Accuracy Round Solutions

EMCC

February 2021

1. Evaluate $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 19^2 - 20^2 + 21^2$.

   Solution. The answer is $\boxed{231}$.

   We can write the sum as $(21^2 - 20^2) + (19^2 - 18^2) + \cdots + (3^2 - 2^2) + 1^2$ which can be simplified using difference of squares to be $21 + 20 + 19 + 18 + ... + 3 + 2 + 1 = \frac{21 \cdot 22}{2} = 231$.

2. Kevin is playing in a table-tennis championship against Vincent. Kevin wins the championship if he wins two matches against Vincent, while Vincent must win three matches to win the championship. Given that both players have a 50% chance of winning each match and there are no ties, the probability that Vincent loses the championship can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

   Solution. The answer is $\boxed{27}$.

   Note that the championship must end in 4 games, so we can simply play out all 4 games (i.e. never end the championship early) since it will never change the winner. Kevin loses if and only if he loses at least 3 games, which happens with probability $\frac{\binom{4}{0}+\binom{4}{1}}{2^4} = \frac{5}{16}$. Thus, Vincent loses with probability $\frac{11}{16}$, so $a + b = 27$.

3. For how many positive integers n less than 2000 is n^{3n} a perfect fourth power?

   Solution. The answer is $\boxed{502}$.

   Note that n^{3n} is a perfect fourth power if and only if at least one of the following is true: n is a multiple of 4, n is a perfect square and n is a multiple of 2, or n is a perfect fourth power. Now, any even perfect squares are also multiples of 4, so we only need to consider the first and last cases. There are $\frac{2000}{4} - 1 = 499$ multiples of 4 from 0 to 2000 exclusive. The remaining values of n must all be perfect fourth powers that are not multiples of 4, so they are precisely the odd fourth powers. These are 1^4, 3^4, and 5^4, so there are a total of 502 values of n for which n^{3n} is a perfect fourth power.

4. Given that a coin of radius $\sqrt{3}\,\text{cm}$ is tossed randomly onto a plane tiled by regular hexagons of side length $14\,\text{cm}$, the chance that it lands strictly inside of a hexagon can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

   Solution. The answer is $\boxed{85}$.

We consider where the center of the coin lands. The coin is completely inside of a hexagon if and only if the nearest edge to the center is at least a distance of $\sqrt{3}$ away, and thus in each hexagon the center has a smaller regular hexagon to land in. We can calculate the similarity ratio between the hexagons using the distance separating opposite sides. In the bigger hexagon it is $14\sqrt{3}$, and for the smaller hexagon, it is reduced by twice the radius, giving $12\sqrt{3}$. Thus, the similarity ratio is $\frac{6}{7}$, giving a probability of $(\frac{6}{7})^2 = \frac{36}{49}$. Finally, $p + q = 85$.

5. Given that A, C, E, I, P, and M are distinct nonzero digits such that $\overline{EPIC} + \overline{EMCC} + \overline{AMC} = \overline{PEACE}$, what is the least possible value of $\overline{PEACE}$?

   Solution. The answer is $\boxed{19439}$.

   First, we cancel some digits both in the summands and the sum to rewrite the problem as $\overline{PIC} + \overline{EM0C} + \overline{MC} = \overline{P000E}$. Let G be the amount that caries over from the units digit addition, in other words, $G = \lfloor \frac{3C}{10} \rfloor$. Notice that $G \le 2$. Clearly, we want $P = 1$, and we know that $P + M + \lfloor \frac{I+C+M+G}{10} \rfloor$ is less than $1+9+3 = 13$ because $I+C+M+G \le 26$, hence the number carrying over to the thousands digit addition is at most 1, so $E = 9$. $M = 7$ or 8. If $M = 7$, $I + M + \lfloor \frac{3C}{10} \rfloor \ge 20$, which is impossible. Thus, $M = 8$. Because $3C$ has units digit 9, $C = 3$, which fixes $I = 2$. Because A is a leading digit and cannot repeat the existing values, $A = 4$. We have $\overline{PEACE} = 19439$.

6. A *palindrome* is a number that reads the same forwards and backwards. Call a number *palindrome-ish* if it is not a palindrome but we can make it a palindrome by changing one digit (we cannot change the first digit to zero). For instance, 4009 is palindrome-ish because we can change the 4 to a 9. How many palindrome-ish four-digit numbers are there?

   Solution. The answer is $\boxed{1620}$.

   We have 2 different forms of palindrome-ish numbers: $\overline{AAAB}$ with B at any position, and $\overline{ABCA}$ with either A outside and inside. For the former type, because we can't have leading $0s$, we have in total $9 \cdot 9 \cdot 4 = 324$. For the latter type, we have $9 \cdot 9 \cdot 8 \cdot 2 = 1296$, which add up to 1620.

7. Given that the heights of triangle ABC have lengths $\frac{15}{7}$, 5, and 3, what is the square of the area of ABC?

   Solution. The answer is $\boxed{75}$.

   Since the product of a side and its corresponding height are equal in a triangle, the sides are in ratio of 3 to 5 to 7. Since a 3-5-7 triangle has area $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{15}{2} \cdot \frac{1}{2} \cdot \frac{5}{2} \cdot \frac{9}{2}} = \frac{15\sqrt{3}}{4}$ by Heron's formula, the heights are $\frac{15\sqrt{3}}{14}$, $\frac{5\sqrt{3}}{2}$, and $\frac{3\sqrt{3}}{2}$. Thus, we need to scale this triangle up by a factor of $\frac{2\sqrt{3}}{3}$, giving us an area of $\frac{15\sqrt{3}}{4} \cdot \left(\frac{2\sqrt{3}}{3}\right)^2 = 5\sqrt{3}$. Our final answer is $(5\sqrt{3})^2 = 75$.

8. Suppose that cubic polynomial $P(x)$ has leading coefficient 1 and three distinct real roots in the interval $[-20, 2]$. Given that the equation $P(x + \frac{1}{x}) = 0$ has exactly two distinct real solutions, the range of values that $P(3)$ can take is the open interval (a, b). Compute $b - a$.

Solution. The answer is $\boxed{570}$.

Since $P(x + \frac{1}{x}) = 0$ has exactly two real solutions, either $P(x)$ has exactly one real root outside the closed interval $[-2, 2]$ or $P(x)$ has roots -2 and 2 (and no roots outside $[-2, 2]$). This is because any roots outside of $[-2, 2]$, say r, will provide two values of x for which $x + \frac{1}{x} = r$, while $r = 2$ or $r = -2$ will provide exactly one value of x each. Now, if $P(x)$ has roots 2 and -2, the last root must be in the open interval $(-2, 2)$ so the polynomial is $P(x) = (x + 2)(x - 2)(x - c)$ for some $-2 < c < 2$. Thus, $P(3) = 5(3 - c)$, which must be in the range $(5, 25)$. Now, if $P(x)$ instead has exactly one root outside the closed interval $[-2, 2]$, then the root must be in the interval $[-20, -2]$, so let it be $-20 \leq c \leq -2$. Suppose the other two roots are d and e, so $-2 < d, e < 2$. Now, $P(x) = (x - c)(x - d)(x - e)$ so $P(3) = (3 - c)(3 - d)(3 - e) < (23)(5)(5) = 575$. Also, $P(3) > (5)(1)(1) = 5$, so $P(3)$ lies in the range $(5, 575)$. Note that this completely includes the interval from the other case, so our answer is $b - a = 570$.

9. Vincent the Bug has 17 students in his class lined up in a row. Every day, starting on January 1, 2021, he performs the same series of swaps between adjacent students. One example of a series of swaps is: swap the 4th and the 5th students, then swap the 2nd and the 3rd, then the 3rd and the 4th. He repeats this series of swaps every day until the students are in the same arrangement as on January 1. What is the greatest number of days this process could take?

   Solution. The answer is $\boxed{210}$.

   Note that each day, the series of swaps leads to the exact same mapping between places in the line. Specifically, for $1 \leq i \leq 17$, the student at position i will move to position $p(i)$ the next day, where p is a bijection and thus permutation from the numbers 1 through 17 to itself. Thus, each connected component of the permutation must be precisely a simple cycle, and a cycle of length l will only reset to the original arrangement after a multiple of l days, so our answer is the least common multiple of all cycle lengths. The sum of these lengths is 17, and a little casework shows that the maximum possible LCM is $\text{lcm}(2, 3, 5, 7) = 210$.

10. The summation

$$\sum_{i=1}^{18} \frac{1}{i}$$

   can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute the number of divisors of b.

   Solution. The answer is $\boxed{320}$.

   Let $N = \text{lcm}(1, 2, 3, \ldots, 18)$. The summation can be rewritten as $\frac{1}{N} \cdot \sum_{i=1}^{18} \frac{N}{i} = \frac{p}{q}$.

   The key idea is to look by each individual prime. For each prime p, define the largest power of p dividing an integer n to be $v_p(n)$. Then, note that the only terms $\frac{N}{i}$ that are multiples of p are those where $v_p(i) = v_p(N)$. These numbers i thus range from $p^{v_p(N)}$ to $c \cdot p^{v_p(N)}$, where c is the largest number such that $c \cdot p^{v_p(N)} \leq 18$. Therefore, $c < p$, because if $c \geq p$, then $p^{v_p(N)+1} \leq 18$, contradicting the definition of v_p.

Now, note that when simplifying the fraction $\frac{\sum_{i=1}^{18} \frac{N}{i}}{N}$, we do not divide any factors of p from the numerator or denominator if and only if $\sum_{i=1}^{c} \frac{1}{i} \not\equiv 0 \pmod{p}$. This is because in this case, the v_p of the numerator will be 0 (it equals the above summation plus other numbers of the form $\frac{N}{i}$ which are multiples of p).

For $p \geq 11$ and $p = 2$, $c = 1$ and the summation $\sum_{i=1}^{c} \frac{1}{i}$ equals 1. This is not 0 mod any of those primes, so none of these primes divide For $p = 7$ and 5, $c = 2$ and the summation equals $\frac{3}{2}$. Therefore, 7 and 5 do not divide the numerator For $p = 3$, $c = 2$ and in fact $\frac{3}{2} \equiv 0 \pmod{3}$, so 3 divides the numerator. However, 9 does not divide the numerator, because the sum of all terms in the numerator which are not a multiple of 9 is $\sum_{i=1}^{6} \frac{N}{3i}$. This equals $\frac{49N}{60}$, and since 27 does not divide N, we know that 9 cannot divide this expression. Hence, 9 cannot divide the numerator. This means that in simplifying our fraction, we have to divide the numerator and denominator by the prime 3 exactly once.

Therefore, $b = \frac{N}{3}$.

Now, we note $N = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, so the number of divisors of b is $5 \cdot 2^6 = 320$.