Chapter 15
Dynamic Optimization

Pierre de Fermat (1601?-1665)   L. Euler   Lev Pontryagin (1908-1988)

15.1 Dynamic Optimization

• In this chapter, we will have a *dynamic* system –i.e., evolving over time (either discrete or continuous time). Our goal: *optimize* the system.

• We will study an optimization problems with the following features:

1) Aggregation over time for the objective function.

2) Variables linked (constrained) across time.

• Analytical solutions are rare, usually numerical solutions are obtained.

• Usual problem: The cake eating problem

There is a cake whose size at time is W_t and a consumer wants to eat in T periods. Initial size of the cake is $W_0 = \varphi$ and $W_T = 0$. What is the optimal strategy, $\{W_t^*\}$?

15.1 Dynamic Optimization

• **Example 1 – Discrete time setting**: Maximize expected utility function, with time-separable preferences, given a budget constraint:

$$Max_{c_t, y_{t+1}} \sum_{t=0}^{T-1} \beta^t u(c_t) \quad \textit{subject to} \quad y_t = (1 + r) \, (y_{t-1} - c_{t-1})$$

• Notation:

$u(.)$ = well-behaved utility function
c_t = period t consumption ($y_T = c_t$)
y_t = period t wealth (y_0 given)
β = discount rate of future consumption.
r = interest rate capital

• We are not looking for a single optimal value C* , but for values c_t that produce an optimal value for the aggregated sum (or aggregate discounted utility over time).

• We are looking for an optimal $\{c_t, y_{t+1}\}$*.

3

15.1 Dynamic Optimization

• **Example 1** is a standard equality constrained optimization problem.

The Lagrangean method can be used, assigning T *multipliers,* μ_t:

$$
\begin{aligned}
L &= \sum_{t=0}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T} \mu_t \{ y_t - (1 + r) \, (y_{t-1} - c_{t-1}) \} \\
&= \sum_{t=0}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T} \mu_t \, y_t + \sum_{t=0}^{T-1} \mu_{t+1} (1 + r) \, (y_t - c_t) \} \\
&= u(c_0) + \mu_1 (1 + r) \, (y_0 - c_0) \\
&\quad + \sum_{t=1}^{T-1} \{ \beta^t u(c_t) - \mu_t \, y_t + \mu_{t+1 \, (1}+r) \, (y_t - c_t) \} \\
&\quad + \mu_T \, y_T
\end{aligned}
$$

• F.o.c.:

$$u'(c_0) = \mu_1 (1 + r) \qquad \qquad \text{(derivative w.r.t. } c_0)$$
$$\beta^t u'(c_t) = \mu_{t+1 \, (1} + r) \quad t = 1, 2, ..., T\text{-1} \quad \text{(derivative w.r.t. } c_t)$$
$$\mu_t = \mu_{t+1 \, (1} + r) \qquad \quad t = 1, 2, ..., T\text{-1} \quad \text{(derivative w.r.t. } y_t)$$
$$\mu_T = 0 \qquad \qquad \qquad \text{(derivative w.r.t. } y_T)_4$$

15.1 Dynamic Optimization

• **Example 1 (continuation)**

• A little bit of algebra gives us:

$$\beta^t u'(c_t) = \mu_t = \beta^{t-1} u'(c_{t-1})/(1 + r)$$

or $\quad \beta\, u'(c_{t+1})(1 + r) = u'(c_t)$

• That is, an optimal consumption plan requires that consumption be allocated through time so that marginal benefit of consumption in period t, $u'(c_t)$, is equal to its marginal cost, which is the interest foregone $(1 + r)$ times the marginal benefit of consumption tomorrow discounted at the rate β.

• $\mu_T = 0 \qquad \Rightarrow$ Any wealth left over has no value.

5

15.1 Dynamic Optimization

• **Example 2 – Continuous time setting** (Ramsey problem): A social planner's problem, optimal economic growth:

$$Max_{C(t)} \int_0^T e^{-rt} C(t)dt \qquad \text{subject to} \quad \dot{K} = F(K) - C - \delta K$$

• Notation:

$K(t)$ = capital (only factor of production)

$F(K)$ = well-behaved output function –i.e., $F_K > 0$, $F_{KK} < 0$, for $K > 0$

$C(t)$ = consumption

$I(t)$ = investments = $F(K) - C(t)$

δ = constant rate of depreciation of capital.

• Again, we are looking for an optimal $\{C(t)\}^*$. Using the techniques discussed in Chapter 12, we cannot solve this problem. We need to adapt them or we need new ones.

6

15.1 Dynamic Optimization

• We will use dynamic optimization methods in different environments:
- Discrete time and Continuous time
- Finite horizons and Infinite horizons
- Deterministic and Stochastic

• Several ways to solve these problems:

1) Discrete time methods (Lagrangean approach, Optimal control theory, Bellman equations, Numerical methods).

2) Continuous time methods (Calculus of variations, Optimal control theory, Bellman equations, Numerical methods).

Usual Applications: Asset-pricing, consumption, investments, I.O., etc.

7

15.2 Optimal Control Theory

• *Optimal control*: Find a control law for a given system such that a certain optimality criterion is achieved.

• Typical situation: We want to minimize a cost functional that is a function of state and control variables.

   Example: Maximization of a utility function that is a function of consumption (control) and wealth (state).

• More general than Calculus of Variations.

• Handles Inequality restrictions on Instruments and State Variables

• Similar to Static Methods (Lagrange Method and KKT Theory)

• Long history in economics: Shell (1967), Arrow (1968) and Shell (1969).

8

15.2 Optimal Control: Discrete time

• First, we discussed the finite horizon case. In this situation, we can describe the dynamic optimization problem as a series of decisions, $\boldsymbol{u} = \{u_0, u_1, ..., u_{T-1}\}$ that influences the future, described as a collection of state variables, $\boldsymbol{x} = \{x_1, x_2, ..., x_T\}$, with the initial state x_0 given.

• At time $t = 0$, we start with an initial state x_0, then, a decision u_0 is made (should be feasible, that is $u_0 \in U_0$). This gives a benefit (utility):

$$f(u_0, x_0)$$

and a new state x_1, determined by some function transition function g(.)

$$x_1 = g(u_0, x_0)$$

• At $t = 1$, another (feasible) decision is made $u_1 \in U_1$, generating $f(u_1, x_1)$ and leading to a new state $x_2 = g(u_1, x_1)$ to begin $t=2$.

• Continue until $t = T - 1$, with final decision u_{T-1}. At T, we are left with a terminal value state, x_T, with a value $V(x_T)$.

9

15.2 Optimal Control: Discrete time

• In each period t, the transition to the new state is driven by the *transition equation*

$$x_{t+1} = g(u_t, x_t)$$

The sequential choices $u_0, u_1, \ldots, u_{T-1}$ and implied transitions leaves a terminal state x_T, which is value at $V(x_T)$.

Notes: The decision u_t is known as the *control variable*, since it is under the control of the decision-maker (any value $u_t \in U_t$ may be chosen). The variable x_t is known as the *state variable*.

• In general, constraints are imposed on the state variable. For example, in economic models, negative values may be infeasible.

10

15.2 Optimal Control: Discrete time

• The setup we used is one of the most common dynamic optimization problems in economics and finance. We have:

– x_t, a state variable, which is usually a *stock* (measured at the beginning of period t, since time is discrete)

– u_t, the control variable, is a *flow* (measured at the end of period t)

– the *objective functional, f(.):* usually involves an intertemporal utility function, which is *additively separable*, *stationary* –i.e., independent of t–, and involves time-discounting.

– $f(.), g(.)$ are well behaved (continuous and smooth) functions.

– there is a non-Ponzi scheme condition: $\lim_{t\to\infty} \beta^t x_t \geq 0$.

– the solution $u_t^* = h(x_t)$ is called the *policy function.* It gives an optimal rule for changing the optimal control, given the state of the economy.

11

15.2 Optimal Control: Discrete time

• Back to our setup. Assuming separability –i.e., $f(.)$ is not influenced by previous or future decisions–, the decision maker problem becomes:

$$Max_{u_t \in U_t} \sum_{t=0}^{T-1} \beta^t f(u_t, x_t) + \beta^T V(x_T) \qquad \text{(P1)}$$
$$\text{subject to} \qquad x_{t+1} = g(u_t, x_t)$$

• Like **Example 1**, we have a standard equality constrained optimization problem. The Lagrangean method can be used, assigning *T multipliers,* μ_t, and allowing $f(.)$ and $g(.)$ to be time dependent:

$$L = \sum_{t=0}^{T-1} \beta^t f_t(u_t, x_t) + \beta^T V(x_T) - \sum_{t=0}^{T-1} \mu_{t+1}\{x_{t+1} - g_t(u_t, x_t)\}$$
$$= f_0(u_0, x_0) + \mu_1 g_0(u_0, x_0)$$
$$+ \sum_{t=1}^{T-1}\{\beta^t f_t(u_t, x_t) + \mu_{t+1} g_t(u_t, x_t) - \mu_t x_t\}$$
$$- \mu_T x_T + \beta^T V(x_T)$$

12

15.2 Optimal Control: Discrete time

• It is common to multiply the constraint by β^{t+1} to form the Lagrangean, which changes the interpretation of the multiplier. Now,

$$\lambda_t \, \beta^t = \mu_t \qquad \Rightarrow \lambda_t = \beta^{-t} \, \mu_t \qquad (\lambda_t: \textit{current value multiplier})$$

Using λ_t and assuming U_t is open for all t and the gradient of the constraints are (linearly) independent, we have the f.o.c.:

$$D_{u_t}L = \beta^t \{D_{u_t}f_t(u_t, x_t) + \beta \, \lambda_{t+1} \, D_{u_t}g_t(u_t, x_t)\} = 0 \qquad t = 0, 1, ..., T\text{-}1$$
$$D_{x_t}L = \beta^t \{D_{x_t}f_t(u_t, x_t) + \beta \, \lambda_{t+1} \, D_{x_t}g_t(u_t, x_t) - \lambda_t\} = 0 \quad t = 1, 2, ..., T\text{-}1$$
$$D_{x_T}L = \beta^T \{-\lambda_T + V'(x_T)\} = 0$$
$$D_{\lambda_{t+1}}L = 0 \qquad \Rightarrow x_{t+1} = g(u_t, x_t) \qquad\qquad\qquad t = 0, 1, ..., T\text{-}1$$

13

15.2 Optimal Control: Discrete time

• **Theorem**:

In the previous dynamic problem (**P1**), with finite horizon, assume:

- U_t is open for all t.

- the gradient of the constraints is (linearly) independent.

- (.) and $g(.)$ are concave and increasing in x_t.

- V is concave and increasing.

Then, $\boldsymbol{u} = \{u_0, u_1, ..., u_{T\text{-}1}\}$ is an *optimal solution* iff $\exists \, \boldsymbol{\lambda} = \{\lambda_1, \lambda_2, ..., \lambda_T\}$ such that

$$D_{u_t}f_t(u_t, x_t) + \beta \, \lambda_{t+1}D_{u_t}g_t(u_t, x_t) = 0 \qquad t = 0, 1, ..., T - 1 \ (1^*)$$
$$D_{x_t}f_t(u_t, x_t) + \beta \, \lambda_{t+1}D_{x_t}g_t(u_t, x_t) = \lambda_t \qquad t = 1, 2, ..., T - 1 \ (2^*)$$
$$x_{t+1} = g(u_t, x_t) \qquad\qquad t = 0, 1, ..., T - 1 \ (3^*)$$
$$V'(x_T) = \lambda_T \qquad\qquad\qquad (4^*)$$

14

15.2 Optimal Control: Discrete time

• **Example**: An oil lease expires in 2 years. There are x_0 barrels of oil in the well. The price of oil is P (USD per barrel), the cost is $\{\kappa * (u_t^2/x_t)\}$, where u_t is the production, x_t is the amount of oil in the well and is a κ constant. Then, the optimal production solves:

$$Max_{u_t \in U_t} \sum_{t=0}^{2} \beta^t (P - \kappa * \frac{u_t}{x_t})u_t \quad \text{s.t} \quad x_{t+1} = x_t - u_t.$$

Then,

$$P - 2\kappa\frac{u_t}{x_t} - \beta\,\lambda_{t+1} = 0 \qquad t = 0, 1$$

$$\kappa\left(\frac{u_t}{x_t}\right)^2 + \beta\,\lambda_{t+1} = \lambda_t \qquad t = 1$$

$$x_{t+1} = x_t - u_t \qquad t = 0, 1$$

$$0 = \lambda_2$$

15

15.2 Optimal Control: Discrete time

• **Example (continuation)**:

$t = 2, \qquad \lambda_2 = 0$

$t = 1 \qquad \kappa * \left(\frac{u_1}{x_1}\right)^2 + \beta\,\lambda_2 = \lambda_1 \qquad \Rightarrow \lambda_1 = \kappa * \left(\frac{u_1}{x_1}\right)^2$

$\qquad\qquad P - 2\kappa\frac{u_1}{x_1} - \beta\,\lambda_2 = 0 \qquad \Rightarrow u_1 = \frac{P}{2\kappa} * x_1$

$\qquad\qquad x_2 = x_1 - u_1$

$t = 0 \qquad P - 2\kappa\frac{u_0}{x_0} - \beta\,\lambda_1 = 0 \qquad \Rightarrow u_0 = \left(\frac{P - \beta\,\lambda_1}{2\kappa}\right) * x_0$

$\qquad\qquad x_1 = x_0 - u_0$

For $x_0 = 8{,}000$; $P = 60$; $\beta = 1$; & $\kappa = 45$, we get:
 $u = [3555.56, 2962.96]$, $x = [8000, 4444.44, 1481.48]$, & $\lambda = [20, 0]$

16

•8

15.2 Optimal Control: Discrete time

• Interpretation of necessary conditions:

- Conditions (1*) and (3*) are standard.

- Condition (2*) describes the evolution of the shadow price of x_t. A marginal change x_t has two effects:

(a) It affects the instantaneous benefit at time t by $D_{x_t} f_t(u_t, x_t)$.

(b) It affects the attainable state at time $t+1$ by $D_{x_t} g_t(u_t, x_t)$, which we value (after discounting) at $\beta \lambda_{t+1} D_{x_t} g_t(u_t, x_t)$.

- Condition (4*) gives the value of the shadow price of x_t at $t = T$. This condition is called *transversality condition*. If there is terminal value for $V(x_T) = V^*$, then optimality requires $\lambda_T = 0$.

• In many models, it is common to impose nonnegative constraints to x_t and/or u_t. In these situation, KKT conditions apply.

17

15.2 Optimal Control: Discrete time

• Form a Lagrangean-type expression, the *Hamiltonian*

$$H_t(u_t, x_t, \lambda_t) = f_t(u_t, x_t) + \beta \lambda_{t+1} g_t(u_t, x_t)$$

The Lagrangean for Problem **P1** becomes:

$$L = H_0(u_t, x_t, \lambda_t) + \sum_{t=1}^{T-1} \beta^t \{H_t(u_t, x_t, \lambda_t) - \lambda_t x_t\} - \beta^T \lambda_T x_T + \beta^T V(x_T)$$

Assuming U_t is open, the f.o.c., called the *Maximum Principle*, are:

$$D_{u_t} H_t(u_t, x_t, \lambda_t) = 0 \qquad\qquad t = 0, 1, ..., T-1$$
$$D_{x_t} H_t(u_t, x_t, \lambda_t) = \lambda_t \qquad\qquad t = 1, 2, ..., T-1$$
$$x_{t+1} = g(u_t, x_t) \qquad\qquad t = 0, 1, ..., T-1$$
$$V'(x_T) = \lambda_T$$

Under the assumptions from the previous theorem, the solution *u* is an *optimal solution*.

William R. Hamilton (1805 – 1865)

15.2 Optimal Control: Maximum Principle

• The Hamiltonian measures the total return in period t:
$$H_t(u_t, x_t, \lambda_t) = f_t(u_t, x_t) + \beta \lambda_{t+1} \, g_t(u_t, x_t).$$

It augments the single period return $f_t(u_t, x_t)$ to account for the future consequences of current decisions, aggregating the direct and indirect effects of the choice of u_t in period t.

The f.o.c. for $t = 0, 1, ..., T - 1$:
$$D_{u_t} H_t(u_t, x_t, \lambda_t) = D_{u_t} f_t(u_t, x_t) + \beta \lambda_{t+1} D_{u_t} g_t(u_t, x_t) = 0$$

characterizes an interior maximum of the Hamiltonian along the optimal path. It is a general principle. If there are constraints, we replace the previous equation by:
$$Max_{u_t \in U_t} \sum_{t=0}^{T-1} H_t(u_t, x_t, \lambda_t) \qquad t = 0, 1, ..., T - 1.$$

19

15.2 Optimal Control: Maximum Principle

• The *Maximum Principle* prescribes that u_t should be chosen to maximize the total benefits in each period t. It transforms a dynamic optimization problem into a sequence of static optimization problems.

• The Hamiltonian that we have defined above:
$$H_t(u_t, x_t, \lambda_t) = f_t(u_t, x_t) + \beta \lambda_{t+1} \, g_t(u_t, x_t)$$
is called the *current value Hamiltonian,* since it measures total benefit at t.

The Hamiltonian can be written in terms of present value
$$H_t(u_t, x_t, \lambda_t) = \beta^t f_t(u_t, x_t) + \mu_{t+1} \, g_t(u_t, x_t),$$
where μ_{t+1} is the present value multiplier (f.o.c. are unchanged).

20

15.2 Optimal Control: Infinite Horizon

• The problem now becomes:

$$Max_{u_t \in U_t} \sum_{t=0}^{\infty} \beta^t f(u_t, x_t) \quad \text{subject to } x_{t+1} = g(u_t, x_t)$$

given x_0. We usually assume $f(.)$ bounded for every t and $\beta < 1$.

We can rewrite the problem as:

$$Max_{u_t \in U_t} \sum_{t=0}^{T-1} \beta^t f(u_t, x_t) + \beta^T V(x_T) \qquad \textbf{(P1)}$$

$$\text{subject to} \quad x_{t+1} = g(u_t, x_t)$$

where

$$V(x_T) = Max_{u_t \in U_t} \sum_{t=T}^{\infty} \beta^{t-T} f(u_t, x_t) \quad \text{s.t.} \quad x_{t+1} = g(u_t, x_t) \quad t = T, T+1,$$

• The infinite horizon problem must satisfy the same optimality conditions as its finite horizon (1*)-(3*). The transversality condition, (4*), should be modified. We will talk about this later.

21

15.3 Optimal Control: Continuous Time

• So far we have worked with discrete time (days, months, years), now we measure variables in continuous time. Now, the problem becomes:

$$Max_{u_t \in U_t} \int_0^T e^{-rt} f(u(t), x(t), t)\, dt \qquad \text{s.t.} \quad \dot{x}(t) = g(u(t), x(t), t)$$

given $x(0) = x_0$.

• An integral replaces the sum in the objective function, a differential equation replaces the difference equation in the transition equation and $\beta^t = e^{-rt}$.

• The Lagrange multipliers, λ, define a functional on the set 1, 2, ..., T. Now, in continuous time, λ becomes a functional on [0, T].

• To form L, it is convenient to multiply each constraint by e^{-rt}. Then,

$$L = \int_0^T e^{-rt} f(u(t), x(t), t)\, dt + e^{-rT} V(x(T)) - \int_0^T e^{-rt} \lambda(t) \{\dot{x}(t) - g(u(t), x(t), t)\}\, dt$$

22

15.3 Optimal Control: Continuous Time

• To form L, it is convenient to multiply each constraint by e^{-rt}. Then,

$$L = \int_0^T e^{-rt} f(u(t), x(t), t) \, dt + e^{-rT} V(x(T)) - \int_0^T e^{-rt} \lambda(t) \{ \dot{x}(t) - g(u(t), x(t), t) \} \, dt$$

• Rearranging terms in L:

$$L = \int_0^T e^{-rt} H(u(t), x(t), \lambda(t), t) \, dt - \int_0^T e^{-rt} \lambda(t) \dot{x}(t) \, dt + e^{-rT} V(x(T))$$

where H is the current value Hamiltonian, now defined as

$$H(u(t), x(t), \lambda(t), t) = f(u(t), x(t), t) + \lambda(t) \, g(u(t), x(t), t)$$

• Next, to simplify L, we integrate by parts $\int_0^T e^{-rt} \lambda(t) \dot{x}(t) \, dt$ with $dv = \dot{x}(t), u = e^{-rt} \lambda(t)$

15.3 Optimal Control: Continuous Time

$$\int_0^T e^{-rt} \lambda(t) \dot{x}(t) \, dt = e^{-rT} \lambda(T) x(T) - \lambda(0) x(0) - \int_0^T e^{-rt} \dot{\lambda}(t) x(t) \, dt + r \int_0^T e^{-rt} \lambda(t) x(t) \, dt$$

Then,

$$L = \int_0^T e^{-rt} \{ H(u(t), x(t), \lambda(t), t) - r\lambda(t) x(t) + \dot{\lambda}(t) x(t) \} dt - e^{-rT} \lambda(T) x(T) + \lambda(0) x(0) + e^{-rT} V(x(T))$$

The necessary conditions for an optimal, after some algebra:

$$D_{u_t} L = D_{u_t} H(u(t), x(t), \lambda(t), t) = 0$$

$$D_{x_t} L = D_{x_t} H(u(t), x(t), \lambda(t), t) + \dot{\lambda}(t) - r\lambda(t) = 0$$

$$D_{x_T} L = -\lambda(T) + V'(x(T)) = 0$$

$$D_{\lambda_t} L = 0 \quad \Rightarrow \dot{x}(t) = g(u(t), x(t), t)$$

15.3 OCT: Continuous Time – Pontryagin's Maximum principle

• Conditions for necessary conditions to work –i.e., $u(t)$ and $x(t)$ maximize:

$$\int_0^T e^{-rt} f(u(t), x(t), t) \, dt \qquad \text{s.t.} \qquad \dot{x}(t) = g(u(t), x(t), t)$$

- Control variable must be piecewise continuous (some jumps, discontinuities OK).

- State variable, $x(t)$, must be continuous and piecewise differentiable.

- $f(.)$ and $g(.)$ first-order differentiable with respect to $x(t)$ and t, but not necessarily with respect to $u(t)$.

- Initial condition finite for state variable, $x(0) = x_0$.

- If no finite terminal value for state variable, then $\lambda(T) = 0$.

25

15.3 OCT: Continuous Time – Pontryagin's Maximum principle

• Similar to the discrete time case, the Maximum Principle requires that the Hamiltonian be maximized along the optimal path. Then, the necessary conditions are:

$$\{u^*(t)\} \text{ maximizes } H(u(t), x(t), \lambda(t), t)$$
$$\dot{\lambda}(t) = r\lambda(t) - D_{x_t} H(u(t), x(t), \lambda(t), t)$$
$$\dot{x}(t) = D_{\lambda_t} H = g(u(t), x(t), t)$$
$$\lambda(T) = V'(x(T)) \qquad \text{(transversality conditions)}$$

These conditions, with the exception of the transversality condition, all apply to the infinite horizon case.

• *Sufficiency*: if $f(.)$ and $g(.)$ are strictly concave, then the necessary conditions are sufficient, meaning that any path satisfying these conditions does in fact solve the problem.

Lev Pontryagin (1908 – 1988, Russia)

15.3 OCT: Continuous Time – Example

Back to Ramsey problem (optimal growth), with infinite horizon:

$Max_{C(t)} \int_0^\infty e^{-rt} u(c(t)) dt$ subject to $\dot{K}(t) = F(K(t)) - c(t) - \delta K(t)$

Hamiltonian becomes:

$$H(u(t), x(t), \lambda(t), t) = u(c(t)) + \lambda(t)\{\dot{K} - F(K(t)) - c(t) - \delta K(t)\}$$

F.o.c.:

$D_{u_t} H = u'(c(t)) - \lambda(t) = 0 \qquad \Rightarrow u'(c(t)) = \lambda(t)$

$D_{x_t} H(u(t), x(t), \lambda(t), t) + \dot{\lambda}(t) - r\lambda(t) = 0$

$\Rightarrow \dot{\lambda}(t) = r\lambda(t) - \lambda(t)\{F'(K(t)) - \delta\} = \lambda(t)\{r + \delta - F'(K(t))\}$

$\dot{K}(t) = F(K(t)) - c(t) - \delta K(t)$

A little bit of work delivers: $u''(c(t))\dot{c}(t) = \lambda(t)\{r + \delta - F'(K(t))\}$

$$\Rightarrow \dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}\{r + \delta - F'(K(t))\}$$
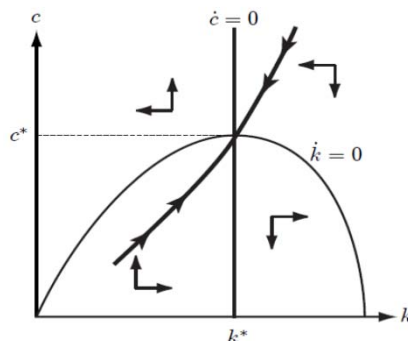
27

15.3 OCT: Continuous Time – Example

From $\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}\{r + \delta - F'(K(t))\}$

$\Rightarrow \dot{c}(t) \lesseqgtr 0 \Leftrightarrow \qquad \{F'(K(t)) - (r + \delta)\} \lesseqgtr 0$

A phase diagram given by: $\dot{K}(t) = F(K(t)) - c(t) - \delta K(t)$

$\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}\{r + \delta - F'(K(t))\}$

In the steady state (c^*, K^*):
$$F'(K(t)) = (r + \delta)$$

That is, the marginal productivity of capital equals the total user cost: interest rate + depreciation.

28

•14

15.3 OCT: Continuous Time – General Problem

• More General Problem: We add more state variables, more control variables (no problem, think of multivariate calculus), a terminal-value function and inequality restrictions.

Objective Functional:

$$Max_{u_t \in U_t} \int_0^{t_1} f(\boldsymbol{u}(t), \boldsymbol{x}(t), t)\, dt + \varphi(\boldsymbol{x}(t_1), t) \qquad \text{subject to}$$

$$\dot{x}_i(t) = g_i(t, x(t), u(t)), \quad x_i(t_0) = x_{i0}, \quad i = 1,..., n$$
$$x_i(t_1) = x_{i1}, \qquad i = 1,..., q$$
$$x_i(t_1)\ \text{free}, \qquad i = q+1,..., r$$
$$x_i(t_1) \geq 0, \qquad i = r+1,..., s$$
$$K(x_1(t),..., x_n(t)) \geq 0, \quad \text{at } t_1$$
$$1 \leq q \leq r \leq s \leq n$$

with $\boldsymbol{x}(t) = n$-dimensional vector, & $\boldsymbol{u}(t) = m$-dimensional vector.

29

15.3 OCT: Continuous Time – General Problem

• First order conditions:

$$D_{u_{j,t}} L = D_{u_{j,t}} H(\boldsymbol{u}(t), \boldsymbol{x}(t), \boldsymbol{\lambda}(t), t) = 0 \qquad\qquad j = 1, 2, ..., m$$
$$D_{x_{i,t}} L = D_{x_{i,t}} H(\boldsymbol{u}(t), \boldsymbol{x}(t), \boldsymbol{\lambda}(t), t) + \dot{\lambda}_i(t) = 0 \qquad i = 1, 2, ..., n$$
$$D_{\lambda_{i,t}} L = 0 \qquad \Rightarrow \dot{x}_i(t) = g_i(u(t), x(t), t) \qquad i = 1, 2, ..., n$$

<u>Note</u>: $H(\boldsymbol{u}(t), \boldsymbol{x}^*(t), \boldsymbol{\lambda}(t), t)$ is maximized by $\boldsymbol{u}(t) = \boldsymbol{u}^*(t)$.

• Transversality conditions, if:

- $\boldsymbol{x}(T)$ is free; $\quad \lambda_i(T) = D_{x_{i,T}} V(\boldsymbol{x}(T))$

- $\boldsymbol{x}(T) \geq 0; \qquad \lambda_i(T) \geq D_{x_{i,T}} V(\boldsymbol{x}(T))$

$$\qquad \Rightarrow x_i(T)\{\lambda_i(T) - D_{x_{i,T}} V(\boldsymbol{x}(T))\} = 0 \qquad \textit{(slackness condition)}$$

- $K(\boldsymbol{x}(T)) = 0; \ \lambda_i(T) = D_{x_{i,T}} V(\boldsymbol{x}(T)) + \psi\, D_{x_{i,T}} K(\boldsymbol{x}(T))$ ($\psi \geq 0; \psi K = 0$)

- $K(\boldsymbol{x}(T)) \geq 0; \ \lambda_i(T) = D_{x_{i,T}} V(\boldsymbol{x}(T)) + \psi\, D_{x_{i,T}} K(\boldsymbol{x}(T))$

30

15.3 OCT: Continuous Time – General Problem

• In the infinite horizon problem, the transversality conditions is:

$$\lim_{t\to\infty} \lambda_j(t)\, x_j(t) = 0$$

or with an initial value Hamiltionian:

$$\lim_{t\to\infty} e^{-rt}\, \lambda_j(t)\, x_j(t) = 0$$

• Sufficiency: Arrow and Kurz (1970)

If the maximized Hamiltonian is strictly concave in the state variables, any path satisfying the conditions above will be sufficient to solve the problem posed.

31

15.3 OCT: Application

• **Example**: Nerlove-Arrow Advertising Model

Let $G(t) \geq 0$ denote the stock of goodwill at time t.

$$\dot{G} = u - \partial G, \qquad G(0) = G_0$$

where $u = u(t) \geq 0$ is the advertising effort at time t measured in dollars per unit time. Sales S are given by

$$S = S(p,G,Z),$$

where p is the price level and Z other exogeneous variables.

Let $c(S)$ be the rate of total production costs, then, total revenue net of production costs is:

$$R(p,G,Z) = p\, S(p,G,Z) - c(S)$$

Revenue net of advertising expenditure is: $R(p,G,Z) - u$

Kenneth Joseph Arrow (1921, USA)

15.3 OCT: Application

• The firm wants to maximize the present value of net revenue streams discounted at a fixed rate ρ:

$$\max_{u \geq 0, p \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} [R(p, G, Z) - u] \, dt \right\}$$

subject to $\dot{G} = u - \partial G, \qquad G(0) = G_0$

• Note that the only place p occurs is in the integrand, which we can maximize by first maximizing R w.r.t p holding G fixed, and then maximized the result with respect to u. Thus,

$$\frac{\partial R(p, G, Z)}{\partial p} = S + p \frac{\partial S}{\partial p} - c_S \frac{\partial S}{\partial p} = 0, \qquad (7.5)$$

• Implicitly, we get $p^*(t) = p(G(t), Z(t))$

• Define $\pi(G, Z) = R(p^*, G, Z)$. Now, J is a function of G and Z only. For convenience, assume Z is fixed.   33

15.3 OCT: Application

 • Solution by the Maximum Principle

$$H = \pi(G) - u + \lambda[u - \delta G]$$

$$\frac{dH}{du} = 0$$

$$\dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{\partial \pi}{\partial G}$$

$$\lim_{t \to +\infty} e^{-\rho t} \lambda(t) = 0.$$

The adjoint variable $\lambda(t)$ is the shadow price associated with the goodwill at time t. The Hamiltonian can be interpreted as the dynamic profit rate which consist of two terms:

(i) the current net profit rate $\pi(G) - u$.

(ii) the value $\lambda \dot{G} = \lambda[u - \delta G]$ of the new goodwill created by advertising at rate u.   34
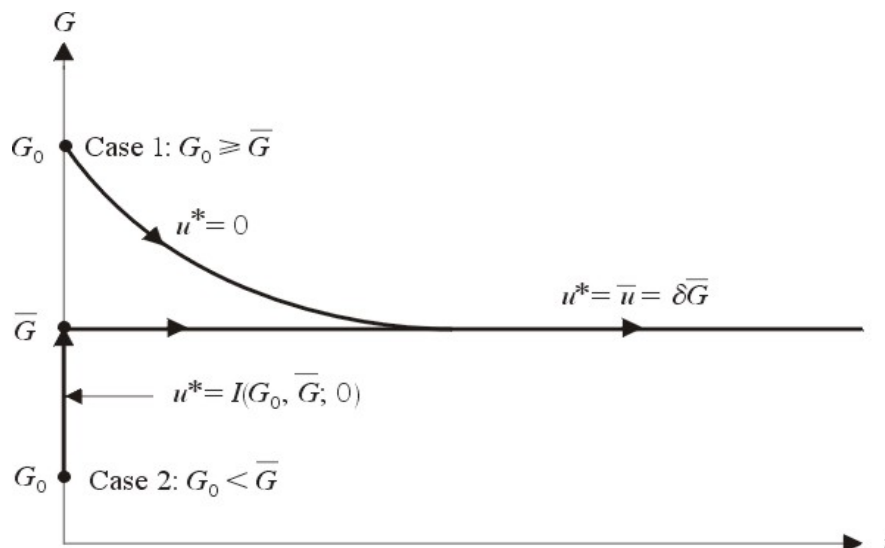
•17

15.3 OCT: Application

The second equation corresponds to the usual equilibrium relation for investment in capital goods:

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{\partial \pi}{\partial G}$$

It states that the marginal opportunity cost of investment in goodwill, $d\lambda := \dot{\lambda}dt$, should equal the sum of the marginal profit $(\partial\pi/\partial G)dt$ from increased goodwill and the capital gain, $\lambda(\varrho + \delta)dt$.

35

15.3 OCT: Application

•18

15.4 Calculus of variations – Classic Example

- The *calculus of variations* involves finding an extremum (maximum or minimum) of a quantity that is expressible as an integral.
- Question: What is the shortest path between two points in a plane? You know the answer -a straight line- but you probably have not seen a proof of this: the calculus of variations provides such a proof.
- Consider two points in the *x-y* plane, as shown in the figure.
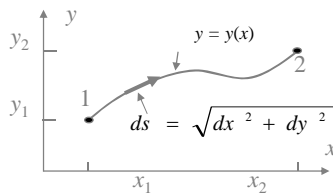
- An arbitrary path joining the points follows the general curve $y = y(x)$, and an element of length along the path is

$$ds = \sqrt{dx^2 + dy^2}.$$

37

15.4 Calculus of variations – Classic Example

- We can rewrite this as: $ds = \sqrt{1 + y'(x)^2}\,dx,$

  which is valid because $dy = \dfrac{dy}{dx}dx = y'(x)dx.$ Thus, the length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,dx.$$

- Note that we have converted the problem from an integral along a path, to an integral over *x*:

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,dx.$$

- We have thus succeeded in writing the problem down, but we need some additional mathematical machinery to find the path for which L is an extremum (a minimum in this case).

38

15.4 Calculus of variations – Classic Example

- In our usual minimizing or maximizing of a function $f(x)$, we would take the derivative and find its zeroes These points of zero slope are *stationary points* –i.e., the function is *stationary* at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope).
- Similarly, we want to be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called *calculus of variations*.
- The methods we will develop are called *variational methods*.
- These principles are common, and of great importance, in many areas of physics (such as quantum mechanics and general relativity) and economics.

39

15.4 Euler-Lagrange Equations

- We will try to find an extremum (to be definite, a minimum) for an as yet unknown curve joining two points x_1 and x_2, satisfying the integral relation:
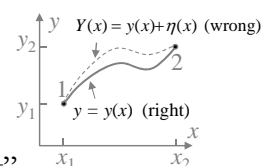
$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx.$$

- The function f is a function of three variables, but because the path of integration is $y = y(x)$, the integrand can be reduced to a function of just one variable, x.
- To start, let's consider two curves joining points 1 and 2, the "right" curve $y(x)$, and a "wrong" curve:

$$Y(x) = y(x) + \eta(x); \quad \eta(x_1) = \eta(x_2) = 0.$$

$Y(x)$ that is a small displacement from the "right" curve, as shown in the figure.

- We call the difference between these curves as some function $h(x)$.

40

15.4 Euler-Lagrange Equations

- There are infinitely many functions $h(x)$, that can be "wrong." We require that they each be longer than the "right" path. To quantify how close the "wrong" path can be to the "right" one, let's write $Y = y + \alpha h$, so that

$$S(\alpha) = \int_{x_1}^{x_2} f[Y, Y'(x), x] dx$$

$$= \int_{x_1}^{x_2} f[y + \alpha \eta, y' + \alpha \eta', x] dx.$$

- Now, we can characterize the shortest path as the one for which the derivative $dS/d\alpha = 0$ when $\alpha = 0$. To differentiate the above equation with respect to α, we need the partial derivative $\partial S / \partial \alpha$ via the chain rule

$$\frac{\partial f(y + \alpha \eta, y' + \alpha \eta', x)}{\partial \alpha} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'},$$

so $dS/d\alpha = 0$ gives

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx = 0$$

41

15.4 Euler-Lagrange Equations

- The second term in the equation can be integrated by parts:

$$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} dx = \left[\eta(x) \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx,$$

but the first term of this relation (the end-point term) is zero because $h(x)$ is zero at the endpoints.

- Our modified equation is:

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx = 0.$$

- This leads us to the *Euler-Lagrange* equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.$$

42

- Key: the modified equation has to be zero for any $h(x)$.

15.4 Euler-Lagrange Equations

- Let's go over what we have shown. We can find a minimum (more generally, a stationary point) for the path S if we can find a function for the path that satisfies:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- The procedure for using this is to set up the problem so that the quantity whose stationary path you seek is expressed as

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx,$$

where $f[y(x),y'(x),x]$ is the function appropriate to your problem.

- Then, write down the Euler-Lagrange equation, and solve for the function $y(x)$ that defines the required stationary path. [43]

15.4 Euler-Lagrange Equations – Example I

- Find the shortest path between two points:

$$L = \int_{1}^{2} ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- The integrand contains our function $f(y, y', x) = \sqrt{1 + y'(x)^2}$.
- The two partial derivatives in the Euler-Lagrange equation are:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}.$$

- The Euler-Lagrange equation gives us: $\quad \dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = \dfrac{d}{dx}\dfrac{y'}{\sqrt{1+y'^2}} = 0.$

- This says that $\quad \dfrac{y'}{\sqrt{1+y'^2}} = C, \quad \text{or} \quad y'^2 = C^2(1 + y'^2).$

   $\Rightarrow y'^2 =$ constant (call it m^2), so $y(x) = mx + b$
   $\Rightarrow$ A straight line is the shortest path. [44]

15.4 Euler-Lagrange Equations – Example II

- Intertemporal utility maximization problem:

$$\max \int_{x_1}^{x_2} [B - u(c(t))]dx \quad s.t. \quad c(t) = f(k(t)) - k'(t)$$

B: bliss level of utility

c(t) = c(k(t),k'(t))

- Use substitution method. Substitute constraint into integrand:

g(x(t),x'(t),t) = B - u(c(k(t),k'(t)) = V(k(t),k'(t))

- Euler-Lagrange equation: $\quad V_k = \dfrac{dV}{dk} = \dfrac{d}{dt}\dfrac{dV}{dk'} = \dfrac{d}{dt}V_{k'}$

$$V_k = -u'(c)f'(k); \quad V_{k'} = V_c \frac{dc}{dk'} = -u'(c)(-1) = u'(c)$$

$$V_k = \frac{d}{dt}V_{k'} \quad \Rightarrow -u'(c)f'(k) = \frac{d}{dt}u'(c)$$

45

15.4 Euler-Lagrange Equations – Example II

- Repeating Euler-Lagrange equation:

$$V_k = \frac{d}{dt}V_{k'} \quad \Rightarrow -u'(c)f'(k) = \frac{d}{dt}u'(c)$$

- If we are given functional forms:

f(k(t)) = k(t)^α

u(c(t)) = ln(c(t))

Then,

$$-u'(c)f'(k) = \frac{d}{dt}u'(c)$$

$$-\frac{1}{c(t)}\alpha k(t)^{\alpha-1} = \frac{d}{dt}(\frac{1}{c(t)}) = -\frac{\dot{c}(t)}{c(t)^2}$$

$$\Rightarrow \alpha k(t)^{\alpha-1} = \frac{\dot{c}(t)}{c(t)}$$

46

15.4 Calculus of Variations: Summary

• In the general economic framework, $F(.)$ will be the objective function, the variable x will be time, and y will be the variable to choose over time to optimize $F(.)$. Changing notation:

$$\max_x \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))dt \qquad \text{s.t. } x(t_0) = x_0, x(t_1) = x_1$$

• Necessary Conditions: Euler-LaGrange Equation

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0.$$

47

15.4 Calculus of Variations: Limitations

• Method gives extremals, but it does not tell maximum or minimum:
  - Distinguishing mathematically between max/min is more difficult.
  - Usually have to use geometry to setup the problem.

• Solution curve $x(t)$ must have continuous second-order derivatives
  - Requirement from integration by parts.

• We find stationary states, which vary only in space, not in time.
  - Very few cases in which systems varying in time can be solved.
  - Even problems involving time (e.g., brachistochrones) do not change in time.

48

15.5 Dynamic Programing (DP)

• "Programming" = planning   (engineering context)

• Idea: A problem can be broken into recursive sub-problems, which we solve and then combine the solutions of the sub-problems.

Example: Shortest route Houston-Chicago.

You know that the shortest route passes includes the segment Houston-Dallas. Then, finding the shortest Houston-Chicago route is simplified to finding the shortest Dallas-Chicago route.

We can reduce calculation if a problem can be broken down into recursive parts.

49

15.5 Dynamic Programing (DP)

• In general, it is quite efficient. Moreover, many problems are presented as DP problems, because DP is computationally efficient.

Example: Calculation of Fibonacci numbers.

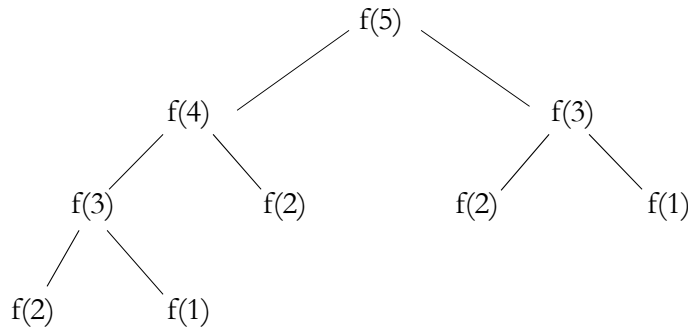We can calculate iteratively $f(n) = f(n\text{-}1) + f(n\text{-}2)$.

This function grows as n grows. The run time doubles as n grows and is order $O(2^n)$. Not efficient; there are a lot of repetitive calculations.

If we setup the problem as a recursive problem, time can be saved: Recursive problems have no memory!

50

15.5 Dynamic Programing (DP)

• We can pose the calculations as a DP problem:

```
                        f(5)
                       /    \
                  f(4)        f(3)
                 /    \      /    \
             f(3)      f(2) f(2)   f(1)
            /    \
         f(2)    f(1)
```

• Dynamic programming calculates from bottom to top.
• Values are stored for later use.
• This reduces repetitive calculation.

15.5 DP: Typical Problem

• DP provides an alternative way to solve intertemporal problems
• Equivalent in many contexts to methods already seen
• We decompose the problem into sub-problems, which we solve and then combine the solutions of the sub-problems.
• Ramsey problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t. } c_t + k_{t+1} - (1-\delta)k_t = f(k_t)$$

which can be rewritten as a recursive problem:

$$V(k_t) = \max_{c_t, k_{t+1}} \sum_{i=0}^{\infty} \beta^i u(k_{t+i}) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta \sum_{i=0}^{\infty} \beta^i u(c_{t+i+1})\}$$

$$= \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$$

$$\text{s.t. } k_{t+1} = f(k_t) - c_t - (1-\delta)k_t = g(k_t, c_t)$$

52

15.5 DP – Bellman (HJB) Equation

$$V(k_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$$

• The previous equation is usually called *Bellman equation*, also called *Hamilton-Jacobi-Bellman* (*HJB*) equation.

• The Bellman Equation expresses the value function as a combination of a flow amount $u(.)$ and a discounted continuation payoff.

• The current value function is $V(k_t)$. The continuation value function is $V(k_{t+1})$. $V(k_t)$ measures the best that can be done given the current state and remaining time.

• The solution to the Bellman Equation is $V(.)$. In general, it is difficult to obtain (usual methods: guess a solution, value-function iterations).

• The key to get $V(.)$ is a proper f.o.c.. Several ways to do it: Lagrangean, tracing the dynamics of $V(k_t)$, and using envelope theorem.

53

15.5 DP – Necessary Conditions

• Forming a general Bellman equation:

$$V_t(x_t) = Max_{u_t}\{f_t(u_t, x_t) + \beta V_{t+1}(x_{t+1}): \ x_{t+1} = g_t(u_t, x_t)\}$$
$$= Max_{u_t}\{f_t(u_t, x_t) + \beta V_{t+1}(x_{t+1}): \ x_{t+1} = g_t(u_t, x_t)\}$$

Assuming $V_{t+1}(x_{t+1})$ is differentiable, and, then, $\lambda_{t+t} = V_{t+1}{}'(x_{t+1})$,

the optimality conditions (f.o.c.) in the Bellman equation become:

$$D_{u_t} f_t(u_t, x_t) + \beta \lambda_{t+t} D_{u_t} g_t(u_t, x_t) = 0 \qquad (*)$$

The same condition we derived using the Lagrangean method in 15.2.

Let $u_t = h_t(x_t)$ be the policy function, the solution to (*), then
$$V_t(x_t) = f_t(h(x_t), x_t) + \beta V_{t+1}(g_t(u_t, x_t))$$

Assuming differentiability for $h(x_t)$ and $V_t(x_t)$:

$$\lambda_t = V_t'(x_t) = D_{u_t}f_t D_{x_t}h_t + D_{x_t}f_t + \beta \lambda_{t+1}(D_{u_t}g_t D_{x_t}h_t + D_{x_t}g_t)$$ 54

15.5 DP – Necessary Conditions

• Assuming differentiability for $h(x_t)$ and $V_t(x_t)$:

$$\lambda_t = V_t'(x_t) = D_{u_t}f_t D_{x_t}h_t + D_{x_t}f_t + \beta\lambda_{t+1}(D_{u_t}g_t D_{x_t}h_t + D_{x_t}g_t)$$

$$= (D_{u_t}f_t + \beta\,\lambda_{t+1}\,D_{u_t}g_t)D_{x_t}\,h_t + D_{x_t}f_t + \beta\,\lambda_{t+1}D_{x_t}h_t$$

$$\Rightarrow \qquad \lambda_t = D_{x_t}f_t + \beta\,\lambda_{t+1}D_{x_t}h_t$$

which, again, is the recursion same recursion we obtained using the Lagrangean method in section 15.2.

• That is, we have the same f.o.c.. The attraction of DP is the recursive method, which is easier to program.

55

15.5 DP – Oil Lease Example

• **Example**: We solved this example in section 15.2, using the Lagrangean method. Recall, the optimal production solves:

$$Max_{u_t \in U_t} \sum_{t=0}^{2} \beta^t (P - \kappa * \frac{u_t}{x_t})u_t \quad \text{s.t} \qquad x_{t+1} = x_t - u_t.$$

Assume $V_{T=2} = 0$. Then,

$$V_1(u_1) = Max_{w \in U_t} \{\left(60 - 45 * \frac{w}{x_1}\right)w + V_{T=2}\}$$

$$\Rightarrow w = u_1^* = \frac{2}{3}x_1 \qquad \Rightarrow V_1(u_1^*) = (60 - 45 * \frac{\frac{2}{3}x_1}{x_1})\frac{2}{3}x_1 = 20x_1$$

Then, $V_0(u_0) = Max_{w \in U_t} \{\left(60 - 45 * \frac{w}{x_0}\right)w + V_{T=1}\}$

$$= Max_{w \in U_t} \{\left(60 - 45 * \frac{w}{x_0}\right)w + 20(x_0 - w)\}$$

$$\Rightarrow w = u_0^* = \frac{4}{9}x_0 = \frac{4}{9} * 8{,}000 = 3{,}555.56$$

56

15.5 DP – Oil Lease Example

• **Example (continuation)**:

$\Rightarrow w = u_0^* = \frac{4}{9}x_0 = \frac{4}{9} * 8{,}000 = 3{,}555.56$

$\Rightarrow x_1 = x_0 - u_0 = 8{,}000 = 3{,}555.56 = 4{,}444.44$

$\qquad \Rightarrow u_1^* = \frac{2}{3}x_1 = \frac{2}{3} * (x_0 - w) = \frac{2}{3} * 4{,}444.44 = 2{,}962.96.$

$\qquad \Rightarrow x_2 = x_1 - u_1 = 4{,}444.44 - 2{,}962.96 = 1{,}481.48$

• $u = [3555.56, 2962.96], \quad x = [8000, 4444.44, 1481.48], \ \& \ \lambda = [20, 0].$

• Same solution as before in section 15.2, as expected. But, here, we start by computing the value function starting from the terminal state.

57

15.5 DP – Recursive Solution

• Now, we solve the Ramsey problem recursively, as a sequence of two-period problems:

- **Step (1):** First solve the problem at t = T

Choose c_T and k_{T+1} to maximize: $\quad \beta^T u(c_T) + V_0(k_{T+1}) \qquad k_T$ given

$\qquad$ subject to $\qquad c_T + k_{T+1} - (1-\delta)k_T = f(k_T)$

$\qquad\qquad\qquad \Rightarrow c_T = c_T(k_T), k_{T+1} = k_{T+1}(k_T)$

- **Step (2):** Now solve the period T-1 problem

$\qquad$ Choose c_{T-1} and k_T to maximize

$\qquad\qquad \beta^{T-1}u(c_{T-1}) + \beta^T u(c_T) + V_0(k_T(k_{T+1}))$

$\qquad$ subject to $\quad c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1}) \qquad k_{T-1}$ given

58

- **Step (3):** Now solve the period T-2 problem

•29

15.5 DP – Recursive Solution

Continue solving backwards to time 0.

The same optimality conditions arise from the problem:

$$V_1(k_T) = \max_{c_T, k_{T+1}} u(c_T) + \beta V_0(k_{T+1})$$

subject to $\quad c_T + k_{T+1} - (1-\delta)k_T = f(k_T)$

Optimality Conditions:

$$u'(c_T) = \lambda_T$$
$$\beta V_0'(k_{T+1}) - \lambda_T \leq 0, \left[\beta V_0'(k_{T+1}) - \lambda_T\right]k_{T+1} = 0$$
$$c_T + k_{T+1} - (1-\delta)k_T = f(k_T)$$

These are the same conditions as in 15.2 (though we redefined λ_T). 59

15.5 DP – Recursive Solution

Envelope Theorem implies $\quad V_1'(k_T) = \lambda_T\left[f'(k_T) + (1-\delta)\right]$

Given the constraint and k_T, $V_1(k_T)$ is the maximized value of

$$u(c_T) + \beta V_0(k_{T+1})$$

Period T-1 problem is equivalent to maximizing

$$u(c_{T-1}) + \beta V_1(k_T)$$

with the same constraint at T-1 and k_{T-1} given:

$$V_2(k_{T-1}) = \max_{c_{T-1}, k_T} u(c_{T-1}) + \beta V_1(k_T)$$

subject to $\quad c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$

60

15.5 DP – Recursive Solution

Optimality Conditions:

$$u'(c_{T-1}) = \lambda_{T-1}$$
$$\beta V_1'(k_T) = \lambda_{T-1}$$
$$c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$$

The envelope theorem can be used to eliminate V_1'

$$u'(c_{T-1}) = \lambda_{T-1}$$
$$\beta \lambda_T [f'(k_T) + (1-\delta)] = \lambda_{T-1}$$
$$c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$$

The period T-1 envelope condition is

$$V_2'(k_{T-1}) = \lambda_{T-1} [f'(k_{T-1}) + (1-\delta)]$$

61

15.5 DP – Bellman's Principle of Optimality

This process can be continued giving the following HJB Equation:

$$V_{J+1}(k_{T-j}) = \max_{c_t, k_t} \{u(c_{T-J}) + \beta V_0(k_{T-j+1})\}$$

s.t. $c_{T-J} + k_{T-J+1} - (1-\delta)k_{T-J} = f(k_{T-J})$ k_{T-j} given

• The general *HJB* equation, can be written as

$$V(k_T) = \max_{c_t, k_t} \{u(c_T) + \beta V(k_{T+1})\}$$

• Bellman's Principle of Optimality

The fact that the original problem can be written in this recursive way leads us to Bellman's Principle of Optimality:

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." 62

15.5 DP – HJB Equation - Summary

• The general *HJB* equation, can be written as

$$V(k_T) = \max_{c_t, k_t} \{u(c_T) + \beta V(k_{T+1})\}$$

• Using the Lagrangean approach we rewrite $V(k_T)$ as:

$$V(k_t) = \max_{c_t, k_t} \{u(c_t) + \beta V(k_{t+1}) + \lambda_t [g(k_t, c_t) - k_{t+1}]\}$$

with f.o.c.: $u'(c_t) + \lambda_t \dfrac{dg(k_t, c_t)}{dc_t} = 0$

$$\beta V'(k_{t+1}) - \lambda_t = 0$$

Envelope Theorem: $V'(k_t) = \lambda_t \dfrac{dg(k_t, c_t)}{dc_t}$

Euler equation: $u'(c_t) + \beta V'(k_{t+1}) \dfrac{dg(k_t, c_t)}{dc_t} = 0$

63

15.5 DP – Solving HJB Equations

• Since *t* is arbitrarily taken, the Euler equation must also hold if we take one period backward:

$$u'(c_{t-1}) + \beta V'(k_t) \frac{dg(k_{t-1}, c_{t-1})}{dc_{t-1}} = 0$$

<u>Note</u>: An Euler equation is a difference or differential equation that is an intertemporal f.o.c. for a dynamic choice problem. It describes the evolution of economic variables along an optimal path.

• If the Euler equation allows us to find a policy function *c = h(k)*, then the HJB equation becomes:

$$V(k_t) = \{u[h(k_t)] + \beta V[f(k_t) - h(k_t) + (1 - \delta)k_t\}$$

• This equation has no explicit solution for generic utility and production functions.

Richard E. Bellman (1920 –1984, USA)

15.5 DP – Solving HJB Equations

• In many situations, we want to go beyond the Euler equation to obtain an explicit solution. Then, we need to find the solution $V_t(x_t)$ of this functional equation.

• Given $V_t(x_t)$, it is straightforward to solve the HJB's equation successively to compute the optimal policy. To solve, we use backward induction. But, in the infinite horizon case we cannot do this.

• There are several ways to solving HJB's equation in infinite horizon cases: (a) guess and verify; (b) value function iteration; & (c) policy function iteration.

• In simple cases, it is possible to guess the functional form of the value function, and then verify that it satisfies Bellman's equation.

65

15.5 DP – Solving HJB Equations - Induction

Example: Ramsey problem, with $u(c) = ln(c);\ f(k) = k^\alpha;\ \&\ \delta=1$.

Then, $u'(c_t) + \beta V'(k_{t+1}) \dfrac{dg(k_t, c_t)}{dc_t} = 0$

$$\frac{1}{c_t} + \beta \frac{\alpha k_{t+1}^{\alpha-1}}{c_{t+1}}(-1) = 0 \qquad \Rightarrow \frac{1}{c_t} = \beta \frac{\alpha k_{t+1}^{\alpha-1}}{c_{t+1}}$$

$$\frac{k_{t+1}}{c_t} = \alpha\beta \frac{k_{t+1}^\alpha}{c_{t+1}} \qquad \Rightarrow \frac{k_t^\alpha - c_t}{c_t} = \alpha\beta \frac{k_{t+1}^\alpha}{c_{t+1}}$$

$$\frac{k_t^\alpha}{c_t} = \alpha\beta \frac{k_{t+1}^\alpha}{c_{t+1}} + 1$$

• This is a 1st-order difference equation. The forward solution, assuming $\alpha\beta$ small (as T→∞):

$$\frac{k_t^\alpha}{c_t} = \frac{1}{1-\alpha\beta} \qquad \Rightarrow c_t = (1-\alpha\beta)k_t^\alpha$$

$$\Rightarrow k_{t+1} = k_t^\alpha - c_t = \alpha\beta k_t^\alpha \qquad 66$$

15.5 DP – Solving HJB Equations – Guess

• Guess a solution

Suppose $u(c) = ln(c)$; $f(k) = Ak^\alpha$; and $\delta = 1$.

Guessed solution: $V(k) = B_0 + B_1 \, ln(k)$.

Then: $V(k) = max \, \{ln(c_t) + \beta \, [B_0 + B_1 \, ln(k_{t+1})]\}$

• After some work from f.o.c., we get:

$$k_{t+1} = \frac{\beta B_1}{1 + \beta B_1} k_t^\alpha$$

$$c_t = h(k_t) = \frac{Ak_t^\alpha}{1 + \beta B_1}$$

67

15.5 DP – Solving HJB Equations – Guess

• By substitution in the HJB equation:

$$\left[B_0 + B_1 \ln(k_t)\right] = \ln(\frac{Ak_t^\alpha B_1}{1 + \beta B_1}) + \beta \left[B_0 + B_1 \ln(\frac{Ak_t^\alpha B_1}{1 + \beta B_1})\right]$$

• After some algebra, we get $\quad c_t^* = (1 - \alpha\beta) \, A \, k_t^\alpha$

$$k_{t+1}^* = \alpha \, \beta \, A \, k_t^\alpha$$

68

15.5 DP – Extension: Uncertainty

• DP can easily be extended to maximization under uncertainty.

$$\max_{\{c_t, k_{t+1}\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \qquad \text{s.t.} \ \ k_{t+1} = \varepsilon_t f(k_t) - c_t$$

where the production $y_t = f(k_t)$ is affected by a random shock, ε_t -an *i.i.d.* process-, realized at the beginning of each period t, so that the value of output is known when consumption takes place.

For example, we introduce a technology input, z_t, subject to a shock:
$$y_t = k_t^{\alpha} z_t^{1-\alpha}, \qquad \text{where } z_t = \kappa + z_{t-1} + \varepsilon_t$$

• The *HJB* equation becomes:

$$V_t(k_t, \varepsilon_t) = Max_{c_t, k_{t+1}} \{ u_t(c_t) + \beta E_t[V_{t+1}(k_{t+1}, \varepsilon_{t+1})] \}$$

69

15.5 DP – Extension: Epstein-Zin Preferences

• Today, Epstein-Zin preferences are used, especially in DGSE (*Dynamic Stochastic General Equilibrium)* models, since they incorporate early/later *resolution of uncertainty* (through IES). Typical problem:

$$V_t = \left\{ (1 - \beta) \, c_t^{1-\rho} + \beta \left[E_t V_{t+1}^{1-\rho} \right]^{\frac{1}{\theta}} \right\}^{\frac{\theta}{1-\rho}},$$

where $\rho \geq 0$ is the parameter that controls risk aversion, $\psi \geq 0$ is the intertemporal elasticity of substitution (IES) and
$$\theta = \frac{1-\rho}{1 - \frac{1}{\psi}}$$

<u>Note</u>: The term $\beta \left[E_t V_{t+1}^{1-\rho} \right]^{\frac{1}{1-\rho}}$ is called *risk-adjusted expectation operator*. When $\rho = 1$, preferences collapse to the state-separable, CRRA utility. When $\rho > \frac{1}{\psi}$, agents prefer an early resolution of uncertainty.

70

15.5 DP – Extension: More Goods and Inputs

• More goods, for example, consumption, c_t, and leisure/labor, l_t are usually introduced, in general, keeping the power utility framework.

$$u_t(c_t, l_t) = c_t^\nu \, l_t^{1-\nu}$$

• More production inputs, for example labor, can also be incorporated in the model, for example:

$$y_t = k_t^\alpha \, (z_t l_t)^{1-\alpha}$$

• These realistic problems are complicated to solve. Many methods.

• One method ("perturbation") takes a two-step approach:

(1) Solve equations under the "no uncertainty case." Then, use Taylor-expansions (1st, 2nd, etc) to approximate value and policy functions.

(2) Add equations with uncertainty and estimate econometric model.   71

Ken Arrow's office

72