Regular Expressions and Non-Regular Languages

Chanathip Namprempre

Computer Science
Reed College

Outline

Regular expressions: Examples

Language	Regular Expression
All strings starting with a 0 or a 1 followed by any number of 0s.	$(0 \cup 1)0^*$
All possible strings of 0s and 1s.	$(0 \cup 1)^*$
All strings ending with 1.	Σ^*1
All strings that either start with a 0 or end with a 1.	$(0\Sigma^*) \cup (\Sigma^*1)$

Precedence: parentheses, star, concatenation, union

Regular expressions: Examples

"A variable in C begins with a letter followed by any number of letters, digits, and underscore."

$$\text{letter}(\text{letter} \cup \text{digit} \cup \_)^*$$

"A real number (in mathematics) is some number of digits, optionally followed by a decimal point and more digits."

$$\text{digit}^*(. \cup \varepsilon)\text{digit}^+$$

These things can be described more precisely using regular expressions.

Formal definition of a regular expression

Definition

We say that R is a regular expression if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$ where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$ where R_1 and R_2 are regular expressions, or
6. (R_1^*) where R_1 is a regular expression.

- This is an inductive definition, aka a recursive definition.
- The notation $L(R)$ denotes the language defined by the regular expression R.

More Examples

Regular expression	Language
0^*10^*	$\{w \mid w$ has exactly a single 1 $\}$
$\Sigma^*1\Sigma^*$	$\{w \mid w$ has at least one 1 $\}$
$\Sigma^*001\Sigma^*$	$\{w \mid w$ contains the string 001 as a substring $\}$
$(\Sigma\Sigma)^*$	$\{w \mid w$ is a string of even length $\}$
$(\Sigma\Sigma\Sigma)^*$	$\{w \mid w$ has length a multiple of 3 $\}$

More Examples

Regular expression	Language
$01 \cup 10$	$\{01, 10\}$
$0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$	$\{w \mid w$ starts and ends with the same symbol $\}$
$(0 \cup \varepsilon)1^*$	$01^* \cup 1^*$
$(0 \cup \varepsilon)(1 \cup \varepsilon)$	$\{\varepsilon, 0, 1, 01\}$
$1^*\emptyset$	$\emptyset$
$\emptyset^*$	$\{\varepsilon\}$

More example

Let D be the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Consider the following regular expression.

$$\{+, -, \varepsilon\}(\mathrm{DD}^* \cup \mathrm{DD}^*.\mathrm{D}^* \cup \mathrm{D}^*.\mathrm{DD}^*)$$

What language do you think this regular expression describes?

Regular expressions and finite automata are EQUIVALENT.

Theorem

A language is regular if and only if some regular expression describes it.

There are two directions that need to be proved:

[$\Longleftarrow$] If a language is described by a regular expression, then it is a regular language.

[$\Longrightarrow$] If a language is a regular language, then there is a regular expression describing it.

Theorem

A language is regular if and only if some regular expression describes it.

1. If a language is described by a regular expression, then it is a regular language.

   Proof idea: Given a regular expression, use it to construct an NFA recognizing the same language.

2. If a language is a regular language, then there is a regular expression describing it.

   Proof idea: Given a DFA recognizing the language, construct a regular expression from the DFA.

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1) Converting Regular Expression to NFA

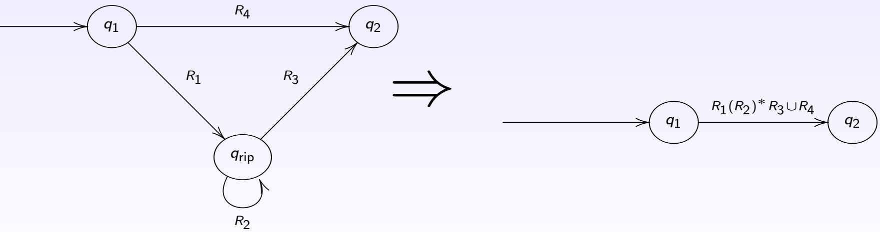
idea

Easy. Start with the recursive definition of regular expression. Construct an NFA for each case.

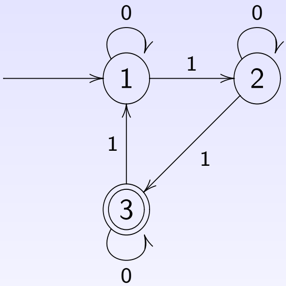
2) Converting DFA to Regular Expression

idea

1. Add a new start state S' and a new accept state F'. This gives us a GNFA.
2. Rip out one state (that isn't S' and F') at a time



2) Converting DFA to Regular Expression: Example



Try ripping in different orders, say, 1,2,3 and 2, 3, 1. Are the answers the same?

One gives you $0^*10^*1(0 \cup (10^*10^*1))^*$. The other gives you $(10^*10^*1 \cup 0)^*10^*10^*$.

Outline

Some languages are not regular! For example,

$$\{0^n1^n \mid n \geq 0\}$$

or the language

$$\{w \mid w \text{ has an equal number of 0s and 1s}\}$$

Intuitively, these language are problematic for FAs because they require infinite memory to count.

BUT the following language is regular!

$\{w \mid w$ has an equal number of occurrences of 01 and 10 as substrings$\}$

Q: How do we tell???

A: Use pumping lemma

Pumping Lemma: Intuition

Consider the following language:

$$L = \{0^n 1^n \mid n \geq 0\}$$

- Suppose towards a contradiction that L was regular.
- Suppose that L is recognized by a DFA D.
- Suppose that D has m states.
- Consider the string $w = 0^{m+1}1^{m+1}$.
- There must be a loop when D processes w.
- We can show that this means that D would accept a string $w' \notin L$.
- Thus, we have a contradiction. So L is not regular.

Pumping Lemma

Theorem (Pumping Lemma)

If L *is a regular language,* then
there is *a number* $p \geq 0$
so that for all $s \in L$ *with* $|s| \geq p$
there is *a parse of* $s = xyz$ *with* $|y| \geq 1$ *and* $|xy| \leq p$
such that for any $i \geq 0, xy^i z \in L$

We can use the Pumping Lemma to prove languages not regular.

Theorem (Contrapositive of Pumping Lemma)

If for any *number* $p \geq 0$
there exists *a string* $s \in L$ *with* $|s| \geq p$
so that for any *parse of* $s = xyz$ *with* $|y| \geq 1$ *and* $|xy| \leq p$
there exists *some* $i \geq 0$ *such that* $xy^i z \notin L$,
then L *is not regular.*

Examples

We can use the contrapositive of the Pumping Lemma to prove these languages non-regular.

1. $C = \{w \mid w \text{ has an equal number of 0s and 1s}\}$.
2. $F = \{ww \mid w \in \{0,1\}^*\}$
3. $E = \{0^i 1^j \mid i > j\}$

Notice that we cannot use the Pumping Lemma (or its contrapositive) to prove that a language is regular.