Speed Test Solutions

1. What is the average of 9999 and 100001?

   Solution: The answer is $\boxed{55000}$.

   The average of 9999 and 100001 equals $\frac{9999+100001}{2} = \frac{110000}{2} = 55000$.

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2. What is $\sqrt{36 \times 49}$?

   Solution: The answer is $\boxed{42}$.

   We have $\sqrt{36 \cdot 49} = \sqrt{36} \cdot \sqrt{49} = 6 \cdot 7 = 42$.

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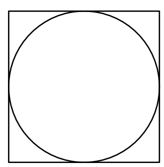
3. A circle is inscribed in a square, as shown below. If the area of the square and the circumference of the circle are both equal to x, what is x?



   Solution: The answer is $\boxed{\pi^2}$.

   Let the side length of the square be s. The diameter of the circle is also s, so we have that $x = s^2 = s\pi$. Dividing by s gives that $s = \pi$, so $x = s^2 = \pi^2$.

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4. Grant studies for 6 hours and 26 minutes, beginning a minutes after midnight and finishing b minutes after 7:00 AM. What is $a - b$?

   Solution: The answer is $\boxed{34}$.

   We have that

$$a + (6 \cdot 60 + 26) - b = 7 \cdot 60,$$

   so $a - b = 60 - 26 = 34$.

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5. If
$$x + (x+1) + (x+2) + (x+3) + (x+4) = 100 + 102 + 104 + 106 + 108,$$
what is x?

Solution: The answer is $\boxed{102}$.

The LHS and RHS of the equation are each the sums of five terms in an arithmetic progression. Therefore, their middle terms should be equal. So, we have $x + 2 = 104$, and $x = 102$.

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6. What is the largest integer dividing 432, 324 and 243?

Solution: The answer is $\boxed{27}$.

By 2026 OTIS Mock AIME #1, we have that $\gcd(432, 324) = 108$. Clearly, $\gcd(108, 243) = 27$.

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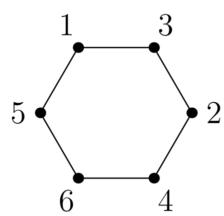
7. The numbers 1 through 6 are written at the vertices of a hexagon below. In a move, Alice can swap any two numbers. What is the fewest number of swaps necessary for the numbers to be sorted in either clockwise or counterclockwise order?



Solution: The answer is $\boxed{1}$.

Swapping the 1 and 4 sorts the numbers in counterclockwise order, so the answer is 1.

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8. What fraction of the numbers in $\{1, 2, 3, \ldots, 899\}$ are perfect squares?

Solution: The answer is $\boxed{\frac{1}{31}}$.

Since $30^2 = 900$, there are 29 perfect squares in $\{1, 2, 3, \ldots, 899\}$. We can use difference of squares to write $899 = 30^2 - 1 = (30+1)(30-1) = 31(29)$, so $\frac{29}{899} = \frac{29}{29(31)} = \frac{1}{31}$.

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9. Alice and Bob are attendees at a party with 10 people total. Each person randomly chooses somebody else at the party to gift a present. What is the probability that somebody gets a present from both Alice and Bob?

Solution: The answer is $\boxed{\frac{8}{81}}$.

Since Alice and Bob cannot gift to themselves, this person must be one of the other 8 attendees. The probability that Alice gives a gift to one of these attendees is $\frac{8}{9}$. Then Bob must choose the same person, which occurs with probability $\frac{1}{9}$, so the answer is $\frac{8}{9} \cdot \frac{1}{9} = \frac{8}{81}$.

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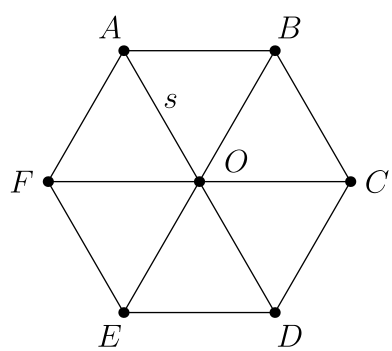
10. An equilateral triangle with area 10 has side length s. What is the area of a regular hexagon with circumradius s?

Solution: The answer is $\boxed{60}$.

We can put six of these equilateral triangles together as such:



Note that $ABCDEF$ is the hexagon we are looking for, as point O is a distance of s from each vertex. Its area is six times that of the triangle, giving us 60.

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11. Jesse is purchasing gas masks. He doesn't know the price of one gas mask, but he knows it costs \$274.32 to buy 9 gas masks. What is the fewest positive number of gas masks that Jesse must buy so that the total is an integer number of dollars?

Solution: The answer is $\boxed{25}$.

By looking at the last two digits, we see that 27432 is a multiple of 4 but not 5. Dividing by 9 does not change this fact. Therefore, the smallest integer which we can multiply $\frac{27432}{9}$ by to get an integer is 25.

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12. Let ℓ be a line with slope 2 and let m be a line with slope -1. Lines ℓ, m and x-axis determine a triangle with an area of 108. How far apart are the x-intercepts of ℓ and m?

Solution: The answer is $\boxed{18}$.

Suppose the two lines meet at a point with y-coordinate h. We can also shift the lines left or right so their intersection has x-coordinate 0. Then, since ℓ has slope 2, it meets the x-axis at $(-\frac{h}{2}, 0)$. Since m has slope -1, it meets the x-axis at $(h, 0)$. So, the area of the triangle is

$$\left(\frac{3h}{2}\right)h.$$

Setting this equal to 108, we find that $h = 12$, so $\frac{3}{2}h = 18$, as desired.

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13. When an unfair coin is flipped 10 times, there is a $\frac{2}{3}$ chance that the total number of heads is even. If, instead, the coin was flipped 20 times, what is the probability that the total number of heads is even?

Solution: The answer is $\boxed{\frac{5}{9}}$.

Split the 20 flips into two groups of 10. We require that there is an even number of heads in both groups or an odd number of heads in both groups. This occurs with probability $\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}$.

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14. What is the sum of the digits in the decimal expansion of 5^{-8}?

Solution: The answer is $\boxed{13}$.

We have that $5^{-8} = \frac{2^8}{10^8} = 256 \cdot 10^{-8}$, so the sum of digits will be $2 + 5 + 6 = 13$.

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15. In isosceles triangle ABC with $AB = AC$, the incircle passes through the centroid. What is AB/BC? (The *centroid* of a triangle is the common intersection point of its three medians.)

Solution: The answer is $\boxed{\frac{5}{2}}$.

Let I be the incenter, let D be the foot from A to' $\overline{BC}$ and let E be the foot from I to $\overline{AC}$. Since $AG : GD = 2 : 1$ and I is the midpoint of $\overline{GD}$, it follows that $AI : ID = 5 : 1$. So, $AI : IE = 5 : 1$. By AA similarity, we have $\triangle AIE \sim \triangle ABD$, so $\frac{AB}{BC} = \frac{AB}{2BD} = \frac{5}{2}$.

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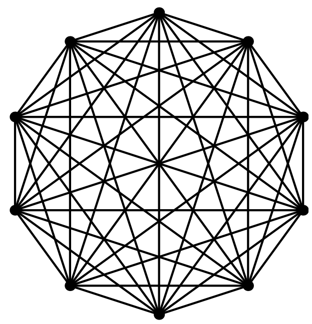
16. Lotad draws the sides and diagonals of a regular 10-sided polygon. Then, she draws a line ℓ that does not pass through any vertices in the polygon. What is the maximum possible number of sides and diagonals that ℓ could intersect?



Solution: The answer is $\boxed{25}$.

The line l will partition the vertices of the polygon into two sets, one on each side of the line. A given side or diagonal will intersect l if and only if its endpoints lie on opposite sides of the line. The number of intersections is thus the product of the sizes of the two sets. Since there are 10 total elements, by AM-GM this product is maximized when there are 5 on each side, which gives a total of $5^2 = 25$ intersections.

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17. A *repdigit* is a positive integer whose digits are all the same. What is the maximum number of distinct digits that the absolute difference of two repdigits could have?

Solution: The answer is $\boxed{4}$.

The difference of 1111 and 22 is 1089, with four distinct digits. It remains to show that this is the best bound possible.

Suppose the larger repdigit consists of repeated A's and the smaller repdigit consists of repeated B's. If $A \geq B$, every digit in the difference must be either $A - B$ or A, so there are at most 2 distinct digits. If $A < B$, every digit in the difference must be either $A - B + 10$, $A - B + 9$, A or $A - 1$, so there are at most 4 distinct digits.

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18. A 7-digit integer has strictly increasing digits from left to right. How many such integers are divisible by 3?

Solution: The answer is $\boxed{12}$.

Zero cannot be the first digit and thus cannot appear in the number due to the strictly increasing condition. We are wish to choose 7 numbers from 1 through 9 which sum to a multiple of 3, or equivalently, choose two numbers summing to a multiple of 3 to leave out, because $1 + 2 + \cdots + 9 = 45$

is divisible by 3. This can occur is one numbers is $1 \pmod 3$ and the other is $2 \pmod 3$, or if they are both divisible by 3, for a total of $3(3) + \binom{3}{2} = 9 + 3 = 12$ cases.

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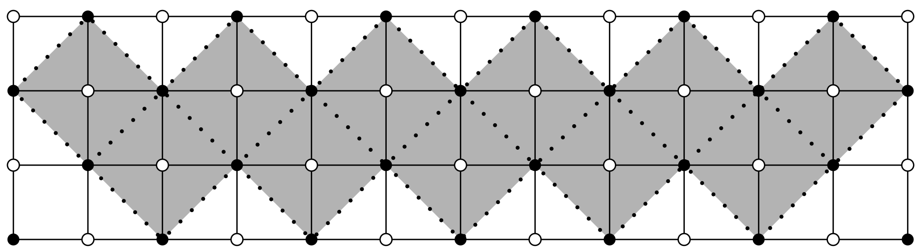
19. In a 3 by 12 grid of unit squares, Albert wishes to draw diagonals in some of the unit squares such that the diagonals form a single cycle that does not cross or touch itself. How many ways can Albert draw these diagonals?

Solution: The answer is $\boxed{132}$.

Albert's path encloses a polyomino made from $\sqrt{2} \times \sqrt{2}$ squares. Moreover, if we color every vertex of the grid black or white in checkerboard fashion, each $\sqrt{2} \times \sqrt{2}$ square has vertices of the same color.



We will count the number of paths on black vertices and multiply by 2 at the end. The set of $\sqrt{2} \times \sqrt{2}$ squares with black vertices forms a tilted staircase, as shown above. To form a polyomino within this 11-cell staircase, it suffices to choose the leftmost cell and the rightmost cell to include in the polyomino. By doing casework on whether these cells are distinct or not, there are $\binom{11}{2} + 11 = 66$ polyominos.

Remembering to multiply by 2 in the case that the vertices are white instead of black, we obtain our final answer of 132.

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20. Let $PA_1A_2 \cdots A_{10}$ be a pyramid such that base $A_1A_2 \cdots A_{10}$ is a regular 10-sided polygon. There exists a sphere tangent to every face of the pyramid, as well as a sphere passing through every vertex of the pyramid. Given that these two spheres share a center, what is $\angle PA_1A_2$, in degrees?

Solution: The answer is $\boxed{81^\circ}$.

We claim that every face has the same circumradius. This would imply that $\angle A_1PA_2 = 18^\circ$ by the inscribed angle theorem, so $\angle PA_1A_2 = 81^\circ$. (Evidently, the triangular faces are all isosceles.)

To show the claim, let O be the coinciding point of the incenter and circumcenter. Let R and r be the circumradius and inradius of the pyramid. For each face $\mathcal{F}$ of the pyramid, let O' be the foot from O to $\mathcal{F}$. So, if X is any vertex of $\mathcal{F}$, we have that $\triangle O'OX$ is a right triangle. We know that $OO' = r$ and that $OX = R$, so by the Pythagorean theorem, $O'X = \sqrt{R^2 - r^2}$. Repeating this argument on all vertices X of $\mathcal{F}$, we see that $\mathcal{F}$ is cyclic with circumradius $\sqrt{R^2 - r^2}$ and center O'. Repeating this argument on all faces $\mathcal{F}$ implies the claim.

Accuracy Test Solutions

1. At a clothing store, sweaters usually cost \$56.22 more than jeans. With a 40% discount, sweaters now cost \$56.22. How many jeans are equal to the price of 30 discounted sweaters?

   Solution: The answer is $\boxed{45}$.

   Let $x = \$56.22$. The second sentence tells us that x is equal to 60% of the price of a non-discounted sweater y, so the first sentence tells us that jeans cost $0.4y$. Then we get that the price of 10 discounted sweaters is $10 \cdot 0.6y = 15 \cdot 0.4y$ so the answer is 15.

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2. Sabrina rolls a fair six-sided die three times. If the product of the the three numbers obtained has fifteen divisors, what is the sum of the three numbers?

   Solution: The answer is $\boxed{16}$.

   The number must be of the form p^{14} or p^4q_2 for distinct primes p and q. The first option is clearly impossible with only three rolls. The second option is only achievable with rolls of 4, 6, and 6 in some order, leading to a sum of 16.

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3. A right triangle $\mathcal{T}$ has side lengths a, b and c. If $abc = 260$ and $a^2 + b^2 + c^2 = 200$, what is the area of $\mathcal{T}$?

   Solution: The answer is $\boxed{13}$.

   Let the hypotenuse be c. Then $(a^2 + b^2) + c^2 = 2c^2 = 200$ implies that $c = 10$. Then the area of the triangle is $\frac{ab}{2} = \frac{260}{2c} = 13$.

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4. Suppose A, B, C and D are (not necessarily distinct) digits such that the two-digit numbers $\underline{AB}$, $\underline{BC}$ and $\underline{CD}$ are consecutive (in that order). What is the four digit number $\underline{ABCD}$?

   Solution: The answer is $\boxed{8890}$.

   Since $\underline{AB}$ and $\underline{BC}$ are consecutive, their units digits B and C cannot be equal. This means that the tens digits of $\underline{BC}$ and $\underline{CD}$ are not equal, so we must have $C = 9$ and $D = 0$. From there we fill in $A = B = 8$ to get 8890, which works.

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5. Michael has a 9×9 square of chocolate. Every minute, he takes a rectangle of chocolate and splits it along a vertical or horizontal line into two pieces of chocolate, each with integer side lengths. When Michael is unable to split any more pieces of chocolate, he stops. What is the average length of the cuts that Michael makes?

Solution: The answer is $\boxed{\frac{9}{5}}$.

Each cut increases the total number of pieces by 1. Since Michael stops with 81 unit squares of chocolate, he makes 80 cuts. The total length of cuts that he makes is the sum of the length of the 8 vertical and 8 horizontal line segments in the interior of the grid. Thus the average cut length is $\frac{(8+8)(9)}{80} = \frac{9}{5}$.

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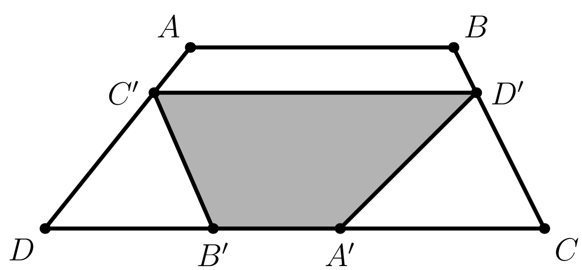
6. Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$. A trapezoid $A'B'C'D'$ similar to $ABCD$ is inscribed inside, as shown below. Given that $AB = 6$, $BD' = 1$ and $D'C = 3$, what is CA'?



Solution: The answer is $\boxed{\frac{9}{4}}$.

Since $\frac{CD'}{CB} = \frac{3}{4}$, the ratio of the heights of the trapezoids is also $\frac{3}{4}$. By the similarity, we get that $B'A' = 6 \cdot \frac{3}{4} = \frac{9}{2}$. The similarity also tells us that $AD \parallel D'A'$ and $BC \parallel C'B'$, and the parallelograms this implies that $C'D' = DA' = B'C$.

To finish, we let $x = CA'$ and note that $C'D'$ and DC are in a $3:4$ ratio as well. We obtain the equation

$$\frac{x + \frac{9}{2}}{2x + \frac{9}{2}} = \frac{3}{4}.$$

Expanding and solving gives $x = \frac{9}{4}$.

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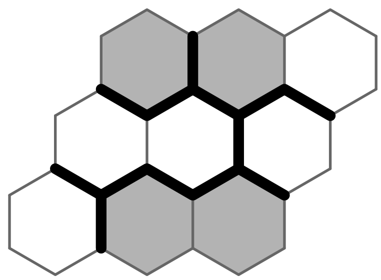
7. In the hexagonal grid below, some edges are bolded, dividing the grid into six regions. Each of the nine cells is to be filled with a distinct single digit integer, such that

   - no consecutive numbers share a **bolded** edge;
   - the one or two digit number formed by each region is a prime number.

   What is the product of the digits in the four shaded cells?



Solution: The answer is $\boxed{672}$.

Label the first row ABC, the second row DEF and the last row GHI.

(a) The digits 4, 6 and 8 can only appear in spots B, D or H since they cannot be the units digit of a prime. In particular, BDH must be some permutation of 468.

(b) The digit 5 cannot appear in CEI, so it must be one of the single digit primes. Both A and G will border two numbers from $\{4, 6, 8\}$, so neither of them can be 5. Therefore, F is 5.

(c) Since B cannot be 4 or 6, it must be 8.

(d) Note that DH is a permutation of 46. So, G cannot be 3 or 7, so it must be 2.

(e) Since A touches 8, it must be 3.

(f) Since D touches 3, it must be 6. Now, H must be 4.

(g) Since E touches 8, it must be 1.

(h) Since the number $\overline{4I}$ is prime, I must be 7.

(i) This leaves us with C, which is 9.

We extract $3 \times 8 \times 4 \times 7 = 672$.

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8. Let $A_1A_2 \dots A_9$ be a regular 9-sided polygon. Two points X and Y are randomly selected inside the polygon. What is the probability that segments A_1X and A_2Y intersect?

Solution: The answer is $\boxed{\frac{7}{18}}$.

Suppose that $A_1A_2 \cdots A_9$ has area 1. Then, by symmetry, the expected value of $[A_1A_2X]$ is $\frac{1}{9}$. So, the probability that Y lies inside of $\triangle A_1A_2X$ is $\frac{1}{9}$. Likewise, the probability that X lies inside of $\triangle A_1A_2Y$ is $\frac{1}{9}$.

These events are mutually exclusive, and in the $\frac{7}{9}$ chance that neither is true, then either A_1A_2XY or A_1A_2YX is a convex quadrilateral.

Again, these two events are mutually exclusive, so they must both occur with a $\frac{7}{18}$ probability. Lines A_1X and A_2Y intersect if and only if A_1A_2XY is convex, which we know is a $\frac{7}{18}$ chance.

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9. If a, b, c and d are positive integers such that $\gcd(a! + b!, c! + d!) = 1320$, what is $a \times b \times c \times d$?

Solution: The answer is $\boxed{4050}$.

Claim: We have $\min(a, b, c, d) = 5$.

Proof: Let m stand as shorthand for $\min(a, b, c, d)$. Since $m!$ divides each of $a!$, $b!$, $c!$ and $d!$, we have $m \leq 5$. Moreover, $5!$ dividing $a! + b!$ implies that $a, b \geq 5$. Likewise, $c, d \geq 5$, so $m = 5$.

Without loss of generality, set a to 5. We can easily compute that the remainders when $5!, \ldots, 10!$ are divided by 11 are 10, 5, 2, 5, 1 and 10, respectively. Since we have $11 \mid a! + b!$, this implies that $b = 9$. We have that

$$5! + 9! = 120(1 + 6 \cdot 7 \cdot 8 \cdot 9) = 120(1 + 54 \cdot 56) = 120 \cdot 55^2.$$

Since $5^2 \nmid 1320$, it follows that $5^2 \nmid c! + d!$. So, $\min(c, d) \leq 9$. This also implies that $\max(c, d) \leq 10$ (because of divisibility by 11). From the fact that $11 \mid c! + d!$, we now see that $\{c, d\} = \{9, 10\}$. Our final answer is $5 \times 9 \times 9 \times 10 = 4050$.

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10. Let a, b and c be positive real numbers and let $P(x) = x^3 - ax^2 + bx - c$ be a polynomial with real roots $\frac{1}{3}$, r and s. If $c^2 = aP(\frac{a}{2})$, what is the maximum possible value of $r^2 - s^2$?

Solution: The answer is $\boxed{\frac{\sqrt{2}}{6}}$.

By Descartes' rule of signs, r and s are positive. Note that

$$K = \sqrt{\frac{a}{2} P\left(\frac{a}{2}\right)}$$

is the area of a triangle with side lengths $\frac{1}{3}$, r and s. Let the circumradius of this triangle be R. By geometric formulas, we have that $\frac{rs}{3} = 4KR$. So, we have

$$\begin{aligned} c &= \sqrt{2}\sqrt{\frac{a}{2} P\left(\frac{a}{2}\right)} \\ \frac{rs}{3} &= K\sqrt{2} \\ 4KR &= K\sqrt{2} \\ R &= \frac{\sqrt{2}}{4}. \end{aligned}$$

So, our problem reduces to this: we have a $\triangle ABC$ with circumradius $\frac{\sqrt{2}}{4}$; we are given that $BC = \frac{1}{3}$, and we wish to maximize $AB^2 - AC^2$. Let D be the foot from A to $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. We have

$$\begin{aligned} AB^2 - AC^2 &= (AD^2 + DB^2) - (AD^2 + DC^2) \\ &= DB^2 - DC^2 \\ &= 2(BC)(DM). \end{aligned}$$

Let O be the circumcenter of $\triangle ABC$. Since M and D are the feet from O and A to $\overline{BC}$, respectively, we have $DM \le OA = \frac{\sqrt{2}}{4}$, so

$$AB^2 - AC^2 \le 2(BC)(DM) \le \frac{\sqrt{2}}{6}.$$

Equality holds when $\overline{OA} \parallel \overline{BC}$.

Team Test Solutions

1. Let N be a two-digit integer. If the sum of the digits of N is a multiple of 6 and the product of the digits of N is a multiple of 7, what is the sum of all possible values of N?

   Solution: The answer is $\boxed{192}$.

   For the product of digits to be a multiple of 7, at least one of the digits must be 0 or 7. Combined with the first condition, we obtain the three possibilities 60, 57, and 75 which sum to 192.

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2. When
$$(0.565656\cdots) + (0.134134134\cdots)$$
   is written as a decimal, what is the sum of the distinct digits that appear after the decimal point?

   Solution: The answer is $\boxed{22}$.

   We can see that the sum will repeat every six digits. This repeating string will be $565656 + 134134 = 699790$, so the sum of the distinct digits is $6 + 9 + 7 + 0 = 22$.

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3. In square $ABCD$ with side length 1, points X and Y lie on the incircle. If $AXCY$ is a rhombus, what is its area?

   Solution: The answer is $\boxed{\frac{\sqrt{2}}{2}}$.

   Because $AX = XC$ and $AY = YC$, both X and Y must lie on the perpendicular bisector of AC, which is BD. Then since BD passes through the center of the square, XY is a diameter of the incircle, so $XY = 1$. Then since the diagonals of a rhombus are perpendicular, its area is $\frac{1}{2} \cdot AC \cdot XY = \frac{\sqrt{2}}{2}$.

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4. Grant thinks of a set of integers. He writes down all sums of distinct pairs of integers from the set. The smallest number he writes is -5 and the largest number he writes is 12. What is the maximum possible sum of the numbers in Grant's set?

   Solution: The answer is $\boxed{17}$.

   The second largest number in Grant's set is at most 5, or else he would write a number greater than 12. Then the third number is at most 4, so the greatest possible sum of the positive numbers in his set is $12 + 4 + 3 + 2 + 1 = 22$. Then the negative numbers in his set must sum to -5 or less in order for him to write -5, so the sum can be no more than $22 - 5 = 17$. The set $\{-3, -2, 1, 2, 3, 4, 5, 7\}$ achieves this.

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5. What positive integer n satisfies $n + \operatorname{lcm}(n, 30) = 576$?

Solution: The answer is $\boxed{96}$.

Since $n = 576 - 30k$ for some integer k, n is a multiple of 6 but not a multiple of 5. This means that $\gcd(n, 30) = 6$, so $\operatorname{lcm}(n, 30) = \frac{30n}{6} = 5n$. Thus $6x = 576$ and $x = 96$.

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6. What is the smallest positive real x such that $\lfloor x^2 \rfloor + \lfloor x \rfloor$ is a positive multiple of 5?

Solution: The answer is $\boxed{2\sqrt{2}}$.

When $x < 2$, $\lfloor x^2 \rfloor + \lfloor x \rfloor \le 3 + 1 = 4$, so we must have $x \ge 2$. Then if $\lfloor x \rfloor = 2$, then $\lfloor x^2 \rfloor = 8$ is the only possible way to achieve a multiple of 5. This gives a minimum value of $2\sqrt{2}$.

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7. Grant thinks of some percentages summing to 100. He rounds each percentage to the nearest integer and writes the results down. If the new percentages sum to 97, what is the fewest number of percentages Grant could have written down?

Solution: The answer is $\boxed{7}$.

Rounding a percentage decreases it by strictly less than 0.5, so to decrease the total by 3, Grant must have rounded down at least 7 percentages.

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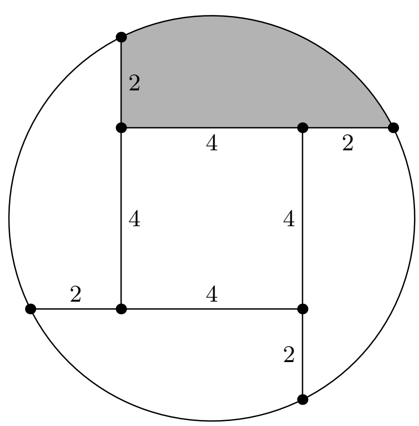
8. Four segments are drawn inside a circle, forming right angles with each other. Given the side lengths in the diagram below, what is the area of the shaded region?



Solution: The answer is $\boxed{5\pi - 4}$.

By symmetry, the two points where two opposing segments meet the circle are diametrically opposite each other. This means that the diameter of the circle is $\sqrt{(2+4+2)^2+4^2} = 4\sqrt{5}$ and the area of the circle is $(2\sqrt{5})^2\pi = 20\pi$. Then if the area of the shaded region is x, we have that $4x + 16 = 20\pi$. Solving, we get $x = 5\pi - 4$.

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9. Let $\mathcal{S}$ be the set of all positive integers n for which each prime divisor of n is less than 2026, and no perfect square greater than 1 divides n. Over all elements of $\mathcal{S}$, what is the average units digit?

Solution: The answer is $\boxed{\frac{15}{4}}$.

The units digit is 0 when both 2 and 5 divide n, which occurs $\frac{1}{4}$ of the time. For the rest of the cases, we will show that the average digit is exactly 5. For each term n not divisible by 19, we pair n with $19n$. Since n does not have units digit 0, the units digits of n and $19n$ will add to 10, so the average in each pair is 5, and thus the overall average in this case is 5. Our final answer is $0 \cdot \frac{1}{4} + 5 \cdot \frac{3}{4} = \frac{15}{4}$.

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10. Grant thinks of a set of 19 integers. He writes down all products of (not necessarily distinct) pairs of integers from the set. What is the fewest number of distinct values that Grant could have written down?

Solution: The answer is $\boxed{35}$.

By thinking of

$$\{-2^9, \dots, -2^1\} \cup \{0\} \cup \{2^1, \dots, 2^9\},$$

Grant writes down $\{-2^{18}, \dots, -2^2\} \cup \{0\} \cup \{2^2, \dots, 2^{18}\}$, for a total of 35 numbers. We now show that he always writes at least 35 numbers.

Given a set of i positive integers $a_1 < \cdots < a_i$, and another set of j positive integers $b_1 < \cdots < b_j$, there are at least $i + j - 1$ products of the form $a_k b_\ell$: just consider the $i + j - 1$ products

$$a_1 b_1 < a_1 b_2 < \cdots < a_1 b_j < a_2 b_j < a_3 b_j < \cdots < a_i b_j.$$

Similarly, given a set of i positive integers and j negative integers, there are at least $i + j - 1$ products between one number from each set.

Now, suppose Grant is thinking of x negative numbers and y negative numbers.

- If 0 is also one of the numbers he is thinking of, $x + y = 18$. He is thinking of at at least $\max(2x-1, 2y-1) \geq 17$ positive numbers and $x + y - 1 = 17$ negative numbers, as well as 0. This amounts to at least 35 numbers.
- If 0 is not one of the numbers he is thinking of, $x + y = 19$. Then, $\max(2x-1, 2y-1) \geq 19$, so this amounts to at least 37 numbers.

So, Grant needs at least 35 numbers.

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11. Every second, a real number p is randomly chosen in the interval $[0, 1]$. Then, with probability p, the process continues; otherwise, the process terminates and no more numbers are chosen. What is the probability that, after the process terminates, an odd number of the chosen numbers are greater than $\frac{1}{2}$?

Solution: The answer is $\boxed{\frac{2}{5}}$.

Let x be the answer to the question; then, there is a $1 - x$ chance that after the process terminates, an even number of the chosen numbers are greater than $\frac{1}{2}$. If the first probability chosen p_1 is less than $\frac{1}{2}$, there is a $\frac{3}{4}$ chance the process terminates, and a $\frac{1}{4}$ chance that the process continues. On the other hand, if $p_1 > \frac{1}{2}$, there is a $\frac{1}{4}$ chance the process terminates, and a $\frac{3}{4}$ chance that the process continues. So, we have

$$x = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)x + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)(1 - x).$$

Solving gives $x = \frac{2}{5}$.

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12. How many triples (a, b, c) of nonnegative integers are there satisfying

$$\begin{cases} a + b & \leq 12, \\ b + c & \leq 12, \\ c + a & \leq 12? \end{cases}$$

Solution: The answer is $\boxed{616}$.

We split into cases based on how many of the variables are equal to each other.

- Case 1: all three variables are distinct.
  We first consider triples with $a > b > c$. Then, for $a = 2, \ldots, 6$ we have $\binom{a}{2}$ choices for b and c, since b and c can be any nonnegative integers less than a. For $a = 7, \ldots, 11$ we have $\binom{13-a}{2}$ choices for b and c, since b and c can be at most $12 - a$. In this case, there are therefore

  $$2\left(\binom{2}{2} + \cdots + \binom{6}{2}\right) = 70$$

  triples with $a > b > c$, and $70 \cdot 3! = 420$ triples altogether.
- Case 2: two of the three variables are equal. First we look at the case when $a = b \neq c$. For $a = 0, \ldots, 6$, c can be any integer from 0 to $12 - a$, excluding a, which gives $12 - a$ choices for c. In this case, there are therefore

  $$12 + 11 + \cdots + 6 = 63$$

  triples with $a = b \neq c$. The $a = c$ and $b = c$ cases are the same, so there are $63 \cdot 3 = 189$ triples altogether.
- Case 3: all three variables are equal. There are evidently 7 triples.

Our final answer is $420 + 189 + 7 = 616$.

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13. There are two lines tangent to the parabolas $y = x^2 + 4x + 9$ and $y = 4x^2 - 8x + 3$. At what point do they meet?

Solution: The answer is $\boxed{(2, -3)}$.

Rewrite the equations in the form

$$\begin{cases} y &= (x+2)^2 + 5 \\ 4y &= (4x-4)^2 - 4. \end{cases}$$

With the second parabola written in this form, it is clear that it is a $\times\frac{1}{4}$ positive dilation of the original parabola. Then, it follows that the tangent lines will meet at this center of dilation.

To find the center of dilation, we consider the vertices of the respective parabolas. The first parabola has vertex $P = (-2, 5)$ while the second parabola has vertex $Q = (1, -1)$. The center of dilation is the point R on ray $\overrightarrow{PQ}$ past Q with $PQ : QR = 3 : 1$, so $R = (2, -3)$, as desired.

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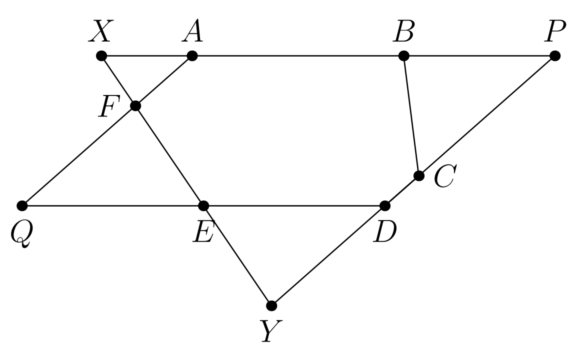
14. Let $ABCDEF$ be a hexagon with $AB = 14$, $BC = 8$, $CD = 3$, $DE = 12$, $EF = 8$, and $FA = 5$. Suppose $AB \parallel DE$ and $CD \parallel AF$. Line EF meets AB at X and CD at Y. Given that $XY = 20$, what is the area of $ABCDEF$?

Solution: The answer is $\boxed{60\sqrt{7}}$.



Construct parallelogram $AQPD$ as shown in the diagram below. As $AQ + QD = PD + AP$ and $AF + DE = 17 = AB + CD$, we get $FQ + QE = CP + PB$. This, along with $\angle FQE = \angle BPC$ and $FE = BC$, is enough to force $\triangle BPC \cong \triangle FQE$.

Hence let $FQ = BP = x$, so $QE = CP = x + 2$. Next, $\triangle XFA \sim \triangle EFQ \sim \triangle EYD$ gives that $EY = \frac{DE \cdot FE}{QE} = \frac{96}{x+2}$ and $XF = \frac{AF \cdot FE}{FQ} = \frac{40}{x}$. Hence $20 = XF + FE + EF = 8 + \frac{40}{x} + \frac{96}{x+2}$ which solves to yield $x = 10$. Hence triangles FQE and BPC have side lengths $8, 10, 12$ so by Heron's formula they each have area $\sqrt{15 \cdot 7 \cdot 5 \cdot 3} = 15\sqrt{7}$. Moreover, triangle AQD has area $15\sqrt{7} \cdot \frac{15}{10} \cdot \frac{24}{12} = 45\sqrt{7}$. Thus hexagon $ABCDEF$ has area

$$[APQD] - [QFE] - [BPC] = 90\sqrt{7} - 15\sqrt{7} - 15\sqrt{7} = 60\sqrt{7}.$$

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15. In an 8 by 8 grid, an *up-right* path is a sequence of 15 cells which begins in the bottom-left cell and ends in the top-right cell, with each cell being directly above or to the right of the previous one. How many sets of 8 distinct up-right paths are there such that exactly one path traverses each of the 36 cells not on the edge of the grid?

Solution: The answer is $\boxed{4096}$.

Consider the 8 cells along the diagonal from the top left to the bottom right cell. By the condition, each of these squares is traversed by exactly one path (the condition doesn't apply to the corners, but there is only one unique path passing through each of those cells). Then we only have to find the number of sets of 8 paths from the bottom left corner to each of the squares on the main diagonal, then we can square that number to account for choosing the paths from the main diagonal to the top right.

Starting at a non-corner square on the main diagonal, we can choose a direction for the previous square along its path, left or down. Then all other non-corner squares must have arrived from the same direction or else the exactly one path condition will fail. The paths from corner squares must be either fully vertical or fully horizontal. Then this reduces to the same situation with a shorter diagonal. The same logic applies 6 times in total, so 6 choices of left or down determine the set of paths uniquely. This gives 64 ways, which we square to get the final answer of 4096.
