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Finance I, Spring 2020 A
Assignment 1

1. Useful facts

a) $\mathbb{E}[\omega' R] = \omega' \bar{R}$, where $\bar{R}$ is the $N \times 1$ vector of expected returns.

$$\mathbb{E}[\omega' \bar{R}] = \mathbb{E}\left[\sum_{i=1}^{N} \tilde{R}_n \omega_n\right] = \sum_{i=1}^{N} \omega_n \mathbb{E}[\tilde{R}_n] = \omega' \bar{R}$$

b) $\frac{\partial \mathbb{E}[R'\omega]}{\partial \omega_n} = R_n$, or the entire vector $\frac{\partial \mathbb{E}[R'\omega]}{\partial \omega} = R$
As in (a),

$$\mathbb{E}[\omega' \bar{R}] = \sum_{i=1}^{N} \omega_n \mathbb{E}[\tilde{R}_n]$$

and thus

$$\frac{\partial \mathbb{E}[R'\omega]}{\partial \omega_n} = \mathbb{E}[\tilde{R}_n] = \bar{R}$$

and so in vector form,

$$\frac{\partial \mathbb{E}[R'\omega]}{\partial \omega} = R$$

c) $V(R'\omega) = cov(R'\omega, R'\omega)$ where V is the variance-covariance matrix of returns, with typical element $\sigma_{i,j} = cov(\tilde{R}_i, \tilde{R}_j)$.
By definition,

$$\begin{aligned}
cov(R'\omega, R'\omega) &= \mathbb{E}\left[\sum_{i=1}^{N} \omega_n \tilde{R}_n \cdot \sum_{i=1}^{N} \omega_n \tilde{R}_n\right] - \mathbb{E}\left[\sum_{i=1}^{N} \omega_n \tilde{R}_n\right]^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j \mathbb{E}[R_i R_j] - \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j \mathbb{E}[R_i]\mathbb{E}[R_j] \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j cov(R_i, R_j) \\
&= \omega' V \omega
\end{aligned}$$

d) $cov(R'\omega, R_n) = \omega' V e_n$, where $e_n = [0, \ldots, 0, 1, 0, \ldots, 0]'$, with 1 in the n^{th} position.
Using the same definition for the covariance as above,

$$\begin{aligned}
cov(R'\omega, R_n) &= \mathbb{E}[R'\omega R_n] - \mathbb{E}[R'\omega]\mathbb{E}[R_n] \\
&= \sum_{i=1}^{N} \omega_i \big(\mathbb{E}[R_i R_n] - \mathbb{E}[R_i]\mathbb{E}[R_n]\big) \\
&= \sum_{i=1}^{N} \omega_i cov(R_i, R_n)
\end{aligned}$$

The final sum above is the n^{th} entry of the matrix $\omega' V$, and thus $cov(R'\omega, R_n) = \omega' V e_n$

e) $\frac{\partial V(R'\omega)}{\partial \omega_n} = 2cov(R'\omega, R_n)$

As in part (c),

$$V(R'\omega) = \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j cov(R_i, R_j)$$

Thus,

$$\frac{\partial V(R'\omega)}{\partial \omega_n} = 2\sum_{i=1}^{N} \omega_i cov(R_i, R_n) = 2cov(R'\omega, R_n)$$

where the final equality follows from (d).

2. The portfolio problem with risky assets is

$$\begin{gathered}\min_{\omega} \frac{1}{2}\omega' V \omega \\ \text{s.t.} \\ \omega'\bar{R} \geq \mu \\ \omega'\mathbf{1} = 1\end{gathered}$$

where $\mathbf{1}$ is the N-vector with 1 in every element. The Lagrangean for this problem is

$$\mathcal{L} = \frac{1}{2}\omega' V \omega + \lambda_1 \left[\mu - \omega'\bar{R}\right] + \lambda_2 \left[1 - \omega'\mathbf{1}\right]$$

with first-order conditions

$$V\omega = \lambda_1 \bar{R} - \lambda_2 \mathbf{1} \tag{1}$$
$$\mu = \omega'\bar{R} \tag{2}$$
$$\omega'\mathbf{1} = 1 \tag{3}$$

To begin, premultiply (1) by V^{-1}, yielding

$$\omega = \lambda_1 V^{-1}\bar{R} + \lambda_2 V^{-1}\mathbf{1}$$

Now, define the following two efficient portfolios:

$$\begin{aligned}\omega_R &= \frac{V^{-1}\bar{R}}{\mathbf{1}'V^{-1}\bar{R}} \\ \omega_1 &= \frac{V^{-1}\mathbf{1}}{\mathbf{1}'V^{-1}\mathbf{1}}\end{aligned}$$

Thus, the form for ω above can be transformed in the following way:

$$\begin{aligned}\omega^* &= \lambda_1 V^{-1}\bar{R} + \lambda_2 V^{-1}\mathbf{1} \\ &= \lambda_1 \left(\frac{\mathbf{1}'V^{-1}\bar{R}}{\mathbf{1}'V^{-1}\bar{R}}\right) V^{-1}\bar{R} + \lambda_2 \left(\frac{\mathbf{1}'V^{-1}\mathbf{1}}{\mathbf{1}'V^{-1}\mathbf{1}}\right) V^{-1}\mathbf{1} \\ &= \left(\lambda_1 \mathbf{1}'V^{-1}\bar{R}\right)\omega_R + \left(\lambda_2 \mathbf{1}'V^{-1}\mathbf{1}\right)\omega_1 \\ &\equiv \alpha\omega_R + (1-\alpha)\omega_1\end{aligned}$$

To show the final equality (i.e., that the weights on portfolios ω_R and w_1 sum to unity), we premultiply both sides by $\mathbf{1}'$ and impose (3):

$$\mathbf{1}'\omega^* = \left(\lambda_1 \mathbf{1}' V^{-1}\bar{R}\right)\mathbf{1}'w_R + \left(\lambda_2 \mathbf{1}'V^{-1}\mathbf{1}\right)\mathbf{1}'w_1$$

Because ω^*, ω_R, and ω_1 are portfolios, the above implies that

$$1 = \left(\lambda_1 \mathbf{1}' V^{-1}\bar{R}\right) + \left(\lambda_2 \mathbf{1}'V^{-1}\mathbf{1}\right)$$

To solve for α, we impose (2):

$$\begin{aligned}\omega^{*\prime}\bar{R} = \mu &\implies \alpha\omega_R'\bar{R} + (1-\alpha)\omega_1'\bar{R} = \mu \\ &\implies \alpha = \frac{\mu - \omega_1'\bar{R}}{\omega_R'\bar{R} - \omega_1'\bar{R}}\end{aligned}$$

3. To find the Global Minimum Variance (GMV) portfolio, the problem is similar:

$$\begin{gathered}\min_{\omega} \frac{1}{2}\omega' V\omega \\ \text{s.t.} \\ \omega'\mathbf{1} = 1\end{gathered}$$

The Lagrangean for this problem is

$$\mathcal{L} = \frac{1}{2}\omega' V\omega + \lambda[1 - \omega'\mathbf{1}]$$

From the first-order condition for ω, we can use the same method as in question 2:

$$\begin{aligned}\omega^* &= \lambda V^{-1}\mathbf{1} \\ &= \lambda\left(\frac{\mathbf{1}'V^{-1}\mathbf{1}}{\mathbf{1}'V^{-1}\mathbf{1}}\right)V^{-1}\mathbf{1} \\ &= \left(\lambda\mathbf{1}'V^{-1}\mathbf{1}\right)\omega_1\end{aligned}$$

Because ω and ω_1 are both portfolios, premultiplying the final line above by $\mathbf{1}'$ shows that $(\lambda\mathbf{1}'V^{-1}\mathbf{1}) = 1$. Thus, ω_1 is the GMV portfolio

4. With both risky assets and one risk-free asset, the problem becomes

$$\begin{gathered}\min_{\omega} \frac{1}{2}\omega' V\omega \\ \text{s.t.} \\ (1 - \omega'\mathbf{1})R_f + \omega'\bar{R} \geq \mu\end{gathered}$$

   a) The Lagrangian for this problem is as follows:

$$\mathcal{L} = \frac{1}{2}\omega' V\omega + \lambda\left[\mu - (1 - \omega'\mathbf{1})R_f - \omega'\bar{R}\right]$$

   with first-order conditions

$$V\omega + \lambda(\mathbf{1}R_f - \bar{R}) = \mu \tag{4}$$
$$(1 - \omega'\mathbf{1})R_f + \omega'\bar{R} \geq \mu \tag{5}$$

To begin, premultiply (4) by V^{-1} (assuming that V is invertible):

$$\omega = \lambda V^{-1}\big(\bar{R} - \mathbf{1}R_f\big) \tag{6}$$

Similarly, from (5),

$$\big(\bar{R} - \mathbf{1}R_f\big)\omega' = \mu - R_f$$

Thus, premultiplying (6) by $(\bar{R} - \mathbf{1}R_f)$ gives

$$\lambda\big(\bar{R} - \mathbf{1}R_f\big)V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)$$

and thus

$$\lambda = \frac{\mu - R_f}{\big(\bar{R} - \mathbf{1}R_f\big)V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}$$

Therefore, the optimal ω is given by

$$\omega = \frac{\mu - R_f}{\big(\bar{R} - \mathbf{1}R_f\big)V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)$$

In order to verify that this ω can be written as $\gamma\omega^*$, we note that by construction, $\gamma\omega^{*\prime} = \gamma$, and thus

$$\gamma = \frac{\mu - R_f}{\big(\bar{R} - \mathbf{1}R_f\big)'V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}\mathbf{1}'V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)$$

b) To verify that the efficient frontier is linear in σ_p, note that the variance of this portfolio is given by

$$\begin{aligned}
\omega' V \omega &= \frac{\mu - R_f}{\big(\bar{R} - \mathbf{1}R_f\big)V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}\big(\bar{R} - \mathbf{1}R_f\big)'V^{-1}V\frac{\mu - R_f}{\big(\bar{R} - \mathbf{1}R_f\big)V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}V^{-1}\big(\bar{R} - \mathbf{1}R_f\big) \\
&= \frac{(\mu - R_f)^2}{\big(\bar{R} - \mathbf{1}R_f\big)'V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}
\end{aligned}$$

And thus the standard deviation is

$$\sigma_p = \sqrt{\omega' V \omega} = \frac{|\mu - R_f}{\sqrt{V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}}$$

Thus, solving for the return μ_p gives

$$\mu_p = R_f \pm \sigma_p\sqrt{V^{-1}\big(\bar{R} - \mathbf{1}R_f\big)}$$

Thus, the portfolio return is linear in the variance, and thus the efficient frontier is linear.

c) The optimal Sharpe Ratio portfolio is the portfolio ω which solves

$$\max_{\omega} \frac{\omega'\bar{R} - R_f}{\sqrt{\omega' V \omega}}$$

The first-order condition for this problem is

$$\frac{d}{d\omega} = -\frac{1}{2}(\omega'\bar{R}) - R_f(\omega' V\omega)^{-\frac{3}{2}}V\omega + \bar{R}(\omega' V\omega)^{-\frac{1}{2}} = 0$$

The portfolio solving this problem is

$$\omega^* = \frac{V^{-1}\left(\bar{R} - \mathbf{1}R_f\right)}{\mathbf{1}'V^{-1}\left(\bar{R} - \mathbf{1}R_f\right)}$$

This is the portfolio where, given N risky assets, the efficient frontier for the assets is tangent to the efficient frontier for the risky assets plus a risk free asset. Thus, this portfolio is also the efficient portfolio that includes only risky assets, when both risky and risk-free assets are available.

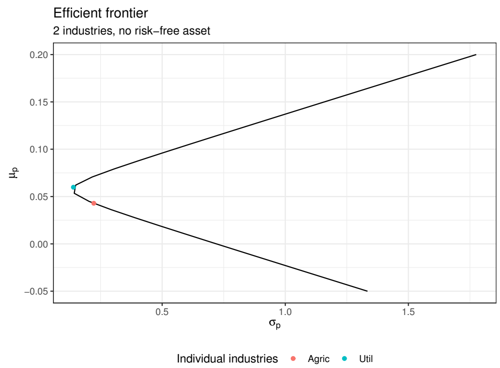
5. The excess returns, variances, standard deviations, and Sharpe ratios for the industries in the data are shown in Table 1:

Table 1: Returns Data, Full Sample

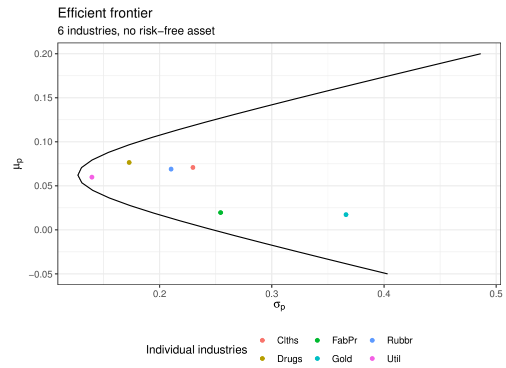
Industry	Excess Return	V	σ	Sharpe
Aero	7.046	2.479	15.746	44.751
Agric	3.288	5.477	23.402	14.051
Autos	2.679	5.156	22.706	11.800
Banks	5.365	7.210	26.851	19.979
Beer	7.040	6.610	25.711	27.383
BldMt	4.980	3.231	17.974	27.705
Books	4.264	6.261	25.022	17.042
Boxes	5.114	4.784	21.873	23.382
BusSv	5.526	3.677	19.175	28.819
Chems	4.366	4.938	22.223	19.646
Chips	4.399	2.368	15.389	28.584
Clths	6.086	4.965	22.281	27.312
Cnstr	4.058	4.656	21.577	18.805
Coal	1.349	7.407	27.216	4.958
Drugs	6.651	4.185	20.457	32.509
ElcEq	6.167	2.634	16.229	37.998
FabPr	0.961	5.273	22.964	4.185
Fin	6.390	7.963	28.219	22.643
Food	6.965	3.390	18.412	37.828
Fun	8.075	2.985	17.277	46.735
Gold	0.727	3.898	19.743	3.685
Guns	7.408	4.416	21.014	35.251
Hardw	4.550	6.558	25.609	17.766
Hlth	3.795	4.800	21.909	17.320
Hshld	5.587	6.316	25.132	22.230
Insur	5.872	6.467	25.430	23.090
LabEq	5.639	5.029	22.426	25.144
Mach	4.883	4.874	22.077	22.118
Market	5.237	5.832	24.150	21.686
Meals	6.574	13.402	36.608	17.957
MedEq	7.780	7.115	26.674	29.168
Mines	4.916	14.067	37.506	13.108
Oil	5.350	3.813	19.527	27.397
Other	0.689	1.947	13.954	4.938
Paper	4.974	2.668	16.335	30.454
PerSv	1.403	5.429	23.299	6.023
RlEst	0.309	3.987	19.968	1.548
Rtail	6.664	6.323	25.146	26.500
Rubbr	5.899	13.098	36.191	16.301
Ships	4.682	7.057	26.565	17.626
Smoke	8.989	6.126	24.751	36.319
Soda	6.965	3.663	19.139	36.393
Softw	0.129	4.038	20.095	0.642
Steel	0.126	4.152	20.376	0.619
Telcm	4.960	3.658	19.126	25.936
Toys	2.110	4.484	21.176	9.964
Trans	5.036	4.505	21.224	23.728
Txtls	4.489	3.670	19.157	23.431
Util	4.983	7.262	26.949	18.492
Whlsl	5.587	5.807	24.098	23.184

6. Efficient frontiers

Figure 1: Efficient Frontiers

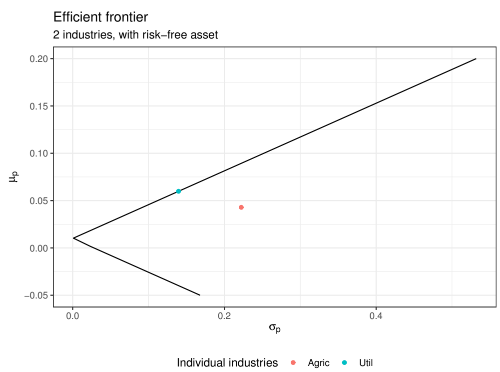


(a) Two-asset case

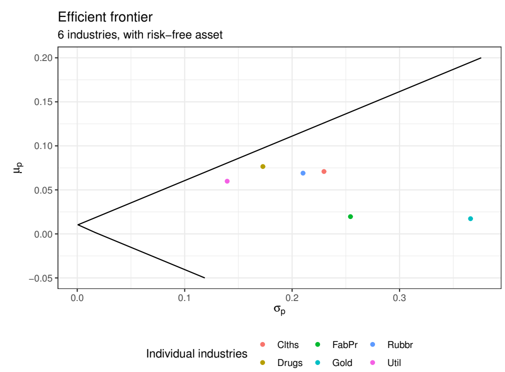


(b) Six-asset case

Figure 2: Efficient Frontiers, with Risk-Free Rate of 1%



(a) Two-asset case



(b) Six-asset case

a) See Figure 1a
b) See Figure 1b
c) See Figure 2

d) The above plots are generated by solving for ω^* as a function of μ, and plotting the efficient frontier in black at each value of μ.

Table 2: Example Weights

μ_p	Gold	Clths	Rubbr	Drugs	Util	FabPr	σ_p
-0.050	0.004	-0.261	0.032	-0.050	-0.463	0.193	0.088
-0.022	0.002	-0.140	0.017	-0.027	-0.249	0.103	0.047
0.006	0.0003	-0.019	0.002	-0.004	-0.034	0.014	0.007
0.033	-0.001	0.102	-0.013	0.019	0.180	-0.075	0.034
0.061	-0.003	0.223	-0.027	0.042	0.394	-0.164	0.075
0.089	-0.005	0.344	-0.042	0.065	0.609	-0.253	0.116
0.117	-0.006	0.465	-0.057	0.088	0.823	-0.342	0.157
0.144	-0.008	0.586	-0.072	0.111	1.037	-0.431	0.198
0.172	-0.010	0.707	-0.087	0.134	1.251	-0.521	0.239
0.200	-0.011	0.828	-0.102	0.157	1.466	-0.610	0.279

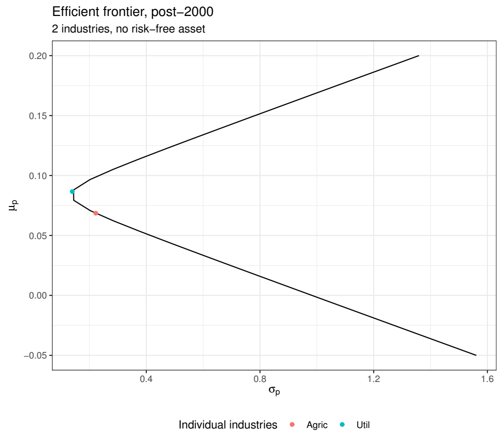
Table 2 shows example weights ω^* (under the industry names) for the efficient frontier shown in Figure 2b. $\omega_n^* < 0$ indicates a short position in asset n. Note that the weights do not sum to one; the remainder of the portfolio is held in the risk-free asset.

7. Table 3 and Figure 3 show the same exercise as in questions 5 and 6, but with the data limited to returns post-2000.

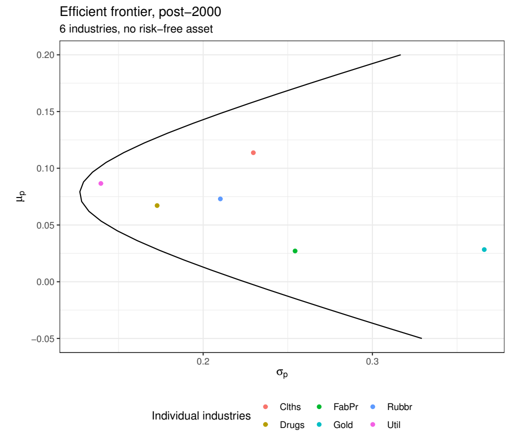
Table 3: Returns Data, Post-2000

Industry	Excess Return	V	σ_p	Sharpe
Aero	9.318	2.479	15.746	59.176
Agric	5.856	5.477	23.402	25.022
Autos	0.330	5.156	22.706	1.452
Banks	4.048	7.210	26.851	15.076
Beer	6.884	6.610	25.711	26.776
BldMt	6.656	3.231	17.974	37.033
Books	-0.583	6.261	25.022	-2.332
Boxes	7.918	4.784	21.873	36.198
BusSv	4.837	3.677	19.175	25.225
Chems	6.417	4.938	22.223	28.877
Chips	1.821	2.368	15.389	11.832
Clths	10.370	4.965	22.281	46.542
Cnstr	7.701	4.656	21.577	35.692
Coal	-3.351	7.407	27.216	-12.311
Drugs	5.713	4.185	20.457	27.924
ElcEq	3.997	2.634	16.229	24.629
FabPr	1.711	5.273	22.964	7.449
Fin	4.256	7.963	28.219	15.082
Food	6.941	3.390	18.412	37.695
Fun	8.808	2.985	17.277	50.979
Gold	1.833	3.898	19.743	9.287
Guns	14.340	4.416	21.014	68.237
Hardw	0.721	6.558	25.609	2.815
Hlth	7.334	4.800	21.909	33.473
Hshld	4.978	6.316	25.132	19.807
Insur	6.783	6.467	25.430	26.673
LabEq	6.617	5.029	22.426	29.504
Mach	7.802	4.874	22.077	35.337
Market	4.399	5.832	24.150	18.215
Meals	9.959	13.402	36.608	27.205
MedEq	9.133	7.115	26.674	34.240
Mines	6.530	14.067	37.506	17.412
Oil	4.558	3.813	19.527	23.344
Other	1.090	1.947	13.954	7.809
Paper	4.988	2.668	16.335	30.537
PerSv	4.130	5.429	23.299	17.728
RlEst	4.544	3.987	19.968	22.755
Rtail	6.367	6.323	25.146	25.320
Rubbr	6.300	13.098	36.191	17.408
Ships	12.983	7.057	26.565	48.873
Smoke	13.890	6.126	24.751	56.118
Soda	9.429	3.663	19.139	49.264
Softw	2.779	4.038	20.095	13.830
Steel	-1.392	4.152	20.376	-6.834
Telcm	0.711	3.658	19.126	3.719
Toys	5.157	4.484	21.176	24.354
Trans	7.318	4.505	21.224	34.477
Txtls	5.396	3.670	19.157	28.166
Util	7.661	7.262	26.949	28.429
Whlsl	6.201	5.807	24.098	25.732

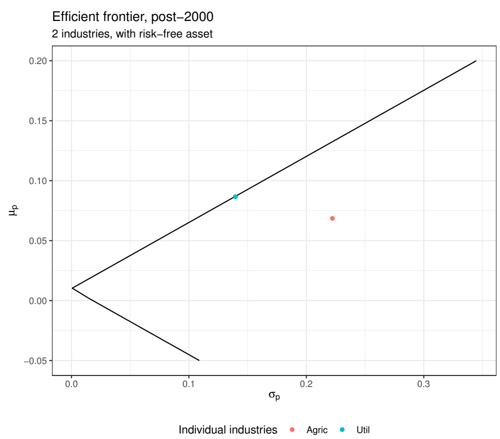
Figure 3: Efficient Frontiers, Post-2000



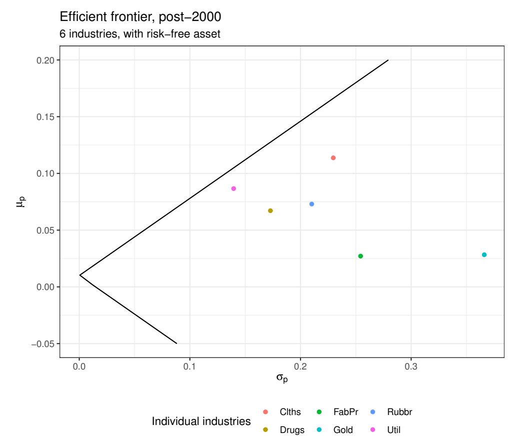
(a) Two-asset case



(b) Six-asset case



(c) Two-asset case



(d) Six-asset case

8. I constructed the Sharpe Ratio-maximal portfolio for all 1176 combinations of two assets in the data (excluding the "market"). Of these, **589** have both portfolio weights nonnegative, and thus the probability that a portfolio of two of these industries does not include a short position is **50.1%**.

References