Pushdown Automata

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Outline

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Pushdown Automata

A pushdown automata is a non-deterministic, finite state machine with access to a stack.

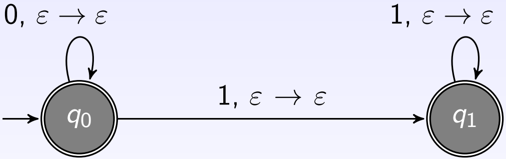
Definition

A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, and F are all finite sets, and

1. Q is a set of states
2. Σ is the input alphabet
3. Γ is the stack alphabet
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to \mathcal{P}(Q \times \Sigma_\varepsilon)$ is the transition function
5. $q_0 \in Q$ is the start state
6. $F \subseteq Q$ is the set of accept states

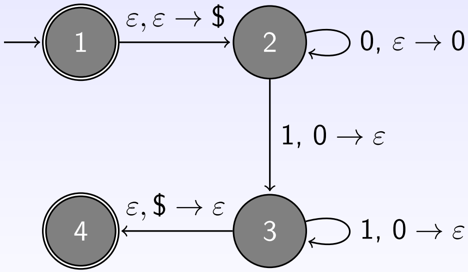
Example 1

$$L_1 = L(0^*1^*)$$



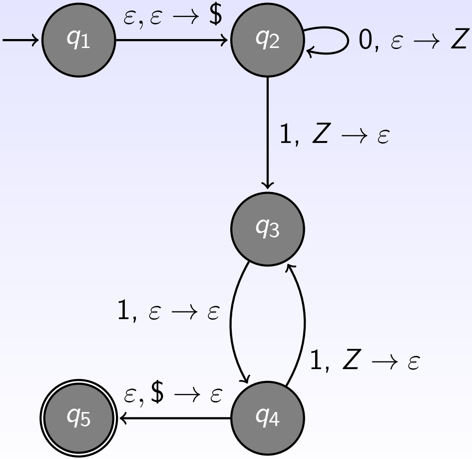
Example 2

$$L_2 = \{0^n 1^n \mid n \geq 0\}$$



Example 3

$$L_3 = \{0^n 1^{2n} \mid n \geq 0\}$$



Formal definition of computation

Let $m \in Z^+$, $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be a pushdown automaton.

Definition

M accepts w if w can be written as $y_1 y_2 \dots y_m$ where each $y_i \in \Sigma_\varepsilon$, and a sequence of states $r_0, r_1, \dots, r_m \in Q$ and string $s_o, s_1, \dots, s_m \in \Gamma^*$ exist with the following three conditions:

1. [M starts in the start state.] $r_0 = q_0$ and $s_0 = \varepsilon$,
2. [M moves according to δ, state, stack, and input symbol.] For $i = 0, \dots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, y_{i+1}, a)$ where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$,
3. [M ends up in an accept state.] $r_m \in F$.

Outline

Equivalence of PDA and CFG

Theorem: PDA and CFG are equivalent

A language is context free if and only if some pushdown automaton recognizes it.

There are two directions that need to be proved:

[$\Longrightarrow$] If a language is a context free language, i.e., has a CFG generating it, then there is a PDA recognizing it.

(Easy)

[$\Longleftarrow$] If a language has a PDA recognizing it, then it has a CFG generating it.

(Hairy)

Constructing a PDA from a CFG

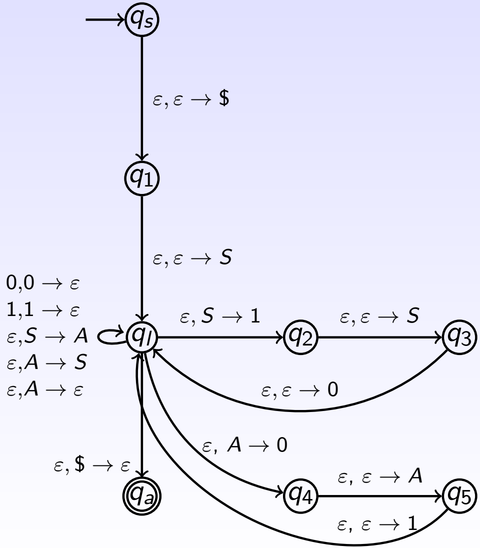
1. Place the marker symbol \$ and the start variable on the stack
2. Repeat forever
    1. If the top of stack is a variable A, nondeterministically select one of the rules for A and substitute A by the string on the right-hand side of the rule, last symbol first.
    2. If the top of stack is a terminal symbol a, read the next symbol from the input and compare it to a. If they match, repeat. Otherwise, reject on this branch of nondeterminism.
    3. If the top of stack is the symbol \$, enter the accept state. Doing so accepts the input if it has all been read.

Try it

$$\begin{array}{rcl} S & \rightarrow & 0S1 \mid A \\ A & \rightarrow & 1A0 \mid S \mid \varepsilon \end{array}$$

Constructing a PDA from a CFG: Example

$$S \to 0S1 \mid A$$
$$A \to 1A0 \mid S \mid \varepsilon$$



Constructing a PDA from a CFG: Example (cont.)

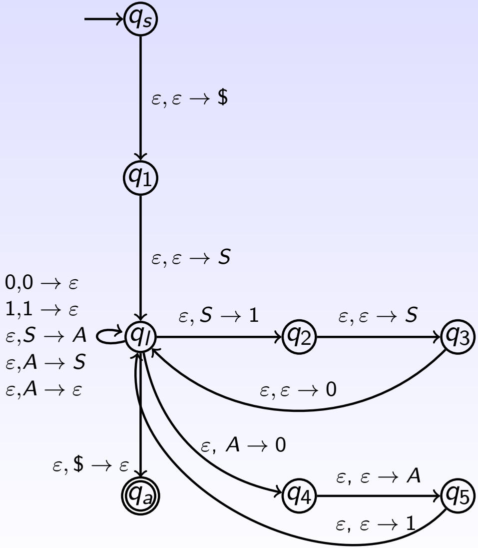
$S \to 0S1 \mid A$

$A \to 1A0 \mid S \mid \varepsilon$

Consider an input 0101

$S \to 0\underset{*}{S}1 \to 0\underset{*}{A}1 \to 01\underset{*}{A}01 \to 0101$

$$
\begin{aligned}
(q_s, 0101, \varepsilon) &\to (q_1, 0101, \$) \\
&\to (q_l, 0101, S\$) \\
&\to (q_2, 0101, 1\$) \\
&\to (q_3, 0101, S1\$) \\
&\to (q_l, 0101, 0S1\$) \quad * \\
&\to (q_l, 101, S1\$) \\
&\to (q_l, 101, A1\$) \quad * \\
&\to (q_4, 101, 01\$) \\
&\to (q_5, 101, A01\$) \\
&\to (q_l, 101, 1A01\$) \quad * \\
&\to (q_l, 01, A01\$) \\
&\to (q_l, 01, 01\$) \\
&\to (q_l, 1, 1\$) \\
&\to (q_l, \varepsilon, \$) \\
&\to (q_a, \varepsilon, \varepsilon)
\end{aligned}
$$



Constructing a PDA from a CFG: More examples

Once you write a PDA, try running it on some input strings. For each input string, compare the derivation from the grammar and the execution via the PDA.

1. $$S \to 0AA$$
   $$A \to 0S \mid 1S \mid 0$$

2. $$S \to 0S1$$
   $$S \to 0 \mid 1 \mid \varepsilon$$

Converting a PDA into a CFG

This is a bit hairy and not too interesting. We skip!