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Finance I, Spring 2020 A
Assignment 2

1. Because the market portfolio ω^m is a convex combination of efficient portfolios, it is itself efficient. Thus, it solves the portfolio problem:

$$\min_{\omega} \frac{1}{2}\omega' V \omega$$
$$\text{s.t.}$$
$$\omega' \bar{R} \geq \mu$$
$$\omega' \mathbf{1} = 1$$

The first-order condition for this problem is

$$\bar{R} = \frac{1}{\lambda_1} V \omega^m - \frac{\lambda_2}{\lambda_1}\mathbf{1} \tag{1}$$

To derive the CAPM, I use (1) twice. First, premultiply (1) by ω^m, yielding

$$\bar{R}^m = \frac{1}{\lambda_1}\sigma_m^2 - \frac{\lambda_2}{\lambda_1} \tag{2}$$

where $\bar{R}^m$ is the return on the market portfolio. Now, introduce ω^z, the "zero-β" portfolio, which is constructed such that $\omega^{z\prime} V \omega^m = 0$; that is, ω^z is orthogonal to the market portfolio. Premultiplying (1) by ω^z yields

$$\bar{R} = -\frac{\lambda_2}{\lambda_1} \tag{3}$$

Substituting (3) into (2) gives

$$\frac{1}{\lambda_1} = \frac{\bar{R}^m - \bar{R}^z}{\sigma_m^2} \tag{4}$$

Substituting (3) and (4) into (1) gives the CAPM:

$$\bar{R} = \bar{R}^z \mathbf{1} + [\bar{R}^m - \bar{R}^z]\frac{V\omega^m}{\sigma_m^2}$$

For any asset n in the market portfolio, this result implies that

$$\bar{R}_n = \bar{R}^z + [\bar{R}^m - \bar{R}^z]\frac{cov(R_n, R_m)}{\sigma_m^2}$$

2. To test the CAPM, I use the specification

$$r_i(t) = \hat{\alpha}_i + \hat{\beta}_i r_m(t) + \hat{\varepsilon}_t$$

where $r_i(t)$ is the return on asset i in period t, and $r_m(t)$ is the market return in period t. I then test the null hypothesis that $\hat{\alpha}_i = 0$, and reject the null hypothesis if the resulting p-value is less than 0.05.

a) The results are reported in Table (1)

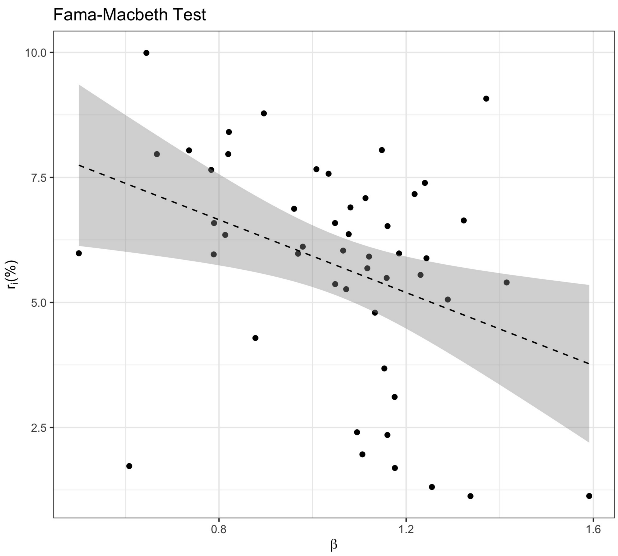
Table 1

Industry	β	α	p_α
Aero	1.148	0.001	0.604
Agric	0.878	-0.0004	0.848
Autos	1.153	-0.003	0.102
Banks	1.077	-0.0004	0.800
Beer	0.736	0.003	0.075
BldMt	1.185	-0.001	0.498
Books	1.072	-0.002	0.177
Boxes	0.979	0.0002	0.919
BusSv	1.160	-0.001	0.348
Chems	1.048	-0.00005	0.969
Chips	1.415	-0.002	0.250
Clths	1.113	-0.0001	0.933
Cnstr	1.289	-0.003	0.071
Coal	1.160	-0.005	0.173
Drugs	0.783	0.002	0.112
ElcEq	1.218	0.0002	0.904
FabPr	1.106	-0.004	0.058
Fin	1.240	-0.0003	0.793
Food	0.667	0.003	0.026
Fun	1.371	0.0004	0.826
Gold	0.608	-0.003	0.419
Guns	0.821	0.003	0.114
Hardw	1.231	-0.003	0.134
Hlth	1.133	-0.002	0.484
Hshld	0.790	0.0003	0.788
Insur	0.961	0.001	0.407
LabEq	1.323	-0.002	0.324
Mach	1.243	-0.001	0.259
Meals	1.034	0.0004	0.810
MedEq	0.896	0.002	0.264
Mines	1.121	-0.002	0.457
Oil	0.813	0.0005	0.780
Other	1.176	-0.007	0.0003
Paper	0.969	-0.00004	0.977
PerSv	1.095	-0.005	0.011
RlEst	1.255	-0.007	0.002
Rtail	1.008	0.001	0.447
Rubbr	1.081	-0.0004	0.787
Ships	1.117	-0.001	0.742
Smoke	0.645	0.005	0.017
Soda	0.820	0.002	0.418
Softw	1.591	-0.006	0.057
Steel	1.337	-0.005	0.005
Telcm	0.789	0.001	0.429
Toys	1.175	-0.003	0.079
Trans	1.065	-0.001	0.625
Txtls	1.158	-0.002	0.320
Util	0.501	0.002	0.114
Whlsl	1.048	-0.001	0.635

For most industries, we cannot reject the null hypothesis that $\hat{\alpha} = 0$. The p-value is only below the critical value of 0.05 for seven industries in the table.

b) Figure (1) shows the Fama-Macbeth test:

Figure 1



If the CAPM held, the relationship between r_i and β would be positive and linear. Instead, as the fitted line shows, the relationship is somewhat negative. This relationship is not strong; the confidence bands (shaded in grey) around this line are wide.

3. In this economy, the states are $s \in \{s_1, s_2\}$, with prices P and payoffs D as follows:

$$P = \begin{bmatrix} 1.575 & 1.35 & 3.425 \end{bmatrix}$$
$$D = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}$$

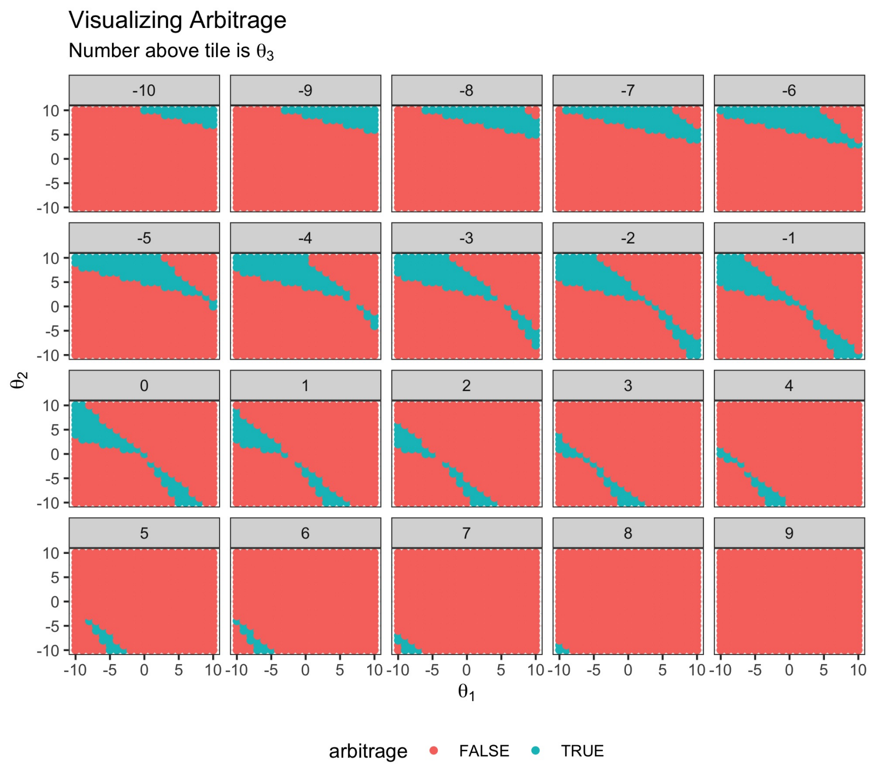
a) This setup allows for arbitrage. Intuitively, the number of states is less than the number of assets, and thus one of the assets must be redundant. This intuition is confirmed by noting that a portfolio of 9/5 asset 1 plus 2/5 asset 2 delivers identical payoffs to asset 3, while costing

$$\frac{9}{5} \cdot 1.575 + \frac{2}{5} 3.375 < 3.425$$

Thus, an agent can extract unlimited profits by buying 9/5 asset 1 plus 2/5 asset 2, and selling asset 3.

b) Figure (2) illustrates the arbitrage in this example:

Figure 2



In Figure (2), the x-axis represents the shares of asset 1, the y-axis the shares in asset 2, and the numbers in shaded gray boxes above the tiles the shares in asset 3. The teal areas are portfolios for which $P\theta \leq 0$ and $D\theta \geq 0$.

4. In this economy, there are three states, $s \in \{s_1, s_2, s_3\}$, each with equal 1/3 probability. The prices and payoffs are as follows:

$$P = \begin{bmatrix} 2.4 & 1.248 & 5.952 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 1 & 11 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

a) This market is not complete. Row-reducing the matrix D shows that

$$D = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank of D is two, and the span of D is $\mathbb{R}^2$.

b) With complete markets, the Arrow-Debreu state prices are given by the vector $A = PD^{-1}$. However, because D is not of full rank, it cannot be inverted. Thus, there exists a continuum of Arrow-Debreu state prices. Given any $a(s_3)$, prices $a(s_1)$ and $a(s_2)$ can be chosen in order to construct the replicating portfolio.

c) Because there is a continuum of A-D state prices, there is a continuum of risk-free rates. To see this, note that $R_f = 1/B$, where $B = \sum_{s=1}^{k} a(s_k)$. Because there is an infinite number of values that any $a(s)$ can take, there will similarly be a continuum of values for the risk-free rate.

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