From higher Bruhat orders to Steenrod cup-i coproducts, arXiv:2309.16481

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- We give a construction which associates a coproduct on the chain complex of the simplex to an element of the higher Bruhat orders.
- The minimal and maximal elements of the higher Bruhat orders recover the Steenrod cup-i coproducts.
- Our construction allows us to give simple geometric proofs of the key properties of these coproducts.

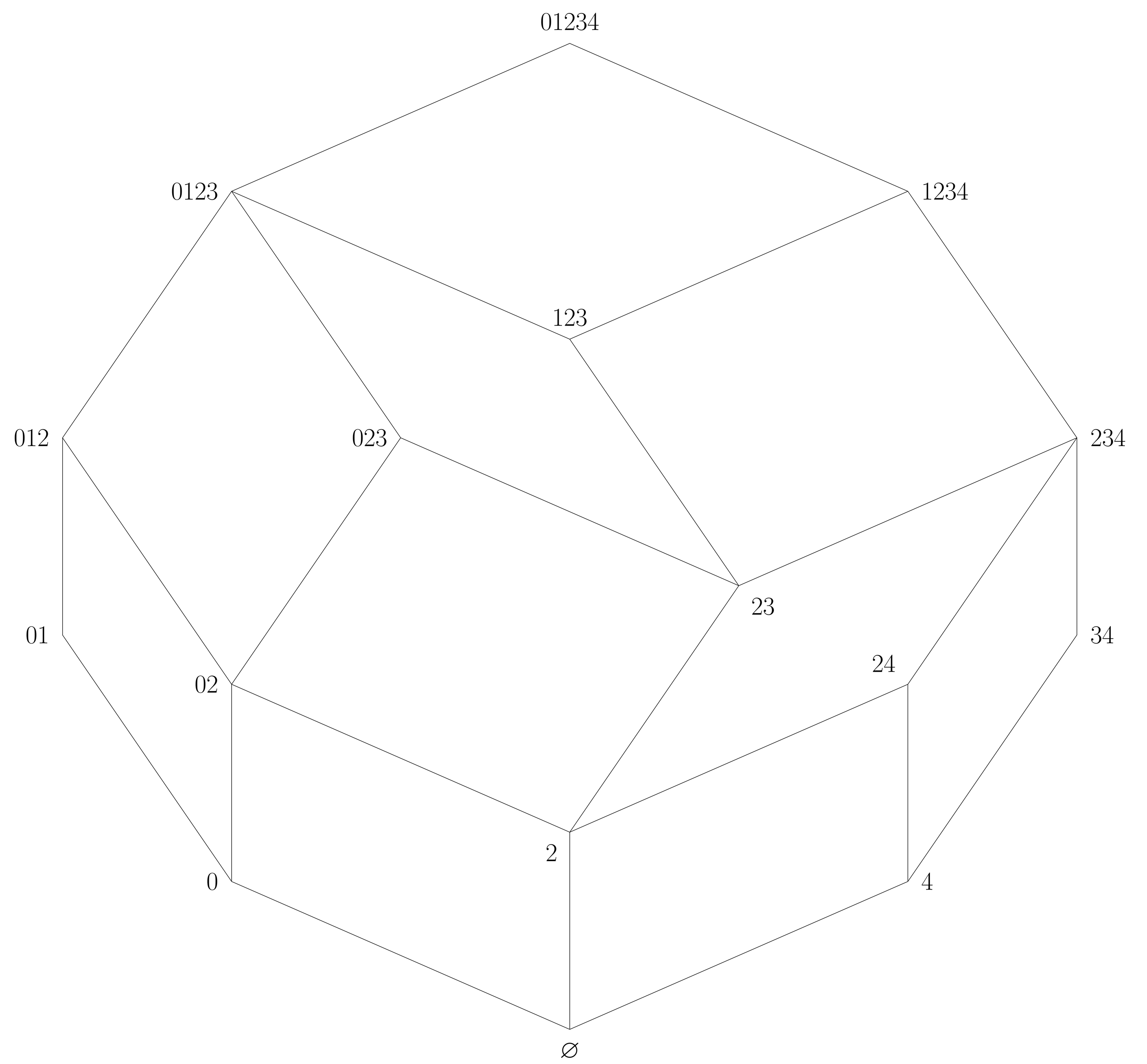


Figure 1: Cubillage of $Z(5,2)$

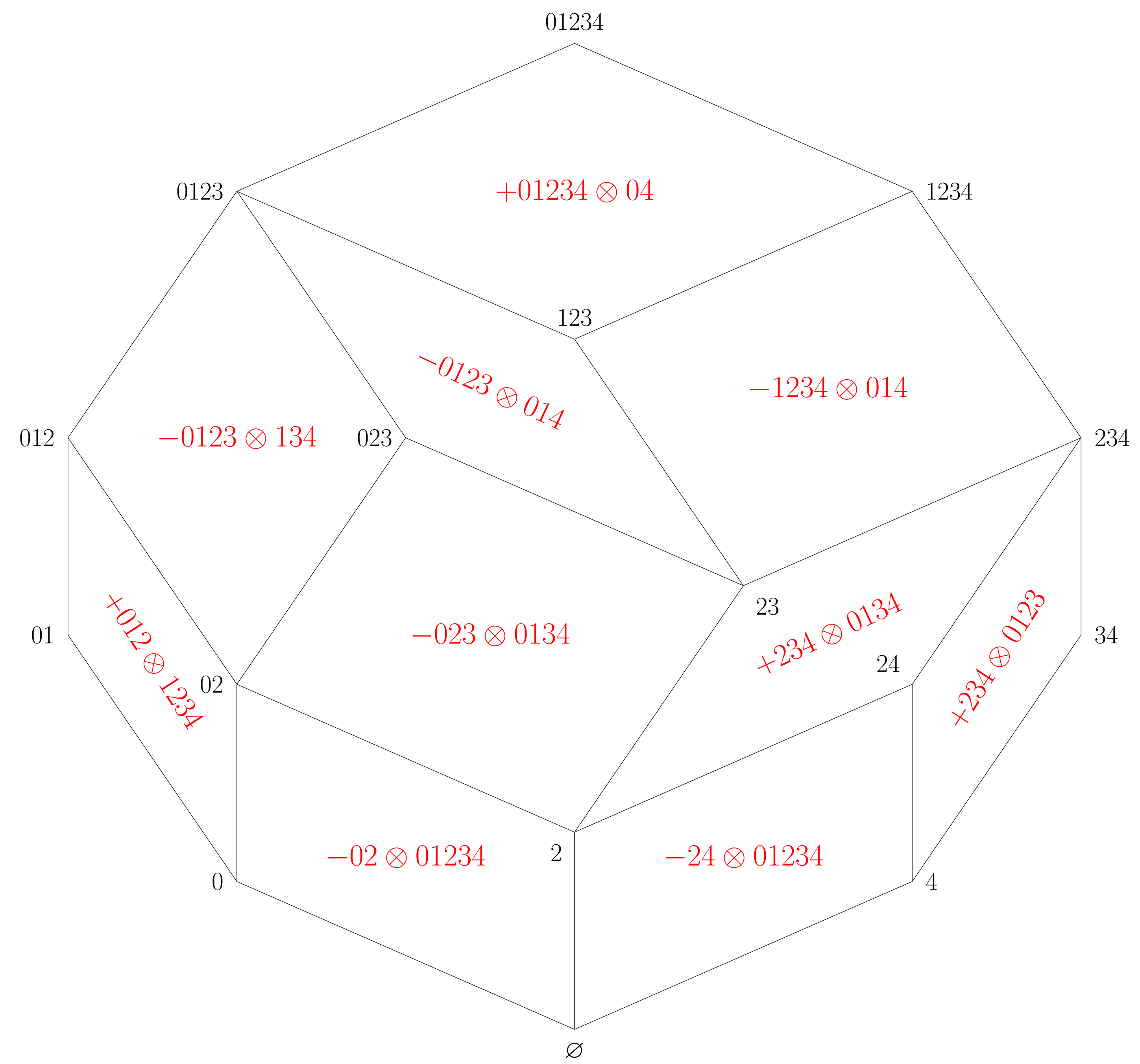


Figure 2: Coproduct defined by the cubillage

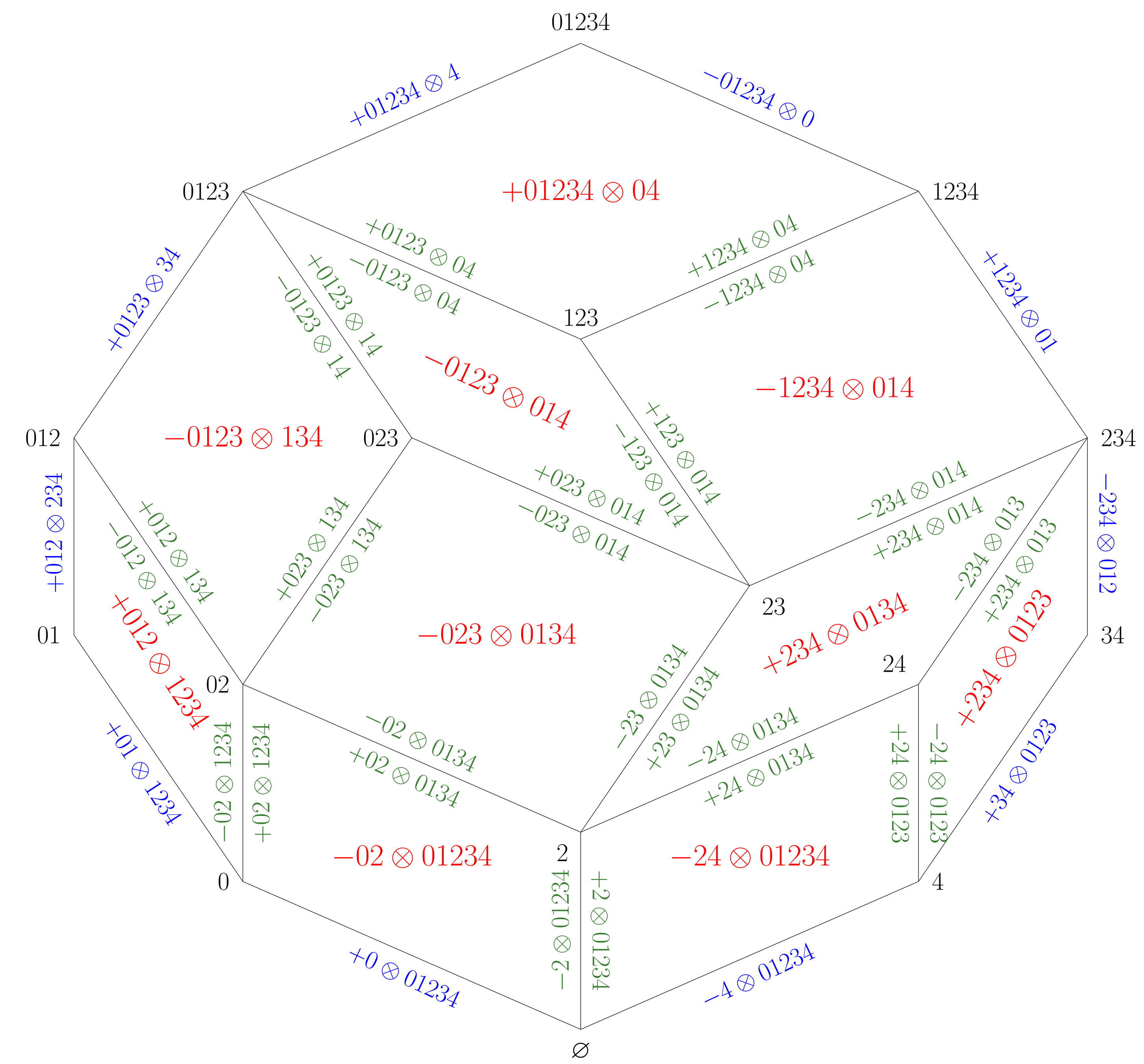


Figure 3: Geometric explanation of homotopy formula (1)

Higher Bruhat orders

Higher Bruhat orders $\mathcal{B}(n,k)$: family of posets introduced by Manin and Schechtman [MS89]. They essentially describe a higher-categorical structure on the weak Bruhat order on the symmetric group S_n.

- $\mathcal{B}(n,1)$: weak Bruhat order on S_n.
- $\mathcal{B}(n,k)$: equivalence classes of maximal chains in $\mathcal{B}(n,k-1)$.
- $\mathcal{B}(n,k)$ can be described in terms of "cubillages" of "cyclic zonotopes" $Z(n,k)$.

Consider $\xi\colon \mathbb{R} \to \mathbb{R}^{i+1}$, $t \mapsto (1,t,t^2,\dots,t^i)$. A **cyclic zonotope** $Z(n,i+1)$ is the Minkowski sum of line segments $\overline{0\xi(t_1)},\dots,\overline{0\xi(t_n)}$ for $t_1,\dots,t_n \in \mathbb{R}$.

A **cubillage** of a cyclic zonotope is a tiling by parallelotopes. We refer to these parallelotopes as "tiles".

References

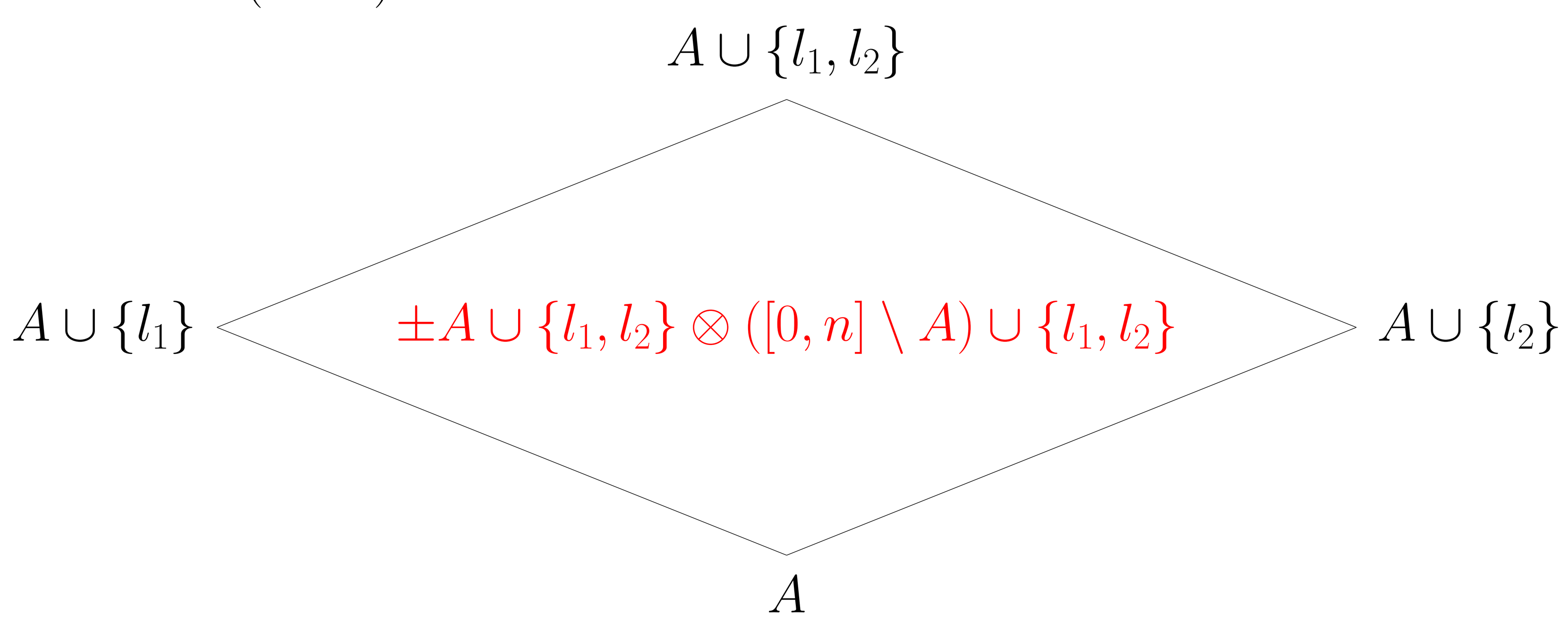
[MS89] Yuriĭ I. Manin and Vadim V. Schechtman. "Arrangements of hyperplanes, higher braid groups and higher Bruhat orders". In: *Algebraic number theory*. Vol. 17. Adv. Stud. Pure Math. Academic Press, Boston, MA, 1989.

[Ste47] Norman E. Steenrod. "Products of cocycles and extensions of mappings". In: *Ann. of Math. (2)* 48 (1947).

Our construction

Let Δ^n be the standard n-simplex, with $C_\bullet(\Delta^n)$ and $C^\bullet(\Delta^n)$ the associated chain complex and cochain complexes.

Given a cubillage $U \in \mathcal{B}(n+1,i+1)$ of $Z(n+1,i+1)$, we assign terms in $C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ to each tile of U. We illustrate this in two dimensions ($i=1$).



We then define a coproduct $\square_i^U\colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ on the top face $[0,n]$ as the sum of these terms over all the tiles in the cubillage U.

There is an analogous description for smaller faces.

Steenrod cup-i coproducts

The well-known cup product $\smile\colon C^\bullet(\Delta^n) \otimes C^\bullet(\Delta^n) \to C^\bullet(\Delta^n)$ is the linear dual of a cup coproduct $\square\colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$.

As $\square$ is not cocommutative, one extends it to an infinite tower of **Steenrod cup-i** coproducts $\square_i\colon C_\bullet(\Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ for $i \geqslant 0$ where $\square_0 = \square$, such that

$$\partial\square_i - (-1)^i\square_i\partial = (1 + (-1)^i T)\square_{i-1}, \qquad (1)$$

where $T\colon X \otimes Y \mapsto Y \otimes X$ is the exchange of tensor factors [Ste47].

One interprets (1) as saying that $\square_i$ gives a homotopy between $\square_{i-1}$ and $T\square_{i-1}$, thereby resolving the lack of cocommutativity of $\square_{i-1}$.

We have $\square_i^U = \square_i$ when U is either the minimal or the maximal element of $\mathcal{B}(n+1,i+1)$.

Our geometric explanation of the homotopy formula is that the terms of $\square_i$ form a cubillage of a cyclic zonotope. The right-hand side of (1) gives the terms on the boundary of the zonotope, which is equal to the left-hand side, since the terms from internal facets of tiles cancel out.

Formula (1) hence holds for any cubillage U.