EMCC 2025

Solutions

COMPETITION

January 25, 2025

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Chapter 1

EMCC 2025 Solutions

1.1 Speed Test Solutions

1. What real number $x \geq -1$ satisfies the equation below?

$$1 + \sqrt{x+1} = x + \sqrt{1+x}$$

Solution: The answer is $\boxed{1}$.

Subtracting $\sqrt{x+1}$ from both sides gives $x = 1$.

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2. What is

$$\sqrt{20+25} \times \sqrt{20 \times 25}?$$

Solution: The answer is $\boxed{150}$.

We have

$$\sqrt{20+25} \times \sqrt{20 \times 25} = \sqrt{45} \times \sqrt{500} = 3\sqrt{5} \cdot 10\sqrt{5} = 150.$$

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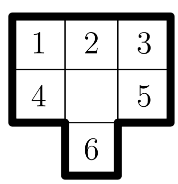
3. A polygon is made from seven squares. One of the labeled squares can be cut out to increase the polygon's perimeter. What is this square's label?



Solution: The answer is $\boxed{2}$.

We need to remove a square with more internal edges than external edges. Squares 1, 3, 4 and 5 all have two internal edges and two external edges, while square 6 has one internal edge and three external edges. Thus, we cannot remove any of these squares. On the other hand, square 2 has three internal edges and one external edge, so we must remove square 2.

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4. The average of two numbers is half of one of the numbers. What is the product of the two numbers?

Solution: The answer is $\boxed{0}$.

Let the two numbers be a and b. By the problem statement, $\frac{a+b}{2} = \frac{a}{2}$, which means $b = 0$, giving $ab = 0$.

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5. Andrew splits the six numbers
$$1, 1, 2, 2, 3, 3$$
into three pairs so that no pair contains two of the same number. For each pair, he multiplies the two numbers in the pair together. What is the sum of his three products?

Solution: The answer is $\boxed{11}$.

The numbers 1 can be paired up with are 2 and 3. However, if both 1's are paired up with 2's, that forces the two 3's to pair up, which is not allowed. Thus, the three pairs must be $(1, 2)$, $(1, 3)$ and $(2, 3)$. Andrew's final answer is
$$1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11.$$

---

6. What is the value of
$$\gcd(1, 9) \times \gcd(2, 8) \times \gcd(3, 7) \times \cdots \times \gcd(9, 1)?$$

Solution: The answer is $\boxed{80}$.

We have that $\gcd(1, 9) = \gcd(9, 1) = 1$, that $\gcd(2, 8) = \gcd(4, 6) = \gcd(6, 4) = \gcd(8, 2) = 2$ and that $\gcd(5, 5) = 5$. Our final answer is
$$1 \cdot 2 \cdot 1 \cdot 2 \cdot 5 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 80.$$

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7. For Ms. Jefferson's spelling test, students will be quizzed on 25 words from a 100-word bank. What is the fewest number of words a student must learn from the bank to guarantee at least 15 correct answers on the quiz?

Solution: The answer is $\boxed{90}$.

If a student wants to get at least 15 correct answers on the quiz, they can risk forgetting at most 10 words, since this is a 25 word quiz. Thus, from the 100-word bank, the student must memorize at least $100 - 10 = 90$ words.

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8. A rectangle has an area of 26. Each side of this rectangle is decreased by 2 units, forming a new rectangle with positive side lengths. What is the sum of the area and perimeter of this new rectangle?

Solution: The answer is $\boxed{22}$.

Let the dimensions of the original rectangle be $a \times b$. Then, we are given $ab = 26$. The new rectangle will have dimensions $(a-2) \times (b-2)$. Summing its area and perimeter, we get $(a-2)(b-2) + 2(a-2) + 2(b-2) = ab - 4 = 22$.

---

9. What positive integer has exactly 5 positive divisors and a units digit of 5?

Solution: The answer is $\boxed{625}$.

Say the integer can be factorized in the form $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Then, the number of the positive divisors is given by $(\alpha_1 + 1) \times (\alpha_2 + 1) \times \cdots \times (\alpha_k + 1)$. But since 5 is prime, we must have $k = 1$ and $\alpha = 4$. So thus, the integer must be of the form p^4, where p is a prime number. Since its units digit is a 5, it must have a prime factor of 5, which means $p = 5$ and that the number is 625.

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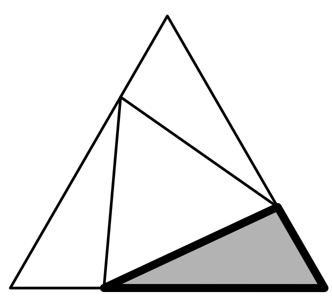
10. An equilateral triangle with an edge length of 5 is inscribed inside an equilateral triangle with an edge length of 7, as shown in the diagram below. What is the perimeter of the shaded triangle?



Solution: The answer is $\boxed{12}$.

There are two other congruent triangles to the shaded one in the diagram. The sum of these three triangles' perimeters must equal the sum of the two equilateral triangles' perimeters, which is $5 \cdot 3 + 7 \cdot 3 = 36$. Since we only want the perimeter of one of the triangles, our final answer is $\frac{36}{3} = 12$.

---

11. Let a and b be integers such that $a < 2025 < b$. If 2025 is closer to b than a, what is the maximum possible value of $a + b$?

Solution: The answer is $\boxed{4049}$.

Since 2025 is closer to b than a, $b-2025 < 2025-a$, which gives $a+b < 4050$. We see that $a+b = 4049$ is attained when $a = 1$ and $b = 4048$.

---

12. An analog clock is stopped at a random time in the day. To the nearest integer percent, what is the chance that the angle between the minute hand and hour hand is obtuse?

Solution: The answer is $\boxed{50\%}$.

Imagine constantly rotating the clock so that the hour hand points upwards. Since this would mean rotating the clock at a constant rate (1 revolution every 12 hours), the minute hand would still appear to rotate at a constant rate. The angle between the hands would be obtuse if and only if the minute hand pointed in the bottom half of the clock, which happens half of the time.

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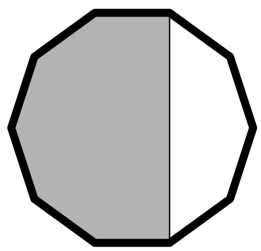
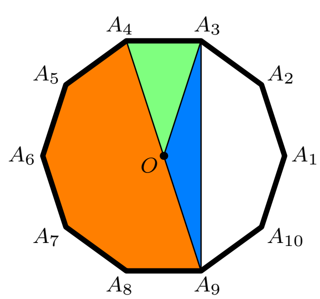
13. In the diagram below, what fraction of the regular 10-sided polygon is shaded?



Solution: The answer is $\boxed{\frac{7}{10}}$.

Split the shaded area of the 10-gon into these three regions:



The orange region, hexagon $A_4A_5A_6A_7A_8A_9$, is exactly half of the 10-gon. The green region, $\triangle A_3A_4O$, is exactly one tenth of the 10-gon. Since $\overline{A_3O}$ is a median in $\triangle A_3A_4A_9$, the area of the blue region, $[A_3OA_9]$, is equal to the area of the green region. So, the total portion of the 10-gon that is shaded is

$$\frac{1}{2} + \frac{1}{10} + \frac{1}{10} = \frac{7}{10}.$$

---

14. At the Las Olas Taqueria, customers may order a bowl with one of 6 proteins and three different toppings from a choice of 24 toppings total. Albert is allergic to some of the proteins and toppings; there are only 2800 distinct bowls he may order under his dietary restrictions. What is the combined number of proteins and toppings that Albert can safely eat?

Solution: The answer is $\boxed{21}$.

Let p and t be the respective number of proteins and toppings. Then, Albert is able to order $\binom{p}{1} \times \binom{t}{3} = 2800$ distinct bowls. Since $t \leq 24$, $\binom{t}{3}$ is not a multiple of 25. Since 2800 is a multiple of 25, it follows that $5 \mid p$, so $p = 5$. So, 7, 5 and 16 are all divisors of $\binom{t}{3}$, from which it follows that t has to equal 16 (since 16, 15 and 14 are multiples of those respective numbers).

We can check now that $5 \cdot \binom{16}{3} = 2800$. So, $p + t = 21$.

---

15. Triangle ABC satisfies $\angle A = 110°$. Points D and E lie on side $\overline{BC}$ such that $AB = BD$ and $AC = CE$. What is the measure of $\angle DAE$, in degrees?

Solution: The answer is $\boxed{35}$.

From the length conditions, we have $\angle BAD = \angle BDA$ and $\angle CAE = \angle CEA$. We now have

$$\begin{aligned}\angle DAE &= \angle BAD + \angle CAE - \angle BAC \\ &= \left(90° - \tfrac{\angle B}{2}\right) + \left(90° - \tfrac{\angle C}{2}\right) - \angle A \\ &= 90° + \tfrac{\angle A}{2} - \angle A \\ &= 35°.\end{aligned}$$

---

16. Harini chooses a geometric sequence of six nonzero real numbers. The sum of these numbers is 20 and the sum of their reciprocals is 25. What is the product of these numbers?

Solution: The answer is $\boxed{\frac{64}{125}}$.

Let the geometric sequence be $a, ar, ar^2, ar^3, ar^4, ar^5$. Then, we have

$$\frac{20}{25} = \frac{a + ar + \cdots + ar^5}{\frac{1}{ar^5} + \frac{1}{ar^4} + \cdots + \frac{1}{a}} = a^2r^5 = \sqrt[3]{a \cdot ar \cdots ar^5}.$$

So, cubing both sides, we have our answer of $\frac{64}{125}$.

---

17. Let x and y be positive reals such that

$$\begin{cases} x^x = \sqrt[y]{2^{20}}, \\ y^y = \sqrt[x]{2^4}. \end{cases}$$

What is $x + y$?

Solution: The answer is $\boxed{5\sqrt{2}}$.

Raising the first equation to the power of y and raising the second equation to the power of x gives us $x^{xy} = 2^{20}$ and $y^{xy} = 2^4$, respectively. Multiplying these two equations together gives us

$$(xy)^{xy} = 2^{24} = 8^8,$$

so $xy = 8$. (In particular, if $0 < xy \leq 1$, then $(xy)^{xy} \leq 1$; if $1 < xy < 8$, then $(xy)^{xy} < 8^8$; if $8 < xy$, then $8^8 < (xy)^{xy}$.)

Substituting this back into our earlier equations, we have that $x^8 = 2^{20}$, so $x = 4\sqrt{2}$. We also have that $y^8 = 2^4$, so $y = \sqrt{2}$. So, $x + y = 5\sqrt{2}$.

---

18. Michael rolls n fair 6-sided dice. What should n be as to maximize the probability that the product of the numbers he rolls is 2025?

Solution: The answer is $\boxed{7}$.

Michael must roll 4 threes, 2 fives, and $n - 6$ ones to result in a product of 2025. The probability of this occurring is

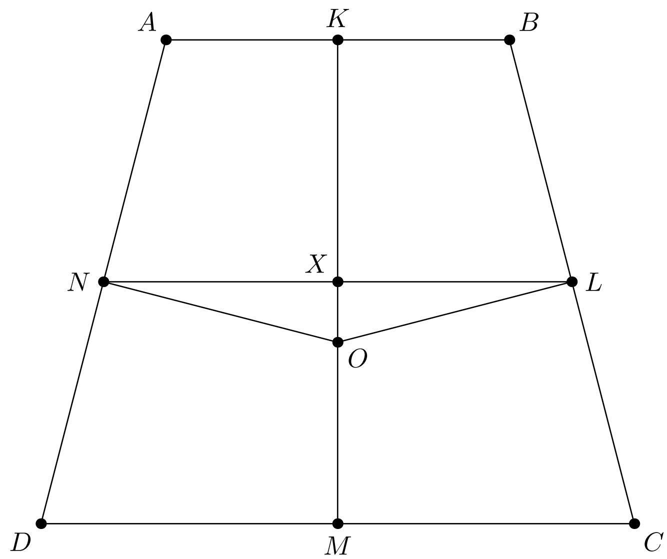
$$\frac{\binom{n}{6} \cdot \binom{6}{2}}{6^n}.$$

Incrementing n from k to $k + 1$ multiplies the above probability by $\frac{k+1}{6(k-5)}$. This value is greater than 1 when $k = 6$, but less than 1 when $k \geq 7$. So, the product is more likely to equal 2025 when $n = 7$ than when $n = 6$, but it is less likely to equal 2025 when $n = 8$ instead of $n = 7$, and will only continue to decrease. Thus, our final answer is $n = 7$.

---

19. Quadrilateral $ABCD$ is inscribed in a circle with center O, such that O lies inside the quadrilateral. If the distances from O to sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are 5, 4, 3 and 4, respectively, what is the area of $ABCD$?

Solution: The answer is $\boxed{16\sqrt{15}}$.



Since O is the same distance from $\overline{AD}$ and $\overline{BC}$, it follows that $AD = BC$, so $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$. Let K, L, M and N be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$, respectively – since O is the circumcenter, its feet to the four sides are K, L, M and N, respectively. Let X be the intersection of $\overline{MK}$ and $\overline{NL}$. Since $\overline{NL}$ is the midline of the trapezoid, X is the midpoint of $\overline{KM}$.

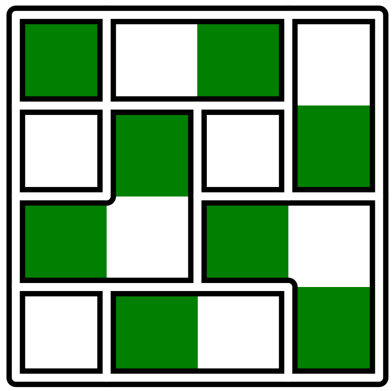
So, we have $OX = MX - OM = \frac{3+5}{2} - 3 = 1$. It is also given that $ON = 4$, so we have

$$NL = 2NX = 2\sqrt{ON^2 - OX^2} = 2\sqrt{15}.$$

Now, the area of this trapezoid is

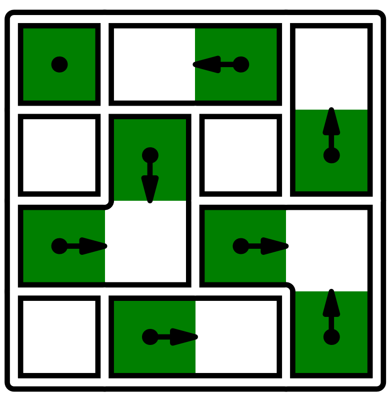
$$\frac{AB + CD}{2} \cdot KM = NL \cdot KM = 2\sqrt{15} \cdot 8 = 16\sqrt{15}.$$

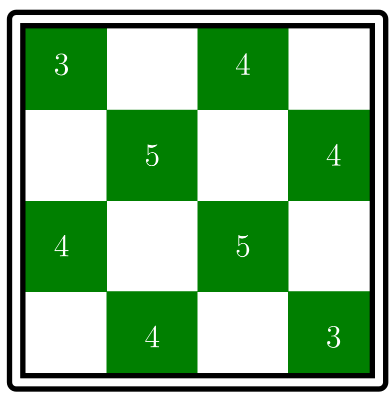
20. How many ways are there to partition a four by four chessboard into polyominos, each of which contains at most one white square? Partitions which differ by a rotation or reflection are considered distinct. (A *polyomino* is any connected region consisting of unit squares. One example of a valid partition is shown below.)



Solution: The answer is $\boxed{57600}$.

Each green square must be part of some polyomino which contains 0 or 1 white cells. If it contains 1 white cell, it must be adjacent to the green square. So, it suffices to assign each green square to either nothing, or one white cell adjacent to itself. The diagram below on the left shows the assignment that results in the polyominos shown in the problem.





The diagram above on the right shows each green cell labeled with the number of ways it can be assigned to a white cell: this is the number of adjacent white cells, plus 1 for when we don't assign the green cell at all.

So, we can constructively make these assignments in

$$3^2 \cdot 4^4 \cdot 5^2 = 57600$$

ways.

EXETER MATH CLUB COMPETITION

1.2 Accuracy Test Solutions

1. What is the integer k in the equation below?

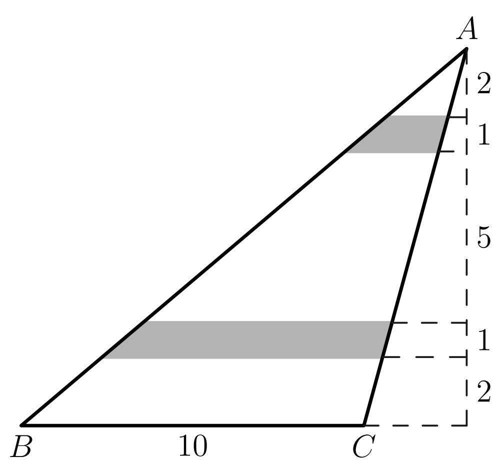
$$2025^{2025} = (20+25)^k$$

Solution: The answer is $\boxed{4050}$.

Note that $2025 = 45^2$, so the equation can be written as $45^{4050} = 45^k$. Thus, $k = 4050$.

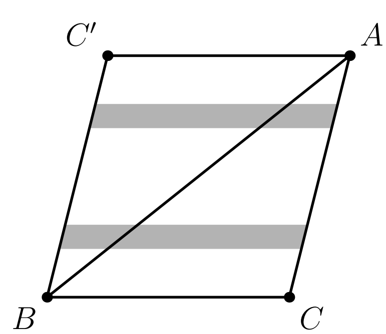
---

2. In triangle ABC with $BC = 10$, Benny draws four lines parallel to BC, dividing the triangle into 5 stripes, with widths as shown in the diagram. He then shades in two of these stripes, as shown. What is the total area of the shaded regions?



Solution: The answer is $\boxed{10}$.

We can draw a congruent image of the figure, reflected over the midpoint of $\overline{AB}$:



This forms a parallelogram. Notice that the shaded strips in the reflected triangle line up with those in the original. Now, each shaded region is a parallelogram with base 10 and height 1, so the two shaded regions have a combined area of 20. This is double what to find, so our final answer is 10.

---

3. Alexandra and Barbara go to a party. Alexandra counts that $\frac{3}{5}$ of the people other than herself are wearing a black shirt, while Barbara counts that $\frac{5}{8}$ of the people other than herself are wearing a black shirt. How many people are at the party in total?

Solution: The answer is $\boxed{41}$.

Let the number of people at the party be x. Then, the number of people wearing a black shirt besides Alexandra must be $\frac{3}{5}(x-1)$ and the number of people wearing a black shirt besides Barbara must be $\frac{5}{8}(x-1)$. The two counts differ by at most 1, so at least one of the following equations hold:

$$\frac{3}{5}(x-1)+1=\frac{5}{8}(x-1), \qquad \frac{3}{5}(x-1)=\frac{5}{8}(x-1), \qquad \frac{3}{5}(x-1)=\frac{5}{8}(x-1)+1.$$

The first equation yields $x=41$, the second has no solutions, and the third has $x=-39$. Thus $x=41$ is the answer.

---

4. The six-digit base-two integer $ABCDEF_{\text{two}}$ and the six-digit base-ten integer $ABCDEF_{\text{ten}}$ are both multiples of 6. What is the value of the six-digit base-six integer $ABCDEF_{\text{six}}$, expressed in base ten?

Solution: The answer is $\boxed{7998}$.

Clearly all digits must be either 0 or 1. Since $ABCDEF_{\text{two}}$ must be divisible by 2, $F=0$. Then we need exactly three of the other digits to 1 for $ABCDEF_{\text{ten}}$ to be divisible by 3. The only choice of three 1s to make $ABCDEF_{\text{two}}$ is $A=C=E=1$, by noting that the place values of A, C, E are $2 \pmod 3$, while the other two are $1 \pmod 3$. Then we can calculate $101010_{\text{six}}=6^5+6^3+6=7998$ in base ten.

---

5. If

$$0.01\times 0.02\times\cdots\times 0.99\times 1.00 = 0.\overbrace{00\ldots0}^{42\text{ zeroes}}\overbrace{933\ldots864}^{k\text{ digits}},$$

what is k?

Solution: The answer is $\boxed{134}$.

Multiply both sides by $(100)^{100}=10^{200}$. Then we have

$$100! = \overbrace{933\ldots864}^{k\text{ digits}}\ \overbrace{00\ldots0}^{158-k\text{ zeroes}},$$

Legendre's Formula tells us that the largest power of 5 dividing 100! is $5^{\lfloor 100/5\rfloor+\lfloor 100/25\rfloor} = 5^{24}$, so 100! ends with 24 zeroes. This gives us $k = 134$.

---

6. Six equilateral triangles are drawn in the coordinate plane such that each triangle has a side parallel to the x-axis. Given that a finite number N of points lie on two or more of the triangles' perimeters, what is the maximum possible value of N?

Solution: The answer is $\boxed{66}$.

Triangles can point either upwards or downwards; let a be the number of upwards pointing triangles and let b be the number of downwards pointing triangles. Two oppositely pointed triangles can meet at up to 6 points, while two triangles with the same orientation meet at up to 2 points. Therefore, the total number of intersections is equal to

$$\binom{6}{2} \cdot 2 + ab \cdot (6-2) \leq 30 + 3 \cdot 3 \cdot 4 = 66.$$

Equality is achieved when $a = b = 3$, the triangles are all congruent and the triangles' centers are negligibly close to one another.

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7. A room contains one person. Then, fifty more people come in, one by one. Each person entering the room forms a friendship with one person already in the room, at random. Once everybody is in the room, what is the expected number of people with exactly one friend? (None of the people are initially friends.)

Solution: The answer is $\boxed{\frac{638}{25}}$.

Number the people from 1 through 51 based on the order they came into the room. To compute the expected number of people with one friend, by linearity of expectation, it suffices to compute the probability that each person has one friend and sum these probabilities together.

Person 1 is guaranteed to befriend person 2, so we just need to compute the probability that she doesn't befriend anybody else. The probability that person 1 doesn't befriend person i is $\frac{i-2}{i-1}$, since there are $i-1$ people in the room when person i walks in, and person i can befriend any of them. So, the probability that person 1 doesn't befriend anybody else is

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{49}{50} = \frac{1}{50}.$$

For $i > 1$, person i automatically befriends somebody when she walks into the room, so we just need to find the probability that she doesn't befriend anybody else. Like before, there is a $\frac{j-2}{j-1}$ chance that person j doesn't befriend person i for $j > i$. Therefore, the probability that person i doesn't form any further friends is $\frac{i-1}{i} \cdot \frac{i+1}{i+2} \cdots \frac{49}{50} = \frac{i-1}{50}$.

Summing over everybody, the total expected value is

$$\frac{1}{50} + \left(\frac{1}{50} + \frac{2}{50} + \cdots + \frac{50}{50}\right) = \frac{1}{50} + \frac{51}{2} = \frac{638}{25}.$$

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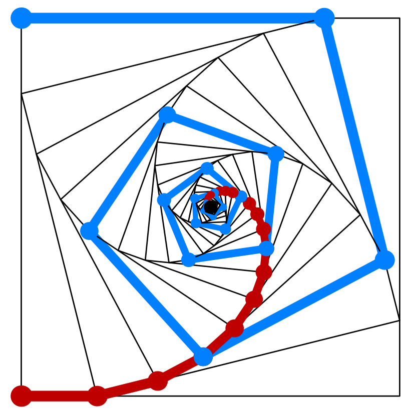
8. In the diagram below, an infinite sequence of nested squares is drawn such that the areas of the squares form a geometric sequence. The highlighted red and blue infinite spirals have lengths of 2 and 11, respectively. What is the side length of the largest square?



Solution: The answer is $\boxed{13 - 5\sqrt{5}}$.

Let $B_1, B_2, \ldots$ denote the points on the blue spiral, let $R_1, R_2, \ldots$ denote the points on the red spiral and let $s_1, s_2, \ldots$ denote the side lengths of the squares. Note that, due to the similarity, $B_1B_2, B_2B_3, \ldots$ and $R_1R_2, R_2R_3, \ldots$ are geometric sequences with the same common ratio (namely, the same common ratio of the geometric sequence $s_1, s_2, \ldots$). Since we know the first sequence sums to 11 while the second sequence sums to 2, it follows that $\frac{B_1B_2}{R_1R_2} = \frac{11}{2}$.

Let X be the leftmost vertex of the second largest square. We have that $\frac{XR_1}{R_1R_2} = \frac{B_1B_2}{R_1R_2} = \frac{11}{2}$, so

$$\frac{XR_2}{R_1B_1} = \frac{XR_2}{XR_1 + R_1R_2} = \frac{\sqrt{11^2 + 2^2}}{11 + 2} = \frac{5\sqrt{5}}{13}.$$

This is the common ratio of $s_1, s_2, \ldots$. Since $s_1 + s_2 + \cdots = (B_1B_2 + R_1R_2) + (B_2B_3 + R_2R_3) + \cdots = 13$, it follows that $s_2 + s_3 + \cdots = \frac{5\sqrt{5}}{13}(s_1 + s_2 + \ldots) = 5\sqrt{5}$, so

$$s_1 = (s_1 + s_2 + \ldots) - (s_2 + s_3 + \ldots) = 13 - 5\sqrt{5}.$$

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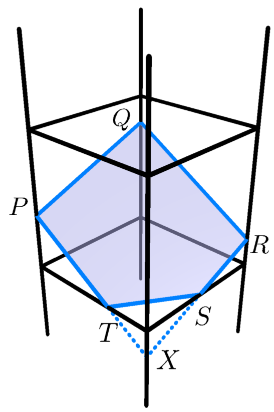
9. The intersection between a plane and a cube is a convex pentagon $ABCDE$ satisfying $AB = BC = 10$, $CD = AE = 8$, and $DE = 3$. What is the surface area of the cube?

Solution: The answer is $\boxed{675}$.

Each side of the pentagon lies on a different face of the cube, so there is exactly one face which the pentagon does not intersect. Without loss of generality, let this face be the top face of the cube.



Since we do not yet know which vertex is labelled A, B, etc., label the pentagon with the placeholders P, Q, R, S and T as shown in the diagram above. Let the plane of the pentagon meet the closest vertical edge to the viewer at X. Equivalently, X is the intersection of lines $\overline{PT}$ and $\overline{RS}$.

We have $PQ = XR > SR$ and $QR = PX > PT$. So, it follows that $PQRST$ corresponds to $ABCDE$, with vertices in that exact order.

Since lines PQ and SR lie on parallel faces of the cube, they don't intersect; combined with the fact that they lie on the plane of the pentagon, it follows that $\overline{PQ} \parallel \overline{SR}$. Similarly, we have $\overline{QR} \parallel \overline{PT}$, so $PQRX$ is a parallelogram. Moreover, since $PQ = QR = 10$, it follows that $PQRX$ is in fact a rhombus. So, $XT = XS = 2$.

So, we have $\triangle XTS \sim \triangle XPR$ with a similarity ratio of $1:5$; therefore, $PR = 3 \cdot 5 = 15$. Since $\overline{PR} \parallel \overline{TS}$, it follows that line PR is a vertical translation of a diagonal of the bottom face of the cube, so the cube has side length $\frac{15}{\sqrt{2}}$. Therefore, its surface area is

$$6 \cdot \left(\frac{15}{\sqrt{2}}\right)^2 = 675.$$

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10. There exists a polynomial P of degree 124 with real coefficients such that for $i = 1, 2, \ldots, 125$, the sum of the coefficients of

$$\underbrace{P(P(P(\ldots(P}_{i \text{ instances of } P}(x))))$$

is $\frac{1}{i+1}$. If $|P(2)|$ can be expressed as $\frac{m}{n}$ for coprime integers m and n, what is the largest nonnegative integer k for which 2^k divides n?

Solution: The answer is $\boxed{120}$.

The sum of coefficients of $\underbrace{P(P(P(\ldots(P}_{i \text{ instances of } P}(x))))$ is just $\underbrace{P(P(P(\ldots(P}_{i \text{ instances of } P}(1))))$. We have that

$$\underbrace{P(P(P(\ldots(P}_{i \text{ instances of } P}(1)))) = P\left(\frac{1}{i}\right) = \frac{1}{i+1}.$$

This means that $(x+1)P(x) = x$ has solutions $1, \frac{1}{2}, \ldots, \frac{1}{125}$, so $(x+1)P(x) - x = a(x-1)\left(x - \frac{1}{2}\right) \cdots \left(x - \frac{1}{125}\right)$ for some constant a. To solve for a, we plug in $x = -1$ which gives us $a = \frac{1}{126}$. Then plug in $x = 2$ to get that $P(2) = \frac{249!!}{3(126)(125!)}$. Applying Legendre's formula gives us $n = 120$.

MATH
EXETER
CLUB
COMPETITION

1.3 Team Test Solutions

1. Students and chaperones board a school bus with a capacity of 50. Each student is assigned to a chaperone; each chaperone is assigned up to 7 students. At most how many students may board the bus?

   Solution: The answer is $\boxed{43}$.

   Suppose there were 44 students, implying there are at most 6 chaperones. Each chaperone can take 7 students, so the chaperones can only account for 42 students, missing 2. It suffices to show 43 is constructible, which it is by 6 chaperones taking 7 students and the last chaperone taking 1.

---

2. How many ordered triples (x, y, z) of real numbers satisfy the system of equations shown below?

$$\begin{cases} x = y \cdot z \\ y = z \cdot x \\ z = x \cdot y \end{cases}$$

   Solution: The answer is $\boxed{5}$.

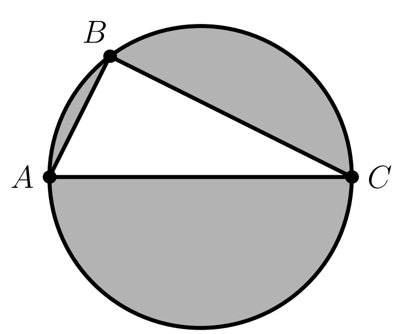
   If any of x, y or z are 0, then they all are, leading to $(0, 0, 0)$ as a solution. Multiplying all 3 equations gives $xyz = (xyz)^2$, so $xyz = 1$ since we have taken care of $xyz = 0$. Multiplying the first 2 equations gives $xy = xyz^2$, so $z^2 = 1$, and similarly we deduce that $x, y, z \in \{-1, 1\}$. From here the only solutions are

$$(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1), (0, 0, 0),$$

   which are the 5 claimed solutions.

---

3. Triangle ABC with $\angle B = 90°$ is inscribed in a circle, as shown in the diagram below. The area of the shaded region is 176π and the area of the triangle is 80π. What is the distance from B to line AC?



Solution: The answer is $\boxed{5\pi}$.

The area of the whole circle is 256π, so its radius is 16. As $\angle ABC = 90°$, we conclude AC is a diameter, so $AC = 32$. Now, if h is the desired distance, by the area formula we get $\frac{h\cdot 32}{2} = 80\pi$, so $h = 5\pi$.

---

4. A billboard reads

Perfect squares such as $\boxed{2}\boxed{0}\boxed{2}\boxed{5}$ are cool.

A prankster wants to pick exactly one digit on the billboard and replace it with any one of the nine other digits, such that the number on the billboard is no longer a perfect square. How many ways can the prankster do this?

Solution: The answer is $\boxed{33}$.

We count the number of ways that the prankster could fail and make a perfect square, and then subtract that from 36, the total number of actions. The last digit can't be changed since 2025 is the only perfect square between 2020 and 2030.

If the prankster changes any other digit, the number on the billboard will remain a multiple of 5; therefore, if it is of the form x^2, x must be a multiple of 5. Therefore we check $5^2, 15^2, \ldots, 95^2$ and find the prankster can only change 2025 to the three numbers $5^2 = 0025$, $55^2 = 3025$, and $95^2 = 9025$. The answer is then $36 - 3 = 33$.

---

5. Phillips Exeter and Phillips Andover pit their finest racehorses against each other in a derby on Front Street. The results of the top horses are as follows:

#	Name	Team
1	Blaze	Andover
2	Storm	Exeter
3	Comet	Exeter
4	Dash	Exeter
5	Echo	Exeter
6	Nova	Andover
7	Spirit	Andover

#	Name	Team
8	Jett	Andover
9	Shadow	Andover
10	Vortex	Exeter
11	Zephyr	Exeter
12	Ember	Exeter
13	**Yash**	Andover
14	Titan	Andover

Exeter: 24 **Andover: 31**

As shown above, the score of each team is the sum of the ranks of their top five horses (so, Exeter is the winner here since they have a lower score). At least how many positions faster would Yash have had to have placed so that Andover's score was lower than Exeter's score?

Solution: The answer is $\boxed{11}$.

If Yash finishes in second place, Andover has a score of $1 + 2 + 7 + 8 + 9 = 27$ while Exeter has a score of $3 + 4 + 5 + 6 + 11 = 29$. But if Yash finishes in third place, Andover and Exeter both tie with a 28, and if Yash ranks lower, Andover outright loses to Exeter. So, Yash must finish in second place, meaning he would have had to have ranked 11 positions faster.

6. Suppose n is a positive integer such that the sum of the digits of n is 100 and the sum of the digits of $11n$ is 2. How many nonzero digits does n have?

Solution: The answer is $\boxed{12}$.

We may assume that n ends in a nonzero digit; otherwise we can remove it without affecting the given information or what we're asked to find.

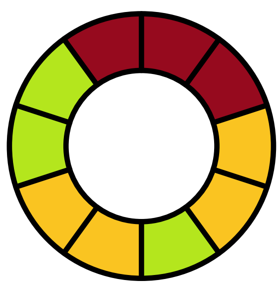
Since $11n$ doesn't end in a 0, it must be of the form $100\dots001$. Since $\frac{1}{11} = 0.09090\dots$, we must have

$$n = \frac{100\dots001}{11} = 9090\dots90 + 1,$$

for some number of digits in each of the "…". We know that the sum of the digits of n is 100, so there must be eleven 9's and a 1. This gives us a total of 12 digits.

---

7. A ring is divided into ten equal sectors. Each sector is randomly shaded with one of ten possible colors, selected uniformly and independently. What is the expected number of connected regions in the resulting figure?
(For example, in the diagram below, three colors are used, resulting in five connected regions.)



Solution: The answer is $\boxed{9.000000001}$.

Let m be the number of borders on the ring that separate two different colors, and let n equal 1 if the whole ring is one color, and 0 otherwise. Then, it is clear that

$$\#\text{ regions} = m + n.$$

By linearity of expectation, it suffices to compute the expected value of m and the expected value of n separately.

To compute the expected value of m, note that each of the ten borders has a $\frac{9}{10}$ chance of connecting two different colors. So, by linearity of expectation once more, the expected value of m is $\frac{9}{10} \cdot 10 = 9$.

The expected value of n is just the probability that the whole ring is one color, which is $\frac{10}{10^{10}} = 10^{-9}$.

So, our final answer is $9 + 10^{-9} = 9.000000001$.

---

8. In $\triangle ABC$, point D lies on side $\overline{BC}$ such that $\overline{AD} \perp \overline{BC}$. Suppose that the lengths AD, AB, AC and BC form an increasing geometric sequence with common ratio r. What is r^2?

Solution: The answer is $\boxed{\frac{1+\sqrt{5}}{2}}$.

Since AD, AB, AC and BC form a geometric sequence, it follows that $AD \cdot BC = AB \cdot AC$. But $AD \cdot BC = 2bh = 2[ABC]$, so $AB \cdot AC$ is also equal to $2[ABC]$. This implies that $\angle A = 90°$. Without loss of generality, let $AB = 1$. Then, $AC = r$ and $BC = r^2$. From the Pythagorean theorem, we now have

$$1 + r^2 = r^4 \implies r = \frac{1+\sqrt{5}}{2}.$$

---

9. Let $\mathcal{P}$ be a regular 40 sided polygon. What fraction of the diagonals of $\mathcal{P}$ are longer than the circumradius of $\mathcal{P}$?

Solution: The answer is $\boxed{\frac{27}{37}}$.

Let A be a fixed point of $\mathcal{P}$ and AB be a random diagonal with endpoint at A. By symmetry, it suffices to find the probability that diagonal AB is longer than the circumradius of $\mathcal{P}$. Let $\overset{\frown}{AB}$ denote the measure of the minor arc between A and B, so segment AB is longer than the circumradius if and only if $\overset{\frown}{AB} \geq 60°$. If B is k points away on $\mathcal{P}$ from A, then $\overset{\frown}{AB} = 9k$, implying that only $2 \leq k \leq 6$ fails. This means 10 out of the 37 points fail, so the answer is $1 - \frac{10}{37} = \frac{27}{37}$.

---

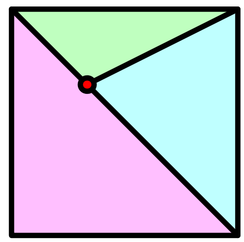
10. Julie draws a square with an area of 6. How many ways can she split it into three triangles with areas 1, 2 and 3?
(Two ways that differ by a rotation and/or reflection are considered distinct.)

Solution: The answer is $\boxed{32}$.

There are two cases.

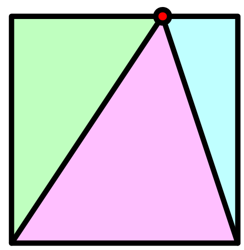
- Suppose the triangle with area 3 is a right isosceles triangle. Then, it suffices to split the other half of the square into the two triangles with an area of 1 and 2. The remaining half of the square is also a (right isosceles) triangle – to split it up into two smaller triangles, we just need to pick one side of the triangle, cut it in the ratio $1:2$ or $2:1$ and draw a cevian from the opposite vertex.

  There are four ways to pick our triangle with area 3, three ways to pick the side of the remaining triangle and two ways to pick where we draw the cevian (either cutting the opposite side $1:2$ or $2:1$). This case thus yields 24 possibilities.



- Otherwise, the triangle with area 3 will have its base coincide with a side of the square, and the opposite vertex will lie strictly in the interior of the opposite sides. Once we pick the base of the triangle, the opposite vertex can cut the opposite side of the square in either a $1 : 2$ ratio or a $2 : 1$ ratio.

  There are four ways to pick the base of our triangle with area 3, and two ways to split the opposite side. This case thus yields 8 possibilities.



So, the final answer is $24 + 8 = 32$.

---

11. Define the function $\operatorname{mod}(x, y) = x - y\lfloor x/y \rfloor$. Bryan randomly and uniformly picks a real number x in the interval $[0, 20)$. What is the probability that

$$\lfloor \operatorname{mod}(x, 2.5) \rfloor = \operatorname{mod}(\lfloor x \rfloor, \lfloor 2.5 \rfloor)?$$

Solution: The answer is $\boxed{\frac{2}{5}}$.

Write $x = 10n + k$ for integer n and real $0 \le n < 10$. Then,

$$\lfloor \operatorname{mod}(10n + k, 2.5) \rfloor = \lfloor \operatorname{mod}(10n + k - 2.5(4n), 2.5) \rfloor = \lfloor \operatorname{mod}(k, 2.5) \rfloor$$

and

$$\operatorname{mod}(\lfloor 10n + k \rfloor, \lfloor 2.5 \rfloor) = \operatorname{mod}(\lfloor k \rfloor, 2).$$

Thus we want $\lfloor \operatorname{mod}(k, 2.5) \rfloor = \operatorname{mod}(\lfloor k \rfloor, 2)$. This is satisfied for

$$k \in [0, 2) \cup [2.5, 3) \cup [3.5, 4) \cup [8, 8.5) \cup [9, 9.5).$$

The lengths of these intervals sum to 4, implying $\frac{4}{10} = \frac{2}{5}$.

---

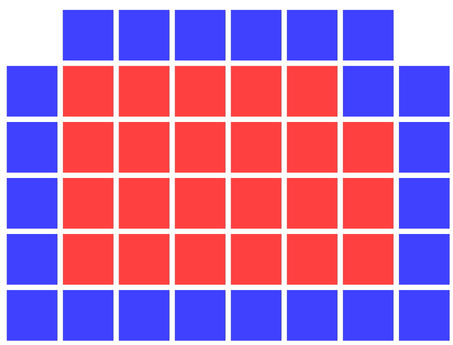
12. In an infinite grid of squares, each square is initially colored white. Oron paints n squares red and n squares blue such that:

- No square is painted both colors.
- Any two red squares are connected by a path of red squares. (Any two consecutive squares in the path must be vertically or horizontally adjacent. In particular, they may not be diagonally adjacent.)
- Any two blue squares are connected by a path of blue squares.
- No red square is adjacent to a white square.

What is the minimum possible value of n?

Solution: The answer is $\boxed{23}$.

This is achieved by the following construction:



Proving this is optimal is left as an exercise.

---

13. Let $\triangle ABC$ be an acute triangle with incenter I such that $AB = AC = 7$. Let D and E be points on sides AB and AC, respectively, such that $AD = AE = 3$. Given that the circumcircle of $\triangle DIE$ is tangent to $\overline{AB}$ and $\overline{AC}$, what is BC?

Solution: The answer is $\boxed{\frac{28}{5}}$.

Clearly, $\triangle IDE$ is isosceles. Combined with the tangency information, we have that

$$\angle BDI = \angle DEI = \angle EDI,$$

so $\overline{DI}$ bisects $\angle BDE$. Similarly, $\overline{EI}$ bisects $\angle DEC$. So, the internal angle bisectors of quadrilateral $BCED$ concur at I, meaning that it has an inscribed, namely the incircle of $\triangle ABC$. In particular, $\overline{DE}$ is tangent to the incircle. So, the inradius r of $\triangle ABC$ is $\frac{2}{7}$ the length h of the altitude from A to $\overline{BC}$. Letting $BC = x$, we have

$$(x + 7 + 7) \cdot r = 2[ABC] = xh \implies x + 14 = \frac{xh}{r} = \frac{7x}{2} \implies x = \frac{28}{5}.$$

---

14. What is the sum of the three smallest positive integers n for which $n^n \times 20^{25}$ is a perfect cube?

Solution: The answer is $\boxed{1700}$.

Let $f(n)$ denote the remainder when n is divided by 3. Clearly, $n^n \times 20^{25}$ is a perfect cube if and only if $n^{f(n)} \times 2^2 \times 5^1$ is a perfect cube, since their ratio is a perfect cube. Now, we split into cases based on $f(n)$:

- If $f(n) = 0$, we need $n^0 \times 2^2 \times 5^1 = 20$ to be a perfect cube. But this is impossible, so there are no solutions in this case.
- If $f(n) = 1$, we need $n \times 2^2 \times 5^1$ to be a perfect cube. So, n is of the form $2 \times 5^2 \times k^3$ for an integer k. The smallest possible values of n of this form (and satisfying $f(n) = 1$) are 400 and 6250.
- If $f(n) = 2$, we need $n^2 \times 2^2 \times 5^1$ to be a perfect cube. This is equivalent to $n^4 \times 2^4 \times 5^2$ being a perfect cube, which is in turn equivalent to $n \times 2 \times 5^2$ being a perfect cube. So, n is of the form $2^2 \times 5 \times k^3$. The smallest possible values of n of this form (and satisfying $f(n) = 2$) are 20 and 1280.

We have our three smallest values: 20, 400 and 1280, which sum to 1700.

---

15. How many ordered triples (x, y, z) of positive real numbers satisfy the system of equations shown below?

$$\begin{cases} xyz &= 100 \\ xy\lfloor z \rfloor &= 99 \\ x\lfloor yz \rfloor &= 98 \end{cases}$$

Solution: The answer is $\boxed{4704}$.

Divide the first two equations to obtain $\frac{z}{\lfloor z \rfloor} = \frac{100}{99}$, which we may write as $\frac{\lfloor z \rfloor + \{z\}}{\lfloor z \rfloor} = \frac{100}{99}$, where $\{z\}$ denotes the fractional part of z, so subtracting 1 we get $\frac{\{z\}}{\lfloor z \rfloor} = \frac{1}{99}$, or $99\{z\} = \lfloor z \rfloor$. Since $\{z\} < 1$ we get $\lfloor z \rfloor < 99$. For each integer value of $\lfloor z \rfloor$ from 1 to 98, we get a valid value of $\{z\} < 1$ and thus a valid value of z.

Similarly, from the first and third equations, we get $\frac{yz}{\lfloor yz \rfloor} = \frac{100}{98} = \frac{50}{49}$, which becomes $49\{yz\} = \lfloor yz \rfloor$, so $\lfloor yz \rfloor < 49$. For each integer value of $\lfloor z \rfloor$ from 1 to 48, we get a valid value for yz.

Now for each distinct pair of possible y, yz we get a unique pair y, z, and by any equation, for each pair y, z we get a unique x. Thus the number of ordered pairs (x, y, z) is equal to the number of pairs (y, yz), which is $98 \cdot 48 = 4704$.

1.4 Guts Test Solutions

1.4.1 Round 1

1. Andrew eats
$$20^{5/2} \cdot 25^{5/20}$$
erasers. How many erasers is this?

Solution: The answer is $\boxed{4000}$.

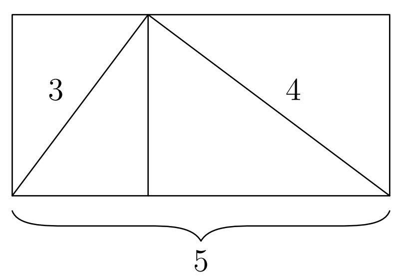
The answer is
$$20^{5/2} \cdot 25^{5/20} = 800\sqrt{5} \cdot \sqrt{5} = 4000.$$

---

2. While eating erasers, Andrew imagines two rectangles with diagonals of lengths 3 and 4, respectively. Their heights are the same and the sum of their widths is 5. What is the sum of their areas?

Solution: The answer is $\boxed{12}$.

Gluing the rectangle along their heights gives us the following diagram:



Since a 3-4-5 triangle is inscribed inside the rectangle, the area of the big rectangle is twice the area of the 3-4-5 triangle, giving us our final answer of $6 \cdot 2 = 12$.

---

3. Andrew repeatedly rolls a fair six-sided die, keeping track of the product of all the numbers he has rolled thus far. Whenever this product is prime, he eats an eraser. What is the probability that Andrew never eats an eraser?

Solution: The answer is $\boxed{\frac{2}{5}}$.

We will compute the complement, the probability that Andrew eats an eraser. For this to happen, his product must be prime at some point, meaning he has rolled some number of 1's and a prime number

p. So, Andrew eats an eraser if and only if his first roll not equal to 1 is prime. This happens with probability $\frac{3}{5}$, making the probability that Andrew does not eat an eraser $\frac{2}{5}$.

1.4.2 Round 2

4. The number 136,279,845 has all digits from 1 to 9, but it is not prime. What digit must be erased and replaced in order to make the number prime?

Solution: The answer is $\boxed{5}$.

The new number cannot be a multiple of 5, so its unit digit must be replaced.

Indeed, replacing the digit with a 1 suffices; 136,279,841 turns out to be a prime. It's nearly impossible to show this by hand, and as such, the problem does not ask for the value of the replacing digit. However, contestants may recognize this prime number, as it was discovered on October 12th, 2024 that $2^{136279841} - 1$ that is prime, making it the largest known prime number as of January 25th, 2025.

---

5. If m and b are real numbers such that the line $y = mx + b$ passes through the points $(-9876, 54321)$ and $(9878, 12345)$, what is $m + b$?

Solution: The answer is $\boxed{33333}$.

Since the midpoint of any two points on a line always lies on the same line, we know that the midpoint of $(-9876, 54321)$ and $(9878, 12345)$ lies on the line. By averaging the respective coordinates, the midpoint is $(1, 33333)$.

But the line $y = mx + b$ always passes through the point $(1, m + b)$, so $m + b = 33333$, as desired.

---

6. Sylvia writes down a string of digits, containing the sub-strings "121", "412", "1234". Interpreted as a base 10 number, what is the smallest possible positive value that Sylvia's string could take?

Solution: The answer is $\boxed{1234121}$.

If our number contained six digits, in order to contain both 1234 and 412, it would have to be 123412. But this doesn't contain 121, so our number must be at least 7 digits long.

The smallest possible value for the first three digits is 121. However, it is impossible for the remaining string to contain both a 412 and 1234. The next smallest possible value for the first three digits is 123, after which a 4 must follow, and then a 121. This gives us our answer 1234121, which does indeed contain the three requested strings.

1.4.3 Round 3

7. Stanley has a cube with a side length of $n > 2$ units. He paints the surface red and cuts the cube into n^3 unit cubes. If seven times as many cubes have exactly one face painted compared to exactly two faces painted, what is n?

Solution: The answer is $\boxed{16}$.

Since $n \geq 1$, the number of cubes with exactly one face painted is $6(n-2)^2$, while the number of cubes with two faces painted is $12(n-2)$. We are given that

$$7 \cdot 12 \cdot (n-2) = 6 \cdot (n-2)^2,$$

and since $n > 2$ we may divide both sides by $n-2$ to obtain

$$84 = 6(n-2) \implies 14 = n-2 \implies n = 16.$$

---

8. An escalator moves upwards at 1 step per second, and half of its steps are visible at any time. If Stanley's sister walks up the escalator, taking 3 steps every second, what fraction of the total steps in the escalator does she step on?

Solution: The answer is $\boxed{\frac{3}{8}}$.

Call Stanley's sister Julie.

Suppose the steps of the escalator were fixed, and instead, the entryways of the escalator were moving downwards at a speed s. Then, Julie walks up the steps at a speed $3s$. Suppose Julie begins at point A, while the top entryway begins at point B. So, the number of steps between points A and B is one half of the total steps in the escalator.

If Julie and the top entryway meet at a point C along the escalator, we must have

$$AC : CB = (3s) : (1s) = 3 : 1.$$

Julie travels along segment AC, which is $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$ of the entire escalator.

---

9. Stanley's cat draws a convex hexagon $\mathcal{H}$. The area of $\mathcal{H}$ is equal to each of its side lengths. If there exists a circle tangent to each side of $\mathcal{H}$, what is this circle's area?

Solution: The answer is $\boxed{\frac{\pi}{9}}$.

Let A be the area of $\mathcal{H}$, let s be the length of one side, let P be the perimeter, and let r be the inradius. We have

$$r = \frac{2A}{P} = \frac{2A}{6s} = \frac{2A}{6A} = \frac{1}{3},$$

so the area of the circle is $\pi r^2 = \frac{\pi}{9}$.

1.4.4 Round 4

10. A frog is at the point $(1,1)$. Each second, the frog jumps one unit up, left, down or right at random. It is bounded by the square with vertices $(0,0)$, $(0,3)$, $(3,3)$ and $(3,0)$. What is the probability that it will touch an edge of the square within the next three jumps?

Solution: The answer is $\boxed{\frac{7}{8}}$.

We use complementary counting to calculate the probability that the frog does *not* touch an edge of the square. Then, it must be contained among the points

$$(1,1), \qquad (1,2), \qquad (2,1) \quad \text{and} \quad (2,2).$$

Call the set of these four points S. For each point in S, there are two other points in S the frog may travel to; the other two points are on the perimeter of the square. So, at each step, there is a $\frac{1}{2}$ chance the frog remains in the boundary. Over three jumps, there is a $(\frac{1}{2})^3 = \frac{1}{8}$ chance the frog is still inside the boundary. So, there is a $\frac{7}{8}$ chance the frog will *not* be inside the boundary after three jumps.

---

11. Clara stands at the point $(0,5)$. She picks a random point with integer coordinates on the circle $x^2+y^2=25$, possibly where she is standing. If d is the distance between Clara and her lattice point, what is the average value of d^2?

Solution: The answer is $\boxed{50}$.

Let P denote the point $(0,5)$, and let Q be a point with integer coordinates on the circle $x^2+y^2=25$. Then, let Q' be the reflection of Q over $(0,0)$. Note that Q also lies on the circle $x^2 = y^2 = 25$, and the coordinates of Q are also integers. So, Clara is just as likely to be standing on Q' as she is to be standing on Q. Moreover, since the radius of the circle is 5, we must have $QQ' = 10$. Since $\angle QPQ' = 90^\circ$, we have by the Pythagorean Theorem that

$$PQ^2 + PQ'^2 = (5 \cdot 2)^2 = 100.$$

If there are k such unordered pairs of antipodal points (Q, Q'), the total sum of all distances d^2 is $100k$. Since there would then be $2k$ points with integer coordinates on the circle, her average is

$$\frac{100k}{2k} = 50.$$

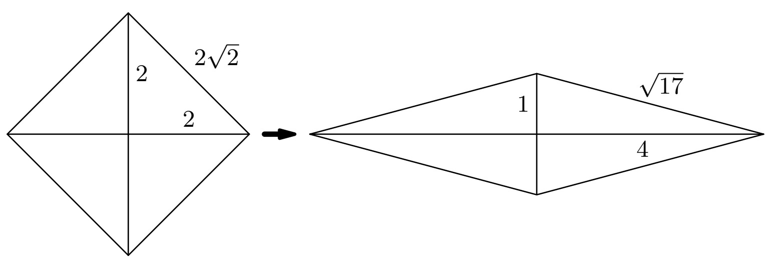
---

12. Benny draws a quadrilateral Q on the coordinate plane. If he stretches the plane vertically (about the x-axis) by a factor of 2, the image of Q is a square; if he instead stretches the plane horizontally (about the y-axis) by a factor of 2, the image of Q is a rhombus. What is the ratio of the side length of the square to the side length of the rhombus?

Solution: The answer is $\boxed{\frac{2\sqrt{34}}{17}}$.

Let Q_1 denote the square and Q_2 denote the rhombus. We are given that the composition of a $\times\frac{1}{2}$ vertical stretch, followed by a $\times 2$ horizontal stretch, results in Q_2. These compositions are equivalent to a $\times 4$ horizontal stretch followed by a $\times\frac{1}{2}$ dilation. Since dilation preserves angles, a rhombus will remain a rhombus. So, we know that a $\times 4$ horizontal stretch (without the dilation) on square Q_1 results in some rhombus Q_2.

Suppose ℓ and m are a pair of perpendicular lines, neither of which have slope 0 or ∞. Then, after a $\times 4$ horizontal stretch, the images of ℓ and m will not be perpendicular. Indeed, each line's slope multiplies by $\frac{1}{4}$, so the product of the images' slopes is $-1 \cdot \frac{1}{4} \cdot \frac{1}{4} = -\frac{1}{16} \neq -1$. Since the diagonals of the square are perpendicular, and remain perpedicular after the $\times 4$ horizontal stretch, it thus follows that they must have had slopes of 0 and ∞, respectively.



Without loss of generality, assume the length of a diagonal in square Q_1 is 4. So, it has side length $2\sqrt{2}$. Then, the lengths of the diagonals in Q_2 are 2 and 8, so it has side length $\sqrt{1^2+4^2} = \sqrt{17}$. Now we compute their ratio:

$$\frac{2\sqrt{2}}{\sqrt{17}} = \frac{2\sqrt{34}}{17}.$$

1.4.5 Round 5

13. Tess plays a gambling game on an 8 by 8 chessboard. Each second, a rook is placed on a randomly chosen empty cell of the chessboard that does not share a row or column with any existing rooks. Once no more rooks can be placed, Tess wins if every rook is on a black square. What is the probability that she wins?

Solution: The answer is $\boxed{\frac{1}{70}}$.

After placing k rooks down, there are $8-k$ unoccupied rows and $8-k$ unoccupied columns. As such, Tess will always place down exactly 8 rooks.

Any configuration of these 8 rooks can be represented by a permutation of $1, 2, \dots, 8$; the first number in the permutation represents the rank of the rook in the first file, the second number represents the rank of the rook in the second file, and so on. As such, there are 8! different ways Tess could place her rooks on the chessboard: there is one way corresponding to each permutation of $1, 2, \dots, 8$.

Each of these permutations is equally likely to occur; one easy way to see this is that all 64 positions on the board are symmetric with respect to each other, in that Tess is equally likely to place a rook on any one of these positions.

Among these 8! possibilities, we need to count how many of them exist where every rook lies on a black square. For the A, C, E and G files, the black squares fall on odd numbered ranks, so in our permutation, the first, third, fifth and seventh indices should be odd numbers. Every odd number must be used exactly once, so there are 4! ways to permute these odd numbers. On the other hand, for the B, D, F and H files, the black squares fall on even numbered ranks, so in our permutation, the second, fourth, sixth and eighth indices should be even numbers. There are 4! ways to permute the even numbers. So, our final answer is

$$\frac{4!4!}{8!} = \frac{1}{70}.$$

---

14. Danny is booking several 7-day vacations at a luxury resort on the Strip. He knows that there, the weather cycles between thunderstorms for 60 consecutive days and sunshine for 10 consecutive days. However, he does not currently know the weather there. What is the fewest number of vacations that Danny must book, at this moment, to guarantee that he experiences at least one vacation with sunshine every day? (No two of his booked vacations may overlap.)

Solution: The answer is $\boxed{18}$.

We ignore the condition that Danny's vacations must be disjoint; we can book a series of overlapping vacations, then shift them forward and backwards by multiples of 70 days, which preserves the weather in each vacation.

Now, note that if any two consecutive vacations that Danny books are more than 4 days apart, he cannot guarantee a completely sunny vacation. Indeed, suppose the first vacation takes place on days 1 through 7 and that the second vacation takes place on days $x+1$ through $x+7$, where $x \geq 5$. It is then possible that the ten days of sunshine occur on days 2 through 11, in which case the first and second vacations both experience rain, and the other vacations will obviously experience rain as well.

So, any two consecutive vacations are at most 4 days apart in the 70-day cycle. This means that Danny must book at least

$$\left\lceil \frac{70}{4} \right\rceil = 18$$

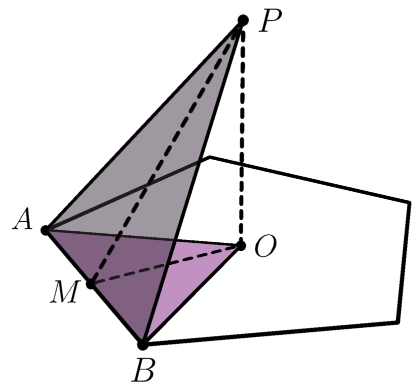
vacations.

---

15. Terry builds a casino in the shape of a pyramid. The base of this pyramid is a regular pentagon with an area of 2025. The five other faces are congruent isosceles triangles with an area of A. If the angle between one of the triangular faces and the pentagonal base is $60°$, what is A?

Solution: The answer is $\boxed{810}$.

Let O be the center of the base pentagon, and let P be the apex of the pyramid. Let A and B be two adjacent vertices of the pentagon, and let M be the midpoint of $\overline{AB}$.



We are given that $\angle PMO = 60°$. Combined with the fact that $\angle MOP = 90°$, it follows that $\triangle PMO$ is a 30-60-90 triangle. So, $\frac{PM}{MO} = 2$.

Since $\triangle ABO$ and $\triangle ABP$ are isosceles, segments MO and MP are altitudes in their respective triangles. Since the area of the pentagon is 2025, we know $[ABO] = \frac{2025}{5} = 405$. So,

$$[ABP] = \frac{AB \cdot PM}{2} = AB \cdot MO = 2[ABO] = 810.$$

1.4.6 Round 6

16. Kevin draws a Christmas tree in the shape of $\triangle ABC$, and draws two point-sized ornaments D and E on segments AB and AC, respectively. Triangle ADE has a perimeter of 11 and an inradius of 1, and quadrilateral $DBCE$ has a perimeter of 31 and an incircle with a radius of 3. What is DE?

Solution: The answer is $\boxed{\frac{11}{3}}$.

We compute $[ADE] = \frac{p \cdot r}{2} = \frac{11}{2}$ and $[DBCE] = p \cdot r2 = \frac{93}{2}$, so $[ABC] = [ADE] + [DBCE] = 52$.

Since the incircle of $DBCE$ is also the incircle of $\triangle ABC$, it follows that $\triangle ABC$ has inradius 3. So, its perimeter is $\frac{2[ABC]}{r} = \frac{104}{3}$. The sum of the perimeters of $\triangle ADE$ and quadrilateral $DBCE$ counts every side of $\triangle ABC$ once and segment DE twice. Thus, subtracting the perimeter of $\triangle ABC$ from this sum and dividing by 2 gives us DE:

$$DE = \frac{11 + 31 - \frac{104}{3}}{2} = \frac{11}{3}.$$

---

17. Harry writes down a three digit positive integer with a nonzero leading digit. Marv notices that if he erases any one of the three digits, the remaining two digits, when read from left to right, form a positive integer that divides Harry's original integer. How many possible integers could Harry have written down?
(It is okay for the remaining two digits to begin with a 0.)

Solution: The answer is $\boxed{14}$.

Let Harry's number be $\overline{ABC}$.

Firstly, from the problem information, we know $\overline{AB}$ is a divisor of $\overline{ABC}$. But clearly, $\overline{AB}$ is a divisor of $\overline{AB0}$, so it must divide their difference: we have that $\overline{AB}$ divides $\overline{C}$. Since $A \neq 0$, the only way this is possible is if $C = 0$.

The problem condition also implies the following:

- We have that $\overline{A0}$ divides $\overline{AB0}$, so $\overline{A0}$ divides $\overline{B0}$; equivalently, the digit A is a divisor of the digit B.
- We have that $\overline{B0}$ divides $\overline{AB0}$, so $\overline{B0}$ divides $\overline{A00}$; equivalently, B is a divisor of $10A$.

Since $A \mid B \mid 10A$, it follows that $B \in \{A, 2A, 5A\}$. We can now easily enumerate the possible numbers that Harry could have written down:

$$110, 220, \ldots, 990, 120, 240, 360, 480, 150.$$

There are 14 numbers in this list in total, and each of them can be easily checked to work.

---

18. A list of positive integers satisfies the following properties:

(A) The mean of the list is 8.

(2) The median of the list is 13.

(D) The mode of the list is 15.

Moreover, the range of the list is 27. What is the fewest possible number of elements that could be in the list?

Solution: The answer is $\boxed{89}$.

First, note that if we have the following 89 numbers:

- $22\times$ ones,
- $22\times$ twos,
- $21\times$ thirteens,
- $23\times$ fifteens,
- $1\times$ twenty-eight,

the four properties are satisfied. So, it suffices to prove that Buzz must always have at least 89 numbers in his list.

Subtract 8 from every number, so that our list of numbers now has a mean (and consequently sum) of 0, a median of 5 and a mode of 7. Moreover, every number in our list is at least -7, so the range condition tells us that the list contains some number at least 20.

Suppose there are k -7's in our list. Pair each one up with a 7 in the list, and delete it. Now, every number in our list is at least -6, there is still at least one 7 in our list, the sum is still 0, the median is still 5 and the maximum is still at least 20.

We now sort the list from least to greatest. The list can be subdivided into the smaller half of n elements, the one or two middle elements, and the larger half of n elements. Let a be the number of -6's in the smaller half, and let b be the number of 7's in the larger half. Then, the sum of the elements in the smaller half is at least $-6a - 5(n-a) = -a - 5n$; the sum of the one or two middle elements is at least 5; the sum of the elements in the larger half is at least $5(n-1-b) + 7b + 20 = 5n + 2b + 15$. So, altogether, the sum of our list is at least

$$(-a - 5n) + 5 + (5n + 2b + 15) = 20 + 2b - a.$$

But since this sum must be 0, it follows that $a \geq 20 + 2b$. When we add back all the -7's and 7's we deleted, we must add back at least $a - b + 1$ of each, so that the mode is indeed 7. Thus, after we've added back these numbers, there are at least

$$\begin{aligned} 2n + 1 + 2(a - b + 1) &\geq 2a + 1 + 2(a - b + 1) \\ &= 4a - 2b + 3 \\ &\geq 4(20 + 2b) - 2b + 3 \\ &\geq 83 + 6b \\ &\geq 89 \end{aligned}$$

numbers in the list.

1.4.7 Round 7

19. In a concert, 6 singers will perform. Each singer randomly chooses a (possibly empty) subset of the other singers, and requests to perform later than all the singers from that set. Let N be the number of orders of the singers such that all of their wishes are satisfied. What is the expected value of N?

Solution: The answer is $\boxed{\frac{45}{2048}}$.

It suffices to count the probability that a random ordering of the singers and a random set of requests agree with each other; multiplying this by 6! gives us our final answer, by the definition of expected value.

The key idea is to choose the random ordering of singers first. Number the singers 1 through 6 based on their order. Now, we need singer 1's request to be empty, singer 2's request to be a subset of $\{1\}$, singer 3's request to be a subset of $\{1, 2\}$, etc. The probability that all 6 singers make a request that agrees with their position in line is

$$\frac{2^0}{2^5} \cdot \frac{2^1}{2^5} \cdots \frac{2^5}{2^5} = \frac{1}{2^{15}}.$$

The final answer is thus $\frac{6!}{2^{15}} = \frac{45}{2048}$.

---

20. How many subsets of $\{1, 2, \ldots, 12\}$ with 5 elements contain no two elements differing by 1, 6 or 11?

Solution: The answer is $\boxed{12}$.

Put the numbers 1 through 12 into a circle, and the elements be $\{a_1, a_2, a_3, a_4, a_5\}$. Let d_i be the distance between a_i and a_{i+1} (where $a_6 = a_1$). Clearly $d_1 + d_2 + d_3 + d_4 + d_5 = 12$, and none of them can be 0 or 1. We also need that no consecutive d's add to 6.

Notice that if one of them is a 4, the other 4 must all must be 2 - but then there will be a pair that differs by 6, which fails. Thus, we must have $d_i \in \{2, 3\}$, and the only arrangement that works is $2, 2, 3, 2, 3$. Since this is asymmetrical, we can rotate this to start at any number, giving us a total of 12 possible subsets.

---

21. Call a positive integer *chromatic* if its leftmost digit is a 1, and each subsequent digit is 0 or 1 greater than the digit immediately to its left. Let N be the sum of all *chromatic* integers less than one billion. What is the largest odd divisor of N?

Solution: The answer is $\boxed{123456789}$.

For convenience, we will consider 0 to be a chromatic integer as well. This won't affect the final answer. It's easy to see that a number is chromatic if and only if it is the sum of the elements in a (possibly empty) subset of

$$\{1, 11, \ldots, 111{,}111{,}111\}.$$

There are $2^9 = 512$ such subsets, and each number appears in half of the subsets—exactly 256. So, our sum is

$$\begin{aligned} &256 \cdot 1 + 256 \cdot 11 + \cdots + 256 \cdot 111{,}111{,}111 \\ &= 256 \cdot 123456789. \end{aligned}$$

Our final answer is then 123456789.

1.4.8 Round 8

22. Evan doodles a triangle ABC with side lengths $AB = 9$, $BC = 16$ and $AC = 15$. Points D and E lie on $\overline{BC}$ with $BD = 4$ and $EC = 7$. Circle ω passes through D and E and is tangent to $\overline{AB}$. If ω intersects $\overline{AC}$ at X and Y, what is XY?

Solution: The answer is $\boxed{8}$.

Let P be the point of tangency of ω to segment AB. Since we know that $BD = 4$, $EC = 7$, and $BC = 16$, we can get that $DE = 5$. Applying Power of a Point at B with respect to ω yields us

$$BP^2 = BD \cdot BE$$

so $BP = 6$. This gets us $AP = 3$.

Then applying Power of a Point to both A and C with respect to ω, we have

$$\begin{aligned} AX \cdot AY &= AP^2 \\ CX \cdot CY &= CD \cdot CE \end{aligned}$$

We can rewrite these two equations as

$$\begin{aligned} AX \cdot (15 - CY) &= 9 \\ (15 - AX) \cdot CY &= 84 \end{aligned}$$

This gets us a two-variable, two-equation system. Subtracting the second equation from the first yields $CY - AX = 5$, so $CY = AX + 5$. Substituting that back into the first equation yields

$$\begin{aligned} 15AX - AX(AX + 5) &= 9 \\ AX^2 - 10AX + 9 &= 0 \\ (AX - 1)(AX - 9) &= 0 \end{aligned}$$

Clearly AX cannot be 9, as $AX \cdot AY = 9$, and $AX < AY$. Thus, we have $AX = 1$, so $AY = 9$. Thus, we get that $XY = 8$, finishing.

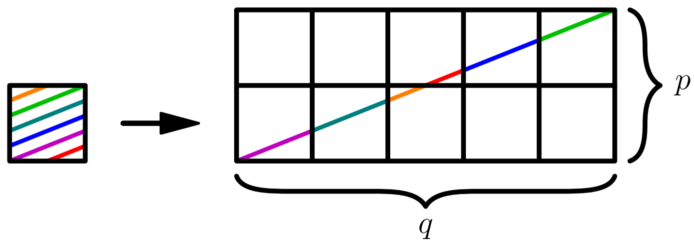
---

23. Let k be a rational number. In the coordinate plane, Celine draws a line of slope k through every lattice point. If exactly 1000 distinct lines pass through the interior of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, how many possible values of k are there? (The interior of a square does not contain its boundary.)

Solution: The answer is $\boxed{1440}$.

Clearly, $k = 0$ does not work. Also if a negative value of k works then reflecting every line over $x = \frac{1}{2}$ creates the lines for when the slope is $-k$, and clearly 1000 lines still pass through the interior of the square. So, without loss of generality, assume $k = \frac{p}{q}$ where p and q are relatively prime *positive* integers.

We claim that the condition is satisfied if and only if $p + q = 1001$ (and $\gcd(p, q) = 1001$). Consider the diagram below:



The above diagram illustrates that we can use the 1000 segments cutting a square and construct them in order to form a line of slope $\frac{p}{q}$ that cuts through 1000 unit squares (with 1000 replaced by 6). Since our line cuts through 1000 unit squares, it must cross 999 gridlines. Of these gridlines, $p - 1$ are horizontal and $q - 1$ are vertical, so we must have $p + q = 1001$, as claimed.

It now suffices to count the number of pairs (p, q) meeting this property. We have

$$\gcd(p, q) = 1 \iff \gcd(p, p + q) = 1 \iff \gcd(p, 1001) = 1.$$

Once we choose p, q is determined as well, as $q = 1001 - p$. We count the number of solutions to $\gcd(p, 1001) = 1$ with Euler's totient formula, which tells us that there are $6 \cdot 10 \cdot 12 = 720$ possible values of p. It therefore follows that there are 720 possible positive values of k.

Since k can be negative as well, our final answer is 1440.

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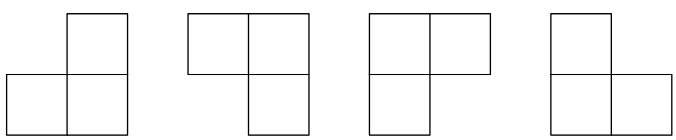
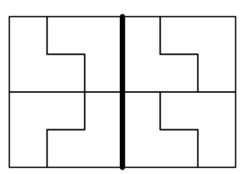
24. Anika draws a 4 by 6 rectangle. How many ways can she completely tile this rectangle with L-shaped triominoes (shown below) and color each triomino red, green or blue, such that any two neighboring triominoes are different colors? (Two triominoes neighbor if they share a positive amount of perimeter.)



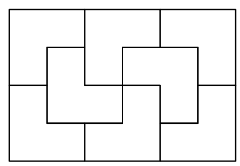
Solution: The answer is $\boxed{1164}$.

We proceed with casework. First, consider the case with four 2×3 boxes, each with two triominoes. Here is an example tiling.



We focus on the two boxes on the left side of the line. It is clear that each box has two ways to be oriented, so there are four different tilings for the two boxes. In two of the cases, the tiles above and below line up along the boundary, like in the example shown, and in each of those cases there are 18 ways (6 ways to choose the left two triominoes, then 3 color combinations for the other two that satisfy the constraints. Then in the other two cases, they don't line up, like the right two boxes. In that case there are again 6 ways to choose the color of the left two boxes, but then the colors of the other two boxes are determined. This gives $2(18 + 6) = 48$ ways to tile the left side. The right side casework in the same, except for the two triominoes touching the center border, there will only be 3 valid color combinations instead of 6, so the right side has 24 ways to be colored. This gives $48(24) = 1152$ for this case.

There are only two tilings that do not fall into the above cases: the tiling shown below and its reflection over the horizontal axis.



Then after choosing the colors of the two leftmost triominoes, which there are 6 ways to do, the color every other triomino is forced.

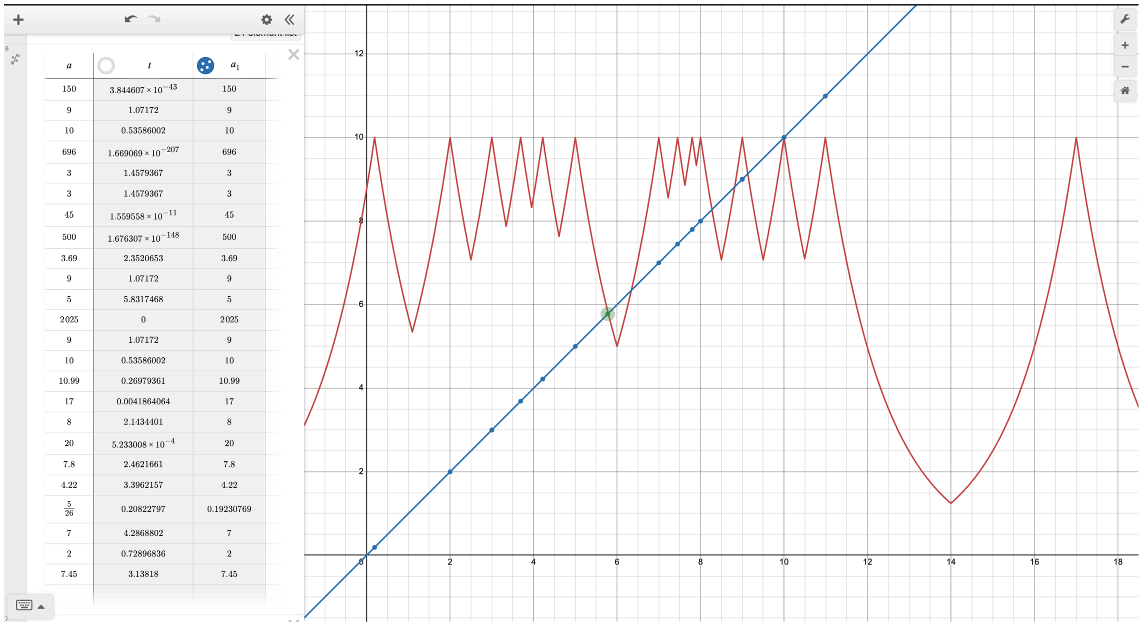
This gives 12 more cases, and our final answer is $1152 + 12 = 1164$.

1.4.9 Estimation Round

25. Estimate the highest tentative score S that a team will earn on this problem. The graders will choose the smallest possible correct answer, accurate to the nearest hundredth. An estimate of E will earn a *tentative score* of $10 \cdot 2^{-|S-E|}$. We will finalize scores by rounding your *tentative score* to the nearest integer.

Solution: The answer is $\boxed{5.78}$.

This problem took some Desmos trickery to pull off. See the screenshot below.



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26. Suppose each pet owned by someone on the EMCC Problem Selection Committee is weighed in kilograms. Estimate r, the real number such that the product of all these numerical masses is 10^r. An estimate of E will earn a score of the closest integer to $10 \cdot 3^{-|r-E|}$.

Solution: The answer is $\boxed{r \approx -2.60}$.

To our surprise, the pet dogs/cats won versus the goldfish and hamsters.

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27. Let $s(n)$ denote the sum of the digits of n. Estimate the smallest positive integer N such that among the list

$$s(1^2), s(2^2), \ldots, s(N^2),$$

there are 2025 more odd numbers than even numbers. An estimate of E will earn a score of the closest integer to $10\left(\min\left(\frac{N}{E}, \frac{E}{N}\right)\right)^2$.

Solution: The answer is $\boxed{68859}$.

Oron, Grant and Benny conjecture the following:

Let b be a positive integer that leaves a remainder of 2 or 4 when divided by 8, and let $s_b(n)$ denote the sum of the digits of n in base b. Prove that for all positive integers n, the following inequality holds:

$$\sum_{i=1}^{n} (-1)^{s_b(i^2)} \leq 0.$$

We tried for five minutes and gave up, though.