Proposition 1. *If $k(\theta) > 0$, then it must be that $\alpha\theta > R$. Furthermore, $\phi > 0$, and $c_1^y = c_1^0 + \frac{\beta\phi}{\lambda_1(1-\alpha)}$.*

Proof. The proof proceeds by contradiction. Assume that for some θ, $k(\theta) > 0$ at the optimum of the planner's problem, but $\alpha\theta < R$. Then, from the first-order condition for k in the planner's problem,

$$R - \alpha\theta = \frac{\mu}{\lambda_1 \theta c_0} - \frac{\phi}{\lambda_1 (c_0 + k)} > 0$$

which implies that

$$\frac{\mu}{\lambda_1 \theta c_0} > \frac{\phi}{\lambda_1 (c_0 + k)} > 0$$

Thus, μ–the multiplier on the incentive constraint–is positive. Then, rearranging the first-order conditions for c_0 and k gives

$$\begin{aligned} R + \frac{\phi}{\lambda_1 (c_0 + k)} &= \frac{c_1^y}{\beta c_0} - \frac{\mu k}{\lambda_1 \theta c_0^2} \\ R + \frac{\phi}{\lambda_1 (c_0 + k)} &= \alpha\theta + \frac{\mu}{\lambda_1 \theta c_0} \end{aligned}$$

which implies

$$R + \frac{\phi}{\lambda_1 (c_0 + k)} = \frac{c_1^y}{\beta c_0} - \frac{k}{c_0}\left(\underbrace{R + \frac{\phi}{\lambda_1 (c_0 + k)} - \alpha\theta}_{>0}\right)$$

The term in parentheses on the right-hand side is equal to $\frac{\mu}{\lambda_1 \theta c_0}$, which by the above, is positive. This implies that

$$\frac{c_1^y}{\beta c_0} > R + \frac{\phi}{\lambda_1 (c_0 + k)}$$

This, however, implies that $\mu < 0$, contradicting the assumption. Thus, $k(\theta) > 0 \implies \alpha\theta > R$. This also shows that if the first-order condition for $k(\theta)$ in the planner's problem holds, $\mu < 0$.

The first-order condition for $c_1(\theta, 0)$, meanwhile, can be rearranged to give

$$c_1^y = c_1^0 + \frac{\beta\phi}{\lambda_1 (1 - \alpha)}$$

By definition, $\phi \geq 0$, with equality if the constraint in REF does not hold. Equation REF ABOVE shows that $k > 0 \implies \phi > 0$; otherwise, the agents would have no incentive to bear the risk of investing. $\square$

Proposition 2. *If $k > 0$, $\tau_k \geq 0$, with equality if $\theta = \overline{\theta}$, and $\tau_b > 0$. If $k = 0$, $\tau_b = 0$.*

Proof. Recall that the intertemporal wedges are given by

$$\begin{aligned}
\tau_k(\theta) =& 1 - \frac{c_1^y}{\alpha\beta\theta c_0} \\
\tau_b(\theta) =& 1 - \frac{\frac{1}{c_0}}{\beta R\left(\frac{\alpha}{c_1^y} + \frac{1-\alpha}{c_1^0}\right)}
\end{aligned}$$

Begin with the case where $k > 0$. Combining the first-order conditions in the planner's problem for c_0 and k gives

$$\frac{c_1^y}{\beta c_0} - \frac{\mu k}{\lambda_1 \theta c_0^2} = \alpha\theta + \frac{\mu}{\lambda_1 \theta c_0}$$

which implies

$$\begin{aligned}
\alpha\theta - \frac{c_1^y}{\beta c_0} &= \frac{\mu}{\lambda_1 \theta c_0}\left(-\frac{k}{c_0} - 1\right) \\
\tau_k = 1 - \frac{c_1^y}{\alpha\beta\theta c_0} &= \frac{\mu}{\lambda_1 \alpha \theta^2 c_0}\left(-\frac{k}{c_0} - 1\right)
\end{aligned}$$

Recall that if $k(\theta) > 0$, $\mu(\theta) < 0$. Because all other variables on the right-hand side of the final equality are nonnegative, the above shows that $\tau_k > 0$. Turning to the wedge on risk-free savings, note that if $k(\theta) > 0$, then period-1 consumption $c_1(\theta)$ becomes a random variable, so by Jensen's inequality,

$$\begin{aligned}
\frac{\alpha}{c_1^y} + \frac{1-\alpha}{c_1^0} &> \frac{1}{\alpha c_1^y + (1-\alpha)c_1^0} \implies \\
\alpha c_1^y + (1-\alpha)c_1^0 &> \frac{1}{\frac{\alpha}{c_1^y} + \frac{1-\alpha}{c_1^0}} \implies \\
\frac{\alpha c_1^y + (1-\alpha)c_1^0}{\beta R c_0} &> \frac{\frac{1}{c_0}}{\beta R\left(\frac{\alpha}{c_1^y} + \frac{1-\alpha}{c_1^0}\right)}
\end{aligned}$$

So in order to establish a positive wedge τ_b, it is sufficient to show that $\beta R c_0 > \alpha c_1^y + (1-\alpha)c_1^0$. From the first-order conditions in the planner's problem,

$$\alpha c_1^y + (1-\alpha)c_1^0 = c_1^y - \frac{\beta\phi}{\lambda_1}$$

and

$$\beta R c_0 = c_1^y - \frac{\beta\mu k}{\lambda_1 \theta c_0} - \frac{\beta\phi c_0}{\lambda_1(c_0 + k)}$$

Combining and multiplying through by $(-\beta/\lambda_1)$, it suffices to show that

$$\phi\left(1 - \frac{c_0}{c_0 + k}\right) \geq \frac{\mu k}{\theta c_0}$$

Note, however, that this inequality holds trivially: by definition, the right hand side is positive, while the left is negative.

In the case where $k(\theta) = 0$, $c_1(\theta, y) = c_1(\theta, 0) \equiv c_1(\theta)$; the planner does not find it optimal for these types to invest, and so she has no need to incentivize them to do so. By Proposition REF1, then, $\phi(\theta) = 0$. Then, the first-order condition for $c_0(\theta)$ in the planner's problem gives

$$1 = \frac{c_1}{\beta R c_0}$$

and thus $\tau_b(\theta) = 0$ for types who do not invest. $\square$