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Game Theory, Spring 2020 A
Assignment 3

1. In the potential sale of B to A, I first consider the strategy for firm B. Firm B knows its value x, and sets a minimum price m, and the sale is completed if firm A's offer y is greater than or equal to m. Because firm B receives a value of x if the sale is not completed, it is easy to see that their equilibrium strategy is to set $m = x$. If firm B sets $m < x$, then if firm A makes an offer such that $m < y < x$, then the sale would be completed, but B would have been better off setting a higher m and receiving x. Conversely, if B sets $m > x$, then the sale would fail if firm A made an offer such that $m > y > x$, but firm B would have been better off had the sale gone through. Thus, $m = x$.

   Firm A, then, aims to choose y to maximize its expected profit, knowing that $m = x$ and $x \sim \mathcal{U}[0, 100]$. Firm A's payoff, knowing these pieces of information, is given by

$$u_A(y|x) = \begin{cases} \frac{3}{2}x - y & x \leq y \\ 0 & x > y \end{cases}$$

   Thus, their expected utility is given by

$$\begin{aligned} \mathbb{E}[u_A|x] &= \left(\frac{3}{2}x - y\right) \cdot \Pr(x \leq y) \\ &= \left(\frac{3}{2}x - y\right) \frac{y}{100} \end{aligned}$$

   The total expected payoff to firm A is given by

$$\int_0^y \left(\frac{3}{2}x - y\right) dx = \frac{3}{4}x^2 - xy\Big|_0^y = -\frac{y^2}{4}$$

   Differentiating the expected payoff with respect to y and setting equal to 0 yields

$$-\frac{1}{2}y = 0 \implies y = 0$$

   Thus, the optimal strategy for firm A, given that firm B plays their equilibrium strategy of $m = x$, is to bid $y = 0$. This seemingly counterintuitive result is a consequence of the payout structure, and the distribution of x.

2. In the two-bidder mixed auction where each player $i \in \{1, 2\}$, the winner pays $\alpha b_i + (1-\alpha)b_j$, $j \neq i$. Players' valuations v_1 and v_2 are both distributed $\mathcal{U}[0, 1]$. In general, expected utility for player i is given by

$$U_i(v_i, b_i, b_j) = \Pr(b_i > b_j) \cdot (v_i - \alpha b_i - (1-\alpha)b_j) + \frac{1}{2}\Pr(b_i = b_j) \cdot (v_i - \alpha b_i - (1-\alpha)b_j)$$

   Because the distributions of the valuations are continuous, $\Pr(b_i = b_j) = 0$, and thus we can restrict attention to the first term in the sum. Assume that a symmetric bidding strategy exists, such that

$$b_i = \beta(v_i)$$

with $\beta(v_i)$ monotone increasing in v_i. To begin, note that because β is monotone, β^{-1} exists. Then,

$$\begin{aligned}
\Pr(b_i > b_j) &= \Pr(b_i > \beta(v_j)) \\
&= \Pr(\beta^{-1}(b_i) > v_j) \\
&= F(\beta^{-1}(b_i))
\end{aligned}$$

Where F is the uniform CDF. Thus, player i's problem is

$$\max_{b_i} F(\beta^{-1}(b_i))(v_i - \alpha b_i - (1-\alpha)b_j)$$

The first-order condition is

$$\frac{f(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}\big(v_i - \alpha b_i - (1-\alpha)b_j\big) - \alpha F(\beta^{-1}(b_i)) = 0$$

Rewriting, this becomes

$$\beta'(v_i)F(v_i) + f(v_i)\beta(v_i) = \frac{f(v_i)v_i - (1-\alpha)f(v_i\beta(v_j))}{\alpha}$$

and thus

$$\frac{d}{dv_i}\beta(v_i)F(v_i) = \frac{f(v_i)v_i - (1-\alpha)f(v_i\beta(v_j))}{\alpha}$$

Integrating yields

$$\beta(v_i) = \frac{1}{\alpha F(v_i)}\int_0^{v_i} f(x)x dx - \frac{1-\alpha}{\alpha}\beta(v_j)\int_0^{v_i} f(x)dx + C$$

Because bidding any amount greater than 0 if $v_i = 0$ is a dominated strategy, it must be that $C = 0$. Substituting this into above, we can integrate and, using the fact that $F(x) = x$, recover

$$\beta(v_i) = v_i\left(\frac{1}{2\alpha} - \frac{1-\alpha}{\alpha}\beta(v_j)\right)$$

The above holds for $i, j \in \{1, 2\}$, and for all $v_i, v_j \in [0, 1]$. Thus, it must hold for the case when $v_i = v_j = v$. In this case,

$$\begin{aligned}
\beta(v) &= v\left(\frac{1}{2\alpha} - \frac{1-\alpha}{\alpha}\beta(v)\right) \\
&= \frac{v}{2(\alpha + (1-\alpha)v)}
\end{aligned}$$

Thus, $\beta(v_i) = 0$ when $v_i = 0$, and β increases monotonically in v_i, as

$$\beta'(v_i) = 2\alpha + (1-\alpha)v$$

Thus, this symmetric equilibrium has the desired properties.

3.

4. In this case of bilateral trade, the buyer offers p_b, while the seller posts p_s. If $p_s \leq p_b$, then the item sells for price $p = \frac{1}{2}(p_s + p_b)$. Assume that both players play a linear equilibrium strategy, that is,

$$p_i = \alpha_i + \beta_i v_i$$

for $i \in \{b, s\}$. In order to calculate the exact equilibrium strategy, I first formulate the best response for the buyer and the seller.

The buyer's payoff is $v_b - \frac{1}{2(p_s+p_b)}$ if $p_s \leq p_b$, and 0 otherwise. Assuming that the seller follows $p_s = \alpha_s + \beta_s v_s$,

$$\Pr(p_s \leq p_b) = F\left(\frac{p_b - \alpha_s}{\beta_s}\right)$$

where F is the uniform CDF on [0,1]. Thus, the seller's expected utility is given by

$$\begin{aligned}
\mathbb{E}[u_b] &= \int_0^{\frac{p_b-\alpha_s}{\beta_s}} \left(v_b - \frac{1}{2}(p_s + p_b)\right) dv_s \\
&= \int_0^{\frac{p_b-\alpha_s}{\beta_s}} \left(v_b - \frac{1}{2}(p_b + \alpha_s + \beta_s v_s)\right) dv_s
\end{aligned}$$

To optimize, using Liebniz's rule, I compute

$$\frac{\partial}{\partial p_b} \int_0^{\frac{p_b-\alpha_s}{\beta_s}} \left(v_b - \frac{1}{2}(p_b + \alpha_s + \beta_s v_s)\right) dv_s$$

and set the result equal to 0 to obtain

$$p_b = \frac{2}{3} v_b + \frac{\alpha_s}{3}$$