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Finance I, Spring 2020 A
Assignment 3

1. Risk aversion and expected utility

a) If $u(c) = -\frac{1}{\alpha}\exp(-\alpha c)$, then

$$A_a(c) = \frac{\alpha \exp(-\alpha c)}{\exp(-\alpha c)} = \alpha$$

b) If $u(c) = \frac{c^\alpha}{\alpha}$, then

$$A_r(c) = \frac{-(\alpha-1)c^{\alpha-2}}{c^{\alpha-1}}c = 1-\alpha$$

c) Consider an insurance premium π as compared to a lottery ε, with $\mathbb{E}\varepsilon = 0$ and $V(\varepsilon) = \sigma_\varepsilon^2$. π must be set such that $u(c-\pi) = \mathbb{E}u(c+\varepsilon)$ To find π, I take a Taylor Series expansion at $u(c)$, obtaining

$$u(c) - u'(c) = u(c) + u'(c)\mathbb{E}\varepsilon + \frac{1}{2}u''(c)\mathbb{E}\varepsilon^2$$

which implies that

$$-u'(c) = \frac{1}{2}u''(c)\sigma_\varepsilon^2$$

and thus

$$\pi = -\frac{1}{2}\frac{u''(c)}{u'(c)}\sigma_\varepsilon^2$$

d) Now, the lottery ε and the insurance premium π are given as proportions of wealth, rather than in dollar amounts. Thus, we are looking for π such that $u(c(1-\pi)) = \mathbb{E}u\big(c(1+\varepsilon)\big)$. To solve for π, I again take a Taylor Series expansion at $u(c)$, obtaining

$$u(c) - cu'(c)\pi = u(c) + cu'(c)\mathbb{E}\varepsilon - \frac{1}{2}c^2u''(c)\sigma_\varepsilon^2$$

and thus

$$\pi = -\frac{1}{2}\frac{u''(c)}{u'(c)}c\sigma_\varepsilon^2$$

e) With wealth $w = 10,000$, $l = 100$, and $g = 110$, an agent refuses a lottery offering $w - l$ and $w + g$ each with equal probability. This implies that

$$\frac{10,000^\alpha}{\alpha} > \frac{1}{2}\frac{9900^\alpha}{\alpha} + \frac{1}{2}\frac{10,110^\alpha}{\alpha}$$

which implies that $\alpha < -8.086$.

f) Given the value for α found in part (e), In order to accept a lottery where $L = 1,000$, it must be the case that

$$\frac{10,000^\alpha}{\alpha} < \frac{1}{2}\frac{9000^\alpha}{\alpha} + \frac{1}{2}\frac{(10,000+G)^\alpha}{\alpha}$$

There is no value of G such that this inequality holds. The same is true for $L = 2,000$.

2. In this two-period economy, the Arrow-Debreu state prices can be found as follows: to begin, I caculate the conditional A-D prices for each of the four states in $z_2 = \{\{uu\}, \{ud\}, \{du\}, \{dd\}\}$. These prices are caculated conditional on being at $z_u = \{\{uu, ud\}\}$. Thus, they use the conditional prices $P_u = [1, 1.9444]'$, and the conditional payoff matrix,

$$D_u = \begin{bmatrix} 1 & 2.5 \\ 1 & 1.5 \end{bmatrix}$$

For example, to caculate $a(z_{uu}|z_u)$, I use the following formula:

$$a(z_{uu}|z_u) = P_u' D_u^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.444$$

The remaining three conditional A-D prices are calculated in the same way:

$$a(z_{ud}|z_u) = P_u' D_u^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.556$$

$$a(z_{du}|z_d) = P_d' D_d^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.314$$

$$a(z_{dd}|z_d) = P_u' D_d^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.646$$

where $P_d = [0.96, 1.2742]'$ and

$$D_d = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

With these values, I can caculate the date-0 state prices. Using the date-0 price vector $P_0 = [0.9487, 1.5285]'$ and substituting the date-1 prices for D_0, I can calculate a replicating portfolio for the A-D securities. The payoff matrix is defined as follows:

$$D_0 = \begin{bmatrix} 1.00 & 1.9444 \\ 0.96 & 1.2472 \end{bmatrix}$$

For example, to calculate $a_0(z_u u)$, I calculate a portfolio θ that generates payoffs of $a(z_{uu}|z_u)$ in the up state and 0 in the down state. Thus,

$$a_0(z_u u) = P_0' D_0^{-1} \begin{bmatrix} a(z_{uu}|z_u) \\ 0 \end{bmatrix} = 0.194$$

$$a_0(z_u d) = P_0' D_0^{-1} \begin{bmatrix} 0 \\ a(z_{ud}|z_u) \end{bmatrix} = 0.297$$

$$a_0(z_d u) = P_0' D_0^{-1} \begin{bmatrix} a(z_{du}|z_d) \\ 0 \end{bmatrix} = 0.137$$

$$a_0(z_d d) = P_0' D_0^{-1} \begin{bmatrix} 0 \\ a(z_{dd}|z_d) \end{bmatrix} = 0.345$$

Lastly, the date-0 A-D state prices for the date-1 states can be calculated as before: the payoff matrix D_0 is the same as above, and thus the replicating portfolio θ is given by

$$\theta = I_2 D_0^{-1}$$

where I_2 is the 2×2 identity matrix. Thus, the date-1 A-D prices are given by

$$\begin{bmatrix} a_0(z_u) \\ a_0(z_d) \end{bmatrix} = P_0' I_2 D_0^{-1} = \begin{bmatrix} 0.436 \\ 0.534 \end{bmatrix}$$

3. With the A-D state prices in hand, I can solve for the stochastic discount factor M_t at each node.