Exeter Math Club Competition
January 27, 2018

Table of Contents

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Jane Street

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Contest Day Acknowledgments

- **Tournament Directors** James Lin, Vinjai Vale
- **Proctors** Yasmina Abukhadra, Chanseo "Brian" Bae, Catherine "Piper" Bau, Penny Brant, Dawson Byrd, Jiaying "Lucy" Cai, Keyu Cao, Evan Chandran, Jaime "Rose" Chen, Sophia Cho, Gordon Chi, Miranda Derossi, Zac Feng, Pavan Garidipuri, Thomas Guo, Nosa Lawani, Clara Lee, Suan Lee, Brian Liu, Bryce Morales, Thomas Mowen, Manuel Paez Jr., William Park, Brian Rhee, Araya "May" Sornwanee, Nathan Sun, Yuyang "Gloria" Sun, Logan Valenti, Maxwell Wang, Celeste Wu, Zheheng "Tony" Xiao, Wendi Yan, Xintong "Tony" Yu, Blane Zhu Qianqia "Queenie" Zhang
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- **Grading System Managers** Michael Chen, Zhuoqun "Alex" Song, Maxwell Wang
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- **Judges** Zuming Feng, Greg Spanier

Chapter 1

EMC^2 2018 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. What is $2018 - 3018 + 4018$?

2. What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

3. What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits a and b? Express your answer as a decimal.

4. A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where n is a positive integer. How many possible values of n are there?

5. When three integers are added in pairs, and the results are 20, 18, and x. If all three integers sum to 31, what is x?

6. A cube's volume in cubic inches is numerically equal to the sum of the lengths of all its edges, in inches. Find the surface area of the cube, in square inches.

7. A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

8. SeaBay, an online grocery store, offers two different types of egg cartons. Small egg cartons contain 12 eggs and cost 3 dollars, and large egg cartons contain 18 eggs and cost 4 dollars. What is the maximum number of eggs that Farmer James can buy with 10 dollars?

9. What is the sum of the 3 leftmost digits of $\underbrace{999\ldots9}_{2018\text{ 9's}}\times 12$?

10. Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

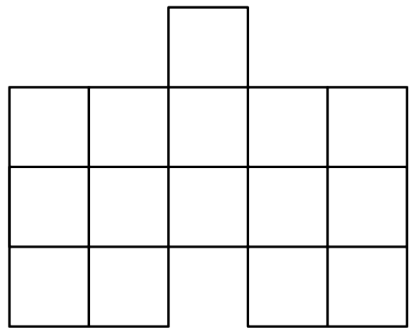
11. Farmer James is ordering takeout from Kristy's Krispy Chicken. The base cost for the dinner is \$14.40, the sales tax is 6.25%, and delivery costs \$3.00 (applied after tax). How much did Farmer James pay, in dollars?

12. Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90^\circ$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

13. Farmer James has 6 cards with the numbers $1 - 6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

14. Farmer James has a 5×5 grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

15. How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



16. On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

17. At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

18. Let x, y, z be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of xyz?

19. Let p be a prime number and x, y be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

20. Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

2. A triangle has side lengths $3, 4$, and 6. A second triangle, similar to the first one, has one side of length 12. Find the sum of all possible lengths of the second triangle's longest side.

3. Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

4. Square $ABCD$ has side length 2. Circle ω is centered at A with radius 2, and intersects line AD at distinct points D and E. Let X be the intersection of segments EC and AB, and let Y be the intersection of the minor arc $\overset{\frown}{DB}$ with segment EC. Compute the length of XY.

5. Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and 4, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

6. Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to N, inclusive. Later, during the live drawing, another group of six numbers from 1 to N will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of N?

7. An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height h. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for h?

8. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer n such that the product $a_1a_2a_3 \cdots a_n$ is greater than 1?

9. In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment BC at E, and AE and BD intersect at P. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find BC.

10. Farmer James places the numbers $1, 2, \ldots, 9$ in a 3×3 grid such that each number appears exactly once in the grid. Let x_i be the product of the numbers in row i, and y_i be the product of the numbers in column i. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 45 minutes.

1. Farmer James goes to Kristy's Krispy Chicken to order a crispy chicken sandwich. He can choose from 3 types of buns, 2 types of sauces, 4 types of vegetables, and 4 types of cheese. He can only choose one type of bun and cheese, but can choose any nonzero number of sauces, and the same with vegetables. How many different chicken sandwiches can Farmer James order?

2. A line with slope 2 and a line with slope 3 intersect at the point (m, n), where $m, n > 0$. These lines intersect the x axis at points A and B, and they intersect the y axis at points C and D. If $AB = CD$, find $\frac{m}{n}$.

3. A multi-set of 11 positive integers has a median of 10, a unique mode of 11, and a mean of 12. What is the largest possible number that can be in this multi-set? (A multi-set is a set that allows repeated elements.)

4. Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

5. $ABCD$ is a square with side length 60. Point E is on AD and F is on CD such that $\angle BEF = 90°$. Find the minimum possible length of CF.

6. Farmer James makes a *trianglomino* by gluing together 5 equilateral triangles of side length 1, with adjacent triangles sharing an entire edge. Two trianglominoes are considered the same if they can be matched using only translations and rotations (but not reflections). How many distinct trianglominoes can Farmer James make?

7. Two real numbers x and y satisfy

$$\begin{cases} x^2 - y^2 = 2y - 2x \text{ , and} \\ x + 6 = y^2 + 2y. \end{cases}$$

   What is the sum of all possible values of y?

8. Let N be a positive multiple of 840. When N is written in base 6, it is of the form $\overline{abcdef}_6$ where a, b, c, d, e, f are distinct base 6 digits. What is the smallest possible value of N, when written in base 6?

9. For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

   (a) If n is divisible by 3, $f(n)$ is not divisible by 3,

   (b) If n is not divisible by 3, $f(n)$ is divisible by 3, and

   (c) $f(f(n)) = n$ holds for exactly 8 distinct values of n in S.

10. Regular pentagon $JAMES$ has area 1. Let O lie on line EM and N lie on line MA so that E, M, O and M, A, N lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of NOM.

11. Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up*.

12. Find the product of all integer values of a such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

13. Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point X is the intersection of the diagonals AC and BD. There exist a positive real number k and a point P inside $ABCD$ which satisfy

$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$

where $[XYZ]$ denotes the area of triangle XYZ. If $PX \parallel AB$, find the value of k.

14. How many positive integers $n < 1000$ are there such that in base 10, every digit in $3n$ (that isn't a leading zero) is greater than the corresponding place value digit (possibly a leading zero) in n?

    For example, $n = 56$, $3n = 168$ satisfies this property as $1 > 0, 6 > 5$, and $8 > 6$. On the other hand, $n = 506$, $3n = 1518$ does not work because of the hundreds place.

15. Find the greatest integer that is smaller than

$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. How many distinct ways are there to *scramble* the letters in $EXETER$?

2. Given that $\dfrac{x-y}{x-z} = 3$, find $\dfrac{x-z}{y-z}$.

3. When written in base 10,
$$9^9 = \overline{ABC420DEF}.$$
Find the remainder when $A+B+C+D+E+F$ is divided by 9.

1.4.2 Round 2

4. How many positive integers, when expressed in base 7, have exactly 3 digits, but don't contain the digit 3?

5. Pentagon $JAMES$ is such that its internal angles satisfy $\angle J = \angle A = \angle M = 90^\circ$ and $\angle E = \angle S$. If $JA = AM = 4$ and $ME = 2$, what is the area of $JAMES$?

6. Let x be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of x?

1.4.3 Round 3

7. Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms D and the average number of students per dorm A. Hen Hao rounds both D and A down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both D and A up to the nearest multiple of 10 and multiplies them, getting an estimate of N students. What is the maximum possible value of N?

8. Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping k eggs into the soup in the kth hour. Find the smallest positive integer n so that after exactly n hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

9. Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How

many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

1.4.4 Round 4

10. Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

11. Find the value of $\dfrac{4 \cdot 7}{\sqrt{4+\sqrt{7}}+\sqrt{4-\sqrt{7}}}$.

12. Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

1.4.5 Round 5

13. Find all ordered pairs of real numbers (x, y) satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

14. An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

15. How many ways are there for Farmer James to color each square of a 2×6 grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

1.4.6 Round 6

16. In a triangle ABC, $\angle A = 45$, and let D be the foot of the perpendicular from A to segment BC. $BD = 2$ and $DC = 4$. Let E be the intersection of the line AD and the perpendicular line from B to line AC. Find the length of AE.

17. Find the largest positive integer n such that there exists a unique positive integer m satisfying
$$\frac{1}{10} \le \frac{m}{n} \le \frac{1}{9}.$$

18. How many ordered pairs (A, B) of positive integers are there such that $A+B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

1.4.7 Round 7

19. Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

20. $P(x)$ is a monic polynomial (a polynomial with leading coefficient 1) of degree 4, such that $P(2^n+1) = 8^n + 1$ when $n = 1, 2, 3, 4$. Find the value of $P(1)$.

21. PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

1.4.8 Round 8

22. How many ways are there to *scramble* the letters in $EGGSEATER$ such that no two consecutive letters are the same?

23. Let $JAMES$ be a regular pentagon. Let X be on segment JA such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point P on segment AE such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

24. Find the minimum value of the function
$$f(x) = \left|x - \frac{1}{x}\right| + \left|x - \frac{2}{x}\right| + \left|x - \frac{3}{x}\right| + \cdots + \left|x - \frac{9}{x}\right| + \left|x - \frac{10}{x}\right|$$
over all nonzero real numbers x.

1.5 Practice Sets

We have arranged 65 of this year's 69 EMCC problems into 11 sets, in approximately increasing order of difficulty. This sorting is based on solve rate during the contest, among other measures. The first two sets consist of 10 problems each, and the other nine consist of 5 problems each. The goal of these Practice Sets is to provide a repository that coaches can use for training at various skill levels.

1.5.1 Set 1

1. (Speed #1) What is $2018 - 3018 + 4018$?

2. (Speed #2) What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

3. (Accuracy #1) On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

4. (Speed 7) A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

5. (Speed #4) A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where n is a positive integer. How many possible values of n are there?

6. (Guts #1) How many distinct ways are there to *scramble* the letters in $EXETER$?

7. (Speed #3) What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits a and b? Express your answer as a decimal.

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1.5.2 Set 2

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20. (Speed #14) Farmer James has a 5×5 grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

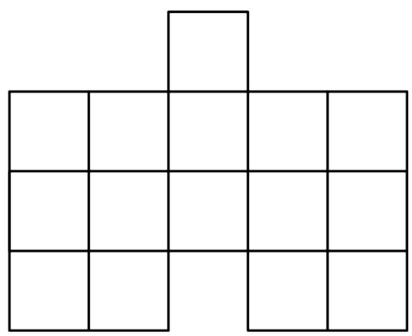
1.5.3 Set 3

21. (Team #4) Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

22. (Speed #9) What is the sum of the 3 leftmost digits of $\underbrace{999\dots9}_{2018\text{ 9's}}\times 12$?

23. (Speed #12) Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90°$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

24. (Speed #15) How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



25. (Guts #6) Let x be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of x?

1.5.4 Set 4

26. (Speed #13) Farmer James has 6 cards with the numbers $1-6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

27. (Accuracy #3) Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

28. (Team #5) $ABCD$ is a square with side length 60. Point E is on AD and F is on CD such that $\angle BEF = 90^\circ$. Find the minimum possible length of CF.

29. (Speed #17) At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

30. (Speed #10) Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

1.5.5 Set 5

31. (Guts #7) Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms D and the average number of students per dorm A. Hen Hao rounds both D and A down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both D and A up to the nearest multiple of 10 and multiplies them, getting an estimate of N students. What is the maximum possible value of N?

32. (Team #7) Two real numbers x and y satisfy

$$\begin{cases} x^2 - y^2 = 2y - 2x \text{ , and} \\ x + 6 = y^2 + 2y. \end{cases}$$

What is the sum of all possible values of y?

33. (Guts #12) Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

34. (Team #2) A line with slope 2 and a line with slope 3 intersect at the point (m, n), where $m, n > 0$. These lines intersect the x axis at points A and B, and they intersect the y axis at points C and D. If $AB = CD$, find $\frac{m}{n}$.

35. (Guts #15) How many ways are there for Farmer James to color each square of a 2×6 grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

1.5.6 Set 6

36. (Guts #8) Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping k eggs into the soup in the kth hour. Find the smallest positive integer n so that after exactly n hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

37. (Guts #11) Find the value of $\dfrac{4 \cdot 7}{\sqrt{4+\sqrt{7}}+\sqrt{4-\sqrt{7}}}$.

38. (Accuracy #4) Square $ABCD$ has side length 2. Circle ω is centered at A with radius 2, and intersects line AD at distinct points D and E. Let X be the intersection of segments EC and AB, and let Y be the intersection of the minor arc $\overset{\frown}{DB}$ with segment EC. Compute the length of XY.

39. (Accuracy #5) Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and 4, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

40. (Speed #16) On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

1.5.7 Set 7

41. (Guts #9) Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

42. (Accuracy #6) Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to N, inclusive. Later, during the live drawing, another group of six numbers from 1 to N will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of N?

43. (Guts #13) Find all ordered pairs of real numbers (x, y) satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

44. (Speed #20) Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

45. (Guts #16) In a triangle ABC, $\angle A = 45$, and let D be the foot of the perpendicular from A to segment BC. $BD = 2$ and $DC = 4$. Let E be the intersection of the line AD and the perpendicular line from B to line AC. Find the length of AE.

1.5.8 Set 8

46. (Speed #18) Let x, y, z be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of xyz?

47. (Speed #19) Let p be a prime number and x, y be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

48. (Guts #14) An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

49. (Guts #10) Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

50. (Accuracy #9) In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment BC at E, and AE and BD intersect at P. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find BC.

1.5.9 Set 9

51. (Accuracy #8) Let $a_1, a_2, a_3, \ldots$ be an infinite geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer n such that the product $a_1 a_2 a_3 \cdots a_n$ is greater than 1?

52. (Team #11) Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up.*

53. (Team #8) Let N be a positive multiple of 840. When N is written in base 6, it is of the form $\overline{abcdef}_6$ where a, b, c, d, e, f are distinct base 6 digits. What is the smallest possible value of N, when written in base 6?

54. (Team #9) For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

    (a) If n is divisible by 3, $f(n)$ is not divisible by 3,

    (b) If n is not divisible by 3, $f(n)$ is divisible by 3, and

    (c) $f(f(n)) = n$ holds for exactly 8 distinct values of n in S.

55. (Team #13) Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point X is the intersection of the diagonals AC and BD. There exist a positive real number k and a point P inside $ABCD$ which satisfy
$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$
where $[XYZ]$ denotes the area of triangle XYZ. If $PX \parallel AB$, find the value of k.

1.5.10 Set 10

56. (Team #10) Regular pentagon $JAMES$ has area 1. Let O lie on line EM and N lie on line MA so that E, M, O and M, A, N lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of NOM.

57. (Team #12) Find the product of all integer values of a such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

58. (Guts #17) Find the largest positive integer n such that there exists a unique positive integer m satisfying
$$\frac{1}{10} \leq \frac{m}{n} \leq \frac{1}{9}.$$

59. (Accuracy #7) An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height h. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for h?

60. (Guts #18) How many ordered pairs (A, B) of positive integers are there such that $A + B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

1.5.11 Set 11

61. (Guts #19) Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

62. (Guts #21) PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

63. (Team #15) Find the greatest integer that is smaller than

$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

64. (Accuracy #10) Farmer James places the numbers $1, 2, \ldots, 9$ in a 3×3 grid such that each number appears exactly once in the grid. Let x_i be the product of the numbers in row i, and y_i be the product of the numbers in column i. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

65. (Guts #23) Let $JAMES$ be a regular pentagon. Let X be on segment JA such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point P on segment AE such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

Chapter 2

EMC^2 2018 Solutions

2.1 Speed Test Solutions

1. What is $2018 - 3018 + 4018$?

   Solution. The answer is $\boxed{3018}$.

   The expression is $2018 - 3018 + 4018 = 2018 + (4018 - 3018) = 2018 + 1000 = 3018$.

2. What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

   Solution. The answer is $\boxed{120}$.

   The least common multiple of 6 and 8 is 24, so this number must be a multiple of 24 above 100, the smallest of which is 120.

3. What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits a and b? Express your answer as a decimal.

   Solution. The answer is $\boxed{2.5}$.

   A fraction that has exactly one digit after the decimal point must have a denominator dividing 10, but can't be 1, so a is either 2 or 5. If a were 5, then $\frac{b}{5} = 5.b > 5$, so $b > 5 \cdot 5$, which cannot be true as b is between 1 and 9. Thus $a = 2$, and any non-integer fraction with denominator 2 must end in .5, so our number is 2.5 or $\frac{5}{2}$.

4. A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where n is a positive integer. How many possible values of n are there?

   Solution. The answer is $\boxed{7}$.

   By the triangle inequality, we have $2 - 1 < \sqrt{n} < 2 + 1$, so $1^2 < n < 3^2$, giving 7 possible integer values for n.

5. When three integers are added in pairs, and the results are 20, 18, and x. If all three integers sum to 31, what is x?

   Solution. The answer is $\boxed{24}$.

   Note that each of the 3 numbers is contained in exactly two of the pairwise sums, meaning that the sum of all 3 pairs will count each element twice. Since the 3 numbers sum to 31, we get $2 \cdot 31 = 20 + 18 + x$, therefore $x = 24$.

6. A cube's volume in cubic inches is numerically equal to the sum of the lengths of all its edges, in inches. Find the surface area of the cube, in square inches.

   Solution. The answer is $\boxed{72}$.

   If the side length of the cube is s, this tells us that $s^3 = 12s$, or $s^2 = 12$, since a cube has 12 edges. Then the surface area is just $6s^2 = 6 \cdot 12 = 72$.

7. A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

Solution. The answer is $\boxed{4}$.

The only possible palindromes between 9:30 and 10:30 are 9:39, 9:49, 9:59, and 10:01, giving 4 palindromes total.

8. SeaBay, an online grocery store, offers two different types of egg cartons. Small egg cartons contain 12 eggs and cost 3 dollars, and large egg cartons contain 18 eggs and cost 4 dollars. What is the maximum number of eggs that Farmer James can buy with 10 dollars?

   Solution. The answer is $\boxed{42}$.

   Since you can buy up to 2 large cartons, there are 3 cases. If you buy 2 large cartons, then you will have two dollars left, so you can buy 0 small cartons, and will have $2 \cdot 18 = 36$ eggs. If you buy 1 large carton, then you will have six dollars left, so you can buy two small cartons, and will have $18 + 2 \cdot 12 = 42$ eggs. If you don't buy any large cartons, then you can buy 3 small cartons, and will have $3 \cdot 12 = 36$ eggs. Therefore, buying 1 large and 2 small cartons gives you 42 eggs, the greatest number possible.

9. What is the sum of the 3 leftmost digits of $\underbrace{999\ldots9}_{2018\text{ 9's}} \times 12$?

   Solution. The answer is $\boxed{11}$.

   The given number is slightly less than $\overline{1000\ldots}$ (with many trailing 0's), so 12 times the number will be slightly less than $\overline{12000\ldots}$, so it will start with $\overline{11999\ldots}$, making the sum of the first three digits $1 + 1 + 9 = 11$.

10. Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

    Solution. The answer is $\boxed{18}$.

    Note that each time Farmer James cuts off a small tetrahedron, none of the original edges are lost, because the middle third of the original edge is never cut off. In addition, he creates 3 new edges, the sides of the newly-created triangular face. Therefore, Farmer James gains 3 edges for each of the 4 times he cuts off a small tetrahedron, gaining 12 edges in total. Since he started with 6 edges, the resulting shape has $6 + 12 = 18$ edges in total.

11. Farmer James is ordering takeout from Kristy's Krispy Chicken. The base cost for the dinner is \$14.40, the sales tax is 6.25%, and delivery costs \$3.00 (applied after tax). How much did Farmer James pay, in dollars?

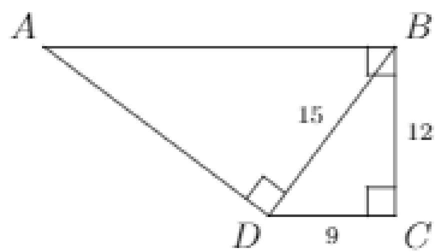
    Solution. The answer is $\boxed{18.30}$.

    A 6.25% sales tax is equal to an extra $\frac{1}{16}$ of the base cost, therefore the total cost is $\$14.40 \cdot \left(1 + \frac{1}{16}\right) + \$3.00 = \$15.30 + \$3.00 = \$18.30$.

12. Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90^\circ$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

Solution. The answer is $\boxed{204}$.

By the Pythagorean Theorem, $BD = \sqrt{9^2 + 12^2} = 15$. Also, $\angle DBA = 90^\circ - \angle CBD = \angle CDB$, so $\triangle DBA \sim \triangle CDB$ in a ratio of 5 to 3. Thus the area of $\triangle DBA$ is $\frac{25}{9}$ times the area of $\triangle CDB$, which itself has area $\frac{1}{2} \cdot 9 \cdot 12 = 54$, so the total area of $ABCD$ is $54\left(1 + \frac{25}{9}\right) = 204$.



13. Farmer James has 6 cards with the numbers $1-6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

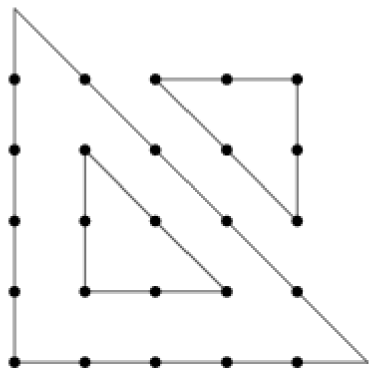
Solution. The answer is $\boxed{120}$.

For a number to be divisible by 6, its digits must sum to a multiple of 3, and since the numbers 1 through 6 already sum to a multiple of 3, Farmer James must discard either 3 or 6. Since the number is even, if he discards 3, there are 3 choices for the last digit: $2, 4$, and 6. The other 4 digits can then be ordered in the front in $4! = 24$ ways, giving $3 \cdot 24 = 72$ possible numbers. Similarly, if he discards 6, there are 2 choices for the last digit, 2 and 4, so there are $2 \cdot 4! = 48$ total possibilities. Therefore, Farmer James can create $72 + 48 = 120$ different numbers.

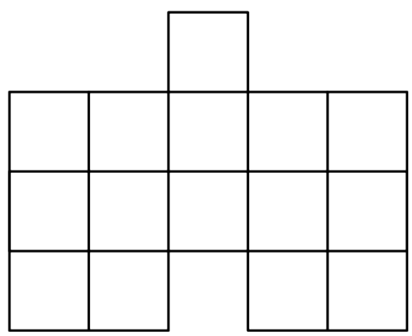
14. Farmer James has a 5×5 grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

Solution. The answer is $\boxed{3}$.

If it were possible to cover all 25 points with 2 triangles, one of them must have at least 13 points on its boundary. It is not hard to see that the maximum number of points that can lie on any triangle is in fact 13, but this can only be achieved one way (up to rotation), and the remaining 12 points clearly cannot be covered by one triangle. They can, however, be covered by two triangles, giving a minimal covering of 3 triangles, shown below.

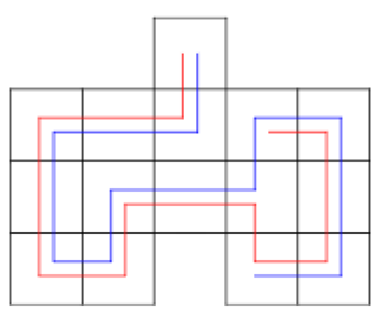


15. How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



Solution. The answer is $\boxed{8}$.

The top square is only adjacent to one other square, so the number that goes there should only be adjacent to one other number, meaning it is either 1 or 15. From this point, we can then draw one of two paths, times two for horizontal reflection, giving a total of $2 \cdot 2 \cdot 2 = 8$ different labellings.

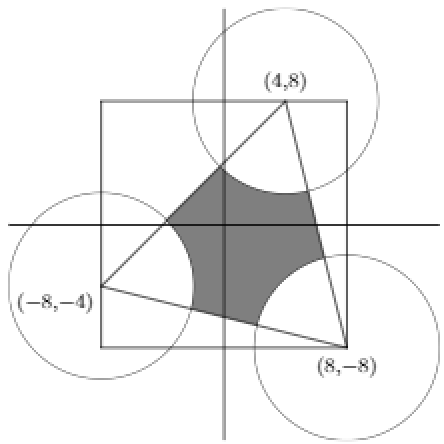


16. On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the

feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

Solution. The answer is $\boxed{120 - 18\pi}$.

By not allowing a feeding station within 6 units of any vertex of this triangle, we are removing three sectors of radius 6, one from each corner. Since the sum of the angles in a triangle is 180 degrees, these sectors added together will equal the area of a semicircle with radius 6, or 18π. We can then find the area of the triangle by bounding it in a larger square with side length 16, as shown, giving the area as $16 \cdot 16 - \frac{12 \cdot 12}{2} - \frac{16 \cdot 4}{2} - \frac{16 \cdot 4}{2} = 120$, thus the remaining area where the feeding station can be placed is $120 - 18\pi$.



17. At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

Solution. The answer is $\boxed{27}$.

If we simply sum up the memberships of each club, we get a total of $8 + 12 + 20 = 40$ people. However, doing so counts people in exactly 2 clubs twice each, and people in all 3 clubs three times each. To fix this, we can first subtract the 10 people who go to at least 2 clubs, leaving $40 - 10 = 30$ people, and ensuring that the people who attend exactly 2 clubs are counted properly. However, those who attend all 3 clubs will still have been counted $3 - 1 = 2$ times after this, so we must additionally subtract the number of people who go to all three clubs, giving the correct value as $30 - 3 = 27$ people.

18. Let x, y, z be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of xyz?

Solution. The answer is $\boxed{28/27}$.

From power rules, we know that $9^{\frac{3}{2}} = 27, 25^{\frac{3}{2}} = 125$, and $128^{\frac{3}{7}} = 8$, therefore $8^{\frac{3}{2}x} = 27$, $27^{\frac{3}{2}y} = 125$, and $125^{\frac{3}{7}z} = 8$, or $\left(\left(8^{\frac{3}{2}x}\right)^{\frac{3}{2}y}\right)^{\frac{3}{7}z} = 8$, thus $\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{7} \cdot xyz = 1$, so $xyz = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{7}{3} = \frac{28}{27}$.

19. Let p be a prime number and x, y be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

    Solution. The answer is $\boxed{38}$.

    Since $9xy$ is divisible by 9, $p(p + 3x + 6y)$ must be as well. However, p and $p + 3x + 6y$ differ by a multiple of 3, so in order for their product to be divisible by 3 (and furthermore by 9), both numbers must be multiples of 3, meaning $p = 3$ (as no other prime is divisible by 3). From here, we have $9xy = 9 + 9x + 18y$, or $xy - x - 2y - 1 = 0$, which we can rewrite as $(x - 2)(y - 1) = 3$. Over all 4 possible integer solutions for x and y, the one having the largest value for $x^2 + y^2$ is $(x, y) = (5, 2)$, giving $p^2 + x^2 + y^2 = 3^2 + 5^2 + 2^2 = 38$.

20. Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

    Solution. The answer is $\boxed{3/5}$.

    If we look at the eggs from left to right after they have been dropped, we can assign each egg a ranking from highest to lowest in terms of their y value (the probability of any two eggs having the exact same x or y value is 0, so we consider when this does not happen). Our bounding rectangle, in order to have the least area, will touch the highest, lowest, furthest left, and furthest right of the 6 eggs. Therefore, in order for an egg to end up in a corner, it must have two of these properties at the same time; in other words, one of the leftmost/rightmost eggs must also be the lowest/highest egg.
    We can calculate this probability by instead finding the odds that this does not happen: there is a $\frac{4}{6}$ chance that the leftmost egg is neither the highest nor lowest egg, and then a $\frac{3}{5}$ chance that the rightmost is neither of these as well: after picking the height of the leftmost egg, there are only 5 remaining ranks for height, and only 3 of these are neither the highest nor the lowest. Therefore the probability that no egg ends up in the corner is $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$, meaning the complementary probability that at least one egg is in a corner is $1 - \frac{2}{5} = \frac{3}{5}$.

2.2 Accuracy Test Solutions

1. On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

   Solution. The answer is $\boxed{58}$.

   The total cost is \$2.50 times 12 of pounds of green herring, added to \$4.00 times 7 pounds of blue herring, or $12 \cdot \$2.50 + 7 \cdot \$4.00 = \$30 + \$28 = \$58$

2. A triangle has side lengths $3, 4$, and 6. A second triangle, similar to the first one, has one side of length 12. Find the sum of all possible lengths of the second triangle's longest side.

   Solution. The answer is $\boxed{54}$.

   The second triangle is either $2, 3$, or 4 times as large as the original, depending on whether the side of length 12 corresponds to the side of length $6, 4$, or 3 in the original, respectively. Therefore the sum of all possible longest side lengths will just be the original longest side, 6, scaled by all three of these possibilities, giving $6 \cdot (2 + 3 + 4) = 54$.

3. Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

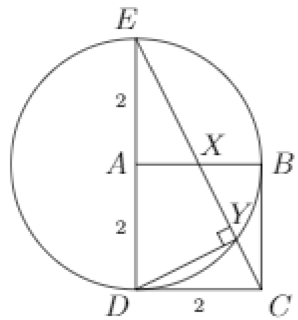
   Solution. The answer is $\boxed{3/2}$.

   Since we are not given any info on the speed or distance PEAcock travels, we can pick our own values, so suppose PEAcock's average speed for the first lap was 15mph, and the lap around the farm is 60 miles long. We know the overall average speed is 20% slower than 15mph, or 12mph, thus the total time taken by PEAcock would be 2 laps times $\frac{60}{12}$ hours per lap, or 10 hours. On the first lap, he travels 60 miles at 15mph, taking $\frac{60}{15} = 4$ hours, so he must have completed the second lap in $10 - 4 = 6$ hours, making his speed $\frac{60}{6} = 10$mph. The ratio of the two speeds is therefore $\frac{15}{10} = \frac{3}{2}$.

4. Square $ABCD$ has side length 2. Circle ω is centered at A with radius 2, and intersects line AD at distinct points D and E. Let X be the intersection of segments EC and AB, and let Y be the intersection of the minor arc $\overset{\frown}{DB}$ with segment EC. Compute the length of XY.

   Solution. The answer is $\boxed{3\sqrt{5}/5}$.

   First, note that EC, the segment containing XY, has length $\sqrt{2^2 + 4^2} = 2\sqrt{5}$. We can find that $\angle EYD = 90°$, since it inscribes diameter ED of the circle. Thus $\angle DYC = 90°$ as well, so $\triangle DYC \sim \triangle EDC$, giving $\frac{CY}{DC} = \frac{DC}{EC} = \frac{2}{2\sqrt{5}}$, or $CY = \frac{\sqrt{5}}{5} \cdot DC = \frac{2\sqrt{5}}{5}$. We can also easily find X to be the midpoint of EC, as $\triangle EAX \cong \triangle CBX$, so $CX = \sqrt{5}$. Thus $XY = CX - CY = \sqrt{5} - \frac{2\sqrt{5}}{5} = \frac{3\sqrt{5}}{5}$.



5. Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and 4, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

   Solution. The answer is $\boxed{21/128}$.

   The possible perfect squares that could be the sum are $4, 9$, and 16. 4 and 16 can both only occur in one way: $1, 1, 1, 1$, and $4, 4, 4, 4$. Next, there are 4 possible (unordered) combinations of numbers that can sum to 9: $\{1, 1, 3, 4\}, \{1, 2, 2, 4\}, \{1, 2, 3, 3\}$, and $\{2, 2, 2, 3\}$. The first 3 can be reordered in 12 ways, while the last can only be reordered in 4 ways, giving 40 total ordered combinations. Finally, after adding the one combination each from 4 and 16, and dividing by the total number of combinations (which is 4^4), we end up with a probability of $\frac{42}{4^4} = \frac{21}{128}$.

6. Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to N, inclusive. Later, during the live drawing, another group of six numbers from 1 to N will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of N?

   Solution. The answer is $\boxed{47}$.

   First, there are $\binom{N}{6}$ combinations of 6 numbers that Farmer James chooses from, each with equal probability. Of these, there are $\binom{N-6}{6}$ ways to have no *winning* numbers, and $\binom{6}{1}\binom{N-6}{5}$ ways to have exactly 1 *winning* number. For the two probabilities to be equal, these two values must be equal. This can be expressed as $\frac{(N-6)!}{6!(N-12)!} = \frac{6(N-6)!}{5!(N-11)!}$, or $\frac{5!(N-11)!}{6!(N-12)!} = \frac{N-11}{6} = 6$, so $N = 47$.

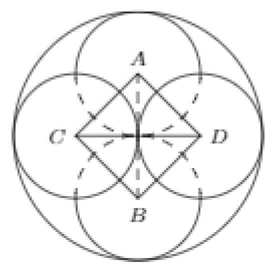
   (Note that during the contest, the problem mistakenly stated $N \geq 6$, allowing the 5 degenerate solutions $6 \leq N \leq 10$, where both probabilities were equal to 0)

7. An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height h. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for h?

   Solution. The answer is $\boxed{2+\sqrt{2}}$.

   We can first place two eggs against the bottom of the cylindrical egg plant, tangent to each other, such that their point of contact lies along the central vertical axis of the cylinder. Next, we place another pair of eggs on top, such that all four eggs are pairwise tangent to each other. Let A and

B be the centers of the lower spheres, and C and D be the centers of the upper spheres. Since each pair of spheres is tangent, $ABCD$ is a regular tetrahedron of side length 2. In addition, all four points are 1 unit away from the central vertical axis of the cylinder, and the top pair and bottom pair are oriented perpendicularly, so the planar distance (if viewed from above) between any two centers from opposite pairs (ex: A and C) would be $\sqrt{2}$. Therefore, the vertical distance between the two pairs is $\sqrt{2^2 - \sqrt{2}^2} = \sqrt{2}$. Since the bottom of the cylinder is 1 unit below the lower pair AB, and the top is 1 unit above the upper pair CD, the total height of the egg plant is $2 + \sqrt{2}$.



(birds eye view of an egg plant)

8. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer n such that the product $a_1 a_2 a_3 \cdots a_n$ is greater than 1?

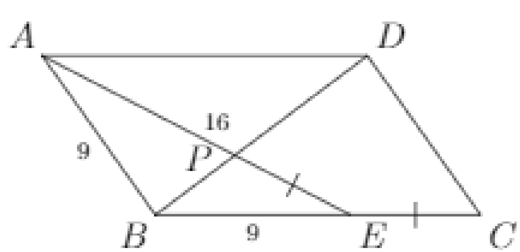
   Solution. The answer is $\boxed{76}$.

   If the constant ratio between consecutive terms is r, then we can write $a_n = a_1 \cdot r^{n-1}$, so the condition given is $(a_1)^{20} \cdot r^{19 \cdot 20} = (a_1)^{18} \cdot r^{17 \cdot 18}$, or $(a_1)^2 = r^{17 \cdot 18 - 19 \cdot 20} = r^{-74}$, thus $a_1 = r^{-37}$. Therefore, the product $a_1 a_2 \cdots a_{75}$ is $r^{(-37)+(-36)+\cdots+36+37} = 1$, and since $a_1 < 1 \implies r > 1$, we get that $a_1 a_2 \cdots a_{76} = (a_1 a_2 \cdots a_{75}) \cdot a_{76} = 1 \cdot r^{38}$ is the first product of consecutive terms that is greater than 1.

9. In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment BC at E, and AE and BD intersect at P. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find BC.

   Solution. The answer is $\boxed{15}$.

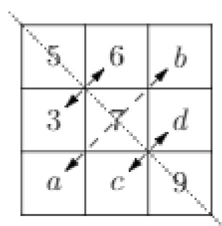
   Since AE bisects $\angle DAB$ and AD is parallel to BC, we have $\angle BAE = \angle DAE = \angle BEA$, so $BE = AB = 9$. Let $EC = EP = x$, then $AD = BC = 9 + x$, so from $\triangle APD \sim \triangle EPB$ we have $\frac{PA}{AD} = \frac{PE}{EB} \Rightarrow \frac{16-x}{9+x} = \frac{x}{9} \Rightarrow x = 6$ or $x = -24$. Since x is positive, we get $BC = 9 + x = 15$.



10. Farmer James places the numbers $1, 2, \ldots, 9$ in a 3×3 grid such that each number appears exactly once in the grid. Let x_i be the product of the numbers in row i, and y_i be the product of the numbers in column i. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

    Solution. The answer is $\boxed{432}$.

    First, note that if 5 and 7 are in the same row, one of the x_i would be divisible by 35, but none of the y_i would be, since 5 and 7 can't be in both the same row and the same column simultaneously. Therefore 5 and 7 are in separate rows and, by the same argument, in separate columns as well. If we then try to place these numbers, there would be 9 spots to first place the 5, and only 4 remaining spots to place the 7. Since we can reorder the rows and columns freely, assume WLOG that the 5 is in the top left, and the 7 is in the center of the grid.
    Then if 9 is placed in the same column as 7 then there will be a column product that is divisible by 63, which forces 3 and 6 to be placed on the same row as 7, and this makes all three column products divisible by 3 but one row having no multiples of 3. Similarly, we can rule out the possibility of 9 sharing a row or column with either 5 or 7. Thus the only remaining possibility is that 9 shares no column or row with 5 or 7, and WLOG it is on the bottom right of the grid.
    It is then clear that 3 and 6 must be symmetric about the diagonal containing $5, 7, 9$, so there are 6 unique ways to place the two of them. Each of these ways result in 3 and 6 sharing rows/columns with exactly two of $5, 7, 9$, so WLOG we can place 3 at the middle left, and 6 at the top center, where they will share rows/columns with 5 and 7. If we consider the row and column containing 5 here, we get $5 \cdot 3 \cdot a = 5 \cdot 6 \cdot b$, or $a = 2b$, for some $a, b \in \{1, 2, 4, 8\}$, and likewise by looking at 7 we get $c = 2d$ for the remaining two elements. There are only two ways to make such a pairing: $(a, b), (c, d) = (2, 1), (8, 4)$ or $(8, 4), (2, 1)$.
    Therefore the total number of possible grids comes out to $9 \cdot 4 \cdot 6 \cdot 2 = 432$.



2.3 Team Test Solutions

1. Farmer James goes to Kristy's Krispy Chicken to order a crispy chicken sandwich. He can choose from 3 types of buns, 2 types of sauces, 4 types of vegetables, and 4 types of cheese. He can only choose one type of bun and cheese, but can choose any nonzero number of sauces, and the same with vegetables. How many different chicken sandwiches can Farmer James order?

   Solution. The answer is $\boxed{540}$.

   There are 3 ways to choose the bun and 4 ways to choose the cheese. There are 4 types of vegetables, each of which can be on the sandwich or not, for 2^4 choices. However, there must be a nonzero number of vegetables, so only 2^4-1 of these choices can work. Similarly, there are 2^2-1 sauce choices. So, there are a total of $3\cdot 4\cdot(2^2-1)\cdot(2^4-1)=540$ different sandwiches that Farmer James can order.

2. A line with slope 2 and a line with slope 3 intersect at the point (m,n), where $m,n>0$. These lines intersect the x axis at points A and B, and they intersect the y axis at points C and D. If $AB=CD$, find $\frac{m}{n}$.

   Solution. The answer is $\boxed{1/6}$.

   We have two lines: $y-n=2(x-m)$ and $y-n=3(x-m)$. Plugging $y=0$ into both of these, we get the two x-intercepts are $(m-\frac{n}{2},0)$ and $(m-\frac{n}{3},0)$, so $AB=\frac{n}{6}$. Plugging $x=0$ into both of these, we get the two y-intercepts are $(0,n-2m)$ and $(0,n-3m)$, so $CD=m$. $AB=CD$ tells us that $\frac{m}{n}=\frac{1}{6}$.

3. A multi-set of 11 positive integers has a median of 10, a unique mode of 11, and a mean of 12. What is the largest possible number that can be in this multi-set? (A multi-set is a set that allows repeated elements.)

   Solution. The answer is $\boxed{71}$.

   First, since the mean of the 11 integers is 12, their sum must be $11\times 12=132$. To find the largest possible value of the largest integer, we must find the smallest possible value of the sum of the 10 smallest integers. The 5 smallest integers must be between 1 and 9, inclusive, and since the 6^{th} smallest must be the median, the 6^{th} to 10^{th} smallest integers are at least 10. In addition, if there are n 11s, then there must be at most $n-1$ 1s, since 11 is the unique mode, and therefore $6-n$ integers that are at least 2. Therefore, the sum of the 5 smallest integers is at least $2(6-n)+1(n-1)=11-n$, and the sum of the 6^{th} to 10^{th} smallest integers is at least $50+n$. This means the total sum is at least 61, and the largest integer is at most $132-61=71$. The value 71 can be achieved with the set $\{1,1,1,2,2,10,11,11,11,11,71\}$.

4. Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

   Solution. The answer is $\boxed{7}$.

   Suppose Farmer James's swimming speed is x miles per hour. Then, when swimming upstream, he will only move at $x-5$ miles per hour, relative to the riverbank, and similarly when swimming downstream, he will move at $x+5$ miles per hour, therefore the total amount of time his round trip takes is $\frac{1}{x-5}+\frac{1}{x+5}=\frac{2x}{x^2-25}$ hours. This is twice as long as it would have taken without the river current, which would be $\frac{1}{x}+\frac{1}{x}=\frac{2}{x}$ hours, thus $2\cdot\frac{2}{x}=\frac{2x}{x^2-25}$, or $2(x^2-25)=x^2$, giving $x^2=50$. Therefore Farmer James's swimming speed is $x=5\sqrt{2}$, which is closest to 7 miles per hour.

5. $ABCD$ is a square with side length 60. Point E is on AD and F is on CD such that $\angle BEF=90°$. Find the minimum possible length of CF.

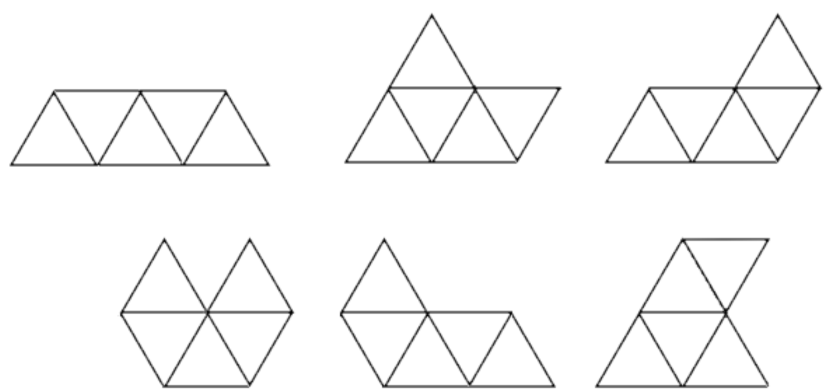
   Solution. The answer is $\boxed{45}$.

   Let $AE=x$. Then, we have $DE=60-x$, and $\triangle BAE\sim\triangle EDF$. Hence, $DF=\frac{x(60-x)}{60}$. Since $DC=60$, we wish to maximize $DF=\frac{900-(x-30)^2}{60}\le\frac{900}{60}=15$, so the minimum value of CF is when $AE=30$, giving $CF=60-DF=60-15=45$.

6. Farmer James makes a *trianglomino* by gluing together 5 equilateral triangles of side length 1, with adjacent triangles sharing an entire edge. Two trianglominoes are considered the same if they can be matched using only translations and rotations (but not reflections). How many distinct trianglominoes can Farmer James make?

   Solution. The answer is $\boxed{6}$.

   There are only 6 possible distinct trianglominoes, as shown below.



7. Two real numbers x and y satisfy

$$\begin{cases} x^2-y^2=2y-2x\ , \text{ and} \\ x+6=y^2+2y. \end{cases}$$

   What is the sum of all possible values of y?

   Solution. The answer is $\boxed{-4}$.

Subtracting the two equations gives $0 = x^2 + x - 6 = (x+3)(x-2)$. So, $x = -3$ or $x = 2$. Plugging these both into the second equation, we have $y^2 + 2y = 3$ or $y^2 + 2y = 8$, which can be factored as $(y-1)(y+3) = 0$ and $(y+4)(y-2) = 0$ respectively, giving all possible y values as $-4, -3, 1, 2$, which sum to -4.

8. Let N be a positive multiple of 840. When N is written in base 6, it is of the form $\overline{abcdef}_6$ where a, b, c, d, e, f are distinct base 6 digits. What is the smallest possible value of N, when written in base 6?

   Solution. The answer is $\boxed{415320_6}$.

   Since 840 is a multiple of 6, we must have $f = 0$. Hence, (a, b, c, d, e) is a permutation of $(1, 2, 3, 4, 5)$. Now, dividing by 6, we get that $abcde_6$ is a multiple of 140. This means that e is even, and, by the definition of base 6, that $e + 6 \cdot d + 6^2 \cdot c + 6^3 \cdot b + 6^4 \cdot a \equiv 0 \pmod 7$. We can write 6 as $-1 \pmod 7$, so we get $a + c + e \equiv b + d \pmod 7$. Note that $a + b + c + d + e = 15$, so we must have $a + c + e = 11$ and $b + d = 4$ or $a + c + e = 4$ and $b + d = 11$. The latter is impossible since $a + c + e \geq 1 + 2 + 3 = 6$, so we must have $b + d = 4$. Hence, either $b = 1$ and $d = 3$ or $d = 1$ and $b = 3$. Also, note that $abcde_6$ is divisible by 4, so de_6 is divisible by 4. Therefore, $e = 2$ as 14_6 and 34_6 are not multiples of 4. Hence, a and c must be 4 and 5, while b and d must be 1 and 3 so the minimum possible value of $abcde_6$ is 41532_6. This is indeed a multiple of 840, so 415320_6 is the answer.

9. For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

   (a) If n is divisible by 3, $f(n)$ is not divisible by 3,

   (b) If n is not divisible by 3, $f(n)$ is divisible by 3, and

   (c) $f(f(n)) = n$ holds for exactly 8 distinct values of n in S.

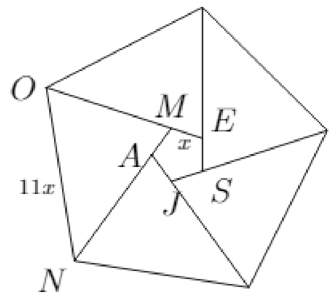
   Solution. The answer is $\boxed{430080}$.

   Note that $f(n) \in \{3, 6, 9, 12\}$ for $n \in \{1, 2, 4, 5, 7, 8, 10, 11\}$, while $f(3), f(6), f(9), f(12)$ are the other possible values. Hence, there are at most 8 possible values for $f(n)$, so these must be the values for $f(f(n))$ as well. Since $f(f(n)) = n$ for 8 values of n, we must have exactly 8 elements in the range of $f(n)$. From this, we know that $f(3), f(6), f(9), f(12)$ are all distinct, and that the set of $f(n) \in \{3, 6, 9, 12\}$ for $n \in \{1, 2, 4, 5, 7, 8, 10, 11\}$ contains 4 distinct elements. There are $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways to choose the values of $f(3), f(6), f(9)$, and $f(12)$. Now, we know $f(f(n)) = n$ for $n \in \{3, 6, 9, 12, f(3), f(6), f(9), f(12)\}$, and there are 4 ways to assign values for each $f(n)$ not already assigned. Hence, the total number of ways is $1680 \cdot 4^4 = 430080$.

10. Regular pentagon $JAMES$ has area 1. Let O lie on line EM and N lie on line MA so that E, M, O and M, A, N lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of NOM.

    Solution. The answer is $\boxed{24}$.

    If we perform the given triangle construction on every side of pentagon $JAMES$, instead of just on side MA, we end up with an external regular pentagon with side length NO. Since $NO = 11 \cdot ME$, this pentagon has an area 11^2 times the original, or 121. This pentagon's area is also composed of 5 instances of the desired triangle, and the original pentagon in the center, thus the area of the desired triangle $\triangle NOM$ is simply $\frac{121-1}{5} = 24$



11. Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up.*

    Solution. The answer is $\boxed{89/144}$.

    Let $T(n)$ be the total number of arrangements for n coin flips, and $S(n)$ be the number that end in a sunny side up flip. Clearly $S(n) = T(n-1)$, since you can simply add a sunny flip to the end of any sequence of flips, and removing the sunny flip from the end of a valid sequence still gives a valid sequence. To find $T(n)$, we can consider a few cases: if our sequence ends with a rainy flip, the previous $n-1$ flips only have to end in a sunny flip, so there are $S(n-1) = T(n-2)$ of these. If our sequence ends with a sunny flip, then there are $S(n) = T(n-1)$ such sequences. So we have $T(n) = T(n-1) + T(n-2)$. Starting with $T(0) = 1, T(1) = 2, T(2) = 3$, we see that this is exactly the Fibonacci recursion, giving our final answer as $\frac{S(10)}{T(10)} = \frac{T(9)}{T(10)} = \frac{89}{144}$.

12. Find the product of all integer values of a such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

    Solution. The answer is $\boxed{-1500}$.

    We will divide this problem into two cases.

    Case 1: The polynomial is the product of a linear term and a cubic (over the integers). Then, either

    $$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^3 + px^2 + qx + 1)(x - 1)$$

    or

    $$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^3 + px^2 + qx - 1)(x + 1)$$

    For the first subcase, substituting $x = 1$ gives $1 + 8 + a + 2 - 1 = 0$, so $a = -10$ is one possibility. For the second subcase, substituting $x = -1$ gives $1 - 8 + a - 2 - 1 = 0$. This gives $a = 10$ as a possibility. (It is not difficult to verify that in both subcases the factorization indeed exists).

    Case 2: The polynomial is the product of two quadratics that are indecomposable over the integers. Then,

    $$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^2 + px + 1)(x^2 + qx - 1)$$

    so $q - p = 2$ and $p + q = 8$ (from looking at the coefficients of x and x^3). Hence, $q = 5$ and $p = 3$, so $x^4 + 8x^3 + ax^2 + 2x - 1 = (x^2 + 3x + 1)(x^2 + 5x - 1)$, which gives $a = 15$ as our final possibility.

    Hence, the product of all possible values of a is $-10 \cdot 10 \cdot 15 = -1500$.

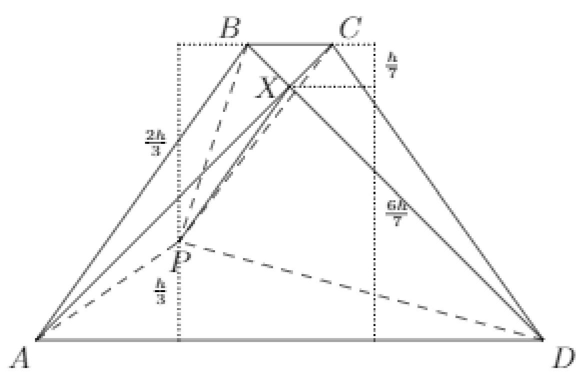
13. Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point X is the intersection of the diagonals AC and BD. There exist a positive real number k and a point P inside $ABCD$ which satisfy

$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$

where $[XYZ]$ denotes the area of triangle XYZ. If $PX \parallel AB$, find the value of k.

Solution. The answer is $\boxed{73/14}$.

Let h be the height of the trapezoid, which is the distance between AD and BC. Now, since $\frac{[PDA]}{[PBC]} = 3$, and $AD = 6BC$, we know that the distance from P to AD is $\frac{h}{3}$, while the distance from P to BC is $\frac{2}{3}h$. Since $PX||AB$ we know that $[PAB] = [XAB]$. We also know that $\triangle XAD \sim \triangle XCB$ with ratio 6, so the distance from X to AD is $\frac{6}{7}h$, while the distance from X to BC is $\frac{h}{7}$. Also, $[XAB] = [XCD]$ by symmetry, so $2[XAB] = [ABCD] - [XAD] - [XBC] = \frac{(AD+BC)\cdot h}{2} - \frac{AD\cdot 6h}{14} - \frac{BC\cdot h}{14} = \frac{6BC\cdot h}{7}$. Hence, $[PAB] = [XAB] = \frac{3BC\cdot h}{7}$. Also, $[PBC] = \frac{BC\cdot h}{3}$ and $[PAD] = BC \cdot h$, so $[PCD] = \frac{7BC\cdot h}{2} - \frac{3BC\cdot h}{7} - \frac{BC\cdot h}{3} - BC \cdot h = \frac{73BC\cdot h}{42}$. Hence, $k = \frac{[PCD]}{[PBC]} = \frac{73}{14}$.



14. How many positive integers $n < 1000$ are there such that in base 10, every digit in $3n$ (that isn't a leading zero) is greater than the corresponding place value digit (possibly a leading zero) in n?

For example, $n = 56$, $3n = 168$ satisfies this property as $1 > 0, 6 > 5$, and $8 > 6$. On the other hand, $n = 506$, $3n = 1518$ does not work because of the hundreds place.

Solution. The answer is $\boxed{88}$.

When multiplying by 3, the digits which will end up larger in the final number (without any carrying from the previous place value) are $1, 2, 3$, and 6. With carrying, however, this list changes to $0, 1, 2, 5, 6$. We can then note that, out of the 4 possibilities for a non-carried place, 3 of them will leave the next place non-carried, while 1 (the 6) will cause the next place to become carried; out of the 5 possibilities for a carried place, 3 of them will leave the next place non-carried, while the other 2 will keep the next place carried.

If we say non-carried is N (note that the last digit of a number is always non-carried, since there is no place value smaller than it), and carried is C, we can then do casework on 3 digit numbers:

$NNN = 4 \cdot 3 \cdot 3 = 36$, since there are 3 numbers that transition from N to N, a necessary condition

for only the last two digits.
$NCN = 4 \cdot 3 \cdot 1 = 12$, as there is 1 number that transitions from N to C, and 3 that transition from C to N. The remaining cases will work similarly.
$CNN = 4 \cdot 1 \cdot 3 = 12$, note that we must remove 1 from the 5 possible carried digits for this case (and the next), because it would create a leading 0.
$CCN = 4 \cdot 2 \cdot 1 = 8$
This gives a total of $36 + 12 + 12 + 8 = 68$ three digit numbers. However, we can use the same method to quickly check one and two digit numbers:
$CN = 4 \cdot 1 = 4$, $NN = 4 \cdot 3 = 12$, total is 16 for two digits.
$N = 4$, total is 4 for one digit.
Thus the final answer is $68 + 16 + 4 = 88$.

15. Find the greatest integer that is smaller than

$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

Solution. The answer is $\boxed{18}$.

We can approximate the sum as $\dfrac{2018}{36 \cdot 38} + \dfrac{2018}{38 \cdot 40} + \cdots + \dfrac{2018}{106 \cdot 108}$, where the error in each term will effectively be negligible (on the order of $\epsilon \leq \frac{2018}{37^2(37^2-1)} < \frac{1}{928}$ for 36 terms), as we are looking for only the greatest integer less than the sum. We can then write this modified sum as $1009\left(\frac{1}{36} - \frac{1}{38}\right) + 1009\left(\frac{1}{38} - \frac{1}{40}\right) + \cdots + 1009\left(\frac{1}{106} - \frac{1}{108}\right)$, which telescopes to $1009\left(\frac{1}{36} - \frac{1}{108}\right) = \frac{1009}{54} \approx 18.7$, making the greatest integer less than the sum 18.

2.4 Guts Test Solutions

2.4.1 Round 1

1. How many distinct ways are there to *scramble* the letters in $EXETER$?

   Solution. The answer is $\boxed{120}$.

   We can choose one of the 6 places to place the X, one of the remaining 5 places to place the T, and one of the remaining 4 places to place the R. The rest of the letters will be filled with Es, which are indistinguishable, thus the total number of orderings is $6 \cdot 5 \cdot 4 = 120$

2. Given that $\dfrac{x-y}{x-z} = 3$, find $\dfrac{x-z}{y-z}$.

   Solution. The answer is $\boxed{-\frac{1}{2}}$.

   By setting $(x, y, z) = (4, 1, 3)$, we get $\frac{x-y}{x-z} = \frac{4-1}{4-3} = 3$ as given, and therefore $\frac{x-z}{y-z} = \frac{4-3}{1-3} = -\frac{1}{2}$. Alternatively, since we are given $\frac{x-y}{x-z} = 3$, $\frac{z-y}{x-z} = \frac{x-y}{x-z} - \frac{x-z}{x-z} = 3 - 1 = 2$. Therefore, $\frac{y-z}{x-z} = -2$, and $\frac{x-z}{y-z} = -\frac{1}{2}$.

3. When written in base 10,
   $$9^9 = \overline{ABC420DEF}.$$
   Find the remainder when $A + B + C + D + E + F$ is divided by 9.

   Solution. The answer is $\boxed{3}$.

   Since 9^9 is divisible by 9, the digits must sum to a multiple of 9. $4 + 2 + 0$ leaves a remainder of 6 when divided by 9, so the sum of the remaining 6 digits, when divided by 9, must leave a remainder of 3.

2.4.2 Round 2

4. How many positive integers, when expressed in base 7, have exactly 3 digits, but don't contain the digit 3?

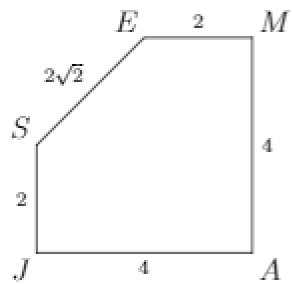
   Solution. The answer is $\boxed{180}$.

   The last two digits in our number can be anything except 3, giving 6 possibilities for each, while the first can be neither 0 nor 3, since numbers can't start with 0, leaving only 5 possibilities, so the total amount of possible numbers is $5 \cdot 6 \cdot 6 = 180$.

5. Pentagon $JAMES$ is such that its internal angles satisfy $\angle J = \angle A = \angle M = 90^\circ$ and $\angle E = \angle S$. If $JA = AM = 4$ and $ME = 2$, what is the area of $JAMES$?

Solution. The answer is $\boxed{14}$.

Since we know $\angle E = \angle S$ and the sum of the angles of a pentagon is $540°$, we get that $\angle E = \angle S = 135°$. Pentagon $JAMES$ is thus a 4×4 square with a 2×2 isosceles right triangle cut from one of the corners, as shown in the diagram below. We can calculate its area to be $4\cdot 4 - \frac{2\cdot 2}{2} = 14$.



6. Let x be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of x?

   Solution. The answer is $\boxed{1}$.

   We can simplify to get $2x - 1 = \sqrt{x}$, and then square both sides, noting that there will be an extraneous solution (where $2x - 1 = -\sqrt{x}$). This gives $4x^2 - 4x + 1 = x$, or $(4x-1)(x-1) = 0$. Plugging in these solutions yields $x = 1$ to be the only non-extraneous solution, so the answer is 1.

2.4.3 Round 3

7. Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms D and the average number of students per dorm A. Hen Hao rounds both D and A down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both D and A up to the nearest multiple of 10 and multiplies them, getting an estimate of N students. What is the maximum possible value of N?

   Solution. The answer is $\boxed{2600}$.

   Suppose Hen Hao rounded his numbers down to $10A$ and $10B$, where $AB = 12$. Then, we are trying to maximize $N = (10A+10)(10B+10)$. This is simply $100AB + 100 + 100(A+B) = 1300 + 100(A+B)$, so we want to maximize the sum of two factors of 12, which occurs at $(A, B) = (1, 12)$, giving $N = 1300 + 100 \cdot 13 = 2600$.

8. Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping k eggs into the soup in the kth hour. Find the smallest positive integer n so that after exactly n hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

Solution. The answer is $\boxed{175}$.

After n hours, Hen Hao will have dropped $\frac{n(n+1)}{2}$ eggs, so $n(n+1)$ must be a multiple of $400 = 16 \cdot 25$. Since n and $n+1$ are relatively prime, either one of them is divisible by 400, giving $n = 399$ as the minimum, or one is divisible by 16, and the other is divisible by 25. By listing out the first few multiples of 25, we can see that no multiple of 16 is adjacent to any of these until 175 and 176, so $n = 175$ is minimal.

9. Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

Solution. The answer is $\boxed{7}$.

Note that if Farmer James has $2019 - 3^k$ pieces at any point, he will eat 1346 of them, and then triple the remaining amount, leaving us with $3(2019 - 3^k - 1346) = 2019 - 3^{k+1}$ pieces for the next day. On the first day, we have $2018 = 2019 - 3^0$ pieces, and the first power of 3 greater than 2019 is $3^7 = 2187$, so Farmer James can FEAST for values of k between 0 and 6, inclusive, giving a total of 7 days.

2.4.4 Round 4

10. Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

Solution. The answer is $\boxed{\frac{27}{256}}$.

The odds of all 3 eggs ending up in any basket is equal among the 3 baskets, so WLOG we can calculate the probability they all end up in basket 1, and multiply that by 3.
In this case, the first egg has to end up in either basket 2 or 3 after two minutes, and since it is guaranteed to already be in basket 2 or 3 after one minute, it must switch from one to the other, a move that has probability $\frac{1}{2}$ of occurring. The final jump into basket 1 also has $\frac{1}{2}$ chance of occurring, giving the first egg a total $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ probability of success.
The second egg also cannot be in basket 1 after two minutes. If it moved to basket 1 originally in the first minute, there is no chance of this happening, whereas if it moved to basket 3, there would be a $\frac{1}{2}$ chance, so the overall probability is $\frac{3}{4}$ that the second egg will be able to make the last jump to basket 1, which again occurs with $\frac{1}{2}$ chance, giving a probability of $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$.
Since the second and third eggs are symmetric, the total probability of success here is $\frac{1}{4} \cdot \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{256}$. Since this is only for the first basket, the final answer will be 3 times this, or $\frac{27}{256}$.

11. Find the value of $\dfrac{4 \cdot 7}{\sqrt{4+\sqrt{7}} + \sqrt{4-\sqrt{7}}}$.

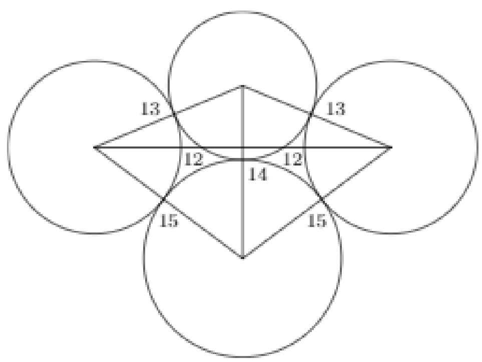
Solution. The answer is $\boxed{2\sqrt{14}}$.

If we square the denominator, we get $(4+\sqrt{7})+(4-\sqrt{7})+2\sqrt{(4+\sqrt{7})(4-\sqrt{7})} = 8+2\sqrt{16-7} = 14$, so the denominator is equal to $\sqrt{14}$, giving the total fraction a value of $\frac{28}{\sqrt{14}} = 2\sqrt{14}$

12. Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

    Solution. The answer is $\boxed{24}$.

    If we connect the centers of the mutually tangent circles with radii $6, 7$, and 8, we end up with a triangle with side lengths $13, 14, 15$. This triangle has an altitude of length 12 to the side composed of the length 6 and 8 radii, and the same configuration will exist on the other side of these two radii, forming a total diagonal distance, between the two centers of the radius 7 circles, of 24. Besides this pair, every other pair of circles is tangent, meaning the distance between the centers is the sum of the radii, which will clearly be less than 24, therefore 24 is the maximum distance.



2.4.5 Round 5

13. Find all ordered pairs of real numbers (x, y) satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

Solution. The answer is $\boxed{(2, 2 \pm \sqrt{3})}$.

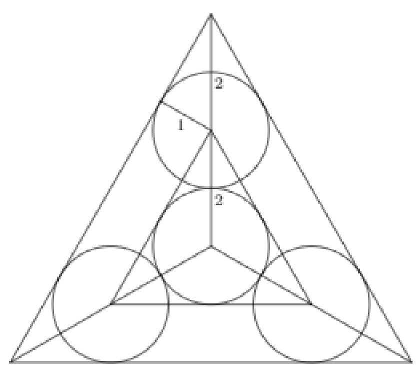
If we divide the two equations, we get $\dfrac{y^2 + \frac{1}{y^2}}{y + \frac{1}{y}} = \frac{7}{2}$, so if we substitute $z = y + \frac{1}{y}$, we find $\frac{z^2-2}{z} = \frac{7}{2}$, or

$2z^2 - 7z - 4 = 0$, which factors as $(2z+1)(z-4) = 0$. Note that from the second (given) equation we have $x > 0$, and thus from the first we have $y > 0$, so $z > 0$ as well, meaning $z = 4$ is the only solution. Since the first equation can be written as $z = 2x$, we find $x = 2$, and the solutions for $y + \frac{1}{y} = 4$ are given by $y^2 - 4y + 1 = 0$, or $y = 2 \pm \sqrt{3}$, giving the solution set $(2, 2-\sqrt{3}), (2, 2+\sqrt{3})$.

14. An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

    Solution. The answer is $\boxed{24\sqrt{3}}$.

    Since the height of the prism is 2, and the radius of each sphere is 1, the centers of all the spheres must be at height 1, otherwise they would intersect with the bottom or top of the prism. Thus this reduces to a 2-dimensional problem: We have four circles of radius one to fit into an equilateral triangle. The optimal arrangement is to have one circle in the center, and every other circle tangent to that central circle and to two other sides. The center to vertex length of this equilateral triangle is two radii plus the hypotenuse of a $30-60-90$ triangle with the short leg as a radius. So, the center to vertex length of the triangle is 4, giving the side length of the triangle as $4\sqrt{3}$, meaning the total area of the triangle is $\frac{(4\sqrt{3})^2\sqrt{3}}{4} = 12\sqrt{3}$, so the volume of the prism is $24\sqrt{3}$.



(birds eye view of an egg plant)

15. How many ways are there for Farmer James to color each square of a 2×6 grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

    Solution. The answer is $\boxed{1458}$.

    For the first column, there are 3 ways to color the top square, and 2 remaining ways to color the bottom square, giving 6 combinations total. Then, given an existing column to the left (say with colors A and B as top and bottom, respectively), among all combinations of colors for the next column, only 3 of them work (BA, BC, CA as top and bottom). Therefore, number of ways to color the grid is simply $6 \cdot (3^5) = 1458$ ways.

2.4.6 Round 6

16. In a triangle ABC, $\angle A = 45$, and let D be the foot of the perpendicular from A to segment BC. $BD = 2$ and $DC = 4$. Let E be the intersection of the line AD and the perpendicular line from B to line AC. Find the length of AE.

    Solution. The answer is $\boxed{6}$.

    Define the foot of the perpendicular from B be F. Notice that since $\angle A = 45$, then $AF = BF$. Since $\angle AEF = \angle BED = \angle BCF$, and $\angle AFE = \angle BFC$, we have that $\triangle AFE \cong \triangle BFC$. So, $AE = BC = 6$.

17. Find the largest positive integer n such that there exists a unique positive integer m satisfying

$$\frac{1}{10} \leq \frac{m}{n} \leq \frac{1}{9}.$$

    Solution. The answer is $\boxed{161}$.

    Take the reciprocal of each half of the inequality, and multiply by m, to get $9m \leq n \leq 10m$. The condition requires a unique solution for m, meaning that the left side of this inequality cannot be satisfied by a greater value of m, so $n < 9(m+1)$. Similarly, we get $n > 10(m-1)$, or $9m+9 > n > 10m-10$ in total. Since the inequalities are strict, this means the interval from $10m-10$ to $9m+9$ must have width at least 2, to leave room for the intermediate value n, so $(9m+9)-(10m-10) \geq 2$, or $m \leq 17$. Thus the maximal n occurs when $m = 17$, giving $9m+9 = 162$ and $10m-10 = 160$, so $n = 161$.

18. How many ordered pairs (A, B) of positive integers are there such that $A+B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

    Solution. The answer is $\boxed{384}$.

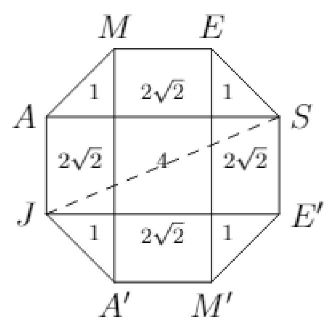
    Note that $A^2 + AB + B = 10000A + B$, so B must be at least 3 digits, otherwise the sum will have two consecutive 0s. Also, since $A + B = 10000$, the last digits of each must sum to 10, while the remaining pairs of corresponding digits (which there are 3 of) must sum to 9 each. Out of the digits from 0 to 9, there are 5 such pairs that sum to 9, and 4 options for the initial pair summing to 10 (since the digits must be distinct). This initial pair also uses up one of the elements for 2 out of 5 pairs that sum to 9, leaving exactly 3 pairs, which are therefore strictly determined by this choice of initial pair. There are then $3! \cdot 2^3 = 48$ ways to permute these pairs among the place values, and then assign one of each pair to either A or B (note that leading 0s are no longer a concern, as both numbers will be at least 3 digits, so 0 will only occur at most once). We must also assign each digit in the initial pair to either A or B, which can be done in 2 ways, giving the total number of ordered pairs as $4 \cdot 2 \cdot 48 = 384$.

2.4.7 Round 7

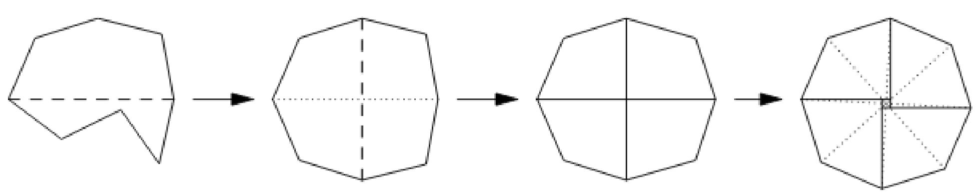
19. Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

Solution. The answer is $\boxed{4+4\sqrt{2}}$.

Notice that if we reflect the pentagon over side JS, we end up with an equilateral octagon (with no other constraints). It is then intuitive (and is proven below, for completeness) that the equilateral octagon with largest possible area is a regular one, so it remains to find the area of a regular octagon with side length 2. To do this, we can split the octagon into 9 regions, as shown. The four corner regions have an area of 1 each, the four edge regions have an area of $2\sqrt{2}$ each, and the central square has area 4, making the total area $8+8\sqrt{2}$. Since our pentagon is half of this octagon, it ends up with area $4+4\sqrt{2}$.



To show that the equilateral octagon with maximal area is in fact regular, we can observe that any for any such octagon, we can choose a diameter, pick the half with larger area, and reflect that over the diameter, replacing the other half and leaving an octagon at least as large, and with horizontal symmetry. We can then again reflect the larger half over the (now) perpendicular diameter, leaving an octagon (at least as large) with 4-way symmetry. If the two diagonals in this octagon are not the same length, we can perform a dissection, creating a new octagon that is *strictly* larger than before; it thus follows that an octagon with maximal area *must* have equal angles from the center to the vertices.



From here, each triangle formed between two adjacent vertices with the center has an angle of 45 degrees across from a side of length 1, and this has maximal area when it is isosceles, thus the maximal octagon occurs when all angles in the octagon are equal, i.e. it is regular

20. $P(x)$ is a monic polynomial (a polynomial with leading coefficient 1) of degree 4, such that $P(2^n+1) = 8^n+1$ when $n = 1, 2, 3, 4$. Find the value of $P(1)$.

Solution. The answer is $\boxed{1025}$.

Note that the cubic polynomial $(x-1)^3+1$ takes 2^n+1 to 8^n+1, therefore to get the correct polynomial, we can simply add this to the monic 4th degree polynomial which takes all given inputs to 0, or $(x-3)(x-5)(x-9)(x-17)$. Thus $P(1) = (1-1)^3+1+(1-3)(1-5)(1-9)(1-17) = 1+1024 = 1025$.

21. PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

Solution. The answer is $\boxed{4070306}$.

Since PEAcock runs 2018 times faster than Hen Hao, if Hen Hao's position is the real number $0 \le x < 1$ (representing his progress along the track), then PEAcock will be at $\{2018x\}$, where $\{x\}$ denotes the fractional part of x. We know that, in the second photo, Hen Hao will now be at $\{2018x\}$, meaning PEAcock will end up at $\{2018\{2018x\}\}$, which must be Hen Hao's original position, giving us $x = \{2018\{2018x\}\} = \{2018^2x\}$. If $2018^2x = x + k$ for some integer k, then $k = (2018^2 - 1)x$, so $x = \frac{k}{2018^2-1}$, where k can range from 0 to $2018^2 - 2$ inclusive, giving $2018^2 - 1$ values. However, we have not yet considered the condition that PEAcock and Hen Hao have their picture taken at distinct locations; if this is not the case, then $x = \{2018x\}$, and then again we get $x = \frac{k}{2017}$, leaving 2017 solutions that we must exclude from our original. Therefore the total number of distinct possible positions for the first photograph is $2018^2 - 1 - 2017 = 2017 \cdot 2018 = 4070306$.

2.4.8 Round 8

22. How many ways are there to *scramble* the letters in $EGGSEATER$ such that no two consecutive letters are the same?

Solution. The answer is $\boxed{10200}$.

The only letters with more than one appearance are the three E's and two G's. (There are 9 letters in total.) We first enforce the requirement that the three E's are not next to each other, and then exclude the case where the two G's are next to each other.

By stars and bars there are $\binom{7}{3} = 35$ ways to place the three E's in the 9-letter sequence so that they are not next to each other, so this gives $35 \cdot \frac{6!}{2} = 12600$ possible scrambles initially. If the two G's are next to each other, then we can consider them as one single letter, so by the same method we see that this accounts for $\binom{6}{3} \cdot 5! = 2400$ scrambles.

Therefore, the total number of scrambles is $12600 - 2400 = 10200$.

23. Let $JAMES$ be a regular pentagon. Let X be on segment JA such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point P on segment AE such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

Solution. The answer is $\boxed{\frac{3\sqrt{5}+7}{2}}$.

First, we prove quadrilateral $XPMA$ is cyclic. To do this, let P' be the point on line EA such

that quadrilateral $XAMP'$ is cyclic. Then, $\angle P'XM = \angle P'AM = 36$, and $\angle P'MX = \angle P'AX = 72$. So, since $P'XM$ is a $36-72-72$ triangle, we know P' fits the problem conditions. Since there can only be one point on line segment AE that fits these conditions, we know $P' = P$. Therefore, quadrilateral $XPMA$ is cyclic. Moreover, $\triangle MXP \sim \triangle MJE$. This tells us that $\triangle MXP$ is just $\triangle MJE$ rotated and dilated about point M (a spiral similarity).
Now, let the intersection of MS and AE be Y. So, P in $\triangle MYE$ corresponds to X in $\triangle MAJ$, and therefore $\frac{EY}{EP} = \frac{JA}{JX}$.
WLOG assume $JA = 1$. Then, the condition $\frac{JX}{XA} = \frac{XA}{JA}$ becomes $XA^2 + XA - 1 = 0$. So, $XA = \frac{\sqrt{5}-1}{2}$, so $\frac{JA}{JX} = \frac{3+\sqrt{5}}{2}$.
It is well-known that the ratio of a diagonal of a regular pentagon to its side is $\frac{\sqrt{5}+1}{2}$ (and in any other $36-36-72$ or $36-36-108$ triangle). Therefore, if Z is the intersection of AE and JM, then $YZ = \frac{EY}{\frac{\sqrt{5}+1}{2}}$, so $AE = EY \cdot (2 + \frac{2}{\sqrt{5}+1})$.
Then, $\frac{AE}{PE} = \frac{EY}{PE} \cdot (2 + \frac{2}{\sqrt{5}+1}) = \frac{3+\sqrt{5}}{2} \cdot (2 + \frac{2}{\sqrt{5}+1}) = \frac{7+3\sqrt{5}}{2}$

24. Find the minimum value of the function

$$f(x) = \left|x - \frac{1}{x}\right| + \left|x - \frac{2}{x}\right| + \left|x - \frac{3}{x}\right| + \cdots + \left|x - \frac{9}{x}\right| + \left|x - \frac{10}{x}\right|$$

over all nonzero real numbers x.

Solution. The answer is $\boxed{2\sqrt{26}}$.

Consider the 11 intervals split by the x intercepts $(1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{10})$ of the 10 absolute values inside $f(x)$. Between all 11 intervals, we can take off the absolute values, leaving a function of the form $f(x) = ax + \frac{b}{x}$. For $x \le \sqrt{6}$, $a \le 0$ and $b > 0$, so all these intervals are monotonically decreasing. Similarly, $x \ge \sqrt{7}$ makes $a > 0$ and $b < 0$, and each of these intervals is monotonically increasing. This function is continuous; therefore, the minimum must be between $\sqrt{6} \le x \le \sqrt{7}$. In this range, we $f(x) = 2x + \frac{13}{x}$, so we can apply AM-GM to give $f(x) \ge 2\sqrt{2x \cdot \frac{13}{x}} = 2\sqrt{26}$, with equality achieved when $x = \sqrt{\frac{13}{2}}$, which is indeed between $\sqrt{6}$ and $\sqrt{7}$.