Speed Round Solutions

EMCC

February 2021

1. Evaluate $20 \cdot 21 + 2021$.

   Solution. The answer is $\boxed{2441}$.

   $20 \cdot 21 + 2021 = 420 + 2021 = 2441$.

2. Let points A, B, C, and D lie on a line in that order. Given that $AB = 5CD$ and $BD = 2BC$, compute $\frac{AC}{BD}$.

   Solution. The answer is $\boxed{3}$.

   Let $CD = x$. Thus, $AB = 5x$, and since $BD = 2BC$, $BC = CD = x$. Hence, $AC = 6x$ and $BD = 2x$ so $\frac{AC}{BC} = \frac{6x}{2x} = 3$.

3. There are 18 students in Vincent the Bug's math class. Given that 11 of the students take U.S. History, 15 of the students take English, and 2 of the students take neither, how many students take both U.S. History and English?

   Solution. The answer is $\boxed{10}$.

   Since 2 of the students take neither class, 16 students take a least one of the two classes. Among these people, a total of $11 + 15 = 26$ classes are taken (only counting U.S. history and English), which means that $26 - 16 = 10$ students take both U.S. History and English.

4. Among all pairs of positive integers (x, y) such that $xy = 12$, what is the least possible value of $x + y$?

   Solution. The answer is $\boxed{7}$.

   The divisors of 12 are $1, 2, 3, 4, 6$, and 12, so the possible sums are $1 + 12 = 13$, $2 + 6 = 8$, and $3 + 4 = 7$. Hence, the answer is 7.

5. What is the smallest positive integer n such that $n! + 1$ is composite?

   Solution. The answer is $\boxed{4}$.

   We compute $1! + 1 = 2$, $2! + 1 = 3$, $3! + 1 = 7$ are all primes, but $4! + 1 = 25$ is composite, so our answer is 4.

6. How many ordered triples of positive integers (a, b, c) are there such that $a + b + c = 6$?

Solution. The answer is $\boxed{10}$.

First Solution: We use stars and bars. Consider 6 stars in a row, and place 2 bars in the 5 slots between consecutive stars (no two in the same slot). There are $\binom{5}{2} = 10$ ways to do this, and each one corresponds to an ordered triple (a, b, c) (the 2 bars split the stars into three groups). Thus, the answer is 10.

Second Solution: The possible unordered triples are: (i) $(1, 1, 4)$; (ii) $(1, 2, 3)$; and (iii) $(2, 2, 2)$, by considering the largest number. There are 3 ordered triples corresponding to the unordered triple (i), 6 corresponding to (ii), and 1 corresponding to (iii), so there are $3 + 6 + 1 = 10$ total ordered triples.

7. Thomas orders some pizzas and splits each into 8 slices. Hungry Yunseo eats one slice and then finds that she is able to distribute all the remaining slices equally among the 29 other math club students. What is the fewest number of pizzas that Thomas could have ordered?

   Solution. The answer is $\boxed{11}$.

   We count up the multiples of 29 and choose the first number that has a remainder of 7 when divided by 8. The first such number is 87. If there were 87 slices of slices distributed to the students, there must have been 88 slices of pizza before Yunseo ate one. Thus, Yunseo must have ordered $\frac{88}{8} = 11$ pizzas.

8. Stephanie has two distinct prime numbers a and b such that $a^2 - 9b^2$ is also a prime. Compute $a + b$.

   Solution. The answer is $\boxed{9}$.

   For $a^2 - 9b^2$ to be positive, it is necessary that $a > 3b$, so $a - 3b$ is a positive integer. $a^2 - 9b^2 = (a - 3b)(a + 3b)$. Since $a - 3b$ and $a + 3b$ are both positive integers, it is necessary that, for their product to be prime, one must equal 1. Since $a + 3b > a - 3b$, we know that $a - 3b = 1$. If b is an odd prime, it follows that $a = 3b + 1 \geq 10$, where a is an even prime. This is impossible. Thus, b must be an even prime, so testing $a = 7, b = 2$ suffices.

9. Let $ABCD$ be a unit square and E be a point on diagonal AC such that $AE = 1$. Compute $\angle BED$, in degrees.

   Solution. The answer is $\boxed{135}$.

   Observe that $AB = AE = 1$. So, triangle ABE is isosceles with base BE. Therefore $\angle BEA = \frac{180° - \angle BAE}{2} = \frac{135°}{2}$. Similarly, $\angle BED = \frac{135°}{2}$. Summing, $\angle BED = \angle BAE + \angle EAD = 135°$.

10. Sheldon wants to trace each edge of a cube exactly once with a pen. What is the fewest number of continuous strokes that he needs to make? A continuous stroke is one that goes along the edges and does not leave the surface of the cube.

    Solution. The answer is $\boxed{4}$.

    We can flatten a cube as shown below.
    Consider the degree of each vertex, which is the number of edges attached to each. Note

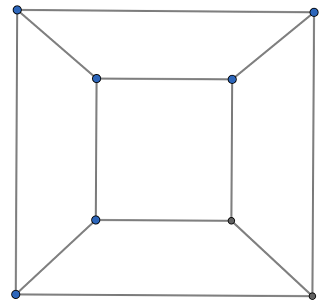

Figure 1: Flattened cube

that all 8 vertices have degree 3, which is odd. Also, for each continuous stroke, suppose we remove the edges that are traced. Then, all of the intermediate vertices (i.e. not the start or end vertex) have their degree reduced by an even number, as we can pair every "arriving" edge with a "leaving" edge. Thus, at most two vertices have their degree reduced by an odd number for each stroke, and since the degrees must all be zero at the end, the number of strokes must be at least half the number of odd-degree vertices. Thus, a minimum of 4 strokes is required, and this can be attained by many different tracings.

11. In base b, 130_b is equal to $3n$ in base ten, and 1300_b is equal to n^2 in base ten. What is the value of n, expressed in base ten?

    Solution. The answer is $\boxed{18}$.

    Rewriting the base b numerals in expanded form and expressing the equivalences in base 10, we have the following system:

    $$b^2 + 3b = 3n$$
    $$b^3 + 3b^2 = n^2$$

    Dividing the second equation by the first, we get $b = \frac{n}{3}$. Substituting back into the first equation, we are left with $b^2 + 3b = 9b$, from which we find $b = 6$ and $n = 18$ in base 10.

12. Lin is writing a book with n pages, numbered $1, 2, \ldots, n$. Given that $n > 20$, what is the least value of n such that the average number of digits of the page numbers is an integer?

    Solution. The answer is $\boxed{108}$.

    Since $n > 20$, the average being 1 is not possible because some of the page numbers will have at least 2 digits. So now let's try to find a value of n such that the average is 2, the next smallest integer.

    Any value of $n > 20$ under 100 would not work since digit counts of only 1 and 2 would result in an average strictly between 1 and 2 (not an integer). Thus, we must have some pages with three digits. In order to have a total average of 2, we need an equal number of pages with 1 and 3 digits, as all other pages will have 2 digits. Since there are 9 one-digit positive integers, we must have 9 three-digit integers, starting at 100 and ending at 108. Therefore, $n = 108$.

13. Max is playing bingo on a 5×5 board. He needs to fill in four of the twelve rows, columns, and main diagonals of his bingo board to win. What is the minimum number of boxes he needs to fill in to win?

Solution. The answer is $\boxed{14}$.

Note that Max needs to fill in four lines, which amounts to 20 boxes. However, whenever two lines intersect, Max "saves" a box when he fills in the intersection. Thus, as long as these intersections are all unique and exist, Max can save up to 6 boxes, so the minimum possible number of boxes he needs to fill in is $20 - 6 = 14$. This is attainable by filling in both long diagonals, along with the first row and second column.

14. Given that x and y are distinct real numbers such that $x^2 + y = y^2 + x = 211$, compute the value of $|x - y|$.

Solution. The answer is $\boxed{29}$.

Observe that $x^2 - y^2 = x - y \implies (x - y)(x + y - 1) = 0$. Since $x \neq y$, we must have $x + y - 1 = 0$. Therefore, $y = 1 - x \implies x^2 - x = 210$. Solving, $x = 15$ and $y = -14$ or vice versa, so the answer is $|15 - (-14)| = 29$.

15. How many ways are there to place 8 indistinguishable pieces on a 4×4 checkerboard such that there are two pieces in each row and two pieces in each column?

Solution. The answer is $\boxed{90}$.

We number the rows 1, 2, 3, and 4, and consider which rows the two pieces in each column are in. Thus, in total we must have 2 of each number and each column must have 2 different numbers. Without loss of generality, assume that the first column has 1 and 2. Then, we consider where the other 1 and other 2 are. If they are together, the two other columns must both be 3 and 4, a total of 3 possibilities. If they are not, then there are 6 choices for which columns they go in, and since the remaining column must be a 3 and 4, then there are 2 choices for where the other 3 and 4 go. Thus, this gives us a total of $6 \cdot 2 + 3 = 15$ cases, and since we assumed that the first column had 1 and 2, there are 6 choices there, giving us a final total of 90.

16. The *Manhattan distance* between two points (a, b) and (c, d) in the plane is defined to be $|a - c| + |b - d|$. Suppose Neil, Neel, and Nail are at the points $(5, 3)$, $(-2, -2)$ and $(6, 0)$, respectively, and wish to meet at a point (x, y) such that their Manhattan distances to (x, y) are equal. Find $10x + y$.

Solution. The answer is $\boxed{11}$.

Rather than solving a system of equations, we consider the locus of points equidistant from two of the points. There are three possible pairs to choose from, and all three loci are drawn in the diagram below.

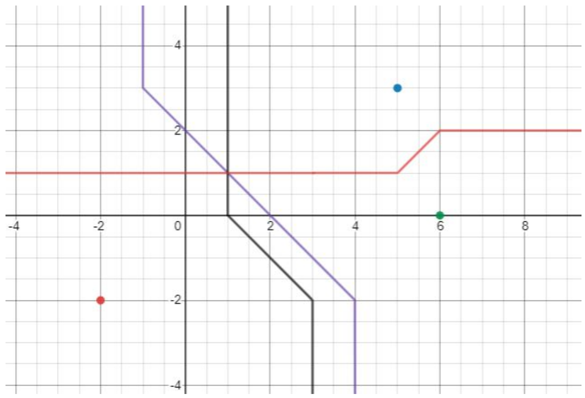

Figure 2: Manhattan Curves of Equidistance

It is trivial to see that the common intersection is $(1, 1)$. Hence the answer is $10 + 1 = 11$.

17. How many positive integers that have a composite number of divisors are there between 1 and 100, inclusive?

    Solution. The answer is $\boxed{67}$.

    Note that 1 is not composite, so we only consider numbers from 2 to 100. The smallest number that has 11 divisors is $2^{10} = 1024$. Therefore, any integer from 2 to 100 inclusive without a composite number of divisors must have 2, 3, 5, or 7 divisors. There are 25 prime numbers less than 100, which are exactly those with 2 divisors. A number has 3 prime divisors if and only if it is a square of a prime number, and since $10^2 = 100$, there are 4 prime numbers less than 10 and thus 4 numbers with 3 divisors. A number has 5 divisors if and only if it is the fourth power of a prime. There are two such primes with fourth powers less than 100, namely 2 and 3. A number has 7 prime divisors if it is a sixth power of a prime. There is one such number between 1 and 100, namely $2^6 = 64$. Therefore, there are $99-25-4-2-1 = 67$ numbers with a composite number of divisors.

18. Find the number of distinct roots of the polynomial

    $$(x-1)(x-2)\cdots(x-90)(x^2-1)(x^2-2)\cdots(x^2-90)(x^4-1)(x^4-2)\cdots(x^4-90).$$

    Solution. The answer is $\boxed{603}$.

    First, we have 360 distinct roots for the degree 4 factors, namely $k^{0.25}$, $-k^{0.25}$, $k^{0.25}i$, and $-k^{0.25}i$ for k from 1 through 90. Now, we have another 180 roots for the degree 2 factors, of the form $k^{0.5}$ and $-k^{0.5}$ for k from 1 through 90, but for $1 \le k \le 9$, these 18 roots are already counted by by the degree 4 factors, so we only have $360 + 180 - 18 = 522$ distinct roots. Lastly, the degree 1 factors give another 90 roots, namely the integers from 1 through 90, but the roots 1 through 9 are already counted. Hence, in total, we have $522+90-9 = 603$ distinct roots.

19. In triangle ABC, let D be the foot of the altitude from A to BC. Let P, Q be points on AB, AC, respectively, such that PQ is parallel to BC and $\angle PDQ = 90°$. Given that $AD = 25$, $BD = 9$, and $CD = 16$, compute $111 \cdot PQ$.

Solution. The answer is $\boxed{1875}$.

Let PQ intersect AD at E. Then, DE is the D-altitude in triangle PDQ. Therefore, $\angle PED = \angle DEQ = 90^\circ$ and $\angle PDE = 90^\circ - \angle EDQ = \angle DQE$. Hence, $\triangle PED \sim \triangle DEQ \implies \frac{PE}{ED} = \frac{DE}{EQ} \implies PE \cdot EQ = DE^2$. Let $AE = x$, then clearly $APE \sim ABD$, hence $PE = BD \cdot \frac{AE}{AD} = \frac{9x}{25}$ and similarly $QE = \frac{16x}{25}$. So, $DE = \frac{12x}{25}$. Then, we know that $x + \frac{12x}{25} = AE + ED = AD = 25$, hence $x = \frac{625}{37}$. This implies that $PQ = PE + EQ = \frac{25x}{25} = \frac{625}{37}$ so our answer is $111 \cdot \frac{625}{37} = 1875$.

20. The simplified fraction with numerator less than 1000 that is closest but not equal to $\frac{47}{18}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute p.

    Solution. The answer is $\boxed{974}$.

    Let the fraction be $\frac{a}{b}$. Then, the difference we are looking to minimize is $|\frac{47}{18} - \frac{a}{b}| = \frac{|47b-18a|}{18b}$. In order to minimize this, we can first consider when the numerator is 1. Then, we get that either $47b - 18a = 1$ or $47b - 18a = -1$, and we wish to maximize b given that a is at most 1000. We can check small numbers to see that the first equation has solution $(a, b) = (13, 5)$, and thus the general form of the solutions is $(a, b) = (47k + 13, 18k + 5)$, for some integer k. The maximal solution such that $a < 1000$ is when $k = 20$, giving $(a, b) = (953, 365)$. Meanwhile, we can get that the second equation has general solutions $(a, b) = (47n + 34, 18n + 13)$, giving us the largest solution again when $k = 20$ as $(a, b) = (974, 373)$, which has a larger b, and is the better solution. If the numerator is greater than 1, meanwhile, we can see that $18b$ would have to be much larger in order to have the difference be less than those we've already seen, thus a would be too large, so our best fraction is $\frac{974}{373}$.