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Optimal Indirect and Capital Taxation

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We consider an environment in which agents' skills are private information and follow arbitrary stochastic processes. We prove that it is typically Pareto optimal for an individual's marginal benefit of investing in capital to exceed his marginal cost of doing so. This wedge is consistent with a positive tax on capital income. We also prove that it is Pareto optimal for the marginal rate of substitution between any two consumption goods to equal the marginal rate of transformation. This lack of a wedge is consistent with uniform taxation of consumption goods within a period.

1. INTRODUCTION

The modern economic analysis of optimal taxation has at least two important lines of research. The first emphasizes the effects of taxation on capital accumulation (see Chari and Kehoe, 1999, for an excellent survey). The basic assumption is that a government faces a dynamic Ramsey problem: it needs to fund a stream of purchases over time using linear taxes on capital and labour income. The hallmark result of this literature is that it is optimal for the government to set capital income tax rates to zero in the long run (Judd (1985), Chamley (1986)).

A second branch of the literature is based on the work of Mirrlees (1971, 1976). Here, the government has access to nonlinear taxation. However, agents have fixed heterogeneous skill levels that are unobservable to others. The goal of taxation in this setting becomes (in part) one of transferring resources from the highly skilled to the less skilled in an efficient way, given that incomes but not skills are observable. An important lesson of this literature is the *uniform commodity taxation theorem* of Atkinson and Stiglitz (1976, 1980). It states that if utility is weakly separable between consumption and leisure, then, despite the presence of the incentive problem, it is socially optimal for all consumption goods to be taxed at the same rate.

In this paper, we re-examine the zero capital taxation and uniform commodity taxation theorems in the context of a large class of dynamic economies. We enlarge the class of economies by allowing for unobservable skills to evolve stochastically over time. We impose *no* restriction on the evolution of skills except that it must be independent across agents.

Besides enlarging the class of economies in this way, we enlarge the choice set of the taxation authority. We do not restrict attention to linear tax schemes (*à la* Ramsey) or piecewise differentiable schemes (*à la* Mirrlees). Instead, we allow the taxation authority to use arbitrary nonlinear tax schemes; in other words, it can achieve any incentive-compatible and physically feasible allocation.

This general class of environments is technically challenging because it features dynamically evolving private information. There is no known way to develop a full characterization of the socially optimal allocations in this environment. In particular, we might well obtain misleading answers if we were to simply substitute first-order conditions for the large number of incentive constraints, and then apply Lagrangian methods.[1]

In the first part of the paper, we re-consider the zero capital income taxation theorem. We specialize the environment to have only one consumption good. We assume also that utility is additively separable in consumption and leisure. We prove that in a Pareto optimal[2] allocation, individual consumption satisfies a "reciprocal" intertemporal first-order condition of the kind derived by Rogerson (1985*a*):

$$1/u'(c_t) = (\beta R_{t+1})^{-1} E_t\{1/u'(c_{t+1})\}. \tag{1}$$

Here, R_{t+1} is the marginal return to investment, u is the agent's momentary utility function, β is the individual discount factor, and E_t is the period t conditional expectation (with respect to the randomness generated by period $(t+1)$ skills).

This "reciprocal first-order condition" has an important consequence. If individual marginal utility $u'(c_{t+1})$ in a Pareto optimum is random from the point of view of period t, then from Jensen's inequality we know that

$$u'(c_t) < \beta R_{t+1} E_t u'(c_{t+1}). \tag{2}$$

(The incentive problem means that it is typically efficient for individual consumption to be stochastic: the planner needs to offer more consumption to high skill types to get them to work more.) Thus, in an optimal tax system, the individual's marginal benefit of purchasing capital is higher than his marginal cost of doing so.

The intuition behind the inequality (2) is as follows. Suppose society considers increasing investment by lowering an individual's period t consumption by ε_t and raising an individual's period $(t+1)$ consumption by $\varepsilon_t R_{t+1}$. Doing so has two immediate consequences on social welfare (measured in utiles): there is a cost $u'(c_t)\varepsilon_t$ and a benefit $\beta\varepsilon_t R_{t+1} E_t u'(c_{t+1})$. However, there is an additional adverse effect on welfare. At an interior optimum, u is locally concave. Hence, the period t conditional covariance between period $(t+1)$ skills and $u(c_{t+1} + \varepsilon_t R_{t+1})$ is lower than the period t conditional covariance between period $(t+1)$ skills and $u(c_{t+1})$. Reducing this covariance provides less incentive for the agent to work in period $(t+1)$; his effort, and societal output, therefore fall in period $(t+1)$.

Thus, lowering consumption in period t and raising consumption in period $(t+1)$ generates the usual benefit $\beta\varepsilon_t R_{t+1} E_t u'(c_{t+1})$, the usual cost $\varepsilon_t u'(c_t)$, and an additional cost due to the incentive problem. In a social optimum, the marginal social cost and the marginal social benefit are equated, which implies that the partial marginal cost $u'(c_t)$ is less than the total marginal benefit $\beta R_{t+1} E_t u'(c_{t+1})$.[3]

We go on to re-consider the uniform commodity taxation theorem. We revert to the general assumption of multiple consumption goods, and assume that utility is weakly separable between consumption and labour. We prove that any Pareto optimal allocation has the property that within a period, the marginal rate of substitution between any two consumption goods, for any

1. Rogerson (1985*b*) provides sufficient conditions for the validity of the first-order approach in a static principal-agent context. However, there are no known generalizations of his conditions in dynamic settings.

2. By Pareto optimal, we mean Pareto optimal relative to the set of all allocations that are both incentive-compatible and physically feasible.

3. See Kocherlakota (1998) and Mulligan and Sala-i-Martin (1999) for a similar intuition.

agent, equals the marginal rate of transformation between those goods. This result implies that if agents can trade consumption goods in a spot market, all consumption goods should be taxed uniformly.

The idea behind the proof of the uniform commodity taxation theorem is as follows. Because utility is weakly separable, consumption only affects the incentive constraints and the planner's objective function through the amount of sub-utility derived from consumption. Hence, as long as resources are scarce, the planner wants to find a way to deliver these sub-utilities that minimizes the resource cost of doing so. This immediately implies the uniform commodity taxation theorem.

Our positive capital taxation and uniform commodity taxation results have predecessors in the literature. For example, Diamond and Mirrlees (1978, 1986) prove in a particular dynamic setting that Pareto optima feature the above kind of intertemporal wedge. They derive their result in a model of disability insurance: they assume that skills are hidden, have a two-point support in all periods, and that the low skill state is an absorbing one. As stated above, Atkinson and Stiglitz (1976) prove the uniform commodity taxation theorem assuming that skills do not change over time.

The main contribution of our analysis over this previous work is our *generality*. There is a large empirical literature on the intertemporal structure of individual wages and skills. The consensus in this literature is that an empirically plausible statistical model of the intertemporal evolution of individual skills should allow for the possibility of both a random fixed component and an autoregressive (possibly unit root) component.[4] There are *no* prior results in the dynamic private information literature that allow for such an elaborate stochastic process. In contrast, we allow an individual's hidden skills to follow *any* stochastic process, and we are still able to establish two important partial characterizations of Pareto optimal allocations.[5]

Our results are about wedges (or the lack thereof) in constrained Pareto optima. The revelation principle tells us that there exists at least one nonlinear tax system that weakly implements these wedges as an equilibrium outcome: namely, the direct mechanism. We present no results about how these wedges might be implemented using tax systems that have a more "decentralized" flavour.[6] We know, though, that there is no refinement of the theory that will enable it to make sharp predictions about the nature of optimal tax systems. As is well known from the Ramsey literature, a wedge between the benefits and costs of saving can be generated in equilibrium using a tax on savings, a tax on consumption that grows over time, or some linear combination of the two. More generally, Chari and Kehoe (1999) emphasize that in the Ramsey taxation literature, in which governments can only use linear taxes, there are typically an infinite number of combinations of various taxes that can be used to implement a particular wedge. This kind of indeterminacy is only more pronounced when the government can use arbitrary *nonlinear*

4. For an illustrative example, see Storesletten, Telmer and Yaron (2001). They argue that the autoregressive component is large and highly persistent.

5. In Section 5, we provide a thorough literature review and identify the key feature of our model that allows us to prove such general theorems: intratemporal and intertemporal consumption marginal rates of substitution are public information.

6. It might appear obvious how to construct such a tax schedule. First, set a marginal labour tax rate for each agent that equates his marginal rate of substitution between consumption and time to his marginal rate of transformation. Second, set a marginal tax on savings that equates his intertemporal marginal rate of substitution to the social intertemporal marginal rate of substitution.

Unfortunately, there is a problem with this approach: there is no guarantee that the resultant tax schedule gives rise to a convex decision problem for the agent. This means that even though his first-order conditions are satisfied by the social optimum, he may not find it optimal to make choices consistent with the social optimum.

Golosov and Tsyvinski (2003) design a simple implementation of the constrained Pareto optimum in the Diamond–Mirrlees disability insurance model. The implementation relies on capital income taxation and asset-based "means testing" for the provision of disability insurance.

taxes. Hence, the robust predictions of any kind of theory of optimal taxation are not about *taxes*, but, like our results, are about *wedges.*

The rest of the paper is structured as follows. In the next section, we describe the class of model environments. In Section 3, we demonstrate the optimality of positive capital income taxation. In Section 4, we generalize the uniform commodity taxation theorem. We defer a complete discussion of the related literature until Section 5; the discussion clarifies why we are able to prove our results in such generality. Finally, we conclude in Section 6.

2. SET-UP

The economy lasts for T periods, where T may be infinity, and has a unit measure of agents. The economy is endowed with K_1^* units of the single capital good. There are J consumption goods, which are produced by capital and labour. The agents have identical preferences. A given agent has von Neumann–Morgenstern preferences, and ranks deterministic sequences according to the function

$$\sum_{t=1}^{T} \beta^{t-1} U(c_t, l_t), \qquad 1 > \beta > 0 \tag{3}$$

where $c_t \in R_+^J$ is the agent's consumption in period t, and $l \in R_+$ is the agent's labour in period t. We assume that U is bounded from above or bounded from below; this guarantees that the utility from any consumption/labour process is well defined as an element of the extended reals.

The agents' skills differ across agents and over time. We model this cross-sectional and temporal heterogeneity as follows. Let Θ be a Borel[7] set in R_+, and let μ be a probability measure over the Borel subsets of Θ^T. At the beginning of time, an element θ^T of Θ^T is drawn for each agent according to the measure μ; the draws are independent across agents. This random vector θ^T is the agent's type; its t-th component θ_t is the agent's skill in period t. We assume that a law of large numbers applies: the measure of agents in the population with type θ^T in Borel set B is given by $\mu(B)$ (see Uhlig, 1996, for a formal justification of this assumption).

What makes the information problem dynamic is that a given agent privately learns his θ_t at the beginning of period t and not before. Thus, at the beginning of period t, an agent knows his history $\theta^t = (\theta_1, \ldots, \theta_t)$ of current and past skill vectors but not his future skill vectors. This implies that his choices in period t can only be a function of this history. We model this by using a standard mathematical formalism: we define a random variable $x : \Theta^T \to R$ to be θ^t-measurable if and only if, given a Borel subset M of R, $x^{-1}(M) = B \times \Theta^{T-t}$, where B is a Borel subset of Θ^t. Then, we restrict an agent's period t decisions to be θ^t-measurable.

This stochastic specification is general along two important dimensions. First, it allows for virtually arbitrary dynamic evolution of an agent's skills. For example, the agent's skills could be constant over time (which is the traditional public finance assumption). Alternatively, the skills could follow stationary or nonstationary stochastic processes over time. The only real restriction is that the skill processes are independent across agents.[8]

What is the economic impact of these skill vectors? An agent with type θ_t produces effective labour y_t according to the function

$$y_t = \theta_t l_t \tag{4}$$

where l_t is the agent's labour input. Effective labour y_t is observable, but actual labour l_t and skill θ_t are not.

7. Not all subsets of R_+ are Borel sets; nonetheless, a casual reader will not be misled about the nature of our results by simply ignoring the word Borel throughout the written text.

8. Note that the specification is general enough to embed both the case in which Θ is an interval (as is generally assumed in the Mirrlees literature) and the case in which Θ is a finite set (as is typically assumed in the dynamic contracting literature).

Along with the consumption goods, there is an accumulable capital good. We define an allocation in this society to be $(c, y, K) = (c_t, y_t, K_{t+1})_{t=1}^T$ where for all t

$$\begin{aligned}&K_{t+1} \in R_+\\&c_t : \Theta^T \to R_+^J\\&y_t : \Theta^T \to R_+\\&(c_t, y_t) \text{ is } \theta^t\text{-measurable.}\end{aligned} \tag{5}$$

Here, $y_t(\theta^T)$ is the amount of effective labour produced by a type θ^T in period t, $c_{jt}(\theta^T)$ is the amount of the j-th consumption good given to a type θ^T in period t, and K_{t+1} is the amount of capital carried over period t into period $(t+1)$. The measurability restriction on (c_t, y_t) guarantees that they depend only on current and past realizations of θ_t, not on the future realizations $(\theta_{t+1}, \dots, \theta_T)$.

We assume that the initial endowment of capital is K_1^*, and define an allocation (c, y, K) to be *feasible* if c_t and y_t are integrable for all t and

$$G\left(\int c_t d\mu, K_{t+1}, K_t, \int y_t d\mu\right) \le 0 \qquad \text{for all } t \tag{6}$$

$$K_1 \le K_1^*. \tag{7}$$

Here, $G : R_+^{J+3} \to R$ is assumed to be strictly increasing and continuously differentiable with respect to its first $(J+1)$ arguments, and strictly decreasing and continuously differentiable with respect to its other two arguments.[9] (In (6), and throughout the remainder of the paper, we use the convention that the range of integration is Θ^T when it is left unspecified.) Thus, we allow for very general specifications of technology; one example technology, when $J = 1$, is that G takes the form

$$G(C, K', K, Y) = C + K' - (1-\delta)K - K^{1/2}Y^{1/2}.$$

Because θ^T is unobservable, allocations must respect incentive-compatibility conditions. A *reporting strategy* σ is a mapping from Θ^T into Θ^T such that for all t, σ_t is θ^t-measurable. Let Σ be the set of all possible reporting strategies, and define

$$\begin{aligned}&W(\cdot; c, y) : \Sigma \to R\\&W(\sigma; c, y) = \sum\nolimits_{t=1}^T \beta^{t-1} \int U(c_t(\sigma), y_t(\sigma)/\theta_t) d\mu\end{aligned} \tag{8}$$

to be the utility from reporting strategy σ, given an allocation (c, y). Let σ^* be the truth-telling strategy ($\sigma^*(\theta^T) = \theta^T$ for all θ^T). Then, an allocation (c, y, K) is *incentive-compatible* if

$$W(\sigma^*; c, y) \ge W(\sigma; c, y) \qquad \text{for all } \sigma \text{ in } \Sigma. \tag{9}$$

An allocation which is incentive-compatible and feasible is said to be incentive-feasible.[10]

9. Our definition of feasibility does not explicitly allow for government purchases. However, our results all go through if we change the second restriction to read

$$G\left(\int c_t d\mu, K_{t+1}, K_t, \int y_t d\mu\right) \le \alpha_t$$

where $\{\alpha_t\}_{t=1}^\infty$ is a deterministic sequence of negative numbers. We can interpret these deterministic fluctuations in the feasible set as being due to deterministic fluctuations in government purchases.

10. We restrict attention to direct mechanisms. By the revelation principle, this is without loss of generality. As well, we restrict attention to mechanisms in which an individual's consumption and output depend only on his own announcements. This is without loss of generality because there is a continuum of agents with independent shock processes.

We assume that the government or social planner has the ability to fully commit *ex ante* to a tax system (or, equivalently, a direct mechanism). This is certainly not without loss of generality. As Roberts (1984) emphasizes, a benevolent government or social planner would find in its interests to alter the tax system *after* the revelation of agents' information. Nonetheless, as the dynamic Ramsey literature shows, the full-commitment case is (at least) a useful benchmark.

We do allow, though, for the possibility that the planner's *ex ante* objective weights agents differently based on their initial skill levels. Specifically, let $\chi_1 : \Theta^T \to R_+$ be θ^1-measurable, and suppose that $\int \chi_1 d\mu = 1$. Then, we define the following programming problem, $P1(K_1')$, for an arbitrary level K_1' of initial capital

$$
\begin{aligned}
V^*(K_1') = \sup\nolimits_{c,y,K} & \sum\nolimits_{t=1}^{T} \beta^{t-1} \int U(c_t, y_t/\theta_t)\chi_1 d\mu \\
& \text{s.t. } G\left(\int c_t d\mu, K_{t+1}, K_t, \int y_t d\mu\right) \le 0 \qquad \text{for all } t \\
& W(\sigma^*; c, y) \ge W(\sigma; c, y) \qquad \text{for all } \sigma \text{ in } \Sigma \\
& K_1 \le K_1' \\
& c_t \ge 0, \qquad y_t \ge 0, \qquad K_t \ge 0 \qquad \text{for all } t \text{ and almost all } \theta^T.
\end{aligned} \tag{10}
$$

We say that (c^*, y^*, K^*) solves $P1(K_1')$ if (c^*, y^*, K^*) lies in the constraint set of $P1(K_1')$ and

$$V^*(K_1') = \sum\nolimits_{t=1}^{T} \beta^{t-1} \int U(c_t^*, y_t^*/\theta_t)\chi_1 d\mu. \tag{11}$$

In the actual model economy, there are initially K_1^* units of capital. Hence, the planner's problem is to solve $P1(K_1^*)$. We assume throughout that there is a solution to $P1(K_1^*)$ and that $|V^*(K_1^*)| < \infty$. Any solution to $P1(K_1^*)$ is a Pareto optimum.[11]

Note that the planner's maximized objective V^* is weakly increasing. In our analysis, we will often require that V^* is strictly increasing. The following lemma shows that, under a mild regularity condition, V^* is strictly increasing if U is additively separable between consumption and leisure. (In the remainder of the paper, as is standard, we use the terms for almost all θ^T and almost everywhere (or a.e.) equivalently.)

Lemma 1. *Let $U(c, l) = u(c) - v(l)$, where u is strictly increasing and continuously differentiable. Suppose that for any (c^*, y^*, K^*) that solves $P1(K_1^*)$, there exists some t and positive scalars c^+, c_+ such that $c^+ \ge c_{jt}^* \ge c_+$ a.e. for all j. Then, $V^*(K_1) < V^*(K_1^*)$ for all $K_1 < K_1^*$.*

Proof. In Appendix. ‖

The proof of the lemma works as follows. Suppose the planner has not used up all initial capital. Then, roughly speaking, the planner can add $\varepsilon/u'(c_{jt}^*(\theta^T))$ to $c_{jt}^*(\theta^T)$ for all θ^T. This guarantees that for all θ^T, $u(c_{jt}^*(\theta^T))$ increases by ε (for ε small). Since all types' utilities are going up by ε, this change does not affect incentive-compatibility. Is the change feasible? The integral of $1/u'(c_{jt}^*(\theta^T))$ must be finite; the upper and lower bounds on c_{jt}^*, along with u's being C^1, are enough to guarantee this.[12]

11. Specifically, any solution to $P1(K_1^*)$ is interim Pareto optimal, conditional on the realization of θ_1. If $\chi = 1$, the solutions to $P1(K_1^*)$ are symmetric *ex ante* Pareto optima.

12. This last issue arises repeatedly in our analysis. We increase consumption for a set of types so as to increase each type's utility by the same amount. Then, we have to be sure that the integral of this increase in consumption is finite.

3. OPTIMAL INTERTEMPORAL WEDGES

To obtain results about the intertemporal characteristics of optimal taxation, we simplify the model. We set the number of consumption goods $J = 1$, and set

$$G(C_t, Y_t, K_t, K_{t+1}) = C_t + K_{t+1} - K_t(1-\delta) - F(K_t, Y_t) \tag{12}$$

where F is strictly increasing and continuously differentiable in its first argument. (These restrictions on J and G do not apply in the next section.) Throughout the section, we assume that the partial derivative U_c exists and is continuous in its first argument over the positive reals. We provide a partial characterization of Pareto optima.

The main result in this section is a restriction on the intertemporal behaviour of individual consumption. The result is similar to (but much more general than) that derived by Rogerson (1985*a*) for optimal contracts in relationships with repeated moral hazard.

We begin by stating the result. We use the notation $E_t x_{t+1}$ to be the period t conditional expectation[13] of a random variable x_{t+1}.

Theorem 1. *Let $U(c,l) = u(c) - v(l)$. Suppose (c^*, y^*, K^*) solves $P1(K_1^*)$, and that there exists $t < T$ and scalars c^+, c_+ such that $c^+ \geq c_t^*, c_{t+1}^*, K_{t+1}^* \geq c_+ > 0$* a.e. *Then*

$$\beta\left(1-\delta+F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right)\right)\Big/ u'(c_t^*) = E_t\{1/u'(c_{t+1}^*)\}. \tag{13}$$

Proof. In Appendix. ‖

There are two ways to read Theorem 1. First, it says that given any positive measure Borel set B in Θ^t, the *average* of $u'(c_t^*)/u'(c_{t+1}^*)$ across the agents who have a period t history in B is equal to $\beta(1-\delta+F_{K,t+1})$. Second, it says that given that an agent knows his period t history lies in B, the agent's *expectation* of $u'(c_t^*)/u'(c_{t+1}^*)$ is equal to $\beta(1-\delta+F_{K,t+1})$. (These two ways of reading Theorem 1 are equivalent because of the law of large numbers.)

Here is a sketch of the proof of Theorem 1, for the case in which Θ is finite. Suppose (c^*, y^*, K^*) is an interior optimum, and fix a positive probability skill history $\overline{\theta}^t$. We consider a perturbation similar to that used by Rogerson (1985*a*), and define a new consumption allocation c' to be the same as c^* except that

$$c_t'(\overline{\theta}^t) = c_t^*(\overline{\theta}^t) - \varepsilon/u'(c_t^*(\overline{\theta}^t)) \tag{14}$$

$$c_{t+1}'(\overline{\theta}^t, \theta) = c_{t+1}^*(\overline{\theta}^t, \theta) + \beta^{-1}\varepsilon/u'(c_{t+1}^*(\overline{\theta}^t, \theta)) \qquad \text{for all } \theta \text{ in } \Theta \tag{15}$$

where ε is small and positive. For the agents with skill history $\overline{\theta}^t$, this change is designed to reduce momentary utility in period t by ε, and increase momentary utility in period $(t+1)$ (given any continuation skill history) by $\beta^{-1}\varepsilon$.

The key to the proof is that, by construction,

$$\sum_{t=1}^{T} \beta^{t-1} u(c_t'(\theta^T)) = \sum_{t=1}^{T} \beta^{t-1} u(c_t^*(\theta^T)) \tag{16}$$

While weaker assumptions will ensure this, it is sufficient to assume that consumption is bounded from above and below and that u is C^1.

13. As is standard (see Billingsley, 1995, Chapter 34), given a θ^{t+1}-measurable random variable x_{t+1}, we define the conditional expectation $E_t x_{t+1}$ to be a θ^t-measurable random variable such that

$$\int_A E_t x_{t+1} d\mu = \int_A x_{t+1} d\mu$$

for any $A \subseteq \Theta^T$ such that $\mu(A) > 0$ and $A = B \times \Theta^{T-t}$ for some Borel set B in Θ^t.

for all (not almost all!) types θ^T. This means that, given that (c^*, y^*) is incentive-compatible, the new plan (c', y^*) must also be incentive-compatible. Intuitively, any sequence of reports generates the same (*ex ante*) utility under (c', y^*) as under (c^*, y^*). Hence, the ranking of reporting strategies must be the same under (c', y^*) as under (c^*, y^*).

Similarly, the new consumption plan (c', y^*) does not change the planner's objective. It follows (from Lemma 1) that the new timing of consumption payments cannot result in extra resources for the planner or (c^*, y^*, K^*) is not optimal. There are $\mu(\overline{\theta}^t)$ agents who have skill history $\overline{\theta}^t$. Hence, the new plan frees up $\mu(\overline{\theta}^t)\varepsilon/u'(c_t^*(\overline{\theta}^t))$ units of consumption in period t. Similarly, it costs $\beta^{-1}\varepsilon\sum_{\theta\in\Theta}\mu(\overline{\theta}^t,\theta)/u'(c_{t+1}^*(\overline{\theta}^t,\theta))$ in period $(t+1)$. By saving the extra period t consumption into period $(t+1)$, the planner has extra resources after paying the costs unless:

$$\mu(\overline{\theta}^t)\left(1-\delta+F_K\left(K_t^*, \int y_{t+1}^* d\mu\right)\right)\Big/ u'(c_t^*(\overline{\theta}^t)) \leq \beta^{-1}\sum_{\theta\in\Theta}\mu(\overline{\theta}^t,\theta)/u'(c_{t+1}^*(\overline{\theta}^t,\theta)). \tag{17}$$

Making the same argument with ε small and negative implies the reverse inequality. In other words, at an optimum, the two sides of (17) are equated, which implies Theorem 1.

It is important to note that even if θ^T is public information (so that there is no incentive problem), Theorem 1 is still valid. In this case, full insurance is possible and $u'(c_t^*)$ is deterministic for all t. Theorem 1 immediately implies the standard first-order condition:

$$u'(c_t^*) = \beta\left(1-\delta+F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right)\right)u'(c_{t+1}^*). \tag{18}$$

Thus, the incentive problem does not create the restriction in Theorem 1. Rather, the incentive problem determines the variance of the marginal utility process that gets plugged into the formula in Theorem 1.

This kind of thinking informs the next two corollaries.

Corollary 1. *Let $U(c,l) = u(c) - v(l)$. Suppose (c^*, y^*, K^*) solves $P1(K_1^*)$, and that there exists $t < T$ and scalars c^+, c_+ such that $c^+ \geq c_t^*, c_{t+1}^*, K_{t+1}^* \geq c_+ > 0$ a.e. Then*

$$u'(c_t^*) \leq \beta\left(1-\delta+F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right)\right)E_t u'(c_{t+1}^*) \text{ a.e.} \tag{19}$$

In addition, suppose that it is not true that $u'(c_{t+1}^*)$ equals $E_t u'(c_{t+1}^*)$ almost everywhere. Then

$$u'(c_t^*) < \beta\left(1-\delta+F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right)\right)E_t u'(c_{t+1}^*) \tag{20}$$

over some subset of Θ^T with positive measure.

Proof. From the definition of a conditional expectation and Theorem 1, we know that

$$\beta(1-\delta+F_{K,t+1}) = E_t\{u'(c_t^*)/u'(c_{t+1}^*)\} \tag{21}$$

$$= u'(c_t^*)E_t\{1/u'(c_{t+1}^*)\} \tag{22}$$

where the latter equality follows from $u'(c_t^*)$ being θ^t-measurable (see, Billingsley, 1995, Theorem 34.3). The corollary then follows from the conditional version of Jensen's inequality. ‖

The first part of the corollary says that the expected marginal utility of investing in capital, conditional on information known to an agent as of time t, is at least as high as the marginal utility of current consumption. The second part of the corollary says that if some individuals do not know their $u'(c^*_{t+1})$ as of time t, then the inequality becomes strict. Note that this lack of predictability is to be expected in general because the planner wants to elicit high labour from high skill types.

Economically, this is the most important result of this section. In a Pareto optimum, an individual's period t shadow return from saving is given by

$$\beta^{-1}u'(c^*_t)/E_t u'(c^*_{t+1}). \tag{23}$$

The period t technological return to capital is given by

$$\left(1-\delta+F_K\left(K^*_{t+1},\int y^*_{t+1}d\mu\right)\right). \tag{24}$$

Corollary 1 shows that if $u'(c^*_{t+1})$ does not equal $E_t u'(c^*_{t+1})$ (which means that the private information constraint is binding for the social planner), it is optimal for the technological return to capital to exceed some individual's period t shadow return.[14] Of course, this is inconsistent with equilibrium if agents can competitively trade capital and consumption without facing taxes. Rather, if the Pareto optimal wedge is to be replicated in the equilibrium, individuals must face taxes. As discussed in the introduction, these taxes can be of several forms: a tax on savings, a growing tax on consumption, or a combination thereof.

We described the intuition behind this result in the introduction. The basic idea is that because u is locally concave, when an individual saves more from period t to period $(t+1)$, he works less in response to any given period $(t+1)$ compensation scheme. This adverse effect of savings on incentives implies that it is optimal for society to deter savings by taxing it.

It is interesting to contrast Corollary 1 with the results concerning optimal linear taxation of capital and labour income in a representative agent economy. Judd (1985) and Chamley (1986) prove for a general specification of u that it is optimal in the long run to eliminate the wedge between expected marginal utility of investing in capital and the marginal utility of current consumption. Indeed, when $u(c) = c^{1-\sigma}/(1-\sigma)$, Chamley proves an even stronger result: it is optimal for the wedge to be zero for all t, not just in the long run. In contrast, we find that for any specification of u, as long as $u'(c^*_{t+1})$ is not known at time t, the wedge in period t should be non-zero.

There are special circumstances in which the inequality in Corollary 1 becomes an equality instead. In particular, if agents have fixed skills over time, then the Pareto optimal allocations display no wedge between the marginal utility of consumption and the expected marginal utility of investment.

Corollary 2. *Suppose that $\mu(A) > 0$ only if $\mu(A) = \mu\{\theta^T \in A|\theta_t = \theta_1$ for all $t\}$. Let $U(c,l) = u(c) - v(l)$. Suppose (c^*, y^*, K^*) solves $P1(K^*_1)$, and that there exists $t < T$ and scalars c^+, c_+ such that $c^+ \geq c^*_t, c^*_{t+1}, K^*_{t+1} \geq c_+ > 0$* a.e. *Then*

$$\beta u'(c^*_{t+1}(\theta^T))\left(1-\delta+F_K\left(K^*_{t+1},\int y^*_{t+1}d\mu\right)\right)\Big/u'(c^*_t(\theta^T)) = 1 \text{ a.e.} \tag{25}$$

This corollary follows from the fact that θ_t is perfectly predictable, given θ_1. In fact, using a similar approach as in Theorem 1, we can prove (at least when Θ is finite) that even if preferences are non-separable between consumption and labour, we obtain a version of Chamley–Judd's classic result for this case of fixed skills.

14. Why does the second part of Corollary 1 only hold with positive probability? Imagine a world in which there are two possible realizations in period 1. If skills are high in period 1, they are high in period 2 with probability 1. If they are low in period 1, they are equally likely to be high or low in period 2. In this kind of world, the inequality in Corollary 1 is strictly positive in period 1 when skills are low, but becomes an equality when the skills are high.

Proposition 1. *Suppose $T = \infty$, Θ is finite, and that $\mu\{\theta^\infty\} > 0$ iff $\theta_t = \theta_1$ for all t. Suppose that $V^*(K_1) < V^*(K_1^*)$ for all $K_1 < K_1^*$. Let a strictly positive allocation (c^*, y^*, K^*) solve $P1(K_1^*)$, and suppose that for all θ_1, the sequence $\{c_t^*(\theta_1), y_t^*(\theta_1), K_t^*\}_{t=1}^\infty$ converges to a positive limit $(c_{ss}(\theta_1), y_{ss}(\theta_1), K_{ss})$. Then*

$$\beta^{-1} = 1 + F_K\left(K_{ss}, \int y_{ss}\, d\mu\right) - \delta. \tag{26}$$

Proof. In Appendix. ∥

4. SUBOPTIMAL INTRATEMPORAL WEDGES

In this section, we prove the uniform commodity taxation theorem. We return to the general set-up described in the first section (with multiple commodities and a general production structure), except that we assume that utility is weakly separable:

$$U(c, l) = U^*(u(c), l), \qquad u : R_+^J \to R_+. \tag{27}$$

We also assume that u is strictly increasing and is continuously differentiable over the positive orthant of R^J. The notation u_j and G_j represents the partial derivatives of those functions with respect to their j-th arguments.

Theorem 2. *Suppose $V^*(K_1) < V^*(K_1^*)$ for all $K_1 < K_1^*$. Let (c^*, y^*, K^*) solve $P1(K_1^*)$ and suppose that there exists some t and scalars c^+, c_+ such that $c^+ > c_{jt}^*(\theta^T) > c_+ > 0$ for all j and for almost all θ^T. Then, if $J > 1$,*

$$\begin{aligned} &u_j(c_t^*(\theta^T))/u_k(c_t^*(\theta^T)) \\ &= G_j\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, \int y_t^* d\mu\right) \Big/ G_k\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, \int y_t^* d\mu\right) \end{aligned} \tag{28}$$

for all j, k and almost all θ^T.

Proof. In Appendix. ∥

Theorem 2 states that in a Pareto optimum, the marginal rate of substitution between two consumption goods is equalized to the marginal rate of transformation between those two goods. It is a direct extension of Atkinson and Stiglitz (1976), who also assume weakly separable preferences but restrict attention to static settings. The key to the proof is that the consumption goods enter both sides of the incentive constraints only through the sub-utility $u(c)$. Hence, it is optimal for the planner to deliver this sub-utility from consumption in a way that minimizes the resource cost of doing so.[15]

Theorem 2 establishes a result about marginal rates of substitution and transformation. It implies that, within a given period, it is suboptimal for agents to face taxes or subsidies that differ across consumption goods. What is the optimal intertemporal behaviour of this common tax rate? There is no sharp answer to this question. Suppose U^* is additively separable between consumption and labour. Then, it is possible to prove multi-good versions of Theorem 1 and Corollary 1 which imply that it is optimal for an individual's marginal benefit of saving to exceed his marginal cost of doing so. As emphasized earlier, this wedge is consistent with a

15. The theorem relies on the assumption that the planner's objective V^* is strictly increasing in initial resources. Lemma 1 guarantees that this assumption is satisfied at least when U is additively separable in its two arguments.

tax on savings, a growing tax on consumption goods, or a combination thereof. All we know for sure is that because of Theorem 2, if there is to be a tax/subsidy on consumption goods in any period, that tax/subsidy must be the same across all goods.

5. RELATED LITERATURE

Our results are valid regardless of the nature of the dynamic evolution of the private information in our model. In this section, we argue that our generality in this regard is due to one key attribute of our framework: any agent's marginal rate of substitution between consumption goods is unaffected by his type, and is therefore public information.

Consider first the Atkinson–Stiglitz theorem on the suboptimality of intratemporal wedges. It is well known that it is possible to perturb our basic model in order to generate optimal intratemporal wedges. For example, Mirrlees (1976) and Cremer, Pestieau and Rochet (2001) show that if agents have different endowments or different tastes, and those attributes are also private information, then the Atkinson–Stiglitz theorem is no longer valid. Alternatively, if preferences are not weakly separable between consumption goods and leisure, then intratemporal wedges may be optimal.

What undoes the Atkinson–Stiglitz result in these cases? With hidden endowments or hidden tastes, an agent's willingness to substitute between any two consumption goods is known only to him. Similarly, without the weak separability assumption, an agent's willingness to substitute depends on his (unobservable) time spent working and is again private information. This type of private information implies that an optimal tax system must screen people based on their willingness to substitute between consumption goods.

In contrast, in our set-up an agent's intratemporal marginal rate of substitution between any two consumption goods is publicly known. Hence, the planner has no efficiency reason to separate people using this attribute, and the Atkinson–Stiglitz result is still valid.[16]

This same issue arises with regard to our intertemporal results. There are now many papers on efficient dynamic insurance in the presence of hidden idiosyncratic shocks to endowments or marginal utilities of consumption (see, among others, Townsend (1982), Green (1987), Thomas and Worrall (1990), Atkeson and Lucas (1992), Khan and Ravikumar (2001)). A key result that runs through this dynamic insurance literature is that in Pareto optimal allocations, the typical agent's shadow interest rate is no larger than the societal shadow interest rate. This result is similar to our Corollary 1.

But, unlike our Corollary 1, the result from this literature with hidden endowments or hidden tastes depends crucially on the nature of the shock process. To see this point, consider a two-period economy with a continuum of agents who have a utility function

$$u(c_1) + u(c_2)$$

over sequences of consumption. The typical agent's endowment is $((1+\theta), (1+\theta)^2)$, where θ is random with positive support; the endowments are private information. The society can borrow and lend from an outside lender at a net rate of return r. In this type of dynamic insurance model, with endowments that are not stochastically independent, in an optimal allocation agents' shadow interest rates are *higher* than r. This wedge is consistent with subsidies for saving, not taxes on saving.

What generates this lack of robustness? In models with hidden endowments or hidden tastes, this intertemporal marginal rate of substitution is private information. It is not surprising that the distortions in the latter kind of model hinge crucially on the dynamic nature of the private

16. See Laffont and Tirole (1994, p. 194) for a similar discussion of the limitations of the Atkinson–Stiglitz theorem.

information. In our model, instead, the intertemporal marginal rate of substitution for any agent is publicly known.

There are many other papers which also assume that agents' intertemporal marginal rates of substitution are public information. For example, Diamond and Mirrlees (1978, 1986) consider a special case of our general set-up. In their model, agents are long-lived and can be disabled or not. Disabled agents are unproductive; able agents have known productivities. Once disabled, the agent stays disabled; the probability of an able agent becoming disabled is exogenous. The informational problem is that the disability status of the agent is known only to the agent. Diamond and Mirrlees prove that in the social optimum, the shadow societal interest rate is higher than the private shadow interest rate. They argue explicitly that this result implies that capital income taxation is socially optimal. As we stress in the introduction, our contribution over their work is that we generalize their positive capital income taxation result to a much larger class of individual skills processes.

There are several papers on the properties of efficient allocations in the presence of repeated moral hazard (see, among others, Rogerson (1985*a*), Phelan and Townsend (1991), Phelan (1994), Atkeson and Lucas (1995)). Again, in these settings the optimal allocations have the property that agents' shadow interest rates are lower than the societal shadow interest rate. The intuition behind this result is essentially the same as that behind Corollary 1. However, in this literature, the idiosyncratic output shocks are restricted to be independently and identically distributed over time; we instead allow for a much wider range of skills processes.

We were originally motivated to write this paper by the work of da Costa and Werning (2001). They examine optimal monetary policy in two models (a cash-credit good framework and a shopping-time set-up) in which agents are privately informed about their fixed skills. In the cash-credit good framework, da Costa and Werning prove that if preferences are weakly separable between consumption and leisure, then the Friedman rule (zero nominal interest rates) is socially optimal. This is essentially an implication of the uniform commodity taxation theorem, and so we conjecture that this result could be established in our more general set-up. They also consider how deviations from weak separability of preferences affect optimal monetary policy.

In a paper written at the same time as ours, but independently, Werning (2001) analyses the properties of optimal capital income taxes in a model economy with unobservable and heterogeneous fixed skills. Like us (Corollary 2), he finds that it is optimal for capital income taxes to be zero in this setting.

6. CONCLUSION

In this paper, we consider the problem of optimal taxation when individual skills are unobservable and evolve stochastically over time. We show that when utility is weakly separable between consumption and leisure, it is optimal to equate the marginal rate of substitution between consumption goods for any agent to the marginal rate of transformation between those goods.

We consider the intertemporal structure of optimal taxation when there is only a single consumption good and utility is additively separable between consumption and leisure. In this case, if the optimal allocation requires future consumption to be random given current information, then individuals face distorted consumption paths. These distortions are consistent with the presence of positive capital income taxes (or, equivalently, with growing consumption taxes).

Given additive separability of preferences between consumption and labour, the uniform commodity taxation theorem is generally valid, but the zero capital income taxation theorem is generally not. The reason for this distinction is that over time, individuals are acquiring information about their types. It is this idiosyncratic uncertainty that generates positive capital income taxes. In particular, if individuals knew their entire sequence of skills in period 1, then we could use exactly the same reasoning as in Theorem 2 (or Corollary 2) to conclude that Pareto optimal allocations are consistent with zero capital income taxation.

We are able to prove the theorems in a highly general setting. Individual skills are independent over a continuum of individuals but follow arbitrary stochastic processes over time. There is one crucial assumption that makes our analysis work: a given agent's (intertemporal or intratemporal) marginal rate of substitution between consumption goods is public information. As we emphasized in Section 5, this assumption is not true if agents have hidden endowments of consumption, have random and unobservable tastes over consumption goods, or if our various separability assumptions are violated.

However, the good news is that we can allow any additional private information as long as individuals' willingness to substitute consumption over time is common knowledge. This means, for example, that we could allow labour and skills to be multidimensional, so that agents can work at different tasks that may be imperfectly substitutable in producing consumption goods. More interestingly, we could allow agents to secretly accumulate human capital, and thereby endogenize skills. Our results would go through in this kind of environment as long as time is the only input into human capital formation.

APPENDIX

In this Appendix, we collect the proofs of the main results.

A.1. *Proof of Lemma* 1

Suppose $V^*(K_1) = V^*(K_1^*)$ for some $K_1 < K_1^*$. Let (c^*, y^*, K^*) solve $P1(K_1)$. It lies in the constraint set of $P1(K_1^*)$, and so also solves $P1(K_1^*)$. Without loss of generality, assume that c_1^* satisfies the uniform boundedness conditions. Define $c'_{11}(\theta^T, \varepsilon)$ to be the solution to the equation

$$u(c'_{11}(\theta^T, \varepsilon), (c^*_{1j}(\theta^T))_{j\neq 1}) - u(c_1^*(\theta^T)) = \varepsilon \qquad \text{for all } \theta^T \tag{A.1}$$

for ε non-negative. Here, $c'_{11}(\theta^T, \varepsilon)$ is the amount of consumption good 1 that gives a type θ^T ε more utiles than c_1^*. Clearly, c'_{11} is θ^1-measurable with respect to θ^T, and is continuous with respect to ε.

From the mean value theorem, for ε small, we know that

$$|c'_{11}(\theta^T, \varepsilon) - c^*_{11}(\theta^T)| = \varepsilon / u_1(c'_{11}(\theta^T, \varepsilon'), (c^*_{1j}(\theta^T))_{j\neq 1}), \qquad 0 < \varepsilon' < \varepsilon \tag{A.2}$$

where u_1 is the partial of u with respect to its first argument. From the regularity conditions on c^*, we know that there exists $M > 0$ such that

$$|c'_{11}(\theta^T, \varepsilon) - c^*_{11}(\theta^T)| < M\varepsilon \qquad \text{for } \varepsilon \text{ small.} \tag{A.3}$$

Hence, for ε small, $c'_{11}(\theta^T, \varepsilon)$ is integrable as a function of θ^T. Moreover, adding ε to initial consumption is feasible for initial capital K_1^*, as long as ε is sufficiently small. That is, for sufficiently small ε,

$$G\left(\int c'_1(\theta^T, \varepsilon) d\mu, K_2^*, K_1^*, \int y^* d\mu\right) < 0 \tag{A.4}$$

where $c'_1(\theta^T, \varepsilon) \equiv (c'_{11}(\theta^T, \varepsilon), (c^*_{1j}(\theta^T))_{j\neq 1})$. Thus, (c', y^*, K^*) is feasible, given initial capital K_1^*.

For all θ^T,

$$u(c'_1(\theta^T, \varepsilon)) - v(y_1^*(\theta^T)/\theta_1) \tag{A.5}$$
$$= u(c_1^*(\theta^T)) + \varepsilon - v(y_1^*(\theta^T)/\theta_1) \tag{A.6}$$
$$\geq u(c_1^*(\theta^{T\prime})) + \varepsilon - v(y_1^*(\theta^{T\prime})/\theta_1) \tag{A.7}$$
$$= u(c'_1(\theta^{T\prime}, \varepsilon)) - v(y_1^*(\theta^{T\prime})/\theta_1) \tag{A.8}$$

which proves that (c', y^*) is incentive-compatible (the inequality is implied by the incentive-compatibility of (c^*, y^*)). It follows that (c^*, y^*) cannot be a solution to $P1(K_1^*)$. ‖

A.2. *A technical lemma*

We use the following notation:

$$\Omega_t = \{A \subseteq \Theta^T | A = B \times \Theta^{T-t}, B \text{ Borel and } B \subseteq \Theta^t\} \tag{A.9}$$

$$L_t^\infty = \{x\ \theta^l\text{-measurable } | \exists A \in \Omega_t,\ \sup_{\theta^T \in A} |x(\theta^T)| < \infty,\ \text{and } \mu(A) = 1\}. \tag{A.10}$$

Let $||\cdot||$ denote the usual ess-sup norm on L_t^∞.

The proofs of Theorems 1 and 2 use two technical results. The first is Theorem 1 of Luenberger (1969, p. 243). This theorem assumes that in an optimization problem with equality constraints, the objective and constraints are continuously Frechet differentiable in the neighbourhood of a local optimum. It then proves that this local optimum must satisfy analogues of the usual Lagrangian first-order conditions.

The second key result is the following lemma. It establishes that as long as c_t^* is bounded from above and below, the constraints in the minimization problems in the proofs of Theorems 1 and 2 are defined by a function that is continuously Frechet differentiable in a neighbourhood of c_t^*.

Lemma A1. *Let $u : R_+ \to R$ be C^1 and let c_t^* be an element of L_t^∞. Suppose there exists scalars c^+ and c_+ such that $c^+ \geq c_t^* \geq c_+ > 0$. Define $U : L_t^\infty \to L_t^\infty$ by*

$$U(c_t)(\theta^T) = u(c_t(\theta^T)). \tag{A.11}$$

Then U is continuously Frechet differentiable in a neighbourhood of c_t^*.

Proof. Note that u' is uniformly continuous over the interval $[c_+/2, 3c^+/2]$. Let $\{\Delta_{nt}\}_{n=1}^\infty$ be an arbitrary sequence in L_t^∞ such that $\lim_{n\to\infty} ||\Delta_{nt}|| = 0$. Then

$$\lim_{n\to\infty} ||u(c_t^* + \Delta_{nt}) - u(c_t^*) - u'(c_t^*)\Delta_{nt}||/||\Delta_{nt}|| \tag{A.12}$$

$$= \lim_{n\to\infty} ||u'(c_t^* + \Delta'_{nt})\Delta_{nt} - u'(c_t^*)\Delta_{nt}||/||\Delta_{nt}||, 0 \leq \Delta'_{nt} \leq \Delta_{nt} \tag{A.13}$$

$$\leq \lim_{n\to\infty} ||u'(c_t^* + \Delta'_{nt}) - u'(c_t^*)||(||\Delta_{nt}||/||\Delta_{nt}||) \tag{A.14}$$

$$= \lim_{n\to\infty} ||u'(c_t^* + \Delta'_{nt}) - u'(c_t^*)|| \tag{A.15}$$

$$= 0. \tag{A.16}$$

The first step follows from the mean value theorem and the last step from the uniform continuity of u' over $[c_+/2, 3c^+/2]$.

It follows that in a neighbourhood of c_t^*, the Frechet derivative of U is well defined and given by $U'(c_t)(\Delta) = u'(c_t)\Delta$ for all Δ in L_t^∞. The norm of this linear operator is given by $||u'(c_t)||$. Let $||c_t - c_t^*|| < c_+/2$ and let $\{\Delta_{nt}\}_{n=1}^\infty$ be a sequence in L_t^∞ such that $\lim_{n\to\infty} ||\Delta_{nt}|| = 0$. Then

$$\lim_{n\to\infty} ||u'(c_t + \Delta_{nt}) - u'(c_t)|| = 0 \tag{A.17}$$

because u' is uniformly continuous over $[c_+/2, c^+/2 + c_+/2]$. So U is continuously Frechet differentiable in a neighbourhood of c_t^*. ‖

We use this technical lemma for the proofs of Theorems 1 and 2.

A.3. *Proof of Theorem* 1

The proof has two distinct parts.

Part 1*: Constructing a minimization problem*

In the first part of the proof, we construct a particular class of two-period deviations from the candidate optimum. The class of possible deviations satisfies two requirements. First, the deviations are required to deliver the same utility to all types as does the candidate optimum. Second, the deviations are required to satisfy resource-feasibility in all periods.

Obviously, the first requirement means that all of these deviations provide the same objective value to the planner. In addition, the first requirement implies that all of the deviations are incentive-compatible. Hence, we now have a necessary condition for the candidate optimum: it must use fewer initial resources than any of these possible deviations.

More precisely, consider the following minimization problem $MIN1$:

$$\begin{aligned}
&\min_{\eta_t,\varepsilon_{t+1},\zeta_t}\left[\zeta_t + \int \eta_t d\mu\right] \text{ s.t.}\\
&\int \varepsilon_{t+1} d\mu = F\left(K^*_{t+1} + \zeta_t, \int y^*_{t+1} d\mu\right) - F\left(K^*_{t+1}, \int y^*_{t+1} d\mu\right) + (1-\delta)\zeta_t\\
&u(c^*_t + \eta_t) + \beta u(c^*_{t+1} + \varepsilon_{t+1}) = u(c^*_t) + \beta u(c^*_{t+1}) \text{ a.e.}\\
&c^*_t + \eta_t \geq 0, c^*_{t+1} + \varepsilon_{t+1} \geq 0, K^*_{t+1} + \zeta_t \geq 0 \text{ a.e.}\\
&\eta_t \in L^\infty_t, \varepsilon_{t+1} \in L^\infty_{t+1}, \zeta_t \in R.
\end{aligned} \tag{A.18}$$

The objective of this problem is to minimize the resources used in period t. The first constraint requires that feasibility be satisfied in period $(t+1)$. The second constraint requires that utility to almost all types be kept the same under the deviation plan as under the candidate optimum.

We claim that $MIN1$ is solved by setting $(\eta_t, \varepsilon_{t+1}, \zeta_t) = 0$. Suppose not, and that there exists some element $(\eta_t, \varepsilon_{t+1}, \zeta_t)$ of the constraint set which generates a negative value for the objective. There exists a subset B of Θ^T such that $\mu(B) = 1$ and

$$u(c^*_t(\theta^T) + \eta_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T) + \varepsilon_{t+1}(\theta^T)) = u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \qquad \text{for all } \theta^T \text{ in } B. \tag{A.19}$$

Define (c', K') so that $c' = c^*$ and $K' = K^*$ except that

$$\begin{aligned}
c'_t(\theta^T) &= c^*_t(\theta^T) + \eta_t(\theta^T) \qquad \text{for all } \theta^T \text{ in } B\\
c'_{t+1}(\theta^T) &= c^*_{t+1}(\theta^T) + \varepsilon_{t+1}(\theta^T) \qquad \text{for all } \theta^T \text{ in } B\\
K'_{t+1} &= K^*_{t+1} + \zeta_t.
\end{aligned} \tag{A.20}$$

We claim that (c', y^*, K') is incentive-feasible, delivers the same value of the planner's objective as (c^*, y^*, K^*) and uses fewer resources. The allocation (c', y^*, K') is obviously feasible because

$$\int c'_t d\mu + K'_{t+1} = \int c^*_t d\mu + K^*_{t+1} + \zeta_t + \int \eta_t d\mu \tag{A.21}$$

$$< \int c^*_t d\mu + K^*_{t+1}. \tag{A.22}$$

We next want to show that the allocation (c', y^*, K') is incentive-compatible. By construction

$$u(c'_t(\theta^T)) + \beta u(c'_{t+1}(\theta^T)) \tag{A.23}$$

$$= u(c^*_t(\theta^T)) + \beta u(c^*_{t+1}(\theta^T)) \qquad \text{for all } \theta^T \tag{A.24}$$

(not just θ^T in B). Then, we know that for any σ in Σ and for all θ^T:

$$\begin{aligned}
&\sum\nolimits_{s=1}^{T} \beta^{s-1} u(c'_s(\sigma(\theta^T)))\\
&= \sum\nolimits_{s=1}^{t-1} \beta^{s-1} u(c^*_s(\sigma(\theta^T))) + \beta^{t-1}[u(c'_t(\sigma(\theta^T))) + \beta u(c'_{t+1}(\sigma(\theta^T)))] + \sum\nolimits_{s=t+2}^{T} \beta^{s-1} u(c^*_s(\sigma(\theta^T)))\\
&= \sum\nolimits_{s=1}^{t-1} \beta^{s-1} u(c^*_s(\sigma(\theta^T))) + \beta^{t-1}[u(c^*_t(\sigma(\theta^T))) + \beta u(c^*_{t+1}(\sigma(\theta^T)))] + \sum\nolimits_{s=t+2}^{T} \beta^{s-1} u(c^*_s(\sigma(\theta^T)))\\
&= \sum\nolimits_{s=1}^{T} \beta^{s-1} u(c^*_s(\sigma(\theta^T))).
\end{aligned} \tag{A.25}$$

This means that for any σ, agents get the same utility from c' as from c^*. It follows that (c', y^*) is incentive-compatible:

$$\int \sum\nolimits_{t=1}^{T} \beta^{t-1}[u(c'_t) - v(y^*_t/\theta_t)] d\mu \tag{A.26}$$

$$= \int \sum\nolimits_{t=1}^{T} \beta^{t-1}[u(c^*_t) - v(y^*_t/\theta_t)] d\mu \tag{A.27}$$

$$\geq \int \sum\nolimits_{t=1}^{T} \beta^{t-1}[u(c^*_t(\sigma)) - v(y^*_t(\sigma)/\theta_t)] d\mu \qquad \text{for any } \sigma \tag{A.28}$$

$$= \int \sum\nolimits_{t=1}^{T} \beta^{t-1}[u(c'_t(\sigma)) - v(y^*_t(\sigma)/\theta_t)] d\mu. \tag{A.29}$$

The inequality comes from the fact that (c^*, y^*) is incentive-compatible.

Hence, (c', y^*, K') uses fewer resources, is incentive-compatible, and delivers the same value of the objective to the planner. This violates Lemma 1. We can therefore characterize (c^*, K^*) using the first-order conditions of $MIN1$.

Part 2*: Deriving the first-order conditions*

The second part of the proof is purely technical: in it, we verify that the theorem's implication is in fact a first-order condition for $MIN1$.

Suppose we enlarge the constraint set by dropping the non-negativity constraints. The non-negative orthant of L_t^∞ has a non-empty interior. Hence, 0 must also be a local minimum of the enlarged minimization problem without the non-negativity constraints.

Note that the Frechet derivative $U'(c_t^*)$ maps L_t^∞ on to L_t^∞. Hence, $(0, 0, 0)$ is a regular point of the constraint set. From Lemma A1 and Luenberger (1969, Theorem 1, p. 243) we can conclude that there exists $z_{t+1}^* \in L_{t+1}^{\infty *}$ (the dual of L_{t+1}^∞) and $\lambda_t^* \in R$ such that 0 is a stationary point of the following Lagrangian:

$$\begin{aligned} L(\zeta_t, \eta_t, \varepsilon_{t+1}) &= \zeta_t + \int \eta_t d\mu + \lambda_t^* \left[\int \varepsilon_{t+1} d\mu - (1-\delta)\zeta_t - F(K_{t+1}^* + \zeta_t, Y_{t+1}^*) \right] \\ &- \langle z_{t+1}^*, u(c_t^* + \eta_t) + \beta u(c_{t+1}^* + \varepsilon_{t+1}) \rangle. \end{aligned} \tag{A.30}$$

(Here, as is standard, we use the notation $\langle z, u \rangle$ to denote the result of applying a linear operator z to the random variable u.) In other words

$$1 - \lambda_t^*(1-\delta) - F_K(K_{t+1}^*, Y_{t+1}^*)\lambda_t^* = 0 \tag{A.31}$$

$$\int \eta_t d\mu - \langle z_{t+1}^*, u'(c_t^*)\eta_t \rangle = 0 \qquad \text{for all } \eta_t \text{ in } L_t^\infty \tag{A.32}$$

$$\lambda_t^* \int \varepsilon_{t+1} d\mu - \langle z_{t+1}^*, \beta u'(c_{t+1}^*)\varepsilon_{t+1} \rangle = 0 \qquad \text{for all } \varepsilon_{t+1} \text{ in } L_{t+1}^\infty. \tag{A.33}$$

It follows that

$$\int \eta_t'/u'(c_t^*) d\mu = \langle z_{t+1}^*, \eta_t' \rangle \qquad \text{for all } \eta_t' \text{ in } L_t^\infty \tag{A.34}$$

$$\beta^{-1}\lambda_t^* \int \varepsilon_{t+1}'/u'(c_{t+1}^*) d\mu = \langle z_{t+1}^*, \varepsilon_{t+1}' \rangle \qquad \text{for all } \varepsilon_{t+1}' \text{ in } L_{t+1}^\infty \tag{A.35}$$

$$\lambda_t^* = [1 - \delta + F_K(K_{t+1}^*, Y_{t+1}^*)]^{-1} \tag{A.36}$$

and so

$$\beta^{-1}[1 - \delta + F_K(K_{t+1}^*, Y_{t+1}^*)]^{-1} \int \eta_t'/u'(c_{t+1}^*) d\mu = \int \eta_t'/u'(c_t^*) d\mu \qquad \text{for all } \eta_t' \text{ in } L_t^\infty. \tag{A.37}$$

Let $\eta_t' = 1_A u'(c_t^*)$, where A is an arbitrary element of Ω_t. Theorem 1 follows. ||

A.4. *Proof of Proposition* 1

We claim that (c^*, K^*) solves the following minimization problem:

$$\begin{aligned} &\min_{c,K} K_1 \\ &\text{s.t.} \int c_t d\mu + K_{t+1} = K_t(1-\delta) + F\left(K_t, \int y_t^* d\mu\right) \qquad \text{for all } t \\ &\textstyle\sum_{t=1}^\infty \beta^{t-1} U\left(c_t(\theta_1), \frac{y_t^*(\theta_1)}{\hat{\theta}_1}\right) = \sum_{t=1}^\infty \beta^{t-1} U\left(c_t^*(\theta_1), \frac{y_t^*(\theta_1)}{\hat{\theta}_1}\right) \qquad \text{for all } \theta_1, \hat{\theta}_1 \\ &K_t \in R_+, c_t \geq 0 \qquad \text{for all } t. \end{aligned} \tag{A.38}$$

Suppose not. Then, there exists non-negative (c', K') such that $K_1' < K_1^*$ and

$$\int c_t' d\mu + K_{t+1}' = K_t'(1-\delta) + F\left(K_t', \int y_t^* d\mu\right) \qquad \text{for all } t \tag{A.39}$$

$$\textstyle\sum_{t=1}^\infty \beta^{t-1} U\left(c_t'(\theta_1), \frac{y_t^*(\theta_1)}{\hat{\theta}_1}\right) = \sum_{t=1}^\infty \beta^{t-1} U\left(c_t^*(\theta_1), \frac{y_t^*(\theta_1)}{\hat{\theta}_1}\right) \qquad \text{for all } \theta_1, \hat{\theta}_1. \tag{A.40}$$

It is clear that (c', y^*, K') is feasible; (c', y^*) is incentive-compatible because we have kept the utility of all announcement/true type pairs the same. This allocation solves $P1(K_1)$, for $K_1 < K_1^*$, which violates the assumption that V^* is strictly increasing.

Now, we can characterize (c^*, y^*, K^*) using the first-order conditions to this problem. Let λ_t be the multiplier on the period t feasibility constraint and let $\gamma(\theta_1, \hat{\theta}_1)$ be the multiplier on the appropriate utility constraint.

Abusing notation slightly, we use $\mu(\theta_1)$ to denote $\mu\{(\theta_1, \theta_1, \theta_1, \ldots)\}$. Differentiating with respect to $c_t(\theta_1)$ for any θ_1, we obtain

$$\sum\nolimits_{\hat{\theta}_1} \gamma(\theta_1, \hat{\theta}_1)\beta^{t-1} U_c\left(c_t^*(\theta_1), \frac{y_t^*(\theta_1)}{\hat{\theta}_1}\right) = \lambda_t \mu(\theta_1) \tag{A.41}$$

where U_c is the partial derivative of U with respect to c. Differentiating with respect to K_{t+1} we obtain

$$\lambda_t = \lambda_{t+1}\left(1 + F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right) - \delta\right). \tag{A.42}$$

The assumption that $(c_t(\theta_1), y_t(\theta_1), K_t)$ converges to a positive limit for all θ_1 guarantees that

$$\lim\nolimits_{t\to\infty} \lambda_t/\lambda_{t+1} = 1/\beta \tag{A.43}$$

$$\lim\nolimits_{t\to\infty} \lambda_t/\lambda_{t+1} = \left(1 + F_K\left(K_{t+1}^*, \int y_{t+1}^* d\mu\right) - \delta\right). \tag{A.44}$$

This implies the proposition.

A.5. *Proof of Theorem* 2

We proceed much as in the proof of Theorem 1. Again, we construct a particular class of deviations from the candidate optimum. In particular, we focus on deviant allocations that deliver the same sub-utility in all states as the optimal allocation.

Thus, we claim that c^* solves the following optimization problem $MIN2$:

$$\begin{aligned}
&\min\nolimits_c G\left(\int c_t d\mu, K_{t+1}^*, K_t^*, \int y_t^* d\mu\right) \\
&\text{s.t. } u(c_t) = u(c_t^*) \text{ a.e.} \\
&\text{s.t. } c_t \in L_t^\infty \\
&\text{s.t. } c_t \geq 0 \text{ a.e.}
\end{aligned} \tag{A.45}$$

Suppose not. Then, there exists a non-negative c_t' in L_t^∞ such that

$$G\left(\int c_t' d\mu, K_{t+1}^*, K_t^*, \int y_t^* d\mu\right) < 0 \tag{A.46}$$

and $u(c_t'(\theta^T)) = u(c_t^*(\theta^T))$ for all θ^T in $A \subseteq \Theta^T$, where $\mu(A) = 1$. Let $c_t''(\theta^T) = c_t'(\theta^T)$ for all θ^T in A and $c_t''(\theta^T) = c_t^*(\theta^T)$ for all θ^T not in A. Let $c'' = (c_t'', c_{-t}^*)$.

Clearly, (c'', y^*, K^*) is feasible. As in Theorem 1, this allocation is also incentive-compatible because

$$W(\sigma^*; c'', y^*) \tag{A.47}$$
$$= W(\sigma^*; c^*, y^*) \tag{A.48}$$
$$\geq \max\nolimits_{\sigma\in\Sigma} W(\sigma; c^*, y^*) \tag{A.49}$$
$$= \max\nolimits_{\sigma\in\Sigma} W(\sigma; c'', y^*). \tag{A.50}$$

Thus, (c'', y^*, K^*) also solves $P1(K_1^*)$. However, because G is strictly increasing in K_{t+1}, and strictly decreasing in K_t, there exists K' such that (c'', y^*, K') solves $P1(K_1)$ for some $K_1 < K_1^*$. But this means that $V^*(K_1) = V^*(K_1^*)$ which is a contradiction.

Thus, c^* solves the above minimization problem. The rest of the proof is simply technical: establishing that the solution to the minimization problem satisfies the first-order conditions in the theorem.

Note that Lemma A1 can easily be extended to the case in which c_t^* is a finite-dimensional random vector. As in the proof of Theorem 1, if we drop the non-negativity constraints from the minimization problem, we know that c_t^* is a local minimum in the resulting problem, and that it is a regular point in the constraint set. From Lemma A1, and Luenberger (1969, Theorem 1, p. 243), we know that there exists $z_t^* \in L_t^{\infty *}$ such that c_t^* is a stationary point of the Lagrangian:

$$L(c_t) = G\left(\int c_t d\mu, K_{t+1}^*, K_t^*, Y_t^*\right) - \langle z_t^*, u(c_t)\rangle. \tag{A.51}$$

In other words

$$0 = G_j\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\int \Delta d\mu - \langle z_t^*, u_j(c_t^*)\Delta\rangle \qquad \text{for all } \Delta \text{ in } L_t^\infty \tag{A.52}$$

$$0 = G_k\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\int \Delta d\mu - \langle z_t^*, u_k(c_t^*)\Delta\rangle \qquad \text{for all } \Delta \text{ in } L_t^\infty. \tag{A.53}$$

It follows that

$$0 = G_j\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\int \{\Delta'/u_j(c_t^*)\}d\mu - \langle z_t^*, \Delta'\rangle \qquad \text{for all } \Delta' \text{ in } L_t^\infty \tag{A.54}$$

$$0 = G_k\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\int \{\Delta'/u_k(c_t^*)\}d\mu - \langle z_t^*, \Delta'\rangle \qquad \text{for all } \Delta' \text{ in } L_t^\infty \tag{A.55}$$

and so

$$0 = \int\left[G_j\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\Big/ u_j(c_t^*) - G_k\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\Big/ u_k(c_t^*)\right]\Delta' d\mu \quad \text{for all } \Delta' \text{ in } L_t^\infty. \tag{A.56}$$

The theorem follows by setting

$$\Delta' = G_j\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\Big/ u_j(c_t^*) - G_k\left(\int c_t^* d\mu, K_{t+1}^*, K_t^*, Y_t^*\right)\Big/ u_k(c_t^*). \tag{A.57}$$

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