Optimal Taxation with Heterogeneous Rates of Return
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# 1 Tax Problem

In this problem I assume that individuals are indexed by type $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$ and endowed with some initial $w_0$. Individuals choose their consumption and savings $k$, and produce output $y = \theta k$. Individuals have utility over consumption, and discount the future at rate $\beta$. The government, unable to observe $\theta$ or $k$, can levy a (possibly nonlinear) tax $T$ on $\theta k$.

To begin, denote

$$\begin{aligned}\mathcal{U}(\theta) &= \max_{k \in [0, w_0]} u(w_0 - k) + \beta u(\theta k - T(\theta k)) \\ &\equiv u(w_0 - k(\theta)) + \beta u(\theta k(\theta - T(\theta k(\theta))) \qquad (1)\end{aligned}$$

The envelope condition applied to (1) gives

$$\mathcal{U}'(\theta) = \beta u'(\theta k - T(\theta k))k(1 - T'(\theta k)) \qquad (2)$$

The first-order condition for the individual's problem in (1), meanwhile, gives

$$1 - T'(\theta k) = \frac{u'(w_0 - k)}{\beta \theta u'(\theta k - T(\theta k))} \qquad (3)$$

Combining (2) and (3) gives the individual optimality condition:

$$\mathcal{U}'(\theta) = u'(w_0 - k)\frac{k}{\theta} \qquad (4)$$

The government's objective is to choose a tax function $T(\theta k(\theta))$ to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} \Psi(\mathcal{U}(\theta)) f(\theta) d\theta \qquad (5)$$

where $\Psi$ is a concave function over utilities representing redistributive motives. The government maximizes (5) subject to (4) and its resource constraint in the second period:

$$\int_{\underline{\theta}}^{\bar{\theta}} c(\theta) dF(\theta) \leq \int_{\underline{\theta}}^{\bar{\theta}} \theta k(\theta) dF(\theta) - E$$

where $E$ is government expenditures. Note that because $c(\theta) = \theta k(\theta) - T(\theta k(\theta))$, the above constraint is equivalent to

$$\int_{\underline{\theta}}^{\bar{\theta}} T(\theta k(\theta)) \geq E \qquad (6)$$

I assume no taxes in the first period, so by the definition of $\mathcal{U}$ in (1), the resource constraint is guaranteed to hold in the first period. Following Mirrlees (1971), Diamond (1998), and Salanie (2011), I formulate the Hamiltonian for the government's problem, with $\mathcal{U}(\theta)$ as the state and $k(\theta)$ as the control:

$$\mathcal{H} = \Psi\left(\mathcal{U}(\theta)\right) f(\theta) + \lambda T(\theta k(\theta)) f(\theta) + \mu(\theta) u'(w_0 - k(\theta)) \frac{k(\theta)}{\theta} \qquad (7)$$

The Pontryagin maximization principle gives three conditions: first, $k(\theta)$ maximizes $\mathcal{H}$, so

$$0 = \frac{\partial \mathcal{H}}{\partial k(\theta)}$$

and from this,

$$-\lambda\theta^2 f(\theta)T'(\theta k(\theta)) = \mu(\theta)\left[u'(w_0 - k(\theta)) - u''(w_0 - k(\theta))k(\theta)\right] \tag{8}$$

The costate equation gives

$$\mu'(\theta) = -\frac{\partial H}{\partial \mathcal{U}(\theta)} = -\Psi'(\mathcal{U}(\theta)) \tag{9}$$

The boundary conditions are

$$\mu(\underline{\theta}) = \mu(\bar{\theta}) = 0$$

Integrating (9), along with the boundary condition at $\bar{\theta}$, gives

$$\mu(\theta) = -\int_{\theta}^{\bar{\theta}} \Psi'\left(\mathcal{U}(t)\right) f(t)dt \tag{10}$$

Thus, the optimality condition for this taxation problem is

$$T'(y) = \frac{1}{\lambda\theta^2 f(\theta)}\left(\int_{\theta}^{\bar{\theta}} \Psi'\left(\mathcal{U}(t)\right) f(t)dt\right)\left[u'(w_0 - k(\theta)) - u''(w_0 - k(\theta))k(\theta)\right] \tag{11}$$

This condition needs work on a few dimensions. First, it lacks the formulation for $\lambda$ in, for example, Diamond (1998). Additionally, it includes the allocations $k(\theta)$ inside of it, while the optimality conditions derived by Diamond (1998) and Salanie (2011) incorporate elasticities instead.

## 2 Updated Mechanism Design Problem

Here, I revisit the mechanism design formulation of this problem, which I did not formulate correctly. Here, I assumed that the planner chooses allocations $y(\theta)$ and $c_1(\theta)$ to maximize

$$\int_{\underline{\theta}}^{\bar{\theta}} \mathcal{U}(\theta) f(\theta) d\theta \tag{12}$$

where

$$\mathcal{U}(\theta) = u\left(w_0 - \frac{y(\theta)}{\theta}\right) + \beta u(c_1(\theta)) \tag{13}$$

In section 1, I assumed that $c_1(\theta) = \theta k(\theta) - T(\theta k(\theta))$. With this definition, along with the assumption that $\Psi(\mathcal{U}) = \mathcal{U}$, the problems in sections 1 and 2 are the same. Note also that the envelope condition applied to (13) gives

$$\begin{aligned}\mathcal{U}'(\theta) &= u'\left(w_0 - \frac{y(\theta)}{\theta}\right)\frac{y(\theta)}{\theta^2} \\ &= u'(w_0 - k)\frac{k}{\theta} \end{aligned} \tag{14}$$

exactly as in (4). The incentive constraints are:

$$\theta \in \arg\max_{\hat{\theta}} u\left(w_0 - \frac{\hat{\theta}k}{\theta}\right) + \beta u(c_1(\hat{\theta})) \quad \forall \theta \in \Theta \tag{15}$$

The constraints in (15) can be interpreted as follows: the planner collects reports $\hat{\theta}$, and allocates output $y(\theta)$ and consumption $c_1(\theta)$. Thus, if an agent of type $\theta$ claims to be of type $\hat{\theta}$, she will receive $c_1(\hat{\theta})$, but in return, she will be required to produce output $y(\hat{\theta})$, requiring investment $\frac{\hat{\theta}k}{\theta}$. The Hamiltonian for the government's problem is

$$\mathcal{H} = \left[u(w_0 - k(\theta)) + \beta u(c_1(\theta))\right] f(\theta) + \lambda \left[\theta k(\theta) - c(\theta)\right] f(\theta) + \mu(\theta) \frac{u'(w_0 - k(\theta))k(\theta)}{\theta} \tag{16}$$

## References


Peter A Diamond. Optimal income taxation: an example with a u-shaped pattern of optimal marginal tax rates. *American Economic Review*, pages 83–95, 1998.

James A Mirrlees. An exploration in the theory of optimum income taxation. *The review of economic studies*, 38(2):175–208, 1971.

Bernard Salanie. *The economics of taxation.* MIT press, 2011.