CSCI/MATH 387: Computability and Complexity Fall 2022
Department of Computer Science Reed College
Quiz 1 **Instructor:** Chanathip Namprempre

Name: Solution

Student ID: ____________________

Instructions:

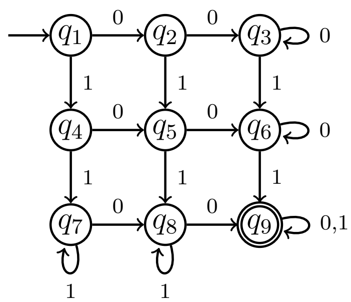
1. Write your name clearly.
2. This quiz comprises 8 problems on 8 pages.
3. Make sure that you have all the pages before you start.
4. You may write your answers with either a pen or a pencil.
5. You may access *only* these items during the quiz:
   (a) basic stationery, e.g., pen, pencil, eraser, etc.
   (b) calculator (no need to reset)
   (c) 1 sheet of letter-sized paper with notes only on *one* side and written by hand in your own handwriting
6. For each question, make clear which area is meant to be graded. Be neat.

Problem	Score
1	out of 5
2	out of 5
3	out of 5
4	out of 10
5	out of 10
6	out of 6
7	out of 7
8	out of 7
Total	out of 55

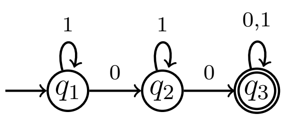
---

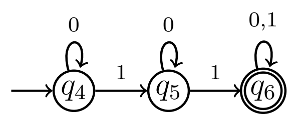
1. (5 points) Let the alphabet Σ be $\{0, 1\}$. Draw a DFA for the following language:

$$L_1 = \{w \mid w \text{ contains at least two 0's and at least two 1's}\}$$

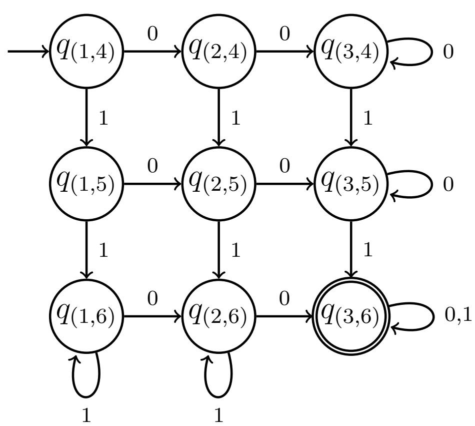


Note: You could construct this machine by applying the product construction to two DFAs. Let D_1 be a DFA for the language $\{w \mid w \text{ contains at least two 0's }\}$, and let D_2 be a DFA for the language $\{w \mid w \text{ contains at least two 1's }\}$. They are depicted as folows. The left machine is D_1 while the right machine is D_2.





Applying the product construction to D_1 and D_2, we obtain the following DFA

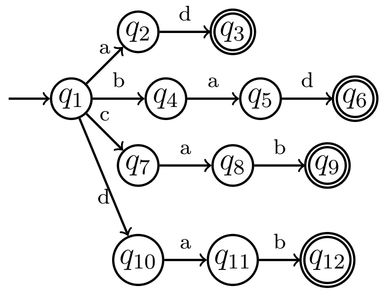


which is essentially the same DFA as the first figure shown above.

---

2. (5 points) Let the alphabet Σ be { a, b, c, d }. Draw an NFA for the following language:

$$L_2 = \{ \text{ ad, bad, cab, dab } \}$$



---

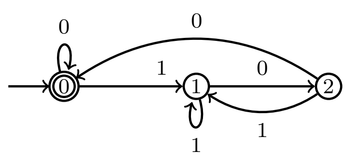
3. (5 points) Let the alphabet Σ be $\{0, 1\}$. Draw a DFA for the following language:

$$L_3 = \{w \mid w \text{ is a binary representation of a natural number that is a multiple of 4 .}\}$$

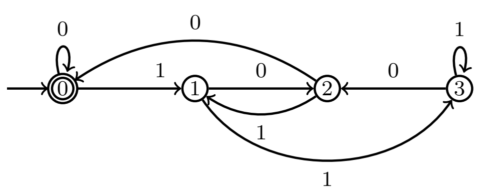
The definition of the set of "natural numbers" that I used is the set $\{0, 1, 2, \ldots\}$. However, as some students pointed out, some definitions use the set $\{1, 2, \ldots\}$. Those of you who are affected by this difference should bring your quiz and come see me.

Solution:

One could notice that a binary number representing a natural number that is a multiple of 4 always ends with 00. This means that the following DFA will suffice.

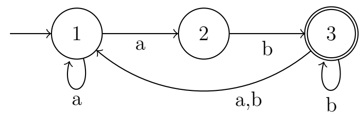


Or one could construct it by using the states to represent the remainder modulo 4.

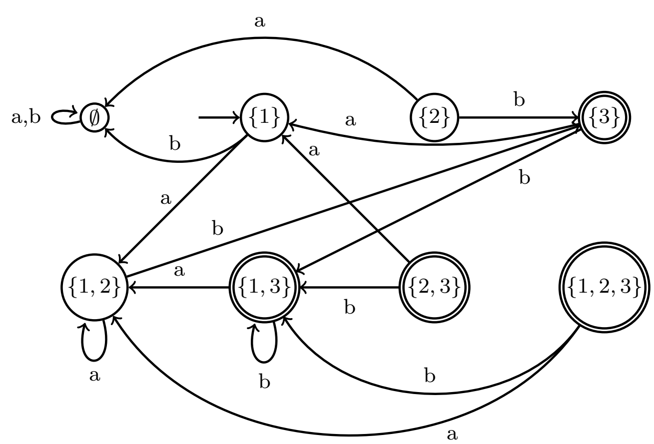


---

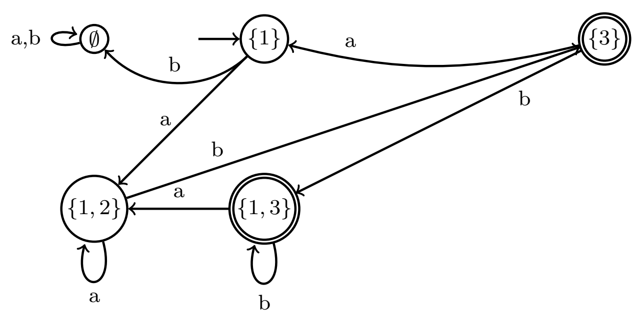
4. (10 points) Let the alphabet Σ be $\{a, b\}$. Consider the following NFA N.



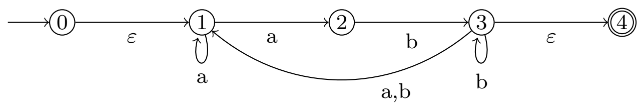
(a) (3 points) Convert N into an equivalent DFA D using the subset construction. Be sure to write down all of the possible states generated according to the subset construction.



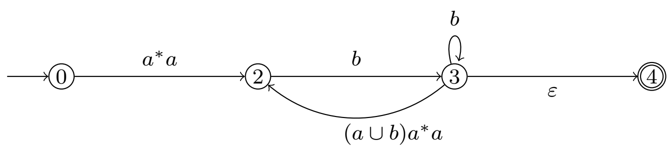
(b) (2 points) Remove all unnecessary states from D while ensuring that the resulting DFA still recognizes the same language as N.

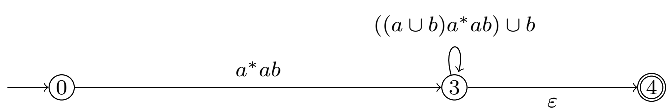


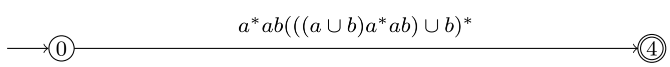
(c) (2 points) Convert N into a GNFA G that recognizes the same language.



(d) (3 points) Convert G into a regular expression by ripping the states 1, 2, and 3 *in that order*. Show your work for a possibility of receiving partial credits in the event that your final regular expression is incorrect.







---

5. (10 points) Recall the set difference operation. Let A, B be sets. Then, the set difference $A - B$ is defined as follows:

$$A - B = \{w \mid w \in A \text{ but } w \notin B\ .\}$$

State a construction that shows that the class of regular languages is closed under the set difference operation. Provide an argument as to why your construction recognizes the appropriate language. You need not provide a complete proof, but make sure that there are no undefined components in your construction.

Proof idea: First, we notice that $A - B = A \cap \overline{B}$. Then, we use the product construction except that we use the non-accept states for B instead of the accept states.

Proof:

Suppose A and B are regular languages. We assume that they are languages over the same alphabet Σ.

Let $D_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$ be the DFA recognizing the language A.

Let $D_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ be the DFA recognizing the language B.

We construct $D = (Q, \Sigma, \delta, q_0, F)$ recognizing the language $A - B$ as follows.

$$\begin{aligned} Q &= Q_A \times Q_B \\ q_0 &= (q_A, q_B) \\ F &= F_A \times (Q_B - F_B) \end{aligned}$$

and, for all $(q_1, q_2) \in Q_A \times Q_B$ and for all $a \in \Sigma$,

$$\delta((q_1, q_2), a) = (\delta_A(q_1, a), \delta_B(q_2, a))\ .$$

---

6. (6 points) Let the alphabet Σ be $\{a, b, c\}$. Consider the following language:

$$L_6 = \{a^n b^m c^r \ \mid \ n \leq m \text{ and } n, m, r \geq 0\}$$

Is L_6 a regular language? Prove your answer.

<u>Solution</u>:

L_6 is not a regular language. We use the pumping lemma for regular languages to prove our answer.

Let $p \geq 0$. Consider $s = a^p b^p c \in L$.

Let $s = xyz$ with $|y| \geq 1$ and $|xy| \leq p$.

Let i, j be integers such that $i \geq 0$ and $j \geq 1$ and $i + j \leq p$ and

$$x = a^i \quad \text{and} \quad y = a^j \quad \text{and} \quad z = a^{p-i-j} b^p c\ .$$

Consider xy^2z. We have

$$\begin{aligned} xy^2z &= a^i a^{2j} a^{p-i-j} b^p c \\ &= a^{i+2j+p-i-j} b^p c \\ &= a^{p+j} b^p c \end{aligned}$$

Since $j \geq 1$, we have that $p + j > p$. So, $xy^2z \notin L_6$. Thus, L_6 is not regular.

---

7. (7 points) Let the alphabet Σ be $\{0, 1\}$. Let A and B be finite regular languages over Σ. Consider the following statement:

$$|A \cup B| \geq |A| \quad \text{and} \quad |A \cup B| \geq |B| \; .$$

(a) (2 points) Is this statement always true? Circle the correct answer.

Yes   No

(b) (5 points) *Prove* your answer.

Solution:

We know that $|A \cup B| = |A| + |B| - |A \cap B|$. We also know that

$$|A \cap B| \leq |A|, \text{ and}$$
$$|A \cap B| \leq |B| \; .$$

Thus,

$$|A \cup B| \geq |A| + |B| - |A| = |B|, \text{ and also}$$
$$|A \cup B| \geq |A| + |B| - |B| = |A|.$$

8. (7 points) Let the alphabet Σ be $\{0,1\}$. Let A and B be finite regular languages over Σ. Consider the following statement:

$$|A \circ B| > |A| \quad \text{and} \quad |A \circ B| > |B| \; .$$

(a) (2 points) Is this statement always true? Circle the correct answer.

Yes $\qquad\qquad$ $\boxed{\text{No}}$

(b) (5 points) *Prove* your answer.

Solution:

We prove by showing a counter-example. Let $A = \{0,1\}$, and let $B = \emptyset$. This means that $|A| = 2$ and $|B| = 0$.

Thus, $A \circ B = \emptyset$, which means that $|A \circ B| = 0$.

Therefore, $|A \circ B| \not> |A|$ and $|A \circ B| \not> |B|$.
