Econometrica, Vol. 82, No. 2 (March, 2014), 601–653

DYNAMIC MECHANISM DESIGN: A MYERSONIAN APPROACH

BY ALESSANDRO PAVAN, ILYA SEGAL, AND JUUSO TOIKKA[1]

We study mechanism design in dynamic quasilinear environments where private information arrives over time and decisions are made over multiple periods. We make three contributions. First, we provide a necessary condition for incentive compatibility that takes the form of an envelope formula for the derivative of an agent's equilibrium expected payoff with respect to his current type. It combines the familiar marginal effect of types on payoffs with novel marginal effects of the current type on future ones that are captured by "impulse response functions." The formula yields an expression for dynamic virtual surplus that is instrumental to the design of optimal mechanisms and to the study of distortions under such mechanisms. Second, we characterize the transfers that satisfy the envelope formula and establish a sense in which they are pinned down by the allocation rule ("revenue equivalence"). Third, we characterize perfect Bayesian equilibrium-implementable allocation rules in Markov environments, which yields tractable sufficient conditions that facilitate novel applications. We illustrate the results by applying them to the design of optimal mechanisms for the sale of experience goods ("bandit auctions").

KEYWORDS: Asymmetric information, stochastic processes, incentives, mechanism design, envelope theorems.

1. INTRODUCTION

WE CONSIDER THE DESIGN OF INCENTIVE-COMPATIBLE MECHANISMS in a dynamic environment in which agents receive private information over time and decisions are made in multiple periods over an arbitrary time horizon. The model allows for serial correlation of the agents' information and for the dependence of this information on past allocations. It covers as special cases problems such as allocation of private or public goods to agents whose valuations evolve stochastically over time, procedures for selling experience goods to consumers who refine their valuations upon consumption, and multiperiod procurement under learning-by-doing. Since the time horizon is arbitrary, the model also accommodates problems where the timing of decisions is a choice variable such as when auctioning off rights for the extraction of a natural resource.

Our main results, Theorems 1–3, provide characterizations of dynamic local and global incentive-compatibility constraints that extend the Myersonian

[1]We thank participants at various conferences, seminars, and institutions where this paper was presented. Special thanks go to Thomas Chaney, Daniel Garrett, Li Hao, Narayana Kocherlakota, Andy Skrzypacz, Xianwen Shi, a co-editor, and anonymous referees for useful suggestions that helped us improve the paper. Pavan also thanks the hospitality of Collegio Carlo Alberto and the University of California Berkeley where part of this work was completed. This paper supersedes previous working papers "Revenue Equivalence, Profit Maximization and Transparency in Dynamic Mechanisms" by Segal and Toikka, "Long-Term Contracting in a Changing World" by Pavan, and "Infinite-Horizon Mechanism Design" by Pavan, Segal, and Toikka.

 DOI: 10.3982/ECTA10269

approach to mechanism design with continuous types (Myerson (1981)) to dynamic environments. We then apply these results to the design of optimal dynamic mechanisms. We focus on quasilinear environments where the agents' new private information is unidimensional in each period.[2] To rule out full surplus extraction à la Cremer and McLean (1988), we assume throughout that this information is independent across agents conditional on their own allocations. In addition to the methodological contribution, our results provide some novel concepts that facilitate a unified view of the existing literature and help to explain what drives distortions in optimal dynamic contracts.

The cornerstone of our analysis is a dynamic envelope theorem, Theorem 1, which, under appropriate regularity conditions, yields a formula for the derivative of an agent's expected equilibrium payoff with respect to his current private information, or type, in any perfect Bayesian incentive-compatible mechanism.[3] This formula characterizes local incentive compatibility constraints, as in Mirrlees' (1971) first-order approach for static environments. It captures the usual direct effect of a change in the current type on the agent's utility as well as a novel indirect effect due to the induced change in the distribution of the agent's future types. The stochastic component of the latter is summarized by *impulse response functions* that describe how a change in the agent's current type propagates through his type process. Theorem 1 thus identifies the impulse response as the notion of stochastic dependence that is relevant for mechanism design. Our definition of the impulse response functions and the proof of Theorem 1 make use of the fact that any stochastic process can be constructed from a sequence of independent random variables. This observation was first used in the context of mechanism design by Eső and Szentes (2007).

The envelope formula of Theorem 1 is independent of the transfers. Thus, applying the formula to the initial period, yields a dynamic payoff equivalence result that generalizes Myerson's (1981) revenue equivalence theorem. More generally, Theorem 2 shows that, given any dynamic allocation rule, the envelope formula can be used to construct payments that satisfy local incentive-compatibility constraints at all truthful histories. In the single-agent case, the allocation rule determines, up to a scalar, the net present value of payments, for any realized sequence of types. This *ex post payoff equivalence* extends to multiple agents under an additional condition and pins down the expected net present value of payments conditional on the agent's own type sequence, where

[2]By reinterpreting monetary payments as "utility from monetary payments," all of our results on incentive compatibility extend to non-quasilinear environments where the agents' utility from monetary payments (or, more generally, from some other instrument available to the designer) is independent of their private information and additively separable from their allocation utility. For example, this covers models typically considered in new dynamic public finance or in the managerial compensation literature.

[3]This envelope theorem may be useful also in other stochastic dynamic programming problems.

the expectation is over the other agents' type sequences.[4] This condition is satisfied if, for example, the evolution of types is independent of allocations.

We then focus on Markov environments so as to characterize global incentive-compatibility constraints. Theorem 3 shows that an allocation rule is implementable in a perfect Bayesian equilibrium if and only if it satisfies *integral monotonicity*. The Markov restriction implies that any allocation rule that satisfies integral monotonicity can be implemented, using payments from Theorem 2, in a *strongly truthful* equilibrium where the agents report truthfully on and off the equilibrium path. This allows us to restrict attention to one-shot deviations from truthtelling, and is the reason for our focus on Markov environments. However, it is instructive to note that, even if an agent's current type is unidimensional, his report can affect allocations in multiple periods. Thus, the static analog of our problem is one with unidimensional types but multidimensional allocations, which explains why the integral-monotonicity condition cannot be simplified without losing necessity.[5]

Theorem 3 facilitates formulating sufficient conditions for implementability that are stronger than necessary, but easier to verify. A special case is the notion of *strong monotonicity* typically considered in the literature, which requires that each agent's allocation be increasing in his current and past reports in every period; it is applicable to models where payoffs satisfy a single-crossing property and where type transitions are independent of allocations and increasing in past types in the sense of first-order stochastic dominance. Having identified the underlying integral-monotonicity condition, we are able to relax both the notion of monotonicity and the requirements on the environment. Heuristically, this amounts to requiring monotonicity only "on average" across time (*ex post monotonicity*) or, weaker still, across time and future types (*average monotonicity*). We use these new conditions to establish the implementability of the optimal allocation rule in some settings where strong monotonicity fails.

The leading application for our results is the design of optimal mechanisms in Markov environments.[6] We adopt the first-order approach that is familiar from static settings where an allocation rule is found by solving a relaxed problem that only imposes local incentive-compatibility constraints and the lowest initial types' participation constraints, and where a monotonicity condition is

[4]The result is useful in the non-quasilinear models of footnote 2. There it determines the ex post net present value of the agent's utility from monetary payments and facilitates computing the cost-minimizing timing of payments.

[5]Implementability has been characterized in static models in terms of analogous conditions by Rochet (1987), and, more recently, by Carbajal and Ely (2013) and Berger, Müller, and Naeemi (2010).

[6]In the Supplemental Material (Pavan, Segal, and Toikka (2014)), we discuss how our characterization of incentive compatibility for Markov environments can be used to derive sufficient conditions for implementability in some classes of non-Markov environments and to extend our results on optimal mechanisms to such environments.

used to verify the implementability of the rule. The envelope formula from Theorem 1 can be used as in static settings to show that the principal's problem is then to maximize expected virtual surplus, which is only a function of the allocation rule. This is a Markov decision problem and, hence, it can be solved using standard methods. We then use integral monotonicity from Theorem 3 to verify that the solution is implementable, possibly by checking one of the sufficient conditions discussed above. When this is the case, the optimal payments can be found by using Theorem 2. If, for each agent, the lowest initial type is the one worst off under the candidate allocation rule (which is the case, for example, when utilities are increasing in own types and transitions satisfy first-order stochastic dominance), then the participation constraints of all initial types are satisfied and the mechanism so constructed is an optimal dynamic mechanism.

The impulse response functions play a central role in explaining the direction and dynamics of distortions in optimal dynamic mechanisms. As in static settings, distortions are introduced to reduce the agents' expected information rents, as computed at the time of contracting. However, because of the serial correlation of types, it is optimal to distort allocations not only in the initial period, but at every history at which the agent's type is responsive to his initial type, as measured by the impulse response function. We illustrate by means of a buyer–seller example that this can lead to the distortions being nonmonotone in the agent's reports and over time. The optimal allocation rule in the example is not strongly monotone and, hence, the new sufficiency conditions derived from integral monotonicity are instrumental for uncovering these novel dynamics.

Similarly to static settings, the first-order approach outlined above yields an implementable allocation rule only under fairly stringent conditions. These conditions are by no means generic, though they include as special cases the ones usually considered in the literature. We provide some sufficient conditions on the primitives that guarantee that the relaxed problem has a solution that satisfies strong monotonicity, but as is evident from above, such conditions are far from being necessary. We illustrate the broader applicability of the tools by solving for optimal "bandit auctions" of experience goods in a setting where bidders update their values upon consumption. The optimal allocation there violates strong monotonicity, but satisfies average monotonicity.

We conclude the Introduction by commenting on the related literature. The rest of the paper is then organized as follows. We describe the dynamic environment in Section 2, and present our results on incentive compatibility and implementability in Section 3. We then apply these results to the design of optimal dynamic mechanisms in Section 4, illustrating the general approach by deriving the optimal bandit auction in Section 5. We conclude in Section 6. All proofs that are omitted in the main text are provided in the Appendix. Additional results can be found in the Supplemental Material (Pavan, Segal, and Toikka (2014)).

1.1. *Related Literature*

The literature on optimal dynamic mechanism design goes back to the pioneering work of Baron and Besanko (1984), who use the first-order approach in a two-period single-agent setting to derive an optimal mechanism for regulating a natural monopoly. They characterize optimal distortions using an "informativeness measure," which is a two-period version of our impulse response function. More recently, Courty and Li (2000) consider a similar model to study optimal advanced ticket sales, and also provide sufficient conditions for a dynamic allocation rule to be implementable. Eső and Szentes (2007) then extend the analysis to multiple agents in their study of optimal information revelation in auctions.[7] They use a two-period state representation to orthogonalize an agent's future information by generating the randomness in the second-period type through an independent shock. We build on some of the ideas and results in these papers, and special cases of some of our results can be found in them. We comment on the connections at the relevant points in the analysis. In particular, we discuss the role of the state representation in the Concluding Remarks (Section 6), having first presented our results.

Whereas the aforementioned works consider two-period models, Besanko (1985) and Battaglini (2005) characterize the optimal infinite-horizon mechanism for an agent whose type follows a Markov process, with Besanko considering a linear AR(1) process over a continuum of states, and Battaglini considering a two-state Markov chain. Their results are qualitatively different: Besanko (1985) finds that the allocation in each period depends only on the agent's initial and current type, and is distorted downward at each finite history with probability 1. In contrast, Battaglini (2005) finds that once the agent's type turns high, he consumes at the efficient level irrespective of his subsequent types. Our analysis shows that the relevant property of the type processes that explains these findings, and the dynamics of distortions more generally, is the impulse response of future types to a change in the agent's initial private information.[8]

Board (2007) is the first to consider a multi-agent environment with infinite time horizon. He extends the analysis of Eső and Szentes (2007) to a setting where the timing of the allocation is endogenous, so that the principal is selling options. Subsequent to the first version of our manuscript, Kakade, Lobel, and Nazerzadeh (2011) consider a class of allocation problems that generalize Board's model as well as our bandit auctions, and show that the optimal mechanism is a virtual version of the dynamic pivot mechanism of Bergemann and Välimäki (2010). We comment on the connection to Kakade et al. in Section 5 after presenting our bandit auction.

[7]See Riordan and Sappington (1987) for an early contribution with many agents.

[8]Battaglini's (2005) model with binary types is not formally covered by our analysis. However, we discuss in the Supplemental Material how impulse responses can be adapted to discrete type models.

That the literature on optimal dynamic mechanisms has focused on relatively specific settings reflects the need to arrive at a tractable optimization problem over implementable allocation rules. In contrast, when designing efficient (or expected surplus-maximizing) mechanisms, the desired allocation rule is known a priori. Accordingly, Bergemann and Välimäki (2010), and Athey and Segal (2013) introduce dynamic generalizations of the static Vickrey–Clarke–Groves and expected externality mechanisms for very general quasilinear private-value environments.[9]

A growing literature considers both efficient and profit-maximizing dynamic mechanisms in settings where each agent receives only one piece of private information, but where the agents or objects arrive stochastically over time as in, for example, Gershkov and Moldovanu (2009) and more recently Board and Skrzypacz (2013). The characterization of incentive compatibility in such models is static, but interesting dynamics emerge from the optimal timing problem faced by the designer. We refer the reader to the excellent recent survey by Bergemann and Said (2011).[10]

Our work is also related to the literature on dynamic insurance and optimal taxation. While the early literature following Green (1987) and Atkeson and Lucas (1992) assumed independent and identically distributed (i.i.d.) types, the more recent literature has considered persistent private information (e.g., Fernandes and Phelan (2000), Kocherlakota (2005), Albanesi and Sleet (2006), or Kapicka (2013)). In terms of the methods, particularly related are those of Farhi and Werning (2013) and Golosov, Troshkin, and Tsyvinski (2011), who used a first-order approach to characterize optimal dynamic tax codes. There is also a continuous-time literature on contracting with persistent private information that uses Brownian motion, in which impulse responses are constant, simplifying the analysis. See Williams (2011) and the references therein.

Our analysis of optimal mechanisms assumes that the principal can commit to the mechanism he is offering and, hence, the dynamics are driven by changes in the agents' information. In contrast, the literature on dynamic contracting with adverse selection and limited commitment typically assumes constant types and generates dynamics through lack of commitment (see, for example, Laffont and Tirole (1988) or, for more recent work, Skreta (2006) and the references therein).[11]

Dynamic mechanism design is related to the literature on multidimensional screening, as noted, for example, by Rochet and Stole (2003). Nevertheless, there is a sense in which incentive compatibility is easier to ensure in a dynamic

[9]Rahman (2010) derived a general characterization of implementable dynamic allocation rules similar to Rochet's (1987) cyclical monotonicity. Its applicability to the design of optimal mechanisms is, however, yet to be explored.

[10]Building on this literature and on the results of the current paper, Garrett (2011) combined private information about arrival dates with time-varying types.

[11]See Battaglini (2007) and Strulovici (2011) for an analysis of limited commitment with changing types.

setting than in a static multidimensional setting. This is because in a dynamic setting, an agent is asked to report each dimension of his private information before learning the subsequent dimensions, so he has fewer deviations available than in the corresponding static setting in which he observes all the dimensions at once. Because of this, the set of implementable allocation rules is larger in a dynamic setting than in the corresponding static multidimensional setting. Nonetheless, our necessary conditions for incentive compatibility are valid also for multidimensional screening problems.

2. THE ENVIRONMENT

Conventions. For any set B, B^{-1} denotes a singleton. If B is measurable, $\Delta(B)$ is the set of probability measures over B. Any function defined on a measurable set is assumed to be measurable. Tildes distinguish random variables from realizations so that, for example, θ denotes a realization of $\tilde{\theta}$. Any set of real vectors or sequences is endowed with the product order unless noted otherwise.

Decisions. Time is discrete and indexed by $t = 0, 1, \dots, \infty$. There are $n \geq 1$ agents, indexed by $i = 1, \dots, n$. In every period t, each agent i observes a signal, or type, $\theta_{it} \in \Theta_{it} = (\underline{\theta}_{it}, \bar{\theta}_{it}) \subseteq \mathbb{R}$, with $-\infty \leq \underline{\theta}_{it} \leq \bar{\theta}_{it} \leq +\infty$, and then sends a message to a mechanism that leads to an allocation $x_{it} \in X_{it}$ and a payment $p_{it} \in \mathbb{R}$ for each agent i. Each X_{it} is assumed to be a measurable space (with the sigma-algebra left implicit). The set of feasible allocation sequences is $X \subseteq \prod_{t=0}^{\infty} \prod_{i=1}^{n} X_{it}$. This formulation allows for the possibility that feasible allocations in a given period depend on the allocations in the previous periods or that the feasible allocations for agent i depend on the other agents' allocations.[12] Let $X_t \equiv \prod_{i=1}^{n} X_{it}$, $X_i^t \equiv \prod_{s=0}^{t} X_{is}$, and $X^t \equiv \prod_{s=0}^{t} X_s$. The sets Θ_t, Θ_i^t, and Θ^t are defined analogously. Let $\Theta_i^{\infty} \equiv \prod_{t=0}^{\infty} \Theta_{it}$ and $\Theta \equiv \prod_{i=1}^{n} \Theta_i^{\infty}$.

In every period t, each agent i observes his own allocation x_{it} but not the other agents' allocations $x_{-i,t}$.[13] The observability of x_{it} should be thought of as a constraint: a mechanism can reveal more information to agent i than x_{it}, but cannot conceal x_{it}. Our necessary conditions for incentive compatibility do not depend on what additional information is disclosed to the agent by the mechanism. Hence it is convenient to assume that the agents do not observe anything beyond θ_{it} and x_{it}, not even their own transfers. (If the horizon is finite, this is without loss as transfers could be postponed until the end.) As for sufficient conditions, we provide conditions under which more information can

[12]For example, the (intertemporal) allocation of a private good in fixed supply $\bar{x}$ can be modelled by letting $X_{it} = \mathbb{R}_+$ and putting $X = \{x \in \mathbb{R}_+^{\infty N} : \sum_{it} x_{it} \leq \bar{x}\}$, while the provision of a public good whose period-t production is independent of the level of production in any other period can be modelled by letting $X = \{x \in \mathbb{R}_+^{\infty N} : x_{1t} = x_{2t} = \cdots = x_{Nt} \text{ all } t\}$.

[13]This formulation does not explicitly allow for decisions that are not observed by any agent at the time they are made; however, such decisions can be accommodated by introducing a fictitious agent who observes them.

be disclosed to the agents without violating incentive compatibility. In particular, we construct payments that can be disclosed in each period, and identify conditions under which the other agents' reports and allocations can also be disclosed.

Types. The evolution of agent i's information is described by a collection of kernels $F_i \equiv \langle F_{it} : \Theta_i^{t-1} \times X_i^{t-1} \to \Delta(\Theta_{it})\rangle_{t=0}^{\infty}$, where $F_{it}(\theta_i^{t-1}, x_i^{t-1})$ denotes the distribution of the random variable $\tilde{\theta}_{it}$, given the history of signals $\theta_i^{t-1} \in \Theta_i^{t-1}$ and allocations $x_i^{t-1} \in X_i^{t-1}$. The dependence on past allocations can capture, for example, learning-by-doing or experimentation (see the bandit auction application in Section 5). The time-t signals of different agents are drawn independently of each other. That is, the vector $(\tilde{\theta}_{1t}, \dots, \tilde{\theta}_{nt})$ is distributed according to the product measure $\prod_{i=1}^{n} F_{it}(\theta_i^{t-1}, x_i^{t-1})$. We abuse notation by using $F_{it}(\cdot|\theta_i^{t-1}, x_i^{t-1})$ to denote the cumulative distribution function (c.d.f.) that corresponds to the measure $F_{it}(\theta_i^{t-1}, x_i^{t-1})$.

Note that we build in the assumption of *independent types* in the sense of Athey and Segal (2013): in addition to independence of agents' signals within any period t, we require that the distribution of agent i's private signal be determined by things he has observed, that is, by $(\theta_i^{t-1}, x_i^{t-1})$. Without these restrictions, payoff equivalence, in general, fails by an argument analogous to that of Cremer and McLean (1988). On the other hand, dependence on other agents' past signals through the implemented observable decisions x_i^{t-1} is allowed.

Preferences. Each agent i has von Neumann–Morgenstern (vNM) preferences over lotteries on $\Theta \times X \times \mathbb{R}^{\infty}$, described by a Bernoulli utility function of the quasilinear form $U_i(\theta, x) + \sum_{t=0}^{\infty} \delta^t p_{it}$, where $U_i : \Theta \times X \to \mathbb{R}$ and $\delta \in (0, 1]$ is a discount factor common to all agents.[14] The special case of a "finite horizon" arises when each $U_i(\theta, x)$ depends only on (θ^T, x^T) for some finite T.

Choice rules. A choice rule consists of an allocation rule $\chi : \Theta \to X$ and a transfer rule $\psi : \Theta \to \mathbb{R}^{\infty} \times \cdots \times \mathbb{R}^{\infty}$ such that for all $t \geq 0$, the allocation $\chi_t(\theta) \in X_t$ and transfers $\psi_t(\theta) \in \mathbb{R}^N$ implemented in period t depend only on the history θ^t (and so will be written as $\chi_t(\theta^t)$ and $\psi_t(\theta^t)$). We denote the set of feasible allocation rules by $\mathcal{X}$. The restriction to deterministic rules is without loss of generality since randomizations can be generated by introducing a fictitious agent and conditioning on his reports. (Below we provide conditions for an optimal allocation rule to be deterministic.)

Stochastic processes. Given the kernels $F \equiv (F_i)_{i=1}^{n}$, an allocation rule $\chi \in \mathcal{X}$ uniquely defines a stochastic process over Θ, which we denote by $\lambda[\chi]$.[15] For

[14] As usual, we may alternatively interpret p_{it} as agent i's utility from his period-t payment (See, e.g., Garrett and Pavan (2013) and the discussion below). Furthermore, Theorem 1 below extends as stated to environments where i's utility is of the form $U_i(\theta, x) + P_i(p_{i0}, p_{i1}, \dots)$ for an arbitrary function $P_i : \mathbb{R}^{\infty} \to \mathbb{R}$. (A model without transfers corresponds to the special case where $P_i \equiv 0$ all i.)

[15] Existence and uniqueness follows by the Tulcea extension theorem (see, e.g., Pollard (2002, Chapter 4, Theorem 49)).

any period $t \geq 0$ and history $\theta^t \in \Theta^t$, we let $\lambda[\chi]|\theta^t$ denote the analogous process where $\tilde{\theta}^t$ is first drawn from a degenerate distribution at θ^t, and then the continuation process is generated by applying the kernels and the allocation rule starting from the history $(\theta^t, \chi^t(\theta^t))$.

When convenient, we view each agent i's private information as being generated by his initial signal θ_{i0} and a sequence of "independent shocks." That is, we assume that for each agent i, there exists a collection $\langle \mathcal{E}_i, G_i, z_i \rangle$, where $\mathcal{E}_i \equiv \langle \mathcal{E}_{it} \rangle_{t=0}^{\infty}$ is a collection of measurable spaces, $G_i \equiv \langle G_{it} \rangle_{t=0}^{\infty}$ is a collection of probability distributions with $G_{it} \in \Delta(\mathcal{E}_{it})$ for $t \geq 0$, and $z_i \equiv \langle z_{it} : \Theta_i^{t-1} \times X_i^{t-1} \times \mathcal{E}_{it} \to \Theta_{it} \rangle_{t=0}^{\infty}$ is a sequence of functions such that for all $t \geq 0$ and $(\theta_i^{t-1}, x_i^{t-1}) \in \Theta_i^{t-1} \times X_i^{t-1}$, if $\tilde{\varepsilon}_{it}$ is distributed according to G_{it}, then $z_{it}(\theta_i^{t-1}, x_i^{t-1}, \tilde{\varepsilon}_{it})$ is distributed according to $F_{it}(\theta_i^{t-1}, x_i^{t-1})$. Given any allocation rule χ, we can then think of the process $\lambda[\chi]$ as being generated as follows: Let $\tilde{\varepsilon}$ be distributed on $\prod_{t=1}^{\infty} \prod_{i=1}^{n} \mathcal{E}_{it}$ according to the product measure $\prod_{t=1}^{\infty} \prod_{i=1}^{n} G_{it}$. Draw the period-0 signals θ_0 according to the initial distribution $\prod_{i=1}^{n} F_{i0}$ independently of $\tilde{\varepsilon}$, and construct types for periods $t > 0$ recursively by $\theta_{it} = z_{it}(\theta_i^{t-1}, \chi^{t-1}(\theta^{t-1}), \varepsilon_{it})$. (Note that we can think of each agent i observing the shock ε_{it} in each period t, yet (θ_i^t, x_i^{t-1}) remains a sufficient statistic for his payoff-relevant private information in period t.) It is a standard result on stochastic processes that such a *state representation* $\langle \mathcal{E}_i, G_i, z_i \rangle_{i=1}^{n}$ exists for any kernels F.[16] For example, if agent i's signals follow a linear autoregressive process of order 1, then the z_i functions take the familiar form $z_{it}(\theta_i^{t-1}, x^{t-1}, \varepsilon_{it}) = \phi_i \theta_{i,t-1} + \varepsilon_{it}$ for some $\phi_i \in \mathbb{R}$. The general case can be handled using the canonical representation introduced in the following example.

EXAMPLE 1—Canonical Representation: Fix the kernels F. For all $i = 1, \dots, n$, and $t \geq 1$, let $\mathcal{E}_{it} = (0, 1)$, let G_{it} be the uniform distribution on $(0, 1)$, and define the generalized inverse F_{it}^{-1} by setting $F_{it}^{-1}(\varepsilon_{it}|\theta_i^{t-1}, x_i^{t-1}) \equiv \inf\{\theta_{it} : F_{it}(\theta_{it}|\theta_i^{t-1}, x_i^{t-1}) \geq \varepsilon_{it}\}$ for all $\varepsilon_{it} \in (0, 1)$ and $(\theta_i^{t-1}, x_i^{t-1}) \in \Theta_i^{t-1} \times X_i^{t-1}$. The random variable $F_{it}^{-1}(\tilde{\varepsilon}_{it}|\theta_i^{t-1}, x_i^{t-1})$ is then distributed according to the c.d.f. $F_{it}(\cdot|\theta_i^{t-1}, x_i^{t-1})$ so that we can put $z_{it}(\theta_i^{t-1}, x_i^{t-1}, \varepsilon_{it}) = F_{it}^{-1}(\varepsilon_{it}|\theta_i^{t-1}, x_i^{t-1})$.[17] We refer to the state representation so defined as the *canonical representation* of F.

Nevertheless, the canonical representation is not always the most convenient, as many processes such as the AR(1) above are naturally defined in terms of other representations; hence, we work with the general definition.

In what follows, we use the fact that, given a state representation $\langle \mathcal{E}_i, G_i, z_i \rangle_{i=1}^{n}$, for any period $s \geq 0$, each agent i's continuation process can be expressed

[16]This observation was first used in a mechanism-design context by Eső and Szentes (2007), who studied a two-period model of information disclosure in auctions.

[17]This construction is standard; see the second proof of the Kolmogorov extension theorem in Billingsley (1995, p. 490).

directly in terms of the history θ_i^s and shocks ε_{it}, $t \geq 0$, by defining the functions $Z_{i,(s)} \equiv \langle Z_{i,(s),t} : \Theta_i^s \times X_i^{t-1} \times \mathcal{E}_i^t \to \Theta_{it} \rangle_{t=0}^\infty$ recursively by $Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \varepsilon_i^t) = z_{it}(Z_{i,(s)}^{t-1}(\theta_i^s, x_i^{t-2}, \varepsilon_i^{t-1}), x_i^{t-1}, \varepsilon_{it})$, where $Z_{i,(s)}^{t-1}(\theta_i^s, x_i^{t-2}, \varepsilon_i^{t-1}) \equiv (Z_{i,(s),\tau}(\theta_i^s, x_i^{\tau-1}, \varepsilon_i^\tau))_{\tau=0}^{t-1}$ with $Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \varepsilon_i^t) \equiv \theta_{it}$ for all $t \leq s$. For example, in the case of a linear AR(1) process, $Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \varepsilon_i^t) = \phi_i^{t-s}\theta_{is} + \sum_{\tau=s+1}^t \phi^{t-\tau}\varepsilon_{i\tau}$ is simply the moving-average representation of the process started from θ_{is}.

2.1. *Regularity Conditions*

Similarly to static models with continuous types, our analysis requires that each agent's expected utility be a sufficiently well behaved function of his private information. In a dynamic model, an agent's expected continuation utility depends on his current type, both directly through the utility function, and indirectly through its impact on the distribution of future types. Hence, we impose regularity conditions on both the utility functions and the kernels.

CONDITION U-D—Utility Differentiable: *For all $i = 1, \dots, n$, $t \geq 0$, $x \in X$, and $\theta \in \Theta$, $U_i(\theta_i, \theta_{-i}, x)$ is a differentiable function of $\theta_i^t \in \Theta_i^t$.*

With a finite horizon $T < \infty$, this condition simply means that $U_i(\theta_i^T, \theta_{-i}^T, x^T)$ is differentiable in θ_i^T.

Next, define the norm $\|\cdot\|$ on $\mathbb{R}^\infty$ by $\|y\| \equiv \sum_{t=0}^\infty \delta^t |y_t|$ and let $\Theta_{i\delta} \equiv \{\theta_i \in \Theta_i^\infty : \|\theta_i\| < \infty\}$.[18]

CONDITION U-ELC—Utility Equi-Lipschitz Continuous: *For all $i = 1, \dots, n$, the family $\{U_i(\cdot, \theta_{-i}, x)\}_{\theta_{-i} \in \Theta_{-i}, x \in X}$ is equi-Lipschitz continuous on $\Theta_{i\delta}$. That is, there exists $A_i \in \mathbb{R}$ such that $|U_i(\theta_i', \theta_{-i}, x) - U_i(\theta_i, \theta_{-i}, x)| \leq A_i \|\theta_i' - \theta_i\|$ for all $\theta_i, \theta_i' \in \Theta_{i\delta}$, $\theta_{-i} \in \Theta_{-i}$, and $x \in X$.*

Conditions U-D and U-ELC are similar to the differentiability and bounded-derivative conditions imposed in static models (cf. Milgrom and Segal (2002)). For example, stationary payoffs $U_i(\theta, x) = \sum_{t=0}^\infty \delta^t u_i(\theta_t, x_t)$ satisfy U-D and U-ELC if u_i is differentiable and equi-Lipschitz in θ_{it} (e.g., linear payoffs $u_i(\theta_t, x_t) = \theta_{it} x_{it}$ are fine provided that x_{it} is bounded).

CONDITION F-BE—Process Bounded in Expectation: *For all $i = 1, \dots, n$, $t \geq 0$, $\theta^t \in \Theta^t$, and $\chi \in \mathcal{X}$, $\mathbb{E}^{\lambda[\chi]|\theta^t}[\|\tilde{\theta}_i\|] < \infty$.*

[18]It is possible to rescale θ_{it} and work with the standard l_1 norm. However, we use the weighted norm to deal without rescaling with the standard economic applications with time discounting. Note also that for a finite horizon, the norm $\|\cdot\|$ is equivalent to the Euclidean norm, and so the choice is irrelevant. For infinite horizon, increasing δ weakens the conditions imposed on the utility function while strengthening the conditions imposed on the kernels.

Condition F-BE implies that for any allocation rule χ and any period-t type history θ^t ($t \geq 0$), the sequence of agent i's future types has a finite norm with $\lambda[\chi]|\theta^t$-probability 1. This allows us to effectively restrict attention to the space $\Theta_{i\delta}$. With a finite horizon, F-BE simply requires that for all $t \geq 0$, the expectation of each $\tilde{\theta}_{i\tau}$, with $t < \tau \leq T$, exists conditional on any θ^t.

CONDITION F-BIR—Process Bounded Impulse Responses: *There exist a state representation* $\langle \mathcal{E}_i, G_i, z_i \rangle_{i=1}^n$ *and functions* $C_{i,(s)} : \mathcal{E}_i \to \mathbb{R}^\infty$, $i = 1, \dots, n$, $s \geq 0$, *with* $\mathbb{E}[\|C_{i,(s)}(\tilde{\varepsilon}_i)\|] \leq B_i$ *for some constant* B_i *independent of* s, *such that for all* $i = 1, \dots, n$, $t \geq s$, $\theta_i^s \in \Theta_i^s$, $x_i \in X_i$, *and* $\varepsilon_i^t \in \mathcal{E}_i^t$, $Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \varepsilon_i^t)$ *is a differentiable function of* θ_{is} *with* $|\partial Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \varepsilon_i^t)/\partial \theta_{is}| \leq C_{i,(s),t-s}(\varepsilon_i)$.

Condition F-BIR is essentially the process analog of Conditions U-D and U-ELC. It guarantees that small changes in the current type have a small effect on future types. We provide a way to check F-BIR as well as examples of kernels that satisfy it in later sections (see, e.g., Example 3).

Finally, we impose the following bounds on the agents' utility functions to ensure that the expected net present value of the transfers we construct exists when the horizon is infinite.

CONDITION U-SPR—Utility Spreadable: *For all* $i = 1, \dots, n$, *there exists a sequence of functions* $\langle u_{it} : \Theta^t \times X^t \to \mathbb{R} \rangle_{t=0}^\infty$ *and constants* L_i *and* $(M_{it})_{t=0}^\infty$, *with* $L_i, \|M_i\| < \infty$, *such that for all* $(\theta, x) \in \Theta \times X$ *and* $t \geq 0$, $U_i(\theta, x) = \sum_{t=0}^\infty \delta^t u_{it}(\theta^t, x^t)$ *and* $|u_{it}(\theta^t, x^t)| \leq L_i |\theta_{it}| + M_{it}$.

The condition is satisfied, for example, if the functions u_{it} are uniformly bounded or take the linear form $u_{it}(\theta^t, x^t) = \theta_{it} x_{it}$ with X_{it} bounded (but Θ_{it} possibly unbounded).

For ease of reference, we combine the above conditions into a single definition.

DEFINITION 1—Regular Environment: The environment is *regular* if it satisfies Conditions U-D, U-ELC, F-BE, F-BIR, and U-SPR.

3. PERFECT BAYESIAN EQUILIBRIUM IMPLEMENTABILITY

Following Myerson (1986), we restrict attention to direct mechanisms where, in every period t, each agent i confidentially reports a type from his type space Θ_{it}, no information is disclosed to him beyond his allocation x_{it}, and the agents report truthfully on the equilibrium path. Such a mechanism induces a dynamic Bayesian game between the agents and, hence, we use perfect Bayesian equilibrium (PBE) as our solution concept.

Formally, a reporting strategy for agent i is a collection $\sigma_i \equiv \langle \sigma_{it} : \Theta_i^t \times \Theta_i^{t-1} \times X_i^{t-1} \to \Theta_{it} \rangle_{t=0}^\infty$, where $\sigma_{it}(\theta_i^t, \hat{\theta}_i^{t-1}, x_i^{t-1}) \in \Theta_{it}$ is agent i's report in period t

when his true type history is θ_i^t, his reported type history is $\hat{\theta}_i^{t-1}$, and his allocation history is x_i^{t-1}. The strategy σ_i is *on-path truthful* if $\sigma_{it}((\theta_i^{t-1}, \theta_{it}), \theta_i^{t-1}, x_i^{t-1}) = \theta_{it}$ for all $t \geq 1$, $\theta_{it} \in \Theta_{it}$, $\theta_i^{t-1} \in \Theta_i^{t-1}$, and $x_i^{t-1} \in X_i^{t-1}$. Note that on-path truthful strategies impose no restrictions on behavior after lies.

The specification of a PBE also includes a belief system Γ, which describes each agent i's beliefs at each of his information sets $(\theta_i^t, \hat{\theta}_i^{t-1}, x_i^{t-1})$ about the unobserved past moves by Nature (θ_{-i}^{t-1}) and by the other agents $(\hat{\theta}_{-i}^{t-1})$. (The agent's beliefs about the contemporaneous types of agents $j \neq i$ then follow by applying the kernels.) We restrict these beliefs to satisfy two natural conditions:

CONDITION B(i): *For all $i = 1, \ldots, n$, $t \geq 0$, and $(\theta_i^t, \hat{\theta}_i^{t-1}, x_i^{t-1}) \in \Theta_i^t \times \Theta_i^{t-1} \times X_i^{t-1}$, agent i's beliefs are independent of his true type history θ_i^t.*

CONDITION B(ii): *For all $i = 1, \ldots, n$, $t \geq 0$, and $(\theta_i^t, \hat{\theta}_i^{t-1}, x_i^{t-1}) \in \Theta_i^t \times \Theta_i^{t-1} \times X_i^{t-1}$, agent i's beliefs assign probability 1 to the other agents having reported truthfully, that is, to the event that $\hat{\theta}_{-i}^{t-1} = \theta_{-i}^{t-1}$.*

Condition B(i) is similar to Condition B(i) in Fudenberg and Tirole (1991, p. 331). It is motivated by the fact that given agent i's reports $\hat{\theta}_i^{t-1}$ and observed allocations x_i^{t-1}, the distribution of his true types θ_i^t is independent of the other agents' types or reports. Condition B(ii) in turn says that agent i always believes that his opponents have been following their equilibrium strategies.[19] Note that under these two conditions, we can describe agent i's beliefs as a collection of probability distributions $\Gamma_{it} : \Theta_i^{t-1} \times X_i^{t-1} \to \Delta(\Theta_{-i}^{t-1})$, $t \geq 0$, where $\Gamma_{it}(\hat{\theta}_i^{t-1}, x_i^{t-1})$ represents agent i's beliefs over the other agents' past types θ_{-i}^{t-1} (which he believes to be equal to the reports) given that he reported $\hat{\theta}_i^{t-1}$ and observed the allocations x_i^{t-1}. We then have the following definitions.

DEFINITION 2—On-Path Truthful PBE; PBIC: An *on-path truthful PBE* of a direct mechanism $\langle \chi, \psi \rangle$ is a pair (σ, Γ) that consists of an on-path truthful strategy profile σ and a belief system Γ such that (i) Γ satisfies Conditions B(i) and B(ii), and is consistent with Bayes' rule on all positive-probability events given σ, and (ii) for all $i = 1, \ldots, n$, σ_i maximizes agent i's expected payoff at each information set given σ_{-i} and Γ.[20] The choice rule $\langle \chi, \psi \rangle$ is *perfect Bayesian incentive compatible (PBIC)* if the corresponding direct mechanism has an on-path truthful PBE.

[19]With continuous types, any particular history for agent i has probability 0 and, hence, B(ii) cannot be derived from Bayes' rule, but has to be imposed. Note that even when the kernels do not have full support, they are defined at all histories and, hence, the continuation process is always well defined.

[20]In particular, the expected allocation utility and the expected net present value of transfers from an on-path truthful strategy are well defined and finite conditional on any truthful history.

When σ is on-path truthful, the requirement in part (i) in the above definition depends only on the allocation rule χ. As a result, hereafter we denote by $\boldsymbol{\Gamma}(\chi)$ the set of beliefs systems that satisfy part (i) in the above definition. (Each element of $\boldsymbol{\Gamma}(\chi)$ is a system of regular conditional probability distributions, the existence of which is well known; see, for example, Dudley (2002).) Note that the concept of PBIC implies, in particular, that truthful reporting is optimal at every truthful history.

3.1. *First-Order Necessary Conditions*

We start by deriving a necessary condition for PBIC by applying an envelope theorem to an agent's problem of choosing an optimal reporting strategy at an arbitrary truthful history.

Fix a choice rule $\langle\chi,\psi\rangle$ and a belief system $\Gamma\in\boldsymbol{\Gamma}(\chi)$. Suppose agent i plays according to an on-path truthful strategy, and consider a period-t history of the form $((\theta_i^{t-1},\theta_{it}),\theta_i^{t-1},\chi_i^{t-1}(\theta^{t-1}))$, that is, when agent i has reported truthfully in the past, the complete reporting history is θ^{t-1}, and agent i's current type is θ_{it}. Agent i's expected payoff is then given by

$$V_{it}^{\langle\chi,\psi\rangle,\Gamma}\big(\theta^{t-1},\theta_{it}\big)\equiv\mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}}\left[U_i\big(\chi(\tilde\theta),\tilde\theta\big)+\sum_{s=0}^{\infty}\delta^t\psi_{is}(\tilde\theta)\right],$$

where $\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}$ is the stochastic process over Θ from the perspective of agent i. Formally, $\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}$ is the unique probability measure on Θ obtained by first drawing θ_{-i}^{t-1} according to agent i's belief $\Gamma_{it}(\theta_i^{t-1},\chi_i^{t-1}(\theta^{t-1}))$, drawing $\theta_{-i,t}$ according to $\prod_{j\neq i}F_{jt}(\theta_j^{t-1},\chi_j^{t-1}(\theta_i^{t-1},\theta_{-i}^{t-1}))$, and then using the allocation rule χ and the kernels F to generate the process from period-t onward. Note that in period 0, this measure is only a function of the kernels, and, hence, we write it as $\lambda_i[\chi]|\theta_{i0}$, and similarly omit the belief system Γ in $V_{i0}^{\langle\chi,\psi\rangle}(\theta_{i0})$.

The following definition (first-order condition for incentive compatibility (ICFOC)) is a dynamic version of the envelope condition familiar from static models.

DEFINITION 3—ICFOC: Fix $i=1,\dots,n$ and $s\geq 0$. The choice rule $\langle\chi,\psi\rangle$ with belief system $\Gamma\in\boldsymbol{\Gamma}(\chi)$ satisfies *ICFOC*$_{i,s}$ if, for all $\theta^{s-1}\in\Theta^{s-1}$, $V_{is}^{\langle\chi,\psi\rangle,\Gamma}(\theta^{s-1},\theta_{is})$ is a Lipschitz continuous function of θ_{is} with the derivative given almost everywhere (a.e.) by

$$\text{(1)}\qquad \frac{\partial V_{is}^{\langle\chi,\psi\rangle,\Gamma}(\theta^{s-1},\theta_{is})}{\partial\theta_{is}}$$
$$=\mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}\left[\sum_{t=s}^{\infty}I_{i,(s),t}\big(\tilde\theta_i^t,\chi_i^{t-1}(\tilde\theta)\big)\frac{\partial U_i(\tilde\theta,\chi(\tilde\theta))}{\partial\theta_{it}}\right],$$

where[21]

$$(2)\qquad I_{i,(s),t}\big(\theta_i^t, x_i^{t-1}\big) \equiv \mathbb{E}\left[\frac{\partial Z_{i,(s),t}(\theta_i^s, x_i^{t-1}, \tilde{\varepsilon}_i^t)}{\partial \theta_{is}} \middle| Z_{i,(s)}^t\big(\theta_i^s, x_i^{t-1}, \tilde{\varepsilon}_i^t\big) = \theta_i^t\right].$$

The choice rule $\langle \chi, \psi \rangle$ satisfies *ICFOC* if there exists a belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ such that $\langle \chi, \psi \rangle$ with belief system Γ satisfies ICFOC$_{i,s}$ for all agents $i = 1, \dots, n$ and all periods $s \geq 0$.

THEOREM 1: *Suppose the environment is regular.*[22] *Then every PBIC choice rule satisfies ICFOC.*

By Theorem 1, the formula in (1) is a dynamic generalization of the envelope formula familiar from static mechanism design (Mirrlees (1971), Myerson (1981)). By inspection, the period-0 formula implies a weak form of *dynamic payoff (and revenue) equivalence*: each agent's period-0 interim expected payoff is pinned down by the allocation rule χ up to a constant. (We provide a stronger payoff-equivalence result in the next section.) Special cases of the envelope formula (1) have been identified by Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), and Eső and Szentes (2007), among others. However, it should be noted that the contribution of Theorem 1 is not so much so in generalizing the formula, but in providing conditions on the utility functions and type processes that imply that ICFOC is indeed a necessary condition for all PBIC choice rules.

Heuristically, the proof of Theorem 1 in the Appendix proceeds by applying an envelope-theorem argument to the agent's problem of choosing an optimal continuation strategy at a given truthful history. The argument is identical across agents and periods, and hence without loss of generality we focus on establishing ICFOC$_{i,0}$ for some agent i by considering his period-0 ex interim problem of choosing a reporting strategy conditional on his initial signal θ_{i0}. Nevertheless, Theorem 1 is not an immediate corollary of Milgrom and Segal's (2002) envelope theorem for arbitrary choice sets. Namely, their result requires that the objective function be differentiable in the parameter (with an appropriately bounded derivative) for *any* feasible element of the choice set. Here it would require that, for any initial report $\hat{\theta}_{i0}$ and any plan $\rho \equiv \langle \rho_t : \prod_{\tau=1}^{t} \Theta_\tau \to \Theta_t \rangle_{t=1}^{\infty}$ for reporting future signals, agent i's payoff be differentiable in θ_{i0}. But this property need not hold in a regular environment. This is because a change in the initial signal θ_{i0} changes the distribution of the agent's future signals, which in turn changes the distribution of his future reports and allocations through the plan ρ and the choice rule $\langle \chi, \psi \rangle$. For some

[21]The $I_{i,(s),t}$ functions are conditional expectations and thus defined only up to sets of measure 0.

[22]Condition U-SPR, which requires utility to be spreadable, is not used in the proof of this theorem.

combinations of $\hat{\theta}_0$, ρ, and $\langle\chi, \psi\rangle$, this may lead to an expected payoff that is nondifferentiable or even discontinuous in θ_{i0}.[23]

To deal with this complication, we transform the problem into one where the distribution of agent i's future information is independent of his initial signal θ_{i0} so that changing θ_{i0} leaves future reports unaltered.[24] This is done by using a state representation to generate the signal process and by asking the agent to report his initial signal θ_{i0} and his future *shocks* ε_{it}, $t \geq 1$. This equivalent formulation provides an additional simplification in that we may assume that agent i reports the shocks truthfully.[25] The rest of the proof then amounts to showing that if the environment is regular, then the transformed problem is sufficiently well behaved to apply arguments similar to those in Milgrom and Segal (2002).

REMARK 1: If the notion of incentive compatibility is weakened from PBIC to the requirement that on-path truthful strategies are a Bayesian–Nash equilibrium of the direct mechanism, then the above argument can still be applied in period 0 to establish ICFOC$_{i,0}$ for all $i = 1, \dots, n$. Thus the weak payoff equivalence discussed above holds across all Bayesian–Nash incentive-compatible mechanisms.

We finish this subsection with two examples that suggest an interpretation of the functions defined by (2) and establish a connection to the literature. To simplify notation, we restrict attention to the case of a single agent and omit the subscript i.

[23]For a simple example, consider a two-period environment with one agent and a single indivisible good to be allocated in the second period as in Courty and Li (2000) (i.e., $X_0 = \{0\}$, $X_1 = \{0, 1\}$). Suppose the agent's payoff is of the form $U(\theta, x) = \theta_1 x_1$, and that $\tilde{\theta}_0$ is distributed uniformly on $(0, 1)$ with $\tilde{\theta}_1 = \tilde{\theta}_0$ almost surely. (It is straigthforward to verify that this environment is regular; e.g., put $Z_1(\theta_0, \varepsilon_1) = \theta_0$ for all $(\theta_0, \varepsilon_1)$ to verify F-BIR.) Consider the PBIC choice rule $\langle\chi, \psi\rangle$ where χ is defined by $\chi_0 = 0$, $\chi_1(\theta_0, \theta_1) = 1$ if $\theta_0 = \theta_1 \geq \frac{1}{2}$, and $\chi_1(\theta_0, \theta_1) = 0$ otherwise, and where ψ is defined by setting $\psi_0 = 0$ and $\psi_1(\theta_0, \theta_1) = \frac{1}{2}\chi_1(\theta_0, \theta_1)$. Now, fix an initial report $\hat{\theta}_0 > \frac{1}{2}$ and fix the plan ρ that reports θ_1 truthfully in period 1 (i.e., $\rho(\theta_1) = \theta_1$ for all θ_1). The resulting expected payoff is $\hat{\theta}_0 - \frac{1}{2} > 0$ for $\theta_0 = \hat{\theta}_0$, whereas it is equal to 0 for all $\theta_0 \neq \hat{\theta}_0$. That is, the expected payoff is discontinuous in the true initial type at $\theta_0 = \hat{\theta}_0$.

[24]In an earlier draft, we showed that when the horizon is finite, the complication can alternatively be dealt with by using backward induction. Roughly, this solves the problem, as it forces the agent to use a sequentially rational continuation strategy given any initial report and, thus, rules out problematic elements of his feasible set.

[25]By PBIC, truthful reporting remains optimal in the restricted problem where the agent can only choose $\hat{\theta}_0$ and, hence, the value function that we are trying to characterize is unaffected. (In terms of the kernel representation, this amounts to restricting each type θ_{i0} to using strategies where given any initial report $\hat{\theta}_{i0} \in \Theta_{i0}$, the agent is constrained to report $\hat{\theta}_{it} = Z_{it}(\hat{\theta}_{i0}, x_i^{t-1}, \varepsilon_i^t)$ in period t.) Note that restricting the agent to report truthfully his future θ_{it} would not work, as the resulting restricted problem is not sufficiently well behaved in general; see footnote 23.

EXAMPLE 2—AR(k) Process: Consider the case of a single agent, and suppose that the signal θ_t evolves according to an autoregressive (AR) process that is independent of the allocations,

$$\tilde{\theta}_t = \sum_{j=1}^{\infty} \phi_j \tilde{\theta}_{t-j} + \tilde{\varepsilon}_t,$$

where $\tilde{\theta}_t = 0$ for all $t < 0$, $\phi_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, and $\tilde{\varepsilon}_t$ is a random variable distributed according to some c.d.f. G_t with support $\mathcal{E}_t \subseteq \mathbb{R}$, with all the $\tilde{\varepsilon}_t$, $t \geq 0$, distributed independently of each other and of $\tilde{\theta}_0$. Note that we have defined the process in terms of the state representation $\langle \mathcal{E}, G, z \rangle$, where $z_t(\theta^{t-1}, x^{t-1}, \varepsilon_t) = \sum_{j=1}^{\infty} \phi_j \theta_{t-j} + \varepsilon_t$. The functions (2) are then time-varying scalars

$$(3) \qquad I_{(s),s} = 1 \quad \text{and}$$
$$I_{(s),t} = \frac{\partial Z_{(s),t}}{\partial \theta_s} = \sum_{\substack{K \in \mathbb{N}, l \in \mathbb{N}^{K+1}: \\ s = l_0 < \cdots < l_K = t}} \left\{ \prod_{k=1}^{K} \phi_{l_k - l_{k-1}} \right\} \quad \text{for} \quad t > s.$$

In the special case of an AR(1) process, as in Besanko (1985), we have $\phi_j = 0$ for all $j > 1$ and, hence, the formula simplifies to $I_{(s),t} = (\phi_1)^{t-s}$. Condition F-BIR requires that there exist $B \in \mathbb{R}$ such that $\|I_{(s)}\| \equiv \sum_{t=0}^{\infty} \delta^t |I_{(s),t}| < B$ for all $s \geq 0$, which in the AR(1) case is satisfied if and only if $\delta|\phi_1| < 1$. For Condition F-BE, write

$$\tilde{\theta}_t = Z_{(0),t}(\theta_0, \tilde{\varepsilon}^t) = I_{(0),t}\theta_0 + \sum_{\tau=1}^{t} I_{(0),t-\tau}\tilde{\varepsilon}_\tau \quad \text{for all } t \geq 0,$$

so that

$$\begin{aligned} \mathbb{E}^{\lambda|\theta_0}\big[\|\tilde{\theta}\|\big] &\leq \|I_{(0)}\||\theta_0| + \sum_{t=1}^{\infty} \delta^t \sum_{\tau=1}^{t} |I_{(0),t-\tau}| \mathbb{E}\big[|\tilde{\varepsilon}_\tau|\big] \\ &= \|I_{(0)}\|\big(|\theta_0| + \mathbb{E}\big[\|\tilde{\varepsilon}\|\big]\big). \end{aligned}$$

Similarly, we have $\mathbb{E}^{\lambda|\theta^s}[\|\tilde{\theta}\|] \leq \sum_{m=0}^{s-1} \|I_{(m)}\||\theta_m| + \|I_{(s)}\|(|\theta_s| + \delta^{-s}\mathbb{E}[\|\tilde{\varepsilon}\|])$. Hence, F-BE is ensured by assuming, in addition to the bound B needed for F-BIR, that $\mathbb{E}[\|\tilde{\varepsilon}\|] < \infty$, which simply requires that the mean of the shocks grows slower than the discount rate (e.g., it is trivially satisfied if ε_t are i.i.d. with a finite mean).

The constants defined by (3) coincide with the impulse responses of a linear AR process. More generally, the $I_{(s),t}$ functions in (2) can be interpreted as *nonlinear impulse responses*. To see this, apply Theorem 1 to a regular single-agent environment with fixed decisions and no payments (i.e., with $X_t = \{\hat{x}_t\}$ and $\psi_t(\theta) = 0$ for all $t \geq 0$ and $\theta \in \Theta$), in which case optimization is irrelevant, and we simply have $V_s^{\langle \chi, \Psi \rangle}(\theta^s) \equiv \mathbb{E}^{\lambda|\theta^s}[U(\tilde{\theta}, \hat{x})]$. Then (1) takes the form:

$$\frac{d\mathbb{E}^{\lambda|\theta^s}[U(\tilde{\theta}, \hat{x})]}{d\theta_s} = \mathbb{E}^{\lambda|\theta^s}\left[\sum_{t=s}^{\infty} I_{(s),t}(\tilde{\theta}^t, \hat{x}) \frac{\partial U(\tilde{\theta}, \hat{x})}{\partial \theta_t}\right].$$

Note that the impulse response functions $I_{(s),t}$ are determined entirely by the stochastic process and satisfy the above equation for any utility function U that satisfies Conditions U-D and U-ELC.[26]

If for all $t \geq 1$, the function z_t in the state representation of the type process is differentiable in θ^{t-1}, we can use the chain rule to inductively calculate the impulse responses as

$$\begin{aligned} &\frac{\partial Z_{(s),t}(\theta^s, x^{t-1}, \varepsilon^t)}{\partial \theta_s} \\ &\quad = \sum_{\substack{K \in \mathbb{N}, l \in \mathbb{N}^{K+1}: \\ s = l_0 < \cdots < l_K = t}} \left\{ \prod_{k=1}^{K} \frac{\partial z_{l_k}(Z^{l_k - 1}(\theta^s, x^{l_k - 2}, \varepsilon^{l_k - 1}), x^{l_k - 1}, \varepsilon_{l_k})}{\partial \theta_{l_{k-1}}} \right\}. \end{aligned} \tag{4}$$

The derivative $\partial z_m / \partial \theta_l$ can be interpreted as the "direct impulse response" of the signal in period m to the signal in period $l < m$. The "total" impulse response $\partial Z_{(s),t} / \partial \theta_s$ is then obtained by adding up the products of the direct impulse responses over all possible causation chains from period s to period t. Applying this observation to the canonical representation yields a simple formula for the impulse responses and a possible way to verify that the kernels satisfy Condition F-BIR.

EXAMPLE 3—Canonical Impulse Responses: Suppose that, for all $t \geq 1$ and $x^{t-1} \in X^{t-1}$, the c.d.f. $F_t(\theta_t | \theta^{t-1}, x^{t-1})$ is continuously differentiable in (θ_t, θ^{t-1}), and let $f_t(\cdot | \theta^{t-1}, x^{t-1})$ denote the density of $F_t(\cdot | \theta^{t-1}, x^{t-1})$. Then the direct impulse responses in the canonical representation of Example 1 take the form,

$$\frac{\partial z_m(\theta^{m-1}, x^{m-1}, \varepsilon_m)}{\partial \theta_l} = -\left.\frac{\partial F_m(\theta_m | \theta^{m-1}, x^{m-1}) / \partial \theta_l}{f_m(\theta_m | \theta^{m-1}, x^{m-1})}\right|_{\theta_m = F_m^{-1}(\varepsilon_m | \theta^{m-1}, x^{m-1})},$$

for $(\theta^{m-1}, x^{m-1}, \varepsilon_m) \in \Theta^{m-1} \times X^{m-1} \times (0, 1)$ and $m \geq l \geq 0$, where we have used the implicit function theorem. Plugging this into equation (4) yields a formula

[26]We conjecture that this property uniquely defines the impulse response functions with $\lambda|\theta^s$-probability 1.

for the impulse responses directly in terms of the kernels. For example, if the agent's type evolves according to a Markov process whose kernels are independent of decisions, the formula simplifies to

$$(5)\qquad I_{(s),t}(\theta^t) = \prod_{\tau=s}^{t}\left(-\frac{\partial F_\tau(\theta_\tau|\theta_{\tau-1})/\partial\theta_{\tau-1}}{f_\tau(\theta_\tau|\theta_{\tau-1})}\right),$$

because then the only causation chain passes through all periods. Two-period versions of this formula appear in Baron and Besanko (1984), Courty and Li (2000), and Eső and Szentes (2007).

As for Condition F-BIR, because the canonical impulse responses are directly in terms of the kernels F, it is straightforward to back out conditions that guarantee the existence of the bounding functions $C_{(s)} : \mathcal{E} \to \mathbb{R}^\infty$, $s \geq 0$. For example, in the case of a Markov process, it is sufficient that there exists a sequence $y \in \Theta_\delta$ such that for all $t \geq 0$, $(\theta_{t-1}, \theta_t) \in \Theta_{t-1} \times \Theta_t$, and $x^{t-1} \in X^{t-1}$, we have $|\frac{\partial F_t(\theta_t|\theta_{t-1}, x^{t-1})/\partial\theta_{t-1}}{f_t(\theta_t|\theta_{t-1}, x^{t-1})}| \leq y_t$. The general case can be handled similarly.

REMARK 2: Baron and Besanko (1984) suggested interpreting $I_{(0),1}(\theta_0, \theta_1) = -\frac{\partial F_1(\theta_1|\theta_0)/\partial\theta_0}{f_1(\theta_1|\theta_0)}$ as a measure of "informativeness" of θ_0 about θ_1. We find the term "impulse response" preferable. First, for linear processes, it matches the usage in the time-series literature. Second, it is more precise. For example, in the two-period case, if $\tilde{\theta}_1 = \tilde{\theta}_0 + \tilde{\varepsilon}_1$ with $\tilde{\varepsilon}_1$ normally distributed with mean zero, then the impulse response is identical to 1 regardless of the variance of $\tilde{\varepsilon}_1$. On the other hand, θ_0 is more informative about θ_1 (in the sense of Blackwell) the smaller the variance of $\tilde{\varepsilon}_1$.

3.2. *Payment Construction and Equivalence*

Similarly to static settings, for any allocation rule (and belief system), it is possible to use the envelope formula to construct transfers that satisfy first-order conditions at all truthful histories: Fix an allocation rule χ and a belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$. For all $i = 1, \ldots, n$, $s \geq 0$, and $\theta \in \Theta$, let

$$(6)\qquad D_{is}^{\chi,\Gamma}(\theta^{s-1}, \theta_{is}) \equiv \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}\left[\sum_{t=s}^{\infty} I_{i,(s),t}(\tilde{\theta}_i^t, \chi_i^{t-1}(\tilde{\theta}))\frac{\partial U_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial\theta_{it}}\right], \quad \text{and}$$

$$Q_{is}^{\chi,\Gamma}(\theta^{s-1}, \theta_{is}) \equiv \int_{\theta'_{is}}^{\theta_{is}} D_{is}^{\chi,\Gamma}(\theta^{s-1}, q)\, dq,$$

where $\theta'_i \in \Theta_{i\delta}$ is some arbitrary fixed type sequence. Define the transfer rule ψ by setting, for all $i = 1, \ldots, n$, $t \geq 0$, and $\theta^t \in \Theta^t$,

$$\begin{aligned}(7)\qquad \psi_{it}(\theta^t) = {}& \delta^{-t} Q_{it}^{\chi,\Gamma}(\theta^{t-1}, \theta_{it}) - \delta^{-t} \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}}\left[Q_{i,t+1}^{\chi,\Gamma}(\tilde{\theta}^t, \tilde{\theta}_{i,t+1})\right] \\ & - \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}}\left[u_{it}(\tilde{\theta}^t, \chi^t(\tilde{\theta}^t))\right].\end{aligned}$$

Recall that by Theorem 1, if $\langle \chi, \psi \rangle$ is PBIC, then agent i's expected equilibrium payoff in period s satisfies $\partial V_{is}^{\langle \chi,\psi\rangle,\Gamma}(\theta^{s-1}, \theta_{is})/\partial \theta_{is} = D_{is}^{\chi,\Gamma}(\theta^{s-1}, \theta_{is})$. Hence, the transfer in (7) can be interpreted as agent i's information rent (over type θ'_i) as perceived in period t, net of the rent he expects from the next period onward and net of the expected flow utility. We show below that these transfers satisfy ICFOC. To address their uniqueness, we introduce the following condition.

DEFINITION 4—No Leakage: The allocation rule χ *leaks no information to agent i* if for all $t \geq 0$ and $\theta_i^{t-1}, \hat{\theta}_i^{t-1} \in \Theta_i^{t-1}$, the distribution $F_{it}(\theta_i^{t-1}, \chi_i^{t-1}(\hat{\theta}_i^{t-1}, \theta_{-i}^{t-1}))$ does not depend on θ_{-i}^{t-1} (and, hence, can be written as $\hat{F}_{it}(\theta_i^{t-1}, \hat{\theta}_i^{t-1})$).

This condition means that observing θ_{it} never gives agent i any information about the other agents' types. Clearly, all allocation rules satisfy it when agent i's type evolves independently from allocations or, trivially, in a single-agent setting. We then have our second main result.[27]

THEOREM 2: *Suppose the environment is regular. Then the following statements are true.*

(i) *Given an allocation rule χ and a belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$, let ψ be the transfer rule defined by* (7). *Then the choice rule $\langle \chi, \psi \rangle$ satisfies ICFOC, and, for all $i = 1, \ldots, n$, $s \geq 0$, $\theta^{s-1} \in \Theta^{s-1}$, and $\theta_{is} \in \Theta_{is}$, $\mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}[\|\psi_i(\tilde{\theta})\|] < \infty$.*

(ii) *Let χ be an allocation rule that leaks no information to agent i, and let ψ and $\bar{\psi}$ be transfer rules such that the choice rules $\langle \chi, \psi \rangle$ and $\langle \chi, \bar{\psi} \rangle$ are PBIC. Then there exists a constant K_i such that for $\lambda[\chi]$-almost every $\theta_i^\infty \in \Theta_i^\infty$,*

$$\mathbb{E}^{\lambda_i[\chi]|\theta_i^\infty}\left[\sum_{t=0}^{\infty} \delta^t \psi_{it}(\theta_i^t, \tilde{\theta}_{-i}^t)\right] = \mathbb{E}^{\lambda_i[\chi]|\theta_i^\infty}\left[\sum_{t=0}^{\infty} \delta^t \bar{\psi}_{it}(\theta_i^t, \tilde{\theta}_{-i}^t)\right] + K_i.$$

REMARK 3: The flow payments $\psi_{it}(\theta^t)$ defined by (7) are measurable with respect to $(\theta_i^t, \chi_i^t(\theta^t))$. Thus, they do not reveal to agent i any information beyond that contained in the allocations x_i. Hence, they can be disclosed to the agent without affecting his beliefs or incentives.

[27]The notation $\lambda_i[\chi]|\theta_i^\infty$ in the proposition denotes the unique measure over the other agents' types Θ_{-i} that is obtained from the kernels F and the allocation χ by fixing agent i's reports at $\hat{\theta}_i^\infty = \theta_i^\infty$.

As noted after Theorem 1, ICFOC$_{i,0}$ immediately pins down, up to a constant, the expected net present value of payments $\mathbb{E}^{\lambda_i[\chi]|\theta_{i0}}[\sum_{t=0}^{\infty}\delta^t\psi_{it}(\tilde{\theta}^t)]$ for each initial type θ_{i0} of each agent i in any PBIC mechanism that implements the allocation rule χ. This extends the celebrated revenue equivalence theorem of Myerson (1981) to dynamic environments. Part (ii) of Theorem 2 strengthens the result further to a form of ex post equivalence. The result is particularly sharp when there is just one agent. Then the no-leakage condition is vacuously satisfied and the net present value of transfers that implement a given allocation rule χ is pinned down up to a single constant *with probability 1*; that is, if $\langle\chi, \psi\rangle$ and $\langle\chi, \bar{\psi}\rangle$ are PBIC, then there exists $K \in \mathbb{R}$ such that $\sum_{t=0}^{\infty}\delta^t\psi_t(\theta^t) = \sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\theta^t) + K$ for $\lambda[\chi]$-almost every θ.

The ex post equivalence of payments is useful for solving mechanism design problems in which the principal cares not just about the expected net present value of payments, but also about how the payments vary with the state θ or over time. For example, this includes settings where $\psi_t(\theta^t)$ is interpreted as the "utility payment" to the agent in period t, whose monetary cost to the principal is $\gamma(\psi_t(\theta^t))$ for some function γ, as in models with a risk-averse agent. In such models, knowing the net present value of the "utility payments" required to implement a given allocation rule allows computation of the cost-minimizing distribution of monetary payments over time (see, for example, Farhi and Werning (2013) or Garrett and Pavan (2013)).

3.3. *A Characterization for Markov Environments*

To provide necessary and sufficient conditions for PBE implementability, we focus on Markov environments, defined formally as follows.

DEFINITION 5—Markov Environment: The environment is *Markov* if, for all $i = 1, \dots, n$, the following conditions hold.

(i) Agent i's utility function U_i takes the form $U_i(\theta, x) = \sum_{t=0}^{\infty}\delta^t u_{it}(\theta_t, x^t)$.

(ii) For all $t \geq 1$ and $x_i^{t-1} \in X_i^{t-1}$, the distribution $F_{it}(\theta_i^{t-1}, x_i^{t-1})$ depends on θ_i^{t-1} only through $\theta_{i,t-1}$ (and is then denoted by $F_{it}(\theta_{it-1}, x_i^{t-1})$), and there exist constants ϕ_i and $(E_{it})_{t=0}^{\infty}$, with $\delta\phi_i < 1$ and $\|E_i\| < \infty$, such that for all $(\theta_{it}, x_i^t) \in \Theta_{it} \times X_i^t$, $\mathbb{E}^{F_{i,t+1}(\theta_{it}, x_i^t)}[|\tilde{\theta}_{t+1}|] \leq \phi_i|\theta_{it}| + E_{i,t+1}$.

This definition implies that each agent i's type process is a Markov decision process, and that his vNM preferences over future lotteries depend on his type history θ_i^t only through θ_{it} (but can depend on past decisions x^{t-1}). The strengthening of Condition F-BE embedded in part (ii) of the definition allows us to establish an appropriate version of the one-stage-deviation principle for the model. Note that every Markov process satisfies the bounds if the sets Θ_{it} are bounded.

The key simplification afforded by the Markov assumption is that, in a Markov environment, an agent's reporting incentives in any period t depend only on his current true type and his past reports, but not on his past true types. In particular, if it is optimal for the agent to report truthfully when past reports have been truthful (as in an on-path truthful PBE), then it is also optimal for him to report truthfully even if he has lied in the past. This implies that we can restrict attention to PBE in *strongly truthful strategies*, that is, in strategies that report truthfully at all histories.

We say that the allocation rule $\chi \in \mathcal{X}$ is *PBE-implementable* if there exists a transfer rule ψ such that the direct mechanism $\langle \chi, \psi \rangle$ has an on-path truthful PBE (i.e., it is PBIC). We say that χ is *strongly PBE-implementable* if there exists a transfer rule ψ such that the direct mechanism $\langle \chi, \psi \rangle$ has a strongly truthful PBE. Given an allocation rule χ, for all $i = 1, \dots, n$, $t \geq 0$, and $\hat{\theta}_{it} \in \Theta_{it}$, we let $\chi \circ \hat{\theta}_{it}$ denote the allocation rule obtained from χ by replacing θ_{it} with $\hat{\theta}_{it}$ (that is, $(\chi \circ \hat{\theta}_{it})(\theta) = \chi(\hat{\theta}_{it}, \theta_{i,-t}, \theta_{-i})$ for all $\theta \in \Theta$). Finally, recall that, by Theorem 1, the functions $D_{it}^{\chi,\Gamma}(\theta^{t-1}, \theta_{it})$ in (6) are the derivative of agent i's expected equilibrium payoff with respect to his current type at any truthful period-t history $(\theta_i^t, \theta_i^{t-1}, \chi_i^{t-1}(\theta^{t-1}))$ in a PBIC choice rule with allocation rule χ and belief system Γ. We then have our third main result.

THEOREM 3: *Suppose the environment is regular and Markov. An allocation rule $\chi \in \mathcal{X}$ is PBE-implementable if and only if there exists a belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ such that for all $i = 1, \dots, n$, $t \geq 0$, $\theta_{it}, \hat{\theta}_{it} \in \Theta_{it}$, and $\theta^{t-1} \in \Theta^{t-1}$, the following integral monotonicity condition holds*:

$$\int_{\hat{\theta}_{it}}^{\theta_{it}} \left[D_{it}^{\chi,\Gamma}\left(\theta^{t-1}, r\right) - D_{it}^{\chi\circ\hat{\theta}_{it},\Gamma}\left(\theta^{t-1}, r\right) \right] dr \geq 0. \tag{8}$$

Furthermore, when condition (8) *holds, χ is strongly PBE-implementable with payments given by* (7).

The static version of Theorem 3 has appeared in the literature on implementability (see Rochet (1987) or Carbajal and Ely (2013) and the references therein). The key step to prove that the result also holds in the dynamic version is to show that the agent's problem of choosing optimally his current report is well behaved. Then we can apply Lemma 1 below to obtain Theorem 3.

LEMMA 1: *Consider a function $\Phi : (\underline{\theta}, \bar{\theta})^2 \to \mathbb{R}$. Suppose that* (a) *for all $\hat{\theta} \in (\underline{\theta}, \bar{\theta})$, $\Phi(\theta, \hat{\theta})$ is a Lipschitz continuous function of θ, and* (b) *$\bar{\Phi}(\theta) \equiv \Phi(\theta, \theta)$ is a Lipschitz continuous function of θ. Then $\bar{\Phi}(\theta) \geq \Phi(\theta, \hat{\theta})$ for all $(\theta, \hat{\theta}) \in (\underline{\theta}, \bar{\theta})^2$ if and only if, for all $(\theta, \hat{\theta}) \in (\underline{\theta}, \bar{\theta})^2$,*

$$\int_{\hat{\theta}}^{\theta} \left[\bar{\Phi}'(q) - \frac{\partial \Phi(q, \hat{\theta})}{\partial \theta} \right] dq \geq 0. \tag{9}$$

PROOF: For all $\theta, \hat{\theta} \in (\underline{\theta}, \bar{\theta})$, let $g(\theta, \hat{\theta}) \equiv \bar{\Phi}(\theta) - \Phi(\theta, \hat{\theta})$. For any fixed $\hat{\theta} \in (\underline{\theta}, \bar{\theta})$, $g(\theta, \hat{\theta})$ is Lipschitz continuous in θ by (a) and (b). Hence, it is absolutely continuous and

$$g(\theta, \hat{\theta}) = \int_{\hat{\theta}}^{\theta} \frac{\partial g(q, \hat{\theta})}{\partial \theta} \, dq = \int_{\hat{\theta}}^{\theta} \left[\bar{\Phi}'(q) - \frac{\partial \Phi(q, \hat{\theta})}{\partial \theta} \right] dq.$$

Therefore, for all $\theta \in (\underline{\theta}, \bar{\theta})$, $\Phi(\theta, \hat{\theta})$ is maximized by setting $\hat{\theta} = \theta$ if and only if (9) holds. *Q.E.D.*

In a static model, the necessity of the integral monotonicity condition (8) readily follows from Theorem 1 and Lemma 1: There, for any fixed message $\hat{\theta}_i$, agent i's expected payoff can simply be assumed to be (equi-)Lipschitz continuous and differentiable in the true type θ_i. By Theorem 1, this implies Lipschitz continuity of his equilibrium payoff in θ_i, and the necessity of integral monotonicity follows by Lemma 1.

In contrast, in the dynamic model, fixing agent i's period-t message $\hat{\theta}_{it}$, the Lipschitz continuity of his expected payoff in the current type θ_{it} (or the formula for its derivative) cannot be assumed but must be derived from the agent's future optimizing behavior (see the counterexample in footnote 23). In particular, we show that in a Markov environment, the fact that the choice rule $\langle \chi, \psi \rangle$ implementing χ satisfies ICFOC (by Theorem 1) implies that the agent's expected payoff under a one-step deviation from truthtelling satisfies a condition analogous to ICFOC with respect to the modified choice rule $\langle \chi \circ \hat{\theta}_{it}, \psi \circ \hat{\theta}_{it} \rangle$ induced by the lie. This step is nontrivial and uses the fact that, in a Markov environment, truthtelling is an optimal continuation strategy following the lie. Since the agent's expected equilibrium payoff at any truthful history is Lipschitz continuous in the current type by Theorem 1, the necessity of integral monotonicity then follows by Lemma 1.

The other key difference pertains to the sufficiency part of the result: In a static environment, the payments constructed using the envelope formula ICFOC guarantee that the agent's payoff under truthtelling is Lipschitz continuous and satisfies ICFOC by construction. Incentive compatibility then follows from integral monotonicity by Lemma 1. In contrast, in the dynamic model, the payments defined by (7) guarantee only that ICFOC is satisfied at truthful histories (by Theorem 2(i)). However, in a Markov environment, it is irrelevant for the agent's continuation payoff whether he has been truthful in the past or not and, hence, ICFOC extends to all histories. Thus, by Lemma 1, integral monotonicity implies that one-stage deviations from truthtelling are unprofitable. Establishing a one-stage-deviation principle for the environment then concludes the proof.[28]

[28]The usual version of the one-stage-deviation principle (for example, Fudenberg and Tirole (1991, p. 110)) is not applicable, since payoffs are a priori not continuous at infinity because flow payments need not be bounded.

REMARK 4: Theorem 3 can be extended to characterize *strong* PBE implementability in non-Markov environments. However, for such environments, the restriction to strongly truthful PBE is, in general, with loss. In the Supplemental Material, we show that this approach nevertheless allows us to verify the implementability of the optimal allocation rule in some specific non-Markov environments.

3.3.1. *Verifying Integral Monotonicity*

The integral monotonicity condition (8) is, in general, not an easy object to work with. This is true even in static models, except for the special class of environments where both the type and the allocation are unidimensional and the agent's payoff is supermodular, in which case integral monotonicity is equivalent to the monotonicity of the allocation rule. Our dynamic problem essentially never falls in this class: Even if the agent's current type is unidimensional, his report, in general, affects the allocation both in the current period and in all future periods, which renders the allocation space multidimensional. For this reason, we provide monotonicity conditions that are stronger than integral monotonicity but easier to verify.[29] Some of these sufficient conditions apply only to environments that satisfy additional restrictions.

CONDITION F-AUT—Process Autonomous: *For all* $i = 1, \dots, n$, $t \geq 1$, *and* $\theta_i^{t-1} \in \Theta_i^{t-1}$, *the distribution* $F_{it}(\theta_i^{t-1}, x_i^{t-1})$ *does not depend on* x_i^{t-1}.

CONDITION F-FOSD—Process First-Order Stochastic Dominance: *For all* $i = 1, \dots, n$, $t \geq 1$, $\theta_{it} \in \Theta_{it}$, *and* $x_i^{t-1} \in X_i^{t-1}$, $F_{it}(\theta_{it}|\theta_i^{t-1}, x_i^{t-1})$ *is nonincreasing in* θ_i^{t-1}.

COROLLARY 1—Monotonicity: *Suppose the environment is regular and Markov. Then any of the following conditions* (*listed in decreasing order of generality*) *imply integral monotonicity* (8).

(i) Single crossing: *For all* $i = 1, \dots, n$, $t \geq 0$, $\theta^{t-1} \in \Theta^{t-1}$, $\hat{\theta}_{it} \in \Theta_{it}$, *and a.e.* $\theta_{it} \in \Theta_{it}$,

$$\left[D_{it}^{\chi,\Gamma}\left(\theta^{t-1}, \theta_{it}\right) - D_{it}^{\chi\circ\hat{\theta}_{it},\Gamma}\left(\theta^{t-1}, \theta_{it}\right)\right] \cdot (\theta_{it} - \hat{\theta}_{it}) \geq 0.$$

(ii) Average monotonicity: *For all* $i = 1, \dots, n$, $t \geq 0$, *and* $(\theta^{t-1}, \theta_{it}) \in \Theta^{t-1} \times \Theta_{it}$, $D_{it}^{\chi\circ\hat{\theta}_{it},\Gamma}(\theta^{t-1}, \theta_{it})$ *is nondecreasing in* $\hat{\theta}_{it}$.

[29]For similar sufficient conditions for static models with a unidimensional type and multidimensional allocation space, see, for example, Matthews and Moore (1987), whose condition is analogous to our strong monotonicity, and Garcia (2005), whose condition is analogous to our ex post monotonicity.

(iii) Ex post monotonicity: *Condition* F-AUT *holds*, *and for all* $i = 1, \ldots, n$, $t \geq 0$, *and* $\theta \in \Theta$,

$$\sum_{\tau=t}^{\infty} I_{i,(t),\tau}\big(\theta_i^\tau\big) \frac{\partial U_i(\theta, \chi(\hat{\theta}_{it}, \theta_{i,-t}, \theta_{-i}))}{\partial \theta_{i\tau}} \tag{10}$$

is nondecreasing in $\hat{\theta}_{it}$.

(iv) Strong monotonicity: *Conditions* F-AUT *and* F-FOSD *hold*, *and for all* $i = 1, \ldots, n$, $t \geq 0$, *and* $\theta_{-i} \in \Theta_{-i}$, $X_{it} \subseteq \mathbb{R}^m$, $U_i(\theta, x)$ *has increasing differences in* (θ_i, x_i) *and is independent of* x_{-i}, *and* $\chi_i(\theta)$ *is nondecreasing in* θ_i.

To see the relationship between the conditions, observe that the most stringent of the four—strong monotonicity—amounts to the requirement that each individual term in the sum in (10) be nondecreasing in $\hat{\theta}_{it}$ (note that under F-FOSD, $I_{i,(t),\tau} \geq 0$). Ex post monotonicity weakens this by requiring only that the sum be nondecreasing, which permits us to dispense with F-FOSD as well as with the assumption that U_i has increasing differences. By recalling the definition of the D functions in (6), we see that average monotonicity in turn weakens ex post monotonicity by averaging over states, which also permits us to dispense with F-AUT. Finally, single crossing relaxes average monotonicity by requiring that the expectation of the sum in (10) changes sign only once at $\hat{\theta}_{it} = \theta_{it}$, as opposed to being monotone in $\hat{\theta}_{it}$. But single crossing clearly implies integral monotonicity, proving the corollary. The following example is an illustration.

EXAMPLE 4: Consider a regular Markov environment with one agent whose allocation utility takes the form $U(\theta, x) = \sum_{t=0}^{\infty} \delta^t \theta_t x_t$, where for all $t \geq 0$, the period-t consumption x_t is an element of some unidimensional set $X_t \subseteq \mathbb{R}$. Suppose that conditions F-AUT and F-FOSD hold. By (6),

$$D_t^{\chi \circ \hat{\theta}_t}\big(\theta^{t-1}, \theta_t\big) = \mathbb{E}^{\lambda|\theta^{t-1},\theta_t}\left[\sum_{\tau=t}^{\infty} \delta^\tau I_{(t),\tau}\big(\tilde{\theta}^\tau\big)\chi_\tau(\hat{\theta}_t, \tilde{\theta}_{-t})\right].$$

Thus, average monotonicity requires that increasing the current message $\hat{\theta}_t$ increases the agent's average discounted consumption, where period-τ consumption is discounted using the discount factor δ as well as the impulse response $I_{(t),\tau}(\tilde{\theta}^\tau)$ of period-τ signal to period-t signal. Ex post monotonicity requires that the discounted consumption $\sum_{t=\tau}^{\infty} \delta^\tau I_{(t),\tau}(\theta^\tau)\chi_t(\hat{\theta}_t, \theta_{-t})$ be increasing in $\hat{\theta}_t$ along every path θ and strong monotonicity requires that increasing $\hat{\theta}_t$ increases consumption $\chi_\tau(\hat{\theta}_t, \theta_{-t})$ in every period $\tau \geq t$ irrespective of the agent's signals in the other periods.

Courty and Li (2000) study a two-period version of Example 4 with allocation $x_1 \in X_1 = [0, 1]$ only in the second period (i.e., $X_t = \{0\}$ for all $t \neq 1$) and provide sufficient conditions for implementability in two cases. The first (their Lemma 3.3) assumes F-FOSD and corresponds to our strong monotonicity. This case is extended to many agents by Eső and Szentes (2007).[30] The second case assumes that varying the initial signal θ_0 induces a mean-preserving spread by rotating $F_1(\cdot|\theta_0)$ about a single point z. Courty and Li show (as their Lemma 3.4) that it is then possible to implement any χ_1 that is nonincreasing in θ_0, nondecreasing in θ_1, and satisfies "no underproduction": $\chi_1(\theta_0, \theta_1) = 1$ for all $\theta_1 \geq z$.[31] This case is covered by our ex post monotonicity, which in period 0 requires that $I_{(0),1}(\theta_0, \theta_1)\chi_1(\hat{\theta}_0, \theta_1)$ be nondecreasing in $\hat{\theta}_0$. To see this, note that by the canonical impulse response formula (5), we have $I_{(0),1}(\theta_0, \theta_1) \leq 0$ (resp., ≥ 0) if $\theta_1 \leq z$ (resp., $\geq z$). Thus $I_{(0),1}(\theta_0, \theta_1)\chi_1(\hat{\theta}_0, \theta_1)$ is weakly increasing in $\hat{\theta}_0$ if $\theta_1 \leq z$, because χ_1 is nonincreasing in θ_0, whereas it is constant in $\hat{\theta}_0$ if $\theta_1 \geq z$, because χ_1 satisfies no underproduction.

The main application of Corollary 1 is the design of optimal dynamic mechanisms, which we turn to in the next section. There, a candidate allocation rule is obtained by solving a suitable relaxed problem and then Corollary 1 is used to verify that the allocation rule is indeed implementable. The results in the literature are typically based on strong monotonicity (for example, Battaglini (2005) or Eső and Szentes (2007)) with the exception of the mean-preserving-spread case of Courty and Li (2000) discussed above. However, there are interesting applications where the optimal allocation rule fails to be strongly monotone, or where the kernels naturally depend on past decisions or fail first-order stochastic dominance. For instance, the optimal allocation rule in Example 5 below fails strong monotonicity but satisfies ex post monotonicity, whereas the optimal allocation rule in the bandit auctions in Section 5 fails ex post monotonicity but satisfies average monotonicity.

REMARK 5: Suppose the environment is regular and Markov. Then for any allocation rule χ that satisfies ex post monotonicity, there exists a transfer rule ψ such that the complete information version of the model (where agents observe each others' types) has a subgame perfect Nash equilibrium in strongly truthful strategies. In other words, ex post monotone allocation rules can be implemented in a periodic ex post equilibrium in the sense of Athey and Segal (2013) and Bergemann and Välimäki (2010). This implies that any such rule

[30] Eső and Szentes derived the result in terms of a state representation. Translated to the primitive types θ_{i0} and θ_{i1} (or v_i and V_i in their notation), their Corollary 1 shows that the allocation rule they are interested in implementing is strongly monotone. Note that Eső and Szentes's display (22) is a special case of our integral monotonicity condition, but is stated in terms of a state representation. However, they used it only in conjunction with strong monotonicity.

[31] Courty and Li also considered the analogous case of "no overproduction," to which similar comments apply.

can be implemented in a strongly truthful PBE of a direct mechanism where all reports, allocations, and payments are public. The transfers that guarantee this can be constructed as in (7) with the measures $\lambda_i[\Gamma]|\theta^{t-1}, \theta_{it}$ replaced by the measure $\lambda|\theta^t$.

4. OPTIMAL MECHANISMS

We now show how Theorems 1–3 can be used in the design of optimal dynamic mechanisms in Markov environments.[32] To this end, we introduce a principal (labeled as "agent 0") whose payoff takes the quasilinear form $U_0(\theta, x) - \sum_{i=1}^{n} \sum_{t=0}^{\infty} \delta^t p_{it}$ for some function $U_0 : \Theta \times X \to \mathbb{R}$. The principal seeks to design a PBIC mechanism to maximize his expected payoff. As is standard in the literature, we assume that the principal makes a take-it-or-leave-it offer to the agents in period zero, after each agent i has observed his initial type θ_{i0}. Each agent can either accept the mechanism, or reject it to obtain his reservation payoff, which we normalize to 0 for all agents and types.[33] The principal's mechanism design problem is thus to maximize his ex ante expected payoff

$$\mathbb{E}^{\lambda[\chi]}\left[U_0\big(\tilde{\theta}, \chi(\tilde{\theta})\big) - \sum_{i=1}^{n} \sum_{t=0}^{\infty} \delta^t \psi_{it}(\tilde{\theta}^t)\right]$$
$$= \mathbb{E}^{\lambda[\chi]}\left[\sum_{i=0}^{n} U_i\big(\tilde{\theta}, \chi(\tilde{\theta})\big) - \sum_{i=1}^{n} V_{i0}^{\langle\chi,\psi\rangle}(\tilde{\theta}_{i0})\right]$$

by choosing a feasible choice rule $\langle \chi, \psi \rangle$ that is PBIC and satisfies

$$V_{i0}^{\langle\chi,\psi\rangle}(\theta_{i0}) \geq 0 \quad \text{for all } i = 1, \ldots, n \text{ and } \theta_{i0} \in \Theta_{i0}. \tag{11}$$

Any solution to this problem is referred to as an *optimal mechanism*.

We restrict attention to regular environments throughout this section, and assume that the initial distribution F_{i0} of each agent i is absolutely continuous with density $f_{i0}(\theta_{i0}) > 0$ for almost every $\theta_{i0} \in \Theta_{i0}$, and, for simplicity, that the

[32] For other possible applications, see, for example, Skrzypacz and Toikka (2013), who considered the feasibility of efficient dynamic contracting in repeated trade and in other dynamic collective choice problems.

[33] If an agent can accept the mechanism, but can then quit at a later stage, participation constraints have to be introduced in all periods $t \geq 0$. However, in our quasilinear environment with unlimited transfers, the principal can ask the agent to post a sufficiently large bond upon acceptance, to be repaid later, so as to make it unprofitable to quit and forfeit the bond at any time during the mechanism. (With an infinite horizon, annuities can be used in place of bonds.) For this reason, we ignore participation constraints in periods $t > 0$. Note that in non-quasilinear settings where agents have a consumption-smoothing motive, bonding is costly, and, hence, participation constraints may bind in many periods (see, for example, Hendel and Lizzeri (2003)).

set of initial types Θ_{i0} is bounded from below (i.e., $\underline{\theta}_{i0} > -\infty$). By Theorem 1, we can use ICFOC$_{i,0}$, as in static settings, to rewrite the principal's payoff as

$$\mathbb{E}^{\lambda[\chi]}\Bigg[\sum_{i=0}^{n} U_i\big(\tilde{\theta}, \chi(\tilde{\theta})\big) - \sum_{i=1}^{n} \frac{1}{\eta_{i0}(\tilde{\theta}_{i0})} \sum_{t=0}^{\infty} I_{i,(0),t}\big(\tilde{\theta}_i^t, \chi_i^{t-1}(\tilde{\theta})\big) \frac{\partial U_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}}\Bigg] - \sum_{i=1}^{n} V_{i0}^{\langle\chi,\psi\rangle}(\underline{\theta}_{i0}),$$

where $\eta_{i0}(\theta_{i0}) \equiv f_{i0}(\theta_{i1})/(1 - F_{i0}(\theta_{i1}))$ is the hazard rate of agent i's period-0 type.[34] The first term above is the *expected (dynamic) virtual surplus*, which is only a function of the allocation rule χ.

The principal's problem is, in general, analytically intractable. Hence, we adopt the "first-order approach" which is typical in the literature. In particular, we consider a relaxed problem where PBIC is relaxed to the requirement that $\langle\chi, \psi\rangle$ satisfy ICFOC$_{i,0}$ for all i and where a participation constraint is imposed only on each agent's lowest initial type, that is, (11) is replaced with

$$V_{i0}^{\langle\chi,\psi\rangle}(\underline{\theta}_{i0}) \geq 0 \quad \text{for all } i = 1, \dots, n. \tag{12}$$

Since subtracting a constant from agent i's period-0 transfer leaves ICFOC$_{i,0}$ unaffected but increases the principal's payoff, the constraints (12) must bind at a solution. It follows that an allocation rule χ^* is part of a solution to our relaxed problem if and only if χ^* maximizes

$$\mathbb{E}^{\lambda[\chi]}\Bigg[\sum_{i=0}^{n} U_i\big(\tilde{\theta}, \chi(\tilde{\theta})\big) - \sum_{i=1}^{n} \frac{1}{\eta_{i0}(\tilde{\theta}_{i0})} \sum_{t=0}^{\infty} I_{i,(0),t}\big(\tilde{\theta}_i^t, \chi_i^{t-1}(\tilde{\theta})\big) \frac{\partial U_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial \theta_{it}}\Bigg]. \tag{13}$$

This problem is, in general, a dynamic programming problem and, in contrast to static settings, it cannot be solved pointwise in general. Note that as the

[34] Recall that the proof of Theorem 1 uses only deviations in which, in terms of a state representation, each agent i reports truthfully future shocks ε_{it}, $t > 0$. Hence, as noted by Eső and Szentes (2007, 2013), this expression gives the principal's payoff also in a hypothetical environment where the shocks are observable to the principal. In other words, the cost to the principal of implementing a given rule χ is the same irrespective of the observability of the shocks. This observation underlies the "irrelevance result" of Eső and Szentes (2013). However, the set of PBIC choice rules is strictly larger when the shocks are observable, so the principal's problems in the two settings are not equivalent in general.

definition of the relaxed problem uses $\text{ICFOC}_{i,0}$, Theorem 1 plays a key role in identifying the candidate allocation rule χ^*.

If the environment is Markov, then Theorem 3 can be used to verify whether the candidate allocation rule χ^* is PBE-implementable (possibly by checking one of the conditions in Corollary 1). In case the answer is affirmative, Theorem 2 provides a formula for constructing a transfer rule ψ such that $\langle \chi^*, \psi \rangle$ is PBIC. We can then subtract the constant $V_{i0}^{\langle \chi^*, \psi \rangle}(\underline{\theta}_{i0})$ from each agent i's initial transfer to get a PBIC choice rule $\langle \chi^*, \psi^* \rangle$ such that $V_{i0}^{\langle \chi^*, \psi^* \rangle}(\underline{\theta}_{i0}) = 0$ for all $i = 1, \dots, n$. Thus it remains to verify that all the other participation constraints in (11) are satisfied. As part of the next result, we show that F-FOSD is a sufficient condition for this provided that each agent's utility is increasing in his own type sequence (endowed with the pointwise order).

COROLLARY 2—Optimal Mechanisms: *Suppose the environment is regular and Markov. Suppose in addition that Condition* F-FOSD *holds and, for all* $i = 1, \dots, n$ *and* $(\theta, x) \in \Theta \times X$, $U_i(\theta, x)$ *is nondecreasing in* θ_i. *Let* χ^* *be an allocation rule that maximizes the expected virtual surplus* (13) *and suppose that, with some belief system* $\Gamma \in \boldsymbol{\Gamma}(\chi^*)$, *it satisfies the integral monotonicity condition* (8) *in all periods. Then the following statements hold.*

(i) *There exists a transfer rule* ψ^* *such that* (a) *the direct mechanism* $\langle \chi, \psi^* \rangle$ *has a strongly truthful PBE with belief system* Γ *and where, for all* $i = 1, \dots, n$, *the flow payments* ψ^*_{it}, $t \geq 0$, *can be disclosed to agent* i; (b) *the period*-0 *participation constraints* (11) *are satisfied; and* (c) *the period*-0 *participation constraints of the lowest initial types* (12) *hold with equality.*

(ii) *The above choice rule* $\langle \chi^*, \psi^* \rangle$ *maximizes the principal's expected payoff across all PBIC choice rules that satisfy participation constraints* (11).

(iii) *If* $\langle \chi, \psi \rangle$ *is optimal for the principal among all PBIC choice rules that satisfy participation constraints* (11), *then* χ *maximizes the expected virtual surplus* (13).

(iv) *The principal's expected payoff cannot be increased by using randomized mechanisms.*

REMARK 6: Statements (ii)–(iv) remain true if PBIC is weakened to the requirement that there exists a Bayesian–Nash equilibrium in on-path truthful strategies. This is because the derivation of the expected virtual surplus (13) uses only $\text{ICFOC}_{i,0}$, which by Remark 1 holds under this weaker notion of incentive compatibility.

PROOF OF COROLLARY 2: Parts (i)(a) and (i)(c) follow by the arguments preceding the corollary. For (i)(b), note that, under F-FOSD, impulse responses are nonnegative almost everywhere and, hence, each $V_{i0}^{\langle \chi^*, \psi^* \rangle}(\theta_{i0})$ is nondecreasing in θ_{i0} by the envelope formula (1) given that U_i is nondecreasing in θ_i.

Parts (ii) and (iii) follow by the arguments preceding the corollary.

Finally, for part (iv) note that a randomized mechanism is equivalent to a mechanism that conditions on the random types of a fictitious agent. Since the expected virtual surplus in this augmented setting is independent of the signals of the fictitious agent and still takes the form (13), it is still maximized by the nonrandomized allocation rule χ^*. Thus, applying parts (i) and (ii) to the augmented setting implies that the deterministic choice rule $\langle\chi^*, \psi^*\rangle$ maximizes the principal's expected payoff. (A similar point was made by Strausz (2006) for static mechanisms.) *Q.E.D.*

Note that F-FOSD and the assumption that the agents' utilities be nondecreasing in own type are only used to establish that each agent i's equilibrium payoff $V_{i0}^{\langle\chi^*,\psi^*\rangle}(\theta_{i0})$ is minimized at $\underline{\theta}_{i0}$. If this conclusion can be arrived at by some other means (for example, by using (1) to solve for the function $V_{i0}^{\langle\chi^*,\psi^*\rangle}$), these assumptions can be dispensed with.

Corollary 2 provides a guess-and-verify approach analogous to that typically followed in static settings. We illustrate its usefulness below by using it to discuss optimal distortions in dynamic contracts and to find optimal "bandit auctions." However, as in static settings, the conditions under which the relaxed problem has an implementable solution are not generic. As pointed out by Battaglini and Lamba (2012), a particularly problematic case obtains when the type of an agent remains constant with high probability, but nevertheless has a small probability of being renewed. In terms of our analysis, the problem is that then the impulse response becomes highly nonmonotone in the current type, which in turn may result in the allocation being so nonmonotone in the current type that integral monotonicity is violated.[35]

It is, of course, possible to reverse-engineer conditions that guarantee that the relaxed problem has an implementable solution, but given the complexity of the problem, such conditions tend to be grossly sufficient. Nevertheless, for completeness, we provide sufficient conditions for an allocation rule that maximizes expected virtual surplus to satisfy strong monotonicity of Corollary 1.

CONDITION U-COMP—Utility Complementarity: *The set X is a lattice, and, for all $i = 0, \dots, n$, $t \geq 0$, and $\theta \in \Theta$, $U_i(\theta, x)$ is supermodular in x and $-\partial U_i(\theta, x)/\partial\theta_{it}$ is supermodular in x.*[36]

[35]For a concrete example, consider a two-period single-agent environment with $\tilde{\theta}_0$ distributed uniformly on $[0, 1]$. Suppose that $\tilde{\theta}_1 = \tilde{\theta}_0$ with probability q and that with the complementary probability, $\tilde{\theta}_1$ is drawn uniformly from $[0, 1]$ independently of $\tilde{\theta}_0$. The period-0 impulse response is then $I_{(0),1}(\theta_0, \theta_1) = \mathbf{1}_{\{\theta_0=\theta_1\}}$. By inspection of the expected virtual surplus (13), the period-1 allocation is thus distorted only if $\theta_1 = \theta_0$.

[36]The assumption that X is a lattice is not innocuous when $n > 1$: For example, it holds when each x_t describes the provision of a one-dimensional public good, but it need not hold if x_t describes the allocation of a private good.

This condition holds weakly in the special case where X_t is a subset of $\mathbb{R}$ in every period t and the payoffs $U_i(\theta, x)$ are additively separable in x_t. More generally, U-COMP allows for strict complementarity across time, for example, as in habit-formation models where higher consumption today increases the marginal utility of consumption tomorrow. On the other hand, U-COMP is not satisfied when allocating private goods in limited supply as in auctions.

CONDITION U-DSEP—Utility Decision-Separable: *We have* $X = \prod_{t=0}^{\infty} X_t$ *and, for all* $i = 0, \dots, n$, *and* $(\theta, x) \in \Theta \times X$, $U_i(\theta, x) = \sum_{t=0}^{\infty} \delta^t u_{it}(\theta^t, x_t)$.

PROPOSITION 1—Primitive Conditions for Strong Monotonicity: *Suppose the environment is regular and Markov, Conditions F-AUT and F-FOSD hold, and for all* $i = 0, \dots, n$, *and* $t \geq 0$, X_{it} *is a subset of a Euclidean space. Suppose that either of the following conditions is satisfied.*

(i) *Condition U-COMP holds and for all* $i = 1, \dots, n$, *agent i's virtual utility*

$$U_i(\theta, x) - \frac{1}{\eta_{i0}(\theta_{i0})} \sum_{t=0}^{\infty} \delta^t I_{i,(0),t}\big(\theta_i^t\big) \frac{\partial U_i(\theta, x)}{\partial \theta_{it}}$$

has increasing differences in (θ, x), *and the same is true of the principal's utility* $U_0(\theta, x)$.

(ii) *Condition U-DSEP holds, and for all* $i = 1, \dots, n$ *and* $t \geq 0$, $X_{it} \subseteq \mathbb{R}$ *and there exists a nondecreasing function* $\varphi_{it} : \Theta_i^t \to \mathbb{R}^m$, *with* $m \leq t$, *such that agent i's virtual flow utility*

$$u_{it}(\theta_t, x_t) - \frac{1}{\eta_{i0}(\theta_{i0})} I_{i,(0),t}\big(\theta_i^t\big) \frac{\partial u_{it}(\theta_t, x_t)}{\partial \theta_{it}}$$

depends only on $\varphi_{it}(\theta_i^t)$ *and* x_{it}, *and has strictly increasing differences in* $(\varphi_{it}(\theta_i^t), x_{it})$, *while the principal's flow utility depends only on* x_t.

Then, if the problem of maximizing expected virtual surplus (13) *has a solution, it has a solution* χ *such that, for all* $i = 1, \dots, n$ *and* $\theta_{-i} \in \Theta_{-i}$, $\chi_i(\theta_i, \theta_{-i})$ *is nondecreasing in* θ_i.

When F-AUT and U-DSEP hold (i.e., types evolve independently of decisions and payoffs are separable in decisions), expected virtual surplus (13) can be maximized pointwise, which explains why condition (ii) of Proposition 1 only involves flow payoffs. Special cases of this result appear in Courty and Li (2000), who provide sufficient conditions for strong monotonicity by means of parametric examples, and in Eső and Szentes (2007), whose Assumptions 1 and 2 imply that $I_{i,(0),1}(\theta_{i0}, \theta_{i1})$ is nonincreasing in both θ_{i0} and θ_{i1}, which together with their payoff functions imply condition (ii) (with $\varphi_{it} = \text{id}$). For a novel setting that satisfies condition (ii), see Example 6 below.

By inspection of Proposition 1, guaranteeing strong monotonicity requires single-crossing and third-derivative assumptions that are familiar from static models. The new assumptions that go beyond them concern the impulse response functions. This is best illustrated by considering even stronger sufficient conditions, which can be stated separately on utilities and processes. For concreteness, suppose that U-DSEP holds and that X_t is one-dimensional (so that either case in the proposition can be applied). Then, in the initial period $t = 0$, it suffices to impose the static conditions: for each agent i, the allocation utility $u_{i0}(\theta_0, x_0)$ and the partial $-\partial u_{i0}(\theta_0, x_0)/\partial\theta_{i0}$ have increasing differences (ID) in allocation and types (the latter being a third-derivative condition on the utility function), and the hazard rate $\eta_{i0}(\theta_{i0})$ is nondecreasing. In periods $t \geq 1$, in addition to imposing the static conditions to current utility flows, it suffices to assume that the impulse response $I_{i\tau}(\theta_i^\tau)$ be nondecreasing in types. This implies that the term that captures the agent's information rent, $-\frac{1}{\eta_{i0}(\theta_{i0})} I_{it}(\theta_i^t)\,\partial u_{it}(\theta^t, x_t)/\partial\theta_{it}$, has ID in allocation and types. Heuristically, nondecreasing impulse responses lead to distortions being decreasing in types, which helps to ensure monotonicity of the allocation.

REMARK 7: We discuss in the Supplemental Material how Corollary 2 and Proposition 1 can be adapted to find optimal mechanisms in some non-Markov environments. Note that the derivation of the expected virtual surplus (13) above makes no reference to Markov environments and, hence, the difference is in verifying that the allocation rule that maximizes it is indeed implementable.

4.1. *Distortions*

A first-best allocation rule maximizes the expected surplus $\mathbb{E}^{\lambda[\chi]}[\sum_{i=0}^{n} U_i(\tilde{\theta}, \chi(\tilde{\theta}))]$ in our quasilinear environment. Similarly to the static setting, a profit-maximizing principal introduces distortions to the allocations to reduce the agents' expected information rents. When the participation constraints of the lowest initial types in (12) bind, the expected rent of agent i is given by

$$\mathbb{E}^{\lambda[\chi]}\left[\sum_{t=0}^{\infty} \frac{1}{\eta_{i0}(\tilde{\theta}_{i0})} I_{i,(0),t}\big(\tilde{\theta}_i^t, \chi_i^{t-1}(\tilde{\theta})\big) \frac{\partial U_i(\tilde{\theta}, \chi(\tilde{\theta}))}{\partial\theta_{it}}\right].$$

Thus the period-0 impulse response functions are an important determinant of the rent and, by implication, of the distortions in optimal dynamic mechanisms. Given the various forms these functions may take, little can be said about the nature of these distortions in general. Indeed, we illustrate by means of a simple class of single-agent environments that the distortion in period t may be a nonmonotone function of the agent's types or, for a fixed type sequence, a nonmonotone function of the time period t. Example 5 also illustrates the use of ex post monotonicity to verify implementability.

EXAMPLE 5—Nonlinear AR Process: Consider a buyer–seller relationship, which lasts for $T+1$ periods, with $T \leq \infty$. The buyer's payoff takes the form $U_1(\theta, x) = \sum_{t=0}^{T} \delta^t (a + \theta_t) x_t$, with $a \in \mathbb{R}_{++}$ and $X_t = [0, \bar{x}]$ for some $\bar{x} \gg 0$. The seller's payoff is given by $U_0(\theta, x) = -\sum_{t=0}^{T} \delta^t \frac{x_t^2}{2}$. The buyer's type evolves according to the *nonlinear* AR process $\theta_t = \phi(\theta_{t-1}) + \varepsilon_t$, where ϕ is an increasing differentiable function, with $\phi(0) = 0$, $\phi(1) < 1$, and $\phi' \leq b$ for some $1 \leq b < \frac{1}{\delta}$, and where the shocks ε_t are independent over time, with support $[0, 1 - \phi(1)]$. By putting $z_t(\theta_{t-1}, \varepsilon_t) = \phi(\theta_{t-1}) + \varepsilon_t$ and using formula (4), we find the period-0 impulse responses $I_{(0),t}(\theta^t) = \prod_{\tau=0}^{t-1} \phi'(\theta_\tau)$.

Since the type process is autonomous, and decisions are separable across time (i.e., F-AUT and U-DSEP hold), the first-best allocation rule simply sets $x_t = a + \theta_t$ for all $t \geq 0$ and $\theta^t \in \Theta^t$. Furthermore, we can maximize expected virtual surplus (13) pointwise to find the allocation rule

$$\chi_t(\theta^t) = \max\left\{0, a + \theta_t - \frac{1}{\eta_0(\theta_0)} \prod_{\tau=0}^{t-1} \phi'(\theta_\tau)\right\}$$
$$\text{for all } t \geq 0 \text{ and } \theta^t \in \Theta^t.$$

We show in the Supplemental Material that if the hazard rate η_0 is nondecreasing, then χ is ex post monotone, and, thus, it is an optimal allocation rule by Corollaries 1 and 2. Note that χ exhibits downward distortions since $\phi' > 0$. Increasing the period-τ type θ_τ for $1 \leq \tau < t$ reduces distortions in period t if ϕ is concave, but increases distortions if ϕ is convex. (Note that in the latter case, χ is not strongly monotone, yet it is PBE-implementable.) When ϕ is neither concave nor convex, the effect is nonmonotone. Similarly, if $\phi'(\theta_{t-1}) < 1$, then the distortion in period t is smaller than that in period $t-1$, whereas if $\phi'(\theta_{t-1}) > 1$, then the period-t distortion exceeds that in period $t-1$.

Finally, note that the period-t allocation $\chi_t(\theta^t)$ is, in general, a nontrivial function of the buyer's types in all periods $0, \ldots, t$. This is in contrast to the special case of a linear function $\phi(\theta_t) = \gamma\theta_t$, $\gamma > 0$, considered by Besanko (1985), where the impulse response is the time-varying scalar $I_t = \gamma^t$ as in Example 2, and where $\chi_t(\theta^t)$ depends only on the initial type θ_0 and the current type θ_t.

The distortions in Example 5 are independent of the agent's current report. However, it is easy to construct examples where distortions are nonmonotone also with respect to the current report.

EXAMPLE 6: Consider the environment of Example 5, but assume now that $T = 1$ and that the buyer's type evolves as follows. The initial type $\tilde{\theta}_0$ is distributed uniformly on $\Theta_0 = [0, 1]$, whereas $\tilde{\theta}_1$ is distributed on $\Theta_1 = [0, 1]$ according to the c.d.f. $F_1(\theta_1|\theta_0) = \theta_1 - 2(\theta_0 - \frac{1}{2})\theta_1(1 - \theta_1)$ with linear density

$f_1(\theta_1|\theta_0) = 1-2(\theta_0 - \frac{1}{2})(1-2\theta_1)$ strictly positive on Θ_1 for all θ_0. For $\theta_0 = 1/2$, $\tilde{\theta}_1$ is distributed uniformly on [0, 1]. For $\theta_0 < (>)\ 1/2$, the density slopes downward (upward). Note that F satisfies F-FOSD. The canonical impulse response formula (5) from Example 3 gives

$$I_{(0),1}(\theta_0, \theta_1) = \frac{2\theta_1(1-\theta_1)}{1-2\left(\theta_0 - \dfrac{1}{2}\right)(1-2\theta_1)}.$$

The allocation rule that solves the relaxed program is then given by

$$\chi_0(\theta_0) = \max\{0, a+\theta_0 - (1-\theta_0)\},$$

$$\chi_1(\theta^1) = \max\left\{0, a+\theta_1 - (1-\theta_0)\frac{2\theta_1(1-\theta_1)}{1-2\left(\theta_0 - \dfrac{1}{2}\right)(1-2\theta_1)}\right\}.$$

Because χ is strongly monotone, it is clearly implementable. By inspection, the second-period allocation is efficient at the extremes (i.e., for $\theta_1 \in \{0, 1\}$), whereas all interior types are distorted downward. Note that the no-distortions-at-the-bottom result is nontrivial, since even the lowest type here consumes a positive amount, so there would be room to distort downward.

In the Supplemental Material, we use monotone comparative statics to give sufficient conditions for the allocation rule that maximizes expected virtual surplus (13) to exhibit downward distortions as in the above examples. However, upward distortions can naturally arise in applications. This was first shown by Courty and Li (2000), who provided a two-period example where the distribution of the agent's second-period type is ordered by his initial signal in the sense of a mean-preserving spread.

In consumption problems such as Example 5 and the one studied by Courty and Li (2000), distortions can be understood purely in terms of the canonical impulse response (see Example 3)

$$I_t(\theta^t) = \prod_{\tau=1}^{t}\left(-\frac{\partial F_\tau(\theta_\tau|\theta_{\tau-1})/\partial\theta_{\tau-1}}{f_\tau(\theta_\tau|\theta_{\tau-1})}\right).$$

If the kernels satisfy F-FOSD, then $I_t(\theta^t)$ is positive, leading to downward distortions as in Example 5. If F-FOSD fails, then $I_t(\theta^t)$ is negative at some θ^t, yielding upward distortions at that history as in Courty and Li (2000). Dynamics can be seen similarly: As in Example 5, an increase in the impulse response increases distortions compared to the previous period, whereas a decrease leads to consumption being more efficient. In particular, if $I_t(\theta^t) \to 0$, then consumption converges to the first best over time.

5. BANDIT AUCTIONS

To illustrate our results, we consider the problem of a profit-maximizing seller who must design a sequence of auctions to sell, in each period $t \geq 0$, an indivisible, nonstorable good to a set of $n \geq 1$ bidders who update their valuations upon consumption, that is, upon winning the auction. This setting captures novel applications such as repeated sponsored search auctions, where the advertisers privately learn about the profitability of clicks on their ads, or repeated procurement with learning-by-doing. It provides a natural environment where the kernels depend on past allocations.

Let $X_{it} = \{0, 1\}$ for all $i = 0, \dots, n$ and $t \geq 0$, and define the set of feasible allocation sequences by $X = \{x \in \prod_{t=0}^{\infty} \prod_{i=0}^{n} X_{it} : \sum_{i=0}^{n} x_{it} = 1 \text{ for all } t \geq 0\}$. The seller's payoff function is then given by $U_0(\theta, x) = -\sum_{t=0}^{\infty} \delta^t \sum_{i=1}^{N} x_{it} c_{it}$, where $c_{it} \in \mathbb{R}$ is the cost of allocating the object to bidder i (with c_{0t} normalized to 0). Each bidder i's payoff function takes the form $U_i(\theta, x) = \sum_{t=0}^{\infty} \delta^t \theta_{it} x_{it}$.

The type process of bidder $i = 1, \dots, n$ is constructed as follows. Let $R_i = (R_i(\cdot|k))_{k \in \mathbb{N}}$ be a sequence of absolutely continuous, strictly increasing c.d.f.'s with mean bounded in absolute value uniformly in k. The first-period valuation θ_{i0} is drawn from Θ_{i0}, with $\underline{\theta}_{i0} > -\infty$, according to an absolutely continuous, strictly increasing c.d.f. F_{i0}. For all $t > 0$, $\theta_i^t \in \Theta_i^t$, and $x_i^{t-1} \in X_i^{t-1}$, if $x_{i,t-1} = 1$, then

$$F_{it}\big(\theta_{it}|\theta_{i,t-1}, x_i^{t-1}\big) = R_i\left(\theta_{it} - \theta_{i,t-1} \,\middle|\, \sum_{\tau=0}^{t-1} x_{i\tau}\right);$$

if, instead, $x_{i,t-1} = 0$, then

$$F_{it}\big(\theta_{it}|\theta_{i,t-1}, x_i^{t-1}\big) = \begin{cases} 0, & \text{if } \theta_{it} < \theta_{i,t-1}, \\ 1, & \text{if } \theta_{it} \geq \theta_{i,t-1}. \end{cases}$$

This formulation embodies the following key assumptions: (i) Bidders' valuations change only upon winning the auction (i.e., if $x_{it} = 0$, then $\theta_{i,t+1} = \theta_{it}$ almost surely); (ii) the valuation processes are time homogenous (i.e., if bidder i wins the object in period t, then the distribution of his period-$t+1$ valuation depends only on his period-t valuation and the total number of times he won in the past).[37]

[37]This kind of structure arises, for example, in a Bayesian learning model with Gaussian signals. That is, suppose each bidder i has a constant but unknown true valuation v_i for the object and starts with a prior belief $v_i \sim N(\theta_{i0}, \tau_i)$ where precision τ_i is common knowledge. Bidder i's initial type θ_{i0} is the mean of the prior distribution, which the seller and the other bidders believe to be distributed according to some distribution F_{i0} bounded from below. In each period in which the bidder wins the auction, he receives a conditionally i.i.d. private signal $s_i \sim N(v_i, \sigma_i)$, and updates his expectation of v_i using standard projection formulae. Take θ_{it} to be bidder i's posterior expectation in period t. Then, $R_i(\cdot|k)$ is the c.d.f. for the change in the posterior expectation due

We start by verifying that the bandit auction environment defined above is regular and Markov. Each bidder i's payoff function U_i clearly satisfies Conditions U-D, U-ELC, and U-SPR since each X_{it} is bounded. As for bidder i's type process, it satisfies the bounds in part (ii) of the Markov definition (and thus F-BE) by the uniform bound on the means of $(R_i(\cdot|k))_{k\in\mathbb{N}}$. To verify F-BIR, we use the canonical representation of the process. That is, for all $t>0$, $\varepsilon_{it}\in(0,1)$, and $(\theta_{i,t-1}, x_i^{t-1})\in\Theta_{i,t-1}\times X_i^{t-1}$, let

$$\begin{aligned} z_{it}\big(\theta_i^{t-1}, x_i^{t-1}, \varepsilon_{it}\big) &= F_{it}^{-1}\big(\varepsilon_{it}|\theta_{i,t-1}, x_i^{t-1}\big) \\ &= \theta_{i,t-1} + \mathbf{1}_{\{x_{i,t-1}=1\}} R_i^{-1}\left(\varepsilon_{it}\middle|\sum_{\tau=0}^{t-1} x_{i\tau}\right). \end{aligned}$$

The Z functions then take the form

$$Z_{i,(s),t}\big(\theta_i^s, x_i^{t-1}, \varepsilon_i^t\big) = \theta_{is} + \sum_{m=s+1}^{t} \mathbf{1}_{\{x_{i,m-1}=1\}} R_i^{-1}\left(\varepsilon_{im}\middle|\sum_{\tau=0}^{m-1} x_{i\tau}\right) \tag{14}$$

and, hence, $\frac{\partial Z_{i,(s),t}}{\partial \theta_{is}}=1$. Therefore, F-BIR holds, and the impulse responses satisfy $I_{i,(s),t}(\theta_i^t, x_i^{t-1})=1$ for all $i=1,\dots,n$, $t\geq s\geq 0$, $\theta_i^t\in\Theta_i^t$, and $x_i^{t-1}\in X_i^{t-1}$.

Since the impulse responses $I_{i,(0),t}$ are identical to 1 for all agents i and all periods t, the envelope formula (1) takes the form $dV_{i0}^{\langle\chi,\psi\rangle}(\theta_{i0})/d\theta_{i0} = \mathbb{E}^{\lambda_i[\chi]|\theta_{i0}}[\sum_{t=0}^{\infty}\delta^t\chi_{it}(\tilde\theta)]$ and the problem of maximizing expected virtual surplus (13) becomes

$$\sup_{\chi\in\mathcal{X}} \mathbb{E}^{\lambda[\chi]}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{n}\left(\tilde\theta_{it}-c_{it}-\frac{1}{\eta_{i0}(\tilde\theta_{i0})}\right)\chi_{it}\big(\tilde\theta^t\big)\right].$$

This is a standard multiarmed bandit problem: The safe arm corresponds to the seller and yields a sure payoff equal to 0; the risky arm $i=1,\dots,n$ corresponds to bidder i and yields a flow payoff $\theta_{it}-c_{it}-[\eta_{i0}(\theta_{i0})]^{-1}$. The solution takes the form of an index policy. That is, define the virtual index of bidder $i=1,\dots,n$ in period $t\geq 0$ given history $(\theta_i^t, x_i^{t-1})\in\Theta_i^t\times X_i^{t-1}$ as

$$\gamma_{it}\big(\theta_i^t, x_i^{t-1}\big) \equiv \max_T \mathbb{E}^{\lambda[\bar\chi_i]|\theta_{it}, x_i^{t-1}}\left[\frac{\displaystyle\sum_{\tau=t}^{T}\delta^\tau\left(\tilde\theta_{i\tau}-c_{it}-\frac{1}{\eta_{i0}(\theta_{i0})}\right)}{\displaystyle\sum_{\tau=t}^{T}\delta^\tau}\right], \tag{15}$$

to the kth signal, which is indeed independent of the current value of θ_{it}. (Standard calculations show that $R_i(\cdot|k)$ is in fact a Normal distribution with mean zero and variance decreasing in k.) Alternative specifications for R_i can be used to model learning-by-doing, habit formation, and so on.

where T is a stopping time and $\bar{\chi}_i$ is the allocation rule that assigns the object to bidder i in all periods. (Note that the virtual index depends on x_i^{t-1} only through $\sum_{\tau=0}^{t-1} x_{i\tau}$.) The index of the seller is identically equal to zero and for convenience we write it as $\gamma_{0t}(\theta_0^t, x_0^{t-1}) \equiv 0$. The following *virtual index policy* then maximizes the expected virtual surplus:[38] For all $i = 1, \dots, n$, $t \geq 0$, $\theta^t \in \Theta^t$, and $x^{t-1} \in X^{t-1}$, let $J(\theta^t, x^{t-1}) \equiv \arg\max_{j \in \{0,\dots,n\}} \gamma_{jt}(\theta_j^t, x_j^{t-1})$ and let

$$\chi_{it}(\theta^t) = \begin{cases} 1, & \text{if } i = \min J(\theta^t, x^{t-1}), \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

PROPOSITION 2—Optimal Bandit Auctions: *Suppose that for all $i = 1, \dots, n$, the hazard rate $\eta_{i0}(\theta_{i0})$ is nondecreasing in θ_{i0}. Let $\langle \chi, \psi \rangle$ be the choice rule where χ is the virtual index policy defined by (16) and where ψ is the transfer rule defined by (7). Then $\langle \chi, \psi \rangle$ is an optimal mechanism in the bandit auction environment.*

The environment is regular and Markov, F-FOSD holds, and each U_i is nondecreasing in θ_i. Hence, the result follows from Corollary 2 once we show that the virtual index policy χ satisfies integral monotonicity. We do this in the Appendix by showing that χ satisfies average monotonicity defined in Corollary 1, which here requires that, for all $i = 1, \dots, n$, $s \geq 0$, and $(\theta^{s-1}, \theta_{is}) \in \Theta^{s-1} \times \Theta_{is}$, bidder i's expected discounted consumption

$$\mathbb{E}^{\lambda_i[\chi \circ \hat{\theta}_{is}, \Gamma] | \theta^{s-1}, \theta_{is}} \left[\sum_{t=s}^{\infty} \delta^t (\chi \circ \hat{\theta}_{is})_{it}(\tilde{\theta}) \right]$$

is nondecreasing in his current bid $\hat{\theta}_{is}$. Heuristically, this follows because a higher bid in period s increases the virtual index of arm i, which results in bidder i consuming sooner in the sense that, for any $k \in \mathbb{N}$, the expected waiting time until he wins the auction for the kth time after period s is then weakly shorter. Note that because of learning, averaging is important: Even if increasing the current bid always makes bidder i more likely to win the auction today, for bad realizations of the resulting new valuation, it leads to a lower chance of winning the auction in the future. However, by F-FOSD, higher current types are also more likely to win in the future on average.

It is instructive to compare the virtual index policy from the optimal bandit auction to the first-best index policy that maximizes social surplus. The first-best policy is implementable by using the team mechanism of Athey and Segal (2013) or the dynamic pivot mechanism of Bergemann and Välimäki (2010),

[38]The optimality of index policies is well known (e.g., Whittle (1982) or Bergemann and Välimäki (2008)).

who considered a similar bandit setting as an application. In the first-best policy, bidder i's index at period-t history (θ_i^t, x_i^{t-1}) is given by

$$g_{it}\big(\theta_i^t, x_i^{t-1}\big) \equiv \max_T \mathbb{E}^{\lambda[\bar{\chi}_i]|\theta_{it}, x_i^{t-1}} \left[\frac{\displaystyle\sum_{\tau=t}^{T} \delta^\tau (\tilde{\theta}_{i\tau} - c_{it})}{\displaystyle\sum_{\tau=t}^{T} \delta^\tau} \right].$$

By inspection of (15), we see that the virtual index $\gamma_{it}(\theta_i^t, x_i^{t-1})$ differs from the first-best index $g_{it}(\theta_i^t, x_i^{t-1})$ only for the presence of the term $\frac{1}{\eta_{i0}(\theta_{i0})}$, which can be interpreted as bidder i's "handicap." In particular, note that the handicaps are determined by the bidders' first-period (reported) types. Thus, the optimal mechanism can be implemented by using the bidders' initial reports to determine their handicaps along with the period-0 allocation and transfers, and by then running a handicapped efficient mechanism in periods $t > 0$, where the indices are computed as if the seller's cost of assigning the good to bidder i was $c_{it} - \frac{1}{\eta_{i0}(\hat{\theta}_{i0})}$.[39] This implies that even ex-ante symmetric bidders are, in general, treated asymmetrically in the future and, hence, the distortions in future periods reflect findings in optimal static auctions with asymmetric bidders. (For example, the first-best and virtual indices will sometimes disagree on the ranking of any given bidders i and j, and, hence, i may win the object in some period t, even if the first-best policy would award it to j.)

We conclude that the optimal mechanism for selling experience goods is essentially a dynamic auction with memory that grants preferential treatment based on the bidders' initial types. These features are markedly different from running a sequence of second-price auctions with a reserve price, and suggest potential advantages of building long-term contractual relationships in repeated procurement and sponsored search.

REMARK 8: Subsequent to the first version of our manuscript, Kakade, Lobel, and Nazerzadeh (2011) considered a class of allocation problems that generalize our bandit auction environment and showed that the optimal mechanism is a virtual version of the dynamic pivot mechanism of Bergemann and Välimäki (2010), the handicap mechanism being a special case. Postulating the model in terms of a state representation, they derived the allocation rule using our first-order approach and established incentive compatibility in period 0 by verifying a condition analogous to our average monotonicity.

Kakade et al.'s proof of incentive compatibility for periods $t > 0$ differs from ours, and relies on the above observation about the optimal mechanism from

[39] Board (2007) and Eső and Szentes (2007) found similar optimal mechanisms in settings where the type processes are autonomous and there is only one good to be allocated.

period 1 onward being an efficient mechanism for a fictitious environment where the seller's cost of assigning the object to bidder i is $c_{it} - \frac{1}{\eta_{i0}(\hat{\theta}_{i0})}$, where $\hat{\theta}_{i0}$ is i's initial *report*. In particular, using an efficient mechanism for this fictitious environment that asks the bidders to re-report their initial types in period 1 gives the existence of a truthful continuation equilibrium from period 1 onward. This approach requires, however, that every agent i's payoff and state representation be separable in the sense that there exist functions α_i, γ_{it}, and β_{it}, $t \geq 0$, such that (i) $u_{it}(Z^t_{i,(0)}(\theta_{i0}, \varepsilon_i), x^t) = \alpha_i(\theta_{i0})\gamma_{it}(x^t) + \beta_{it}(\varepsilon^t_i, x^t)$ for all t or that (ii) $u_{it}(Z^t_{i,(0)}(\theta_{i0}, \varepsilon_i), x^t) = \alpha_i(\theta_{i0})\beta_{it}(\varepsilon^t_i, x^t)$ for all t. While our bandit auction environment satisfies condition (i) by inspection of (14), neither condition is satisfied in nonlinear environments such as our Example 5, for which our approach of verifying integral monotonicity in every period is applicable. Nonetheless, Kakade et al. can accommodate non-Markov environments that are separable in the above sense. Thus, the approaches are best viewed as complementary.

6. CONCLUDING REMARKS

We extend the standard Myersonian approach to mechanism design to dynamic quasilinear environments. Our main results characterize local incentive-compatibility constraints, provide a method of constructing transfers to satisfy them, address the uniqueness of these transfers, and give necessary and sufficient conditions for the implementability of allocation rules in Markov environments. These results lend themselves to the design of optimal dynamic mechanisms along the familiar lines of finding an allocation rule by maximizing expected (dynamic) virtual surplus and then verifying that the allocation rule is implementable by checking appropriate monotonicity conditions.

The analysis permits a unified view of the existing literature by identifying general principles and highlighting what drives similarities and differences in the special cases considered. The generality of our model offers flexibility that facilitates novel applications, such as the design of sales mechanisms for the provision of new experience goods, or bandit auctions.

Our limited use of a state representation, also known as the *independent-shocks (IS) approach*, deserves some comments given its prominent role, for example, in the works of Eső and Szentes (2007, 2013) or Kakade, Lobel, and Nazerzadeh (2011). Representing the type processes by means of independent shocks is always without loss of generality and, as explained after Theorem 1, it provides a convenient way to identify primitive conditions under which the envelope formula is a necessary condition for incentive compatibility. However, the IS approach is not particularly useful for establishing (necessary and) sufficient conditions for implementability in Markov environments, because the transformation to independent shocks does not, in general, preserve the "Markovness" of the environment. Hence, after the transformation, it is not sufficient to consider one-stage deviations from strongly truthful strategies (see the Supplemental Material for a counterexample). Accordingly, our analysis of implementability in Markov environments in Section 3.3 makes no reference to the IS approach.

Eső and Szentes (2007, 2013) emphasize that the cost to the principal of implementing a given allocation rule is the same as in an hypothetical environment where she can observe the agents' future independent shocks. This result can be seen from the proof of our Theorem 1, where the agents have been constrained to report the future orthogonalized shocks truthfully. An implication of the theorem is that the expected present value of the payments implementing a given allocation rule is unique, up to a scalar, even when the shocks are observable. A corollary of this revenue-equivalence result is that, if the same allocation rule remains implementable when the shocks are the agents' private information, the expected present value of the payments implementing the said rule must be the same irrespective of the observability of the shocks. However, the principal may be able to implement a strictly larger set of allocation rules if she can observe the shocks and, hence, the optimal mechanisms for the two settings are different in general. The optimal mechanisms coincide if the relaxed program discussed in Section 4 yields an implementable allocation rule. Whenever this is not the case, the observability of the shocks is relevant for the principal's payoff and for the agents' information rents.

The most important direction for future work pertains to the generality of our results on optimal dynamic mechanisms. In particular, our results are restricted to settings where the first-order approach yields an implementable allocation rule. The extent to which this affects qualitative findings about the properties of optimal mechanisms is an open question. For some progress in this direction, see Garrett and Pavan (2013), who work directly with the integral monotonicity condition to show that, in the context of managerial compensation, the key properties of optimal contracts extend to environments where the first-order approach is invalid.

APPENDIX: PROOFS

PROOF OF THEOREM 1: We start by establishing ICFOC$_{i,0}$ for all $i = 1, \dots, n$. Let the type processes be generated by the state representation $\langle \mathcal{E}_i, G_i, z_i \rangle_{i=1}^n$, and consider a fictitious environment in which, in each period $t \geq 1$, each agent $i = 1, \dots, n$ observes the shock ε_{it} and computes $\theta_{it} = Z_{i,(0),t}(\theta_{i0}, x_i^{t-1}, \varepsilon_i^t)$. Consider a direct revelation mechanism in the fictitious environment in which each agent i reports θ_{i0} in period 0 and ε_{it} in each period $t \geq 1$, and that implements the decision rule $\hat{\chi}_t(\theta_0, \varepsilon^t) = \chi_t(Z_{(0)}^t(\theta_0, \hat{\chi}^{t-1}(\theta_0, \varepsilon^{t-1}), \varepsilon^t))$ and payment rule $\hat{\psi}_t(\theta_0, \varepsilon^t) = \psi_t(Z_{(0)}^t(\theta_0, \hat{\chi}^{t-1}(\theta_0, \varepsilon^{t-1}), \varepsilon^t))$ in each period t (defined recursively on t with $Z_{(0),t} \equiv (Z_{i,(0),t})_{i=1}^n$, $Z_{(0)}^t = (Z_{(0),s})_{s=0}^t$, and $Z_{(0)} = (Z_{(0),s})_{s=0}^\infty$).

Suppose that all agents other than i report truthfully in all periods. Agent i's payoff when the other agents' initial signals are $\theta_{-i,0}$, agent i's true period-0 signal is θ_{i0}, his period-0 report is $\hat{\theta}_{i0}$, and all future shocks ε are reported truthfully is given by

$$\begin{aligned}
&\hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}, \theta_{-i,0}, \varepsilon) \\
&\quad \equiv U_i\big(Z\big(\theta_{i0}, \theta_{-i,0}, \hat{\chi}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon), \varepsilon\big), \hat{\chi}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon)\big) \\
&\qquad + \sum_{t=0}^{\infty} \delta^t \hat{\psi}_{it}\big(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon^t\big).
\end{aligned}$$

Since $\langle \chi, \psi \rangle$ is PBIC in the original environment, truthful reporting by each agent at all truthful histories is a PBE of the mechanism $\langle \hat{\chi}, \hat{\psi} \rangle$ in the fictitious environment. This implies that agent i cannot improve his expected payoff by misreporting his period-0 type and then reporting the subsequent shocks truthfully. That is, for any $\theta_{i0} \in \Theta_{i0}$,

$$\begin{aligned}
&V_i^{\langle \chi, \Psi \rangle}(\theta_{i0}) = \sup_{\hat{\theta}_{i0} \in \Theta_{i0}} W(\hat{\theta}_{i0}, \theta_{i0}) = W(\theta_{i0}, \theta_{i0}), \\
&\quad \text{where} \quad W(\hat{\theta}_{i0}, \theta_{i0}) \equiv \mathbb{E}\big[\hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon})\big].
\end{aligned}$$

The following lemma shows that the objective function W in the above maximization problem is well behaved in the parameter θ_{i0}:

LEMMA A.1: *Suppose that the environment is regular. Then, for all* $i = 1, \ldots, n$ *and* $\hat{\theta}_{i0} \in \Theta_{i0}$, $W_i(\hat{\theta}_{i0}, \cdot)$ *is equi-Lipschitz continuous and differentiable, with the derivative at* $\theta_{i0} = \hat{\theta}_{i0}$ *given by*

$$\begin{aligned}
&\frac{\partial W_i(\hat{\theta}_{i0}, \hat{\theta}_{i0})}{\partial \theta_{i0}} \\
&\quad = \mathbb{E}\Bigg[\sum_{t=0}^{\infty} \frac{\partial U_i(Z_{(0)}(\hat{\theta}_{i0}, \tilde{\theta}_{-i,0}, \hat{\chi}(\hat{\theta}_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon}), \tilde{\varepsilon}), \hat{\chi}(\hat{\theta}_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon}))}{\partial \theta_{it}} \\
&\qquad \cdot \frac{\partial Z_{i,(0),t}(\hat{\theta}_{i0}, \hat{\chi}_i^{t-1}(\hat{\theta}_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon}^{t-1}), \tilde{\varepsilon}_i^t)}{\partial \theta_{i0}}\Bigg].
\end{aligned}$$

PROOF: Let us focus on those ε for which $\partial Z_{i,(0),t}(\theta_{i0}, \hat{\chi}_i^{t-1}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon_i^{t-1}), \varepsilon_i^t)/\partial \theta_{i0} < C_{it}(\varepsilon_i)$ for all i, t, θ_0, with $\|C_{i,(0)}(\varepsilon_i)\| < \infty$, and $\|Z_{i,(0)}(\theta_{i0}, \hat{\chi}_i(\theta_{i0}, \theta_{-i,0}, \varepsilon), \varepsilon_i)\| < \infty$, which under Conditions F-BE and F-BIR occurs with probability 1, and temporarily drop arguments ε, $\theta_{-i,0}$, $\hat{\theta}_{i0}$, $x = \hat{\chi}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon)$, and subscripts i, (0) to simplify notation.

The classical chain rule (using Conditions U-D and F-BIR) yields that, for any given T,

$$\Delta_T(\theta_0, h) \equiv \frac{1}{h}U\bigl(Z^T(\theta_0+h), Z^{>T}(\theta_0)\bigr) - \frac{1}{h}U\bigl(Z(\theta_0)\bigr) - \sum_{t=0}^{T}\frac{\partial U(Z(\theta_0))}{\partial\theta_t}Z_t'(\theta_0) \to 0 \quad \text{as} \quad h \to 0. \tag{17}$$

Note that

$$\frac{1}{h}U\bigl(Z^T(\theta_0+h), Z^{>T}(\theta_0)\bigr) \to \frac{1}{h}U\bigl(Z(\theta_0+h)\bigr) \quad \text{as} \quad T\to\infty$$

uniformly in h since, using U-ELC, the difference is uniformly bounded by

$$A\frac{1}{h}\sum_{t=T+1}^{\infty}\delta^t\bigl|Z_t(\theta_0+h) - Z_t(\theta_0)\bigr| \le A\sum_{t=T+1}^{\infty}\delta^t C_t$$

and the right-hand side converges to zero as $T\to\infty$ since $\|C\|<\infty$.

Also, the series in (17) converges uniformly in h by the Weierstrass M-test, since, using Conditions U-ELC and F-BIR,

$$\sum_{t=0}^{T}\left|\frac{\partial U(Z(\theta_0))}{\partial\theta_t}\right|\bigl|Z_t'(\theta_0)\bigr| \le \sum_{t=0}^{T}A\delta^t C_t \to A\|C\| \quad \text{as} \quad T\to\infty.$$

Hence, we have

$$\Delta_T(\theta_0, h) \to \frac{1}{h}\bigl[\hat{U}(\theta_0+h) - \hat{U}(\theta_0)\bigr] - \sum_{t=0}^{\infty}\frac{\partial U(Z(\theta_0))}{\partial\theta_t}Z_t'(\theta_0) \quad \text{as} \quad T\to\infty$$

uniformly in h. By uniform convergence, we interchange the order of limits and use (17) to get

$$\begin{aligned}
&\lim_{h\to 0}\left[\frac{1}{h}\bigl[\hat{U}(\theta_0+h) - \hat{U}(\theta_0)\bigr] - \sum_{t=0}^{\infty}\frac{\partial U(Z(\theta_0))}{\partial\theta_t}Z_t'(\theta_0)\right] \\
&\quad = \lim_{h\to 0}\lim_{T\to\infty}\Delta_T(\theta_0, h) \\
&\quad = \lim_{T\to\infty}\lim_{h\to 0}\Delta_T(\theta_0, h) = 0.
\end{aligned}$$

This yields (putting back all the missing arguments)

$$\frac{\partial \hat{U}_i(\hat{\theta}_{i0}, \theta_0, \varepsilon)}{\partial \theta_{i0}} = \sum_{t=0}^{\infty} \frac{\partial U_i(Z_{(0)}(\theta_0, \varepsilon, \hat{\chi}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon)), \hat{\chi}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon))}{\partial \theta_{it}} \times \frac{\partial Z_{i,(0),t}(\theta_{i0}, \varepsilon_i^t, \hat{\chi}_i^{t-1}(\hat{\theta}_{i0}, \theta_{-i,0}, \varepsilon^{t-1}))}{\partial \theta_{i0}}.$$

Next, note that, being a composition of Lipschitz continuous functions, $\hat{U}_i(\hat{\theta}_{i0}, \cdot, \theta_{-i,0}, \varepsilon)$ is equi-Lipschitz continuous in θ_{i0} with constant $A\|C_{(0),i}(\varepsilon)\|$. Since, by F-BIR, $\mathbb{E}[\|C_{i,(0)}(\tilde{\varepsilon})\|] < \infty$, by the dominated convergence theorem, we can write

$$\begin{aligned}
\frac{\partial W_i(\hat{\theta}_{i0}, \theta_{i0})}{\partial \theta_{i0}} &= \lim_{h\to 0} \mathbb{E}\left[\frac{\hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}+h, \tilde{\theta}_{-i,0}, \tilde{\varepsilon}) - \hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon})}{h}\right] \\
&= \mathbb{E} \lim_{h\to 0} \left[\frac{\hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}+h, \tilde{\theta}_{-i,0}, \tilde{\varepsilon}) - \hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon})}{h}\right] \\
&= \mathbb{E}\left[\frac{\partial \hat{U}_i(\hat{\theta}_{i0}, \theta_{i0}, \tilde{\theta}_{-i,0}, \tilde{\varepsilon})}{\partial \theta_{i0}}\right].
\end{aligned}$$

Hence, for any $\hat{\theta}_{i0}$, $W_i(\hat{\theta}_{i0}, \cdot)$ is differentiable and equi-Lipschitz in θ_{i0} with Lipschitz constant $A\mathbb{E}[\|C_{i,(0)}(\tilde{\varepsilon})\|]$ and with the derivative at $\theta_{i0} = \hat{\theta}_{i0}$ given by the formula in the lemma. *Q.E.D.*

The equi-Lipschitz continuity of $W_i(\hat{\theta}_{i0}, \cdot)$ established in Lemma A.1 implies that the value function $\sup_{\hat{\theta}_{i0}\in\Theta_{i0}} W(\hat{\theta}_{i0}, \theta_{i0})$, which coincides with the equilibrium payoff $V_i^{\langle\chi,\Psi\rangle}(\theta_{i0})$, is Lipschitz continuous.[40] Furthermore, by Theorem 1 of Milgrom and Segal (2002), at any differentiability point of $V_i^{\langle\chi,\Psi\rangle}(\theta_{i0})$, we have $\frac{dV_i^{\langle\chi,\Psi\rangle}(\theta_{i0})}{d\theta_{i0}} = \frac{\partial W_i(\theta_{i0},\theta_{i0})}{\partial\theta_{i0}}$. Using Lemma A.1, the law of iterated expectations, and the definition of $I_{i,(0),t}$ in (2) then yields ICFOC$_{i,0}$. The ICFOC$_{i,s}$ for $s > 0$ then follows by the same argument, since agent i's problem at a truthful period-s history is identical to the period-0 problem except for the indexing by the history. *Q.E.D.*

PROOF OF THEOREM 2: For part (i), we show first that the flow transfers in (7) are well defined by showing that the discounted sum (over t) of each of

[40]Since for each θ_{i0}, θ'_{i0}, $|V_i^{\langle\chi,\Psi\rangle}(\theta'_{i0}) - V_i^{\langle\chi,\Psi\rangle}(\theta_{i0})| \leq \sup_{\hat{\theta}_{i0}\in\Theta_{i0}} |W_i(\hat{\theta}_{i0}, \theta'_{i0}) - W_i(\hat{\theta}_{i0}, \theta_{i0})| \leq M|\theta'_{i0} - \theta_{i0}|$, where $M > 0$ is the constant of equi-Lipschitz continuity of W. This argument is similar to the first part of Milgrom and Segal's (2002) Theorem 2.

the three terms in ψ_{it} has a finite expected net present value (NPV) under the measure $\lambda_i[\chi, \Gamma]|\theta^{s-1}, \theta_{is}$ for all $i = 1, \dots, n$, $s \geq 0$, and $(\theta^{s-1}, \theta_{is}) \in \Theta^{s-1} \times \Theta_{is}$ (which implies that the series $\sum_{t=0}^{\infty} \delta^t \psi_{it}(\theta^t)$ converges with probability 1 under $\lambda_i[\chi, \Gamma]|\theta^{s-1}, \theta_{is}$). For the first term, using U-ELC and F-BIR, we have

$$\text{(18)} \qquad \left|D_{it}^{\chi,\Gamma}\left(\theta^{t-1}, \theta_{it}\right)\right| \leq \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{t-1},\theta_{it}}\left[\sum_{\tau=t}^{\infty}\left|I_{i,(t),\tau}\left(\tilde{\theta}_i, \chi_i(\tilde{\theta})\right)\right| A_i \delta^\tau\right] \leq \delta^t A_i B_i,$$

where $A_i > 0$ is the constant of equi-Lipschitz continuity of U_i, and where $B_i > 0$ is the bound on the impulse responses in Condition F-BIR. This means that

$$\text{(19)} \qquad \left|\delta^{-t} Q_{it}^{\chi,\Gamma}\left(\theta^{t-1}, \theta_{it}\right)\right| \leq A_i B_i \left|\theta_{it} - \theta'_{it}\right| \leq A_i B_i \left(|\theta_{it}| + \left|\theta'_{it}\right|\right).$$

Hence, the expected NPV of the first term is finite by Condition F-BE and $\|\theta'_i\| < \infty$. For the second term, using (19) for $t+1$ and the law of iterated expectations, the expected NPV of its absolute value is bounded by

$$A_i B_i \left(\sum_{\tau=0}^{t-1} \delta^{\tau+1} \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{\tau-1},\theta_{i\tau}}[\tilde{\theta}_{i\tau+1}] + \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}\left[\|\tilde{\theta}_i\|\right] + \|\theta'_i\|\right),$$

which is finite by Condition F-BE and $\|\theta'_i\| < \infty$. Finally, the expected NPV of the third term is finite by Conditions U-SPR and F-BE.

We then show that $\text{ICFOC}_{i,s}$ holds for all i and s. Rewrite the time-s equilibrium expected payoff given history $(\theta^{s-1}, \theta_{is})$ using Fubini's theorem and the law of iterated expectations as

$$\begin{aligned}
&V_{is}^{\langle\chi,\Psi\rangle,\Gamma}\left(\theta^{s-1}, \theta_{is}\right) \\
&\quad = \lim_{T\to\infty} \sum_{t=0}^{T} \delta^t \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}\left[u_{it}\left(\tilde{\theta}^t, \chi^t\left(\tilde{\theta}^t\right)\right) + \psi_{it}\left(\tilde{\theta}^t\right)\right] \\
&\quad = \sum_{t=0}^{s-1} \delta^t \left(\mathbb{E}^{\Gamma_i(\theta_i^{s-1}, \chi_i^{s-1}(\theta^{s-1}))}\left[u_{it}\left(\theta_i^t, \tilde{\theta}_{-i}^t, \chi^t\left(\theta_i^t, \tilde{\theta}_{-i}^t\right)\right)\right] + \psi_{it}\left(\theta^t\right)\right) \\
&\qquad + Q_{is}^{\chi,\Gamma}\left(\theta^{s-1}, \theta_{is}\right) - \lim_{T\to\infty} \mathbb{E}^{\lambda_i[\chi,\Gamma]|\theta^{s-1},\theta_{is}}\left[Q_{i,T+1}^{\chi,\Gamma}\left(\tilde{\theta}^T, \tilde{\theta}_{i,T+1}\right)\right].
\end{aligned}$$

(The expectations of the other terms for $t \geq s$ cancel out by the law of iterated expectations.) The second line is independent of θ_{is}, and the limit on the last line equals zero by (19), Condition F-BE, and $\|\theta'_i\| < \infty$. By (6) and (18), the remaining term $Q_{is}^{\chi,\Gamma}(\theta^{s-1},\theta_{is})$ is Lipschitz continuous in θ_{is} and its derivative equals $D_{is}^{\chi,\Gamma}(\theta^{s-1},\theta_{is})$ a.e., which is the right-hand side of (1).

For part (ii), we start by considering a single-agent environment and then extend the result to multiple agents under the no-leakage condition.

Consider the single-agent case, where beliefs are vacuous, and omit the agent index to simplify notation. For any PBIC choice rules $\langle\chi,\psi\rangle$ and $\langle\chi,\bar{\psi}\rangle$ with the same allocation rule χ, for all $s\geq 0$ and $\theta^s\in\Theta^s$, ICFOC$_s$ and the law of iterated expectations imply

$$\begin{aligned}
&V_s^{\langle\chi,\psi\rangle}(\theta^s)-V_{s-1}^{\langle\chi,\psi\rangle}(\theta^{s-1})\\
&\quad=V_s^{\langle\chi,\psi\rangle}(\theta^s)-\mathbb{E}^{F_s(\theta^{s-1},\chi^{s-1}(\theta^{s-1}))}\big[V_s^{\langle\chi,\psi\rangle}(\theta^{s-1},\tilde{\theta}_s)\big]\\
&\quad=\mathbb{E}^{F_s(\theta^{s-1},\chi^{s-1}(\theta^{s-1}))}\left[\int_{\tilde{\theta}_s}^{\theta_s}D_s^{\chi}(\theta^{s-1},q)\,dq\right]\\
&\quad=V_s^{\langle\chi,\bar{\psi}\rangle}(\theta^s)-V_{s-1}^{\langle\chi,\bar{\psi}\rangle}(\theta^{s-1}).
\end{aligned}$$

Substituting the definitions of expected payoffs and rearranging terms yields

$$\begin{aligned}
&\mathbb{E}^{\lambda[\chi]|\theta^s}\left[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right]-\mathbb{E}^{\lambda[\chi]|\theta^s}\left[\sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\tilde{\theta})\right]\\
&\quad=\mathbb{E}^{\lambda[\chi]|\theta^{s-1}}\left[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right]-\mathbb{E}^{\lambda[\chi]|\theta^{s-1}}\left[\sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\tilde{\theta})\right].
\end{aligned}$$

By induction, we then have, for all $T\geq 1$ and $\theta^T\in\Theta^T$,

$$\begin{aligned}
&\mathbb{E}^{\lambda[\chi]|\theta^T}\left[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right]-\mathbb{E}^{\lambda[\chi]|\theta^T}\left[\sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\tilde{\theta})\right]\\
&\quad=\mathbb{E}^{\lambda[\chi]}\left[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right]-\mathbb{E}^{\lambda[\chi]}\left[\sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\tilde{\theta})\right]\\
&\quad\equiv K.
\end{aligned}\tag{20}$$

Since the payoff from truthtelling in a PBIC mechanism is well defined, we have the following lemma.

LEMMA A.2: *Suppose ψ is the transfer rule in a PBIC mechanism. Then for $\lambda[\chi]$-almost all θ, $\mathbb{E}^{\lambda[\chi]|\theta^T}[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})]\to\sum_{t=0}^{\infty}\delta^t\psi_t(\theta)$ as $T\to\infty$.*

PROOF: By the law of iterated expectations,

$$\begin{aligned}&\mathbb{E}^{\lambda[\chi]}\left[\left|\mathbb{E}^{\lambda[\chi]|\tilde{\theta}^T}\left[\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right]-\sum_{t=0}^{\infty}\delta^t\psi_t(\tilde{\theta})\right|\right]\\&\quad=\mathbb{E}^{\lambda[\chi]}\left[\left|\mathbb{E}^{\lambda[\chi]|\tilde{\theta}^T}\left[\sum_{t=T+1}^{\infty}\delta^t\psi_t(\tilde{\theta}^t)\right]-\sum_{t=T+1}^{\infty}\delta^t\psi_t(\tilde{\theta}^t)\right|\right]\\&\quad\le 2\mathbb{E}^{\lambda[\chi]}\left[\sum_{t=T+1}^{\infty}\delta^t\left|\psi_t(\tilde{\theta}^t)\right|\right].\end{aligned}$$

By PBIC, $\mathbb{E}^{\lambda[\chi]}[\|\psi(\tilde{\theta})\|]<\infty$ and, hence, the term on the second line goes to zero as $T\to\infty$. *Q.E.D.*

By Lemma A.2, we can take the limit $T\to\infty$ in (20) to get

$$\sum_{t=0}^{\infty}\delta^t\psi_t(\theta)-\sum_{t=0}^{\infty}\delta^t\bar{\psi}_t(\theta)=K\quad\text{for }\lambda[\chi]\text{-almost all }\theta.$$

To extend the result to multiple agents under the no-leakage condition, observe that if $\langle\chi,\psi\rangle$ and $\langle\chi,\bar{\psi}\rangle$ are PBIC, then they remain PBIC also in the "blind" setting where agent i does not observe his allocation x_i. (Hiding the allocation x_i from agent i simply amounts to pooling some of his incentive constraints.) Furthermore, if the allocation rule χ leaks no information to agent i so that observing the true type θ_i does not reveal any information about θ_{-i}, then we can interpret the blind setting as a single-agent setting in which agent i's allocation in period t is simply his report $\hat{\theta}_{it}$ and his utility is $\hat{U}_i(\theta_i,\hat{\theta}_i)=\mathbb{E}^{\lambda_i[\chi]|\hat{\theta}_i}[U_i(\theta_i,\tilde{\theta}_{-i},\chi(\hat{\theta}_i,\tilde{\theta}_{-i}))]$, where $\lambda_i[\chi]|\hat{\theta}_i$ denotes the probability measure over the other agents' types when agent i's reports are fixed at $\hat{\theta}_i$. (Intuitively, the other agents' types can be viewed as being realized only after agent i has finished reporting, and $\hat{U}_i$ is the expectation taken over such realizations.) Applying to this setting the result established above for the single-agent case, we see that agent i's expected payment $\mathbb{E}^{\lambda_i[\chi]|\theta_i}[\sum_{t=0}^{\infty}\delta^t\psi_{it}(\theta_i,\tilde{\theta}_{-i})]$ is pinned down, up to a constant, by the allocation rule χ with probability 1. *Q.E.D.*

PROOF OF THEOREM 3: Given a choice rule $\langle\chi,\psi\rangle$ and belief system $\Gamma\in\boldsymbol{\Gamma}(\chi)$, for all $i=1,\dots,n$, $t\ge 0$, and $(\theta_i^t,(\theta_i^{t-1},\hat{\theta}_{it}),\theta_{-i}^{t-1})\in\Theta_i^t\times\Theta_i^t\times\Theta_{-i}^{t-1}$, let

$$\Phi_i(\theta_{it},\hat{\theta}_{it})\equiv V_{it}^{\langle\chi\circ\hat{\theta}_{it},\psi\circ\hat{\theta}_{it}\rangle,\Gamma}\left(\theta^{t-1},\theta_{it}\right).$$

That is, $\Phi_i(\theta_{it},\hat{\theta}_{it})$ denotes agent i's expected payoff from reporting $\hat{\theta}_{it}$ at the period-t history $(\theta_i^t,\theta_i^{t-1},\chi_i^{t-1}(\theta_i^{t-1},\theta_{-i}^{t-1}))$ and then reverting to truthful reporting period $t+1$ onward. Because the environment is Markov, $\Phi_i(\theta_{it}, \hat{\theta}_{it})$ depends on agent i's past reports, but not on his past true types, and, hence, it gives agent i's payoff from reporting $\hat{\theta}_{it}$ at the history $((\bar{\theta}_i^{t-1}, \theta_{it}), \theta_i^{t-1}, \chi_i^{t-1}(\theta_i^{t-1}, \theta_{-i}^{t-1}))$ for any past true types $\bar{\theta}_i^{t-1} \in \Theta_i^{t-1}$. We then let $\bar{\Phi}_i(\theta_{it}) \equiv \Phi_i(\theta_{it}, \theta_{it})$ for all $\theta_{it} \in \Theta_{it}$ denote the payoff from reporting truthfully in all periods $s \geq t$.

Necessity: Fix $i = 1, \dots, n$ and $t \geq 0$. Suppose that the allocation rule $\chi \in \mathcal{X}$ with belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ can be implemented in an on-path truthful PBE. Then there exists a transfer rule ψ such that the choice rule $\langle \chi, \psi \rangle$ with belief system Γ is PBIC, and thus satisfies ICFOC$_{i,t}$ by Theorem 1. This implies that $\bar{\Phi}_i(\cdot)$ satisfies condition (b) in Lemma 1 with $\bar{\Phi}_i'(\theta_{it}) = D_{it}^{\chi,\Gamma}(\theta^{t-1}, \theta_{it})$ for a.e. θ_{it}. Thus, it remains to establish condition (a).

LEMMA A.3: *Suppose the environment is regular and Markov. Fix $i = 1, \dots, n$ and $t \geq 0$. If the choice rule $\langle \chi, \psi \rangle$ with belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ satisfies* ICFOC$_{i,t+1}$, *then for all $\hat{\theta}_{it} \in \Theta_{it}$, the choice rule $\langle \chi \circ \hat{\theta}_{it}, \psi \circ \hat{\theta}_{it} \rangle$ with belief system Γ satisfies* ICFOC$_{i,t}$.

PROOF: Note first that, because the environment is Markov and $\langle \chi, \psi \rangle$ with belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ satisfies ICFOC$_{i,t+1}$, the choice rule $\langle \hat{\chi}, \hat{\psi} \rangle \equiv \langle \chi \circ \hat{\theta}_{it}, \psi \circ \hat{\theta}_{it} \rangle$ with belief system Γ clearly satisfies ICFOC$_{i,t+1}$; this is because agent i's payoff does not depend on whether the previous period report $\hat{\theta}_{it}$ has been truthful or not. To show that it also satisfies ICFOC$_{i,t}$, we can use a state representation and the law of iterated expectations to write agent i's expected payoff from truthtelling under choice rule $\langle \hat{\chi}, \hat{\psi} \rangle$, for all $(\theta^{t-1}, \theta_{it})$, as

$$
\begin{aligned}
&V_{it}^{\langle \hat{\chi}, \hat{\psi} \rangle, \Gamma}(\theta^{t-1}, \theta_{it}) \\
&\quad = \mathbb{E}_{\tilde{\theta}_{-i}^t} \mathbb{E}_{\tilde{\varepsilon}_i^{t+1}} \big[V_{i,t+1}^{\langle \hat{\chi}, \hat{\psi} \rangle, \Gamma}\big((\theta_i^t, \tilde{\theta}_{-i}^t), Z_{i,(t),t+1}\big(\theta_{it}, \hat{\chi}_i^t(\theta_i^t, \tilde{\theta}_{-i}^t), \tilde{\varepsilon}_i^{t+1}\big)\big)\big],
\end{aligned}
$$

where $\tilde{\theta}_{-i}^t$ is generated by drawing θ_{-i}^{t-1} according to agent i's belief $\Gamma_i(\theta_i^{t-1}, \chi_i^{t-1}(\theta^{t-1}))$ and then drawing $\tilde{\theta}_{-i,t}$ from $\prod_{j \neq i} F_{jt}(\theta_j^{t-1}, \chi_j^{t-1}(\theta^{t-1}))$. To differentiate this identity with respect to the true period-t type θ_{it}, note first that by the chain rule, we have

$$
\begin{aligned}
&\frac{d}{d\theta_{it}} \big[V_{i,t+1}^{\langle \hat{\chi}, \hat{\psi} \rangle, \Gamma}\big(\theta^t, Z_{i,(t),t+1}\big(\theta_{it}, \hat{\chi}_i^t(\theta_i^t, \theta_{-i}^t), \varepsilon_{i,t+1}\big)\big)\big] \\
&\quad = \mathbb{E}^{\lambda_i[\hat{\chi}, \Gamma] | \theta^t, \theta_{i,t+1}} \left[\frac{\partial U_i(\tilde{\theta}, \hat{\chi}(\tilde{\theta}))}{\partial \theta_{it}} \right] \\
&\qquad + D_{i,t+1}^{\hat{\chi}, \Gamma}(\theta^t, \theta_{i,t+1}) \frac{\partial Z_{i,(t),t+1}(\theta_{it}, \chi_i^t(\theta_i^t, \theta_{-i}^t), \varepsilon_{i,t+1})}{\partial \theta_{it}}.
\end{aligned}
$$

To see this, note that the first term follows because the environment is Markov and $\langle\hat{\chi},\hat{\psi}\rangle$ does not depend on θ_{it} so that

$$\frac{\partial V_{i,t+1}^{\langle\hat{\chi},\hat{\psi}\rangle,\Gamma}(\theta^t,\theta_{i,t+1})}{\partial\theta_{it}}=\mathbb{E}^{\lambda_i[\hat{\chi},\Gamma]|\theta^t,\theta_{i,t+1}}\left[\frac{\partial U_i(\tilde{\theta},\hat{\chi}(\tilde{\theta}))}{\partial\theta_{it}}\right].$$

The second term follows because, by ICFOC$_{i,t+1}$, $\partial V_{i,t+1}^{\langle\hat{\chi},\hat{\psi}\rangle,\Gamma}(\theta^t,\theta_{i,t+1})/\partial\theta_{i,t+1}=D_{i,t+1}^{\hat{\chi},\Gamma}(\theta^t,\theta_{i,t+1})$. Furthermore, by U-ELC, ICFOC$_{i,t+1}$, and F-BIR, all the derivatives above are bounded. Thus, by the dominated convergence theorem, we can pass the derivative through the expectation to get

$$\begin{aligned}&\frac{dV_{it}^{\langle\hat{\chi},\hat{\psi}\rangle,\Gamma}(\theta^{t-1},\theta_{it})}{d\theta_{it}}\\&\quad=\mathbb{E}^{\lambda_i[\hat{\chi},\Gamma]|\theta^{t-1},\theta_{it}}\Bigg[\frac{\partial U_i(\tilde{\theta},\hat{\chi}(\tilde{\theta}))}{\partial\theta_{it}}+I_{i,(t),t+1}\big(\tilde{\theta}_i^{t+1},\hat{\chi}_i^t\big(\theta_i^t,\theta_{-i}^t\big)\big)\\&\qquad\times\sum_{\tau=t+1}^{\infty}I_{i,(t+1),\tau}\big(\tilde{\theta}_i^\tau,\hat{\chi}_i^{\tau-1}(\tilde{\theta})\big)\frac{\partial U_i(\tilde{\theta},\hat{\chi}(\tilde{\theta}))}{\partial\theta_{i\tau}}\Bigg]\\&\quad=\mathbb{E}^{\lambda_i[\hat{\chi},\Gamma]|\theta^{t-1},\theta_{it}}\left[\sum_{\tau=t}^{\infty}I_{i,(t),\tau}\big(\tilde{\theta}_i^\tau,\hat{\chi}_i^{\tau-1}(\tilde{\theta})\big)\frac{\partial U_i(\tilde{\theta},\hat{\chi}(\tilde{\theta}))}{\partial\theta_{i\tau}}\right],\end{aligned}$$

where we have first used (2) to express the expectation in terms of impulse responses, and then the fact that Markovness implies $I_{i,(t),t+1}(\theta_i^{t+1},x_i^t)I_{i,(t+1),\tau}(\theta_i^\tau,x_i^{\tau-1})=I_{i,(t),\tau}(\theta_i^\tau,x_i^{\tau-1})$. Therefore, the choice rule $\langle\hat{\chi},\hat{\psi}\rangle$ with belief system Γ satisfies ICFOC$_{i,t}$. *Q.E.D.*

Since $\langle\chi,\psi\rangle$ with belief system Γ satisfies ICFOC by Theorem 1, Lemma A.3 implies that for all $\hat{\theta}_{it}\in\Theta_{it}$, $\langle\chi\circ\hat{\theta}_{it},\psi\circ\hat{\theta}_{it}\rangle$ with belief system Γ satisfies ICFOC$_{i,t}$. Therefore, for any fixed $\hat{\theta}_{it}$, $\Phi_i(\theta_{it},\hat{\theta}_{it})$ is Lipschitz continuous in θ_{it} with derivative given by $D_{it}^{\chi\circ\hat{\theta}_{it},\Gamma}(\theta^{t-1},\theta_{it})$ for a.e. θ_{it}. Hence, also condition (a) of Lemma 1 is satisfied. Since i and t were arbitrary, we conclude that the integral monotonicity condition (8) is a necessary condition for on-path truthful PBE implementability.

Sufficiency: Suppose the allocation rule $\chi\in\mathcal{X}$ with belief system $\Gamma\in\boldsymbol{\Gamma}(\chi)$ satisfies integral monotonicity. Define the transfer rule ψ by (7). By Theorem 2, the choice rule $\langle\chi,\psi\rangle$ with belief system Γ satisfies ICFOC. Thus, the above arguments show that, for all $i=1,\dots,n$, $t\geq 0$, and any period-t history of agent i, the functions $\{\Phi_i(\cdot,\hat{\theta}_{it})\}_{\hat{\theta}_{it}\in\Theta_{it}}$ and $\bar{\Phi}_i(\cdot)$ satisfy conditions (a) and (b) of Lemma A.3. This implies that a one-step deviation from the strong truthtelling strategy is not profitable for agent i at *any* history in any period. The following version of the one-stage deviation principle then rules out multistep deviations.

LEMMA A.4: *Suppose the environment is regular and Markov. Fix an allocation rule $\chi \in \mathcal{X}$ with belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ and define the transfer rule ψ by (7). If a one-stage deviation from strong truthtelling is not profitable at any information set, then arbitrary deviations from strong truthtelling are not profitable at any information set.*

The proof of this lemma consists of showing that despite payoffs being not a priori continuous at infinity, the bounds implied by U-SPR and part (ii) of the definition of Markov environments guarantee that under the transfers defined by (7), continuation utility is well behaved. We relegate the argument to the Supplemental Material.

We conclude that integral monotonicity is a sufficient condition for the allocation rule $\chi \in \mathcal{X}$ with belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$ to be implementable in a strongly truthful PBE. *Q.E.D.*

PROOF OF PROPOSITION 1: Case (i). We construct a nondecreasing solution $\chi_s(\theta)$ sequentially for $s = 0, 1, \dots$. Suppose we have a solution χ in which $\chi^{s-1}(\theta)$ is nondecreasing. Consider the problem of choosing the optimal continuation allocation rule in period s given type history θ^s and allocation history $\chi^{s-1}(\theta^{s-1})$. Using the state representation $\langle \mathcal{E}_i, G_i, z_i \rangle_{i=1}^n$ from period s onward, we can write the continuation rule for $t \geq s$ as a collection of functions $\hat{\chi}_t(\varepsilon)$ of the shocks ε.

First, note that, by assumption, X is a lattice. Hence there is a way of extending the join and meet operations so that $\prod_{t \geq s} X_t$ is a lattice. This means that the set of feasible shock-contingent plans $\hat{\chi}$ is also a lattice under pointwise meet and join operations (i.e., for each ε). Next, note that, under the assumptions in the proposition, each agent i's virtual utility

$$U_i\big(Z_s\big(\theta^s, \varepsilon\big), \chi^{s-1}\big(\theta^{s-1}\big), x^{\geq s}\big) \\ - \frac{1}{\eta_{i0}(\theta_{i0})} \sum_{t=0}^{\infty} \frac{\partial U_i(Z_s(\theta^s, \varepsilon), \chi^{s-1}(\theta^{s-1}), x^{\geq s})}{\partial \theta_{it}} I_{i,(0),t}\big(Z^t_{i,(s)}\big(\theta^s, \varepsilon\big)\big)$$

is supermodular in $x^{\geq s}$ and has increasing differences in $(\theta^s, x^{\geq s})$ (note that $Z_s(\theta^s, \varepsilon)$ is nondecreasing in θ^s by F-FOSD and that $\chi^{s-1}(\theta^{s-1})$ is nondecreasing in θ^{s-1} by construction). Therefore, summing over i and taking expectation over ε, we obtain that the expected virtual surplus starting with history θ^s is supermodular in the continuation plan $\hat{\chi}$ and has increasing differences in $(\theta^s, \hat{\chi})$. Topkis's theorem then implies that the set of optimal continuation plans is nondecreasing in θ^s in the strong set order. In particular, focus on the first component $\chi_s \in X_s$ of such plans. By Theorem 2 of Kukushkin (2009), there exists a nondecreasing selection of optimal values, $\hat{\chi}_s(\theta^s)$. Therefore, the relaxed program has a solution in which $\chi^s(\theta^s) = (\chi^{s-1}(\theta^{s-1}), \hat{\chi}_s(\theta^s))$ is nondecreasing in θ^s.

Case (ii). In this case, the solution to the relaxed problem is a collection of independent rules χ_t, $t \geq 0$, with each χ_t satisfying, for λ-almost every θ^t,

$$\chi_t(\theta^t) \in \arg\max_{x_t \in X_t}\left[\sum_{i=0}^{n} u_{it}(\theta_t, x_t) - \sum_{i=1}^{n} \frac{1}{\eta_{i0}(\theta_{i0})} I_{i,(0),t}(\theta_i^t) \frac{\partial u_{it}(\theta_t, x_t)}{\partial \theta_{it}}\right]. \tag{21}$$

For any t, we can put $\chi_t(\theta^t) = \bar{\chi}_t(\varphi_{1t}(\theta_1^t), \ldots, \varphi_{nt}(\theta_n^t))$ for some $\bar{\chi}_t : \mathbb{R}^{n\times m} \to X_t$. Now fix $i \geq 1$ and, for $x_{it} \in X_{it}$, let $X_t(x_{it}) \equiv \{x_t' \in X_t : x_{it}' = x_{it}\}$. Then (21) implies

$$\bar{\chi}_{it}(\varphi_t) \in \arg\max_{x_{it} \in X_{it}}\left[\bar{u}_{it}(\varphi_{it}, x_{it}) + g_{it}(\varphi_{-i,t}, x_{it})\right],$$

where $\bar{u}_{it}(\varphi_{it}, x_{it})$ is the virtual utility of agent i and

$$g_{it}(\varphi_{-i,t}, x_{it}) \equiv \max_{x_t' \in X_t(x_{it})}\left[u_{0t}(x_t') + \sum_{j\neq i} \bar{u}_{jt}(\varphi_{jt}, x_{jt}')\right].$$

Since $\bar{u}_{it}(\varphi_{it}, x_{it}) + g_{it}(\varphi_{-i,t}, x_{it})$ has strictly increasing differences in (φ_{it}, x_{it}), by the monotone selection theorem of Milgrom and Shannon (1994), $\bar{\chi}_{it}(\varphi_{it}, \varphi_{-i,t})$ must be nondecreasing in φ_{it} and so $\chi_{it}(\theta_i^t, \theta_{-i}^t)$ is nondecreasing in θ_i^t. *Q.E.D.*

PROOF OF PROPOSITION 2: Fix a belief system $\Gamma \in \boldsymbol{\Gamma}(\chi)$. We show that the virtual index policy given by (16) satisfies average monotonicity: For all $i = 1, \ldots, n$, $s \geq 0$, and $(\theta^{s-1}, \theta_{is}) \in \Theta^{s-1} \times \Theta_{is}$,

$$\mathbb{E}^{\lambda_i[\chi\circ\hat{\theta}_s,\Gamma]|\theta^{s-1},\theta_{is}}\left[\sum_{t=s}^{\infty} \delta^t (\chi \circ \hat{\theta}_{is})_{it}(\tilde{\theta})\right]$$

is nondecreasing in $\hat{\theta}_{is}$. We show this for $s = 0$. The argument for $s > 0$ is analogous but simpler since θ_{is} does not affect the term $\eta_{i0}^{-1}(\theta_{i0})$ in the definition of the virtual index (15) when $s > 0$.

We can think of the processes being generated as follows: First, draw a sequence of innovations $\omega_i = (\omega_{ik})_{k=1}^{\infty}$ according to $\prod_{k=1}^{\infty} R_i(\cdot|k)$ for each i, independently across $i = 1, \ldots, n$, and draw initial types θ_{i0} according to F_{i0} independently of the innovations ω_i and across i. Letting $K_t \equiv \sum_{\tau=1}^{t} x_\tau$, bidder i's type in period t can then be described as

$$\theta_{it} = \theta_{i0} + \sum_{k=1}^{K_t} \omega_{ik}.$$

Clearly this representation generates the same conditional distributions (and hence the same process) as the kernels defined in the main text.[41]

Next, fix an arbitrary bidder $i = 1, \ldots, n$ and a state $(\theta_0, \omega) \in \Theta_0 \times (\mathbb{R}^n)^\infty$, and take a pair $\theta'_{i0}, \theta''_{i0} \in \Theta_0$ with $\theta''_{i0} > \theta'_{i0}$. We show by induction on k that for any $k \in \mathbb{N}$, the kth time that i wins the object if he initially reports θ''_{i0} (and reports truthfully in each period $t > 0$) comes weakly earlier than if he reports θ'_{i0}. As the realization $(\theta_0, \omega) \in \Theta_0 \times (\mathbb{R}^n)^\infty$ is arbitrary, this implies that the expected time to the kth win is decreasing in the initial report, which in turn implies that the virtual policy χ satisfies average monotonicity.

As a preliminary observation, note that the period-t virtual index of bidder i is increasing in the (reported) period-0 type θ_{i0} since the handicap $\eta_{i0}^{-1}(\theta_{i0})$ is decreasing in θ_{i0} and (in the case $t = 0$) $\mathbb{E}^{\lambda[\bar{\chi}_i]|\theta_{i0}}[\theta_{i\tau}]$ is increasing in θ_{i0} for all $\tau \geq 0$.

Base Case. Suppose, toward a contradiction, that the first win given initial report θ'_{i0} comes in period t', whereas it comes in period $t'' > t'$ given report $\theta''_{i0} > \theta'_{i0}$. As the realization (θ_0, ω) is fixed, the virtual indices of bidders $-i$ in period t' are the same in both cases. But $\gamma_{it'}((\theta''_{i0}, \theta_{i0}, \ldots, \theta_{i0}), 0) > \gamma_{it'}((\theta'_{i0}, \theta_{i0}, \ldots, \theta_{i0}), 0)$, implying that i must win in period t' also with initial report θ''_{i0}, which contradicts $t'' > t'$.

Induction Step. Suppose the claim is true for some $k \geq 1$. Suppose, toward a contradiction, that the $(k+1)$th win given report θ'_{i0} comes in period t', whereas it comes in period $t'' > t'$ given $\theta''_{i0} > \theta'_{i0}$. Then observe that (i) in both cases, i wins the auction k times prior to period t'. Furthermore, since the realization (θ_0, ω) is fixed, this implies that (ii) bidder i's current type θ_{it} is the same in both cases and (iii) the number of times each bidder $j \neq i$ wins the object prior to period t' is the same in both cases, and, hence, the virtual indices of bidders $-i$ in period t' are the same in both cases. By (i) and (ii), i's virtual index in period t' is identical in both cases except for the initial report. That bidder i's period-t' index is increasing in the initial report, along with (iii), implies that i must then win in period t' also with initial report θ''_{i0}, contradicting $t'' > t'$. Hence, the claim is true for $k+1$. *Q.E.D.*

REFERENCES

ALBANESI, S., AND C. SLEET (2006): "Dynamic Optimal Taxation With Private Information," *Review of Economic Studies*, 73 (1), 1–30. [606]

ATHEY, S., AND I. SEGAL (2013): "An Efficient Dynamic Mechanism," *Econometrica*, 81 (6), 2463–2485. [606,608,625,636]

[41]The difference is that in this representation, the innovation to bidder i's value if he wins the auction for the kth time in period t is given by the kth element of the sequence ω_i, whereas in the representation in the main text, it is given by (a function of) the tth element of the sequence ε_i. In terms of the latter case, a higher current message increases discounted consumption only on average, whereas in the former, it increases discounted consumption for each realization of ω_i, since the bidder always experiences the same innovation sequence irrespective of the timing of consumption.

ATKESON, A., AND R. E. LUCAS, JR. (1992): "On Efficient Distribution With Private Information," *Review of Economic Studies*, 59 (3), 427–453. [606]

BARON, D., AND D. BESANKO (1984): "Regulation and Information in a Continuing Relationship," *Information Economics and Policy*, 1, 447–470. [605,614,618]

BATTAGLINI, M. (2005): "Long-Term Contracting With Markovian Consumers," *American Economic Review*, 95 (3), 637–658. [605,625]

——— (2007): "Optimality and Renegotiation in Dynamic Contracting," *Games and Economic Behavior*, 60 (2), 213–246. [606]

BATTAGLINI, M., AND R. LAMBA (2012): "Optimal Dynamic Contracting," Report, Princeton University. [629]

BERGEMANN, D., AND M. SAID (2011): "Dynamic Auctions," in *Wiley Encyclopedia of Operations Research and Management Science*. Hoboken, NJ: Wiley. [606]

BERGEMANN, D., AND J. VÄLIMÄKI (2008): "Bandit Problems," in *New Palgrave Dictionary of Economics*, ed. by S. Durlauf and L. Blume. New York: Palgrave Macmillan. [636]

——— (2010): "The Dynamic Pivot Mechanism," *Econometrica*, 78 (2), 771–789. [605,606,625, 636,637]

BERGER, A., R. MÜLLER, AND S. H. NAEEMI (2010): "Path-Monotonicity and Truthful Implementation," Report, Maastricht University. [603]

BESANKO, D. (1985): "Multiperiod Contracts Between Principal and Agent With Adverse Selection," *Economics Letters*, 17, 33–37. [605,614,616,632]

BILLINGSLEY, P. (1995): *Probability and Measure* (Third Ed.). New York: Wiley. [609]

BOARD, S. (2007): "Selling Options," *Journal of Economic Theory*, 136, 324–340. [605,637]

BOARD, S., AND A. SKRZYPACZ (2013): "Revenue Management With Forward-Looking Buyers," Report, Stanford University and UCLA. [606]

CARBAJAL, J. C., AND J. ELY (2013): "Mechanism Design Withouth Revenue Equivalence," *Journal of Economic Theory*, 148 (1), 104–133. [603,621]

COURTY, P., AND H. LI (2000): "Sequential Screening," *Review of Economic Studies*, 67, 697–717. [605,614,615,618,625,630,633]

CREMER, J., AND R. P. MCLEAN (1988): "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56 (6), 1247–1257. [602,608]

DUDLEY, R. (2002): *Real Analysis and Probability*. Cambridge: Cambridge University Press. [613]

ESŐ, P., AND B. SZENTES (2007): "Optimal Information Disclosure in Auctions and the Handicap Auction," *Review of Economic Studies*, 74 (3), 705–731. [602,605,609,614,618,625,627,630, 637-639]

——— (2013): "Dynamic Contracting With Adverse Selection: An Irrelevance Result," Report, LSE and Oxford University. [627,638,639]

FARHI, E., AND I. WERNING (2013): "Insurance and Taxation Over the Life Cycle," *Review of Economic Studies*, 80 (2), 596–635. [606,620]

FERNANDES, A., AND C. PHELAN (2000): "A Recursive Formulation for Repeated Agency With History Dependence," *Journal of Economic Theory*, 91 (2), 223–247. [606]

FUDENBERG, D., AND J. TIROLE (1991): *Game Theory*. Cambridge, MA: MIT Press. [612,622]

GARCIA, D. (2005): "Monotonicity in Direct Revelation Mechanisms," *Economic Letters*, 88 (1), 21–26. [623]

GARRETT, D. (2011): "Durable-Good Sales With Changing Values and Unobservable Arrival Dates," Report, Northwestern University. [606]

GARRETT, D., AND A. PAVAN (2013): "Dynamic Managerial Compensation: On the Optimality of Seniority-Based Schemes," Report, Northwestern University. [608,620,639]

GERSHKOV, A., AND B. MOLDOVANU (2009): "Dynamic Revenue Maximization With Heterogenous Objects: A Mechanism Design Approach," *American Economic Journal: Microeconomics*, 1 (2), 168–198. [606]

GOLOSOV, M., M. TROSHKIN, AND A. TSYVINSKI (2011): "Optimal Dynamic Taxes," Report, Yale University. [606]

GREEN, E. J. (1987): "Lending and the Smoothing of Uninsurable Income," in *Contractual Agreements for Intertemporal Trade*, ed. by E. Prescott and N. Wallace. Minneapolis: University of Minnesota Press. [606]

HENDEL, I., AND A. LIZZERI (2003): "The Role of Commitment in Dynamic Contracts: Evidence From Life Insurance," *Quarterly Journal of Economics*, 118 (1), 299–327. [626]

KAKADE, S., I. LOBEL, AND H. NAZERZADEH (2011): "Optimal Dynamic Mechanism Design and the Virtual Pivot Mechanism," Report, Wharton School, University of Pennsylvania. [605,637, 638]

KAPICKA, M. (2013): "Efficient Allocations in Dynamic Private Information Economies With Persistent Shocks: A First-Order Approach," *Review of Economic Studies*, 80 (3), 1027–1054. [606]

KOCHERLAKOTA, N. R. (2005): "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation," *Econometrica*, 73 (5), 1587–1621. [606]

KUKUSHKIN, N. S. (2009): "On the Existence of Monotone Selections," Report. [648]

LAFFONT, J.-J., AND J. TIROLE (1988): "The Dynamics of Incentive Contracts," *Econometrica*, 56 (5), 1153–1175. [606]

MATTHEWS, S., AND J. MOORE (1987): "Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multidimensional Screening," *Econometrica*, 55 (2), 441–467. [623]

MILGROM, P., AND I. SEGAL (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70 (2), 583–601. [610,614,615,642]

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62 (1), 157–180. [649]

MIRRLEES, J. A. (1971): "An Exploration in the Theory of Optimum Income Taxation," *Review of Economic Studies*, 38 (114), 175–208. [602,614]

MYERSON, R. (1981): "Optimal Auction Design," *Mathematics of Operation Research*, 6 (1), 58–78. [602,614,620]

——— (1986): "Multistage Games With Communication," *Econometrica*, 54 (2), 323–358. [611]

PAVAN, A., I. SEGAL, AND J. TOIKKA (2014): "Supplement to 'Dynamic Mechanism Design: A Myersonian Approach'," *Econometrica Supplemental Material*, 82, http://www.econometricsociety.org/ecta/supmat/10269_proofs.pdf. [603,604]

POLLARD, D. (2002): *A User's Guide to Measure Theoretic Probability*. Cambridge: Cambridge University Press. [608]

RAHMAN, D. (2010): "Dynamic Implementation," Report, University of Minnesota. [606]

RIORDAN, M., AND D. SAPPINGTON (1987): "Awarding Monopoly Franchises," *American Economic Review*, 77 (3), 375–387. [605]

ROCHET, J.-C., AND L. STOLE (2003): "The Economics of Multidimensional Screening," in *Advances in Economics and Econometrics: Theory and Applications—Eight World Congress of Econometric Society*, ed. by M. Dewatripont, L. P. Hansen, and S. Turnovsky. Cambridge: Cambridge University Press, 150–197. [606]

ROCHET, J. C. (1987): "A Necessary and Sufficient Condition for Rationalizability in a Quasi-Linear Context," *Journal of Mathematical Economics*, 16, 191–200. [603,606,621]

SKRETA, V. (2006): "Sequentially Optimal Mechanisms," *Review of Economic Studies*, 73 (4), 1085–1111. [606]

SKRZYPACZ, A., AND J. TOIKKA (2013): "Mechanisms for Repeated Bargaining," Report, MIT and Stanford. [626]

STRAUSZ, R. (2006): "Deterministic versus Stochastic Mechanisms in Principal-Agent Models," *Journal of Economic Theory*, 127 (1), 306–314. [629]

STRULOVICI, B. (2011): "Renegotiation-Proof Contracts With Moral Hazard and Persistent Private Information," Report, Northwestern University. [606]

WHITTLE, P. (1982): *Optimization Over Time*, Vol. 1. Chichester: Wiley. [636]

WILLIAMS, N. (2011): "Persistent Private Information," *Econometrica*, 79 (4), 1233–1275. [606]

Dept. of Economics, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, U.S.A.; alepavan@northwestern.edu,

Dept. of Economics, Stanford University, 579 Serra Mall, Stanford, CA 94305, U.S.A.; ilya.segal@stanford.edu,

and

Dept. of Economics, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge, MA 02142, U.S.A.; toikka@mit.edu.

Manuscript received August, 2011; final revision received December, 2013.