# Solving Mirrleesian Optimal Taxation Problems with Infinitely Many Types Using Finite Element Method

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## 1 Static Model

Consider the following environment. Individual preferences are[1]

$$U(c,l) = u(c) - v(l)$$

Technology

$$y = \theta l.$$

Here $l$ is hours worked (or effort), $\theta$ is labor ability (or productivity), and $y$ is effective labor services.

**Assumption.** *Only $y$ is observable by taxing authorities. Therefore, taxes cannot be functions of $\theta$ or $l$.*

Labor ability/productivity $\theta$ has CDF $F(\theta)$ (PDF $f(\theta)$) on $\left[\underline{\theta},\overline{\theta}\right]$ where $\bar{\theta}$ can be infinity. In what follows, I refer to $\theta$ as individual's 'type'.

*Note.* In some of the derivations it is useful/convenient to use $y/\theta$ to denote hours worked.

Let $T(y)$ be a tax function. Then individual of type $\theta$ faces the following decision problem

$$U(\theta) = \max_{c,l} u(c) - v\left(\frac{y}{\theta}\right)$$
$$c = y - T(y)$$

Note that

$$\theta u'(c) = \left(1 - T'(y)\right) v'\left(\frac{y}{\theta}\right)$$

Also, by envelope theorem

$$\dot{U}(\theta) = \frac{y}{\theta^2} v'\left(\frac{y}{\theta}\right)$$

Which we can rewrite as

$$U'(\theta) = \frac{l(\theta)}{\theta} v'(l(\theta)) \tag{1}$$

Constraint (1) is call *implementability constraint* (or *incentive compatibility constraint*).

Suppose there is government that has expenditure $G$. The government budget constraint is

$$G = \int_\theta T(y(\theta)) f(\theta)\, d\theta.$$

[1] There is nothing special about additive separability. All procedures can be extended to more general preferences.

Finally allocation in the economy is feasible if

$$\int_{\theta} c(\theta) f(\theta) d\theta + G = \int_{\theta} y(\theta) f(\theta) d\theta.$$

**Theorem 1.** *Any feasible allocation $(c(\theta), l(\theta))$ can be implemented via some income tax function $T(y)$ iff it satisfy implementability constraint (1).*

*Proof.* The necessity is obvious (outlined above). The sufficiency is by construction of a tax function. $\square$

This theorem transforms the problem of finding optimal policy function, $T(y)$, (which is a very complicated problem) to a constrained maximization problem over allocations (which can be solved using standard methods).

## 1.1 Planning Problem

Consider the problem of a government who seeks to find policies that maximize weighted average of welfare in the economy. Suppose government assigns weight $g(\theta)$ to individual of type $\theta$.

Instead of writing this maximization problem over the set of policy functions, we write the following maximization problem over the set of *implementable allocations*.

$$\max \int_{\underline{\theta}}^{\overline{\theta}} U(\theta) g(\theta) f(\theta) d\theta$$

s.t.

$$G + \int_{\underline{\theta}}^{\overline{\theta}} (c(\theta) - \theta l(\theta)) f(\theta) d\theta = 0 \quad ; \lambda$$

$$U(\theta) = u(c(\theta)) - v(l(\theta)) \quad ; f(\theta) \eta(\theta)$$

$$U' = \frac{l(\theta)}{\theta} v'(l(\theta)) \quad ; \mu(\theta) f(\theta)$$

First order conditions:

$$-\lambda + u'(c(\theta))\eta(\theta) = 0 \tag{2}$$

$$\theta\lambda - \eta(\theta) v'(l(\theta)) + \frac{\mu(\theta)}{\theta} \left(v'(l(\theta)) + l(\theta) v''(l(\theta))\right) = 0 \tag{3}$$

Hamiltonian:

$$g(\theta) - \eta(\theta) + \mu'(\theta) + \frac{f'(\theta)}{f(\theta)} \mu(\theta) = 0 \tag{4}$$

Boundary conditions:

$$\mu(\overline{\theta}) = \mu(\underline{\theta}) = 0$$

Use (2) to eliminate $\eta(\theta)$

$$\theta - \frac{v'(l(\theta))}{u'(c(\theta))} + \frac{\mu(\theta)}{\lambda\theta} \left(v'(l(\theta)) + l(\theta) v''(l(\theta))\right) = 0$$

$$g(\theta) + \frac{f'(\theta)}{f(\theta)} \mu(\theta) - \frac{\lambda}{u'(c(\theta))} + \dot{\mu}(\theta) = 0$$

We need to solve the following system of equations:

$$G+\int_{\underline{\theta}}^{\overline{\theta}}\left(c(\theta)-\theta l(\theta)\right)f(\theta)d\theta=0 \tag{5}$$

$$U(\theta)=u(c(\theta))-v\left(l(\theta)\right) \tag{6}$$

$$U'(\theta)=\frac{l(\theta)}{\theta}v'\left(l(\theta)\right) \tag{7}$$

$$\theta-\frac{v'\left(l(\theta)\right)}{u'(c(\theta))}+\frac{\mu(\theta)}{\lambda\theta}\left(v'\left(l(\theta)\right)+l(\theta)v''\left(l(\theta)\right)\right)=0 \tag{8}$$

$$g(\theta)+\frac{f'(\theta)}{f(\theta)}\mu(\theta)-\frac{\lambda}{u'(c(\theta))}+\mu'(\theta)=0 \tag{9}$$

$$\mu(\overline{\theta})=\mu(\underline{\theta})=0 \tag{10}$$

To solve for the following five: $c(\theta),l(\theta),U(\theta),\mu(\theta),\lambda$.

Note that this is in fact an ODE in $U\left(\theta\right)$ and $\mu\left(\theta\right)$ with boundary conditions $\mu(\overline{\theta})=\mu(\underline{\theta})=0$. So we can use method of weighted residual to solve it.

## 1.2 Example:

Consider the following example

$$U(c,l)=\frac{c^{1-\sigma}}{1-\sigma}-\psi\frac{l^{\gamma}}{\gamma}$$

So equations (7), (8) and (9) become

$$U'\left(\theta\right)=\frac{\psi l\left(\theta\right)^{\gamma}}{\theta}$$

$$\theta-\psi l\left(\theta\right)^{\gamma-1}c\left(\theta\right)^{\sigma}+\frac{\mu(\theta)}{\lambda\theta}\psi\gamma l\left(\theta\right)^{\gamma-1}=0$$

$$g(\theta)+\frac{f'(\theta)}{f(\theta)}\mu(\theta)-\lambda c(\theta)^{\sigma}+\mu'(\theta)=0$$

Take $\lambda$ as given. We want to solve the following system of equations

$$\begin{aligned}U' &= \psi\frac{l^{\gamma}}{\theta}\\ \mu' &= \lambda c^{\sigma}-g-\frac{f'}{f}\mu\end{aligned}$$

where $l$ and $c$ are solutions to the following equation

$$\begin{aligned}\frac{c^{1-\sigma}}{1-\sigma}-\psi\frac{l^{\gamma}}{\gamma}-U &= 0\\ \theta-\psi l^{\gamma-1}c^{\sigma}+\frac{\mu}{\lambda\theta}\psi\gamma l^{\gamma-1} &= 0\end{aligned}$$

We approximate $\mu$ and $U$ with

$$\begin{aligned} U(\theta) &= \sum_{n=1}^{N} \alpha_n \psi_n(\theta) \\ \mu(\theta) &= \sum_{n=1}^{N} \beta_n \psi_n(\theta) \end{aligned}$$

where $\phi_n(\theta)$ is the tent function on $[\theta_{n-1}, \theta_{n+1}]$.

Define

$$R_\alpha(\theta) = U'(\theta) - \psi \frac{l(\theta; U, \mu, \lambda)^\gamma}{\theta},$$
$$R_\beta(\theta) = \mu'(\theta) - \left(\lambda c(\theta; U, \mu, \lambda)^\sigma - g - \frac{f'}{f}\mu\right).$$

We form the following system equations

$$\int_{\underline{\theta}}^{\bar{\theta}} \psi_n(\theta) R_\alpha(\theta) \, d\theta = 0 \quad n = 1, \ldots, N$$
$$\int_{\underline{\theta}}^{\bar{\theta}} \psi_n(\theta) R_\beta(\theta) \, d\theta = 0 \quad n = 1, \ldots, N$$
$$G + \int_{\underline{\theta}}^{\overline{\theta}} (c(\theta; U, \mu, \lambda) - \theta l(\theta; U, \mu, \lambda)) f(\theta) d\theta = 0$$

This is a system of $2N+1$ equations to solve for $\alpha_n$, $\beta_n$ and $\lambda$. The good news is each equations (except the last one) in only relevant only on one interval $[\theta_n, \theta_{n+1}]$.

Here is how the algorithm works:

1. Start with a guess of $\lambda$, $\alpha_n$ and $\beta_n$.

2. For $\theta \in [\theta_n, \theta_{n+1}]$, find $U(\theta)$, $\mu(\theta)$.

3. Solve for $c(\theta; U, \mu, \lambda)$ and $l(\theta; U, \mu, \lambda)$ such that

$$\begin{aligned} \frac{c^{1-\sigma}}{1-\sigma} - \psi \frac{l^\gamma}{\gamma} - U &= 0, \\ \theta - \psi l^{\gamma-1} c^\sigma + \frac{\mu}{\lambda\theta} \psi \gamma l^{\gamma-1} &= 0. \end{aligned}$$

   These are just promise keeping and FOC w.r.t to $l$.

4. Evaluate $R_\alpha(\theta)$, $R_\beta(\theta)$ and feasibility.

5. Evaluate the derivative of the above equations w.r.t $\alpha_n$, $\beta_n$ and $\lambda$.

6. Do the newton update.

The following is useful in doing steps 2, 3, 4 and 5.

Let $\epsilon = 2(\theta - \theta_n)/(\theta_{n+1} - \theta_n) - 1$ and $\Delta_n = \theta_{n+1} - \theta_n$. Then on the interval $[\theta_n, \theta_{n+1}]$

$$\begin{aligned} U(\theta) &= 0.5\alpha_n (1-\epsilon) + 0.5\alpha_{n+1}(1+\epsilon) \\ \mu(\theta) &= 0.5\beta_n (1-\epsilon) + 0.5\beta_{n+1}(1+\epsilon) \end{aligned}$$

and

$$\begin{aligned} U'(\theta) &= \frac{-\alpha_n + \alpha_{n+1}}{\Delta_n} \\ \mu'(\theta) &= \frac{-\beta_n + \beta_{n+1}}{\Delta_n} \end{aligned}$$

Therefore, we need to solve the system of $2N$ nonlinear equations for $\alpha_n$and $\beta_n$

$$\begin{aligned} \frac{-\alpha_n + \alpha_{n+1}}{\Delta_n} & -\phi & \frac{l(\theta; \alpha_n, \alpha_{n+1}, \beta_n, \beta_{n+1})^\gamma}{\theta} = 0 \\ \frac{-\beta_n + \beta_{n+1}}{\Delta_n} & - & \left(\lambda c(\theta; \alpha_n, \alpha_{n+1}, \beta_n, \beta_{n+1})^\sigma - g - (0.5\beta_n (1-\epsilon) + 0.5\beta_{n+1}(1+\epsilon)) \frac{f'}{f}\right) = 0 \end{aligned}$$

With conditions that $\alpha_1 = \alpha_N = 0$.

Derivative of the $n$th equation with respect to

- $\alpha_n$

$$-1/\Delta_n - \frac{\gamma\psi l^{\gamma-1}}{\theta}\frac{\partial l}{\partial \alpha_n}$$
$$-\sigma\lambda c^{\sigma-1}\frac{\partial c}{\partial \alpha_n}$$

- $\alpha_{n+1}$

$$1/\Delta_n - \frac{\gamma\psi l^{\gamma-1}}{\theta}\frac{\partial l}{\partial \alpha_{n+1}}$$
$$-\sigma f\lambda c^{\sigma-1}\frac{\partial c}{\partial \alpha_{n+1}}$$

- $\beta_n$

$$-\frac{\gamma\psi l^{\gamma-1}}{\theta}\frac{\partial l}{\partial \beta_n}$$
$$-1/\Delta_n - \left(\sigma\lambda c^{\sigma-1}\frac{\partial c}{\partial \beta_n} - 0.5(1-\epsilon)\frac{f'}{f}\right)$$

- $\beta_{n+1}$

$$-\frac{\gamma\psi l^{\gamma-1}}{\theta}\frac{\partial l}{\partial \beta_{n+1}}$$
$$1/\Delta_n - \left(\sigma\lambda c^{\sigma-1}\frac{\partial c}{\partial \beta_{n+1}} - 0.5(1+\epsilon)\frac{f'}{f}\right)$$

Now, we can use the promise keeping and IC to find $\frac{\partial c}{\partial}$ and $\frac{\partial l}{\partial}$

$$\begin{bmatrix} c^{-\sigma} & -\psi l^{\gamma-1} \\ -\sigma\psi l^{\gamma-1}c^{\sigma-1} & \psi(\gamma-1)\left(\frac{\mu\gamma}{\lambda\theta} - c^\sigma\right) l^{\gamma-2} \end{bmatrix} \begin{bmatrix} \frac{\partial c}{\partial U} \\ \frac{\partial l}{\partial U} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ll} c^{-\sigma} & -\psi l^{\gamma-1} \\ -\sigma\psi l^{\gamma-1}c^{\sigma-1} & \psi\left(\gamma-1\right)\left(\frac{\mu\gamma}{\lambda\theta}-c^{\sigma}\right)l^{\gamma-2} \end{array}\right]\left[\begin{array}{c} \frac{\partial c}{\partial\mu} \\ \frac{\partial l}{\partial\mu} \end{array}\right]=\left[\begin{array}{l} 0 \\ -\frac{\psi\gamma}{\lambda\theta}l^{\gamma-1} \end{array}\right]$$

Finally

$$\frac{\partial U}{\partial\alpha_n}=0.5\left(1-\epsilon\right)$$

$$\frac{\partial U}{\partial\alpha_{n+1}}=0.5\left(1+\epsilon\right)$$

$$\frac{\partial\mu}{\partial\beta_n}=0.5\left(1-\epsilon\right)$$

$$\frac{\partial\mu}{\partial\beta_{n+1}}=0.5\left(1+\epsilon\right)$$