

**Investigating the Effects of Upper Lapse Rate and
Surface Heat Flux on an idealized Convective
Atmospheric Boundary Layer Entrainment Layer
using Large Eddy Simulation**

by

Niamh Chaparro

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1. discussion

1.1 Comparison of general Set-up

Sullivan and Patton (2011) found that the shapes of the vertical average potential temperature ($\bar{\theta}$) and average vertical heat flux ($\overline{w'\theta'}$) profiles, as well as the measured **CBL!** height vary depending on grid size. The resolution at which convergence begins is listed in Table 1.1. Until this point the $\bar{\theta}$ and $\overline{w'\theta'}$ profiles are such that the **EL!** is a larger portion of the **CBL!** and measured **CBL!** height is higher. Overall they concluded that vertical resolution was more critical. This compliments the conclusion Brooks and Fowler (2012) when discussing their resolution test. That is, to capture the steep vertical gradients in the **EL!** requires high resolution.

As Turner discusses in his 1986 review of turbulent entrainment, the important of smaller scale processes, ie at the molecular level are relatively unimportant. Large scale engulfment and trapping between thermals dominates. If $\overline{\theta'^2}$ is calculated based on differences from horizontally averaged θ , ie if the ergodic assumption is drawn upon, then it can be a measure of horizontal variance. Although Sullivan and Patton (2011) found the vertical distance over which $\overline{\theta'^2}$ varies more or less converged at the resolution shown in Table 1.1 the maximum continued to increase up to their finest grid spacing (5, 5, 2). This question of whether the mixing within the **EZ!** is adequately resolved serves as motivation for **DNS!** studies such as that of Garcia and Mellado (2014). These authors found the entrainment ratio $\frac{\overline{w'\theta'}_{z_f}}{\overline{w'\theta'}_s}$ to be about 0.1 which is lower than for example what FedConzMir04

(2004) observed, but close to what was seen in this study. Although they did not directly measure **EZ!** depth, based on their heat flux profiles the region of negative flux seems to be about .3 of the height of minimum flux, again this is comparable to what I saw. Among these author’s other conclusions was that the production and destruction rates of **TKE!**, as well as the entrainment ratio used to calculate the entrainment rate, were effectively independent of molecular scale processes.

Table 1.1: Grid spacing around the **EL!** used in comparable **LES!** studies. Those used for resolution tests are not listed here. For Sullivan and Patton’s 2011 resolution study I list the grid sizes at which profiles within the **EL!** and **CBL!** height evolution began to converge.

Publication	EZ! $\Delta x, \Delta y, \Delta z$ (m)	Horizontal main	Do-
Sullivan et al. (1998)	33, 33, 10	5 x 5 km	
Federovich et al. (2004)	100, 100, 20	5 x 5 km	
Brooks and Fowler (2012)	50, 50, 12	5 x 5 km	Sul- livan and Patton (2011)
20, 20, 8	5 x 5 km		
This study	25, 25, 5	3.4 x 4.8 km	

The **FFT!** energy spectra of horizontal slices at the top of the **ML!** show a substantial resolved inertial subrange giving confidence in the choice of horizontal grid size used here. In the **EL!** where turbulence is intermittent, the dominant energy containing structures are smaller, and decay to the grid-size is steeper. This confirms the assertion of Garcia and Mellado (2014) that the **EZ!** is separated into two sublayers in terms of turbulence scales.

The horizontal domain in this study is smaller than those used in the other

studies listed in Table 1.1. Sullivan et al. (1998) carried out one run on a smaller domain with higher resolution, noticed it resulted in lower **CBL!** height and concluded this was due to restricted plume size. However, given the results of Sullivan and Patton (2011) it could have been an effect of the grid. Visualizations of horizontal and vertical slices clearly showed multiple thermals with diameter increasing with increased **CBL!** height, but remaining less than or on the order of 100 meters.

Brooks and Fowler tried basing heights on average profiles but got lots of scatter and concluded that it was best to use statistics of locally determined heights rather than the mean profile. Sullivan and moeng’s heights based on average profiles produced very jagged oscillating versus time plots. Whereas the heights based on average profiles here, using an ensemble of cases, varied smoothly in time. This could be attributed to a smoother average based on more horizontal points (10*128*192 vs 100*100 vs 151*151).

A comparison of how the **Ri!**s were calculated will be left for a later section, but the range was dependent on variation in γ and less so on $\overline{w'\theta'}$. Brooks and Fowler (2012) and Sullivan et al. (1998) imposed a θ jump of varying strength topped by a constant γ . Whereas Federovich et al. (2004) initialized with a layer of uniform θ topped by a constant γ which was different for each run. Thesis authors did not vary average surface heat flux ($(\overline{w'\theta'})_s$). They also used a timescale based on γ rather than the convective timescale τ . The results of this study support this, in that the effects of varying $(\overline{w'\theta'})_s$ seem to be offset by h and $\tau = \frac{h}{w^*}$ depends solely on γ .

1.2 Local ML! heights

Sullivan et al. (1998) determined local **CBL!** height by locating the point of maximum gradient. Analysis of the resulting distributions showed dependence of standard deviation and skewness on Richardson number. The normalized standard deviation decreased with increased **Ri!** whereas skew-

Table 1.2: Grid spacing around the **EL!** used in comparable **LES!** studies. Those used for resolution tests are not listed here. For Sullivan and Patton’s 2011 resolution study I list the grid sizes at which profiles within the **EL!** and **CBL!** height evolution began to converge.

Publication	$\overline{w'\theta'}_s$ W/m ²	γ K/km	Jump	Ri! range
Sullivan et al. (1998)	20 - 450	3	.436 - 5.17	Federovich et al. (2004)
300	1 - 10	na	10 - 40	
Brooks and Fowler (2012)	10 -100	3	1 - 10	This study
60 - 150	2.5 - 10	na	10 - 30	

ness was almost bimodal; being negative at high **Ri!** and positive and low **Ri!**. Initially in this study, I applied a similar method and found distributions with lower **Ri!** to have positive skew. Upon exhaustive inspection of local vertical θ profiles, it became evident that at certain horizontal points high gradients well into the free atmosphere exceeded those closer to the location of the **CBL!** height reasonably identified by eye.

Locating the **ML!** height using the multi-linear regression method employed proved more reliable, based on inspection of hundreds of local vertical θ profiles. For a number of these profiles, ie those at points outside an actively impinging thermal, it was impossible even by eye to locate a reliable **CBL!** height based on a maximum in the vertical gradient but there as a discernible **EZ!**. Conversely at points within an active thermal the **ML!** tops were characterized by a sharp vertical θ gradient and the absence of an **EZ!** similar to the zero-order model, since here entrainment has yet to begin. All inspected local profiles showed a clear **ML!** and h_0^l . The distributions were seen to broaden with increased $(\overline{w'\theta'})_s$ and narrow with increased γ . When normalized by the height of the maximum average vertical potential gradient (h) what apparently remains is the effect of γ on the lower limit

or lowest percentile. That is, the lowest **ML** heights become lower with at lower γ resulting in an apparant increased negative skew. So, there is an overall narrowing of the scaled distributions with increased γ . This supports decreased of scaled **EZ** depth with increased **Ri** of Section ?? since the **Ri**s tend to group according to γ . The approximate range of the scaled **EZ** based on the h_0^l distributions is about 0.2 - 0.4 of h whereas that calculated by Brooks and Fowler (2012 based on distributions of local maximums in gradient was smaller (.05 - .2). This could be partially explained by the local maximum gradient of the tracer profile being within the **EZ** of points for example outside an actively impinging plume and so higher than the **ML** top defined here.

Potential temperature and vertical velocity fluctuations (θ' and w') at several vertical levels around the **EL** were plotted as 2 dimensional histograms. Increased $\overline{w'\theta'}_s$ causes an increase in positive temperature fluctuations and vertical velocity as thermals become more vigorous, causing a higher h so and a deeper **EZ** over which relatively warmer air is pulled down. The convective velocity scales (θ^* and w^*) were applied to isolate the effects of γ , although it is accounted for indirectly via h (see Section ??). As shown in Sorbjan's (1996) θ' is influenced by γ . For example at h there is an apparant increase in the spread, as well as a shift towards the positive. So downward moving positive potential temperature fluctuations representing air from the **FA** are more positive and negative fluctuations representing thermals are less negative, when scaled. The former can easily be attributed to relatively warmer air over a shorter vertical distance (**EZ**) being brought down but an explanation for the latter remains elusive. I conclude that with increased γ there is a positive deviation from the convective temperature scale. Also, there is an apparant damping of the scaled velocities associated with positive temperature fluctuations. So the w^* scales these less effectively with increased γ .

At z_f and h , ie within the **EL** the quadrants of largest magnitude were upward and downward moving relatively cool, thermal, air ($w'^-\theta'^+$ and

$w'^+\theta'^+$). The $w'^-\theta'^-$, $w'^+\theta'^-$ and $w'^-\theta'^+$ quadrants do approximately cancel as Sullivan et al. (1998) concluded. The downward moving warm quadrant ($w'^-\theta'^+$) at h represents warmer free atmosphere air that is being entrained. So its magnitude at a certain point in time is an indication of the increased at a successive time in the region below. In Figure ?? the magnitude increases with respect to time is grouped according to $(\overline{w'\theta'})_s$. Indeed it is an increasing proportion of $(\overline{w'\theta'})_s$ and Figures ?? and ?? show that its increase is primarily due to the increased positive temperature variance (θ'^+). While the velocity of downward warm quadrant w'^- quickly approaches a constant proportion of w^* , the magnitude of temperature fluctuation approaches a constant proportion of $\gamma\Delta h$ or $\gamma(h_1 - h)$ rather than the convective temperature scale θ^* . This is similar to the buoyancy scale for the upper **EZ!** used by Garcia and Mellado's (2014). So the positive variance in temperature is related to difference in temperature over Δh (or $h_1 - h$) of the initial lapse rate γ . A broad explanation is that air from h_1 (ie at potential temperature $\overline{\theta}_0(h_1)$) is brought down by the motions described in ?? to h at which point much of the air is still at $\overline{\theta}_0(h)$. Garcia and Mellado's (2014) describe this region as the upper sub layer within the **EZ!** where the buoyancy and turbulence are strongly influenced by γ . The relationship between the variance and vertical gradient in temperature is clearly shown in the plots of each in Sorbjan's (1996) and Sullivan et al.'s (1998) and Garcia and Mellado's (2014). Their peaks within the acsEL seem to coincide. In the mixed layer vigorous horizontal and vertical motion renders both close to zero. The increased in average potential temperature in the lower **EZ!** (and upper **ML!**) is in part due to the entrainment of warm air from aloft which can be represented by the magnitude of $(\theta'_h)^+$ coming down from above and this depends on γ . Another perspective is noticing the increased magnitude of the minimum $\overline{w'\theta'}$, seen here and in both Sorbjan's (1996) and FedConzMir04 (2004), in the **EZ!** which then leads to increased $-\frac{\partial \overline{w'\theta'}}{\partial z}$ in the lower **EZ!**, since the profiles are otherwise similar, and so increased warming per Equation ??.

1.2.1 Relationship of Entrainment Layer Depth to Richardson Number

The **EZ!** is inhomogenous and on average is a region of transition as clearly represented by the $\bar{\theta}$ profile and where thermals overturn or recoil initiating entrainment as represented by the heat flux profile. The $\bar{\theta}$ profile partially characterises the thermodynamic state of the **CBL!** as well defining its three layer structure. It is directly comparable to both bulk models and local θ profiles which in turn are comparable to a sounding unlike a heat flux profile which is an inherently average quantity. Neither of the two comparable **LES!** (see Table ??) studies define the **EL!** based on the vertical $\frac{\partial \bar{\theta}}{\partial z}$ profile. So, for to enable direct comparison heights were based on the heat flux profile. Based in this framework Federovich et al.'s (2004) show decreasing scaled **EZ!** with increaseing **Ri!** (decreasing **Ri!**⁻¹) and an exponent of of $-\frac{1}{2}$. They say the decrease in the overall depth is due to a slight decrease in the scaled top limit over time. However based on their plot, it seems to go from about .5 to .2. Brooks and Fowler (2012) found no clear **Ri!** dependence of the scaled **EL!** depth based on the heat flux profile, but their definition was based on the lower part, i.e, the portion that according to Federovich et al.'s (2004) does not vary in time. In this study when based on the heat flux profiles as Federovich et al.'s (2004) did, there is no clear dependence on **Ri!**. This is supported by the similarity in time and accross runs of the vertical turbulent heat flux profiles when scaled by $(\overline{w'\theta'})_s$. The most obvious possible cause for this disagreement with the findings of citeauthorFedConzMir04's (2004) is the difference in grid size shown in Table 1.1. Inspection of their heat flux profiles confirms a relatively deeper region of negatflux as compared with those seen here (.4 vs .25). Their surface heat flux was twice the highest used here, but their range of **Ri!** is comparable to that of this study, which serves as confirmation that γ is the more influential parameter.

Here, when heights are defined on the vertical potential temperature profile cruves representing Equation ??

Publication	EL!	CBL! height	$\Delta\theta$
Federovich et al. (2004)	$z_{f1} - z_{f0}$	z_f	$\frac{\bar{\theta}(z_{f1})}{\bar{\theta}(z_{f0})}$ —
Brooks and Fowler (2012)	$2 \times (z_f - z_{f0})$	z_f	average of local values

$$\frac{\Delta h}{h} \propto Ri^b \quad (??)$$

are grouped according to γ and show an (or slope) exponent which increases in magnitude with increasing γ . Overall there is a clear narrowing of the scaled **EZ!** depth with increased **Ri!** (decreased **Ri!**⁻¹). When the $\frac{\partial \bar{\theta}}{\partial z}$ profile is scaled by γ the relative magnitudes of the gradient in the upper **EZ!** are reversed. This reflects as a higher **EZ!** bottom limit (h_0) and so a narrower **EZ!** with increased γ . The resulting scaled **EL!** depth as defined in Figure ?? shows a dependence on Richardson number (**Ri!**) which appears to have an exponent of $-\frac{1}{2}$, as predicted and seen by Boers (1989), at lower **Ri!** (higher **Ri!**⁻¹) possibly increasing to -1 at higher **Ri!**. Although based on different height definitions Federovich et al. (2004) concluded an $a = -\frac{1}{2}$ and Brooks and Fowler's (2012) plots show curves with an apparant exponent less in magnitude than -1 . The curves seem to separate out for each run. In this study, before scaling the $\frac{\partial \bar{\theta}}{\partial z}$ profile curves separate out, but in the reverse order. I see runs with higher stability exhibiting larger **EL!** depths. Brooks and Fowler (2012)'s runs with initially lower **Ri!** (higher **Ri!**⁻¹) have larger **EL!** depths than runs with initially higher **Ri!**s, even where **Ri!** values overlap.

Neither study addresses a change in exponent with increased **Ri!**. It is possible that there is a change in entrainment mechanism. Sullivan et al. (1998) observed enfolding and engulfment at lower **Ri!**. Whereas at higher **Ri!** when motion is more restricted, entrainment seemed to occur via trapping of thinner wisps at the edge of an upward moving thermal. Turner (1986)

also distinguishes between entrainment by convective overturning and recoil. Garcia and Mellado's (2014) refer to a change in entrainment rate due to the effects of increased stability on the upper **EZ!** sublayer. In this study, although the narrowing of the **EZ!** as defined here depends predominantly on the magnitude of the potential temperature gradient in the lower **EZ!** and upper **ML!** the scaled magnitude upper limit based on the $\frac{\partial \bar{\theta}}{\partial z}$ does appear to decrease slightly in time.

1.3 Relationship of Entrainment Rate to Richardson Number (Q3)

The magnitudes of **Ri!** numbers determined in this and the comparable studies are primarily influenced by the magnitude of the θ jump. So with different definitions you can have the same conditions represented by a different **Ri!** value. Here, I define it in two ways as both Federovich et al. (2004) and Garcia and Mellado (2014) did. I do this based on the flux profile for the purpose of direct comparison with these studies and to observe how the change in definition effects Equation ??.

$$\frac{w_e}{w^*} \propto Ri^a \quad (??)$$

As in these studies, the larger jump ie that taken accross the **EZ!** ($\Delta\theta$) yields a larger a as Federovich et al. (2004) conclude. Although Garcia and Mellado (2014) interpret their curves as both asymptoting to a straight line ($a = -1$) as the upper **EZ!** sublayer narrows. Based on their plots alone, ie in the absence of their justification based on the derivation of the entrainment relation, I would not conclude this asymptotic behavior. Rather for $\Delta\theta$ I see a curve with increasing exponent exceeding magnitude -1 at higher **Ri!**. For $\delta\theta$ I see a curve with exponent less in magnitude than -1 .

That there is an analogous distinction between plots of Equation ?? using

$\Delta\theta$ vs $\delta\theta$ when all heights are defined on the $\frac{\partial\bar{\theta}}{\partial z}$ profile lends some credence to this framework. Scatter is least when the θ jump is defined accross the **EL!**. In this plot $-\frac{3}{2}$ fits at higher **Ri!** (lower **Ri!**⁻¹) and -1 seems to fit at lower **Ri!**. Combined with the apparant change in b for Equation ?? this could be interpreted as an indication of a change in entrainment regime at increased **Ri!**.

1.4 Conclusion

The magnitude and variance of local height increase with increasing $\overline{w'\theta'}_s$ and decrease with increasing γ . Similarly for the vertical velocity fluctuations in the **EZ!**. However increased γ results in an increase in the positive temperature fluctuations at h . The magnitude of these positive temperature perturbations at points where vertical velocity is negative represents downward moving entrained air. Below h in the lower **EZ!** the temperature gradient increases with increasing γ . So, the growth of the idealized dry **CBL!** is driven by $\overline{w'\theta'}_s$ and suppressed by γ . But warming is due to $\overline{w'\theta'}_s$ and the entrainment of air from aloft the temperature of which in turn depends on γ .

Both the **EZ!** depth and **CBL!** height based on the average $\frac{\partial\bar{\theta}}{\partial z}$ profile show dependence on **Ri!** as seen in other studies and justified theoretically. This profile serves to characterize the **CBL!** and links bulk models to soundings via an **LES!**. So it is a valid way of defining the **CBL!** and its **EZ!**. Plots of Equations exhibit changes in exponent. A change in entrainment mechanism or regime with increased **Ri!** has been observed in measurement as well as **LES!** and justified theoretically. I suggest the change in exponents represent this.

Throughout this study threads the influence of γ . The convective time scale $\tau = \frac{w^*}{h}$ and **Ri!** group according to this parameter. This justifies the use of the brunt vaisalla time scale as well as the constant heat flux with varying

lapse rate of Federovich et al. (2004). I conclude once the effects of $\overline{w'\theta'}_s$ are accounted for through h , γ remains the dominant parameter in idealized **CBL!** entrainment.

References

- Boers, R., 1989: A parametrization of the depth of the entrainment zone. *Journal of Applied Meteorology*, 107–111.
- Brooks, I. M. and A. M. Fowler, 2012: An evaluation of boundary-layer depth, inversion and entrainment parameters by large-eddy simulation. *Boundary-Layer Meteorology*, **142**, 245–263.
- Federovich, E., R. Conzemus, and D. Mironov, 2004: Convective entrainment into a shear-free, linearly stratified atmosphere: Bulk models reevaluated through large eddy simulation. *Journal of the Atmospheric Sciences*, **61**, 281 – 295.
- Garcia, J. R. and J. P. Mellado, 2014: The two-layer structure of the entrainment zone in the convective boundary layer. *Journal of the Atmospheric Sciences*, doi:10.1175/JAS-D-130148.1.
- Sorbj an, Z., 1996: Effects caused by varying the strength of the capping inversion based on a large eddy simulation of the shear free convective boundary layer. *Journal of the Atmospheric Sciences*, **53**, 2015 – 2023.
- Sullivan, P. P., C.-H. Moeng, B. Stevens, D. H. Lenschow, and S. D. Mayor, 1998: Structure of the entrainment zone capping the convective atmospheric boundary layer. *Journal of the Atmospheric Sciences*, **55**, 3042–3063, doi:10.1007/s10546-011-9668-3.
- Sullivan, P. P. and E. G. Patton, 2011: The effect of mesh resolution on convective boundary layer statistics and structures generated by large eddy simulation. *Journal of the Atmospheric Sciences*, **58**, 2395–2415, doi:10.1175/JAS-D-10-05010.1.
- Turner, J. S., 1986: Turbulent entrainment: the development of the entrainment assumption and its application to geophysical flows. *J. Fluid Mech.*, **173**, 431–471.

A. Appendices

A.1 Potential Temperature: θ

$$\theta = T \left(\frac{p_0}{p} \right)^{\frac{R_d}{c_p}} \quad (\text{A.1})$$

p_0 and P are a reference pressure and pressure respectively.

$$\frac{c_p}{\theta} \frac{d\theta}{dt} = \frac{c_p}{T} \frac{dT}{dt} - \frac{R_d}{p} \frac{dp}{dt} \quad (\text{A.2})$$

If changes in pressure are negligible compared to overall pressure, as in the case of that part atmosphere that extends from the surface to 2Km above it.

$$c_p \frac{d\theta}{\theta} = c_p \frac{dT}{T} - \frac{R_d}{p} \frac{dp}{p} \quad (\text{A.3})$$

$$\frac{d\theta}{\theta} = \frac{dT}{T} \quad (\text{A.4})$$

and if

$$\frac{\theta}{T} \approx 1 \quad (\text{A.5})$$

then small changes in temperature are approximated by small changes in potential temperature

$$d\theta \approx dT \text{ or } \theta' \approx T' \quad (\text{A.6})$$

and at constant pressure change in enthalpy (H) is

$$dH = c_p dT. \quad (\text{A.7})$$

A.2 Second Law of Thermodynamics

$$\frac{ds}{dt} \geq \frac{q}{T} \quad (\text{A.8})$$

For a reversible process

$$\frac{ds}{dt} = \frac{q}{T} \quad (\text{A.9})$$

Using the first law and the equation of state for an ideal gas

$$\frac{q}{T} = \frac{1}{T} \left(\frac{dh}{dt} - \alpha \frac{dp}{dt} \right) = \frac{c_p}{T} \frac{dT}{dt} - \frac{R_d}{p} \frac{dp}{dt} \quad (\text{A.10})$$

so

$$\frac{ds}{dt} = \frac{q}{T} = \frac{c_p}{\theta} \frac{d\theta}{dt} \quad (\text{A.11})$$

For a dry adiabatic atmosphere

$$\frac{ds}{dt} = \frac{c_p}{\theta} \frac{d\theta}{dt} = 0 \quad (\text{A.12})$$

A.3 Reynolds Decomposition and Simplification of Conservation of Enthalpy (or Entropy) for a dry Atmosphere

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = \nu_\theta \frac{\partial^2 \theta}{\partial x_i^2} - \frac{1}{c_p} \frac{\partial Q^*}{\partial x_i} \quad (\text{A.13})$$

ν and Q^* are the thermal diffusivity and net radiation respectively. If we ignore these two effects then (adiabatic?)

$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = 0 \quad (\text{A.14})$$

$$\theta = \bar{\theta} + \theta', \theta = \bar{u}_i + u_i' \quad (\text{A.15})$$

$$\frac{\partial \bar{\theta}}{\partial t} + \frac{\partial \theta'}{\partial t} + \bar{u}_i \frac{\partial \bar{\theta}}{\partial x_i} + u_i' \frac{\partial \bar{\theta}}{\partial x_i} + \bar{u}_i \frac{\partial \theta'}{\partial x_i} + u_i' \frac{\partial \theta'}{\partial x_i} = 0 \quad (\text{A.16})$$

Averaging and getting rid of average variances and their linear products

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{u}_i \frac{\partial \bar{\theta}}{\partial x_i} + u_i' \frac{\partial \theta'}{\partial x_i} = 0 \quad (\text{A.17})$$

Ignoring mean winds

$$\frac{\partial \bar{\theta}}{\partial t} + u_i' \frac{\partial \theta'}{\partial x_i} = 0 \quad (\text{A.18})$$

using flux form

$$\frac{\partial \bar{\theta}}{\partial t} + \frac{\partial (u_i' \theta')}{\partial x_i} - \theta' \frac{\partial u_i'}{\partial x_i} = 0 \quad (\text{A.19})$$

under the bousinesq assumption $\Delta \dot{u}_i = 0$

$$\frac{\partial \bar{\theta}}{\partial t} = - \frac{\partial (u_i' \theta')}{\partial z} \quad (\text{A.20})$$

ignoring horizontal fluxes

$$\frac{\partial \bar{\theta}}{\partial t} = - \frac{\partial (w' \theta')}{\partial z} \quad (\text{A.21})$$

A.4 Reynolds averaged Turbulence Kinetic Energy Equation

$$\frac{\partial \bar{e}}{\partial t} + \bar{U}_j \frac{\partial \bar{e}}{\partial x_j} = \delta_{i3} \frac{g}{\theta} \left(\overline{u_i' \theta'} \right) - \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial \left(\overline{u_j' e'} \right)}{\partial x_j} - \frac{1}{\bar{\rho}} \frac{\partial \left(\overline{u_i' p'} \right)}{\partial x_i} - \epsilon \quad (\text{A.22})$$

e is turbulence kinetic energy (TKE). p is pressure. ρ is density. ϵ is viscous dissipation.