Public Key Encryption

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Agenda: Public Key Encryption

1. Basic math
2. Syntax of PKE
3. Security Definitions
4. Constructions

Outline

Basic Math

RSA Math

Syntax of Public Key Encryption

Security Definitions of PKE

PKE Schemes

Basic math

- $\mathbf{Z}_N = \{0, \ldots, N-1\}$
- $\mathbf{Z}_N^* = \{x \in \mathbf{Z}_N | x \text{ and } N \text{ are co-prime.}\}$
- $\mathbf{Z}_N^* =$ set of invertible elements in $\mathbf{Z}_N$
- Easy operations modulo N: addition, multiplication, exponentiation, inversion

Basic math

Group theory basic

Suppose G is a group with operation $\cdot$.

- Order of $G = |G|$
- Order of $x \in G$:
  $$\mathrm{ord}_G(x) = |\langle x \rangle| = \{\text{ smallest } a > 0 \text{ such that } x^a = 1 \text{ in } G\} \quad .$$
  So,
  $$x^{\mathrm{ord}_G(x)} = 1.$$
- For any $x \in G$,
  $$\langle x \rangle = \{x^0, x^1, \ldots, x^{\mathrm{ord}_G(x)-1}\} \text{ is a subgroup of } G.$$
- For any subgroup S of G, $|S| \Big| |G|$.
- For any element $x \in G$, $\mathrm{ord}_G(x) \Big| |G|$
- For any element $x \in G$, $x^i = x^{i \bmod |G|}$

Let p be a prime.

- $\mathbf{Z}_p^* = \{1, 2, \ldots, p-1\}$
- $\mathbf{Z}_p^*$ is cyclic.
- Fermat's theorem: $\forall x \in \mathbf{Z}_p^*$, $x^{p-1} = 1$ in $\mathbf{Z}_p$.

  Fermat's theorem can be used to test whether a number p is prime. If p is chosen at random, there's a very small chance that p would pass this test yet isn't prime.
- There's a generator $g \in \mathbf{Z}_p^*$ such that $\{1, g, g^2, g^3, \ldots, g^{p-2}\} = \mathbf{Z}_p^*$.
- Not everything in $\mathbf{Z}_p^*$ is a generator.
- Lagrange Thm: $\forall x \in \mathbf{Z}_p^*, \mathrm{ord}_p(x) \Big| p-1$

Recall definition

$\mathbf{Z}_N^* =$ set of invertible elements in $\mathbf{Z}_N$

Let N be an integer.

- Euler's phi function: $\phi(N) = |Z_N^*|$
- Euler's theorem: For any integer N, $\forall x \in \mathbf{Z}_N^*, x^{\phi(N)} = 1$.
- If $N = pq$ where p and q are distinct primes, then $\phi(N) = (p-1)(q-1)$.

Let N be an integer.

- Solving linear equations in $\mathbf{Z}_N$ is easy. For $a, b \in \mathbf{Z}_N$,
$$ax + b = 0$$
Solution: $x = -b \cdot a^{-1}$ in $\mathbf{Z}_N$. Use Euclidean algorithm.

- Solving higher degree polynomial in $\mathbf{Z}_N$ is more complicated.

Let p be an odd prime, and let g be a generator for $\mathbf{Z}_p^*$.

- In $\mathbf{Z}_p^*$, the map $x \to x^2$ is a 2-to-1 function.
- QR: $x \in \mathbf{Z}_p$ is a quadratic residue (QR) iff it has a square root in $\mathbf{Z}_p$
- Legendre/Jacobi symbol of x over $p = J_p(x) = x^{(p-1)/2}$
- $\forall x, y \in \mathbf{Z}_p^*, a, b \in \mathbf{Z}_{p-1}$,

$$
\begin{aligned}
&J_p(x) \in \{1, -1\} \\
&x \text{ is a QR iff } J_p(x) = 1 \\
&J_p(xy) = J_p(x) \cdot J_p(y) \\
&J_p(x^{-1}) = J_p(x) \\
&J_p(g^{ab}) = 1 \text{ iff } J_p(g^a) = 1 \text{ or } J_p(g^b) = 1
\end{aligned}
$$

Try it with $p = 11$

a	1	2	3	4	5	6	7	8	9	10
$a^2 \bmod 11$										
$J_{11}(a)$										
a^{-1}										
$J_{11}(a^{-1})$										

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RSA

Notation

Let $N, e \geq 1$ be integers.
The RSA function associated to N, e is $RSA_{N,e} : \mathbf{Z}_N^* \to \mathbf{Z}_N^*$ defined by

$$RSA_{N,e}(x) = x^e \bmod N \text{ for all } x \in \mathbf{Z}_N^* \ .$$

Claim

Let $N \geq 2$, $e, d \in \mathbf{Z}_{\phi(N)}^*$ be integers such that

$$ed \equiv 1 \pmod{\phi(N)} \ .$$

i.e., $[d = e^{-1}$ in $\mathbf{Z}_{\phi(N)}^*]$. Then,

$RSA_{N,e}$ is a permutation over $\mathbf{Z}_N^*$; $\quad RSA_{N,d}^{-1} = RSA_{N,e}$;
$RSA_{N,d}$ is a permutation over $\mathbf{Z}_N^*$; $\quad RSA_{N,e}^{-1} = RSA_{N,d}$.

RSA : $RSA_{N,e}$ and $RSA_{N,d}$ are inverses of each other

Let $x \in \mathbf{Z}_N^*$

Then,

$$\begin{aligned} RSA_{N,d}(RSA_{N,e}(x)) &= (x^e)^d \\ &= x^{ed \bmod \phi(N)} \\ &= x^1 = x \end{aligned}$$

The second equation holds because $\phi(N)$ is the order of the group $\mathbf{Z}_N^*$.

Similarly, we can show that $RSA_{N,e}(RSA_{N,d}(y)) = y$ for all $y \in \mathbf{Z}_N^*$.

RSA (cont.)

Notice

$RSA_{N,e}(\cdot)$ and $RSA_{N,d}(\cdot)$ are efficiently computable.

Intuition for security : ***one-wayness of RSA***

Given N, e, y, it's hard to compute $RSA_{N,e}^{-1}(y)$ without d.

Modulus Generator

Definition

A modulus generator with associated security parameter $k \geq 2$ is a randomized algorithm taking no inputs & returning integers N, p, q such that

1. p, q are distinct, odd primes.
2. $N = pq$
3. $2^{k-1} \leq N < 2^k$

RSA Generator : K_{rsa}

Definition

An RSA generator with associated security parameter $k \geq 2$ is a randomized algorithm taking no inputs & returning $((N, e), (N, p, q, d))$ such that N, e, p, q, d are integers and

1. p, q are distinct, odd primes.
2. $N = pq$
3. $2^{k-1} \leq N < 2^k$
4. $e, d \in \mathbf{Z}^*_{\phi(N)}$
5. $ed \equiv 1 \pmod{\phi(N)}$

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Syntax of PKE

Syntax

A public key encryption scheme $\mathsf{PKE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ is a triple of algorithms.

alg	input	output	notation	maybe randomized?	maybe stateful?
$\mathcal{K}$	-	key pk, sk	$(pk, sk) \xleftarrow{\$} \mathcal{K}$	yes	no
$\mathcal{E}$	$(pk, sk) \in Keys(\mathsf{PKE})$ $M \in \{0,1\}^*$	ciphertext $C \in \{0,1\}^* \cup \{\perp\}$	$C \xleftarrow{\$} \mathcal{E}_{pk}(M)$	yes	yes
$\mathcal{D}$	$(pk, sk) \in Keys(\mathsf{PKE})$ $C \in \{0,1\}^*$	plaintext $M \in \{0,1\}^* \cup \{\perp\}$	$M \leftarrow \mathcal{D}_{sk}(C)$	no	no

Correctness

For all $(pk, sk) \in Keys(\mathsf{PKE})$ and all $M \in \{0,1\}^*$,

$$\Pr\left[\, C = \perp \text{ OR } \mathcal{D}_{sk}(C) = M \;:\; C \xleftarrow{\$} \mathcal{E}_{pk}(M) \,\right] = 1 \,.$$

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Privacy notion for PKE: Indistinguishability against CPA

Let $\mathsf{PKE} = (\mathsf{KG}, \mathcal{E}, \mathcal{D})$ be a PKE scheme, and let A be an adversary with access to an oracle.

Subroutine `Initialize` $\quad b \xleftarrow{\$} \{0,1\}$; $(pk, sk) \xleftarrow{\$} \mathsf{KG}$ $\quad$Return pk Subroutine `Enc`(M_0, M_1) $\quad$If $\lvert M_0 \rvert \neq \lvert M_1 \rvert$ then return $\perp$ $\quad$Return $\mathsf{Enc}_{pk}(M_b)$ Subroutine `Finalize`(d) $\quad$Return $(d = b)$	Experiment $\mathbf{Exp}^{\mathrm{ind\text{-}cpa}}_{\mathsf{PKE}}(A)$ $\quad pk \xleftarrow{\$}$ `Initialize` $\quad d \xleftarrow{\$} A^{\mathtt{Enc}}(pk)$ $\quad$Return `Finalize`(d)

ind-cpa advantage of A mounting a CPA against PKE:

$$\mathbf{Adv}^{\mathrm{ind\text{-}cpa}}_{\mathsf{PKE}}(A) = 2 \cdot \Pr\left[\, \mathbf{Exp}^{\mathrm{ind\text{-}cpa}}_{\mathsf{PKE}}(A) \Rightarrow \mathsf{true} \,\right] - 1 \; .$$

Privacy notion for PKE: Indistinguishability against CCA

Let $\mathsf{PKE} = (\mathsf{KG}, \mathcal{E}, \mathcal{D})$ be a PKE scheme, and let A be an adversary with access to an oracle.

Subroutine $\mathtt{Initialize}$

$b \xleftarrow{\$} \{0,1\}$; $(pk, sk) \xleftarrow{\$} \mathsf{KG}$

$S \leftarrow \emptyset$; Return pk

Subroutine $\mathtt{Enc}(M_0, M_1)$

If $|M_0| \neq |M_1|$ then return $\perp$

$C \xleftarrow{\$} \mathsf{Enc}(pk, M_b)$

$S \leftarrow S \cup \{C\}$; Return C

Subroutine $\mathtt{Dec}(C)$

If $C \in S$ then return $\perp$

Return $\mathsf{Dec}(sk, C)$

Subroutine $\mathtt{Finalize}(d)$

Return $(d = b)$

Experiment $\mathbf{Exp}^{\text{ind-cca}}_{\mathsf{PKE}}(A)$

$\mathtt{Initialize}$

$d \xleftarrow{\$} A^{\mathtt{Enc},\mathtt{Dec}}$

Return $\mathtt{Finalize}(d)$

ind-cca advantage:

$$\mathbf{Adv}^{\text{ind-cca}}_{\mathsf{PKE}}(A) = 2 \cdot \Pr\left[\mathbf{Exp}^{\text{ind-cca}}_{\mathsf{PKE}}(A) \Rightarrow \mathsf{true}\right] - 1 \,.$$

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ElGamal PKE modulo prime

Let p be a prime and g a generator of $\mathbf{Z}_p^*$. ElGamal PKE is $(\mathrm{KG}, \mathcal{E}, \mathcal{D})$ as follows.

Alg KG	**Alg** $\mathcal{E}_X(M)$	**Alg** $\mathcal{D}_x(Y, W)$
$x \xleftarrow{\$} \mathbf{Z}_{p-1}$	$y \xleftarrow{\$} \mathbf{Z}_{p-1}$; $Y \leftarrow g^y$	$K \leftarrow Y^x$
$X \leftarrow g^x$	$K \leftarrow X^y$	$M \leftarrow W \cdot K^{-1}$
Return (X, x)	$W \leftarrow K \cdot M$	Return M
	Return (Y, W)	

In $\mathbf{Z}_p^*$, ElGamal PKE is NOT IND-CPA secure.

Hint: Use $M_0 = g$ and $M_1 = 1$. What are the Jacobi symbols of these messages?

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$x \xleftarrow{\$} \mathbf{Z}_{p-1}$	$y \xleftarrow{\$} \mathbf{Z}_{p-1}$; $Y \leftarrow g^y$	$K \leftarrow Y^x$
$X \leftarrow g^x$	$K \leftarrow X^y$	$M \leftarrow W \cdot K^{-1}$
Return (X, x)	$W \leftarrow K \cdot M$	Return M
	Return (Y, W)	

In $\mathbf{Z}_p^*$, ElGamal PKE is NOT IND-CPA secure.

Hint: Use $M_0 = g$ and $M_1 = 1$. What are the Jacobi symbols of these messages?

ElGamal PKE modulo prime: NOT IND-CPA

If we ask the encryption oracle $\texttt{Enc}(g, 1)$ and call what we get back (Y, W_0) if it's a left oracle and (Y, W_1) if it's a right oracle, then we have

$$\begin{aligned} X &= g^x \\ Y &= g^y \\ W_0 &= g^{xy} \cdot g \\ W_1 &= g^{xy} \cdot 1 \end{aligned}$$

$J_P(X)$	$J_p(Y)$	$J_p(g^{xy})$	$J_p(W_0)$	$J_p(W_1)$
1	1	1	-1	1
-1	1			
1	-1			
-1	-1			

Algorithm $A(X)$:

$\quad (Y, W) \xleftarrow{\$} \texttt{Enc}(g, 1)$; $J^* \leftarrow J_p(W)$

$\quad$ switch $(J_p(X), J_p(Y))$:

$\quad\quad$ case $(1, 1)$: case $(-1, 1)$: case $(1, -1)$:

$\quad\quad\quad$ If $J^* = -1$ then return ? else return ?

$\quad\quad$ case $(-1, -1)$:

$\quad\quad\quad$ If $J^* = -1$ then return ? else return ?

ElGamal PKE is ok in certain other groups

However, ElGamal PKE is secure in <u>any group where DDH is hard</u>.
e.g., prime-order subgroups of $\mathbf{Z}_p^*$, elliptic curve groups of prime order

DHIES PKE

Let $G = \langle g \rangle$ be a group of order m, $H : \{0,1\}^* \to \{0,1\}^k$ be a hash function, and $\mathrm{SE} = (\mathrm{KG}_{se}, \mathcal{E}_{se}, \mathcal{D}_{se})$ be a symmetric AE scheme with k-bit keys. DHIES PKE is $(\mathrm{KG}, \mathcal{E}, \mathcal{D})$ as follows.

Alg KG	**Alg** $\mathcal{E}_X(M)$	**Alg** $\mathcal{D}_x(Y, C)$
$x \stackrel{\$}{\leftarrow} \mathbf{Z}_m$	$y \stackrel{\$}{\leftarrow} \mathbf{Z}_m$; $Y \leftarrow g^y$	$Z \leftarrow Y^x$
$X \leftarrow g^x$	$Z \leftarrow X^y$	$K \leftarrow H(Y \\| Z)$
Return (X, x)	$K \leftarrow H(Y \\| Z)$	$M \leftarrow \mathcal{D}_{se}(C)$
	$C \stackrel{\$}{\leftarrow} \mathcal{E}_{se}(K, M)$	Return M
	Return (Y, C)	

Textbook RSA

Textbook RSA is insecure!

Alg KG $(N, p, q, e, d) \xleftarrow{\$} K_{rsa}$ $pk \leftarrow (N, e)$ $sk \leftarrow (N, d)$ Return (pk, sk)	**Alg** $\mathcal{E}_{pk}(M)$ $C \leftarrow M^e \bmod N$ Return C	**Alg** $\mathcal{D}_{sk}(C)$ $M \leftarrow C^d \bmod N$ Return M

Adversary gets $C = M^e \pmod{N}$. Suppose M is 64 bits long. If $M = M_1 \cdot M_2$ where $M_1, M_2 < 2^{34}$ (This happens with prob. approx 20%), then

$$C/M_1^e = M_2^e \pmod{N}$$

Meet in the middle attack:

1. Build table $C/1^e, C/2^e, \ldots C/2^{34e}$
2. For $M_2 = 0, \ldots, 2^{34}$, test if M_2^e is in table.
3. Output matching $M = M_1 \cdot M_2$

Time: much less than 2^{64}

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Alg KG	**Alg** $\mathcal{E}_{pk}(M)$	**Alg** $\mathcal{D}_{sk}(C)$
$(N, p, q, e, d) \xleftarrow{\$} K_{rsa}$ $pk \leftarrow (N, e)$ $sk \leftarrow (N, d)$ Return (pk, sk)	$C \leftarrow M^e \bmod N$ Return C	$M \leftarrow C^d \bmod N$ Return M

Adversary gets $C = M^e \pmod N$. Suppose M is 64 bits long. If $M = M_1 \cdot M_2$ where $M_1, M_2 < 2^{34}$ (This happens with prob. approx 20%), then

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Alg KG	**Alg** $\mathcal{E}_{pk}(M)$	**Alg** $\mathcal{D}_{sk}(C)$
$(N, p, q, e, d) \xleftarrow{\$} K_{rsa}$	$C \leftarrow M^e \bmod N$	$M \leftarrow C^d \bmod N$
$pk \leftarrow (N, e)$	Return C	Return M
$sk \leftarrow (N, d)$		
Return (pk, sk)		

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If $M = M_1 \cdot M_2$ where $M_1, M_2 < 2^{34}$ (This happens with prob. approx 20%), then

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Meet in the middle attack:

1. Build table $C/1^e, C/2^e, \ldots C/2^{34e}$
2. For $M_2 = 0, \ldots, 2^{34}$, test if M_2^e is in table.
3. Output matching $M = M_1 \cdot M_2$

Time: much less than 2^{64}

SRSA

Let $H : \{0,1\}^* \rightarrow \{0,1\}^k$ be a hash function, and
$\mathrm{SE} = (\mathrm{KG}_{se}, \mathcal{E}_{se}, \mathcal{D}_{se})$ be a symmetric AE scheme with k-bit keys.
SRSA PKE is $(\mathrm{KG}, \mathcal{E}, \mathcal{D})$ as follows.

Alg KG	**Alg** $\mathcal{E}_{pk}(M)$	**Alg** $\mathcal{D}_{sk}(C_1, C_2)$
$(N, p, q, e, d) \xleftarrow{\$} K_{rsa}$	$x \xleftarrow{\$} \mathbf{Z}_N^*$	$x \leftarrow C_1^d \bmod N$
$pk \leftarrow (N, e)$	$K \leftarrow H(x)$	$K \leftarrow H(x)$
$sk \leftarrow (N, d)$	$C_1 \leftarrow x^e \bmod N$	$M \leftarrow \mathcal{D}_{se}(K, C_2)$
Return (pk, sk)	$C_2 \leftarrow \mathcal{E}_{se}(K, M)$	Return M
	Return (C_1, C_2)	

Hybrid encryption

SRSA follows a common paradigm:

PKE = KEM + DEM

1. Use a trapdoor function (TDF) and a hash to encapsulate an ephemeral symmetric key
2. Use AE to encapsulate the payload

Alg KG
$(N, p, q, e, d) \xleftarrow{\$} K_{rsa}$
$pk \leftarrow (N, e)$
$sk \leftarrow (N, d)$
Return (pk, sk)

Alg $\mathcal{E}_{pk}(M)$
$x \xleftarrow{\$} \mathbf{Z}_N^*$
$K \leftarrow H(x)$
$C_1 \leftarrow x^e \bmod N$
$C_2 \leftarrow \mathcal{E}_{se}(K, M)$
Return (C_1, C_2)

Alg $\mathcal{D}_{sk}(C_1, C_2)$
$x \leftarrow C_1^d \bmod N$
$K \leftarrow H(x)$
$M \leftarrow \mathcal{D}_{se}(K, C_2)$
Return M

One-wayness of RSA against known-exponent attacks

Definition

Let K_{rsa} be an RSA generator with security parameter k.
Let A be an algorithm.

Experiment $\mathbf{Exp}^{\text{ow-kea}}_{K_{rsa}}(A)$

$((N, e), (N, p, q, d)) \xleftarrow{\$} K_{rsa}$

$x \xleftarrow{\$} \mathbf{Z}_N^*; y \leftarrow x^e \bmod N$

$x' \xleftarrow{\$} A(N, e, y)$

If $x = x'$ then 1 else 0

$$\mathbf{Adv}^{\text{ow-kea}}_{K_{rsa}}(A) = \Pr\left[\, \mathbf{Exp}^{\text{ow-kea}}_{K_{rsa}}(A) = 1 \,\right] .$$

One-wayness of RSA against chosen-exponent attacks

Definition

Let K_{mod} be an modulus generator with security parameter k.
Let A be an algorithm.

Experiment $\mathbf{Exp}^{\text{ow-cea}}_{K_{mod}}(A)$

$(N, p, q) \xleftarrow{\$} K_{mod}$

$y \xleftarrow{\$} \mathbf{Z}_N^*$

$(x, e) \xleftarrow{\$} A(N, y)$

If $x^e \equiv y \pmod{N}$ and $e > 1$ then return 1 else return 0

$$\mathbf{Adv}^{\text{ow-cea}}_{K_{mod}}(A) = \Pr\left[\, \mathbf{Exp}^{\text{ow-cea}}_{K_{mod}}(A) = 1 \,\right] .$$

PKCS1 encryption

PKCS1 padding (02 is the mode number):

$$\underbrace{\boxed{02}\boxed{\quad\text{random pad}\quad}\boxed{\mathit{FF}}\boxed{\qquad\qquad\text{msg}\qquad\qquad}}_{\text{RSA-encrypted}}$$

- The entire thing is the value that gets RSA-encrypted.
- "02" is written as a 16-bit binary string.
- "random pad" doesn't contain *FF*.
- Widely deployed, e.g. in HTTPS.

Bleichenbacher attack: An attacker tests to see if 16 MSBs of plaintext is 02.

Bleichenbacher attack (simplified)

Bleichenbacher attack uses the server as a padding oracle.

- Success: the first two bytes are 02.
- Failure: the first two bytes are not 02.

Simplified attack:

- Suppose $N = 2^n$
- Suppose instead of revealing whether the first 2 bytes are 02, the server reveals whether the MSB is 1.
- Suppose adversary snoops a ciphertext C
- Adversary sends C and gets MSB
- Adversary sends $2^e C$ and gets 2nd most MSB
  [$2^e C = (2M)^e$ so we shift M to the left 1 position]
- adversary sends $4^e C$ and gets 3rd most MSB
  [$4^e C = (4M)^e$ so we shift M to the left 2 positions]
- ...

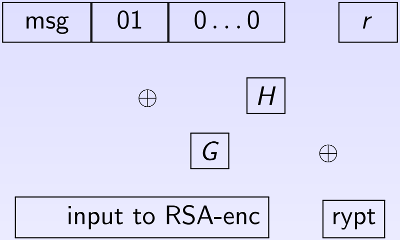
Preventing Bleichenbacher attack: HTTPS

Bleichenbacher attack uses the server as a padding oracle.

So for HTTPS (RFC 5246):

1. Generate a random string R of 46 bytes
2. Decrypt the ciphertext to get M
3. If PKCS1 padding check fails for M, then the decryption is R.

OAEP: Optimal Asymmetric Encryption Padding [BR94]



Theorem

If RSA is a trapdoor permutation, then RSA-OAEP is CCA secure in the random oracle model.

- OAEP+ replaces $010\cdots0$ with $W(m,r)$ where W is a hash function. This works for any trapdoor permutation, not just RSA.
- SAEP+ replaces $010\cdots0$ with $W(m,r)$ where W is a hash function and removes G. This works for RSA.