Exeter Math Club Competition
January 26, 2019

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- **Cleanup Crew** Victor Luo, Thomas Guo, Sanath Govindarajan

Chapter 1

EMC^2 2019 Problems

1.1 Speed Test

There are 20 problems, worth 3 points each, to be solved in 25 minutes.

1. Given that $a + 19b = 3$ and $a + 1019b = 5$, what is $a + 2019b$?

2. How many multiples of 3 are there between 2019 and 2119, inclusive?

3. What is the maximum number of quadrilaterals a 12-sided regular polygon can be *quadrangulated* into? Here *quadrangulate* means to cut the polygon along lines from vertex to vertex, none of which intersect inside the polygon, to form pieces which all have exactly 4 sides.

4. What is the value of $|2\pi - 7| + |2\pi - 6|$, rounded to the nearest hundredth?

5. In the town of EMCCxeter, there is a 30% chance that it will snow on Saturday, and independently, a 40% chance that it will snow on Sunday. What is the probability that it snows exactly once that weekend, as a percentage?

6. Define $n!$ to be the product of all integers between 1 and n inclusive. Compute $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$.

7. There are 2019 people standing in a row, and they are given positions $1, 2, 3, \ldots 2019$ from left to right. Next, everyone in an odd position simultaneously leaves the row, and the remaining people are assigned new positions from 1 to 1009, again from left to right. This process is then repeated until one person remains. What was this person's original position?

8. The product 1234×4321 contains exactly one digit not in the set $\{1, 2, 3, 4\}$. What is this digit?

9. A quadrilateral with positive area has four integer side lengths, with shortest side 1 and longest side 9. How many possible perimeters can this quadrilateral have?

10. Define $s(n)$ to be the sum of the digits of n when expressed in base 10, and let $\gamma(n)$ be the sum of $s(d)$ over all natural number divisors d of n. For instance, $n = 11$ has two divisors, 1 and 11, so $\gamma(11) = s(1) + s(11) = 1 + (1 + 1) = 3$. Find the value of $\gamma(2019)$.

11. How many five-digit positive integers are divisible by 9 and have 3 as the tens digit?

12. Adam owns a large rectangular block of cheese, that has a square base of side length 15 inches, and a height of 4 inches. He wants to remove a cylindrical cheese chunk of height 4, by making a circular hole that goes through the top and bottom faces, but he wants the surface area of the leftover cheese block to be the same as before. What should the diameter of his hole be, in inches?

    Addendum on 1/26/19: the hole must have non-zero diameter.

13. Find the smallest prime that does not divide $20! + 19! + 2019!$.

14. Convex pentagon $ABCDE$ has angles $\angle ABC = \angle BCD = \angle DEA = \angle EAB$ and angle $\angle CDE = 60^\circ$. Given that $BC = 3$, $CD = 4$, and $DE = 5$, find EA.

    Addendum on 1/26/19: $ABCDE$ is specified to be convex.

15. Sophia has 3 pairs of red socks, 4 pairs of blue socks, and 5 pairs of green socks. She picks out two individual socks at random; what is the probability she gets a pair with matching color?

16. How many real roots does the function $f(x) = 2019^x - 2019x - 2019$ have?

17. A 30−60−90 triangle is placed on a coordinate plane with its short leg of length 6 along the x-axis, and its long leg along the y-axis. It is then rotated 90 degrees counterclockwise, so that the short leg now lies along the y-axis and long leg along the x-axis. What is the total area swept out by the triangle during this rotation?

18. Find the number of ways to color the unit cells of a 2×4 grid in four colors such that all four colors are used and every cell shares an edge with another cell of the same color.

19. Triangle $\triangle ABC$ has centroid G, and X, Y, Z are the centroids of triangles $\triangle BCG$, $\triangle ACG$, and $\triangle ABG$, respectively. Furthermore, for some points D, E, F, vertices A, B, C are themselves the centroids of triangles $\triangle DBC$, $\triangle ECA$, and $\triangle FAB$, respectively. If the area of $\triangle XYZ = 7$, what is the area of $\triangle DEF$?

20. Fhomas orders three 2-gallon jugs of milk from EMCCBay for his breakfast omelette. However, every jug is actually shipped with a random amount of milk (not necessarily an integer), uniformly distributed between 0 and 2 gallons. If Fhomas needs 2 gallons of milk for his breakfast omelette, what is the probability he will receive enough milk?

1.2 Accuracy Test

There are 10 problems, worth 9 points each, to be solved in 45 minutes.

1. A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. Two hexos are considered the same if one can be rotated/reflected to match the other. Find the number of different hexos.

2. The sequence $1, 2, 3, 3, 3, 4, 5, 5, 5, 5, 5, 6, \dots$ consists of numbers written in increasing order, where every even number $2n$ is written once, and every odd number $2n+1$ is written $2n+1$ times. What is the 2019^{th} term of this sequence?

3. On planet EMCCarth, months can only have lengths of 35, 36, or 42 days, and there is at least one month of each length. Victor knows that an EMCCarth year has n days, but realizes that he cannot figure out how many months there are in an EMCCarth year. What is the least possible value of n?

4. In triangle ABC, $AB = 5$ and $AC = 9$. If a circle centered at A passing through B intersects BC again at D and $CD = 7$, what is BC?

5. How many nonempty subsets S of the set $\{1, 2, 3, \dots, 11, 12\}$ are there such that the greatest common factor of all elements in S is greater than 1?

6. Jasmine rolls a fair 6-sided die, with faces labeled from 1 to 6, and a fair 20-sided die, with faces labeled from 1 to 20. What is the probability that the product of these two rolls, added to the sum of these two rolls, is a multiple of 3?

7. Let $\{a_n\}$ be a sequence such that a_n is either $2a_{n-1}$ or $a_{n-1} - 1$. Given that $a_1 = 1$ and $a_{12} = 120$, how many possible sequences $a_1, a_2, \dots, a_{12}$ are there?

8. A tetrahedron has two opposite edges of length 2 and the remaining edges have length 10. What is the volume of this tetrahedron?

9. In the garden of EMCCden, there is a tree planted at every lattice point $-10 \le x, y \le 10$ except the origin. We say that a tree is *visible* to an observer if the line between the tree and the observer does not intersect any other tree (assume that all trees have negligible thickness). What fraction of all the trees in the garden of EMCCden are visible to an observer standing at the origin?

10. Point P lies inside regular pentagon ζ, which lies entirely within regular hexagon η. A point Q on the boundary of pentagon ζ is called *projective* if there exists a point R on the boundary of hexagon η such that P, Q, R are collinear and $2019 \cdot \overline{PQ} = \overline{QR}$. Given that no two sides of ζ and η are parallel, what is the maximum possible number of projective points on ζ?

1.3 Team Test

There are 15 problems, worth 20 points each, to be solved in 60 minutes.

1. Three positive integers sum to 16. What is the least possible value of the sum of their squares?
2. Ben is thinking of an odd positive integer less than 1000. Ben subtracts 1 from his number and divides by 2, resulting in another number. If his number is still odd, Ben repeats this procedure until he gets an even number. Given that the number he ends on is 2, how many possible values are there for Ben's original number?
3. Triangle ABC is isosceles, with $AB = BC = 18$ and has circumcircle ω. Tangents to ω at A and B intersect at point D. If $AD = 27$, what is the length of AC?
4. How many non-decreasing sequences of five natural numbers have first term 1, last term 11, and have no three terms equal?
5. Adam is bored, and has written the string "EMCC" on a piece of paper. For fun, he decides to erase every letter "C", and replace it with another instance of "EMCC". For example, after one step, he will have the string "EMEMCCEMCC". How long will his string be after 8 of these steps?
6. Eric has two coins, which land heads 40% and 60% of the time respectively. He chooses a coin randomly and flips it four times. Given that the first three flips contained two heads and one tail, what is the probability that the last flip was heads?
7. In a five person rock-paper-scissors tournament, each player plays against every other player exactly once, with each game continuing until one player wins. After each game, the winner gets 1 point, while the loser gets no points. Given that each player has a 50% chance of defeating any other player, what is the probability that no two players end up with the same amount of points?
8. Let $\triangle ABC$ have $\angle A = \angle B = 75^\circ$. Points D, E, and F are on sides BC, CA, and AB, respectively, so that EF is parallel to BC, $EF \perp DE$, and $DE = EF$. Find the ratio of $\triangle DEF$'s area to $\triangle ABC$'s area.
9. Suppose a, b, c are positive integers such that $a+b = \sqrt{c^2 + 336}$ and $a-b = \sqrt{c^2 - 336}$. Find $a+b+c$.
10. How many times on a 12-hour analog clock are there, such that when the minute and hour hands are swapped, the result is still a valid time? (Note that the minute and hour hands move continuously, and don't always necessarily point to exact minute/hour marks.)
11. Adam owns a square S with side length 42. First, he places rectangle A, which is 6 times as long as it is wide, inside the square, so that all four vertices of A lie on sides of S, but none of the sides of A are parallel to any side of S. He then places another rectangle B, which is 7 times as long as it is wide, inside rectangle A, so that all four vertices of B lie on sides of A, and again none of the sides of B are parallel to any side of A. Find the length of the shortest side of rectangle B.
12. Find the value of $\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$, where the exponents are the odd natural numbers, in increasing order.
13. Jamesu and Fhomas challenge each other to a game of Square Dance, played on a 9×9 square grid. On Jamesu's turn, he colors in a 2×2 square of uncolored cells pink. On Fhomas's turn, he colors in a 1×1 square of uncolored cells purple. Once Jamesu can no longer make a move, Fhomas gets to color

in the rest of the cells purple. If Jamesu goes first, what the maximum number of cells that Fhomas can color purple, assuming both players play optimally in trying to maximize the number of squares of their color?

14. Triangle ABC is inscribed in circle ω. The tangents to ω from B and C meet at D, and segments AD and BC intersect at E. If $\angle BAC = 60^\circ$ and the area of $\triangle BDE$ is twice the area of $\triangle CDE$, what is $\frac{AB}{AC}$?
Addendum on 1/26/19: $\angle A$ changed to $\angle BAC$.

15. Fhomas and Jamesu are now having a number duel. First, Fhomas chooses a natural number n. Then, starting with Jamesu, each of them take turns making the following moves: if n is composite, the player can pick any prime divisor p of n, and replace n by $n-p$; if n is prime, the player can replace n by $n-1$. The player who is faced with 1, and hence unable to make a move, loses. How many different numbers $2 \le n \le 2019$ can Fhomas choose such that he has a winning strategy, assuming Jamesu plays optimally?

1.4 Guts Test

There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.

1.4.1 Round 1

1. [6] What is the smallest number equal to its cube?

2. [6] Fhomas has 5 red spaghetti and 5 blue spaghetti, where spaghetti are indistinguishable except for color. In how many different ways can Fhomas eat 6 spaghetti, one after the other? (Two ways are considered the same if the sequence of colors are identical)

3. [6] Jocelyn labels the three corners of a triangle with three consecutive natural numbers. She then labels each edge with the sum of the two numbers on the vertices it touches, and labels the center with the sum of all three edges. If the total sum of all labels on her triangle is 120, what is the value of the smallest label?

1.4.2 Round 2

4. [7] Adam cooks a pie in the shape of a regular hexagon with side length 12, and wants to cut it into right triangular pieces with angles 30°, 60°, and 90°, each with shortest side 3. What is the maximum number of such pieces he can make?

5. [7] If $f(x) = \frac{1}{2-x}$ and $g(x) = 1 - \frac{1}{x}$, what is the value of $f(g(f(g(\cdots f(g(f(2019)))\cdots))))$, where there are 2019 functions total, counting both f and g?

6. [7] Fhomas is buying spaghetti again, which is only sold in two types of boxes: a 200 gram box and a 500 gram box, each with a fixed price. If Fhomas wants to buy exactly 800 grams, he must spend \$8.80, but if he wants to buy exactly 900 grams, he only needs to spend \$7.90! In dollars, how much more does the 500 gram box cost than the 200 gram box?

1.4.3 Round 3

7. [9] Given that
$$\begin{cases} a + 5b + 9c = 1, \\ 4a + 2b + 3c = 2, \\ 7a + 8b + 6c = 9, \end{cases}$$
what is $741a + 825b + 639c$?

8. [9] Hexagon $JAMESU$ has line of symmetry MU (i.e., quadrilaterals $JAMU$ and $SEMU$ are reflections of each other), and $JA = AM = ME = ES = 1$. If all angles of $JAMESU$ are 135 degrees except for right angles at A and E, find the length of side US.

9. [9] Max is parked at the 11 mile mark on a highway, when his pet cheetah, Min, leaps out of the car and starts running up the highway at its maximum speed. At the same time, Max starts his car and starts driving down the highway at $\frac{1}{2}$ his maximum speed, driving all the way to the 10 mile mark before realizing that his cheetah is gone! Max then immediately reverses directions and starts driving back up the highway at his maximum speed, finally catching up to Min at the 20 mile mark. What is the ratio between Max's max speed and Min's max speed?

1.4.4 Round 4

10. [11] Kevin owns three non-adjacent square plots of land, each with side length an integer number of meters, whose total area is 2019m^2. What is the minimum sum of the perimeters of his three plots, in meters?

11. [11] Given a 5×5 array of lattice points, how many squares are there with vertices all lying on these points?

12. [11] Let right triangle ABC have $\angle A = 90^\circ$, $AB = 6$, and $AC = 8$. Let points D, E be on side AC such that $AD = EC = 2$, and let points F, G be on side BC such that $BF = FG = 3$. Find the area of quadrilateral $FGED$.

1.4.5 Round 5

13. [13] Given a (not necessarily simplified) fraction $\frac{m}{n}$, where $m, n > 6$ are positive integers, when 6 is subtracted from both the numerator and denominator, the resulting fraction is equal to $\frac{4}{5}$ of the original fraction. How many possible ordered pairs (m, n) are there?

14. [13] Jamesu's favorite anime show has 3 seasons, with 12 episodes each. For 8 days, Jamesu does the following: on the n^{th} day, he chooses n consecutive episodes of exactly one season, and watches them in order. How many ways are there for Jamesu to finish all 3 seasons by the end of these 8 days? (For example, on the first day, he could watch episode 5 of the first season; on the second day, he could watch episodes 11 and 12 of the third season, etc.)

15. [13] Let O be the center of regular octagon $ABCDEFGH$ with side length 6. Let the altitude from O meet side AB at M, and let BH meet OM at K. Find the value of $BH \cdot BK$.

1.4.6 Round 6

16. [15] Fhomas writes the ordered pair $(2, 4)$ on a chalkboard. Every minute, he erases the two numbers (a, b), and replaces them with the pair $(a^2 + b^2, 2ab)$. What is the largest number on the board after 10 minutes have passed?

17. [15] Triangle BAC has a right angle at A. Point M is the midpoint of BC, and P is the midpoint of BM. Point D is the point where the angle bisector of $\angle BAC$ meets BC. If $\angle BPA = 90^\circ$, what is $\frac{PD}{DM}$?

18. [15] A square is called *legendary* if there exist two different positive integers a, b such that the square can be tiled by an equal number of non-overlapping a by a squares and b by b squares. What is the smallest positive integer n such that an n by n square is *legendary*?

1.4.7 Round 7

19. [18] Let $S(n)$ be the sum of the digits of a positive integer n. Let $a_1 = 2019!$, and $a_n = S(a_{n-1})$. Given that a_3 is even, find the smallest integer $n \geq 2$ such that $a_n = a_{n-1}$.

20. [18] The local EMCC bakery sells one cookie for p dollars (p is not necessarily an integer), but has a special offer, where any non-zero purchase of cookies will come with one additional free cookie. With \$27.50, Max is able to buy a whole number of cookies (including the free cookie) with a single purchase and no change leftover. If the price of each cookie were 3 dollars lower, however, he would be able to buy double the number of cookies as before in a single purchase (again counting the free cookie) with no change leftover. What is the value of p?

21. [18] Let circle ω be inscribed in rhombus $ABCD$, with $\angle ABC < 90^\circ$. Let the midpoint of side AB be labeled M, and let ω be tangent to side AB at E. Let the line tangent to ω passing through M other than line AB intersect segment BC at F. If $AE = 3$ and $BE = 12$, what is the area of $\triangle MFB$?

1.4.8 Round 8

22. [21] Find the remainder when $1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1!$ is divided by 2019.

23. [21] Two circles ω_1 and ω_2 have radii 1 and 2, respectively and are externally tangent to one another. Circle ω_3 is externally tangent to both ω_1 and ω_2. Let M be the common external tangent of ω_1 and ω_3 that doesn't intersect ω_2. Similarly, let N be the common external tangent of ω_2 and ω_3 that doesn't intersect ω_1. Given that M and N are parallel, find the radius of ω_3.

24. [21] Mana is standing in the plane at $(0, 0)$, and wants to go to the EMCCiffel Tower at $(6, 6)$. At any point in time, Mana can attempt to move 1 unit to an adjacent lattice point, or to make a knight's move, moving diagonally to a lattice point $\sqrt{5}$ units away. However, Mana is deathly afraid of negative numbers, so she will make sure never to decrease her x or y values. How many distinct paths can Mana take to her destination?

1.5 Practice Sets

We have arranged this year's 69 EMCC problems into 12 sets, in approximately increasing order of difficulty. This sorting is based on solve rate during the contest, among other measures. The first two sets consist of 10 problems each, and the others consist of 5 problems each (except for Set 12). The goal of these Practice Sets is to provide a repository that coaches can use for training at various skill levels.

Coming soon!

Chapter 2

EMC^2 2019 Solutions

2.1 Speed Test Solutions

1. Given that $a + 19b = 3$ and $a + 1019b = 5$, what is $a + 2019b$?

   Solution. The answer is $\boxed{7}$.

   Note that the numbers $a+19b$, $a+1019b$, and $a+2019b$ are in arithmetic sequence, because they differ by $1000b$ each. Since we know the first two terms are 3 and 5, the third term must be 7.

2. How many multiples of 3 are there between 2019 and 2119, inclusive?

   Solution. The answer is $\boxed{34}$.

   Since 2019 is a multiple of 3, and our range has width 100, our multiples of 3 can be written as $2019, 2019 + 3, \ldots, 2019 + 99$, so there are a total of $\frac{99}{3} + 1 = 34$ multiples of 3.

3. What is the maximum number of quadrilaterals a 12-sided regular polygon can be *quadrangulated* into? Here *quadrangulate* means to cut the polygon along lines from vertex to vertex, none of which intersect inside the polygon, to form pieces which all have exactly 4 sides.

   Solution. The answer is $\boxed{5}$.

   When we cut a quadrilateral off of our polygon, three of the quadrilateral's sides originally belonged to the polygon, while the fourth side was created by our cut. The polygon itself therefore loses three sides to the quadrilateral, and gains one from the cut made, losing two sides overall. We can continue cutting off quadrilaterals until the original polygon only has 4 sides left, hence becoming a quadrilateral itself, at which point we are done. Starting from 12 sides, we can therefore cut 4 times before the original polygon is left with $12 - 2 \cdot 4 = 4$ sides. These 4 additional quadrilaterals, plus the remnant of the original polygon, give 5 total quadrilaterals.

4. What is the value of $|2\pi - 7| + |2\pi - 6|$, rounded to the nearest hundredth?

   Solution. The answer is $\boxed{1.00}$.

   Since $6 < 2\pi < 7$, we have $|2\pi - 7| + |2\pi - 6| = (7 - 2\pi) + (2\pi - 6) = 1$.

5. In the town of EMCCxeter, there is a 30% chance that it will snow on Saturday, and independently, a 40% chance that it will snow on Sunday. What is the probability that it snows exactly once that weekend, as a percentage?

   Solution. The answer is $\boxed{46\%}$.

   The probability that it snows on Saturday but not Sunday is $0.3 \cdot 0.6 = 0.18$, and the probability that it snows on Sunday but not Saturday is $0.4 \cdot 0.7 = 0.28$, so the total probability is $18\% + 28\% = 46\%$.

6. Define $n!$ to be the product of all integers between 1 and n inclusive. Compute $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$.

   Solution. The answer is $\boxed{\frac{2019}{2017}}$.

   Notice that the product $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$ can be rewritten as $\frac{2019!}{2018!} \times \frac{2016!}{2017!}$. Now, we see that the first fraction is equal to 2019 while the second is $\frac{1}{2017}$, making our final answer is $\frac{2019}{2017}$.

7. There are 2019 people standing in a row, and they are given positions $1, 2, 3, \ldots 2019$ from left to right. Next, everyone in an odd position simultaneously leaves the row, and the remaining people are assigned new positions from 1 to 1009, again from left to right. This process is then repeated until one person remains. What was this person's original position?

   Solution. The answer is $\boxed{1024}$.

   This process is equivalent to repeatedly removing people with odd positions, and dividing the positions of the remaining people by 2. The last person remaining, therefore, must have started with the highest number of 2's in their prime factorization, meaning their position was $2^{10} = 1024$.

8. The product 1234×4321 contains exactly one digit not in the set $\{1, 2, 3, 4\}$. What is this digit?

   Solution. The answer is $\boxed{5}$.

   By approximation, we can quickly check that both 13×44 and 12×43 begin with the digit 5. Therefore, the first digit of our product (which is simply a scaled up version of 12.34×43.21) must also be 5. Because this is not in the set $\{1, 2, 3, 4\}$, it is our answer. (The actual product is 5332114.)

9. A quadrilateral with positive area has four integer side lengths, with shortest side 1 and longest side 9. How many possible perimeters can this quadrilateral have?

   Solution. The answer is $\boxed{10}$.

   We can consider an analog of the triangle inequality, but for quadrilaterals: clearly no one side can be longer than, or equal to (if we want positive area), the sum of the other 3 sides. If the sum of our two unknown sides is s, this gives us $1 + s > 9$, so $s \geq 9$. Since both unknown sides are at most 9, however, we also know that $s \leq 18$, giving us 10 possible perimeters, $1 + 9 + s$ for each $9 \leq s \leq 18$.

10. Define $s(n)$ to be the sum of the digits of n when expressed in base 10, and let $\gamma(n)$ be the sum of $s(d)$ over all natural number divisors d of n. For instance, $n = 11$ has two divisors, 1 and 11, so $\gamma(11) = s(1) + s(11) = 1 + (1 + 1) = 3$. Find the value of $\gamma(2019)$.

    Solution. The answer is $\boxed{32}$.

    Since $2019 = 3 \cdot 673$, we compute

    $$\gamma(2019) = s(2019) + s(673) + s(3) + s(1) = 12 + 16 + 3 + 1 = 32.$$

11. How many five-digit positive integers are divisible by 9 and have 3 as the tens digit?

    Solution. The answer is $\boxed{1000}$.

    If we remove the tens digit of 3, we have that the resulting four-digit integer must leave a remainder of 6 when divided by 9 (since the remainder of the integer when divided by 9 is equal to the remainder of the sum of the digits when divided by 9). Since there are 9000 four-digit integers, there must be $\frac{9000}{9} = 1000$ integers with remainder 6 when divided by 9, and hence 1000 five-digit integers that satisfy the original condition.

12. Adam owns a large rectangular block of cheese, that has a square base of side length 15 inches, and a height of 4 inches. He wants to remove a cylindrical cheese chunk of height 4, by making a circular hole that goes through the top and bottom faces, but he wants the surface area of the leftover cheese block to be the same as before. What should the diameter of his hole be, in inches?

    Addendum on 1/26/19: the hole must have non-zero diameter.

    Solution. The answer is $\boxed{8}$.

    Suppose that the radius of the hole is r inches. Then, the surface area lost from the top and bottom faces is $2 \cdot \pi r^2$ square inches, and the surface area gained from the wall of the cylindrical hole is $4 \cdot 2\pi r = 8\pi r$. Setting these equal gives $2\pi r^2 = 8\pi r$, or $r = 4$ (or $r = 0$, but that would mean that the hole is non-existent), so the diameter is $2r = 8$.

13. Find the smallest prime that does not divide $20! + 19! + 2019!$.

    Solution. The answer is $\boxed{23}$.

    Any prime $p \leq 19$ must divide $20! + 19! + 2019$, because p divides each of $20!$, $19!$, and $2019!$. Hence the smallest prime not dividing $20! + 19! + 2019!$ must be at least 23. Now, note that 23 divides $2019!$, but $20! + 19! = 19! \times (20 + 1) = 19! \times 21$ is not a multiple of 23, so 23 does not divide the expression, and is the answer.

14. Convex pentagon $ABCDE$ has angles $\angle ABC = \angle BCD = \angle DEA = \angle EAB$ and angle $\angle CDE = 60^\circ$. Given that $BC = 3$, $CD = 4$, and $DE = 5$, find EA.

    Addendum on 1/26/19: $ABCDE$ is specified to be convex.

    Solution. The answer is $\boxed{2}$.

    Since the sum of the five interior angles is $180^\circ \cdot (5 - 2) = 540^\circ$, the four equal angles must have measure $\frac{540-60}{4} = 120$ degrees. If we extend BA and DE to intersect at F, then AEF is an equilateral triangle, and moreover $BCDF$ is an isosceles trapezoid. Also, it is not difficult to see that $BCDF$ can be cut into an equilateral triangle with side length 4 and a parallelogram with two sides 3 and 4, so $DF = 3 + 4 = 7$, and thus $AE = EF = DF - DE = 7 - 5 = 2$.

15. Sophia has 3 pairs of red socks, 4 pairs of blue socks, and 5 pairs of green socks. She picks out two individual socks at random; what is the probability she gets a pair with matching color?

    Solution. The answer is $\boxed{\frac{22}{69}}$.

    There are $2(3 + 4 + 5) = 24$ socks in total, so there are $\binom{24}{2} = 276$ possible pairs. Out of these pairs, there are $\binom{6}{2} + \binom{8}{2} + \binom{10}{2} = 88$ pairs that have matching colors, making the probability $\frac{88}{276} = \frac{22}{69}$.

16. How many real roots does the function $f(x) = 2019^x - 2019x - 2019$ have?

    Solution. The answer is $\boxed{2}$.

    We note that $f(-1) = \frac{1}{2019}, f(0) = -2018, f(1) = -2019, f(2) = 2019 \cdot 2016$, so by continuity there must be a root between -1 and 0 and a root between 1 and 2. We now show that there are at most two solutions. This is because the function 2019^x is *convex*, meaning that it increases at an increasing rate, and a line can intersect a convex function at most twice.

17. A $30-60-90$ triangle is placed on a coordinate plane with its short leg of length 6 along the x-axis, and its long leg along the y-axis. It is then rotated 90 degrees counterclockwise, so that the short leg now lies along the y-axis and long leg along the x-axis. What is the total area swept out by the triangle during this rotation?

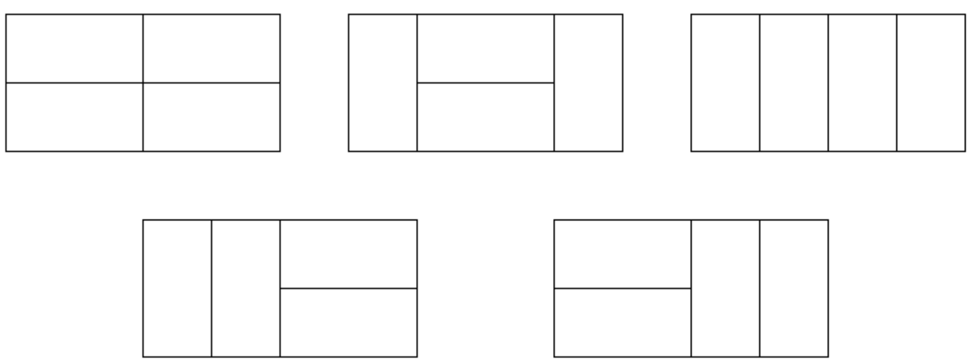
    Solution. The answer is $\boxed{33\pi + 9\sqrt{3}}$.

    Let O be the origin of the coordinate plane, $OA = 6$ be the shorter leg, and $OB = 6\sqrt{3}$ be the longer leg. It is not difficult to see that the rotation must be centered around the origin. If we assume that the triangle starts in the first quadrant, then, during the rotation, everything swept in the second quadrant will be swept by OB, and hence the area swept in the second quadrant is a quarter circle with radius $6\sqrt{3}$, and area $\frac{\pi(6\sqrt{3})^2}{4} = 27\pi$. In the first quadrant, the area swept out is covered by both a quarter circle with radius 6, and the original triangle OAB. We can find the combined area by noticing that the triangle and quarter circle must intersect at C, the midpoint of AB, which means that the area consists of the $\frac{1}{6}$ circular sector OAC with area $\frac{\pi(6)^2}{6} = 6\pi$, and the triangle OBC with area equal to half the original area of OAB, or $\frac{1}{2} \cdot \frac{6 \cdot 6\sqrt{3}}{2} = 9\sqrt{3}$. Adding all three parts up, we get the total area is $27\pi + 6\pi + 9\sqrt{3} = 33\pi + 9\sqrt{3}$.

18. Find the number of ways to color the unit cells of a 2×4 grid in four colors such that all four colors are used and every cell shares an edge with another cell of the same color.

    Solution. The answer is $\boxed{120}$.

    Since each cell shares an edge with another cell of the same color, there must be at least two cells of each color, but since there are only 8 cells in total and all colors are used, there must be *exactly* two cells of each color, making a "domino" shape. There are 5 ways to partition the grid into four dominoes:

    and there are $4! = 24$ ways to color the dominoes, so there are $5 \cdot 24 = 120$ colorings in total.

19. Triangle $\triangle ABC$ has centroid G, and X, Y, Z are the centroids of triangles $\triangle BCG$, $\triangle ACG$, and $\triangle ABG$, respectively. Furthermore, for some points D, E, F, vertices A, B, C are themselves the centroids of triangles $\triangle DBC$, $\triangle ECA$, and $\triangle FAB$, respectively. If the area of $\triangle XYZ = 7$, what is the area of $\triangle DEF$?

Solution. The answer is $\boxed{1008}$.

It is a well known fact that, in triangle ABC, a given vertex A is 3 times as far from base BC as the centroid G is. We will utilize this fact to find our answer. Let M be the midpoint of BC, and suppose that $AM = d$. Then, GM must be $\frac{d}{3}$, by our fact, and likewise XM must be $\frac{d}{9}$, so $XG = GM - XM = \frac{2d}{9}$. We also find that $DM = 3d$, so $DG = DM - GM = \frac{8}{3}d$. We also know that the centroid of a triangle lies along its median, i.e A lies on DM, G lies on AM, and X lies on GM, thus D, G, X are collinear. This, combined with the fact that $DG = \frac{8}{3}d = 12 \cdot \frac{2}{9}d = 12XG$ (and indeed $EG = 12YG$, $FG = 12ZG$, since we could have picked any midpoint to start our calculation), means that $\triangle DEF$ is $\triangle XYZ$ scaled around G by a factor of -12 (here the negative refers to the fact that each points ends up on the opposite side of G from where it started), so the area of $\triangle DEF$ is $12^2 \cdot 7 = 1008$.

20. Fhomas orders three 2-gallon jugs of milk from EMCCBay for his breakfast omelette. However, every jug is actually shipped with a random amount of milk (not necessarily an integer), uniformly distributed between 0 and 2 gallons. If Fhomas needs 2 gallons of milk for his breakfast omelette, what is the probability he will receive enough milk?

    Solution. The answer is $\boxed{\frac{5}{6}}$.

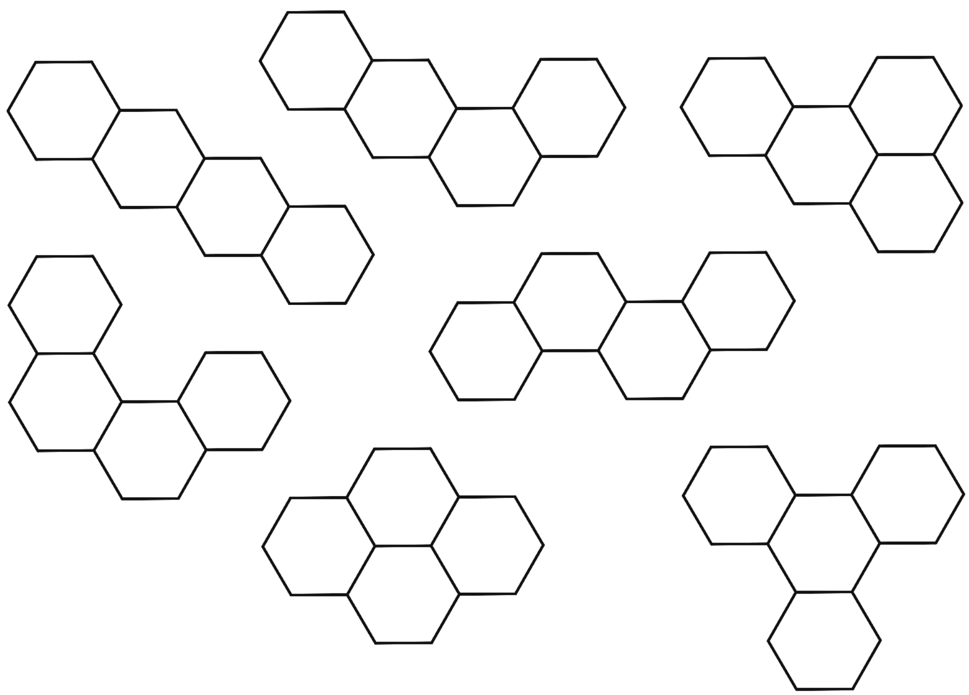
    We need to compute the probability of $x + y + z \geq 2$ when x, y, z are real numbers uniformly chosen between 0 and 2. If we visualize this in a cube, then we are picking a random point in a cube with volume $2^3 = 8$, and avoiding points in the region $x+y+z < 2$. Note that the plane $x+y+z = 2$ slices through the three vertices of the cube next to the origin, so the volume that it slices off is the volume of a tetrahedron, which is $\frac{1}{6} \cdot 2 \cdot 2 \cdot 2 = \frac{4}{3}$. Therefore, the probability that $x + y + z \geq 2$ is $\frac{8-\frac{4}{3}}{8} = \frac{5}{6}$.

2.2 Accuracy Test Solutions

1. A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. Two hexos are considered the same if one can be rotated/reflected to match the other. Find the number of different hexos.

   Solution. The answer is $\boxed{7}$.

   The seven possible ways are shown below:



2. The sequence $1, 2, 3, 3, 3, 4, 5, 5, 5, 5, 5, 6, \ldots$ consists of numbers written in increasing order, where every even number $2n$ is written once, and every odd number $2n+1$ is written $2n+1$ times. What is the 2019^{th} term of this sequence?

   Solution. The answer is $\boxed{89}$.

   Note that numbers $2n-1$ and $2n$ are written $(2n-1)+1 = 2n$ times in total, so the first $2n$ numbers are written $2+4+6+\cdots+2n = \frac{n(2n+2)}{2} = n(n+1)$ times in total, and the largest n such that $n(n+1) \leq 2019$ is $n = 44$. Therefore, because the first 88 numbers are written $44 \cdot 45 = 1980$ times in total, and 90 will be written as the $45 \cdot 46 = 2070$th number, the 2019th term must be 89.

3. On planet EMCCarth, months can only have lengths of 35, 36, or 42 days, and there is at least one month of each length. Victor knows that an EMCCarth year has n days, but realizes that he cannot figure out how many months there are in an EMCCarth year. What is the least possible value of n?

Solution. The answer is $\boxed{323}$.

In order for Victor to be unable to figure out the number of months, there must be two combinations having a different number of total months, but the same number of days. Now, we can consider two cases: either these combinations have a different number of 35 day months, or they have the same number. In the second case, we can only swap between 36 and 42 day months, so the least number we need is $\text{lcm}(36, 42) = 252$, and when added to $35 + 36 + 42 = 113$ (because we need at least one of each month) gives us a total of 365 days. In the first case, however, note that both of the other month lengths are multiples of 6, and therefore if one combination uses 35 day months, it must use at least 6 at a time; the least such swap is $6 \cdot 35 = 5 \cdot 42$, giving us $210 + 113 = 323$ days. Since $323 < 365$, the first case is the optimal one, giving us our answer of 323 days.

4. In triangle ABC, $AB = 5$ and $AC = 9$. If a circle centered at A passing through B intersects BC again at D and $CD = 7$, what is BC?

   Solution. The answer is $\boxed{8}$.

   Let M be the midpoint of BD, then since ABD is an isosceles triangle, AM is perpendicular to BD. Suppose that $DM = x$ and $AM = h$, then using Pythagorean theorem on AMD and AMC we have $x^2 + h^2 = 5^2$ and $(x + 7)^2 + h^2 = 9^2$. Subtracting the first equation from the second gives $14x + 49 = 81 - 25 = 56 \Rightarrow x = \frac{1}{2}$, so $BC = CD + BD = 7 + 2x = 8$.

   Alternatively, it is well known that the power of a point is magnitude of the difference between the square of its distance from the center of the circle and the square of the radius of the circle. Therefore, the power of C is $AC^2 - AB^2 = 9^2 - 5^2 = 56$. Since the power of C is also equal to $CD \cdot CB = 7CB$, CB must be 8.

5. How many nonempty subsets S of the set $\{1, 2, 3, \ldots, 11, 12\}$ are there such that the greatest common factor of all elements in S is greater than 1?

   Solution. The answer is $\boxed{79}$.

   We use the Principle of Inclusion-Exclusion:

   (a) If all elements are divisible by 2, then S is a subset of $\{2, 4, 6, 8, 10, 12\}$, which gives $2^6 - 1 = 63$ possibilities.

   (b) If all elements are divisible by 3, then S is a subset of $\{3, 6, 9, 12\}$, which gives $2^4 - 1 = 15$ possibilities.

   (c) If all elements are divisible by 5, then S is a subset of $\{5, 10\}$, which gives $2^2 - 1 = 3$ possibilities.

   (d) If all elements are divisible by 7 or 11, then $S = \{7\}$ or $S = \{11\}$, which gives 2 possibilities.

   (e) If all elements are divisible by 6, then S is a subset of $\{6, 12\}$, which gives $2^2 - 1 = 3$ possibilities. *These are double-counted by cases (a) and (b).*

   (f) If all elements are divisible by 10, then $S = \{10\}$, which gives 1 possibility. *These are also double-counted by cases (a) and (c).*

   Hence, in total there are $63 + 15 + 3 + 2 - 3 - 1 = 79$ possible subsets.

6. Jasmine rolls a fair 6-sided die, with faces labeled from 1 to 6, and a fair 20-sided die, with faces labeled from 1 to 20. What is the probability that the product of these two rolls, added to the sum of these two rolls, is a multiple of 3?

Solution. The answer is $\boxed{\frac{13}{60}}$.

Suppose that the 6-sided die rolls a and the 20-sided die rolls b, then we need $ab + a + b$ to be a multiple of 3, or $ab + a + b + 1 = (a + 1)(b + 1)$ to be one more than a multiple of 3. This happens in two ways:

(a) $a + 1$ and $b + 1$ are both one more than a multiple of 3, so $a \in \{3, 6\}$ and $b \in \{3, 6, 9, 12, 15, 18\}$, which gives $2 \cdot 6 = 12$ ways.

(b) $a+1$ and $b+1$ are both two more than a multiple of 3, so $a \in \{1, 4\}$ and $b \in \{1, 4, 7, 10, 13, 16, 19\}$, which gives $2 \cdot 7 = 14$ ways.

Since there are $6 \cdot 20 = 120$ possible outcomes, the probability is $\frac{12+14}{120} = \frac{13}{60}$.

7. Let $\{a_n\}$ be a sequence such that a_n is either $2a_{n-1}$ or $a_{n-1} - 1$. Given that $a_1 = 1$ and $a_{12} = 120$, how many possible sequences $a_1, a_2, \ldots, a_{12}$ are there?

Solution. The answer is $\boxed{4}$.

We will consider all such sequences constructed via -1 and *doubling* moves, i.e. following a_n by $a_n - 1$ or $2a_n$, respectively. An important observation to make is that, if at some point there is a -1 move, and there are a total of k doubling moves afterwards, this -1 will end up having a net effect of subtracting 2^k. In other words, if among our 11 moves from a_1 to a_{12}, there are k of the -1 moves, and $11 - k$ of the doubling moves, then our final result will be k powers of 2 subtracted off of 2^{11-k}. As our final result must be 120, we need $2^{11-k} \geq 128$, so k is at most 4.

Now, we perform some brief case checking: If $k = 4$, we need to subtract four powers of 2 from $2^{11-4} = 128$ to get 120, or in other words, we want four powers of 2 that sum to 8. Note that the ordering of our subtracted powers of 2 is necessarily from greatest to least, as we move from a_1 to a_{12}, since it is impossible for an earlier -1 move to be followed by fewer doubling moves than a -1 move occurring later in the sequence. Thus there are two distinct possibilities in this case: $(2^2, 2^1, 2^0, 2^0)$ and $(2^1, 2^1, 2^1, 2^1)$.

If $k = 3$, we need three powers of 2 summing to $2^{11-3} - 120 = 136$. Here again we can find two solutions: $(2^6, 2^6, 2^3)$ and $(2^7, 2^2, 2^2)$.

If $k = 2$, we will need two powers of 2 summing to $2^{11-2} - 120 = 392$; if we write out $392 = 2^8 + 2^7 + 2^3$ in binary, however, it is clear we cannot accomplish this with only two powers of 2. Likewise, $k = 1, 0$ are similarly impossible, due to the lack of sufficient powers of 2 to subtract off.

In conclusion, we found two cases for $k = 4$ and two for $k = 3$, giving 4 sequences in total. If anyone is curious, these sequences are:

$$1, 2, 4, 8, 16, 32, 31, 62, 61, 122, 121, 120$$

$$1, 2, 4, 8, 16, 32, 64, 63, 62, 61, 60, 120$$

$$1, 2, 4, 3, 2, 4, 8, 16, 15, 30, 60, 120$$

$$1, 2, 1, 2, 4, 8, 16, 32, 31, 30, 60, 120$$

8. A tetrahedron has two opposite edges of length 2 and the remaining edges have length 10. What is the volume of this tetrahedron?

Solution. The answer is $\boxed{\frac{14\sqrt{2}}{3}}$.

Let $ABCD$ be the tetrahedron, with $AB = 2$. Let M be the midpoint of side AB, and N be the midpoint of CD. Then, we can split tetrahedron $ABCD$ into triangular pyramids $ACDM$ and $BCDM$, each of which have height 1, and base triangle MCD. Since AMC is a right triangle, we have $MC = \sqrt{AC^2 - AM^2} = \sqrt{99}$. Furthermore, MCN is a right triangle, so $MN = \sqrt{MC^2 - CN^2} = \sqrt{98} = 7\sqrt{2}$. The area of triangle MCD is therefore $\frac{1}{2}MN \cdot CD = 7\sqrt{2}$. Therefore, the volumes of $ACDM$ and $BCDM$ are each $\frac{1}{3} \cdot 7\sqrt{2} \cdot 1$, so the total volume is $\frac{14\sqrt{2}}{3}$.

9. In the garden of EMCCden, there is a tree planted at every lattice point $-10 \le x, y \le 10$ except the origin. We say that a tree is *visible* to an observer if the line between the tree and the observer does not intersect any other tree (assume that all trees have negligible thickness). What fraction of all the trees in the garden of EMCCden are visible to an observer standing at the origin?

Solution. The answer is $\boxed{\frac{32}{55}}$.

There are $21^2 - 1 = 440$ trees in the garden. We divide the garden by the four lines $x = 0, y = 0, x = y, x = -y$ into eight symmetric sections. The observer can see one tree on each of the eight directions along the four lines, and it suffices to count the number of trees visible in each section. We consider the section $\{(x, y) \mid 1 \le x < y \le 10\}$. It is not difficult to see that a tree is visible if and only if $\gcd(x, y) = 1$, so we count row by row in increasing order of x:

- $x = 1$: $y = 2, 3, \ldots, 10$, for 9 trees.
- $x = 2$: $y = 3, 5, 7, 9$, for 4 trees.
- $x = 3$: $y = 4, 5, 7, 8, 10$, for 5 trees.
- $x = 4$: $y = 5, 7, 9$, for 3 trees.
- $x = 5$: $y = 6, 7, 8, 9$, for 4 trees.
- $x = 6$: $y = 7$, for 1 tree.
- $x = 7$: $y = 8, 9, 10$, for 3 trees.
- $x = 8$: $y = 9$, for 1 tree.
- $x = 9$: $y = 10$, for 1 tree.

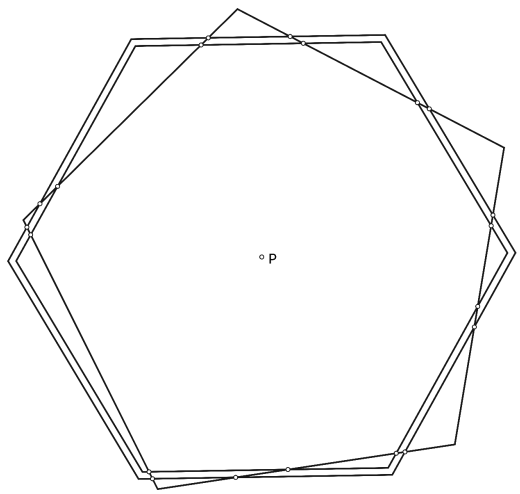
Hence one can see $9+4+5+3+4+1+3+1+1 = 31$ trees in the sector, so one can see $8 \cdot 31 + 8 = 256$ trees in total, and the fraction is therefore $\frac{256}{440} = \frac{32}{55}$.

10. Point P lies inside regular pentagon ζ, which lies entirely within regular hexagon η. A point Q on the boundary of pentagon ζ is called *projective* if there exists a point R on the boundary of hexagon η such that P, Q, R are collinear and $2019 \cdot \overline{PQ} = \overline{QR}$. Given that no two sides of ζ and η are parallel, what is the maximum possible number of projective points on ζ?

Solution. The answer is $\boxed{20}$.

We construct two dilations of η: η' is η dilated about P with a ratio of $\frac{1}{2020}$, and η'' is η dilated about P with a ratio of $-\frac{1}{2018}$ (η'' will be slightly larger than η'). If Q is a projective point and R is on the other side of Q as P, then we have $\overline{PR} = 2020\overline{PQ}$, so the corresponding point R' on η' coincides with Q. Similarly, if R is on the same side as P, then we have $\overline{PR} = 2018\overline{PQ}$, so the corresponding point R'' on η'' coincides with Q. This means that Q is projective if and only if it also lies on η' or η'' (i.e. it is an intersection between ζ and η' or η'').

To count the maximum number of intersections, note that each side of ζ can intersect η' or η'' at most twice (since a line can intersect the boundary of a convex shape at most twice). Thus there can be at most $(2 \cdot 5) \cdot 2 = 20$ intersections and so at most 20 projective points. The construction of such a configuration is not difficult:



2.3 Team Test Solutions

1. Three positive integers sum to 16. What is the least possible value of the sum of their squares?

   Solution. The answer is $\boxed{86}$.

   Note that if two of the three numbers differ by two or more, say a and b with $b - a \geq 2$, then we will have $(a+1)^2 + (b-1)^2 = a^2 + b^2 + 2(a - b + 1) < a^2 + b^2$. Since $a > 0$, we have $b = a + 2 > 2$ so $b - 1 > 0$ is a positive integer, so taking the new triplet with $a + 1$ and $b - 1$ will give a smaller sum of squares. Hence, for the triplet minimizing the sum of squares, the three numbers must all be in an interval consisting of two consecutive integers, so the triplet must be $(5, 5, 6)$ which gives $5^2 + 5^2 + 6^2 = 86$.

2. Ben is thinking of an odd positive integer less than 1000. Ben subtracts 1 from his number and divides by 2, resulting in another number. If his number is still odd, Ben repeats this procedure until he gets an even number. Given that the number he ends on is 2, how many possible values are there for Ben's original number?

   Solution. The answer is $\boxed{8}$.

   Since Ben's number reduces to 2 eventually, his initial number can only be 2 with the inverse of Ben's process applied at least once. Since the inverse of Ben's process is multiplying by 2 and then adding 1, the only possible answers are $5, 11, 23, 47, 95, 191, 383, 767$, for a total of 8 numbers.

3. Triangle ABC is isosceles, with $AB = BC = 18$ and has circumcircle ω. Tangents to ω at A and B intersect at point D. If $AD = 27$, what is the length of AC?

   Solution. The answer is $\boxed{12}$.

   Because of the tangency condition, we know that $\angle BAC = \angle BCA = \angle BAD = \angle ABD$, so $\triangle ABC$ is similar to $\triangle ADB$. Therefore, $\frac{AD}{AB} = \frac{AB}{AC}$, so $AC = \frac{18^2}{27} = 12$.

4. How many non-decreasing sequences of five natural numbers have first term 1, last term 11, and have no three terms equal?

   Solution. The answer is $\boxed{255}$.

   For any non-decreasing sequence of five natural numbers that satisfy the condition, say (a, b, c, d, e), consider $(a, b+1, c+2, d+3, e+4)$. Since $a \leq b \leq c \leq d \leq e$, we have $a < b+1 < c+2 < d+3 < e+4$, and $a = 1$ and $e + 4 = 15$, so there are $\binom{13}{3} = 286$ ways to pick $b + 1$, $c + 2$, and $d + 3$ from the integers ranging from 2 to 14. Now, suppose that we have some sequence (a, b, c, d, e) with three terms equal to a value k. If $1 < k < 11$, then we must have $b = c = d = k$ so there are 9 such sequences, one for each of value of k. Now, if $k = 1$, we must have $b = c = 1$ and d can be any value from 1 to 11, giving 11 sequences. Similarly, if $k = 11$, we must have $c = d = 11$ and b can be any value from 1 to 11, giving another 11 sequences. In total, we have $9 + 11 + 11 = 31$ sequences with three terms equal so the number of sequences satisfying the conditions is $286 - 31 = 255$.

5. Adam is bored, and has written the string "EMCC" on a piece of paper. For fun, he decides to erase every letter "C", and replace it with another instance of "EMCC". For example, after one step, he will have the string "EMEMCCEMCC". How long will his string be after 8 of these steps?

Solution. The answer is $\boxed{1534}$.

Replacing each "C" with "EMCC" will double the number of "C"'s in the string, so on the ith step, Adam will replace 2^i "C"'s with "EMCC". Each one of these steps increases the number of characters by 3, so the total length of the string after 8 steps will be $4+3\cdot(2^1+2^2+2^3+2^4+2^5+2^6+2^7+2^8) = 4+3\cdot(2^9-2) = 4+3\cdot 510 = 1534$.

6. Eric has two coins, which land heads 40% and 60% of the time respectively. He chooses a coin randomly and flips it four times. Given that the first three flips contained two heads and one tail, what is the probability that the last flip was heads?

   Solution. The answer is $\boxed{\frac{13}{25}}$.

   If the coin was the 40% one, then the probability of getting two heads and one tail would be $3\cdot(\frac{2}{5})^2\cdot(\frac{3}{5})$, as the three possibilities are HHT, HTH, and THH, each of which has probability $(\frac{2}{5})^2\cdot(\frac{3}{5})$. If the coin was the 60% one, the probability of getting two heads and one tail would be $3\cdot(\frac{2}{5})\cdot(\frac{3}{5})^2$ instead. Taking the ratio of these two probabilities, it is $\frac{2}{3}$ times as likely for the chosen coin to be the 40% one than the 60% one, so the coin is 40% with probability $\frac{2}{5}$ and 60% with probability $\frac{3}{5}$. Hence, the probability that the fourth flip is heads is $\frac{2}{5}\cdot\frac{2}{5}+\frac{3}{5}\cdot\frac{3}{5}=\frac{13}{25}$.

7. In a five person rock-paper-scissors tournament, each player plays against every other player exactly once, with each game continuing until one player wins. After each game, the winner gets 1 point, while the loser gets no points. Given that each player has a 50% chance of defeating any other player, what is the probability that no two players end up with the same amount of points?

   Solution. The answer is $\boxed{\frac{15}{128}}$.

   Since each player wins between zero and four games (inclusive), the only way for no two players to have the same amount of points is if one player wins i games for every integer $0\le i\le 4$. There are 5 ways to choose who wins all 4 of their games, 4 ways to choose which of the 4 remaining players wins their remaining 3 games, 3 ways to choose which of the 3 remaining players win their remaining 2 games, 2 ways to choose which of the 2 remaining players wins their remaining game, and 1 way to choose who loses all of their games (as there is only 1 player left). In total, this gives $5\cdot4\cdot3\cdot2\cdot1=120$ ways, and each way has probability $(\frac{1}{2})^{10}$ as each of the 10 games have a $\frac{1}{2}$ chance of ending with a specific winner. Thus, the probability that no two players end up with the same amount of points is $\frac{120}{2^{10}}=\frac{120}{1024}=\frac{15}{128}$.

8. Let $\triangle ABC$ have $\angle A=\angle B=75^\circ$. Points D, E, and F are on sides BC, CA, and AB, respectively, so that EF is parallel to BC, $EF\perp DE$, and $DE=EF$. Find the ratio of $\triangle DEF$'s area to $\triangle ABC$'s area.

   Solution. The answer is $\boxed{\frac{2}{9}}$.

   Note that the conditions given tell us that DEF is an isosceles right triangle with right angle at E. Therefore, if we drop an altitude from D onto BC, intersecting BC at G, we will have $DEFG$ be a square. Now, we can "glue" together triangles BDG and FEC by constructing a point P on BC such that $EP||DB$ (so $\triangle BDG\cong\triangle PEF$). This gives us two triangles, ADE and EPC, similar to the original triangle ABC by angle-angle similarity. Finally, note that $EC=2EF$, as CFE is a $30-60-90$ triangle, and $EF=ED=EA$, so $CE=2EA$, meaning that ADE and EPC are $\frac{1}{3}$ and

$\frac{2}{3}$ as large as ABC, respectively. Thus these two triangles take up $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$ of the area, and the remaining $\frac{4}{9}$ is taken up by our square (after we again separate EPC into EFC and BDG), which is twice the area of DEF, meaning DEF is $\frac{2}{9}$ the area of ABC.

9. Suppose a, b, c are positive integers such that $a+b = \sqrt{c^2+336}$ and $a-b = \sqrt{c^2-336}$. Find $a+b+c$.

   Solution. The answer is $\boxed{56}$.

   Squaring both equations and subtracting, we have $4ab = 672$, so $ab = 168 = 2^3 \cdot 3 \cdot 7$. Adding the squared equations gives $2(a^2+b^2) = 2c^2$ so $a^2+b^2 = c^2$. Since a, b, and c are positive integers, the only pairs (a, b) such that $ab = 168$ and a^2+b^2 is a perfect square are $(7, 24)$ and $(24, 7)$, so $c = 25$ and $a+b+c = 7+24+25 = 56$.

10. How many times on a 12-hour analog clock are there, such that when the minute and hour hands are swapped, the result is still a valid time? (Note that the minute and hour hands move continuously, and don't always necessarily point to exact minute/hour marks.)

    Solution. The answer is $\boxed{143}$.

    On a normal clock, the minute hand rotates 12 times faster (in terms of angles) than the hour hand. Therefore, if a minute hand is at an angle that is 12 times the angle of the hour hand, then the hour hand, minute hand pair is a valid time. Consider a third hand that rotates 12 times faster than the minute hand (assume that they all start out pointing at 12 simultaneously). At a given time, let the hour, minute, and third hands be at positions h, m, and t, respectively. By definition, (h, m) and (m, t) are valid times. We want to count the number of times when (m, h) also happens to be a valid time. Since there is only one possible valid time with a given hand at a given position, since (m, t) is a valid time, (m, h) can be valid if and only if $h = t$ (or differ by a whole number of revolutions, i.e. they point to the same angle).

    The third hand rotates $12 \cdot 12 = 144$ times as fast as the hour hand, and they both start facing 12, so over a 12 hour period, the hour hand makes one complete revolution while the third hand makes 144 complete revolutions. The difference of their angles is a strictly increasing function that ranges from 0 to 143 full revolutions, so there are exactly 144 times during this 12 hour interval for which the angles differ by an integer number of revolutions. However, we double count the position when all hands are facing 12, so there are 143 times for which swapping the minute and hour hands will give a valid time.

11. Adam owns a square S with side length 42. First, he places rectangle A, which is 6 times as long as it is wide, inside the square, so that all four vertices of A lie on sides of S, but none of the sides of A are parallel to any side of S. He then places another rectangle B, which is 7 times as long as it is wide, inside rectangle A, so that all four vertices of B lie on sides of A, and again none of the sides of B are parallel to any side of A. Find the length of the shortest side of rectangle B.

    Solution. The answer is $\boxed{\frac{29}{4}}$.

    First, it is not difficult to see that the sides of A are parallel to the diagonals of S, and cut S into two congruent pairs of isosceles right triangles, and the similarity ratio between the two pairs is 6. This means that the legs of the triangles are 6 and 36 respectively, and the sides of A are $6\sqrt{2}$ and $36\sqrt{2}$.

    Similarly, the sides of B cut A into two congruent pairs of right triangles with similarity ratio of 7. Suppose that the legs of the smaller right triangles are x and y (with $x < y$), then we have $7x+y = 6\sqrt{2}$

and $x + 7y = 36\sqrt{2}$. Solving this system of equations gives $x = \frac{1}{8}\sqrt{2}$ and $y = \frac{41}{8}\sqrt{2}$, so the length of the shorter side of B is $\sqrt{x^2 + y^2} = \frac{\sqrt{2(1^2+41^2)}}{8} = \frac{\sqrt{841}}{4} = \frac{29}{4}$.

12. Find the value of $\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$, where the exponents are the odd natural numbers, in increasing order.

    Solution. The answer is $\boxed{27}$.

    Let $x = \sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$. We would be able to calculate x if we can express x recursively, or in terms of itself.

    We can observe that:

$$3^2x = \sqrt{3^4 * 3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}} = \sqrt{3^3\sqrt{3^4 * 3^3\sqrt{3^5\sqrt{\cdots}}}} = \sqrt{3^3\sqrt{3^5\sqrt{3^4 * 3^5\sqrt{\cdots}}}} = \sqrt{3^3\sqrt{3^5\sqrt{3^7\sqrt{\cdots}}}}$$

    Therefore, we can express x in terms of 3^2x, by the relation $x = \sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}} = \sqrt{3 * (3^2x)}$, which gives $x = 27$.

13. Jamesu and Fhomas challenge each other to a game of Square Dance, played on a 9×9 square grid. On Jamesu's turn, he colors in a 2×2 square of uncolored cells pink. On Fhomas's turn, he colors in a 1×1 square of uncolored cells purple. Once Jamesu can no longer make a move, Fhomas gets to color in the rest of the cells purple. If Jamesu goes first, what the maximum number of cells that Fhomas can color purple, assuming both players play optimally in trying to maximize the number of squares of their color?

    Solution. The answer is $\boxed{49}$.

    Let the 9×9 grid consist of squares (x, y) for $1 \leq x, y \leq 9$. Call a square special if both its x and y coordinates are even, so there are $4 \cdot 4 = 16$ special squares in total. Note that every 2×2 square that Jamesu can place will contain exactly one of these special squares, so if Fhomas always colors in an empty special square, Jamesu will be unable to move on his 9th move as all 16 special squares will be filled in. This strategy gives Jamesu $8 \cdot 4 = 32$ squares, leaving Fhomas with the remaining $81 - 32 = 49$ squares. On the other hand, we can split the 9×9 square into 16 2×2's (that form an 8×8 square) and each one of Fhomas's moves can cover at most one of these 2×2 squares, so Jamesu is guaranteed to be able to fill in at least 8 2×2 squares. Hence, 49 squares is the maximum number of squares that Fhomas can guarantee.

14. Triangle ABC is inscribed in circle ω. The tangents to ω from B and C meet at D, and segments AD and BC intersect at E. If $\angle BAC = 60°$ and the area of $\triangle BDE$ is twice the area of $\triangle CDE$, what is $\frac{AB}{AC}$?
    Addendum on 1/26/19: $\angle A$ changed to $\angle BAC$.

    Solution. The answer is $\boxed{\sqrt{2}}$.

    Since we know that BD and CD are tangent to ω, and $\angle BAC = 60°$, it follows from inscribed angles that $\angle BCD = \angle CBD = 60°$, or, in other words, $\triangle BCD$ is equilateral. Now, we can scale up ABC about A to draw a new triangle $AB'C'$, such that D now lies on $B'C'$. This gives us the useful property that $\angle C'DC = \angle B'DB = 60°$ (since $B'C' || BC$), so by angle-angle-angle similarity, we have $\triangle BAC \sim \triangle CDC' \sim \triangle B'DB$. Now, we can see that $\frac{BE}{EC} = \frac{B'D}{DC'}$ by scaling, and $\frac{B'D}{DC'} = \frac{B'D}{BD} \cdot \frac{CD}{DC'}$ (as $BD = CD$), which in turn is equal to $\frac{BA}{CA} \cdot \frac{BA}{CA} = \frac{BA^2}{CA^2}$ by the aforementioned similarities. As we are given that $\frac{BE}{EC} = \frac{[BDE]}{[CDE]} = 2$, it follows that $\frac{BA^2}{CA^2} = 2$, so $\frac{AB}{AC} = \sqrt{2}$ as desired.

(Note that this line AE, constructed via tangents, is commonly known as the *symmedian* of triangle ABC, and indeed the general result $\frac{AB^2}{AC^2} = \frac{BE}{EC}$ is one of the symmedian's many useful properties, although this knowledge is not necessary to solve the problem.)

15. Fhomas and Jamesu are now having a number duel. First, Fhomas chooses a natural number n. Then, starting with Jamesu, each of them take turns making the following moves: if n is composite, the player can pick any prime divisor p of n, and replace n by $n - p$; if n is prime, the player can replace n by $n - 1$. The player who is faced with 1, and hence unable to make a move, loses. How many different numbers $2 \le n \le 2019$ can Fhomas choose such that he has a winning strategy, assuming Jamesu plays optimally?

Solution. The answer is $\boxed{1015}$.

We want to find which numbers are *winning*, and which are *losing*; in other words, which numbers, when given to a player, guarantee that they will eventually win through playing optimally, and which numbers it is impossible to win from (given the opponent plays optimally). Clearly a number is losing if *every* possible move on it results in a winning number, and is winning if *at least one* possible move results in a losing number, as the optimal play will be to pick that move.

We can observe that 1 and 3 are losing numbers, while 2 is a winning number. Furthermore, note that every possible move on an odd number will necessarily result in an even number (i.e. the move is somewhat forced, as should be with a losing number). From these observations, we can begin with the assumption that all odd numbers are losing, and all even numbers are winning. Now, in order for an even number n to be winning, our move must turn it into a losing, or odd, number. This is possible if n has any odd prime divisor p, since we can make the move $n \to n - p$.

Therefore, our assumption fails if n has no odd divisors, i.e. it is a power of 2 larger than 2 (note that 2 itself is still winning, since we can move to the odd number 1, which is losing). Given such a number $n = 2^k$, our only possible move is $2^k \to 2^k - 2$, so we are forced to move to a winning even number (either 2 if $k = 2$, or a non-power of 2 otherwise), so 2^k is losing.

Knowing this, we have to consider whether it is then possible for an odd number to be winning, through being able to make a move to 2^k, a losing even number. However, notice that if some odd prime p divides an odd n, then it also divides $n - p$; in other words, any move on an odd composite number cannot result in only even prime factors. Thus, the only way to move to a power of 2 from an odd number is if the number is a prime of the form $2^k + 1$. We can verify that there are four such primes in our range: $3, 5, 17, 257$; we do not count 3, however, as 2 is winning.

The final thing we must check, then, is that our assumed strategy of moving from even to odd will not accidentally land on one of these 3 winning odd primes. However, the only way to move from an even number to a prime p is if our original number was in fact $2p$. Since $p = 2^k + 1$, these evens would be $2^{k+1} + 2$. In these cases, we can instead make the move $2^{k+1} + 2 \to 2^{k+1}$, since 2^{k+1} is a losing number. Thus it is always possible for an even non-power of two to move to a losing number, so we do not have to check any further cases.

In conclusion, a number is losing if it is odd and not one of $5, 17, 257$, or if it is a power of 2 besides 2 itself. Since Fhomas wants to give Jamesu a losing number, he can choose from among $\lfloor \frac{2019}{2} \rfloor - 3 = 1006$ odds, and 9 evens (one of $\{2^2, \cdots, 2^{10}\}$), giving 1015 total choices.

2.4 Guts Test Solutions

2.4.1 Round 1

1. [6] What is the smallest number equal to its cube?

   Solution. The answer is $\boxed{-1}$.

   If $x = x^3$, then either $x = 0$ or $x^2 = 1$. The latter has two solutions $x = -1$ and $x = 1$, so $x = -1$ is the smallest number equal to its cube.

2. [6] Fhomas has 5 red spaghetti and 5 blue spaghetti, where spaghetti are indistinguishable except for color. In how many different ways can Fhomas eat 6 spaghetti, one after the other? (Two ways are considered the same if the sequence of colors are identical)

   Solution. The answer is $\boxed{62}$.

   Each of the 2^6 possible color sequences of length 6 work, except for $RRRRRR$ and $BBBBBB$ because there are only 5 red and 5 blue spaghetti, so there are a total of $2^6 - 2 = 62$ ways for Fhomas to eat the spaghetti.

3. [6] Jocelyn labels the three corners of a triangle with three consecutive natural numbers. She then labels each edge with the sum of the two numbers on the vertices it touches, and labels the center with the sum of all three edges. If the total sum of all labels on her triangle is 120, what is the value of the smallest label?

   Solution. The answer is $\boxed{7}$.

   Let the smallest label be x. Then, the three vertices have labels x, $x + 1$, and $x + 2$ so the three sides have labels $2x + 1$, $2x + 2$, and $2x + 3$. Hence, the center has label $6x + 6$ so the sum of all labels is $x + x + 1 + x + 2 + 2 \cdot (6x + 6) = 15x + 15 = 120$, so $x = 7$.

2.4.2 Round 2

4. [7] Adam cooks a pie in the shape of a regular hexagon with side length 12, and wants to cut it into right triangular pieces with angles 30°, 60°, and 90°, each with shortest side 3. What is the maximum number of such pieces he can make?

   Solution. The answer is $\boxed{48}$.

   Adam can first partition the hexagon into six equilateral triangles with side length 12, then partition each triangle into four smaller equilateral triangles with side length 6, and the partition each smaller triangle into two right triangles with the desired properties, for $6 \cdot 4 \cdot 2 = 48$ triangles in total. Because this fills the entire hexagon, it must be the greatest number of pieces.

5. [7] If $f(x) = \frac{1}{2-x}$ and $g(x) = 1 - \frac{1}{x}$, what is the value of $f(g(f(g(\cdots f(g(f(2019)))\cdots))))$, where there are 2019 functions total, counting both f and g?

Solution. The answer is $\boxed{-\frac{1}{1008}}$.

Note that $g(f(x)) = 1 - \frac{1}{f(x)} = 1-(2-x) = x-1$, so $g(f(\ldots f(g(f(2019)))\ldots)) = 2019-1009 = 1010$ where there are 2018 functions total. Hence, the expression is equal to $f(1010) = -\frac{1}{1008}$.

6. [7] Fhomas is buying spaghetti again, which is only sold in two types of boxes: a 200 gram box and a 500 gram box, each with a fixed price. If Fhomas wants to buy exactly 800 grams, he must spend \$8.80, but if he wants to buy exactly 900 grams, he only needs to spend \$7.90! In dollars, how much more does the 500 gram box cost than the 200 gram box?

   Solution. The answer is $\boxed{\$1.30}$.

   Let the 200 gram box cost a cents and the 500 gram box cost b cents. Then, $4a = 880$ so $a = 220$, but $b + 2a = 790$ so $b = 350$. Hence, $b - a = 130$ so the 500 gram box costs \$1.30 more than the 200 gram box.

2.4.3 Round 3

7. [9] Given that

$$\begin{cases} a + 5b + 9c = 1, \\ 4a + 2b + 3c = 2, \\ 7a + 8b + 6c = 9, \end{cases}$$

   what is $741a + 825b + 639c$?

   Solution. The answer is $\boxed{921}$.

   Adding 100 times the third equation, 10 times the second equation, and 1 times the first equation, we get $741a + 825b + 639c = 921$.

8. [9] Hexagon $JAMESU$ has line of symmetry MU (i.e., quadrilaterals $JAMU$ and $SEMU$ are reflections of each other), and $JA = AM = ME = ES = 1$. If all angles of $JAMESU$ are 135 degrees except for right angles at A and E, find the length of side US.

   Solution. The answer is $\boxed{2-\sqrt{2}}$.

   If we connect MS, we note that triangle EMS is an isosceles right triangle with $MS = \sqrt{2}$. Moreover, since $\angle AME = \angle ESU = 135^\circ$, we have that $\angle UMS = \frac{135^\circ}{2} - 45^\circ = 22.5^\circ$ and $\angle MSU = 135^\circ - 45^\circ = 90^\circ$, so it suffices to find US in triangle SUM given these information.

   Pick X on SM such that USX is an isosceles right triangle, then we have $\angle MUX = 22.5^\circ = \angle UMX$, which implies that $MX = UX = \sqrt{2}SX$ and thus $(1+\sqrt{2})SX = SM = \sqrt{2}$. We can then compute $US = SX = \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2}$, as desired.

9. [9] Max is parked at the 11 mile mark on a highway, when his pet cheetah, Min, leaps out of the car and starts running up the highway at its maximum speed. At the same time, Max starts his car and

starts driving down the highway at $\frac{1}{2}$ his maximum speed, driving all the way to the 10 mile mark before realizing that his cheetah is gone! Max then immediately reverses directions and starts driving back up the highway at his maximum speed, finally catching up to Min at the 20 mile mark. What is the ratio between Max's max speed and Min's max speed?

Solution. The answer is $\boxed{\frac{4}{3}}$.

Assume that it takes m minutes for Min to travel a mile at maximum speed and M minutes for Max to travel a mile at maximum speed, then it took $9m$ minutes for Min to go from 11 miles to 20 miles, and it took $2M + 10M$ minutes for Max to go from 11 miles to 10 miles and then to 20 miles. Hence $9m = 12M$, which means that $M = \frac{3}{4}m$, so Max is $\frac{4}{3}$ times as fast as Min when at max speed.

2.4.4 Round 4

10. [11] Kevin owns three non-adjacent square plots of land, each with side length an integer number of meters, whose total area is 2019m^2. What is the minimum sum of the perimeters of his three plots, in meters?

Solution. The answer is $\boxed{228}$.

Let the side lengths be a, b, c meters with $a \geq b \geq c$. It is not difficult to see that total perimeter is minimized when $a + b + c$ is minimized, which is also minimized when the three numbers are as far apart as possible. (See solution to team round problem 1.) Note that since 2019 is congruent to 3 modulo 4, while any perfect square is congruent to 0 or 1 modulo 4, we need all three of a, b, c to be odd. The largest odd a with $a^2 \leq 2019$ is $a = 43$, which leaves $b^2 + c^2 = 2019 - 1849 = 170 = 13^2 + 1^2$. Hence, the sum is minimized when $a = 43, b = 13, c = 1$, for total perimeter of $4(a + b + c) = 228$.

11. [11] Given a 5×5 array of lattice points, how many squares are there with vertices all lying on these points?

Solution. The answer is $\boxed{50}$.

We count up by side length:

- Length 1: $4^2 = 16$ squares.
- Length $\sqrt{2}$: $3^2 = 9$ squares.
- Length 2: $3^2 = 9$ squares.
- Length $\sqrt{5}$: $2(2^2) = 8$ squares (accounting for orientation).
- Length $2\sqrt{2}$: 1 square.
- Length 3: $2^2 = 4$ squares.
- Length $\sqrt{10}$: 2 squares (accounting for orientation).
- Length 4: 1 square.

Hence, there are $16 + 9 + 9 + 8 + 1 + 4 + 2 + 1 = 50$ squares.

12. [11] Let right triangle ABC have $\angle A = 90°$, $AB = 6$, and $AC = 8$. Let points D, E be on side AC such that $AD = EC = 2$, and let points F, G be on side BC such that $BF = FG = 3$. Find the area of quadrilateral $FGED$.

Solution. The answer is $\boxed{\frac{51}{5}}$.

We have $DC = 6, EC = 2, FC = 7, GC = 4$, so we have

$$[FGED] = [FDC] - [EGC] = \left(\frac{6}{8} \cdot \frac{7}{10} - \frac{2}{8} \cdot \frac{4}{10}\right)[ABC] = \frac{17}{40} \cdot \frac{6 \cdot 8}{2} = \frac{51}{5}.$$

2.4.5 Round 5

13. [13] Given a (not necessarily simplified) fraction $\frac{m}{n}$, where $m, n > 6$ are positive integers, when 6 is subtracted from both the numerator and denominator, the resulting fraction is equal to $\frac{4}{5}$ of the original fraction. How many possible ordered pairs (m, n) are there?

Solution. The answer is $\boxed{14}$.

We are given that $\frac{m-6}{n-6} = \frac{4}{5} \cdot \frac{m}{n} = \frac{4m}{5n}$. Cross-multiplying, $5mn - 30n = 4mn - 24m$, which simplifies to $mn + 24m - 30n = 0$. This can be factored as $(m - 30)(n + 24) = -720$, and m, n are both positive integers so the minus sign must come from $m - 30$. Hence, $(30 - m)(n + 24) = 720$ and $n > 6$ so $n + 24 > 30$. Thus, $30 - m < 30 - 6 = 24$ is some divisor of 720 that is less than 24, so $30 - m \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20\}$. Each of these corresponds to some integer value of m between 10 and 29, and we also have $n + 24 = \frac{720}{30-m} \geq 36$ so they also correspond to an integer value of n greater than or equal to 12. Hence, there are 14 possible ordered pairs (m, n).

14. [13] Jamesu's favorite anime show has 3 seasons, with 12 episodes each. For 8 days, Jamesu does the following: on the n^{th} day, he chooses n consecutive episodes of exactly one season, and watches them in order. How many ways are there for Jamesu to finish all 3 seasons by the end of these 8 days? (For example, on the first day, he could watch episode 5 of the first season; on the second day, he could watch episodes 11 and 12 of the third season, etc.)

Solution. The answer is $\boxed{1440}$.

It is not difficult to see that the sixth, seventh, and eighth day must be spend watching different seasons. We now separate into cases:

(a) The fifth day is the same season as the seventh day. Then we can have either $((1, 2, 3, 6), (5, 7), (4, 8))$ or $((2, 4, 6)(5, 7), (1, 3, 8))$. The former gives $3! \cdot 4! \cdot 2! \cdot 2! = 576$ ways and the latter gives $3! \cdot 3! \cdot 2! \cdot 3! = 432$ ways.

(b) The fifth day is the same season as the sixth day, then we have $((1, 5, 6), (2, 3, 7), (4, 8))$, which gives $3! \cdot 3! \cdot 3! \cdot 2! = 432$ ways.

Therefore, there are $576 + 432 + 432 = 1440$ ways in total.

15. [13] Let O be the center of regular octagon $ABCDEFGH$ with side length 6. Let the altitude from O meet side AB at M, and let BH meet OM at K. Find the value of $BH \cdot BK$.

   Solution. The answer is $\boxed{36}$.

   Note that triangle HAB is isosceles since $HA = AB$, and that AKB is isosceles by symmetry about OM. Since both triangles share the $\angle HBA$, they are similar and thus we get $\frac{HB}{AB} = \frac{AB}{KB} \Rightarrow BH \cdot BK = BA^2 = 36$.

2.4.6 Round 6

16. [15] Fhomas writes the ordered pair $(2, 4)$ on a chalkboard. Every minute, he erases the two numbers (a, b), and replaces them with the pair $(a^2 + b^2, 2ab)$. What is the largest number on the board after 10 minutes have passed?

   Solution. The answer is $\boxed{\frac{6^{1024}+2^{1024}}{2}}$.

   Let (a_n, b_n) denote the two numbers on the board after n minutes (so $a_0 = 2, b_0 = 4$). We also define $s_n = a_n + b_n$ and $d_n = a_n - b_n$, then we have

$$s_{n+1} = (a_n^2 + b_n^2) + (2a_nb_n) = (a_n + b_n)^2 = s_n^2 \quad \text{and} \quad d_{n+1} = (a_n^2 + b_n^2) - (2a_nb_n) = (a_n - b_n)^2 = d_n^2.$$

   Therefore, we have $s_{10} = s_9^2 = s_8^4 = \cdots = s_0^{1024} = 6^{1024}$ and similarly $d_{10} = d_0^{1024} = (-2)^{1024} = 2^{1024}$. Since $d_{10} > 0$, the larger of the two numbers is a, which is equal to $\frac{s_{10}+d_{10}}{2} = \frac{6^{1024}+2^{1024}}{2}$.

17. [15] Triangle BAC has a right angle at A. Point M is the midpoint of BC, and P is the midpoint of BM. Point D is the point where the angle bisector of $\angle BAC$ meets BC. If $\angle BPA = 90°$, what is $\frac{PD}{DM}$?

   Solution. The answer is $\boxed{\frac{\sqrt{3}}{2}}$.

   Since the altitude of A onto BM bisects BM at P, we must have $AB = AM$, and since $AM = BM = CM$, we get that ABM is equilateral, so $\angle B = 60°$ and $\angle C = 30°$.

   Now let $AB = 1, AC = \sqrt{3}$, and $BC = 2$. We have that $BM = 1, BP = 1/2$, and $BD = \frac{2}{1+\sqrt{3}} = \sqrt{3}-1$ from angle-bisector theorem. Therefore we have

$$\frac{PD}{DM} = \frac{BD - BP}{BM - BD} = \frac{\sqrt{3} - \frac{3}{2}}{2 - \sqrt{3}} = \left(\sqrt{3} - \frac{3}{2}\right)(2 + \sqrt{3}) = \frac{\sqrt{3}}{2}.$$

18. [15] A square is called *legendary* if there exist two different positive integers a, b such that the square can be tiled by an equal number of non-overlapping a by a squares and b by b squares. What is the smallest positive integer n such that an n by n square is *legendary*?

   Solution. The answer is $\boxed{10}$.

For ease of reference, call n *legendary* if an n by n square is legendary. WLOG assume that $a < b$. Suppose that both types of squares are used m times, then we must have $n^2 = m(a^2 + b^2)$. When $a = 1, b = 2$, we have n being a multiple of 5. If $n = 5$, then we need $m = 5$, but one cannot fit five 2 by 2 squares in a 5 by 5 square. If $n = 10$, then we need $m = 20$, and it is not difficult to see that one can fit twenty 2 by 2 squares on the top 8 by 10 rectangle and twenty 1 by 1 squares in the bottom 2 by 10 rectangle. Hence $n = 10$ is legendary.

To show that no n less than 10 is legendary, we first note that since obviously $m > 1$ (one cannot combine two squares into a larger square), we must have $n \geq 2b$ to fit at least two b by b squares, which means that $n^2 \geq 4b^2 > 2(a^2 + b^2)$ so $m \geq 3$. The case $(a, b) = (1, 2)$ has been covered above, $(a, b) = (1, 3), (2, 3), (1, 4)$ forces n to be a multiple of 10, 13, or 17, which will not yield smaller legendary numbers, and $(a, b) = (2, 4)$ is pointless because we can simply cut each square into four to yield $(a, b) = (1, 2)$. These force $n^2 \geq 3 \cdot (3^2 + 4^2) = 75$, or $n \geq 9$, so it suffices to check $n = 9$. Since $m \geq 3$, we have $a^2 + b^2 \leq 81/3 = 27$, but we also have $a^2 + b^2 \geq 25$ so $m = 3$. However, it is not difficult to check (by hand or noting the remainder modulo 4) that 27 cannot be expressed as sum of two squares, so $n = 9$ is not legendary. Therefore, 10 is the smallest legendary number.

2.4.7 Round 7

19. [18] Let $S(n)$ be the sum of the digits of a positive integer n. Let $a_1 = 2019!$, and $a_n = S(a_{n-1})$. Given that a_3 is even, find the smallest integer $n \geq 2$ such that $a_n = a_{n-1}$.

    Solution. The answer is $\boxed{5}$.

    Note that $2019! < 10000^{2019} = 10^{8076}$ so 2019! has fewer than 8076 digits. Hence, $a_2 = S(a_1) \leq 9 \cdot 8076 < 100000$, so $a_3 = S(a_2) < 9 \cdot 5 = 45$. Since a_3 is even and $S(n) \equiv n \pmod 9$, we know that $a_3 \equiv a_2 \equiv a_1 \equiv 0 \pmod 9$, so $a_3 = 0$ or $a_3 \geq 18$. However, $S(n) = 0$ if and only if $n = 0$, so $a_3 \neq 0$. Thus, $18 \leq a_3 \leq 45$ and $a_3 \equiv 0 \pmod 9$ so $a_4 = 9$. It follows that $a_5 = 9$ so 5 is the smallest value of n such that $a_n = a_{n-1}$.

20. [18] The local EMCC bakery sells one cookie for p dollars (p is not necessarily an integer), but has a special offer, where any non-zero purchase of cookies will come with one additional free cookie. With \$27.50, Max is able to buy a whole number of cookies (including the free cookie) with a single purchase and no change leftover. If the price of each cookie were 3 dollars lower, however, he would be able to buy double the number of cookies as before in a single purchase (again counting the free cookie) with no change leftover. What is the value of p?

    Solution. The answer is $\boxed{\$5.50}$.

    Suppose that without the special offer Max bought x cookies, then after the decrease in price Max would be able to by $2(x + 1) - 1 = 2x + 1$ cookies without the special offer. Since $x = \frac{27.5}{p}$ and $2x + 1 = \frac{27.5}{p-3}$, so we have the equation

    $$\frac{55}{p} + 1 = \frac{27.5}{p - 3}.$$

    This simplifies to a quadratic $p^2 + 24.5p - 165 = 0$, which can be factored as $(p - 5.5)(p + 30) = 0$. Therefore the only positive solution is $p = 5.5$.

21. [18] Let circle ω be inscribed in rhombus $ABCD$, with $\angle ABC < 90°$. Let the midpoint of side AB be labeled M, and let ω be tangent to side AB at E. Let the line tangent to ω passing through M other than line AB intersect segment BC at F. If $AE = 3$ and $BE = 12$, what is the area of $\triangle MFB$?

    Solution. The answer is $\boxed{27}$.

    Let the center of ω be O, then since AOB is a right angle and OE is perpendicular to AB, we have that $AE/OE = OE/BE$, implying the radius $OE = \sqrt{AE \cdot BE} = 6$.

    If we drop a perpendicular from A to BC intersecting at G, then note that both M and O lie on the perpendicular bisector of AF, so by symmetry since AM is tangent to ω, so is GM. This means that $F = G$; in other words, AFB is a right triangle. Since AF is equal to the diameter (which is 12), we have $BF = \sqrt{AB^2 - AF^2} = 9$, so $[MFB] = [AFB]/2 = \frac{9 \cdot 12}{4} = 27$.

2.4.8 Round 8

22. [21] Find the remainder when $1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1!$ is divided by 2019.

    Solution. The answer is $\boxed{1}$.

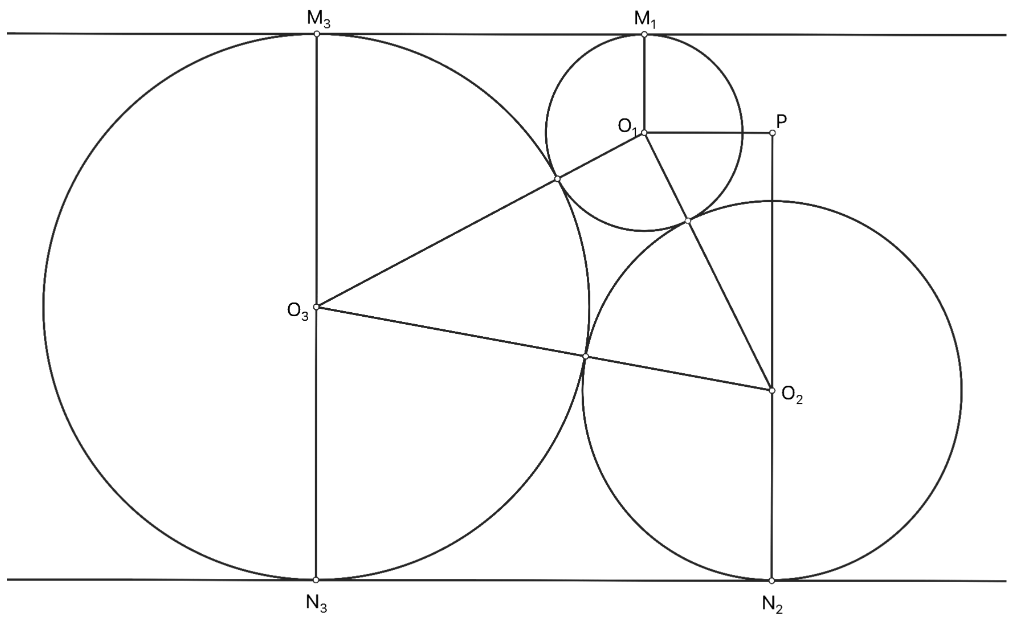
    Observe that $2019 = 3 \cdot 673$, so $n!$ is a multiple of 2019 for all $n \geq 673$. Therefore we have

$$\begin{aligned}
1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1! &\equiv -(1009 \cdot 1009! + 1008 \cdot 1008! + \cdots + 1 \cdot 1!) \\
&= -((1010! - 1009!) + (1009! - 1008!) + \cdots + (2! - 1!)) \\
&= -(1010! - 1!) \equiv -(0 - 1) \equiv 1 \pmod{2019}
\end{aligned}$$

23. [21] Two circles ω_1 and ω_2 have radii 1 and 2, respectively and are externally tangent to one another. Circle ω_3 is externally tangent to both ω_1 and ω_2. Let M be the common external tangent of ω_1 and ω_3 that doesn't intersect ω_2. Similarly, let N be the common external tangent of ω_2 and ω_3 that doesn't intersect ω_1. Given that M and N are parallel, find the radius of ω_3.

    Solution. The answer is $\boxed{2\sqrt{2}}$.

    After setting up our drawing, we can see that the condition requires that the distance between lines M and N equal the diameter of ω_3, which we will call $2r$. Now, label the center of each circle ω_n by O_n. Additionally, label the tangency points of ω_2, ω_3 to N as N_2 and N_3 respectively, and define M_1 and M_3 similarly. Finally, to make comparing lengths easier, we will construct a right triangle O_1PO_2 with right angle at P, such that $O_1P || M, N$ and $O_2P || M_3N_3$.



From here, we can see that $M_3N_3 = M_1O_1 + PO_2 + O_2N_2$, giving $PO_2 = M_3N_3 - M_1O_1 - O_2N_2 = 2r - 1 - 2 = 2r - 3$. Additionally, by the Pythagorean theorem,

$$N_3N_2 = \sqrt{O_3O_2^2 - (O_3N_3 - O_2N_2)^2} = \sqrt{(r+2)^2 - (r-2)^2} = 2\sqrt{2}\sqrt{r}$$

and similarly $M_3M_1 = \sqrt{(r+1)^2 - (r-1)^2} = 2\sqrt{r}$. Now, we can equate horizontal lengths to get $N_3N_2 = M_3M_1 + O_1P$, giving $O_1P = (2\sqrt{2} - 2)\sqrt{r}$.

Finally, we use the Pythagorean theorem on triangle O_1PO_2, giving

$$O_1O_2^2 = (2r-3)^2 + ((2\sqrt{2}-2)\sqrt{r})^2 = 4r^2 - 8\sqrt{2}r + 9$$

after expansion. However, we already know $O_1O_2 = 1+2 = 3$, so we can simplify to get $4r^2 - 8\sqrt{2}r = 0$, or $4r(r - 2\sqrt{2}) = 0$. Since $r > 0$, we must have $r = 2\sqrt{2}$.

24. [21] Mana is standing in the plane at $(0, 0)$, and wants to go to the EMCCiffel Tower at $(6, 6)$. At any point in time, Mana can attempt to move 1 unit to an adjacent lattice point, or to make a knight's move, moving diagonally to a lattice point $\sqrt{5}$ units away. However, Mana is deathly afraid of negative numbers, so she will make sure never to decrease her x or y values. How many distinct paths can Mana take to her destination?

Solution. The answer is $\boxed{5810}$.

We can observe that making a knights move is equivalent to first choosing to move either up or right, and then adding an extra "diagonal" movement (1 unit both up and right). We can therefore do casework on the number of knights moves. Since a knights move will add 3 to the sum of Mana's x and y values, and she needs to reach a destination with $x + y = 12$, we can make at most 4 knights moves.

If we do make 4 knights moves, we can ignore the extra diagonal movements, reducing our problem to moving from $(0,0)$ to $(2,2)$ via up/right movements. Since we have 4 moves, and must assign 2 of them the "up" direction, there are $\binom{4}{2} = 6$ ways to accomplish this.

If we make 3 knights moves, ignoring the diagonal movements again reduces our journey to moving from $(0,0)$ to $(3,3)$ via up/right moves. Again there are $\binom{6}{3}$ ways to choose these moves. However, we now have two different types of moves: knights moves that were *reduced* to up/right moves by subtracting off an extra diagonal movement, and moves that were already 1 unit up/right to begin with, in our original path. Out of our 6 moves, we must therefore identify which 3 are original moves, and which are reduced knights moves; there are again $\binom{6}{3}$ ways to do this, bringing the total in this case to $\binom{6}{3} \cdot \binom{6}{3} = 400$ paths.

For 2 knights moves, we again reduce to a 4×4 square, giving $\binom{8}{4}$ possible move choices, and we identify our two reduced moves in $\binom{8}{2}$ ways, giving $\binom{8}{4} \cdot \binom{8}{2} = 70 \cdot 28 = 1960$ paths.

For 1 knight move, we reduce to a 5×5 square, and assign one move as reduced, giving $\binom{10}{5} \cdot \binom{10}{1} = 2520$ paths.

Finally, for 0 knights moves, we simply have $\binom{12}{6}$ ways to assign 6 out of our 12 moves as being upwards, giving 924 paths. So in total, Mana has $6 + 400 + 1960 + 2520 + 924 = 5810$ possible paths to choose from.