$ be a quadratic polynomial with positive integer coefficients. Compute the least possible value of p such that 220 divides p and the equation $P(x^3) = P(x)$ has at least four distinct integer solutions.

    Solution. The answer is $\boxed{681560}$.

    Since $1^3 = 1$, $0^3 = 0$, and $(-1)^3 = -1$, we know that $P(x^3) - P(x) = 0$ must have 1,0, and -1 as roots.

    Therefore, we may expand $P(x^3) - P(x)$ and factor out $x^3 - x$:

    $$P(x^3) - P(x) = (x^6 + px^3 + q) - (x^2 + px + q) = (x^3 - x)(x^3 + x + p).$$

    Our fourth integer root must come from $x^3 + x + p$, so we wish to find the least $p > 0$ which is a multiple of 220 and which equals $-x^3 - x$ for some integer $x \neq 0, \pm 1$. Setting $y = -x$, this is equivalent to finding the least $p > 0$ which is a multiple of 220 and equals $y^3 + y$. This is true if and only if $y^3 + y$ is divisible by 4, 5, and 11 (as $220 = 2^2 \cdot 5 \cdot 11$).

    In order for 11 to divide $y^3 + y$, it must divide either y or $y^2 + 1$. Observe that $y^2 + 1$ can never be congruent to zero modulo 11:

    In addition, y and $y^2 + 1$ must produce two factors of 2 when combined. Because $y^2 + 1$ can never be a multiple of 4 (see table below), we know that y must be even. However, this means that $y^2 + 1$ is odd, and so y must be a multiple of 4.

    Combining these two divisibility relations, 44 divides x. By inspection, $44^3 + 44$ is not a multiple of 5 but $88^3 + 88$ is. Hence the minimum possible value of p is $88^3 + 88 = 681560$.

y	0	1	2	3	4	5	6	7	8	9	10
y^2+1	1	2	5	10	6	4	4	6	10	5	2

Table 2.1: Table of values modulo 11.

y	0	1	2	3
y^2+1	1	2	1	2

Table 2.2: Table of values modulo 4.

13. Everyone at a math club is either a truth-teller, a liar, or a piggybacker. A truth-teller always tells the truth, a liar always lies, and a piggybacker will answer in the style of the previous person who spoke (i.e., if the person before told the truth, they will tell the truth, and if the person before lied, then they will lie). If a piggybacker is the first one to talk, they will randomly either tell the truth or lie. Four seniors in the math club were interviewed and here was their conversation:

    Neil: There are two liars among us.

    Lucy: Neil is a piggybacker.

    Kevin: Excluding me, there are more truth-tellers than liars here.

    Neil: Actually, there are more liars than truth-tellers if we exclude Kevin.

    Jacob: One plus one equals three.

    Define the base-4 number $M = \overline{NLKJ}_4$, where each digit is 1 for a truth-teller, 2 for a piggybacker, and 3 for a liar (N corresponds to Neil, L to Lucy, K corresponds to Kevin, and J corresponds to Jacob). What is the sum of all possible values of M, expressed in base 10?

    Solution. The answer is $\boxed{282}$.

    We do casework on Neil. Suppose that Neil is a piggybacker. Because Neil disagrees with Kevin on whether or not there are more truth-tellers or liars, they cannot both be telling the truth. Since Neil is a piggybacker, both him and Kevin must be lying, which can only happen when the number of truth-tellers and liars among Neil, Lucy, and Jacob are equal. Note that Lucy and Jacob cannot both be piggybackers, because Lucy was telling the truth when she said "Neil is a piggybacker", implying that Neil is telling the truth when he says "there are two liars". This is impossible if there are three piggybackers. Thus Lucy is the truth-teller and Jacob is the liar. Finally, since Lucy was telling the truth and Kevin lies immediately afterwards, Kevin must be a liar.

    Now suppose that Neil is a liar. Considering Neil's statement that "there are more liars than truth-tellers if we exclude Kevin", we know that there are at least as many truth-tellers as liars when Kevin is excluded. Since Neil is a liar, one of Lucy or Jacob must be a truth-teller. However, Lucy cannot be the truth-teller because her claim that Neil is a piggybacker is false. Jacob also cannot be the truth-teller because one plus one does not equal three. Therefore, this case is impossible.

    Finally, we suppose that Neil is a truth-teller. This means that both Lucy and Jacob are liars, because they both tell a lie after Neil speaks (and therefore cannot be piggybackers). Since Neil reveals that there are exactly two liars, Kevin must be a piggybacker.

    So the two cases are $\overline{NLKJ} = 2133$ and $\overline{NLKJ} = 1323$. In base ten, these numbers are 159 and 123, for a sum of 282.

14. An equilateral triangle of side length 8 is tiled by 64 equilateral triangles of unit side length to form a triangular grid. Initially, each triangular cell is either living or dead. The grid evolves over time under the following rule: every minute, if a dead cell is edge-adjacent to at least two living cells, then that cell becomes living, and any living cell remains living. Given that every cell in the grid eventually evolves to be living, what is the minimum possible number of living cells in the initial grid?

    Solution. The answer is $\boxed{22}$.

    We define the edge of a living cell to be a *boundary edge* if it is either on the edge of the grid, or it borders a dead cell. If there are n cells alive at the start, then there are at most $3n$ boundary edges at the start. Once every cell becomes alive, there are $3 \cdot 8 = 24$ boundary edges in the final grid.

    The key insight is that every time a cell transforms from dead to living, at least one boundary edge is lost. If the dead cell bordered two living cells, the two living cells each lose a boundary edge and the final edge of the dead cell becomes a new boundary edge (for a net loss of $2 - 1 = 1$ boundary edge). If the dead cell bordered three living cells, then a total of three boundary edges are lost. This means that since $64 - n$ dead cells need to become living, we must lose at least $64 - n$ boundary edges, so $3n - (64 - n) \geq 24$. Solving, we get $n \geq 22$.

    There are many possible constructions of 22 initial cells, of which the following is just one example (filled-in triangles represent living cells, while empty triangles represent dead cells):

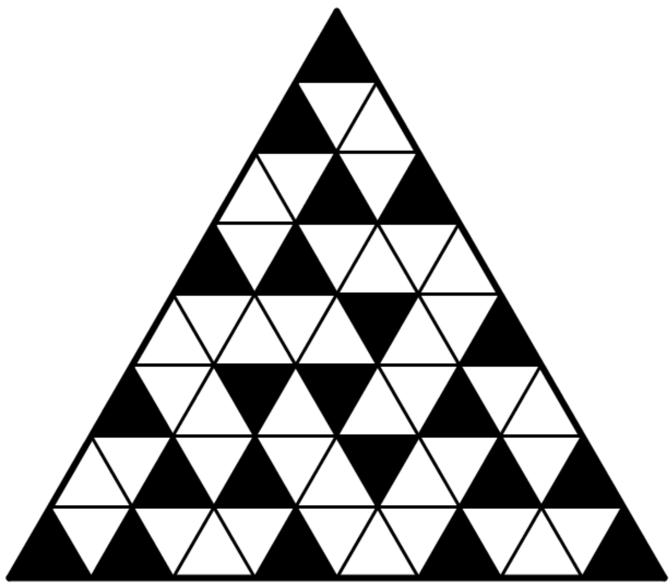


Figure 2.7: A possible construction.

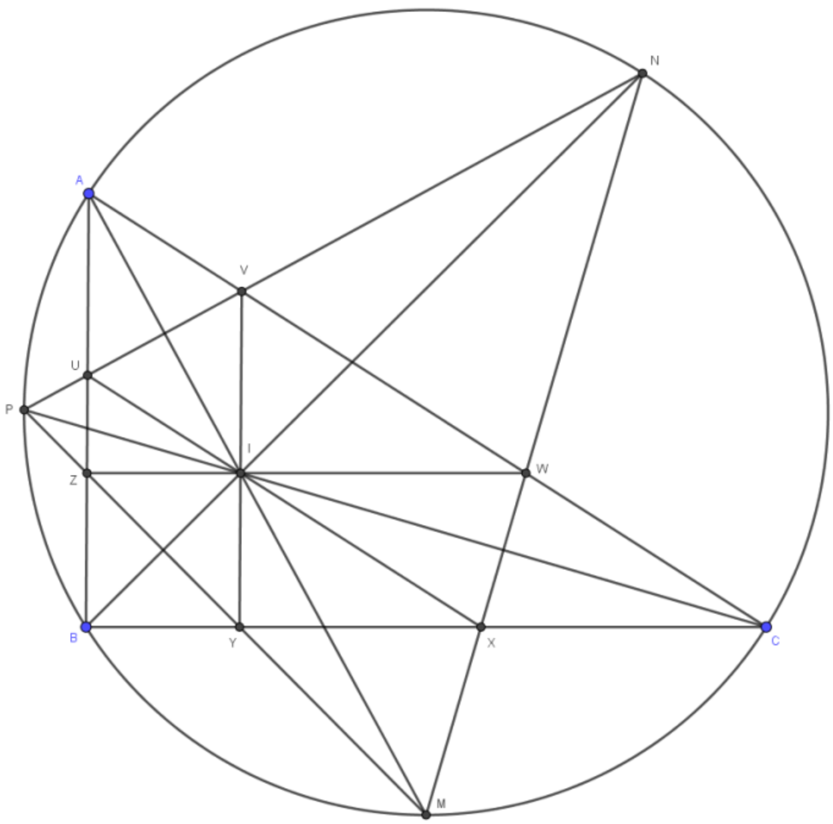
15. In triangle ABC, $AB = 7$, $BC = 11$, and $CA = 13$. Let Γ be the circumcircle of ABC and let M, N, and P be the midpoints of minor arcs $\widehat{BC}$, $\widehat{CA}$, and $\widehat{AB}$ of Γ, respectively. Given that $\mathcal{K}$ denotes the area of ABC and $\mathcal{L}$ denotes the area of the intersection of ABC and MNP, the ratio $\frac{\mathcal{L}}{\mathcal{K}}$ can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $a + b$.

Solution. The answer is $\boxed{1611}$.

Let I denote the incenter of ABC. We'll use $l_1 \cap l_2$ to denote the intersection of lines l_1 and l_2. Set $U = AB \cap NP$, $V = AC \cap NP$, $W = AC \cap NM$, $X = BC \cap NM$, $Y = BC \cap MP$, and $Z = AB \cap MP$.

Lemma: $MC = MI = MB$.

Proof: We can calculate $\angle MIB = 180^\circ - \angle AIB = \angle IAB + \angle IBA = \frac{\angle A}{2} + \frac{\angle B}{2}$ and $\angle MBI = \angle MBC + \angle CBI = \angle MAC + \angle CBI = \frac{\angle A}{2} + \frac{\angle B}{2}$, hence $\angle MIB = \angle MBI \implies MB = MI$. By symmetry, $MC = MI$ hence $MB = MC = MI$. This proves the lemma.



By symmetry, we must also have that $PA = PI = PB$ and $NA = NI = NC$. Because of this, we see that NP is in fact the perpendicular bisector of AI. Therefore, $UA = UI$, $VA = VI$, and $UV \perp AI$. Yet since $\angle IAU = \angle IAV$, by ASA congruence $\triangle AIU \cong \triangle AIV$. Hence, $UA = UI = VI = VA$, so $IUAV$ is a rhombus. Similarly, $IYBZ$ and $IWCX$ are rhombi.

This implies that $IU \parallel AC \parallel IX$, hence U and X are on the line through I parallel to AC. Similarly, V, Y are on the line through I parallel to AB and W, Z are on the line through I parallel to BC.

With these key synthetic observations, we may begin the computations. Set $a = BC, b = CA$, and $c = AB$ for simplicity. Using $[\mathcal{F}]$ to denote the area of $\mathcal{F}$, we may write the area of hexagon $UVWXYZ$ as $[IUV]+[IVW]+[IWX]+[IXY]+[IYZ]+[IZU]$. However, because $IUAV$, $IYBZ$, and $IWCX$ are rhombi, we know that $[IUV] = \frac{1}{2}[IUAV]$, $[IYZ] = \frac{1}{2}[IYBZ]$, and $[IWX] = \frac{1}{2}[IWCX]$.

Now observe that the sides of IYX are parallel to the corresponding sides of ABC. So, $\angle IYX = \angle ABC$ and $\angle IXY = \angle ACB$, hence $\triangle IYX \sim \triangle ABC$. In addition, letting $D = AM \cap BC$, we see that $\frac{IY}{AB} = \frac{DI}{DA} = \frac{DX}{DC} = \frac{DY}{DB}$, and so D is the center of the dilation taking IXY to ABC. Then by the Angle Bisector Theorem, $\frac{DI}{IA} = \frac{DB}{BA} = \frac{DC}{CA} = \frac{DB+DC}{BA+CA} = \frac{a}{b+c}$. It follows that $\frac{IY}{AB} = \frac{DI}{DA} = \frac{a}{a+b+c}$. So, the ratio $\frac{[IYX]}{[ABC]} = \frac{a^2}{(a+b+c)^2}$. Similarly, $\frac{[IWV]}{[ABC]} = \frac{b^2}{(a+b+c)^2}$ and $\frac{[IUZ]}{[ABC]} = \frac{c^2}{(a+b+c)^2}$.

To conclude, letting $[IUV] + [IYZ] + [IWX] = S_1$ and $[IVW] + [IXY] + [IZU] = S_2$, we have established that $2S_1 + S_2 = [ABC]$ and $S_2 = \frac{a^2+b^2+c^2}{(a+b+c)^2}[ABC]$. Adding the equations and dividing by 2, $S_1 + S_2 = \frac{1}{2}\left(1 + \frac{a^2+b^2+c^2}{(a+b+c)^2}\right)[ABC]$. Since by definition $\mathcal{L} = S_1 + S_2$ and $\mathcal{K} = [ABC]$, we have that

$\frac{\mathcal{L}}{\mathcal{K}} = \frac{1}{2}\left(1 + \frac{a^2+b^2+c^2}{(a+b+c)^2}\right)$. Finally, we plug in our numbers and get a ratio of $\frac{1}{2}\left(1 + \frac{7^2+11^2+13^2}{(7+11+13)^2}\right) = \frac{650}{961}$. Hence the answer is $650 + 961 = 1611$.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [6] Let $ABCDEF$ be a regular hexagon. How many acute triangles have all their vertices among the vertices of $ABCDEF$?

   Solution. The answer is $\boxed{2}$.

   The only two such triangles are ACE and BDF, since all other triangles have an angle of degree measure either 90° or 120°.

2. [6] A rectangle has a diagonal of length 20. If the width of the rectangle is doubled, the length of the diagonal becomes 22. Given that the width of the original rectangle is w, compute w^2.

   Solution. The answer is $\boxed{28}$.

   Let l be the length of both rectangles. Then by the Pythagorean Theorem, $w^2 + l^2 = 20^2$ and $4w^2 + l^2 = 22^2$. Subtracting the two equations, $3w^2 = 84$, hence $w^2 = 28$.

3. [6] The number $\overline{2022A20B22}$ is divisible by 99. What is $A + B$?

   Solution. The answer is $\boxed{6}$.

   Using the divisibility rules for 9 and 11, we see that $2 + 2 + B + 0 + 2 + A + 2 + 2 + 0 + 2$ is a multiple of 9 and $2 - 2 + B - 0 + 2 - A + 2 - 2 + 0 - 2$ is a multiple of 11. Therefore, $A + B \equiv 6 \pmod 9$ and $B - A \equiv 0 \pmod{11}$. Because $0 \le A, B \le 9$, we see that $B = A$. So, $A = B = 3$ (as $A + B = 15$ yields non-integer solutions). So, the answer is $3 + 3 = 6$.

2.4.2 Round 2

4. [7] How many two-digit positive integers have digits that sum to at least 16?

   Solution. The answer is $\boxed{6}$.

   Listing such two-digit numbers out, we find $79, 88, 89, 97, 98$, and 99, yielding 6 numbers.

5. [7] For how many integers k less than 10 do there exist positive integers x and y such that $k = x^2 - xy + y^2$?

   Solution. The answer is $\boxed{5}$.

   Assume without loss of generality that $x \ge y$. Notice that $10 > x^2 - xy + y^2 = (x - y)^2 + xy$, so $x - y \le 3$ and $xy < 10$, and (x, y) can only be $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, $(4, 1)$, and $(4, 2)$. By checking these cases, the only possible k are $1, 3, 4, 7, 9$, hence the answer is 5.

6. [7] Isosceles trapezoid $ABCD$ is inscribed in a circle of radius 2 with $AB \parallel CD$, $AB = 2$, and one of the interior angles of the trapezoid equal to $110°$. What is the degree measure of minor arc $\widehat{CD}$?

Solution. The answer is $\boxed{140}$.

Let O be the center of the circumcircle of $ABCD$. Then $OA = OB = AB$ hence OAB is equilateral. Therefore, the measure of minor arc $\widehat{AB}$ is $60°$.

Now, if $\angle ABC = 110°$, then $\widehat{ADC} = 220°$, hence $\widehat{BC} = \widehat{AD} = 160°$. But then $\widehat{DABC} = 380° > 360°$, which is a contradiction.

Therefore, $\angle BCD = 110°$. Hence, let $\widehat{BC} = \widehat{DA} = x°$ and $\widehat{CD} = y°$, where all these arcs are the minor arcs. Then, $x + y = 220°$ and $2x + y = 300°$. Subtracting, $x = 80°$ and so $y = 140°$, thus the answer is 140.

2.4.3 Round 3

7. [9] In rectangle $ALEX$, point U lies on side EX so that $\angle AUL = 90°$. Suppose that $UE = 2$ and $UX = 12$. Compute the square of the area of $ALEX$.

Solution. The answer is $\boxed{4704}$.

Let P be the foot of the perpendicular from U to side AL. Then $\angle APU = \angle UPL = 90°$ and $\angle PAU = 90° - \angle ULA = \angle PUL$, hence $\triangle APU \sim \triangle UPL$. In addition, $APUX$ and $LEUX$ are both rectangles. Therefore, $\frac{AP}{UP} = \frac{PU}{PL}$, $AP = 12$, and $PL = 2$. Hence, $UP^2 = 24$. Therefore, the area of $ALEX$ is $AL \cdot UP = (12 + 2) \cdot \sqrt{24}$, so the square of the area of $ALEX$ is $14^2 \cdot 24 = 4704$.

8. [9] How many digits does 20^{22} have?

Solution. The answer is $\boxed{29}$.

Note that $2^{10} = 1024$ and so $2^{20} = 1048576$ is between 10^6 and $2 \cdot 10^6$. Then $4 \cdot 10^6 < 2^{22} < 8 \cdot 10^6$. Hence, $20^{22} = 10^{22} \cdot 2^{22}$, so $4 \cdot 10^{28} < 20^{22} < 8 \cdot 10^{28}$. Observe that $4 \cdot 10^{28}$ and $8 \cdot 10^{28}$ both have 29 digits. Therefore, 20^{22} must have exactly 29 digits as well, so the answer is 29.

9. [9] Compute the units digit of $3 + 3^3 + 3^{3^3} + \ldots + 3^{3^{\cdot^{\cdot^{3}}}}$, where the last term of the series has 2022 3s.

Solution. The answer is $\boxed{0}$.

Consider 3^{3^a} where a is an odd positive integer. Since $a \equiv 1 \pmod 2$, we know $3^a \equiv 3 \pmod 4$, so $3^{3^a} \equiv 7 \pmod{10}$. Since any positive power of 3 is odd, the expression is equivalent to $3+7+7+\cdots+7 \pmod{10}$ where 2021 7s are summed. This is congruent to $3 + 2021 \cdot 7 \equiv 3 + 7 \equiv 0 \pmod{10}$.

2.4.4 Round 4

10. [11] Given that $\sqrt{x-1}+\sqrt{x}=\sqrt{x+1}$ for some real number x, the number x^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. The answer is $\boxed{7}$.

Squaring both sides, $2x-1+2\sqrt{x^2-x}=x+1$, or $2-x=2\sqrt{x^2-x}$. Squaring again, $4-4x+x^2=4x^2-4x$, so $3x^2=4$. Therefore, $x^2=\frac{4}{3}$, hence the answer is $4+3=7$.

Note that $x=\sqrt{\dfrac{4}{3}}$ is not an extraneous solution for the following reason. Consider the function $f(x)=\sqrt{x–1}+\sqrt{x}-\sqrt{x+1}$, whose domain is all real numbers greater than or equal to 1. Then because f is continuous and $f(1)=1-\sqrt{2}<0$ and $f(2)=1+\sqrt{2}–\sqrt{3}>0$, it follows that there must be a value of x between 1 and 2 such that $f(x)=0$.

11. [11] Eric the Chicken Farmer arranges his 9 chickens in a 3-by-3 grid, with each chicken being exactly one meter away from its closest neighbors. At the sound of a whistle, each chicken simultaneously chooses one of its closest neighbors at random and moves $\frac{1}{2}$ of a unit towards it. Given that the expected number of pairs of chickens that meet can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers, compute $p+q$.

Solution. The answer is $\boxed{8}$.

Note that no three chickens can meet at a point, since they must meet at the midpoint of a unit segment which they begin at a vertex, but each segment has only two endpoints. Thus the expected number of pairs of chickens that meet equals the expected number of edges for which the chickens at the endpoint of the edge meet.

Now, for each edge of the grid, we calculate the probability that the two chickens on this edge meet. Observe that by linearity of expectation, the expected number of edges for which the chickens at their endpoints meet equals the sum of all of these probabilities.

Now, for the 8 edges that lie completely on the boundary of the square grid, the probability is $\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$. For the other 4 edges, the probability is $\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$. Thus, the sum of the probabilities over all the edges is $8\cdot\frac{1}{6}+4\cdot\frac{1}{12}=\frac{5}{3}$. Hence the answer is $5+3=8$.

12. [11] For a positive integer n, let $s(n)$ denote the sum of the digits of n in base 10. Find the greatest positive integer n less than 2022 such that $s(n)=s(n^2)$.

Solution. The answer is $\boxed{1999}$.

First, because $s(n)\equiv n \pmod 9$ for every nonnegative integer n, we know that $n\equiv n^2 \pmod 9 \implies n\equiv 0,1 \pmod 9$. Now, we may proceed to check such numbers. We see that 2017^2 ends in 289, 2016^2 ends in 256, 2008^2 ends in 64 and begins with 4, and 2007^2 ends in 49, hence $s(n^2)>s(n)$ for each of these numbers. However, $1999^2=3996001$, and so $s(1999)=s(1999^2)=28$. Therefore, the answer is 1999.

2.4.5 Round 5

13. [13] Find the number of six-digit positive integers that satisfy all of the following conditions:

    (i) Each digit does not exceed 3.
    (ii) The number 1 cannot appear in two consecutive digits.
    (iii) The number 2 cannot appear in two consecutive digits.

    Solution. The answer is $\boxed{1624}$.

    We define a_k to be the number of k-digit positive integers that satisfy the above and have units digit 1 or 2. Similarly define b_k to be the number of k-digit positive integers that satisfy the above and have units digit 0 or 3. Note that to construct a $(k+1)$-digit number, we can append a digit to the end of a valid k-digit number. There is one way to append a 1 or a 2 after a 1 or a 2, and two ways to append a 1 or a 2 after a 0 or a 3, so we get the recursion. $a_{k+1} = a_k + 2b_k$. Similarly, since there are two ways to append a 0 or a 3 after any number, $b_{k+1} = 2a_k + 2b_k$. Thus the following table can be constructed.

k	a_k	b_k
1	2	1
2	4	6
3	16	20
4	56	72
5	200	256
6	712	912

    The desired number of six-digit numbers is $a_6 + b_6 = 1624$.

14. [13] Find the sum of all distinct prime factors of 103040301.

    Solution. The answer is $\boxed{161}$.

    Remark that $103040301 = 104060401 - 1020100 = 10201^2 - 1010^2 = 11211 \times 9191 = 3 \times 3737 \times 9191 = 3 \times 37 \times 13 \times 7 \times 101^2$. Therefore, the answer is $3 + 7 + 13 + 37 + 101 = 161$.

15. [13] Let $ABCA'B'C'$ be a triangular prism with height 3 where bases ABC and $A'B'C'$ are equilateral triangles with side length $\sqrt{6}$. Points P and Q lie inside the prism so that $ABCP$ and $A'B'C'Q$ are regular tetrahedra. The volume of the intersection of these two tetrahedra can be expressed in the form $\frac{\sqrt{m}}{n}$, where m and n are positive integers and m is not divisible by the square of any prime. Find $m + n$.

    Solution. The answer is $\boxed{35}$.

    The intersection of the two tetrahedra comes in the form of two congruent regular tetrahedra placed face to face. The height of a regular tetrahedron with side length s is $\frac{\sqrt{6}}{3}s$, so the height of each of $ABCP$ and $A'B'C'Q$ is 2. Now, the total height of the intersection is the overlap of the two heights of $ABCP$ and $A'B'C'Q$, or 1, thus the height of each of the small overlap tetrahedra is $\frac{1}{2}$. This means the side length of each of the small tetrahedra is $\frac{\sqrt{6}}{4}$. Now, the volume of a regular tetrahedron with side length s is $\frac{\sqrt{2}}{12}s^3$, so the volume of each of the small overlap tetrahedra is $\frac{\sqrt{3}}{64}$. The volume of the overlap is double that, or $\frac{\sqrt{3}}{32}$, giving $m + n = 35$.

2.4.6 Round 6

16. [15] Let $a_0, a_1, \ldots$ be an infinite sequence such that $a_n^2 - a_{n-1}a_{n+1} = a_n - a_{n-1}$ for all positive integers n. Given that $a_0 = 1$ and $a_1 = 4$, compute the smallest positive integer k such that a_k is an integer multiple of 220.

Solution. The answer is $\boxed{19}$.

Rearranging the given recurrence, $a_n(a_n - 1) = a_{n-1}(a_{n+1} - 1)$, hence

$$\frac{a_n - 1}{a_{n-1}} = \frac{a_{n+1} - 1}{a_n}$$

Therefore, $\frac{a_k - 1}{a_{k-1}}$ is a constant c for all $k \geq 1$. Letting $n = 1$, we see that $c = 3$. Therefore, $a_n = 3a_{n-1} + 1$.

Solving this recurrence, we note $a_n + \frac{1}{2} = 3(a_{n-1} + \frac{1}{2})$. Therefore $a_n + \frac{1}{2} = \frac{3^{n+1}}{2}$, so $a_n = \frac{3^{n+1}-1}{2}$.

So, we need to find the least k such that $3^k - 1$ is a multiple of 440.

To do this, let $f(q)$ be the least m such that $3^m \equiv 1 \pmod{q}$. If $3^k \equiv 1 \pmod{q}$, then we claim that $f(q)$ must divide k. Indeed, otherwise we can write $\gcd(f(q), k) = af(q) + bk$ for integers a and b by Bezout's Lemma, and thence $3^{\gcd(f(q),k)} \equiv 1 \pmod{q}$ yet $\gcd(f(q), k) < f(q)$, contradicting the minimality of $f(q)$.

So, calculating $f(8) = 2$, $f(5) = 4$, and $f(11) = 5$ by inspection, we conclude that k must be a multiple of $2, 4$, and 5, hence the least possible value of k is 20, which works. Hence the answer is 19.

17. [15] Vincent the Bug is on an infinitely long number line. Every minute, he jumps either 2 units to the right with probability $\frac{2}{3}$ or 3 units to the right with probability $\frac{1}{3}$. The probability that Vincent never lands exactly 15 units from where he started can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p + q$?

Solution. The answer is $\boxed{3437}$.

We will use complementary counting and determine the probability that at some point, Vincent is able to land exactly 15 units from his starting point. Since 15 is a multiple of 3, we know that the number of 2-unit jumps Vincent must make is a multiple of 3. Now we may split into cases.

Case 1: Vincent makes 0 jumps of 2 units. Then to reach 15 units, he must make 5 jumps of 3 units. The probability this occurs is $(\frac{1}{3})^5 = \frac{1}{243}$.

Case 2: Vincent makes 3 jumps of 2 units. Then to reach 15 units, he must make 3 jumps of 3 units. The probability this occurs is $\binom{6}{3} \cdot (\frac{2}{3})^3(\frac{1}{3})^3 = \frac{160}{729}$.

Case 3: Vincent makes 6 jumps of 2 units. Then to reach 15 units, he must make 1 jump of 3 units. The probability this occurs is $\binom{7}{1} \cdot (\frac{2}{3})^6(\frac{1}{3}) = \frac{448}{2187}$.

Adding up these cases, the probability of landing 15 units away at some point is $\frac{1}{243} + \frac{160}{729} + \frac{448}{2187} = \frac{937}{2187}$, so our desired probability is $1 - \frac{937}{2187} = \frac{1250}{2187}$ and the answer is $1250 + 2187 = 3437$.

18. [15] Battler and Beatrice are playing the "Octopus Game." There are 2022 boxes lined up in a row, and inside one of the boxes is an octopus. Beatrice knows the location of the octopus, but Battler does not. Each turn, Battler guesses one of the boxes, and Beatrice reveals whether or not the octopus

is contained in that box at that time. Between turns, the octopus teleports to an adjacent box and secretly communicates to Beatrice where it teleported to. Find the least positive integer B such that Battler has a strategy to guarantee that he chooses the box containing the octopus in at most B guesses.

Solution. The answer is $\boxed{4040}$.

Let's call a box *valid* if it is possible for an octopus to exist in that box. At the start of the game, all boxes are valid, and Battler has a winning strategy if and only if he can make every box invalid. We can reinterpret the rules of the game as follows: every turn, Battler chooses one box to make invalid. When the octopus moves, the following occurs: a box becomes valid if and only if it is adjacent to a valid box, otherwise it becomes invalid.

The key idea is parity. We will define the terms *odd-valid* and *even-valid* to refer to boxes which are valid for an octopus which started in an odd-numbered box and even-numbered box respectively. For example, the odd-valid boxes are $\{1, 3, \ldots, 2021\}$ at the start, but become $\{2, 4, 6, \ldots, 2022\}$ after moving once (assuming that Battler does not make make any boxes invalid through his guesses). Note that a box can never be both odd-valid and even-valid. We say that a guess *targets odds* if it targets an odd-valid box, and *targets evens* similarly. We will try to show that Battler needs to target odds at least 2020 times to make it so that there are no odd-valid boxes remaining.

Note that Battler can only eliminate all odd-valid boxes by guessing at either 2 or 2021. This is because the only way for there to be exactly one odd-valid box before Battler makes his guess is if the only odd-valid box before the octopus moves is either 1 or 2022, which forces 2 and 2021 respectively to be the last odd-valid box.

Without loss of generality, suppose that the last odd-valid box is 2. In that case, we will keep track of the greatest odd-valid box, which is always at least 2021 before Battler starts targeting odds. Note that right before Battler makes a guess, the odd-valid boxes always form an arithmetic sequence of common difference 2 from the least odd-valid box to the greatest odd-valid box (if Battler tries to split this sequence somewhere in the middle, the subsequent octopus movement will reconnect the odd-valid boxes). This means that when Battler guesses the greatest odd-valid box, the new greatest odd-valid box is temporarily 2 less than the old one, but increases by 1 after octopus movement, for a net decrease of 1. Therefore, it takes $2021 - 2 = 2019$ guesses to make the greatest odd-valid box equal to 2, and one more guess to eliminate the final odd-valid box.

Because Battler needs to target odds 2020 times, and he needs to target evens another 2020 times, the minimum is $2020 + 2020 = 4040$. This is clearly achievable by always guessing the greatest odd-valid box until all odd-valid boxes are gone, and then guessing the least even-valid box until all even-valid boxes are gone (i.e., guess the following boxes in order: $2021, 2020, \ldots, 3, 2, 2, 3, \ldots, 2020, 2021$).

2.4.7 Round 7

19. [18] Given that $f(x) = x^2 - 2$ the number $f(f(f(f(f(f(f(2.5)))))))$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b. Find the greatest positive integer n such that 2^n divides $ab+a+b-1$.

Solution. The answer is $\boxed{129}$.

Write $2.5 = 2 + \frac{1}{2}$. Note that $f(a + \frac{1}{a}) = (a + \frac{1}{a})^2 - 2 = a^2 + \frac{1}{a^2}$. Thus,

$$f(f(f(f(f(f(f\left(2 + \frac{1}{2}\right))))))) = 2^{2^7} + \frac{1}{2^{2^7}}$$

So, $a = 2^{256}+1$ and $b = 2^{128}$. Then $ab+a+b-1 = 2^{384}+2^{128}+2^{256}+1+2^{128}-1 = 2^{129}(1+2^{127}+2^{255})$. Thus, the maximal n is 129.

20. [18] In triangle ABC, the shortest distance between a point on the A-excircle ω and a point on the B-excircle Ω is 2. Given that $AB = 5$, the sum of the circumferences of ω and Ω can be written in the form $\frac{m}{n}\pi$, where m and n are relatively prime positive integers. What is $m+n$? (Note: The A-excircle is defined to be the circle outside triangle ABC that is tangent to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and to the side BC. The B-excircle is defined similarly for vertex B.)

    Solution. The answer is $\boxed{23}$.

    Let ω be tangent to BC at D and Ω be tangent to BC at E. Then $DE = CD + CE = \frac{AB+BC-CA}{2} + \frac{AC+AB-BC}{2} = AB = 5$.

    Now, let $P \in \omega$ and $Q \in \Omega$ be points such that PQ is minimized. Then $OP + PQ + QO' \geq OO'$, where O is the center of ω and O' is the center of Ω. Furthermore, equality can hold when P and Q lie on segment OO'. Letting r be the radius of ω and R be the radius of Ω, we know $2 = OO' - r - R$.

    However, let X be a point such that $DOXE$ is a rectangle. Then O, E, and X are collinear because OE and EX are both perpendicular to AB. Thus, using the Pythagorean Theorem, $OO'^2 = OX^2 + XD^2$, or $OO'^2 = DE^2 + (r+R)^2 = 25 + (r+R)^2$.

    Combining these two equations, $25 + (r+R)^2 = (2+r+R)^2$, hence $25 = 4 + 4(r+R)$. Solving $r + R = \frac{21}{4}$, so the sum of the circumferences of ω and Ω is $2\pi(r+R) = \frac{21}{2}\pi$. Thus the answer is $21 + 2 = 23$.

21. [18] Let $a_0, a_1, ...$ be an infinite sequence such that $a_0 = 1$, $a_1 = 1$, and there exists two fixed integer constants x and y for which a_{n+2} is the remainder when $xa_{n+1}+ya_n$ is divided by 15 for all nonnegative integers n. Let t be the least positive integer such that $a_t = 1$ and $a_{t+1} = 1$ if such an integer exists, and let $t = 0$ if such an integer does not exist. Find the maximal value of t over all possible ordered pairs (x, y).

    Solution. The answer is $\boxed{60}$.

    First consider the sequence modulo 5. Notice that the sequence can be described as a function mapping a pair (x, y) to the very next consecutive pair of numbers in the sequence $(y, ax+by)$. Notice that this function is linear, meaning if both x, y are scaled by some factor, the entire sequence is scaled by that factor.

    So consider the numbers in the minimum period of the sequence. Each input pair $(x, y) \neq (0, 0)$ has four scaled versions: $(x, y), (2x, 2y), (3x, 3y), (4x, 4y)$ respectively. Since the ratio between consecutive appearances of these element is fixed, if the first one that appears is (cx, cy) then if $c^n \equiv 1 \mod 5$ where n is the number of these elements that appear. Thus $n \mid 4$ so $n = 1, 2, 4$. Notice if for any x we have all four appear, then every other element in the series also will have four copies, one between each of the elements $(x, y), (2x, 2y), (3x, 3y), (4x, 4y)$ in whatever order the appear in the sequence, so in this case $4 \mid t$ and $t \leq 24$ since clearly $(0, 0)$ cannot be a part of any sequence (after reaching $(0, 0)$, we cannot return to $(1,1)$). Otherwise, if no number appears four times, then if any number's scaled versions appear twice by the same logic every number in the series' scaled versions appear twice, so $2 \mid t$and $t \leq 12$ since at most half the elements in each set of scaled elements is permitted. Finally, there is the case when no set of scaled elements has its elements appear more than once in the sequence, in which case $t \leq 6$ since there are only 6 different scaled series. To maximize the set of divisors in all primes factors of t, we must have $t = 16, 20, 24$. Similarly, we can obtain that either $2 \mid t$ and $t \leq 8$ or $t \leq 4$ in the mod 3 case, so to maximize the set of divisors $t = 6, 8$.

Since both cases are independent of each other by the Chinese Remainder Theorem, the period of the sequence modulo 15 is the least common multiple of its periods mod 3 and mod 5, which can be checked to be largest when it is $lcm(20, 6) = 60$. The pair $(a, b) = (11, 1)$ achieves this maximal period, so we are done.

2.4.8 Round 8

22. [21] A *mystic square* is a 3 by 3 grid of distinct positive integers such that the least common multiples of the numbers in each row and column are the same. Let M be the least possible maximal element in a mystic square and let N be the number of mystic squares with M as their maximal element. Find $M + N$.

Solution. The answer is $\boxed{7796}$.

We will first show that M is 20, which can be achieved as follows:

20	2	3
1	12	5
6	10	4

To show that it is impossible to go under 20, we will let the common LCM be $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$. Now for any given index i, we note that in order for n to be the LCM of every row, there must be a multiple of $p_i^{e_i}$ in that row, and furthermore that these multiples must be distinct. The largest of these multiples is at least $3p_i^{e_i}$, and since we want the largest element in our mystic square to be less than 20, we get the inequality $3p_i^{e_i} < 20$ or $p_i^{e_i} \leq 6$. Thus the largest prime powers we can use in our mystic square are 3, 4, and 5. Of the numbers less than 20, only these nine numbers do not have larger prime powers as factors: $\{1, 2, 3, 4, 5, 6, 10, 12, 15\}$. However, it is impossible to form a mystic square with these numbers because we have only two multiples of 4: 4 and 12, and thus we cannot have a multiple of 4 in every row.

To count the number of ways to have 20 as our maximal element, we note that the only numbers we can use are $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20\}$, and that we must have exactly one of $\{4, 12, 20\}$ in each row and column. There are 36 ways to do this, so we can factor this out and assume WLOG that these three numbers are arranged along the diagonal as in the example above.

Now we just need to have a multiple of 3 in every row and column, and a multiple of 5 in every row and column. There are two multiples of just 3 remaining: $\{3, 6\}$, and two multiples of just 5: $\{5, 10\}$, and one multiple of both: $\{15\}$. There are also two numbers which do not matter and we can use to fill in the rest of the grid: $\{1, 2\}$. However, we should note that we already have a multiple of 3 in the grid as 12, and a multiple of 5 in 20. What's nice is that 3 and 5 play symmetric roles and we can exploit this symmetry to simplify our casework.

Case 1: We do not use 15. In this case, there are 2 ways to position the remaining multiples of 3 and 2 ways to position the remaining multiples of 5. In the diagram below we indicate cells that must be multiples of 3 with 3∗, cells that must be multiples of 5 with 5∗, and cells that can be anything with ∗.

20	*	3*
*	12	5*
3*	5*	4

Finally, we note that the two remaining numbers can be put into the ∗-marked cells in 2 ways, for a total of 8 possibilities.

Case 2: We use 15, and it shares a row/column with both 20 and 12. This case has two symmetries, so WLOG our diagram is like this:

20	15	?
?	12	?
?	?	4

The two remaining multiples of 5 must be in the bottom two rows, and there must be one in the rightmost column, which means that the middle cell in the rightmost column must contain a multiple of 5 (and a similar argument shows that the bottom left cell must be a multiple of 3):

20	15	?
?	12	5*
3*	?	4

Now it's clear where the other multiples of 3 and 5 belong:

20	15	3*
*	12	5*
3*	5*	4

Once again there are 8 possibilities for this case (as there are two possible values for ∗), which we multiply by 2 due to the symmetry mentioned at the start to yield 16 possibilities.

Case 3: The remaining 4 locations to put 15 are all symmetrical due to the symmetry of the grid and the symmetrical roles that 3 and 5 play. These locations share a row/column with exactly one of 12 and 20, but not the other, helping one of either 3 or 5 more than the other. WLOG we will consider a shape like this:

20	?	15
?	12	?
?	?	4

Multiples of 5 must occupy the bottom two rows, and one must also be in the center column, so the bottom center cell must be a multiple of 5. Since there must also be a multiple of 3 in the bottom row, the bottom left cell must be a multiple of 3. So we get this:

20	?	15
?	12	?
3*	5*	4

However, now we come across an interesting nuance. Both cells in the middle row can contain our final multiple of 5. Then the remaining two cells can be anything. So there are 2 ways to choose our first multiple of 5, 2 ways to choose the multiple of 3 in the bottom left, 2 ways to choose the location of our final multiple of 5, and $3 \cdot 2$ ways to choose the elements of the remaining two cells from our remaining three numbers. This yields a total of 48 possibilities, which when multiplied by 4 to account for the symmetrical scenarios, gives 192 possibilities.

The total number of such mystic squares is thus $36(8+16+192) = 7776$, and our answer is $M+N = 20+7776 = 7796$.

23. [21] In triangle ABC, $AB = 27$, $BC = 23$, and $CA = 34$. Let X and Y be points on sides AB and AC, respectively, such that $BX = 16$ and $CY = 7$. Given that O is the circumcenter of BXY, the value of CO^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Solution. The answer is $\boxed{1962}$.

Let point D lie on BC so that that $BD = 16$ and $DC = 7$. Let I be the incenter of triangle ABC. Then $BD = BX$ and $CD = CY$, so by SAS congruence, $\triangle BDI \cong \triangle BXI$. Hence, $ID = IX$. Similarly, $ID = IY$. Now, let Z denote the foot of the perpendicular from I to BC. Since Z is the tangency point of the incircle of ABC with BC, we know $BZ = \frac{23+27-34}{2} = 8$. Thus $DZ = 8$ as well. Therefore, $\triangle IZB \cong \triangle IZD$, hence $IB = ID$. Therefore $IB = ID = IX = IY$, hence I is the circumcenter of BXY. Therefore, $O = I$. So $CO^2 = CI^2 = CZ^2 + ZI^2$. Now, $CZ = BC - BZ = 15$ and ZI is the inradius. Thus using the two area formulas, $A = rs$ and Heron's formula, $(\frac{23+27+34}{2})ZI = \sqrt{42 \cdot 19 \cdot 15 \cdot 8}$, hence $ZI^2 = \frac{380}{7}$. In conclusion, $CO^2 = 15^2 + \frac{380}{7} = \frac{1955}{7}$, so the answer is $m+n = 1962$.

24. [21] Alan rolls ten standard fair six-sided dice, and multiplies together the ten numbers he obtains. Given that the probability that Alan's result is a perfect square is $\frac{a}{b}$, where a and b are relatively prime positive integers, compute a.

Solution. The answer is $\boxed{15019}$.

Let $x = 2, y = 3, z = 5$ be primes. Then $\frac{1+x+y+x^2+z+xy}{6}$ represents the potential prime factorization of the numbers one dice can roll where the coefficient represents the probability of that number occurring, where the exponent of x, y, z is the exponent of the respective primes. Then it follows that $f(x,y,z) = \left(\frac{1+x+y+x^2+z+xy}{6}\right)^{10}$ represents in the same way each prime factorization of the potential product of ten dice with the respective probability of it occurring as the coefficient. Then taking $g(x,y,z) = \frac{f(x,y,z)+f(-x,y,z)}{2}$, all monomials with an odd exponent of $x = 2$ will be canceled. Taking this further, taking $h(x,y,z) = \frac{g(x,y,z)+g(x,-y,z)}{2}$ cancels all monomials with an odd exponent of $y = 3$. Finally, taking $p(x,y,z) = \frac{h(x,y,z)+h(x,y,-z)}{2}$ cancels all monomials with an odd exponent of $z = 5$. So $p(x,y,z)$ consists only of those monomials in $f(x,y,z)$ that have even exponents of $2, 3, 5$, which is equivalent to those which correspond to perfect squares. By taking $p(1,1,1)$ we get the sum of the probabilities any of these numbers occur, which is equal to

$$\sum_{i,j,k \in \{-1,1\}} \frac{f(i,j,k)}{8} = \frac{6^{10} + 4^{10} + 2^{10} + 0^{10} + 2^{10} + 0^{10} + 2^{10} + 0^{10}}{8 \cdot 6^{10}}$$

$$= \frac{3^{10} + 2^{10} + 3}{8 \cdot 3^{10}}$$

Notice that the numerator is not a multiple of 3. The largest power of 2 that divides $3^{10} + 3$ is 4 by checking $3^9 + 1 \equiv 0 \pmod 4$ and $3^9 + 1 \equiv 4 \pmod 8$. So $a = \frac{3^{10}+2^{10}+3}{4} = 15019$.