Non-Context-Free Languages and the Pumping Lemma

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Outline

Some languages are not context-free! For example,

$$\{0^n 1^n 0^n \mid n \geq 0\}$$

or the language

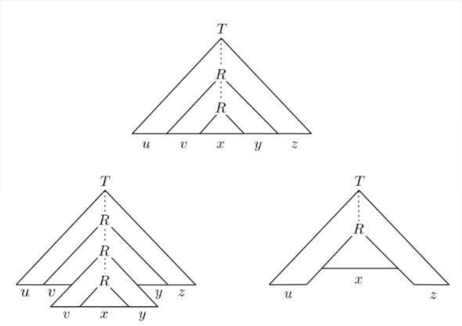
$$\{ww \mid w \in (0 \cup 1)^*\}$$

Intuitively, these language are problematic for PDAs because the latter only has access to a stack.

Pumping Lemma

Theorem (Pumping Lemma)

If L is a context-free language, then
there is a number $p \geq 0$
so that for all $s \in L$ with $|s| \geq p$
there is a parse of $s = uvxyz$ with $|vy| \geq 1$ and $|vxy| \leq p$
such that for any $i \geq 0, uv^i xy^i z \in L$



Applying pumping lemma to a CFL: Example

S	$\rightarrow$	NP VP
NP	$\rightarrow$	Det Nom
VP	$\rightarrow$	V NP
Nom	$\rightarrow$	Adj Nom
Nom	$\rightarrow$	N

Try deriving

- the angry bear chased the frightened little squirrel
- the angry crazy bear chased the frightened little squirrel
- the bear chased the frightened little squirrel

Example from Stuart Shieber

https://dash.harvard.edu/bitstream/handle/1/2026618/Shieber_EvidenceAgainst.pdf?sequence=2

Pumping Lemma

Theorem (Pumping Lemma)

If *L is a context-free language,* then
there is *a number* $p \geq 0$
so that for all $s \in L$ *with* $|s| \geq p$
there is *a parse of* $s = uvxyz$ *with* $|vy| \geq 1$ *and* $|vxy| \leq p$
such that for any $i \geq 0, uv^ixy^iz \in L$

We can use the Pumping Lemma to prove languages not context-free.

Theorem (Contrapositive of Pumping Lemma)

If for any *number* $p \geq 0$
there exists *a string* $s \in L$ *with* $|s| \geq p$
so that for any *parse of* $s = uvxyz$ *with* $|vy| \geq 1$ *and* $|vxy| \leq p$
there exists *some* $i \geq 0$ *such that* $uv^ixy^iz \notin L$,
then *L is not context-free.*

Pumping Lemma: Intuition

Consider the following language:

$$L = \{0^n 1^n 0^n \mid n \geq 0\}$$

1. Suppose towards a contradiction that L was context-free.
2. Suppose that L is recognized by a grammar G.
3. Suppose that G has $|V|$ variables and the most number of symbols on the right hand side of any rules is $b \geq 2$.
4. Let $p = b^{|V|+1}$.
5. Consider the string $w = 0^p 1^p 0^p$.
6. Since $p = b^{|V|+1} \geq b^{|V|} + 1$, the parse tree must be at least of height $|V| + 1$.
7. There must be a repeat of at least one variable when G derives w.
8. We can show that this means that G would derive a string $w' \notin L$.
9. Thus, we have a contradiction. So L is not context-free.

Examples

We can use the contrapositive of the Pumping Lemma to prove these languages non-context-free.

1. $L_1 = \{0^n 1^n 0^n \mid n \geq 0\}$
2. $L_2 = \{ww \mid w \in \{0,1\}^*\}$
3. $L_3 = \{r\#s \mid r, s \in \{0,1\}^* \text{ and } r \text{ is a substring of } s\}$

Notice that we cannot use the Pumping Lemma (or its contrapositive) to prove that a language is context-free.