Rice's Theorem

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Properties

Consider the class of all r.e. languages.

Take a subset $\mathcal{C}$ of this class.

We can look at $\mathcal{C}$ as specifying a property P of a language.

A language L has the property P iff $L \in \mathcal{C}$.

Example

1. $\mathcal{C}_1 = \{L \mid L \text{ is non-empty }\}$
2. $\mathcal{C}_2 = \{L \mid L \text{ is r.e. }\}$

Consider a language $L = \{0, 1, 00, 11, 01\}$. Does L have each of the properties above?

Intuition behind Rice's Theorem

Consider a set $\mathcal{C}$ of r.e. languages. Recall that $L(M) = \{w \mid M \text{ accepts } w\}$.

Consider the language $L_{\mathcal{C}} = \{\langle M \rangle \mid L(M) \in \mathcal{C}\}$.

If $\mathcal{C}$ is such that $L_{\mathcal{C}}$ is nothing (try $\mathcal{C} = \emptyset$) or is everything (try $\mathcal{C} =$ the set of all r.e. languages), then $L_{\mathcal{C}}$ is clearly decidable.

These $\mathcal{C}$'s represent trivial properties.

Rice's Theorem says that, for all nontrivial properties $\mathcal{C}$, the language $L_{\mathcal{C}}$ is undecidable!

Nontriviality of a property

Definition (Triviality of a property)

A property $\mathcal{C}$ is trivial iff $\mathcal{C} = \emptyset$ or $\mathcal{C} =$ set of all r.e. languages.

Lemma (Testing for nontriviality of a property)

A property $\mathcal{C}$ is nontrivial iff there is a language in $\mathcal{C}$ and a language not in $\mathcal{C}$.

Rice's Theorem

Theorem (Rice's Theorem)

If $\mathcal{C}$ is a set of r.e. languages that is nontrivial,
then $L_{\mathcal{C}} = \{\langle M \rangle \mid L(M) \in \mathcal{C}\}$ is undecidable.

Example

1. $\mathcal{C}_1 = \{L \mid L \text{ is non-empty }\}$
2. $\mathcal{C}_2 = \{L \mid L \text{ is r.e. }\}$

What is $L_{\mathcal{C}}$ for each example above?
Which property is nontrivial?

Example: Proving E_{TM} undecidable using Rice's Theorem

Theorem

$\mathrm{E}_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ *is undecidable.*

Proof.

Let $\mathcal{C} = \{L \mid L = \emptyset\}$.

So $\mathrm{E}_{\mathrm{TM}} = L_{\mathcal{C}} = \{\langle M\rangle \mid L(M) \in \mathcal{C}\}$.

Since the language $\emptyset \in \mathcal{C}$ but the language $\{0\} \notin \mathcal{C}$, we have that $\mathcal{C}$ is nontrivial.

So E_{TM} is undecidable. □

More examples

Try using Rice's Theorem to prove that these languages are undecidable.

1. $\mathrm{AcceptsAll}_{\mathrm{TM}}$
2. $\mathrm{AcceptsSome}_{\mathrm{TM}}$
3. $\mathrm{Regular}_{\mathrm{TM}}$
4. $\{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is finite }\}$
5. $\{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is context-free }\}$
6. $\{\langle M\rangle \mid M \text{ is a TM and } M \text{ accepts all even length inputs }\}$
7. $\{\langle M\rangle \mid M \text{ is a TM and } M \text{ accepts at least 100 distinct strings }\}$

Rice's Theorem is not always applicable

You cannot use Rice's Theorem to prove these languages undecidable.

If you try, you will fail to come up with either a language property or a nontrivial language property or a property $\mathcal{C}$ such that $L_{\mathcal{C}}$ is the language in question.

1. $\{\langle M\rangle \mid\ M$ is a TM and M has at least 5 states $\}$

   [The property being considered is not a language property. In fact, this language is decidable!]

2. $\{\langle M\rangle \mid\ L(M)$ is accepted by a TM with an even number of states $\}$

   [The property being considered is trivial. In fact, this language is decidable!]

3. $\{\langle M\rangle \mid\ M$ is a TM and M accepts ε in a perfect square number of steps $\}$

   [The property being considered is not a language property. Nonetheless, this language is undecidable. Try a reduction from $\text{BlankTape}_{\text{TM}}$.]

Proof of Rice's Theorem

Theorem

If $\mathcal{C}$ is a set of r.e. languages that is nontrivial,
then $L_{\mathcal{C}} = \{\langle M \rangle \mid L(M) \in \mathcal{C}\}$ is undecidable.

Proof.

Suppose $\mathcal{C}$ is a set of r.e. languages and is nontrivial.
Case 1: Suppose $\emptyset \notin \mathcal{C}$.
We prove that $\mathrm{BlankTape}_{\mathrm{TM}} \leq_m L_{\mathcal{C}}$.
Case 2: Suppose $\emptyset \in \mathcal{C}$.
We prove that $\mathrm{BlankTape}_{\mathrm{TM}} \leq_m \overline{L_{\mathcal{C}}}$. □