45-721
Seminar in Finance I – Valuation
Mini 3 – 2020

Assign #1
Due: 2020-01- 28

Notes ...

Assignment should be in a PDF and handed in via the link on the canvas page.

Assignments are the main way to learn the material here. You are welcome to complete the assignments pen and paper (scan it to a PDF for submitting). But if you want some good training, start using LaTeX. Working out the answer and already having it as paper-ready is good practice for summer papers (and later research). If you want a "hybrid" to add a sketch of something, just use a scanner/phone and insert the graphic.[1] Main idea here – get fast at getting your ideas onto a page that you can share. Please "do" the assignments. Directly looking up the answers is not helpful as solutions to past assignments is really not helpful (and definitely not allowed). Researching a bit is encouraged. Be sure to list any material you consulted (no need to be overly formal about this. However, if you want my advice: start building bibtex library now. You will use it repeatedly.) Talking to your colleagues in the class about the assignments and material is helpful and is encouraged. However, the final product should be your own.

And this is all fresh and as-we-go. That means something might be crazy, impossible, or (more likely) poorly stated. Feel free to email or stop by with questions. Also note the compile date in the left corner. Check the canvas page `https://canvas.cmu.edu/courses/8061` for updates.

Notation

[1] My go-to tool is (1) a screen-grab that converts to pdf. On a Mac I use `Screenshot Pro`. And then (2) I copy those `screen.pdf` file into `../Graphics/somename.pdf`. And then (3) I stick the graphic into the document
`\pgfimage[width=0.8 \paperwidth]{Graphics/somename}`

Try to use the same notation we used in class. Returns $\tilde{R} : \Omega \to \mathbb{R}_+^N$ where w is the state space with associated probabilities $p(s)$ for state $s \in \Omega$. I try to use the ˜to remind ourselves of random variables. But you will see it gets dropped as we go. For now, $\tilde{R}$ has mean $E[\tilde{R}] = \bar{R}$ and variance $V[\tilde{R}] = V$. When needed, R_f is the a risk-free asset such that $\tilde{R}_f(s) = R_f$ (i.e. $V[R_f] = 0$). So with the risk free asset there are $N + 1$ assets. Portfolios are w and greek letters, α, γ, μ are all scalar parameters.[2]

Question 1 [*Useful Facts*]

Prove these / convince yourself they are true (very short answers is all that is required)

- $E[w'R] = E[R'w] = w'\bar{R}$
- $dE[R'w]/dw_n = R_n$ or the whole N-vector $dE[R'w]/dw = \bar{R}$.
- $Var(R'w) = cov(R'w, R'w) = w'Vw$.
- $cov(R'w, R_n) = w'V[0, 0, ..., 0, 1, 0...0]'$ (the one is in the n-th spot).
- $dVar(R'w)/dw_n = 2cov(R'w, R_n)$ or write the whole Nx1 vector as $dVar(R'w)/dw = 2Vw$.

Question 2 [*Risky Assets*]

$$\begin{aligned} \min_{\omega} \quad & 0.5\omega' V \omega \\ s.t. \quad & w'\bar{R} \geq \mu \\ & w'\mathbf{1} = 1 \end{aligned}$$

Verify that the solution to the above can be written as:

$$w = \alpha w_R + (1 - \alpha) w_1$$

[2] I am just full of advice here: Try to keep your notation clustered in an easy way (like greeks for scalars) and consistent with standard as possible.

Where w_R and w_1 are also efficient portfolios.

Question 3 [*GMV*]

Solve / characterize the GMV or global minimum variance portfolio

$$\begin{aligned} \min_{\omega} \quad & 0.5\omega' V \omega \\ s.t \quad & w'\mathbf{1} = 1 \end{aligned}$$

Hint: Recall the w_1 portfolio we defined as part of solving the general problem.

Question 4 [*Risky Assets + Risk Free*]

$$\begin{aligned} \min_{\omega} \quad & 0.5\ \omega' V \omega \\ s.t. \quad & (1 - w'\mathbf{1})R_f + w'\bar{R} \geq \mu \end{aligned}$$

[a] Verify that the solution to the above can be written as:

$$w = \gamma w^*$$

where w^* satisfies:

$$\begin{aligned} w^{*\prime}\mathbf{1} &= 1 \\ w^* &= \alpha w_R + (1-\alpha) w_1 \end{aligned}$$

Note this implies that w^* does not depend on μ. (i.e., solve for α).

[b] Show that the "efficient frontier" (defined as the optimal (σ_p, μ_p) set) is linear. That is:

$$\mu_p = E[R_p] = a + b\sigma_p$$

(note that this is standard deviation; not variance)

[c] Under what conditions does w^* correspond to the maximal Sharpe Ratio? That is: when does this hold

$$w^* = \text{arg}\max_{w} \frac{w\prime\bar{R} - R_f}{(w'Vw)^{\frac{1}{2}}}$$

(Doing this carefully might be tricky. It might be obvious. I am not sure. Do not get stuck here)

Question 5 [*With some data – estimate mean and variance*]

You can calculate everything here using Excel. However, if you want to do more complex portfolio optimization, you typically move to a more matrix-friendly language like matlab or R. Use what works best for you. (There are all sorts of packages and add-ins for excell, R, matlab, etc. that will do all these calculations like "magic." Do not use there here. You want to code this up so you can see how it works. Moving forward you can then use any of those packages and know what they are doing.)

Download:
`http://homer.tepper.cmu.edu/47721/Industry49_data.csv`

The data is monthly returns (percent per month, compounded continuously) for 49 different industries. The data runs from January 1975 to roughly now. The data source is CRSP and compiled by Ken French see his website for all the details if you like (He has detailed SIC definitions of the industries). The data headers have `R_*` are the monthly continuously compounded returns. We want to work with the columns that are `eR_*` the excess returns (return above the risk free rate) as described below.

See the note on estimation at the end of the assignment.

For the "market" and for each industry (and remember to use the excess return data, denoted: **eR_***:

[a] Use "Excess Return" data to estimate the expected excess return on each industry portfolio. Convert to annual (multiply by 12). Then you can add the current risk free rate of (say) 1% and we have our estimate for $\bar{R}$.

[b] Use "Excess Return" to estimate the Variance and Correlation; Convert to annual (multiply by 12).

[c] Calculate the Sharpe Ratio for each industry and market in the data. Recall: the Sharpe Ratio (below) is excess return divided by the standard deviation.

$$\frac{E[r_i] - r_f}{\sigma_i}$$

Question 6 [*Risky Assets*]

[a] Pick two industries, plot the mean-variance frontier. (Use the solutions you in the algebra above your estimates from [a]). When you are picking two industries to plot, see if you can pick two industries so the frontier looks "pretty". (i.e., if you pick two industries with very similar expected returns, your frontier will be too narrow. It is also nice if you can pick two industries so that the sharpe-ratio maximal portfolio has only positive weights)

[b] Pick three, four, five,... plot the mean-variance frontier. How does it change as you add assets. Why? (You can add as many industries as you like. But at minimum get it up to five)

Now include a risk-free asset with $R^f = 1\%$:

[c] Repeat [a] and [b] (use the same industries) and plot the (linear) efficient frontier.

[d] Show the portfolio weights in w^*. What does $w_n^* < 0$ mean? Is it sensible?

Question 7 [*Robustness*]

Repeat questions 5 and 6 using estimates from different sub-samples of the data (Focus on a few industries if that makes it less tedious). Notice that the solutions (the optimal w are very sensitive to the data and the estimates. The two main reasons for this: (1) Estimates of mean returns are very noisy since the variance of stock returns is so large. (2) The moments we are estimating are not constant. All of these things are changing over time. We will revisit the dynamics of the equity premium later in the mini.

Question 8 [*Feasibility*]

In the optimal portfolio, it is troubling if $w_n^* < 0$. Since this is the optimal portfolio we all want to hold we have a problem. We cannot all be short. To get a feel for how big a deal this might be, try the following: If you select two industries at random, what is the chance that the sharpe-raito maximal portfolio does not involve a short position. (Easiest way to do this is a quick simulation that chooses, say 100, industry pairs and calculates the Sharpe-ratio maximal portfolio. (If you want to just do all $49 \times 48/2$ combinations, feel free).

If you want to see a cool paper related to this, see: Brennan, Thomas J., and Andrew W. Lo. "Impossible frontiers." Management Science 56.6 (2010): 905-923. `https://pubsonline.informs.org/doi/pdf/10.1287/mnsc.1100.1157`

Estimation and statistics – Quick reminder / background

When using data to estimate moments of a distribution, you need a statistical model. Here, let $\tilde{r}_t$ be the return for month t and let $\tilde{R}_t$ be the return for the twelve month period (year). We are interested in estimating $E[R_t]$ and $V[R_t]$ (where you can think of these as scalars for one stock or a vector/matrix for many stocks. To use the data to estimate these quantities we need a statistical model. Here, the model we use is $\tilde{r} \sim IID(\bar{r}, \Sigma)$ – assuming the data is drawn from an identical distribution each period with constant mean and variance. For returns data, this is a sensible model and we will use it here.

We are working with continuous compounding returns ($r = \log((P_{t+1}+D_{t+1})/P_t)$). So $V_0 = 1$, $V_t = V_{t-1}\exp(r_t)$. This has the nice property that $\log V_T = \sum_{t=1}^{T} r_t$. This has the nice property that $\log V_T = \sum_{t=1}^{T} r_t$. So $\tilde{R}_{year_1} = \tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + ... + \tilde{r}_{12}$. From this,

$$\begin{aligned}
E[\tilde{R}_{year_1}] &= E\left[\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + ... + \tilde{r}_{12}\right] \\
&= E[\tilde{r}_1] + E[\tilde{r}_2] + E[\tilde{r}_3] + ... + E[\tilde{r}_{12}] \\
&= \bar{r} + \bar{r} + \bar{r} + ... + \bar{r} \\
&= 12\bar{r}
\end{aligned}$$

Note where we are using the IID assumption. Now, the same with variance

$$\begin{aligned}
V[\tilde{R}_{year_1}] &= V\left[\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3 + ... + \tilde{r}_{12}\right] \\
&= V[\tilde{r}_1] + V[\tilde{r}_2] + V[\tilde{r}_3] + ... + V[\tilde{r}_{12}] \\
&= V + V + V + ... + V \\
&= 12V
\end{aligned}$$

Here you can see we are using the assumption that returns in February are not correlated with returns in March. Lastly, to convert to standard deviation, take the square root. (You can prove that with this IID assumption, monthly correlations are equal to annual correlations.)

We are now looking at mean-variance (or mean-standard-deviation) portfolios. The goal is to use the data to form a portfolio of now. What we really want to know is the current expected returns, variances, and co-variances. Since we cannot see these, we estimate them with the data. The current risk-free rate is about 1% per year (we will work with annual rates.). Since the risk-free rate has varied over the sample of data we have (note the high risk-free rate from the Volker[3] era from the 1980s), it is helpful to calculate means, variances, and co-variances from "excess return" data. That is: stock returns above the risk-free rate. Statistically, we are assuming that excess returns are stationary and independent and identically distributed.

So is the assumption that returns are IID sensible? Yes. As a first cut at the data it is reasonable. In particular, there is not much serial correlation in returns month to month. However, a more sophisticated statistical assumption would address: (1) Variance is not constant across time ("stochastic volatility"), (2) There is some serial correlation in returns ("momentum"), and (3) Risk premia or conditional expected returns are not constant (the "equity risk premium" – the excess return on all stocks in excess of bonds – is low in booms and high in recessions).

[3] Paul Volker was chair of the Federal Reserve Board and ushered in the period of interest-rate targeting as a way of communicating and implementing monetary policy. Prior to Volker, the Fed targeted the harder-to-measure quantity of money as it set out maintaining the stability of the price system. In 1982, inflation rates above 10% were common.