Computational Number Theory

Chanathip Namprempre

Faculty of Engineering
Thammasat University

Agenda

1. Important sets of numbers, Euler Phi function, Greatest Common Divisor
2. Groups, exponentiation
3. Group order
4. Running time of computation on large numbers
5. Subgroups
6. Order of a group element
7. Subgroup generated by an element, subgroup order
8. Generators
9. Discrete Logs
10. Cyclic groups
11. DL and friends: DL, CDH, DDH
12. Elliptic curve groups
13. Diffie-Hellman key exchange protocol

Material mostly from Mihir Bellare (https://cseweb.ucsd.edu/ mihir/cse207/slides/s-cnt.pdf)

Important sets of numbers and Euler Phi function

$$\begin{aligned}
\mathbf{Z} &= \{\ldots, -2, -1, 0, 1, 2, \ldots\} \\
\mathbf{N} &= \{0, 1, 2, \ldots\} \\
\mathbf{Z}^+ &= \{1, 2, \ldots\} \\
\mathbf{Z}_N &= \{0, \ldots, N-1\} \\
\mathbf{Z}_N^* &= \{a \in \mathbf{Z}_N \ : \ \gcd(a, N) = 1\} \\
\phi(N) &= |\mathbf{Z}_N^*|
\end{aligned}$$

- Try $\gcd(12, 20), \gcd(21, 16)$
- Try $\mathbf{Z}_{15}, \mathbf{Z}_{15}^*, \phi(15)$

div and mod

Quotient Remainder Theorem

For any $a, N \in \mathbf{Z}$ with $N > 0$, there exists unique $q, r \in \mathbf{N}$ such that

- $a = Nq + r$ and
- $0 \leq r < N$

- We say a div $N = q$ and a mod $N = r$.
- We also say

$$a \equiv b \pmod{N}$$

  if and only if a mod $N = b$ mod N.

Groups

Group $(G, \text{ op })$

Let G be a non-empty set, and op be a binary operation on G. We say that G is a group with respect to op iff the following properties hold:

Closure For all $x, y \in G$, $x \text{ op } y \in G$.

Associativity For all $x, y, z \in G$, $(x \text{ op } y) \text{ op } z = x \text{ op } (y \text{ op } z)$.

Identity There exists $1 \in G$ such that, for every $x \in G$, $x \text{ op } 1 = 1 \text{ op } x = x$.

Invertibility For every $x \in G$, there exists $x^{-1} \in G$ such that $x \text{ op } x^{-1} = 1$.

Try $(\mathbf{Z}, +)$, $(\mathbf{N}, +)$, $(\mathbf{Z}^+, +)$, $\mathbf{Z}_N$, MOD-ADD, $\mathbf{Z}_N$, MOD-MULT, $\mathbf{Z}_N^*$, MOD-MULT

Exponentiation

- If we write op as $\cdot$ and represent the identity element as 1, then exponentiation is written and means what you naturally think it means.
  - $a^0 = 1$
  - $a^n = a \cdots a$ altogether n terms
  - $a^{-n} = a^{-1} \cdots a^{-1}$ altogether n terms
  - $a^{i+j} = a^i \cdot a^j$
  - $a^{-1} = (a^i)^{-1} = (a^{-1})^i$

Computational shortcuts

To compute $5 \cdot 8 \cdot 10 \cdot 16 \bmod 21$,

- don't do this: $5 \cdot 8 \cdot 10 \cdot 16 = 6400$ then modulo 21
- do this instead: $((5 \cdot 8 \bmod 21) \cdot 10 \bmod 21) \ldots$

Group orders

Group order

The order of a group G is $|G|$.

Try $\mathbf{Z}_{21}^*$. Write out all the elements in this group and count them.

Fact

If G is a group of order m and if $a \in G$, then $a^m = 1$.

- Try $5^{12} \in \mathbf{Z}_{21}^*$
- Try $8^{12} \in \mathbf{Z}_{21}^*$

Group orders (cont.)

Corollary

If G is a group of order m and if $a \in G$, then for any $i \in \mathbf{Z}$,

$$a^i = a^{i \bmod m} .$$

- Try $5^{74} \bmod 21$

Running time of computation on large numbers

Operating on large numbers, we no longer assume that basic operations (like addition and multiplication) take constant time. The running time needs be measured in the size of the inputs.

Algorithm	Input	Output	Time
INT-DIV	a, N	q, r	quadratic
MOD	a, N	$a \bmod N$	quadratic
EXT-GCD	a, N	(d, a', N')	quadratic
MOD-ADD	a, b, N	$a + b \bmod N$	linear
MOD-MULT	a, b, N	$ab \bmod N$	quadratic
MOD-INV	a, N	$a^{-1} \bmod N$	quadratic
MOD-EXP	a, n, N	$a^n \bmod N$	cubic
EXP_G	a, n	$a^n \in G$	$O(\lvert n \rvert)$ G-ops

Extended gcd

EXT-GCD(a, N) returns (d, a', N') such that

$$d = \gcd(a, N) = a \cdot a' + N \cdot N' \ .$$

Examples: EXT-GCD(25,10), EXT-GCD(10,25), EXT-GCD(250,12)

Extended Euclidean algorithm

Assume that $(a, N) \neq (0, 0)$. We know that

$$\gcd(a, N) = \gcd(N, a \bmod N) \ .$$

So we can compute GCD using a recursive algorithm as follows.

Algorithm EXT-GCD(a, N)
    If $N = 0$ then Return $(a, 1, 0)$
    $(q, r) \leftarrow$ INT-DIV(a, N)
    $(d, x, y) \leftarrow$ EXT-GCD(N, r)
    $a' \leftarrow y$; $N' \leftarrow x - qy$
    Return (d, a', N')

The running time is $O(|a| \cdot |N|)$, so quadratic time.

Modular Inverse

For a, N such that $\gcd(a, N) = 1$, we want to compute $a^{-1} \bmod N$, i.e.,

$$\text{unique } a' \in \mathbf{Z}_N^* \text{ such that } aa' \equiv 1 \pmod{N} .$$

Let $(d, a', N') \leftarrow$ EXT-GCD(a, N), then

$$d = 1 = \gcd(a, N) = a \cdot a' + N \cdot N' .$$

Since $N \cdot N' \equiv 0 \pmod{N}$, we have $a \cdot a' \equiv 1 \pmod{N}$.

Algorithm MOD-INV(a,N):
    $(d, a', N') \leftarrow$ EXT-GCD(a, N)
    Return $a' \bmod N$

Modular Exponentiation

Let G be a group and $a \in G$. For $x \in \mathbf{N}$, compute

$$a^x = \underbrace{a \cdots a}_{x}$$

Suppose we compute a^x by collecting one term at a time, i.e.,

$y \leftarrow 1$
For $i \in \{1, 2, \ldots, x\}$ do
    $y \leftarrow y \cdot a$
Return y

What is the running time of this algorithm?

Square-and-Multiply Exponentiation

Suppose the binary length of x is 4, so x has the form $b_3 b_2 b_1 b_0$. Then,

$$\begin{aligned} x &= 2^3 b_3 + 2^2 b_2 + 2^1 b_1 + 2^0 b_0 \\ &= 8b_3 + 4b_2 + 2b_1 + b_0 \end{aligned}$$

To compute a^x, we square and multiply iteratively as follows:

$$\begin{aligned} y_4 &= 1 \\ y_3 &= y_4^2 \cdot a^{b_3} = a^{b_3} \\ y_2 &= y_3^2 \cdot a^{b_2} = a^{2b_3+b_2} \\ y_1 &= y_2^2 \cdot a^{b_1} = a^{4b_3+2b_2+b_1} \\ y_0 &= y_1^2 \cdot a^{b_0} = a^{8b_3+4b_2+2b_1+b_0} \end{aligned}$$

We get $a^x = y_0$. What is the running time of this algorithm?

Subgroups

Subgroup

If G is a group and $S \subseteq G$, then S is a subgroup of G iff S is a group under G's operation.

- Try $G = \mathbf{Z}_{11}^*$ and $S = \{1, 2, 3\}$. Is S a subgroup?

Order of a group element

Let G be a finite group.

Order of a group element

The order of $g \in G$, denoted $o(g)$, is the smallest integer $n \geq 1$ such that $g^n = 1$.

Try $G = \mathbf{Z}_{11}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

i	0	1	2	3	4	5	6	7	8	9	10
$2^i \bmod 11$											
$5^i \bmod 11$											

What are $o(2)$ and $o(5)$?

Subgroup generated by $g \in G$

For $g \in G$, define

$$\langle g \rangle = \{g^0, g^1, \ldots, g^{o(g)-1}\} .$$

$\langle g \rangle$ is a subgroup of G, and its order is $o(g)$.

Lagrange's Theorem

The order of a subgroup S of G always divides the order of G.

Try $G = \mathbf{Z}_{11}^*$ and list elements of $\langle 2 \rangle$ and $\langle 5 \rangle$. What are $o(2)$ and $o(5)$?

Cyclic Group

Cyclic group

A group G is cyclic iff there exists g such that $\langle g \rangle = G$.

Such an element g is called a generator.

Discrete Log problem

Consider $\mathbf{Z}_{11}^*$. We know that 2 is a generator of $\mathbf{Z}_{11}^*$.

i	0	1	2	3	4	5	6	7	8	9
$2^i \bmod 11$	1	2	4	8	5	10	9	7	3	6

We say that, for example,

$$\mathrm{Dlog}_2(8) = 3 \, .$$

Let $G = \langle g \rangle$ be a cyclic group. Then, given $a \in G$, find

$$\mathrm{Dlog}_g(a) \, .$$

Discrete Log Problem: game formulation

Let $G = \langle g \rangle$ be a cyclic group of order m, and let A be an adversary.

DLP game

Game $\mathrm{DL}_{G,g}(A)$:

$x \xleftarrow{\$} \mathbf{Z}_m$

$X \leftarrow g^x$

$x' \leftarrow A(X)$

Return $(x = x')$

The dl-advantage of A is

$$\mathbf{Adv}^{\mathrm{dl}}_{G,g}(A) = \Pr\left[\,\mathrm{DL}_{G,g}(A) \Rightarrow \mathrm{true}\,\right] .$$

Computational Diffie-Hellman problem

Consider $\mathbf{Z}_{11}^*$. We know that 2 is a generator of $\mathbf{Z}_{11}^*$.

i	0	1	2	3	4	5	6	7	8	9
$2^i \bmod 11$	1	2	4	8	5	10	9	7	3	6

Let $G = \langle g \rangle$ be a cyclic group. Then, given $X = g^x$ and $Y = g^y$, find

$$Z = g^{xy} \ .$$

Computational Diffie-Hellman Problem: game formulation

Let $G = \langle g \rangle$ be a cyclic group of order m, and let A be an adversary.

CDH game

Game $\mathrm{CDH}_{G,g}(A)$:

$x \xleftarrow{\$} \mathbf{Z}_m \; ; \; y \xleftarrow{\$} \mathbf{Z}_m$

$X \leftarrow g^x \; ; \; Y \leftarrow g^y$

$Z \leftarrow A(X, Y)$

Return $(Z = g^{xy})$

The cdh-advantage of A is

$$\mathbf{Adv}^{\mathrm{cdh}}_{G,g}(A) = \Pr\left[\, \mathrm{CDH}_{G,g}(A) \Rightarrow \mathrm{true} \,\right] \; .$$

Decisional Diffie-Hellman problem

Consider $\mathbf{Z}_{11}^*$. We know that 2 is a generator of $\mathbf{Z}_{11}^*$.

i	0	1	2	3	4	5	6	7	8	9
$2^i \bmod 11$	1	2	4	8	5	10	9	7	3	6

Let $G = \langle g \rangle$ be a cyclic group. Then, given $X = g^x$, $Y = g^y$, and $Z \in G$, figure out whether

$$Z = g^{xy} \ .$$

Decisional Diffie-Hellman Problem: game formulation

Let $G = \langle g \rangle$ be a cyclic group of order m, and let A be an adversary.

DDH game

Game $\mathrm{DDH}_{G,g}(A)$:

$b \in \{0, 1\}$

$x \xleftarrow{\$} \mathbf{Z}_m \; ; \; y \xleftarrow{\$} \mathbf{Z}_m$

$X \leftarrow g^x \; ; \; Y \leftarrow g^y$

If $b = 0$ then $Z \xleftarrow{\$} G$ else $Z \leftarrow g^{xy}$

$d \leftarrow A(X, Y, Z)$

Return $(d = b)$

The ddh-advantage of A is

$$\mathbf{Adv}^{\mathrm{ddh}}_{G,g}(A) = 2 \cdot \Pr[\, \mathrm{DDH}_{G,g}(A) \Rightarrow \mathrm{true} \,] - 1 \; .$$

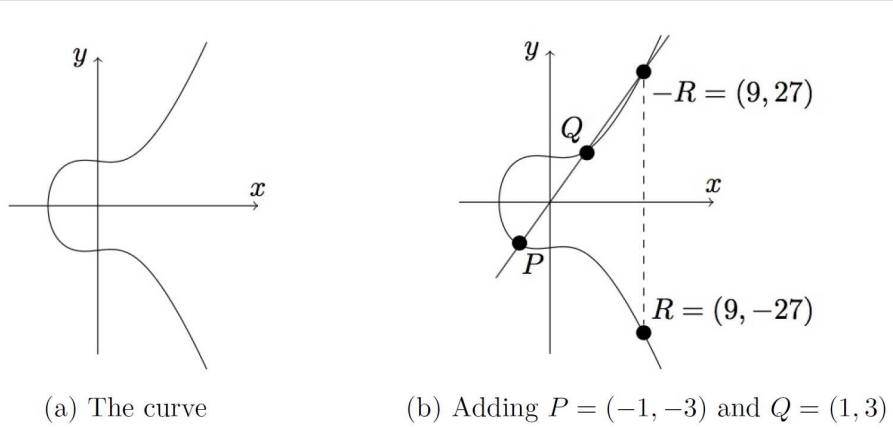
Hardness of DLP

Problem: If we use $G = Z_p^*$ where p is a prime, we need to make p **very large** to ensure that DLP is hard in G.

Solution: Use an elliptic curve group $E/\mathbb{F}_p$. It turns out that, for an elliptic curve group, DLP is hard even for **small** p.

Elliptic Curve Group

This figure below shows the curve $y^2 = x^3 - x + 9$ over the reals.



(a) The curve (b) Adding $P = (-1, -3)$ and $Q = (1, 3)$

Elliptic Curve over Finite Field

Let $p > 3$ be a prime. An elliptic curve E over $\mathbb{F}_p$, denoted $E/\mathbb{F}_p$, is an equation

$$y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{F}_p$ satisfy $4a^3 + 27b^2 \neq 0$.

Try `https://graui.de/code/elliptic2/`!

The condition $4a^3 + 27b^2 \neq 0$ ensures that $x^3 + ax + b = 0$ does not have a double root. This is to avoid certain degeneracies.

Source: Boneh-Shoup book 2023

Finite Field

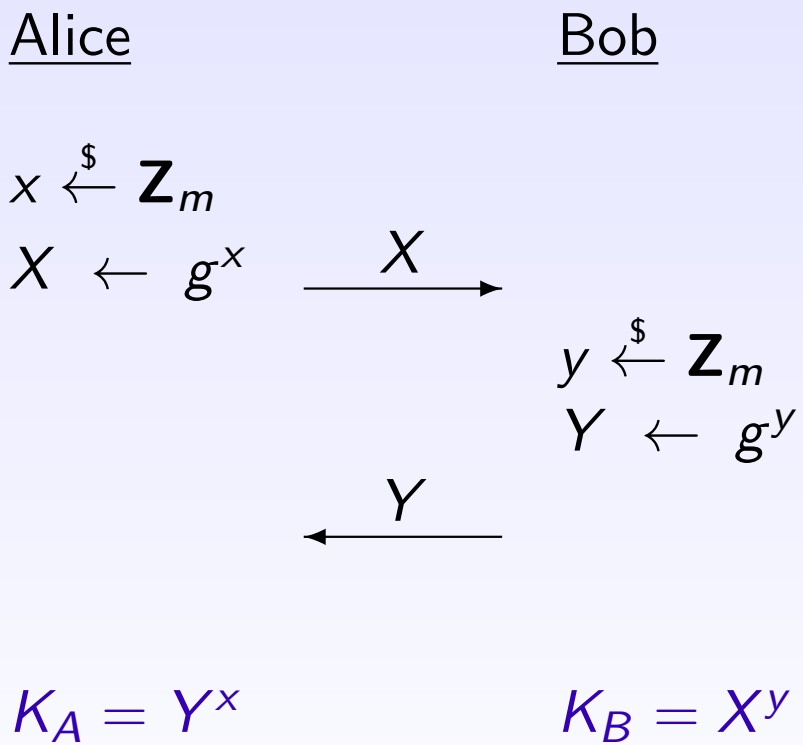
Definition

A field is a set $\mathbb{F}$ together with two binary operations on $\mathbb{F}$ called addition and multiplication. These operations are required to satisfy the following properties, referred to as field axioms (in these axioms, a, b, and $c \in \mathbb{F}$):

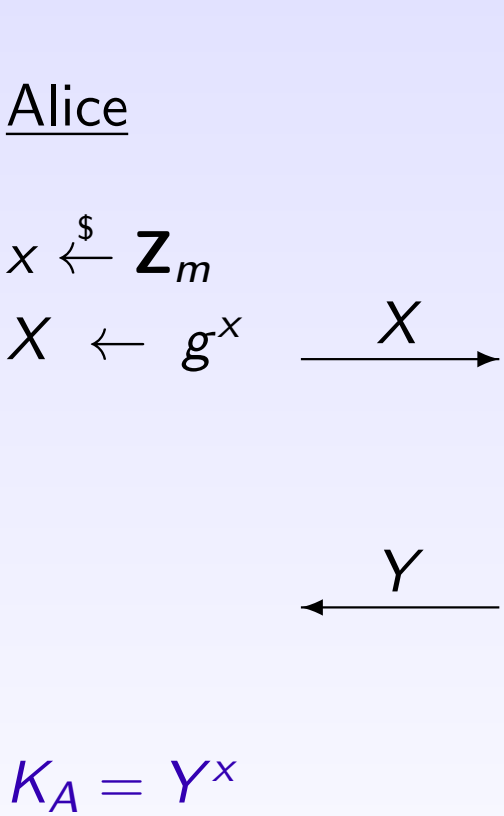
- Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- Commutativity of addition and multiplication: $a + b = b + a$, and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: there exist two different elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$.
- Additive inverses: for every a in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted $-a$, such that $a + (-a) = 0$.
- Multiplicative inverses: for every $a \neq 0$ in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted by a^{-1}, such that $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Diffie-Hellman Key Exchange protocol

Let G be a cyclic group of order m, and let g be a generator.



Relative hardness of discrete-log-based problems



Break DH KE

↑ easy

solve DDH

↑ easy

solve CDH

↑ easy

solve DLP