COMPUTER SCIENCE 20, SPRING 2015
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1. (a) Prove that a graph is 2-colorable if and only if it is bipartite.
   (b) Prove that any tree is 2-colorable.

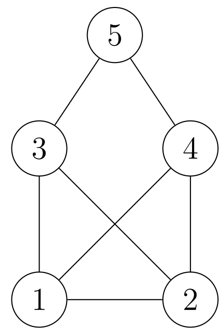
   Solution:

   (a) Let's start by showing that if a graph is 2-colorable, it is bipartite. A bipartite graph is a graph whose vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to one in the other. Because an edge cannot connect two vertices within a set (definition of bipartite), we can color the first set one color, and the second set another color and no vertices colored the same color have adjacentedges. Next, let's prove the converse that if a graph is 2-colorable, it is also bipartite. If a graph is 2-colorable, say red and green, then all vertices in the graph can be separated into two sets, one containing all red vertices and one containing all green vertices. No two vertices within a set are connected to each other (else it would not be 2-colorable since they can't both be a single color if they're connected), the sets are disjoint since no vertex can be both red and green. Therefore, we've shown that there are two sets that are disjoint such that every edge connects a vertex in one set to one in the other, making the graph bipartite.

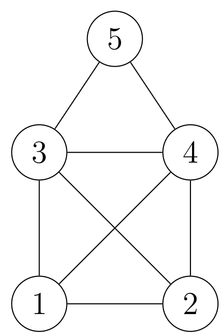
   (b) Let's start by proving that a tree is bipartite. Starting with an (arbitrary) "root" node, for any vertex with a unique path length that's even (where length is defined by the number of edges from the root node), we can call that set S. For any vertex with a unique path length that's odd, we can call this set T. We know that S and T must be disjoint since a unique path length from the root node cannot be both odd and even at the same time. We also know that for every edge, it is connected to an even vertex in S and for an odd vertex in T since an odd number is just one away from an even number. Therefore, since we've proved that the sets S and T are both disjoint and that every edge connects a vertex in one set to one in the other, we've proved that our tree is indeed bipartite. Then, in conjunction with 1a, we know that if a tree is bipartite, then it's also 2-colorable.

2. Determine, with justification, the chromatic numbers for the graphs below:

   (a)



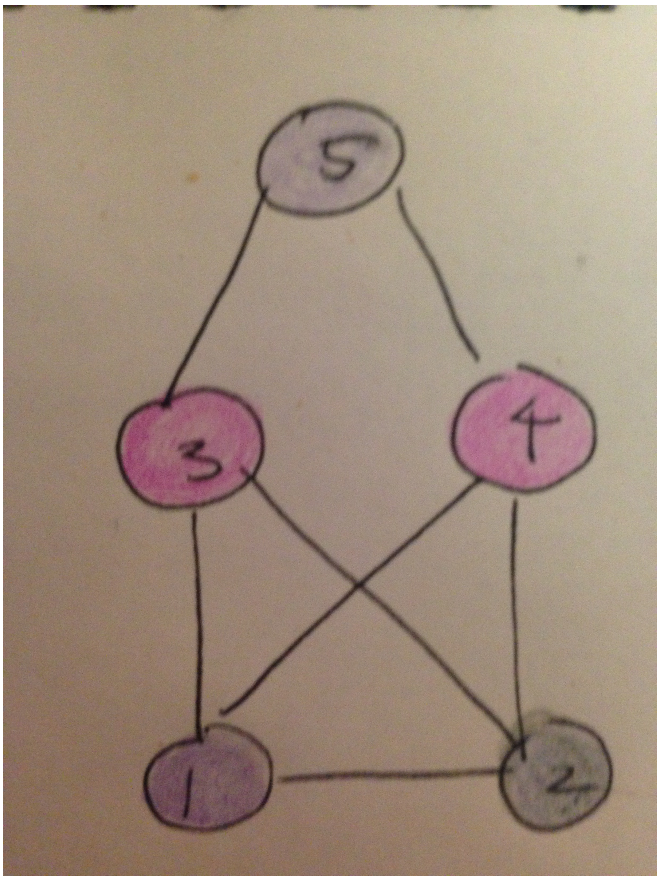
(b)



(c) K_n, the complete graph on n vertices.

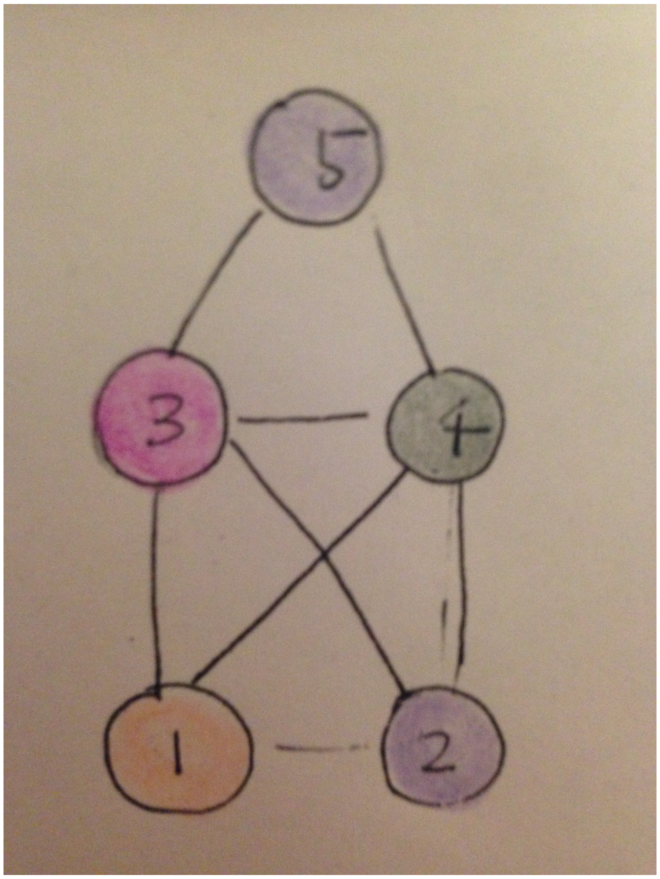
Solution:

(a) The chromatic number for the graph here is 3.



We've shown that it's possible to color the full graph with three colors (above), and that we can't use less than 3 colors since there is a complete graph of 3 vertices that all connect to each other (we can say the set of vertices 1, 2, 3 or 1,2,4) . Therefore, the chromatic number is 3.

(b) The chromatic number for the graph here is 4.



We've shown that it's possible to color the full graph with three colors (above), and that we can't use less than 3 colors since there is a complete graph of 4 vertices that all connect to each other (vertices 1, 2, 3, and 4). Therefore, the chromatic number is 4.

(c) The complete graph K_n on n vertices requires n colors. Since it is a complete graph, all vertices connect to one another and are adjacent. Therefore, no vertices can share a color, and n colors are needed. We can also prove by induction: Base case: looking at a graph with 1 vertex,it does indeed require 1 color. Inductive hypothesis: a complete graph K_n on n vertices requires n colors. Inductive step: let's now look at a graph of $K_(n+1)$. If we add one vertex v to our complete graph of n vertices, in order for it to be a complete graph, v must share an edge with all n other vertices. Therefore, given that it is adjacent to all n vertices, it cannot share a color with any other vertex. We must then use one more color so altogether, so we use a total of n colors (from our inductive hypothesis about a complete graph of n vertices) + 1 (from the vertex we just added). Therefore, our graph of $K_(n+1)$ requires $n+1$ colors meaning our proof is complete.

3. Let d be the maximum degree of any vertex in a graph G. Prove that the chromatic number of G is at most $d+1$.

   Solution: Let's prove by induction.

   Base case: in a graph with only 1 vertex, then n = 1. the maximum in-degree here is 0, so d=0. In this case, the number of colors needed is 1 color, and indeed 1 is maximally d+1 since d = 0.

   Inductive hypothesis: let's assume that in a graph with n vertices and max in-degree d, the maximum number of colors needed is $d+1$.

   Inductive step: Now let's look at the case with $n+1$ vertices. When we add vertex v, we know that v has maximally d vertices it is adjacent to since we know that in-degree is maximally d (given). Even in the worst case scenario where each adjacent vertex is a different color, we still only use d colors. Given our inductive hypothesis, where we use $d+1$ colors for a graph of n vertices, we find that we still have 1 color left for v itself. Thus, in a graph with $n+1$ vertices and max in-degree d, the maximum number of colors needed is $d+1$ and our induction is complete.

4. Fill in the following table—check off the box corresponding to functions f and g and relation ρ iff $f(x)$ is $\rho(g(x))$. For example, you would check off the lower-right box if you think that $\sqrt{x} \sim \log(x)$.

$f(x)$	$g(x)$	O	o	Ω	Θ	$\sim$
x	$x+100$					
$(0.1)^x$	1					
2^x	3^x					
$\log(x)$	$\log(x^2)$					
$x!$	2^x					
$x!$	x^x					

   Solution:

$f(x)$	$g(x)$	O	o	Ω	Θ	$\sim$
x	$x+100$	Y	N	Y	Y	Y
$(0.1)^x$	1	Y	Y	N	N	N
2^x	3^x	Y	Y	N	N	N
$\log(x)$	$\log(x^2)$	Y	N	Y	Y	N
$x!$	2^x	N	N	Y	N	N
$x!$	x^x	Y	Y	N	N	N

5. Show that O is a transitive relation. That is, show that if h is $O(g)$ and g is $O(f)$, then h is $O(f)$.

   Solution: Let's start with a proof by contradiction: let's assume that $h = O(g) \wedge g = O(f) \to \neg(h = O(f))$. $h = O(g)$ means that $\lim_{x\to\infty} \frac{h(x)}{g(x)}$ ¡ ∞ and $g = O(f)$ means that $\lim_{x\to\infty} \frac{g(x)}{f(x)}$ ¡ ∞. According to limit properties, the limit of the products is equal to the product of the limits. Therefore $(\lim_{x\to\infty} \frac{h(x)}{g(x)})(lim_{x\to\infty} \frac{g(x)}{f(x)}) = \lim_{x\to\infty} \frac{h(x)}{g(x)} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{h(x)}{f(x)}$. We know that $(\lim_{x\to\infty} \frac{h(x)}{g(x)})(lim_{x\to\infty} \frac{g(x)}{f(x)})$ is ¡ ∞ since the product of two numbers that are not ∞ cannot result in ∞. But this means that $\lim_{x\to\infty} \frac{h(x)}{f(x)}$ is also ¡ ∞. But now there is a contradiction since our original assumption stated that $h = O(g) \wedge g = O(f) \to \neg(h = O(f))$. $h = O(g)$. The $\neg(h = O(f))$. $h = O(g)$ means that $\lim_{x\to\infty} \frac{h(x)}{f(x)}$ is not less than ∞. Because of this contradiction, our proof is complete: O is a transitive relation.