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Game Theory, Spring 2020 A
Assignment 1

1. In the second-price auction, bidders have valuations $v_i$ and submit bids $b_i$. In this setup, the strategy of bidding one's valuation, $b_i^* = v_i$, weakly dominates all other strategies.

   *Proof.* Following the notation in Fudenberg and Tirole, let $r_i = \max_{j \neq i} s_j$ denote the highest bid competing with $s_i$. Generally speaking, player i has three strategies: $s_i = v_i$, $s_i < v_i$, and $s_i > v_i$.

   First, consider the strategy $s_i > v_i$. If $r_i > s_i > v_i$, then player $i$ loses and gains utility 0, which is equivalent to what he would gain by bidding $s_i = v_i$. Similarly, if $r_i = s_i > v_i$, then they player wins with probability 1/2, in which case he gains utility $v_i - r_i < 0$, and loses with probability 1/2, and thus his expected utility is negative, leaving him worse off than if he had bid $s_i = v_i$. If $r_i \leq v_i < s_i$, then player $i$ wins, but gains utility $v_i - r_i > 0$, which is exactly the benefit to playing $s_i = v_i$. Lastly, if $v_i < r_i < s_i$, then the player wins, but gains $v_i - r_i < 0$, and thus this strategy is dominated by bidding $s_i = v_i$.

   Now, consider the strategy $s_i < v_i$. If $s_i < v_i \leq r_i$, then player $i$ loses and gains utility 0, which he would have gained if he had bid $s_i = v_i$. If $s_i < r_i \leq v_i$, then player $i$ loses and gains nothing, and thus would have been better off bidding $s_i = v_i$, a strategy which would have positive expected utility. Lastly, if $r_i \leq s_i < v_i$, then player $i$ wins and gains $v_i - r_i$, which is equivalent to the benefit to bidding $s_i = v_i$.

   Thus, both strategies $s_i < v_i$ and $s_i > v_i$ are weakly dominated by strategy $s_i = v_i$. □

2. In the travelers' dilemma, players $i \in \{1, 2\}$ submit claims $c_i$, with $c_i$ an integer such that $180 \leq c_i \leq 300$.

   a) The process of elimination of weakly dominated strategies can be defined as follows. The initial set of strategies is given by

   $$S_i^0 = S_i = \{180, 181, \ldots, 300\}$$

   and

   $$\Sigma_i^0 = \Sigma_i$$

   the set of mixed strategies. At any stage $n$ in this process, the sets of remaining strategies are given by

   $$S_i^n = \{s_i \in S_i^{n-1} | \nexists \sigma_i \in \Sigma_i^{n-1} \text{ s.t. } u_i(\sigma_i, s_{-i}) \leq u_i(s_i, s_{-i}) \forall s_{-i}\}$$

   with at least one strict inequality, and

   $$\Sigma_i^n = \{\sigma_i \in \Sigma_i^{n-1} | \sigma_i(s_i) > 0 \text{ only if } s_i \in S^n\}$$

   In this problem, the process of elimination of weakly dominated strategies occurs in two steps begins with player one eliminating the strategy $c_1 = 300$ because, if $c_2 = 299$, then player one receives $299 - R$, while player 2 receives $299 + R$. Thus, in this case, player one would be better off playing $c_1 = 298$, and receiving $298 + R > 298$. Player two–who is aware of this incentive that player one has–eliminates the strategy $c_2 = 299$, because this strategy is similarly dominated by that of submitting $c_2 = 297$.
   This process continues until both players reach the Nash Equilibrium: $c_1 = c_2 = 180$.

b) If the elimination of weakly dominated strategies yields a single strategy profile $s^*$, then this profile will be a Nash equilibrium.

*Proof.* Assume that $s^*$ is not a Nash Equilibrium. Thus, because the set of strategies $\mathcal{S}$ is compact and the utility function bounded, $\exists i$ and $\exists s_i$ such that $s_i = \arg\max_{a \in \mathcal{S}} u_i(a, s^*_{-i})$. Furthermore, $s_i$ must have been eliminated at some stage $n$ in the process of elimination of weakly dominated strategies, and thus $\exists\ \sigma_i \in \Sigma_i$ such that

$$u_i(\sigma_i, s_{-i}) \geq u(s_i, s_{-i}) \quad \forall s_{-i} \in S^{n-1}_{-i}$$

If $\sigma_i = s^*_i$, then the statement is trivially true. However, if $\sigma_i \neq s^*_i$, then because $s^*_{-i} \in S^{n-1}_{-i}$,

$$u_i(\sigma_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$$

However, this implies that $\exists$ some $i$ and $\exists s'_i$ in the support of $\sigma_i$ such that

$$u_i(s'_i, s^*_{-i}) > u_i(s_i, s^*_{-i})$$

which contradicts the choice of $s_i$ to deliver the largest possible deviation from $s^*_i$. Thus, $s^*_i$ is a Nash Equilibrium. □

Note that the equilibrium given by iterated domination of weakly dominated strategies will not be unique. As a counterexample, consider the following game:

| | $L$ | $R$ |
|---|---|---|
| $U$ | $(0,0)$ | $(1,1)$ |
| $D$ | $(0,0)$ | $(0,0)$ |

This game has two Nash Equilibria: $(U, R)$ and $(D, L)$. However, elimination of weakly dominated strategies will eliminate the first column and the second row, leaving only $(U, R)$ as an equilibrium.

3. The game of dividing cookies has one unique Nash Equilibrium in pure strategies. Denote the number of cookies by $N$, and the number of students by $M$. The Nash Equilibrium strategy profile is that in which each student submits the same bid for $N/M$ cookies. No student has an incentive to deviate from this strategy: if they bid less, while every other student bids $N/M$, then they will receive their bid, which is less than $N/M$, and thus they would have been better off bidding $N/M$. If they submit a bid greater than $N/M$, then every other student who bid $N/M$ receives their bid exactly, and the student who bid more receives $N/M$, which is exactly what they would have received if they had bid $N/M$. Thus, this is the unique Nash Equilibrium.

4. In this game, players 1 and 2 submit bids $s$ and $t$, respectively. Their payoffs are as follows:

$$u_1(s,t) = 2\alpha st - s^2$$
$$u_2(s,t) = t^3 - 3st$$

a) From the first-order conditions for the two players, we get their reaction functions:

$$\Gamma_1(t) = \alpha t$$
$$\Gamma_2(s) = \pm\sqrt{s}$$

These functions are plotted below:

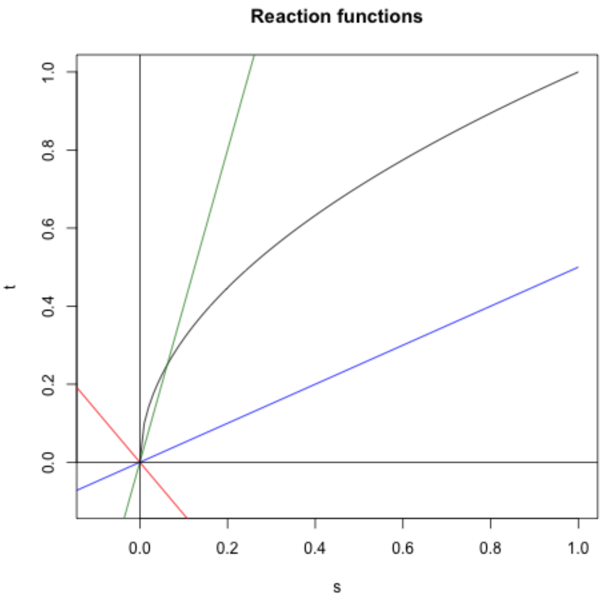


Figure 1: $\Gamma_2(s)$ is shown in black. $\Gamma_1(t)$ for $\alpha$ values of $\frac{-3}{4}$, $\frac{1}{4}$, and 2 are shown in red, green, and blue, respectively.

b) The Nash Equilibria are the points at which the reaction functions intersect. If $\alpha = -\frac{3}{4}$ (the red line), the Nash Equilibrium is $(s,t) = (0,0)$. For $\alpha = \frac{1}{4}$ (the green line), the equilibria are $(s,t) = (0,0)$ and $(s,t) = (\frac{1}{16}, \frac{1}{4})$. For $\alpha = 2$ (the blue line), the equilibrium is again $(s,t) = (0,0)$

5. In this Bertrand economy, denote the sellers $s_1$ and $s_2$, who post prices $p_1$ and $p_2$ respectively.

a) The Nash Equilibrium in pure strategies is $p_1 = p_2 = 0$. In brief, if either seller posts a nonzeroa price, then the other can simply undercut him by an infinitesimally small amount, and take all of the profits. Thus, neither has an incentive to deviate from a price of zero.

b) Even allowing for mixed strategies, the only Nash Equilibrium is $p_1 = p_2 = 0$. By definition, an equilibrium mixed strategy must have pure strategies in its support which are not dominated. However, in this setup, every nonzero price is a dominated strategy, as it can be undercut.

c) If a buyer with valuation one is introduced to the problem, then there is still only one Nash Equilibrium: $p_1 = p_2 = 0$. The buyer's valuation puts a ceiling on the prices that can be charged, but the ceiling never binds, as each firm has an incentive to undercut the other.