Multivariable Functions

(Functions of several variables)

1. Functions of several variables

For examples:

The area of a triangle has the formula: $A=\frac{1}{2}bh$

where b = length of the base and h = height .

The volume of a rectangular box: $V = Lwh$

where L = length , w = width , h = height .

The arithmetic mean: $\bar{x} = \frac{1}{n}\left(x_1 + x_2 + \ldots + x_n\right)$

where $x_1, x_2, \ldots, x_n$ are n real numbers.

From the above example , we have that

A is a function of 2 variables: b and h,

V is a function of 3 variables: L, w, h,

$\bar{x}$ is a function of n variables: $x_1, x_2, \ldots, x_n$.

Notation: $z = f(x,y)$

It means that z is a function of x and y. The variables x and y are called *independent variables* or inputs while the variable z is called a *dependent variable* or output. Analogously, $w = f(x,y,z)$ means w is a function of x, y and z. Also, $u = f(x_1, x_2, \ldots, x_n)$ refers to u as a function in terms of variables $x_1, x_2, \ldots, x_n$.

Definition 1: A function f of two variables x and y is the assignment of each point (x, y) in its domain $D \subseteq \mathbb{R}^2$ to some real number $f(x, y)$.

Definition 2: A function f of three variables x, y and z is the assignment of a point (x, y, z) in its domain $D \subseteq \mathbb{R}^3$ to some real number $f(x, y, z)$.

Example 1 Find the domain and draw the graph of the domain of the functions below:

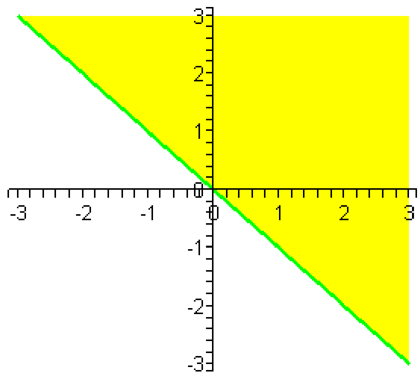
1.1 $f(x,y) = \sqrt{x+y}$.

1.2 $f(x,y) = \sqrt{x}+\sqrt{y}$.

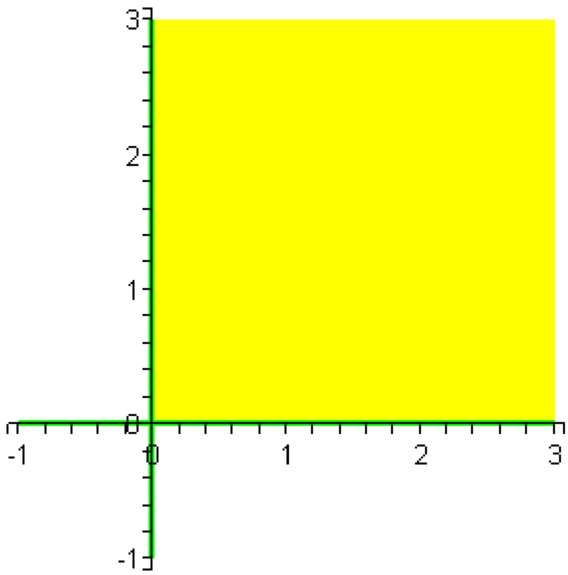
1.3 $f(x,y) = \ln(9-x^2-9y^2)$.

Solution

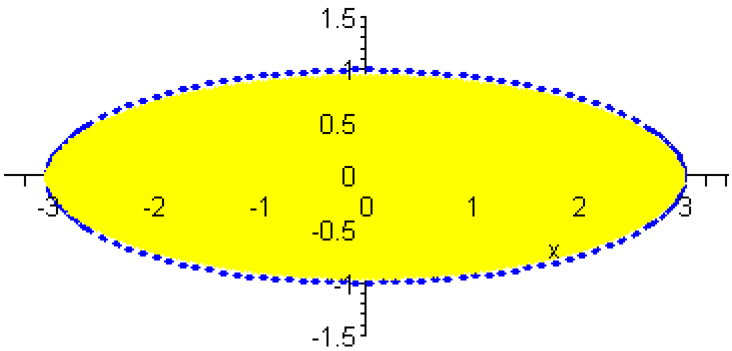
1.1



1.2



1.3



Example 2 Let $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2-16}}$. Find its domain.

Solution

Three Dimensional Spaces

1. Rectangular Coordinate System in Three-dimensional Space

It consists of three orthogonal axes called x-axis, y-axis and z-axis. The intersection point of all axes is called the *origin*.

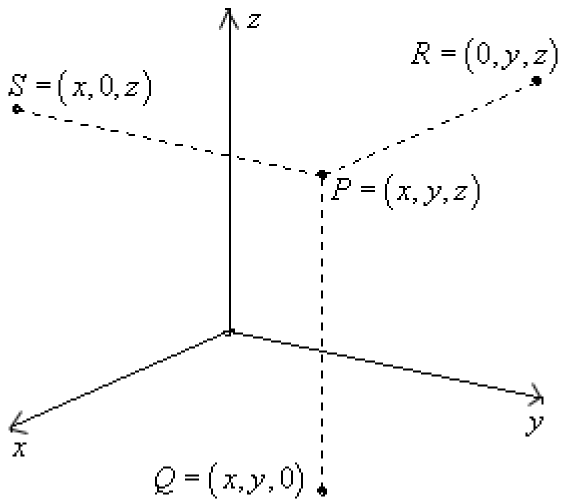


Figure 1 Three dimensional space

Every pair of (x, y, z) forms a coordinate plane: xy-plane, xz-plane and yz-plane. These three planes divide the space into eight parts. Each of which is called *octant*. The first octant is the octant that all x, y, z are positive. The other octants show different signs of x, y, z.

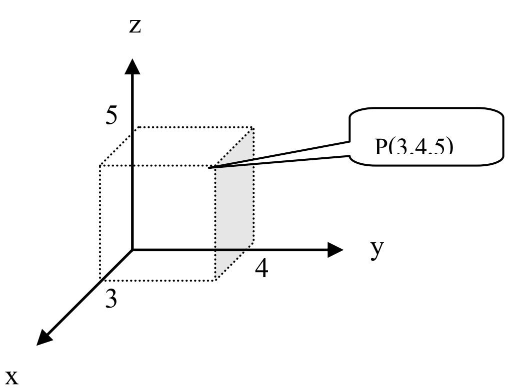
The rectangular coordinate system (x, y, z) refers to the distance from the point to each plane. In particular, the x-coordinate measures the distance from the point to yz-plane while the y-coordinate and z-coordinate indicate the distance from the point to the xz-plane and the xy-plane, respectively.

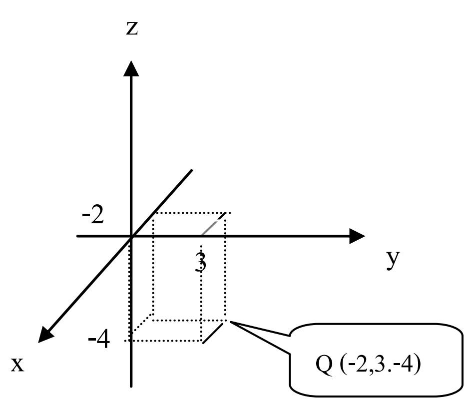
Example1 Locate the following points in the three-dim space.

a. $P(3,4,5)$

b. $Q(-2,3,-4)$

Solution

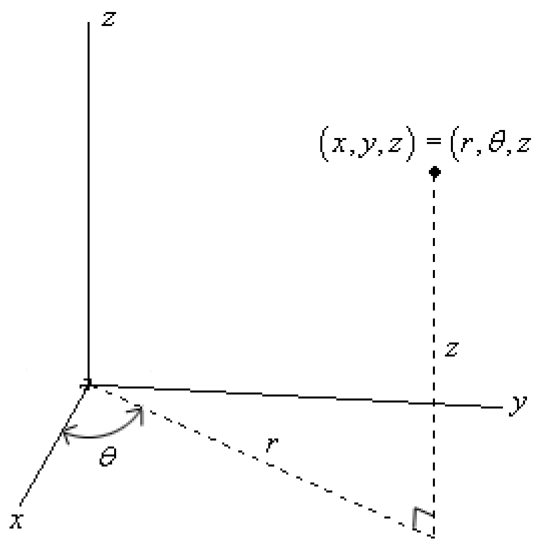




2. Cylindrical and Spherical Coordinates

Beside the rectangular coordinate system, we can locate a point in three dimensional space by cylindrical and/or spherical coordinates as follow:

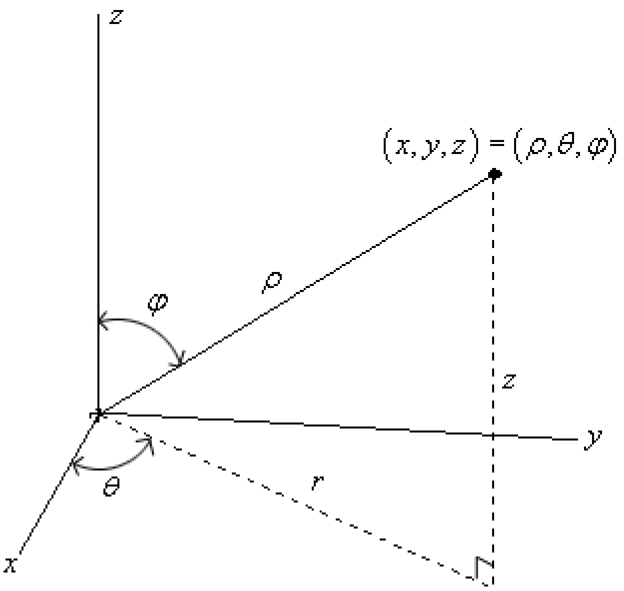
- Cylindrical coordinate, we use the coordinate (r, θ, z).
- Spherical coordinate, we use the coordinate (ρ, θ, ϕ).



Cylindrical coordinate

Relationship between rectangular and cylindrical coordinates:

$$\boxed{\begin{array}{c} x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ x^2 + y^2 = r^2\ ,\ \tan\theta = \dfrac{y}{x} \\ \text{where}\quad r \geq 0,\ \ 0 \leq \theta \leq 2\pi . \end{array}}$$



Spherical coordinate

The relationship between rectangular and spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \;\; y = \rho \sin\phi \sin\theta, \;\; z = \rho \cos\phi$$

$$x^2 + y^2 + z^2 = \rho^2 \,, \; \tan\theta = \frac{y}{x} \,, \;\; \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

where $\rho \ge 0, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$.

Example 1 Find the rectangular coordinates of these two points.

a. $(r, \theta, z) = \left(4, \frac{\pi}{3}, -3\right)$ b. $(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$

Solution a.

$$\boxed{\begin{array}{c} x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ x^2 + y^2 = r^2\ ,\ \tan\theta = \dfrac{y}{x} \end{array}}$$

b.

$$\boxed{\begin{array}{c} x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi \\ x^2 + y^2 + z^2 = \rho^2\ ,\ \tan\theta = \dfrac{y}{x},\ \cos\phi = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \end{array}}$$

Example 2 Covert the following equations:

a. $z = x^2 + y^2 - 2x + y$ to cylindrical coordinate system.

b. $z = x^2 + y^2$ to spherical coordinate system.

Solution: a.

$$\boxed{\begin{array}{c} x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x} \end{array}}$$

b.

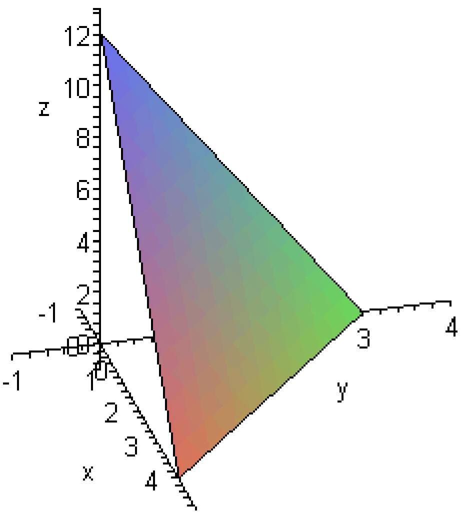
$$\boxed{\begin{array}{c} x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi \\ x^2 + y^2 + z^2 = \rho^2,\ \tan\theta = \dfrac{y}{x},\ \ \cos\phi = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \end{array}}$$

2. Graphs of multivariable functions

A graph of two variable functions $z = f(x,y)$ on three dimensional space is just a surface.

Example 3 Draw a graph of $f(x,y) = 12 - 3x - 4y$ on XYZ-space.

Solution



x − axis : $(4,0,0)$

y − axis : $(0,3,0)$

z − axis : $(0,0,12)$

2.1 Level Curves

By using the same method, we are not able to draw a graph of $w = f(x, y, z)$ since its graph will be in four dimensions. How can we solve this problem?

Let us go back to a function of two variables. Consider a geological map which is a picture of area in 2 and 3 dimensions. Figures A and C below show the three dimensional pictures of mountain area with contour lines (traces). Figure B shows a 2-dimensional picture with different heights indicated.

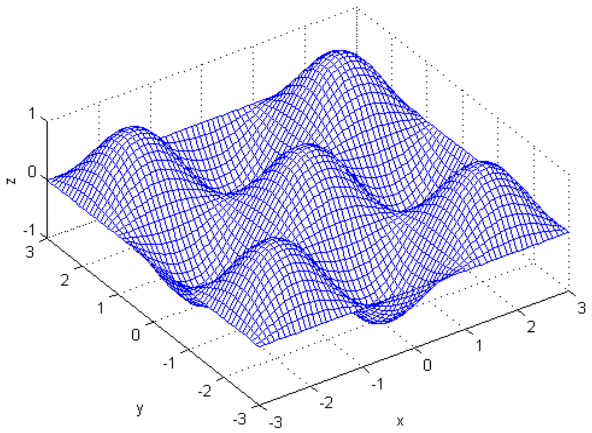


Figure A

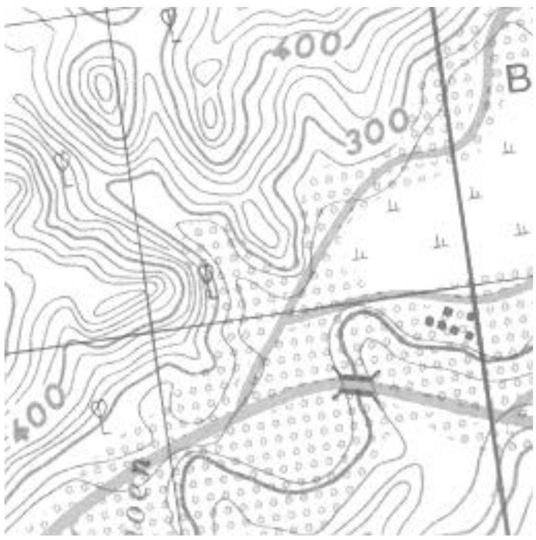


Figure B

Figure D shows several curves of area at different heights in 3 dimensions. Figure E is the projection of curves in figure D onto the xy-plane. This shows how we obtain a map as in figure B.

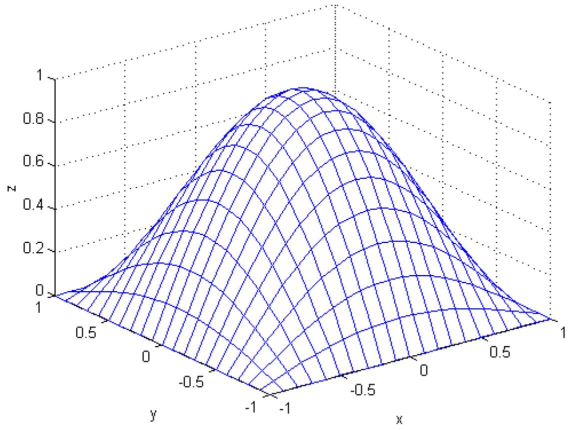


Figure C

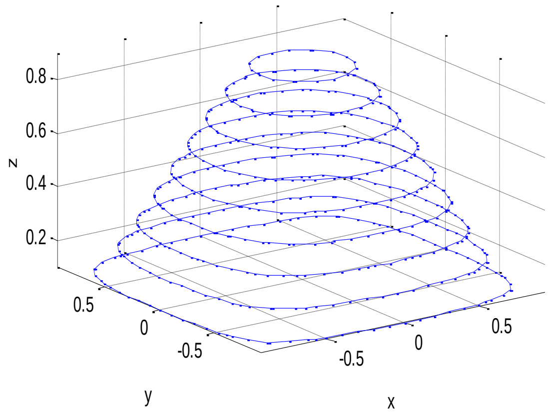


Figure D

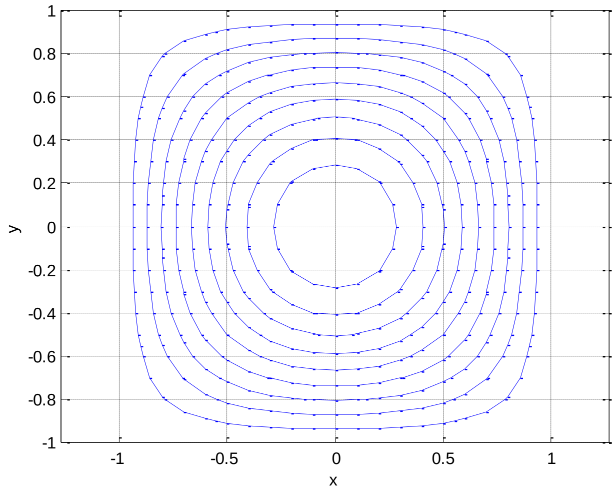
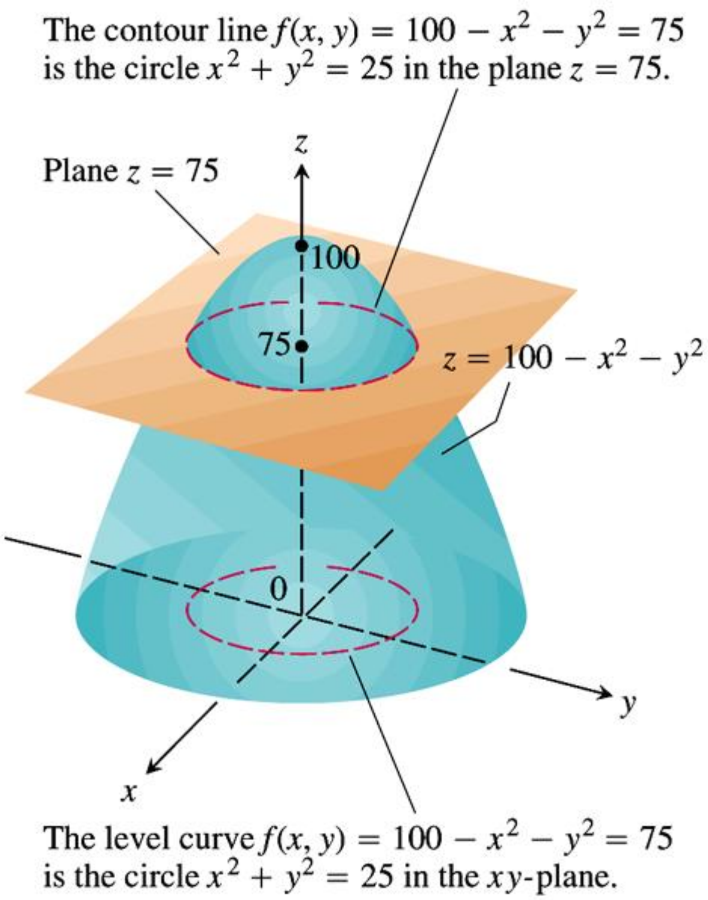


Figure E

The curves in figure E are called *level curves*. Each curve represents $f(x, y) = k$ on the xy-plan.

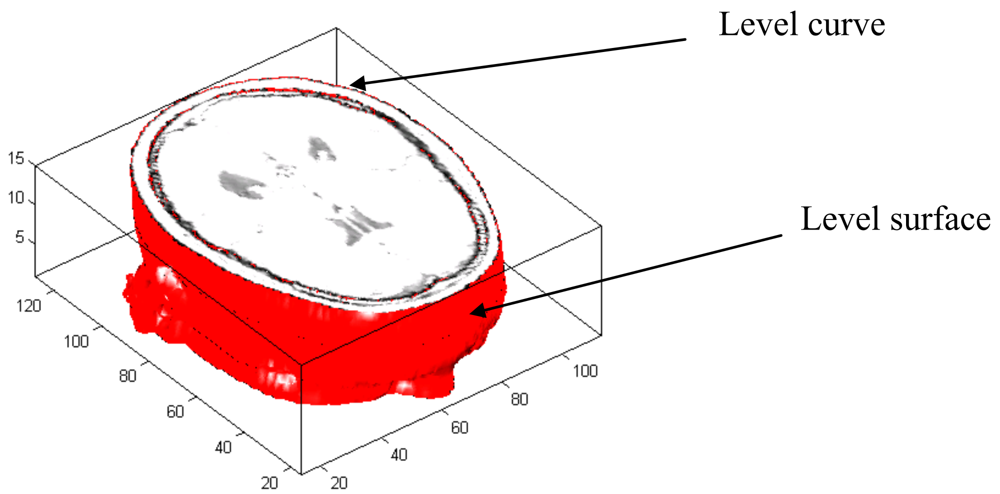
Level curves have been widely used in the atmospheric map to indicate the area with some fixed condition such as constant pressure level and constant temperature. We call the level curve for a constant pressure "isobar," and call the level curve for a constant temperature "isotherm."

Example 4 Draw several level curves of $f(x,y) = 100 - x^2 - y^2$ when $k = 100$, $k = 75$, $k = 0$.



2.2 Level Surface

In the case of 4 variables function $w = f(x, y, z)$, we are not able to draw a graph in 3 dimensional space. For example, we take the MRI picture of someone's brain at a given time. Here, the time becomes another variable. We can see the change by looking at the brain at several different times. At time c, we have $f(x, y, z) = c$. This given equation forms the area called "level surface," as shown here.



Exercise

1. Let $f(x,y) = 3x + y^2$. Evaluate the following:

   (a) $f(2,3)$ (b) $f(2,\sqrt{2})$ (c) $f(0,0)$.

2. Find the domain of the following functions and draw a graph of its domain.

   (a) $f(x,y) = \dfrac{x^2 + y^2 + 8}{(x-4)(y-3)}$

   (b) $g(x,y) = e^{\sqrt{x}} + \ln y$

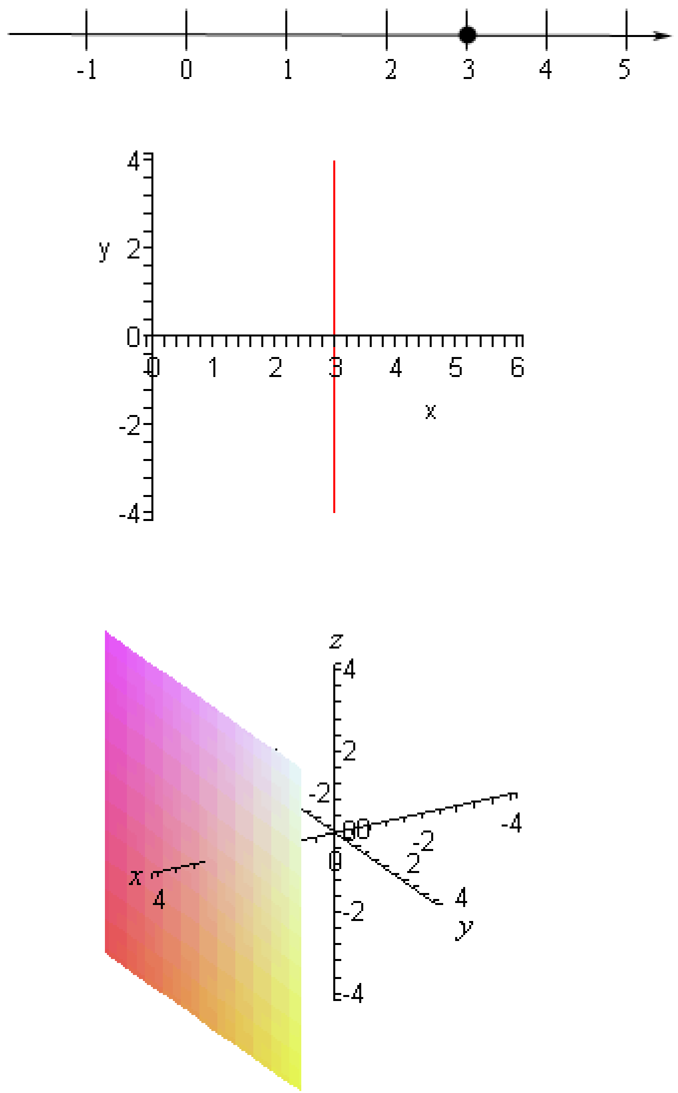
3. Graph of Planar Surface

Let A, B, C, D be some constants.

Definition: A plane is a set of all points in the three dimensional space satisfying the equation $Ax + By + Cz + D = 0$.

Example 1 Draw the graphs of $x=3$ in one-, two- and three-dimensional spaces.

Solution:



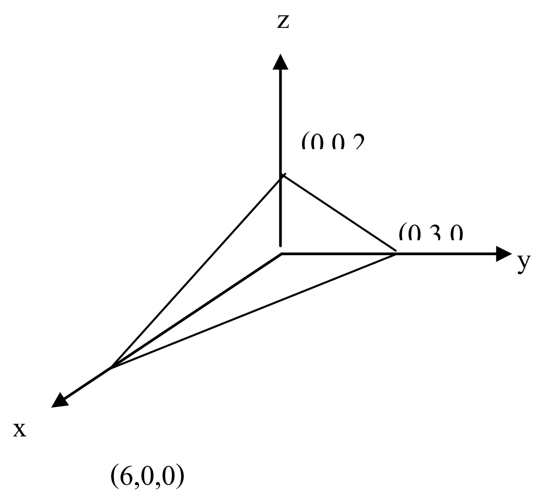
To sketch a planar graph of $Ax+By+Cz+D=0$, we generally draw a plane only on the octant where the plane lies on by finding the intercepts of the plane and each axis. Then connect all intercepted points with lines.

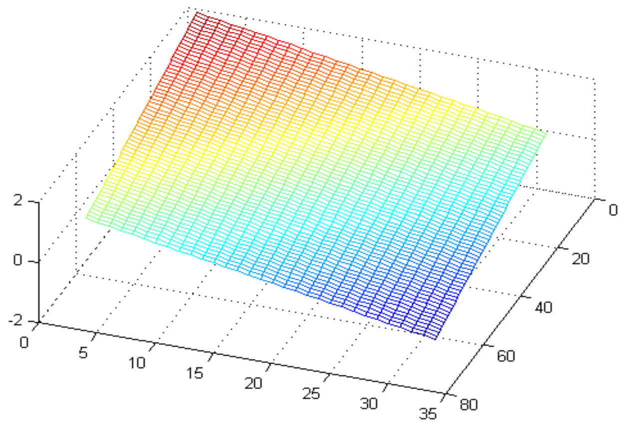
Three intercepted points:

1. If $x=0,\ y=0$, the point $(0,0,z)$ is the z-intercept.
2. If $x=0,\ z=0$, the point $(0,y,0)$ is the y-intercept.
3. If $y=0,\ z=0$, the point $(x,0,0)$ is the x-intercept.

Example 2 Draw a graph of $x+2y+3z=6$.

Solution:





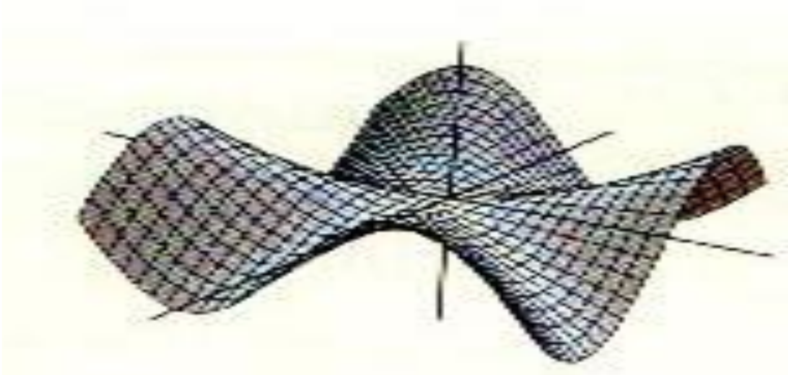
4. Graphs of Quadratic Surfaces

A graph of quadratic surface is a set of all points in three dimensional space satisfying the following equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, I, J$ are constants.

To draw a graph of quadratic surface, we cut the surface by the planes parallel to xy-, yz- and xz-planes making several traces on the surface. These traces form the graph of a surface.



The figure above shows traces of function: $z = \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}$.

The important quadratic surfaces you should know are the following:

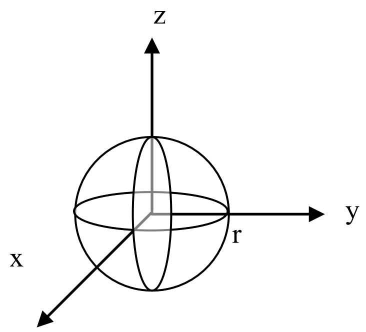
1. Sphere

2. Ellipsoid

3. Hyperboloid of one sheet
4. Hyperboloid of two sheets
5. Elliptic cone
6. Elliptic paraboloid
7. Hyperbolic paraboloid
8. Cylinder

4.1 Sphere

The equation of a sphere centered at (0,0,0) with radius r has the form: $x^2 + y^2 + z^2 = r^2$.

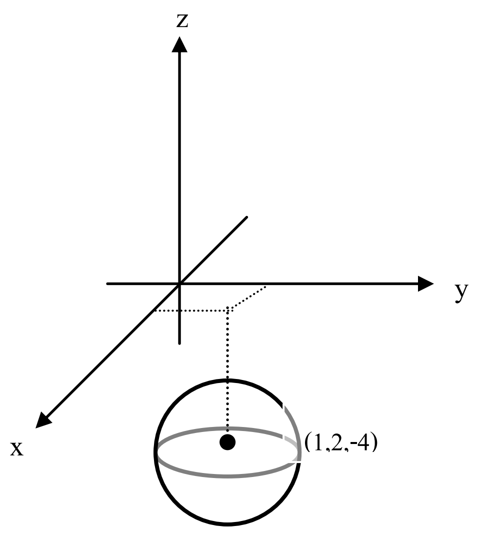


Similarly, the equation of a sphere centered at (x_0, y_0, z_0) with radius r has the form:

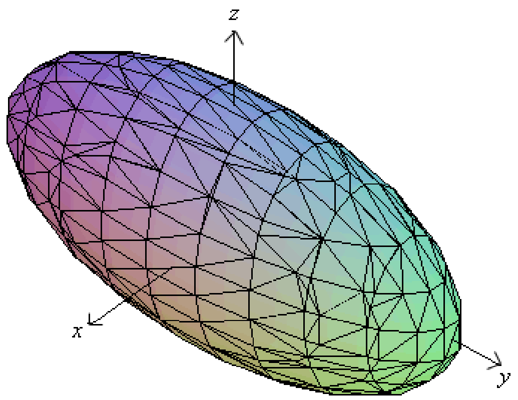
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.$$

Example 1: Draw a graph of

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0.$$



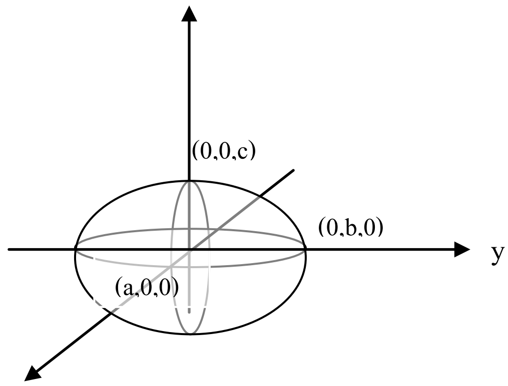
4.2 Ellipsoid



The equation of an ellipsoid centered at $(0,0,0)$ has the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



Similarly, the equation of an ellipsoid centered at (x_0,y_0,z_0) has the form:

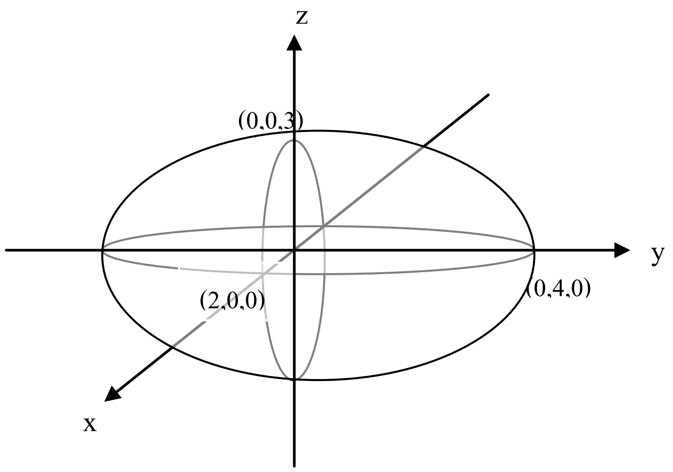
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1.$$

Example Draw a graph of quadratic surface of

$$\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

and find the equation of a graph after cutting the surface by the plane $x=k$, where k is some constant.

Solution



The plane $x=k$ is parallel to the yz-plane, after cutting the surface by this plane, we get the equation: $\frac{y^2}{16}+\frac{z^2}{9}=1-\frac{k^2}{4}$ which is an ellipse on the plane $x=k$; $-2\le k\le 2$.

4.3 Hyperboloid of one sheet

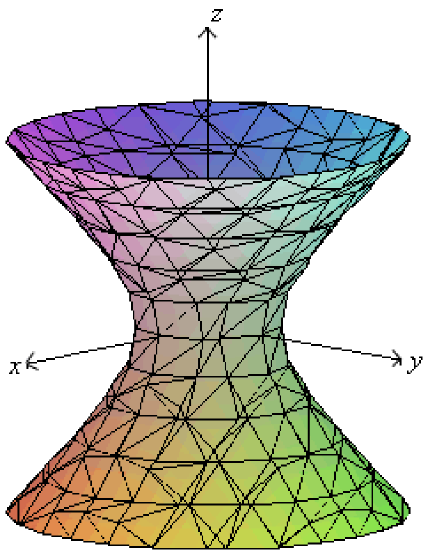
The equation of a hyperboloid of one sheet centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a, b, c are some constants.



The graph of quadratic surface of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Remark: If the hyperboloid of one sheet lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1.$$

Similarly, if it lies along the line parallel to the y-axis, its equation has the form: $\dfrac{(x-x_0)^2}{a^2}-\dfrac{(y-y_0)^2}{b^2}+\dfrac{(z-z_0)^2}{c^2}=1.$

If it lies along the line parallel to the x-axis, its equation has the form: $-\dfrac{(x-x_0)^2}{a^2}+\dfrac{(y-y_0)^2}{b^2}+\dfrac{(z-z_0)^2}{c^2}=1.$

4.4 Hyperboloid of two sheets

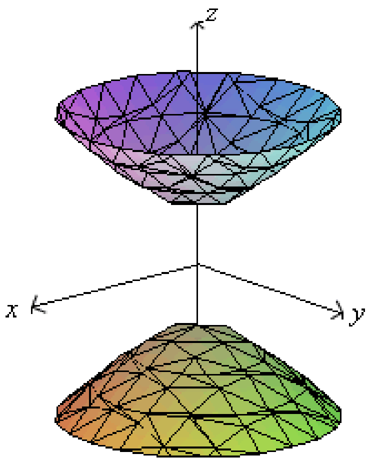
The equation of a hyperboloid of two sheets centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

Lie along y-axis: $-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

Lie along x-axis: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

where a, b, c are some constants.



The graph of quadratic surface $-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

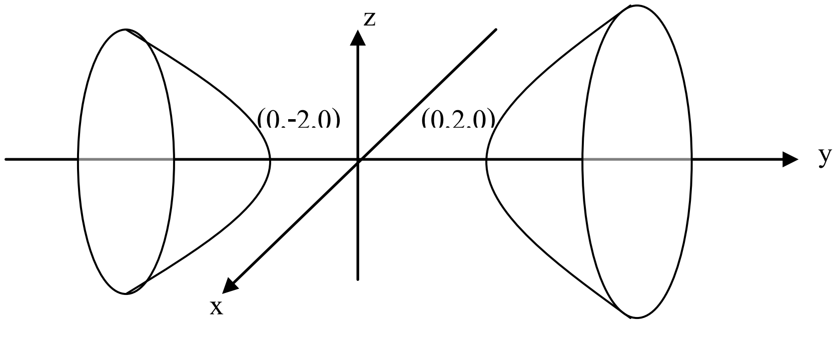
Remark:

If the hyperboloid of two sheets lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$-\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1.$$

Similarly, in the case of hyperboloid of two sheets lying along the y-axis or x-axis, we get the similar form of the equation.

Example: Draw a surface of $4x^2 - y^2 + 2z^2 + 4 = 0$ and find the equations of the traces after cutting the surface by each plane.



4.5 Elliptic cones

The equation of an elliptic cone centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

where a,b,c are some constants.

Remark:

If the elliptic cone lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=0.$$

Similarly, in the case of elliptic cone lying along the y-axis or the x-axis, we get the similar form of the equation.

Example: Draw a graph of the quadratic surface

$$x^2 + y^2 - z^2 - 2x + 6z - 8 = 0.$$

4.6 Paraboloid

The equation of a paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis and open on positive z: $$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

open on negative z: $$z = -\frac{x^2}{a^2} - \frac{y^2}{b^2}$$

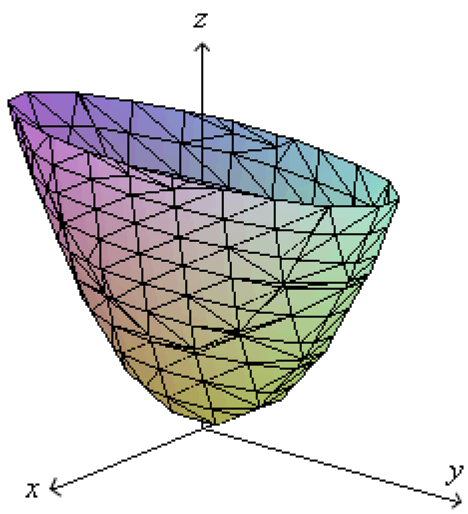
Lie along y-axis and open on positive y: $$y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$$

open on negative y: $$y = -\frac{x^2}{a^2} - \frac{z^2}{c^2}$$

Lie along x-axis and open on positive x: $$x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

open on negative x: $$x = -\frac{y^2}{b^2} - \frac{z^2}{c^2}$$

where a, b, c are some constants.



The graph of quadratic surface $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

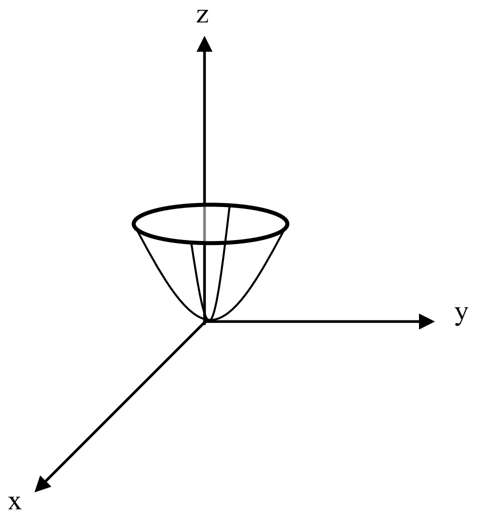
Remark:

If the paraboloid lies along the line parallel to z-axis, centered at (x_0, y_0, z_0) and opens on positive z, its equation is of the form:

$$(z - z_0) = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}$$

Similarly, in the case of paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Example Draw a graph of $z = \frac{x^2}{4} + \frac{y^2}{9}$.



4.7 Hyperbolic Paraboloid

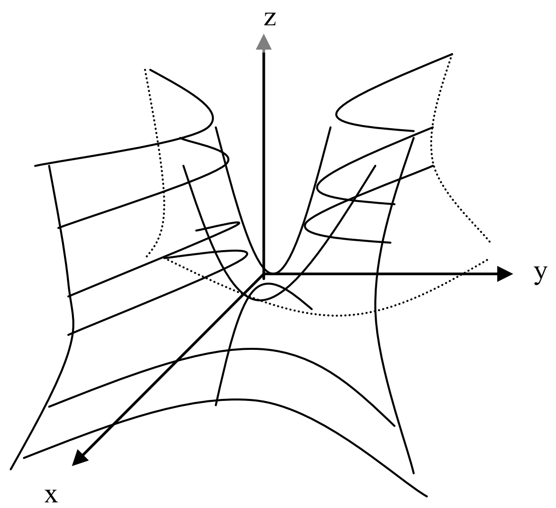
The equation of a hyperbolic paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis : $z = -\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Lie along y-axis : $y = -\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$, $y = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$

Lie along x-axis : $x = -\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$, $x = \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$

where a, b, c are some constants.



The graph of quadratic surface $z = -\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Remark: If the hyperbolic paraboloid lies along the line parallel to z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

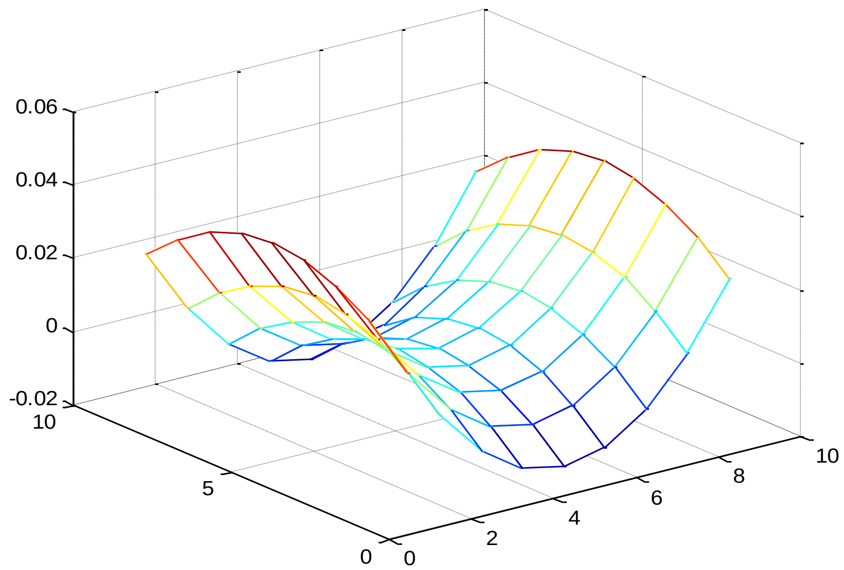
$$(z-z_0) = -\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

or $$(z-z_0) = \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2}$$

Analogously, in the case of hyperbolic paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Note that in the case of a hyperbolic paraboloid, its center can also be called as a *saddle point*.

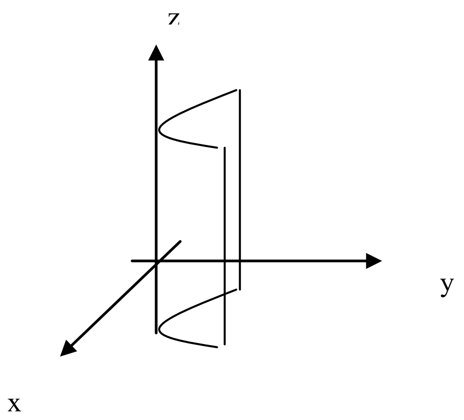
Example: Draw the quadratic surface of $z = \dfrac{x^2}{4} - \dfrac{y^2}{9}$.


0.06
0.04
0.02
0
-0.02
10
5
0
0
2
4
6
8
10

4.8 Cylinders

There are several types of cylinders we need to learn such as cyclic cylinders, elliptic cylinders, hyperbolic cylinders, and parabolic cylinders.

The equation forms of each cylindrical surface are in 2-variables. Since we consider a graph in 3-dimensional space, the missing variable has values in $(-\infty, \infty)$. Thus, the cylinder will lie along the axis of missing variable. For example, the function $y = ax^2$ where a is a positive constant. This equation forms a parabolic cylinder lying along z-axis as shown in the figure below:



Parabolic cylinder of $y = ax^2$ when $a > 0$.

Problem: Draw the following graphs in the 3-dimensional space.

a. $z = \sqrt{y^2 + 1}$ b. $y - z^2 = 0$

c. $y^2 - x^2 = 1$ d. $25x^2 + 9z^2 = 1$

Solution: