Complexity Theory I

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Outline

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Complexity Theory

Complexity theory focuses on decidable problems.

We talk about time complexity first.

Some problems take little time to solve. Some require much more time.

We are interested in categorizing decidable problems into different groups according to the amount of time required to solve them.

A simple example

Consider the following language:

$$L = L(0^*1^*)$$

Certainly, L is decidable. (In fact, it can be recognized by a DFA.)

Questions

- Can you write a deterministic TM recognizing L?
- How much time does your TM take?

Try it on these inputs: 01, 10000, 0000011

Clearly, the running time of your TM varies depending on the length of the input.

Question

Let n be the input length, how much time does your TM take?

Running time of a TM

Definition

Let M be a deterministic TM that halts on all inputs. The running time or time complexity of M is the function $f : \mathrm{N} \to \mathrm{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n.

Let $f : \mathrm{N} \to \mathrm{N}$, and let M be a TM.
The following statements are equivalent:

- $f(n)$ is the running time of M
- M runs in time $f(n)$
- M is an $f(n)$ time TM

Asymptotic Analysis

In this course, we only care about the running time of a TM on large inputs. This means that we ignore constant terms.

So we use asymptotic analysis when analyzing the running time of our TMs.

In this type of analysis, we use asymptotic notation.

Big-O Notation

Definition

Let f and g be two functions from N to R^+. We say that $f(n) = O(g(n))$ if there exist positive integers c and n_0 so that, for every integer $n \geq n_0$,

$$f(n) \leq cg(n) \; .$$

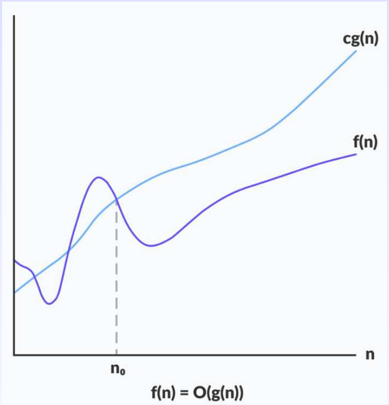
Let $f, g : \mathrm{N} \to \mathrm{R}^+$. The following statements are equivalent:

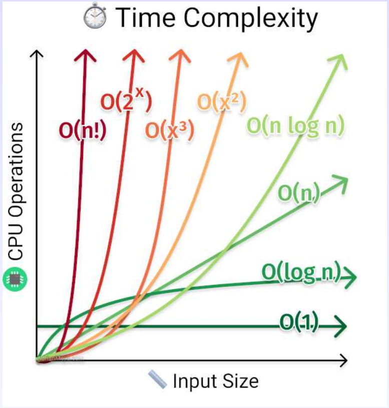
- $f(n) = O(g(n))$
- $g(n)$ is an (asymptotic) upper bound for $f(n)$

Intuitively, $f(n) = O(g(n))$ means that f is no more than g if we disregard differences up to a constant factor.

Examples

$$5n^3 + 2n^2 + 6 = O(n^3) \; ; \; 5n^3 + 2n^2 + 6 = O(n^4) \; ; \; 5n^3 + 2n^2 + 6 \neq O(n^2)$$





Sources:

`https://www.programiz.com/dsa/asymptotic-notations`

`https://thecodingbay.com/learn-everything-about-big-o-notation/`

Small-O Notation

Definition

Let f and g be two functions from N to R^+. We say that $f(n) = o(g(n))$ if there, for any real number $c > 0$, there exists a positive integer n_0 so that, for every integer $n \geq n_0$,

$$f(n) < cg(n) .$$

Intuitively, $f(n) = O(g(n))$ means that f is strictly less than g if we disregard differences up to a constant factor.

Examples

$$n \log n = o(n^2) \; ; \; \sqrt{n} = o(n) \; ; \; n \neq o(n)$$

Deterministic Time Complexity Class: Definition

We can group problems by the time it takes a deterministic TM to solve them.

Definition

Let $t : \mathrm{N} \to \mathrm{N}$ be a function. We define the time complexity class $\mathrm{TIME}(t(n))$ as follows:

$$\mathrm{TIME}(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time TM }\} .$$

Example

$$L(0^*1^*) \in \mathrm{TIME}(n)$$

Can it be the case that $L \in \mathrm{TIME}(\log n)$?

[This is like asking whether it was possible to construct a TM deciding L in time $O(\log n)$.]

Deterministic Time Complexity Class: More examples

Example

$$\{0^n1^n \mid n \geq 0\} \in \mathrm{TIME}(n)$$

Can you write a deterministic TM M deciding L?

- Does your machine M use 1 tape? What is its running time?
- Does your machine M use 2 tapes? What is its running time?

Bottom line

If you change the computation model, you can often change the running time.

Running time of 1-tape TM compared to multitape TM

Usually, we can make things faster if we use more tapes.

BUT the improvement will always be within a polynomial factor.

Here is the reason:

Theorem

Let $t(n) : \mathrm{N} \to \mathrm{N}$ *where* $t(n) \geq n$*. Then,*

every k*-tape TM running in time* $t(n)$ *can be simulated by a single-tape, deterministic TM running in time* $O(k \cdot t^2(n))$.

Deterministic models are polynomially equivalent

In complexity theory, polynomial differences in running time are considered to be small whereas exponential differences are considered to be large.

Examples: Compare n^3 and 2^n. Try with $n = 1000$.

Fact

All reasonable deterministic computational models are polynomially equivalent.

Translation: Any one of them can simulate another with only a polynomial increase in running time.

When we focus on problems that are solvable in polynomial time, our results do not depend on the model of computation being used.

Bottom line

We can focus on fundamental properties of computation rather than the exact model in which the computation is performed.

Running time of Nondeterministic TM

Definition

Let N be a nondeterministic TM that is a decider. The running time of N is the function $f : \mathrm{N} \rightarrow \mathrm{N}$ where $f(n)$ is the maximum number of steps that N uses on any branch of its computation on any input of length n.

Example

$$\{ww^{\mathcal{R}} \mid w \in \{0,1\}^*\}$$

Can you write an NTM to solve this problem? What is its running time?

Nondeterministic TM can give us upto exponential speedup

Theorem

Let $t(n) : \mathrm{N} \rightarrow \mathrm{N}$ where $t(n) \geq n$. Then,

every NTM running in time $t(n)$ can be simulated by a single-tape, deterministic TM running in time $2^{O(t(n))}$.

For proving this, we use the following fact.

Fact

A full b-ary tree with height h has a total of $\sum_{i=0}^{h} b^i = \frac{b^{h+1}-1}{b-1}$ nodes.

Non-Deterministic Time Complexity Class: Definition

We can group problems by the time it takes NTMs to solve them.

Definition

Let $t : \mathrm{N} \to \mathrm{N}$ be a function. We define the time complexity class $\mathrm{NTIME}(t(n))$ as follows:

$$\mathrm{NTIME}(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time NTM}\} .$$

Outline

Complexity class P

Definition (P)

P is the class of languages that are decidable in polynomial time on a deterministic single-tape TM. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}((n^k)) \; .$$

We are interested in P because it is robust under composition.

[If a subroutine runs in polynomial-time and we call it polynomial number of times, we still run in polynomial time.]

Some languages in P

1. Let PATH $= \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$.

$$\text{PATH} \in \text{P}$$

2. Let RELPRIME $= \{\langle x, y\rangle \mid x$ and y are relatively prime $\}$.

$$\text{RELPRIME} \in \text{P}$$

3. Let L be a context-free language.

$$\text{L} \in \text{P}$$

Intuition

In complexity theory, problems in P are considered "easy" to solve.

Theorem

PATH $\in$ P

Proof.

$M =$ "On input $\langle G, s, t\rangle$ where G is a directed graph with nodes s and t:

1. Place a mark on node s
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of G. If an edge (a, b) is found going from a marked node a to an unmarked node b, mark node b.
4. If t is marked, accept. Otherwise, reject.

$\square$

Closure properties of P

Theorem

Let L, L_1, L_2 be languages.

1. *If $L \in \mathrm{P}$, then $\overline{L} \in \mathrm{P}$.*
2. *If $L_1, L_2 \in \mathrm{P}$, then $L_1 \cup L_2 \in \mathrm{P}$.*
3. *If $L_1, L_2 \in \mathrm{P}$, then $L_1 \cap L_2 \in \mathrm{P}$.*

Outline

The class NP

Definition (Complexity class NP)

NP is the class of languages that are decidable in polynomial time on a non-deterministic TM. In other words,

$$\mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k)\ .$$

Alternative Definition for NP: polynomial time verifier

Definition (Polynomially Verifiable Languages)

A verifier for a language A is an algorithm V where

$$A = \{x \mid V \text{ accepts } \langle x, c\rangle \text{ for some string } c\}.$$

A polynomial time verifier is a verifier that runs in polynomial time in the length of its input.

A polynomially verifiable language is a language with a polynomial time verifier.

Alternative Definition for NP

Definition (Complexity class NP)

A language L is in NP iff there exists a polynomial time verifier V.

That is, there exists $V(\cdot, \cdot)$ and a polynomial $p(\cdot)$ such that

1. For any input x, c, algorithm $V(x, c)$ halts in $p(|x|)$ steps.
2. If $x \in L$, then there exists c such that $V(x, c)$ accepts, and
3. If $x \notin L$, then for all c, algorithm $V(x, c)$ rejects.

The value c is called a certificate (or a witness or a proof) of membership of $x \in L$.

Equivalence between the two definitions of NP

The following theorem states that these two views are equivalent.

Theorem

A language L is polynomially verifiable if and only if it is decided by some non-deterministic polynomial time TM.

Proof.

[$\Rightarrow$] L has a polynomial verifier V. We construct an NTM N running on input w by first guessing a certificate c and then simulating V on $\langle w, c\rangle$.

[$\Leftarrow$] There is an NTM N deciding L. We construct a polynomial verifier V as follows: on input $\langle w, c\rangle$, V simulates N using c to figure out which path in N's execution tree to take. □

Examples of NP problems

1. HAMPATH $= \{\langle G, s, t\rangle \mid G$ is a directed graph with a hamiltonian path from s to $t\}$.

   Certificate: a hamiltonian path from s to t
   [hamiltonian path = path that goes through each node once.]

2. COMPOSITES $= \{x \mid x = pq$, for integers $p, q > 1\}$.

   Certificate: p and q

Theorem

HAMPATH $\in$ NP

Proof.

"On input $\langle G, s, t \rangle$,

1. Let m be the number of nodes in G
2. Non-deterministically write a sequence $v_1, v_2, \ldots, v_m$ of m nodes
3. Accept if
    1. $v_1 = s$
    2. $v_m = t$
    3. each $(v_i, v_i + 1)$ is an edge in G and no v_i repeats.
4. Reject if any condition fails.

$\square$

Closure properties of NP

Theorem

Let L_1, L_2 be languages.

1. *If $L_1, L_2 \in \mathrm{NP}$, then $L_1 \cup L_2 \in \mathrm{NP}$.*
2. *If $L_1, L_2 \in \mathrm{NP}$, then $L_1 \cap L_2 \in \mathrm{NP}$.*

<u>Question</u>: Do you think the class NP is closed under complement?

The co-class of NP: The class coNP

coNP is the class of languages whose complements are in NP.

Definition

$L \in$ coNP iff $\overline{L} \in$ NP.

If NP is closed under complement, then NP $=$ coNP.

But this is an open question (i.e. there is no answer either way). Why do you think it's so hard to answer?