Mechanism Design - Problem Set 1
Solution

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1. **Auctions with Discrete Types.** In class, we showed that when the type distribution is continuous, revenue equivalence holds. Verify whether this is true when the type space is discrete.

Solution. The revenue equivalence theorem states the revenue from an auction only depends on its allocation, i.e., who buys the object, and the outside options of the buyers. This does not hold when the type space is discrete. To see this, suppose that we have one buyer! whose valuation is $v_i \in \{0, 1, \cdots, n\}$ for n being some integer. Then consider two "auctions" for the allocation $x(v) = \mathbf{1}[v \geq m]$ for some $m < n$:

$$p_1(v) = m\mathbf{1}[v \geq m]$$
$$p_2(v) = (m - 1/2)\mathbf{1}[v \geq m]$$

Both pricing functions make the allocation incentive compatible yet they imply different revenues.

2. **Optimal Taxation.** In class, we describe the problem of optimal taxation for a government that faces a population of workers (consisted of a unit continuum of workers. Note that we can allude to a version of law of large number and assume that this is equivalent to having one worker) with heterogeneous productivities, $\theta \in \mathbb{R}_+$, whose preferences over consumption and income are given by $c - v(y/\theta)$ with $v(l) = \psi l^\gamma/\gamma$ for $\gamma > 1, \psi > 0$. Suppose that $\theta \sim F(\theta)$ has a continuous distribution with $F(\cdot)$ as its c.d.f. The government is interested in choosing a non-linear tax function $T(\cdot)$ and workers choose how much income to earn while their productivity is only privately known to them. Suppose that the objective of the government is to maximize revenue from taxation.

   1. Define a direct mechanism.
   2. Use the revelation principle to cast the problem as a mechanism design problem.
   3. Show that any direct mechanism can be implemented with a tax function. This is often referred to as the taxation principle.

4. Use the envelope formulation to simplify the incentive constraints.
5. Use optimization technique introduced in lecture 1 to solve for optimal direct mechanism. *Hint: Ignore the monotonicity constraint.*
6. Suppose that $\text{Supp}(F) = \left[\underline{\theta}, \overline{\theta}\right]$. What is optimal marginal tax rate for the most productive worker?
7. Now suppose that $\text{Supp}(F) = [\underline{\theta}, \infty]$. What is the optimal marginal tax rate as $\theta \to \infty$?

Solution. A direct mechanism: $(c(\theta), y(\theta))_{\theta \in \text{Supp}(F)}$. The mechanism design problem is given by

$$\max \int [y(\theta) - c(\theta)]\, dF(\theta)$$

subject to

$$\begin{aligned} c(\theta) - v(y(\theta)/\theta) &\geq c(\theta') - v(y(\theta')/\theta), \forall \theta, \theta' \\ c(\theta) - v(y(\theta)/\theta) &\geq \underline{U} \end{aligned}$$

The last constraint is needed so that the government does not fully exploit workers.

Consider an incentive compatible direct mechanism given by $c(\theta), y(\theta)$. Incentive compatibility implies that if for two values of θ and θ', $y(\theta) = y(\theta')$, then $c(\theta) = c(\theta')$. Thus we can define a tax function

$$T(\hat{y}) = \begin{cases} y(\theta) - c(\theta) & \exists \theta, \hat{y} = y(\theta) \\ \infty & \nexists \theta, \hat{y} = y(\theta) \end{cases}$$

The property we mentioned before implies that this function is well-defined. The payoff of choosing $\hat{y}$ for a worker of type θ is given by

$$\hat{y} - T(\hat{y}) - v(\hat{y}/\theta)$$

Obviously if $\hat{y} \neq y\left(\hat{\theta}\right)$ for all $\hat{\theta}$, this payoff is negative infinity. If for some $\hat{\theta}, \hat{y} = y\left(\hat{\theta}\right)$, then given the definition of $T(\cdot)$, the above payoff is given by

$$y\left(\hat{\theta}\right) - T\left(y\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right) = c\left(\hat{\theta}\right) - v\left(y\left(\hat{\theta}\right)/\theta\right)$$

Incentive compatibility implies that the above is maximized at $\hat{\theta} = \theta$. This means that $T(\cdot)$ implements the initial allocation. Note that given this definition and assuming that allocations are differentiable with respect to θ, we have

$$1 - T'(y(\theta)) = \frac{1}{\theta} v'(y(\theta)/\theta)$$

We can write the optimal taxation problem as

$$\max \int [y(\theta) - c(\theta)]\, dF$$

subject to

$$
\begin{aligned}
U'(\theta) &= \frac{y(\theta)}{\theta^2} v'(y(\theta)/\theta) \\
U(\theta) &= c(\theta) - v(y(\theta)/\theta) \\
y(\theta) &: \text{increasing} \\
U(\theta) &\geq \underline{U}
\end{aligned}
$$

Similar to what we have done in class, we can solve for consumption from the envelope condition and have

$$
c(\theta) = v(y(\theta)/\theta) + \int_{\underline{\theta}}^{\theta} \frac{y\left(\hat{\theta}\right)}{\hat{\theta}^2} v'\left(\frac{y\left(\hat{\theta}\right)}{\hat{\theta}}\right) d\hat{\theta} + \underline{U}
$$

Replacing this into the objective function and ignoring the monotonicity constraint, we have

$$
\max_{y(\cdot)} \int_0^{\infty} \left[y(\theta) - v\left(\frac{y(\theta)}{\theta}\right) - \int_{\underline{\theta}}^{\theta} \frac{y\left(\hat{\theta}\right)}{\hat{\theta}^2} v'\left(\frac{y\left(\hat{\theta}\right)}{\hat{\theta}}\right) d\hat{\theta} - \underline{U} \right] dF(\theta)
$$

This can be solved by calculating Frechet derivatives of the above along the $\delta_\theta(\theta')$ for each θ and setting it equal to 0. Before that, we can use integration by parts and write it as

$$
\int_0^{\infty} \left[y(\theta) - v\left(\frac{y(\theta)}{\theta}\right) - \frac{y(\theta)}{\theta^2} v'\left(\frac{y(\theta)}{\theta}\right) \frac{1-F(\theta)}{f(\theta)} - \underline{U} \right] dF(\theta)
$$

Then the optimality condition is given by

$$
1 - \frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right) - \left\{ \frac{1}{\theta^2} v'\left(\frac{y(\theta)}{\theta}\right) + \frac{y(\theta)}{\theta^3} v''\left(\frac{y(\theta)}{\theta}\right) \right\} \frac{1-F(\theta)}{f(\theta)} = 0
$$

Given that $v(l) = \psi l^\gamma / \gamma$, we have

$$
\frac{1}{\theta^2} v'\left(\frac{y(\theta)}{\theta}\right) + \frac{y(\theta)}{\theta^3} v''\left(\frac{y(\theta)}{\theta}\right) = \psi \frac{y(\theta)^{\gamma-1}}{\theta^{1+\gamma}} \gamma = \gamma \frac{1}{\theta^2} v'\left(\frac{y(\theta)}{\theta}\right)
$$

So we can write the above as

$$
1 - \frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right) - \gamma \frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right) \frac{1-F(\theta)}{\theta f(\theta)} = 0
$$

This implies that

$$
\frac{1}{\frac{1}{\theta} v'\left(\frac{y(\theta)}{\theta}\right)} - 1 = \gamma \frac{1-F(\theta)}{\theta f(\theta)} \rightarrow \frac{1}{1-T'(y(\theta))} - 1 = \gamma \frac{1-F(\theta)}{\theta f(\theta)}
$$

Suppose that θ has a bounded distribution. Then at $\theta = \overline{\theta}$, we have

$$\lim_{\theta \to \overline{\theta}} \frac{1 - F(\theta)}{\theta f(\theta)} = 0$$

Hence, we must have

$$T'\left(y\left(\overline{\theta}\right)\right) = 0$$

With unbounded distribution, it is possible that $\frac{1-F(\theta)}{\theta f(\theta)}$ converges to a positive limit. For example, if F represents a pareto distribution satisfying

$$F(\theta) = 1 - \left(\frac{\underline{\theta}}{\theta}\right)^a \to \lim_{\theta \to \infty} \frac{\theta^{-a}}{a\theta^{-}} = \frac{1}{a}$$

In this case, marginal tax rate is given by

$$\frac{1}{1 - T'(\infty)} = \gamma\frac{1}{a} + 1 \to T'(\infty) = \frac{\frac{\gamma}{a}}{1 + \frac{\gamma}{a}} = \frac{\gamma}{\gamma + a}$$

3. Delegation. In a lot of economic settings of interest, contracting must be done without the use of monetary transfers. For example, consider the interaction between an employer (principal, P) and an employee (agent, A). Suppose that A can take an observable action $a \in [0, 2]$. The payoff for P from the action is $-(a - \theta)^2$ where $\theta \in [1, 2]$ is the state of the world. Suppose that $\theta \sim U[1, 2]$. On the other hand, suppose that A has a payoff of $-(a - \theta + b)^2$ for some $b \in (0, 1)$. The key assumption is that A knows the state while the employer does not. The choice of P is which actions to allow A to take. In other words, P chooses an action set $M \subset \mathbb{R}_+$ while A is free to choose whatever action she wants within the set M.

1. Use the relevation principle to cast the problem of P – finding the set M to maximize her expected payoff – as a direct mechanism design problem.
2. Similar to the auction model, simplify A's incentive constraint using an envelope and monotonicity condition.
3. Give two examples of incentive compatible mechanisms? How do you implement these using delegation sets?
4. Now focus on mechanisms that maximize the payoff of P. What can you say about the payoff of A when $\theta = 1$? Prove your statement.
5. Prove that the solution of P's problem is of the form $a(\theta) = 1, \theta \le \theta^*$ and $a(\theta) = \theta - b$ for $\theta \ge \theta^*$. Find θ^*.

Solution. *Note: The claim in the last part is wrong. In what follows I provide the correct answer.*

A direct mechanism in this case is simply a recommended action $a(\theta) \in [1, 2]$. Incentive compatbility is then given by

$$-(a(\theta) - \theta + b)^2 \ge -(a(\theta') - \theta + b)^2, \forall \theta, \theta'$$

If we rewrite the above for θ', θ, we have

$$-\left(a\left(\theta'\right)-\theta'+b\right)^2 \geq -\left(a\left(\theta\right)-\theta'+b\right)^2$$

Adding the two implies that

$$\left(a\left(\theta\right)-\theta'+b\right)^2-\left(a\left(\theta\right)-\theta+b\right)^2 \geq \left(a\left(\theta'\right)-\theta'+b\right)^2-\left(a\left(\theta'\right)-\theta+b\right)^2$$

or

$$\left(2a\left(\theta\right)-\theta'-\theta+2b\right)\left(\theta-\theta'\right) \geq \left(2a\left(\theta'\right)-\theta'-\theta+2b\right)\left(\theta-\theta'\right)$$

If $\theta > \theta'$, the above implies that $a\left(\theta\right) \geq a\left(\theta'\right)$. The envelope condition is

$$\begin{aligned} U\left(\theta\right) &= -\left(a\left(\theta\right)-\theta+b\right)^2 \\ U'\left(\theta\right) &= 2\left(a\left(\theta\right)-\theta+b\right) \end{aligned}$$

Examples of two incentive compatible mechanisms are

$$\begin{aligned} a\left(\theta\right) &= \begin{cases} \hat{a} & \theta < \hat{a}+b \\ \theta-b & \theta \geq \hat{a}+b \end{cases} \\ a\left(\theta\right) &= \hat{a}, \forall \hat{a} \in [1,2] \\ a\left(\theta\right) &= \theta-b \end{aligned}$$

The sets for the bottom ones are clear. For the top one $M=[\hat{a},2]$.
To characterize optimal allocations, we proceed by proving a sequence of lemmas:
Lemma 1. If $a\left(\theta\right) > \theta-b$, then for all $\theta' \in \left[\theta, \min\left\{a\left(\theta\right)+b, 2\right\}\right]$, $a\left(\theta'\right) = a\left(\theta\right)$.
Proof. Somewhat obvious! If $\theta' \in \left[\theta, \min\left\{a\left(\theta\right)+b, 2\right\}\right]$, then $a\left(\theta'\right) \geq a\left(\theta\right)$ by incentive compatibility but $\theta'-b \leq a\left(\theta\right) \leq a\left(\theta'\right)$. This implies that if $a\left(\theta\right) < a\left(\theta'\right)$, $a\left(\theta\right)$ is closer to the peak of $-\left(a-\theta'+b\right)^2$ which violates I.C. Hence the claim.**Q.E.D.**
Lemma 2. If $a\left(\theta\right) < \theta-b$, then for all $\theta' \in \left[\max\left\{1, a\left(\theta\right)+b\right\}, \theta\right]$, $a\left(\theta'\right) = a\left(\theta\right)$.
Proof of this is the same as the previous lemma **Q.E.D.**
Lemma 3. Consider an allocation $a\left(\theta\right)$ that is optimal for P. If $a\left(\theta\right)$ has a jump at θ, then for all $\theta' \in [1,\theta)$, $a\left(\theta'\right) = a\left(\theta-\right)$. Moreover, $a\left(\theta-\right) \leq 1-b$.
Proof. Suppose the first claim does not hold. Consider a point θ at which $a\left(\theta\right)$ has a jump, i.e., $a\left(\theta+\right) > a\left(\theta-\right)$. Then note that θ should be indifferent between $a\left(\theta+\right)$ and $a\left(\theta-\right)$. Otherwise, incentive compatibility is violated. This implies that $\theta-b \in \left(a\left(\theta-\right), a\left(\theta+\right)\right)$ and moreover, $a\left(\theta+\right)-\left(\theta-b\right) = \left(\theta-b\right)-a\left(\theta-\right)$. From the above lemmas, we must have that $\theta' \in \left[\max\left\{1, a\left(\theta-\right)+b\right\}, \theta\right]$, $a\left(\theta'\right) = a\left(\theta-\right)$ and $\theta' \in \left[\theta, \min\left\{a\left(\theta+\right)+b, 2\right\}\right]$, $a\left(\theta'\right) = a\left(\theta+\right)$. By the contrary assumption at the beginning, it cannot be

that $1 > a(\theta) + b$. Hence, for all $\theta' \in [a(\theta-) + b, \theta]$, $a(\theta') = a(\theta-)$. Now, consider the following alternative mechanism

$$\hat{a}(\theta') = \begin{cases} \theta' - b & \forall \theta' \in [a(\theta-) + b, \min\{2, a(\theta+) + b\}] \\ a(\theta) & \text{otherwise} \end{cases}$$

Obviously, this new allocation is incentive compatible. Let $\theta_1 = a(\theta-) + b$ and $\theta_2 = \min\{a(\theta+) + b, 2\}$. We only need to compare the payoff of P from the interval $[\theta_1, \theta_2]$ for $\hat{a}$ to that of a. Suppose for now that $\theta_2 < 2$. In this case, $\theta_2 = 2\theta - \theta_1$.Then

$$v_p(\hat{a}; \theta_1, \theta_2) = -\int_{\theta_1}^{\theta_2} (\theta' - \theta' + b)^2 \, d\theta' = -b^2 (\theta_2 - \theta_1)$$
$$v_p(a; \theta_1, \theta_2) = -\int_{\theta_1}^{\theta_2} (\theta' - a(\theta'))^2 \, d\theta'$$

Since $-x^2$ is a concave function, Jensen's inequality implies that

$$-\frac{\int_{\theta_1}^{\theta_2} (\theta' - a(\theta))^2 \, d\theta'}{\theta_2 - \theta_1} < -\left(\frac{\int_{\theta_1}^{\theta_2} [\theta' - a(\theta')] \, d\theta'}{\theta_2 - \theta_1}\right)^2$$

and

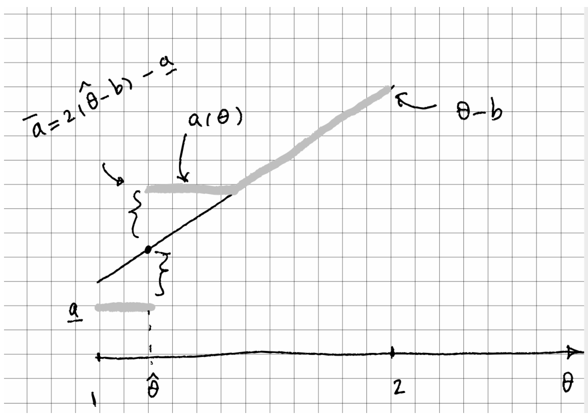
$$\begin{aligned}
-\left(\frac{\int_{\theta_1}^{\theta_2} [\theta' - a(\theta')] \, d\theta'}{\theta_2 - \theta_1}\right)^2 &= -\left(\frac{\frac{\theta_2^2 - \theta_1^2}{2} - a(\theta-)(\theta - \theta_1) - a(\theta+)(\theta_2 - \theta)}{\theta_2 - \theta_1}\right)^2 \\
&= -\left(\frac{\frac{\theta_2^2 - \theta_1^2}{2} - (\theta_1 - b)(\theta - \theta_1) - (\theta_2 - b)(\theta_2 - \theta)}{\theta_2 - \theta_1}\right)^2 \\
&= -\left(\frac{\frac{\theta_2^2 - \theta_1^2}{2} - (\theta_1 - b)\frac{\theta_2 - \theta_1}{2} - (\theta_2 - b)\frac{\theta_2 - \theta_1}{2}}{\theta_2 - \theta_1}\right)^2 \\
&= -\left(\frac{\frac{\theta_2^2 - \theta_1^2}{2} - (\theta_1 + \theta_2 - 2b)\frac{\theta_2 - \theta_1}{2}}{\theta_2 - \theta_1}\right)^2 \\
&= -\left(\frac{2b\frac{\theta_2 - \theta_1}{2}}{\theta_2 - \theta_1}\right)^2 = -b^2(\theta_2 - \theta_1)
\end{aligned}$$

This implies that $\hat{a}$ improves the payoff P. The case where $\theta_2 = 2$ can be shown even easier since the bound above becomes even tighter. This implies that a cannot be optimal which is a contradiction.

To prove the second claim, note that if $a(\theta-) > 1 - b$, then there exist $\theta_1 < \theta$ such that $a(\theta) = \theta - b$. Repeat the above deivation for θ_1, θ_2. **Q.E.D.**

Lemma 4. An optimal $a(\theta)$ cannot have a jump.

Proof. By lemma 3, we know that an optimal mechanism can only have jump at one point and then the value $a\left(\theta-\right)$ at this point is less than $1-b$ and a is flat below θ. This implies that if an optimal $a\left(\theta\right)>\theta-b$ can only hold in one interval; if not, there will be two jumps. Now, suppose that an optimal $a\left(\cdot\right)$ has a jump at $\hat{\theta}$. Then a must have the form depicted in the Figure below.



Consider a perturbation of a of the form

$$\hat{a}\left(\theta\right)=\begin{cases}\underline{a}-\varepsilon & \theta<\hat{\theta}-\varepsilon\\ \overline{a} & \theta\in\left[\hat{\theta}-\varepsilon,\tilde{\theta}\right]\\ a\left(\theta\right) & \text{otherwise}\end{cases}$$

Then the payoff of P is given by

$$-\int_{1}^{\hat{\theta}-\varepsilon}\left(\theta-\underline{a}+\varepsilon\right)^{2}d\theta-\int_{\hat{\theta}-\varepsilon}^{\tilde{\theta}}\left(\theta-\overline{a}\right)^{2}d\theta-\int_{\tilde{\theta}}^{1}\left(\theta-a\left(\theta\right)\right)^{2}d\theta$$

Taking a derivative of the aboev at $\varepsilon=0$, we have

$$\left(\hat{\theta}-\underline{a}\right)^{2}-2\int_{1}^{\hat{\theta}}\left(\theta-\underline{a}\right)d\theta-\left(\hat{\theta}-\overline{a}\right)^{2}=$$

$$\left(\hat{\theta}-\underline{a}\right)^{2}-2\left(\hat{\theta}-1\right)\left(\frac{\hat{\theta}+1}{2}-\underline{a}\right)-\left(\hat{\theta}-\overline{a}\right)^{2}$$

If a is optimal the above has to be 0. Note that $\hat{\theta}-b-\underline{a}=\overline{a}-\left(\hat{\theta}-b\right)\rightarrow x=\hat{\theta}-\underline{a}=\overline{a}-\hat{\theta}+2b$. So the above is given by

$$x^{2}-\left(\hat{\theta}-1\right)\left(1-\hat{\theta}+2x\right)-\left(x-2b\right)^{2}$$

Some algebra shows that the above is equal to

$$2\left[x-b-\frac{\hat{\theta}-1}{2}\right]\left(2b-\left(\hat{\theta}-1\right)\right)=0$$

Note that

$$\begin{aligned} x - b - \frac{\hat{\theta} - 1}{2} &= \hat{\theta} - \underline{a} - b - \frac{\hat{\theta} - 1}{2} \\ &= \frac{\hat{\theta} + 1}{2} - \underline{a} - b \\ &> \frac{\hat{\theta} + 1}{2} - 1 + b - b = \frac{\hat{\theta} - 1}{2} > 0 \end{aligned}$$

This implies that $\hat{\theta} = 1 + 2b$. When this is the case, we can replace $a(\theta)$ with $\max\{a(\theta), \overline{a}\}$ and the payoff of P does not change. This is because $-\int_1^{\hat{\theta}} (\theta - x)^2 d\theta$ is maximized at $x = \frac{1+\hat{\theta}}{2}$ and is symmetric around this peak. When $\hat{\theta} = 1 + 2b$, $\hat{\theta} - b = \frac{1+\hat{\theta}}{2}$. Since $\underline{a}$ and $\overline{a}$ are equidistance from $\hat{\theta} - b = \frac{1+\hat{\theta}}{2}$, replacing $\underline{a}$ with $\overline{a}$ does not change the value of the objective.So we can assume that the optimal a does not have a jump. Q.E.D

Lemma 5. Any optimal mechanism satisfies $a(\theta) \geq \theta - b$.

Proof. Suppose to the contrary that $a(\theta) < \theta - b$ for some θ. Then, we know from Lemma 2, that $a(\theta') = a(\theta)$ for all $\theta' \in [\min\{a(\theta) - b, 1\}, \theta]$. Take $\hat{\theta}$ to be the supremum value of θ so that $a(\theta) = a\left(\hat{\theta}-\right)$. If $\hat{\theta} < 2$, then there must be a jump at $\hat{\theta}$ which cannot happen by Lemma 4. Hence, $\hat{\theta} = 2$. Now consider the highest $\tilde{\theta}$ such that $a\left(\tilde{\theta}\right) = \tilde{\theta} - b$. If it does not exist set it equal to 1. Clearly a deviation to $\hat{a}(\theta') = \theta - b$ for values of $\theta' \geq \tilde{\theta}$ is incentive compatible and increases payoff of P. Q.E.D.

Lemma 6. Any optimal mechanism must be of the form $a(\theta) = \max\{\hat{a}, \theta - b\}$.

Proof. Given the above lemmas this is somewhat easy. As we have shown in Lemma 5, $a(\theta) \geq \theta - b$. Now, we show that if for some θ, $a(\theta) > \theta - b$, then for all $\theta' < \theta$, $a(\theta') = a(\theta)$ and thus optimal mechanism has the form mentioned before. To see that, suppose that there exists $\theta' < \theta$ with $a(\theta') < a(\theta)$. If $a(\theta') = \theta' + b$, define $\theta_1 = \sup\left\{\hat{\theta} | \hat{\theta} \in [\theta', \theta], a\left(\hat{\theta}\right) = \hat{\theta} - b\right\}$ and $\theta_2 = \inf\sup\left\{\hat{\theta} | \hat{\theta} \in [\theta', \theta], a\left(\hat{\theta}\right) = a(\theta)\right\}$. Obviously $\theta_1 < \theta_2$. Now take $\tilde{\theta} \in (\theta_1, \theta_2)$. By definition of θ_1 and θ_2, $a(\theta) > a\left(\tilde{\theta}\right) > \tilde{\theta} - b$. From Lemma 1, $a\left(\hat{\theta}\right) = a\left(\tilde{\theta}\right)$ for all $\hat{\theta} \in \left[\tilde{\theta}, \min\left\{2, a\left(\tilde{\theta}\right) + b\right\}\right]$. Given the definition of θ_2, we must have that $\hat{\theta} = \min\left\{2, a\left(\tilde{\theta}\right) + b\right\} < \theta_2 \leq 2$. Hence, $a\left(\hat{\theta}\right) = \hat{\theta} - b$ with $\hat{\theta} > \theta_1$ which is a contardiction to the definition of θ_1. This proves the claim. Q.E.D.

Lemma 7. A mechanism of the form $a(\theta) = \max\{\hat{a}, \theta - b\}$ is optimal for P when $\hat{a} = 1 + b$.

Proof. To see this, note that P's payoff is given by

$$-\int_1^2 (\theta - \max\{\hat{a}, \theta - b\})^2 d\theta = -\int_1^{\hat{a}+b} (\theta - \hat{a})^2 d\theta - \int_{\hat{a}+b}^2 b^2 d\theta$$

Taking a derivative of the above with respect to $\hat{a}$ and setting it equal to 0, we have

$$\int_1^{\hat{a}+b} (\theta - \hat{a})\, d\theta = 0 \rightarrow \hat{a} = \frac{1+\hat{a}+b}{2} \rightarrow \hat{a} = 1+b$$

Q.E.D.

4. Revelation Principle and Commitment. Consider a monopolist selling a good to a buyer whose wants to purchase multiple units of the good and whose payoff is given by $\theta q - p$ where $\theta \in \{1,2\}$ with $\Pr(\theta = 2) = 1/2$. Suppose that the seller's cost of production is $\frac{1}{2}q^2$. The seller does not know θ while the buyer does. Buyer has an outside option of 0.

1. Suppose that the seller and the buyer play the following game: The seller offers a menu of options, i.e., prices and quantities, to the buyer and the buyer chooses her appropriate option. Use revelation principle to case this problem as a direct revelation mechanism.
2. Find the direct revelation mechanism that maximizes seller's profit.
3. Now consider an alternative game between the seller and the buyer: The seller announces a menu, the buyer makes her choice. After that the seller has the option to revise her offer at a cost c. Show that for c small enough, you cannot use the revelation principle to describe any outcome in the resulting game.
4. In this environment, how would you modify the revelation principle in order to describe the set of arbitrary outcomes of this game.
5. What is the optimal value of profit in this case?

Solution. *Note: For making the problem more interesting and "correct"(!), I assume that the probability of $\theta = 1$ is 3/4!. When the probability of $\theta = 1$ is 1/2, optimal mechanism involves no trade with low type and no information rent for the high type. Hence, the claim in part 3 is wrong!*

A direct mechanism is (q_i, p_i). IC is

$$\theta_i q_i - p_i \geq \theta_i q_{-i} - p_{-i}$$

Profit maximization is given by

$$\max \frac{1}{4}\left(p_H - \frac{1}{2}q_H^2\right) + \frac{3}{4}\left(p_L - \frac{1}{2}q_L^2\right)$$

subject to

$$\begin{aligned} \theta_i q_i - p_i &\geq \theta_i q_{-i} - p_{-i} \\ \theta_i q_i - p_i &\geq 0 \end{aligned}$$

Note that as usual, the downward IC constraint must be binding. Moreover, the IR for $\theta = 1$ should be binding. Thus we have

$$\begin{aligned} p_L &= q_L \\ p_H &= 2q_H - 2q_L + p_L = 2q_H - q_L \end{aligned}$$

Thus seller's problem can be written as

$$\max \frac{3}{4}\left(q_L - \frac{1}{2}q_L^2\right) + \frac{1}{4}\left(2q_H - q_L - \frac{1}{2}q_H^2\right)$$

The optimal allocation is given by $q_L = \frac{1}{2}, q_H = 2$. Total prices are given by $p_L = 0.5, p_H = 3.5$.
Suppose that we can use the revelation principle in its previos form. Then for c small enough, the seller would want to deviate and given each buyer θ units at a price of θ. Since the H type has information rents, i.e., her payoff is given by 0.5, under the allocation above, and she gets nothing if she reveals her type, truth-telling is not an equilibrium.
To think about communication in this environment, we need to allow for general ones. Note that once we depart from the revelation principle there is a need for randomization both in the message sending strategy. Suppose that each type of buyer can choose a mixed strategy $\sigma(\theta) \in \Delta(M)$ over possible messages $M = \{m_1, \cdots, m_M\}$ where . Observing a message $m_k \in M$, the seller chooses (p_k, q_k). Given $\{M, (p_k, q_k)\}$, the buyer's strategy must satisfy

$$m_k \in \text{Supp}\left(\sigma(\theta)\right) : \theta q_k - p_k \geq \theta q_{k'} - p_{k'}, \forall m_{k'} \in M$$

Moreover, for any message realization, the seller should not have any incentive to reoptimize her strategies. For each signal realization m, the seller updates his prior about the buyer to

$$f_H(m_k) = \frac{\sigma_k(H)}{\sigma_k(H) + 3\sigma_k(L)}, f_L(m_k) = 1 - f_H(m_k)$$

The value of updating the menu is thus given by

$$v(f) = \max f_H\left(p_H - \frac{1}{2}q_H^2\right) + f_L\left(p_L - \frac{1}{2}q_L^2\right) - c$$

subject to

$$IC, IR$$

So the seller's problem is given by

$$\max \sum_k \left[\frac{3}{4}\sigma_k(L) + \frac{1}{4}\sigma_k(H)\right]\left(p_k - \frac{1}{2}q_k^2\right)$$

subject to

$$m_k \in \text{Supp}\left(\sigma(\theta)\right) : \theta q_k - p_k \geq \theta q_{k'} - p_{k'}, \forall m_{k'} \in M$$

$$\sum_k \sigma_k(\theta)\left[\theta q_k - p_k\right] \geq 0$$

$$p_k - \frac{1}{2}q_k^2 \geq v\left(\frac{\sigma_k(H)}{\sigma_k(H) + 3\sigma_k(L)}, \frac{3\sigma_k(L)}{\sigma_k(H) + 3\sigma_k(L)}\right)$$

The reason we impose the last constraint is because we are assuming it is optimal not to revise the allocation. This is of course an assumption and should be verified later. First let us describe $v(\cdot)$. Given what we showed earlier, we can easily show that

$$\begin{aligned} q_L &= \max\{f_L - f_H, 0\} \\ p_L &= \max\{f_L - f_H, 0\} \\ q_H &= 2 \\ p_H &= 4 - \max\{f_L - f_H, 0\} \end{aligned}$$

and hence

$$\begin{aligned} v(f) &= f_H(4 - \max\{f_L - f_H, 0\} - 2) + f_L\left(\max\{f_L - f_H, 0\} - \frac{1}{2}(\max\{f_L - f_H, 0\})^2\right) - c \\ &= 2f_H + (f_L - f_H)\max\{f_L - f_H, 0\} - \frac{1}{2}(\max\{f_L - f_H, 0\})^2 - c \\ &= 2f_H + \frac{1}{2}\max\{f_L - f_H, 0\}^2 - c \end{aligned}$$

Moreover, note that there are at most three price quantity pairs being offered. One only for low type, one only for high type and one that is chosen by both. To see this, note that if two (p_k, q_k) are chosen by both types the two types should both be indifferent between them and thus the p-q pairs should coincide. Moreover, if a contract is only chosen by the high type, it implies that the seller is very tempted to deviate and any such (p_k, q_k) must satisfy

$$p_k - \frac{1}{2}q_k^2 \geq 2 - c$$

Since H can always guarantee a payoff higher than L, we can assume that only the IR for L is binding. So the seller's problem can be written as

$$\max \sum_{k\in\{L,M,H\}} \left[\frac{3}{4}\sigma_k(L) + \frac{1}{4}\sigma_k(H)\right]\left(p_k - \frac{1}{2}q_k^2\right)$$

subject to

$$\begin{aligned} m_k \in \text{Supp}(\sigma(\theta)) : \theta q_k - p_k &\geq \theta q_{k'} - p_{k'}, \forall m_{k'} \in M \\ \sum_k \sigma_k(\theta_L)[\theta q_k - p_k] &\geq 0 \\ p_H - \frac{1}{2}q_H^2 &\geq 2 - c \\ p_L - \frac{1}{2}q_L^2 &\geq \frac{1}{2} - c \\ p_M - \frac{1}{2}q_M^2 &\geq \frac{2\sigma_M(H)}{\sigma_M(H) + 3\sigma_M(L)} + \frac{1}{2}\max\left\{\frac{3\sigma_M(L) - \sigma_M(H)}{\sigma_M(H) + 3\sigma_M(L)}, 0\right\}^2 - c \end{aligned}$$

Note that there are two possibilities: 1.$\sigma_H(H) = 0$ and 2. $\sigma_H(H) > 0$. Under the first assumption we have

$$\max \sum_{k\in\{L,M\}} \left[\frac{3}{4}\sigma_k(L) + \frac{1}{4}\mathbf{1}[k = M]\right]\left(p_k - \frac{1}{2}q_k^2\right)$$

subject to

$$\begin{aligned}
2q_M - p_M &\geq 2q_L - p_L \\
\sigma_M(L)\left[q_M - p_M - (q_L - p_L)\right] &= 0 \\
q_L - p_L &\geq q_M - p_M \\
q_L - p_L &\geq 0 \\
p_L - \frac{1}{2}q_L^2 &\geq \frac{1}{2} - c \\
p_M - \frac{1}{2}q_M^2 &\geq \frac{2}{1+3\sigma_M(L)} + \frac{1}{2}\max\left\{\frac{3\sigma_M(L)-1}{1+3\sigma_M(L)}, 0\right\}^2 - c
\end{aligned}$$

If $\sigma_M(L) = 0$, then

$$p_L - \frac{1}{2}q_L^2 \geq \frac{1}{2} - c, q_L - \frac{1}{2}q_L^2 \leq \frac{1}{2} \rightarrow 0 \leq q_L - p_L \leq c$$
$$p_M - \frac{1}{2}q_M^2 \geq 2 - c, 2q_M - \frac{1}{2}q_M^2 \leq 2 \rightarrow 2q_M - p_M \leq c$$

In this case, the IR of L must be binding, other wise an increase in p_L and p_M increases the payoff of seller. From the IC of the H we also need $2q_L - p_L \leq c$. Since Ir of L is binding, this implies that $q_L \leq c$. Moreover, p_M can be increased so that the IC of H holds with equality. Hence, the best that can be done is to set $q_L = c, p_L = c$ and $q_M = 2, p_M = 2q_M - c$. The profit under this allocation is

$$\frac{3}{4}\left(c - \frac{1}{2}c^2\right) + \frac{1}{4}(4 - c - 2) = \frac{1}{2} + \frac{1}{2}c\left(1 - \frac{3}{4}c\right) > \frac{1}{2}, c < \frac{4}{3}$$

When we start increasing $\sigma_M(L)$ above 0, there are two additional constraints: First low type has to be indifferent

$$q_M - p_M = q_L - p_L$$

and the seller would not want to deviate

$$p_M - \frac{1}{2}q_M^2 \geq \frac{2}{1+3\sigma_M(L)} + \frac{1}{2}\max\left\{\frac{3\sigma_M(L)-1}{1+3\sigma_M(L)}, 0\right\}^2 - c$$

As before, the IR for L must be binding. Hence

$$q_M = p_M = q_L = p_L$$

That is there only one point on the menu which thus must satisfy $p = q = 1$. The lowest value of the RHS of the constraint that keeps seller from deviating is achieved at $\sigma_M(L) = 1$ which implies that

$$p_M - \frac{1}{2}q_M^2 = \frac{1}{2} \geq \frac{5}{8} - c$$

As long as $c \geq 1/8$ this is satisfied. Otherwise, there is no such allocation. In any case, profits for the seller would be 1/2 which is for small c less than the profit when $\sigma_M(L) = 0$.

Now, suppose that $\sigma_H(H) > 0$. As before, we must have that $p_L = q_L$. Moreover, if there is any uncertainty in equilibrium about the buyer's type, we also must have that $p_M = q_M = p_L = q_L$. So we can assume that there are two points on the menu (p_H, q_H) and (p_M, q_M) and the L type chooses (p_M, q_M) for sure. In this case, the seller's optimization is given by

$$\max \sum_{k \in \{H,M\}} \left[\frac{1}{4}\sigma_k(H) + \frac{3}{4}\mathbf{1}[k = M]\right]\left(p_k - \frac{1}{2}q_k^2\right)$$

subject to

$$\begin{aligned}
2q_H - p_H &= 2q_M - p_M \\
q_M - p_M &= 0 \\
p_H - \frac{1}{2}q_H^2 &\geq 2 - c \\
p_M - \frac{1}{2}q_M^2 &\geq \frac{2\sigma_M(H)}{\sigma_M(H) + 3} + \frac{1}{2}\max\left\{\frac{3 - \sigma_M(H)}{\sigma_M(H) + 3}, 0\right\}^2 - c
\end{aligned}$$

We have

$$p_H = -q_M + 2q_H$$

So the optimization can be written as

$$\max\left(\frac{1}{4}\sigma_M(H) + \frac{3}{4}\right)\left(q_M - \frac{1}{2}q_M^2\right) + \frac{1 - \sigma_M(H)}{4}\left(2q_H + q_M - \frac{1}{2}q_H^2\right)$$

subject to

$$\begin{aligned}
2q_H - q_M - \frac{1}{2}q_H^2 &\geq 2 - c \\
q_M - \frac{1}{2}q_M^2 &\geq \frac{2\sigma_M(H)}{\sigma_M(H) + 3} + \frac{1}{2}\max\left\{\frac{3 - \sigma_M(H)}{\sigma_M(H) + 3}, 0\right\}^2 - c
\end{aligned}$$

Obiously if the above has a solution, it must satisfy $q_H = 2$. So it becomes

$$\max\left(\frac{1}{4}\sigma_M(H) + \frac{3}{4}\right)\left(q_M - \frac{1}{2}q_M^2\right) + \frac{1 - \sigma_M(H)}{4}(2 + q_M)$$

subject to

$$
\begin{aligned}
2 - q_M &\geq 2 - c \\
q_M - \frac{1}{2} q_M^2 &\geq \frac{2\sigma_M(H)}{\sigma_M(H) + 3} + \frac{1}{2} \max\left\{ \frac{3 - \sigma_M(H)}{\sigma_M(H) + 3}, 0 \right\}^2 - c
\end{aligned}
$$

As this can be seen, the most that can be traded in the M contract is c. Can this be sustained? Well it depends on whether

$$
c - \frac{1}{2} c^2 \geq \frac{2\sigma_M(H)}{\sigma_M(H) + 3} + \frac{1}{2} \max\left\{ \frac{3 - \sigma_M(H)}{\sigma_M(H) + 3}, 0 \right\}^2 - c
$$

for some value of $\sigma_M(H)$. The lowest possible value of RHS is $\frac{1}{2} - c$ which is achieved at $\sigma_M(H) = 0$ but again this is a fully separating allocation. So we have showed that a fully separating allocation with $\{(c, c), (2, 4 - c)\}$ is optimal.