# Symmetric Encryption Revisited

Chanathip Namprempre

Computer Science
Reed College

# Agenda: Symmetric Encryption Revisted

1. Modes of operation
2. Security definitions for confidentiality
   - IND-CPA: definition and example attacks
   - IND-CPA security of CTR and CBC modes
   - IND-CCA: definition and example attacks

# Recall Syntax of Symmetric Encryption

## Syntax

A symmetric encryption scheme $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ is a triple of algorithms.

| alg | input | output | notation | maybe randomized? | maybe stateful? |
|---|---|---|---|---|---|
| $\mathcal{K}$ | - | key $K$ | $K \xleftarrow{\$} \mathcal{K}$ | yes | no |
| $\mathcal{E}$ | $K \in Keys(\mathcal{SE})$<br>$M \in \{0,1\}^*$ | ciphertext<br>$C \in \{0,1\}^* \cup \{\perp\}$ | $C \xleftarrow{\$} \mathcal{E}_K(M)$ | yes | yes |
| $\mathcal{D}$ | $K \in Keys(\mathcal{SE})$<br>$C \in \{0,1\}^*$ | plaintext<br>$M \in \{0,1\}^* \cup \{\perp\}$ | $M \leftarrow \mathcal{D}_K(C)$ | no | no |

## Correctness

For all $K \in Keys(\mathcal{SE})$ and all $M \in \{0,1\}^*$,

$$\Pr\left[\, C = \perp \text{ OR } \mathcal{D}_K(C) = M \ : \ C \xleftarrow{\$} \mathcal{E}_K(M) \,\right] = 1\ .$$

# Modes of operation

OTP is impractical. Most symmetric encryption schemes use block ciphers as building block.

Let $E$ be a block cipher.

**idea**

$$C \leftarrow E_K(M)$$

- But oftentimes, $M$ is longer than the block length and/or isn't of the length multiple of the block length!
- So we need to figure out how to chop up $M$ and/or pad it.
- There are many methods to do this. These methods are called modes of operation.

# Electronic Code Book mode (ECB): key generation

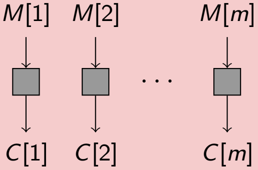
Encryption scheme in ECB mode is **deterministic** and **stateless**.

Let $E : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ be a block cipher.

$\mathcal{K} : K \xleftarrow{\$} \{0,1\}^k$ ; return $K$

[This key generation algorithm will be used for all modes of operation.]

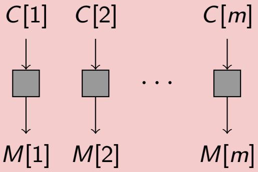
# ECB: encryption and decryption




$\mathcal{E}_K(M)$ :

if $(|M| \bmod n \neq 0$ or $|M| = 0)$
then return $\perp$
Break $M$ into n-bit blocks $M[1] \ldots M[m]$
for $i \leftarrow 1$ to $m$ do
  $C[i] \leftarrow E_K(M[i])$
$C \leftarrow C[1] \ldots C[m]$
return $C$

---

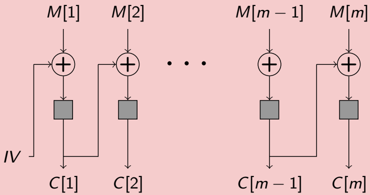



$\mathcal{D}_K(C)$ :

if $(|C| \bmod n \neq 0$ or $|C| = 0)$
then return $\perp$
Break $C$ into n-bit blocks $C[1] \ldots C[m]$
for $i \leftarrow 1$ to $m$ do
  $M[i] \leftarrow E_K^{-1}(C[i])$
$M \leftarrow M[1] \ldots M[m]$
return $M$

# Cipher Block Chaining mode ($CBC\$$): encryption and decryption

Encryption scheme in $CBC\$$ mode is **randomized** and **stateless**.



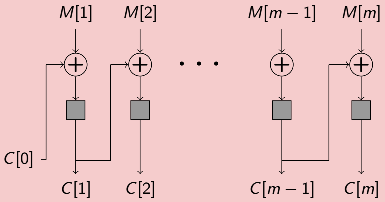

$\mathcal{E}_K(M)$ :
if ($|M| \bmod n \neq 0$ or $|M| = 0$)
then return $\perp$
Break $M$ into n-bit blocks $M[1], \ldots, M[m]$
$C[0] \leftarrow IV \xleftarrow{\$} \{0,1\}^n$
for $i \leftarrow 1$ to $m$ do
$C[i] \leftarrow E_K(C[i-1] \oplus M[i])$
return $C[0] \| \ldots \| C[m]$

$\mathcal{D}_K(C)$ :

??

# Cipher Block Chaining mode (*CBCC*): encryption and decryption

Encryption scheme in *CBCC* mode is **deterministic** and **stateful**.




$\mathcal{E}_K(M)$ :
- if $(|M| \bmod n \neq 0$ or $|M| = 0)$
- then return $\perp$
- Break $M$ into n-bit blocks $M[1], \ldots, M[m]$
- static $ctr \leftarrow 0$ ; $C[0] \leftarrow ctr$ ; $ctr \leftarrow ctr + 1$
- for $i \leftarrow 1$ to $m$ do
  - $C[i] \leftarrow E_K(C[i-1] \oplus M[i])$
- return $C[0] \| \ldots \| C[m]$

$\mathcal{D}_K(C)$ :

??

# Counter mode

**idea**

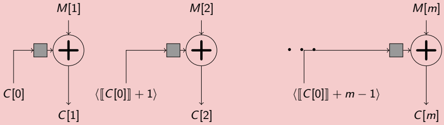
Try to be like OTP but use block cipher to generate the pad.

As usual, there are two versions:

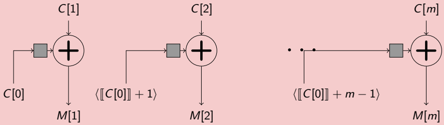
apply the block cipher to a random value $\Longrightarrow$ $CTR\$$
apply the block cipher to a counter $\Longrightarrow$ $CTRC$

# Counter mode ($CTRC$): encryption and decryption

Encryption scheme in $CTRC$ mode is **deterministic** and **stateful**.




$\mathcal{E}_K(M)$ :
- If $(|M| = 0)$ or $(|M| \bmod n \neq 0)$
  - then return $\perp$
- Parse $M$ into $n$-bit blocks $M[1], \ldots, M[m]$
- static $ctr \leftarrow 0$ ; $C[0] \leftarrow ctr$
- $ctr \leftarrow ctr + m$
- if $ctr - 1 > 2^n - 1$ then return $\perp$
- For $i = 1$ to $m$ do
  - $C[i] \leftarrow E_K(\langle [\![ C[0] ]\!] + i - 1 \rangle) \oplus M[i]$
- Return $C[0] \| \ldots \| C[m]$

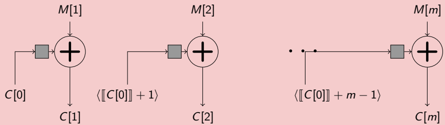



$\mathcal{D}_K(C)$ :

??

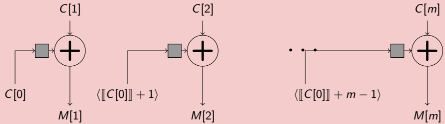
# Counter mode ($CTR\$$): encryption and decryption

Encryption scheme in $CTR\$$ mode is **randomized** and **stateless**.




$\mathcal{E}_K(M)$ :
If $(|M| = 0)$ or $(|M| \bmod n \neq 0)$
then return $\perp$
Parse $M$ into $n$-bit blocks $M[1], \ldots, M[m]$
$C[0] \leftarrow IV \xleftarrow{\$} \{0,1\}^n$
For $i = 1$ to $m$ do
$C[i] \leftarrow E_K(\langle [\![C[0]]\!] + i - 1 \rangle) \oplus M[i]$
Return $C[0] \| \ldots \| C[m]$

---




$\mathcal{D}_K(C)$ :

??

# Security definitions for confidentiality

# Issues in confidentiality

Setting:

- First pick a key: $K \xleftarrow{\$} \mathcal{K}$
- sender and receiver know $K$
- adversary $A$ does not know $K$
- adversary $A$ can capture ciphertexts

What's considered insecure?

# Definition for confidentiality: attempt 1

**key recovery**

From the ciphertexts, $A$ can get $K$.

- For sure, this is true:
  $A$ breaks key recovery $\Rightarrow$ scheme is insecure
- What about the inverse?
- counterexample: can you think of an encryption scheme secure under key recovery but does nothing to hide the message?

# Definition for confidentiality: attempt 2

**plaintext recovery**

From the ciphertexts, $A$ can get $M$.

What if the message format is such that some bits are more important than others?

In this case, what if $A$ can't get the whole message $M$ but can get at those important bits?

# Definition for confidentiality: attempt 3

**partial information recovery**

From the ciphertexts, A can get partial information about $M$.

But which bits do we want to protect???

- 1st bit?
- $i$-th bit?
  For example, suppose
  the $i$-th bit of the plaintext is 0 iff we want to sell stock
- sum of all bits?

Bottom line :
We don't want to make assumptions about data format!

# Definition for confidentiality

- We need to approach this more directly:
  $\boxed{\text{Q}}$ : What would an <u>ideal encryption scheme</u> look like?

  $\boxed{\text{A}}$ : An angel delivers your messages, i.e.
  no partial information gets leaked!

- We want to approximate this.
  [but we can't help but leak the length of $M$]

  So we aim for this :

A secure scheme shouldn't let $A$ relate ciphertexts of messages of the same length.

# Examples of insecure scheme: ECB

$A$ can get information even if $A$ can't break the block cipher.

**example**

$0^n =$ don't fire missile
$1^n =$ fire missile

Suppose the two commands are to fire missiles.

1. $A$ sees the first ciphertext $C_0$ followed by a missile.
2. $A$ sees the second ciphertext $C_1$, which looks exactly the same as $C_0$.
3. What would $A$ do??

Bottom line :
For ECB, ciphertexts of messages with the same contents look exactly the same!

# Definition for confidentiality: first lesson

A secure encryption scheme cannot be both deterministic and stateless.

- one message should correspond to many possible ciphertexts.
- This is not what's historically done.

# Indistinguishability against chosen-plaintext attacks IND-CPA

Idea

- Pick a hidden bit $b$ at random.
- Let $A$ choose two messages.
- One of the messages will get encrypted.
- The resulting ciphertext is given to $A$.
- $A$ guesses what $b$ is.

# IND-CPA

Subroutine *Initialize*

$b \stackrel{\$}{\leftarrow} \{0,1\}$ ; $K \stackrel{\$}{\leftarrow} KG$

Subroutine $\texttt{Enc}(M_0, M_1)$

If $|M_0| \neq |M_1|$ then return $\perp$

Return $\mathrm{Enc}_K(M_b)$

Subroutine *Finalize*$(d)$

Return $(d = b)$

Experiment $\mathbf{Exp}_{\mathsf{SE}}^{\text{ind-cpa}}(A)$

*Initialize*

$d \stackrel{\$}{\leftarrow} A^{\texttt{Enc}}$

Return *Finalize*$(d)$

## ind-cpa advantage

The **ind-cpa advantage** of an adversary $A$ mounting a chosen-ciphertext attack against SE is

$$\mathbf{Adv}_{\mathsf{SE}}^{\text{ind-cpa}}(A) = 2 \cdot \Pr\left[\, \mathbf{Exp}_{\mathsf{SE}}^{\text{ind-cpa}}(A) \Rightarrow \mathsf{true} \,\right] - 1 \,.$$

# IND-CPA: observations

- $\mathcal{SE}$ is secure against IND-CPA if an adversary restricted to **practical** amount of resources can't obtain **significant** advantage.
- resources are
  1. time
     - the running time of A (over all coins of A and all return values)
     - size of A's code
     - time spent by A to read bits returned from oracle (return values in unit time)
  2. number of bits queried
     [length of query $(M_0, M_1)$ = max length of $M_0$ and $M_1$]
  3. number of queries submitted

Bottom line : IND-CPA captures confidentiality.

# IND-CPA: observations

As we'll see,

$$
\begin{array}{lcl}
\text{IND-CPA} & \Rightarrow & \text{key recovery is hard.} \\
 & \Rightarrow & \text{message recovery is hard.} \\
 & \Rightarrow & \text{partial information recovery is hard.} \\
 & & \ldots
\end{array}
$$

# Example IND-CPA attacks

# Proposition : ECB is insecure.

Let $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$. Let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the ECB encryption scheme based on $E$. Then, there exists an ind-cpa adversary $A$ such that,

$$Adv^{\text{ind-cpa}}_{\mathcal{SE}}(A) = 1$$

and $A$ runs in time $O(n)$ and asks 1 query totalling $2n$ bits.

**Notice**

ECB is bad *even if* E is a perfectly good block cipher!
This is a design flaw!

# Proposition :
# Any deterministic and stateless schemes are insecure.

Let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the deterministic, stateless symmetric encryption scheme.

Assume that there's an integer $m$ such that the plaintext space of the scheme cantains at least 2 distinct strings of length $m$. Then, there is an adversary $A$ such that

$$Adv_{\mathcal{SE}}^{\mathrm{ind\text{-}cpa}}(A) = 1$$

and $A$ runs in time $O(m)$ and asks 2 queries totalling $2m$ bits.

# Proposition : CBCC is insecure.

Let $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$.
Let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the CBCC scheme based on $E$.
Then, there exists $A$ such that

$$Adv_{\mathcal{SE}}^{\text{ind-cpa}}(A) = 1$$

and A runs in time $O(n)$ and asking 2 queries totalling $2n$ bits.

# Indistinguishability against chosen-ciphertext attacks IND-CCA: idea

- Similar to IND-CPA except that we also let $A$ ask for decryption of ciphertexts of its choice.
- But to prevent a trivial attack, we do not let $A$ ask for the decryption of the ciphertexts that it got back from the encryption oracle.
- Similar to IND-CPA, we also allow multiple adaptive queries.

# IND-CCA: Left-or-right indistinguishability against chosen-ciphertext attacks: formal definition

Subroutine *Initialize*

$b \stackrel{\$}{\leftarrow} \{0,1\}$ ; $K \stackrel{\$}{\leftarrow} KG$ ; $S \leftarrow \emptyset$

Subroutine $\mathtt{Enc}(M_0, M_1)$

If $|M_0| \neq |M_1|$ then return $\perp$

Return $\mathrm{Enc}_K(M_b)$

Subroutine $\mathtt{Dec}(C)$

If $C \in S$ then return $\perp$

Return $\mathrm{Dec}_K(C)$

Subroutine *Finalize*$(d)$

Return $(d = b)$

Experiment $\mathbf{Exp}_{\mathrm{SE}}^{\mathrm{ind\text{-}cca}}(A)$

*Initialize*

$d \stackrel{\$}{\leftarrow} A^{\mathtt{Enc},\mathtt{Dec}}$

Return *Finalize*$(d)$

## ind-cca advantage

The **ind-cca advantage** of an adversary $A$ mounting a chosen-ciphertext attack against SE is

$$\mathbf{Adv}_{\mathrm{SE}}^{\mathrm{ind\text{-}cca}}(A) = 2 \cdot \Pr\left[\mathbf{Exp}_{\mathrm{SE}}^{\mathrm{ind\text{-}cca}}(A) \Rightarrow \mathrm{true}\right] - 1 .$$

# Example IND-CCA attacks

## Proposition : *CTR*\$ is insecure against chosen-ciphertext attacks.

Let $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^l$ be a family of functions. Let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the *CTR*\$ encryption scheme based on $E$. Then, there exists an ind-cca adversary $A$ such that,

$$Adv_{\mathcal{SE}}^{\text{ind-cca}}(A) = 1$$

and $A$ runs in time $O(n + l)$ plus the time for one application of $E$ and asks 1 query totalling $l$ bits to the encryption oracle and 1 query totalling $n + l$ bits to the decryption oracle.

**Note**

*CTR*\$ is secure against IND-CPA but insecure against IND-CCA.

# Proposition : $CBC\$$ is insecure against chosen-ciphertext attacks.

Let $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ be a family of functions. Let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the $CBC\$$ encryption scheme based on $E$. Then, there exists an ind-cca adversary $A$ such that,

$$Adv_{\mathcal{SE}}^{\text{ind-cca}}(A) = 1$$

and $A$ runs in time $O(n)$ plus the time for one application of $E$ and asks 1 query totalling $n$ bits to the encryption oracle and 1 query totalling $2n$ bits to the decryption oracle.

**Note**

$CBC\$$ is secure against IND-CPA but insecure against IND-CCA.

# Proving positive results

# *CTR\$* and *CTRC* are secure under IND-CPA

## proposition: *CTRC* is secure under IND-CPA

Let $E : \mathcal{K} \times \{0,1\}^l \to \{0,1\}^L$ be a family of functions and let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the *CTRC* encryption scheme. Let $A$ be an ind-cpa adversary that runs in time at most $t$ and asks at most $q$ queries, these totalling at most $\sigma$ $L$-bit blocks. Then there exists a prf adversary $B$ such that

$$Adv_{\mathcal{SE}}^{\mathrm{ind\text{-}cpa}}(A) \leq Adv_{E}^{\mathrm{prf}}(B) \; .$$

Furthermore $B$ runs in time at most $t' = t + O(q + (l + L)\sigma)$ and asks at most $q' = \sigma$ oracle queries.

## proposition: *CTR\$* is secure under IND-CPA

Let $E : \mathcal{K} \times \{0,1\}^l \to \{0,1\}^L$ be a family of functions and let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the *CTR\$* encryption scheme. Let $A$ be an ind-cpa adversary that runs in time at most $t$ and asks at most $q$ queries, these totalling at most $\sigma$ $L$-bit blocks. Then there exists a prf adversary $B$ such that

$$Adv_{\mathcal{SE}}^{\mathrm{ind\text{-}cpa}}(A) \leq Adv_{E}^{\mathrm{prf}}(B) + \frac{0.5\sigma^2}{2^l} \; .$$

Furthermore $B$ runs in time at most $t' = t + O(q + (l + L)\sigma)$ and asks at most $q' = \sigma$ oracle queries.

# $CBC\$$ is secure under IND-CPA

**proposition**

Let $E : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ be a family of functions and let $\mathcal{SE} = (\mathcal{K}, \mathcal{E}, \mathcal{D})$ be the $CBC\$$ encryption scheme. Let $A$ be an ind-cpa adversary that runs in time at most $t$ and asks at most $q$ queries, these totalling at most $\sigma$ $n$-bit blocks. Then there exists a prf adversary $B$ such that

$$Adv_{\mathcal{SE}}^{\text{ind-cpa}}(A) \leq Adv_{E}^{\text{prf}}(B) + \frac{\sigma^2}{2^{n+1}} \, .$$

Furthermore $B$ runs in time at most $t' = t + O(q + n\sigma)$ and asks at most $q' = \sigma$ oracle queries.