An algebraic interpretation of the higher Stasheff–Tamari orders

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Overview

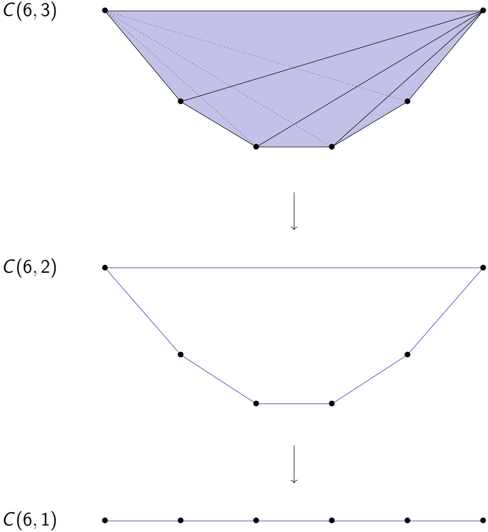
Algebra	Convex geometry	
Higher Auslander algebras of type A	Cyclic polytopes	
Tilting modules	$2d$-dim triangulations	Oppermann–Thomas '12
Riedtmann–Schofield orders	Higher Stasheff–Tamari orders	Buan–Krause '04 ($d = 1$), W '20
d-maximal green sequences	$(2d+1)$-dim triangulations	W '20
Orders on d-maximal green sequences	Higher Stasheff–Tamari orders	W '20

Plan

1. The higher Stasheff–Tamari orders
2. Higher Auslander–Reiten theory
3. Results

Cyclic polytopes

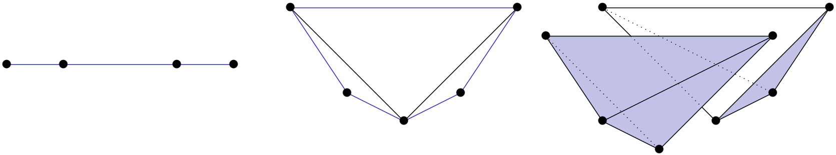
The *cyclic polytope* $C(n, \delta)$ is the convex hull of n points $\{p_{t_1}, \ldots, p_{t_n}\} \subset \mathbb{R}^\delta$ on the curve $p_t = (t, t^2, \ldots, t^\delta)$, where $\{t_1, \ldots, t_n\} \subset \mathbb{R}$.



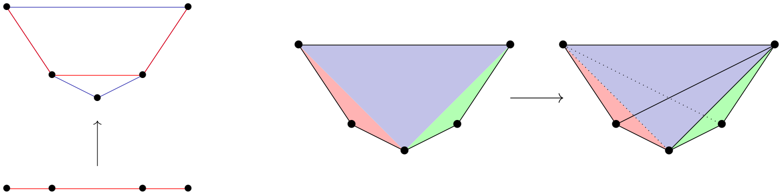
[ER96, Figure 2]

Triangulations and sections

A *triangulation* of $C(n,\delta)$ is a subdivision of $C(n,\delta)$ into δ-simplices whose vertices are vertices of $C(n,\delta)$.



Triangulations $\mathcal{T}$ give sections $s_{\mathcal{T}} \colon C(n,\delta) \to C(n,\delta+1)$.

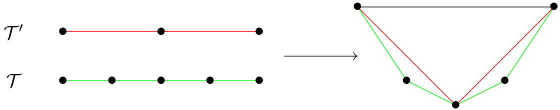


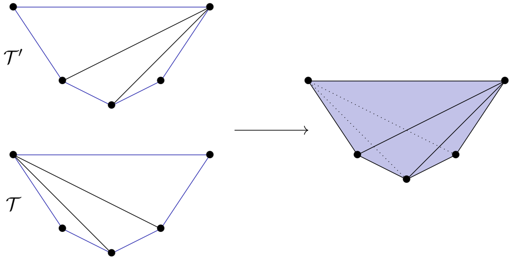
The second higher Stasheff–Tamari order $\mathcal{S}_2(n, \delta)$

Defined by Edelman and Reiner.

Given $\mathcal{T}, \mathcal{T}'$ triangulations of $C(n, \delta)$,

$$\mathcal{T} \leqslant_2 \mathcal{T}' \iff s_{\mathcal{T}}(x)_{\delta+1} \leqslant s_{\mathcal{T}'}(x)_{\delta+1} \quad \forall x \in C(n, \delta).$$

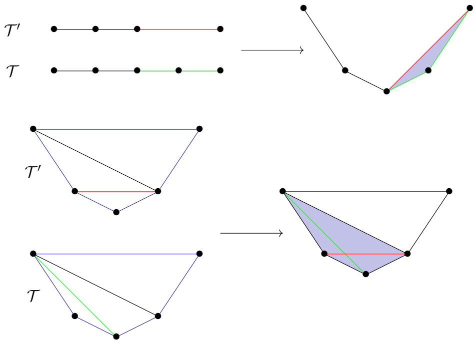




The first higher Stasheff–Tamari order $\mathcal{S}_1(n, \delta)$

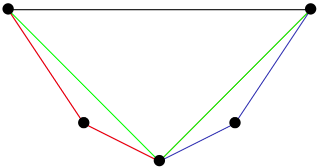
Defined first by Kapranov and Voevodsky and then by Edelman and Reiner in a different way. Thomas showed the two definitions were the same.

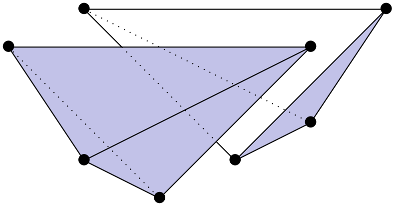
We have that $\mathcal{T} \lessdot_1 \mathcal{T}'$ ($\mathcal{T}'$ is an *increasing flip* of $\mathcal{T}$) if $\mathcal{T} \leqslant_2 \mathcal{T}'$ and the region bounded by $s_{\mathcal{T}}(C(n, \delta))$ and $s_{\mathcal{T}'}(C(n, \delta))$ is the interior of a $(\delta + 1)$-simplex.



Rambau's Theorem

$$\left\{\begin{matrix}\text{Triangulations of} \\ C(n,\delta+1)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Maximal chains in} \\ \mathcal{S}_1(n,\delta)\end{matrix}\right\} \Big/ \sim$$





What is known and what is conjectured

Edelman and Reiner conjectured that $\mathcal{S}_1(n, \delta) = \mathcal{S}_2(n, \delta)$ in their 1996 paper.

This problem is still open, though Edelman and Reiner proved it for $\delta \leqslant 3$.

$\mathcal{S}_1(n, \delta)$ and $\mathcal{S}_2(n, \delta)$ are known to be lattices for $\delta \leqslant 3$.

Edelman, Rambau, and Reiner found a counter-example to $\mathcal{S}_2(n, \delta)$ always being a lattice.

A counter-example to $\mathcal{S}_1(n, \delta)$ always being a lattice was found in [Wil20].

Higher Auslander–Reiten theory

Introduced by Iyama as a higher-dimensional generalisation of classical Auslander–Reiten theory.

Given a finite-dimensional K-algebra Λ over a field K, a functorially finite subcategory $\mathcal{M}$ of $\operatorname{mod}\Lambda$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \operatorname{mod}\Lambda \mid \forall M \in \mathcal{M}, \operatorname{Ext}_\Lambda^{1,\dots,d-1}(X, M) = 0\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \forall M \in \mathcal{M}, \operatorname{Ext}_\Lambda^{1,\dots,d-1}(M, X) = 0\}.\end{aligned}$$

If $\operatorname{add} M$ is a d-cluster-tilting subcategory, then M is called a *d-cluster-tilting module*.

If Λ has a d-cluster-tilting module M and $\operatorname{gl.dim}\Lambda \leqslant d$, then Λ is called *d-representation-finite*, following Iyama–Oppermann. In this case, $\operatorname{add} M$ is unique.

The higher Auslander algebras of type A

$A_n^1 :=$ path algebra of linearly oriented A_n.

A_n^d has a unique basic d-cluster-tilting module $M^{(d,n)}$ and

$$\operatorname{End}_{A_n^d} M^{(d,n)} \cong A_n^{d+1}.$$

A_3^1

$$\begin{tikzcd}
 & & 3 \\
 & 2 \arrow[ur] & \\
1 \arrow[ur] & &
\end{tikzcd}$$

A_3^2

$$\begin{tikzcd}
 & & 15 \arrow[dr] & & \\
 & 14 \arrow[ur] \arrow[dr] & & 25 \arrow[dr] & \\
13 \arrow[ur] & & 24 \arrow[ur] & & 35
\end{tikzcd}$$

A_3^3

$$\begin{tikzcd}
 & & 137 \arrow[rr] & & 147 \arrow[rr] \arrow[dr] & & 157 \arrow[dr] & & \\
 & 136 \arrow[ur] \arrow[rr] & & 146 \arrow[ur] \arrow[dr] & & 247 \arrow[rr] & & 257 \arrow[dr] & \\
135 \arrow[ur] & & & & 246 \arrow[ur] & & & & 357
\end{tikzcd}$$

The results of Oppermann and Thomas

Oppermann and Thomas show that there are bijections

$$\left\{\begin{matrix}\text{Basic tilting } A_n^d\text{-modules} \\ \text{in add } M^{(d,n)}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Triangulations of} \\ C(n+2d, 2d)\end{matrix}\right\},$$

$$\left\{\begin{matrix}\text{Basic cluster-tilting} \\ \text{objects in } \mathcal{O}_{A_n^d}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Triangulations of} \\ C(n+2d+1, 2d)\end{matrix}\right\}.$$

$\mathcal{O}_{A_n^d}$ is the $(d+2)$-angulated cluster category, defined analogously to the usual cluster category of Buan, Marsh, Reineke, Reiten, and Todorov.

We instead consider the subcategory $\mathcal{C}_{A_n^d} := \operatorname{add}(M^{(d,n)} \oplus A_n^d[d])$ of $\mathcal{D}^b(\operatorname{mod} A_n^d)$.

An object $T \in \mathcal{C}_{A_n^d}$ is *cluster-tilting* if $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A_n^d)}(T, T[d]) = 0$ and T has as many isoclasses of indecomposable summands as A_n^d.

Even-dimensional orders

Theorem (W '20)

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d, 2d)$ corresponding to tilting A_n^d-modules T and T'. We then have that

1. $\mathcal{T} \lessdot_1 \mathcal{T}'$ *if and only if T' is a left mutation of T; and*
2. $\mathcal{T} \leqslant_2 \mathcal{T}'$ *if and only if* ${}^{\perp}T \subseteq {}^{\perp}T'$.

Left mutation: $T = E \oplus X, T' = E \oplus Y$,

$$0 \to X \to E_1 \to \cdots \to E_d \to Y \to 0,$$

where $E_i \in \operatorname{add} E$.

$${}^{\perp}T = \{X \in \operatorname{add} M^{(d,n)} \mid \operatorname{Ext}^i(X, T) = 0 \ \forall i > 0\}.$$

(Analogous statement holds for triangulations of $C(n+2d+1, 2d)$ and cluster-tilting objects for A_n^d.)

Odd-dimensional triangulations

A *d-maximal green sequence* for A_n^d is a sequence of left mutations of cluster-tilting objects from A_n^d to $A_n^d[d]$ in the subcategory $\mathcal{C}_{A_n^d}$.

Given a d-maximal green sequence G, we write $\Sigma(G)$ for the set of indecomposable summands occurring in G.

We write $G \sim G'$ if $\Sigma(G) = \Sigma(G')$ and write $\widetilde{\mathcal{MG}}_d(A_n^d)$ for the set of $\sim$-equivalence classes of d-maximal green sequences of A_n^d.

Theorem (W '20)

There is a bijection between $\widetilde{\mathcal{MG}}_d(A_n^d)$ and triangulations of $C(n+2d+1, 2d+1)$.

(Analogous statement holds for maximal chains of tilting modules.)

Odd-dimensional orders

Theorem (W '20)

Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$. We then have that

1. *$\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G}' \in [G']$ such that $\widehat{G}'$ is an increasing elementary polygonal deformation of $\widehat{G}$; and*
2. *$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

Increasing elementary polygonal deformations:

$$\begin{tikzcd}
 & & & \overbrace{\cdots\cdots}^{d+1} \arrow[dr] & & & \\
\bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[ur] \arrow[dr] & \Uparrow & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\
 & & & \underbrace{\cdots\cdots}_{d+2} \arrow[ur] & & &
\end{tikzcd}$$

Corollary (W '20)

The two orders on $\widetilde{\mathcal{MG}}_1(A_n)$ are equal and are lattices.

Thank you for listening!

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