The Wealth Distribution in Bewley Economies with Capital Income Risk

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First draft: October 2013 - This draft: March 2015*

Abstract

We study the wealth distribution in Bewley economies with idiosyncratic capital income risk. We show analytically that under rather general conditions on the stochastic structure of the economy, a unique ergodic distribution of wealth displays a fat tail.

Key Words: Wealth distribution; Bewley economies; Pareto distribution; fat tails; capital income risk

JEL Numbers: E13, E21, E24

*Thanks to Ben Moll for kindly providing us with very useful detailed comments and to Yi Lu for precious suggestions regarding various computational aspects. We also thank Ziwei Deng, Basant Kapur, JungJae Park, Jianguo Xu, and Jie Zhang. Shenghao Zhu acknowledges the financial support from National University of Singapore (R-122-000-175-112). Corresponding author's email: alberto.bisin@nyu.edu.

1 Introduction

Bewley economies, as e.g., in Bewley (1977, 1983) and Aiyagari (1994),[1] represent one of the fundamental workhorses of modern macroeconomics, its main tool when moving away from the study of efficient economies with a representative agent to allow e.g., for incomplete markets.[2] In these economies each agent faces a stochastic process for labor earnings and solves an infinite horizon consumption-saving problem with incomplete markets. Typically, agents are restricted to save by investing in a risk-free bond and face a borrowing limit. The postulated process for labor earnings determines the dynamics of the equilibrium distributions for consumption, savings, and wealth.[3]

Models of Bewley economies have been successful in the study of several macroeconomic phenomena of interest. Calibrated versions of this class of models have been used to study welfare costs of inflation (Imrohoroglu, 1992), asset pricing (Mankiw, 1986 and Huggett, 1993), unemployment benefits (Hansen and Imrohoroglu, 1992), fiscal policy (Aiyagari, 1995 and Heathcote, 2005), and partial consumption insurance (Heathcote, Storesletten, and Violante, 2008a, 2008b; Storesletten, Telmer, and Yaron, 2001; and Krueger and Perri, 2008).[4]

On the other hand, standard and plausible parametrizations of Bewley economies are hardly able to reproduce the observed distribution of wealth in many countries; see e.g., Aiyagari (1994) and Huggett (1993). More specifically, they cannot reproduce the high inequality and the fat right tail that empirical distributions of wealth tend to display.[5] This is because at high wealth levels, the incentives for precautionary savings taper off and the right tail of the wealth distribution remains thin; see Carroll (1997) and Quadrini (1999) for a discussion of these issues.[6]

[1]The *Bewley economy* terminology is rather generally adopted and has been introduced by Ljungqvist and Sargent (2004).

[2]The assumption of complete markets is generally rejected in the data; see e.g., Attanasio and Davis (1996), Fisher and Johnson (2006) and Jappelli and Pistaferri (2006).

[3]More recent specifications of the model allow for aggregate risks and an equilibrium determination of labor earnings and interest rates; see Huggett (1993), Aiyagari (1994), Rios-Rull (1995), Krusell and Smith (2006, 2008); see also Ljungqvist and Sargent (2004), Ch. 17, for a review of results.

[4]See Heathcoate-Storesletten-Violante (2010) for a recent survey of the quantitative implications of Bewley models.

[5]Large top wealth shares in the U.S. since the 60's are documented e.g., by Wolff (1987, 2004) and, more recently, by Kopczuk and Saez (2014) using estate tax return data; Piketty and Zucman (2014) find large and increasing wealth-to-income ratios in the U.S. and Europe in 1970-2010 national balance sheets data. Fat tails for the distributions of wealth are also well documented, for example by Nirei-Souma (2004) for the U.S. and Japan from 1960 to 1999, by Clementi-Gallegati (2004) for Italy from 1977 to 2002, and by Dagsvik-Vatne (1999) for Norway in 1998. Restricting to the Forbes 400 richest U.S. individuals during 1988-2003, Klass et al. (2007) also find that the top end of the wealth distribution obeys a Pareto law.

[6]Stochastic labor earnings can in principle generate some skewness in the distribution of wealth, especially if the earnings process is itself skewed and persistent. Extensive evidence for the skewedness of the income distribution has been put forth in a series of papers by Emmanuel Saez and Thomas

In the present paper we analytically study the wealth distribution in the context of Bewley economies extended to allow for idiosyncratic capital income risk.[7] To this end we provide first an analysis of the standard *Income Fluctuation problem*, as e.g., in Chamberlain-Wilson (2000), extended to account for capital income risk.[8] As in Aiyagari (1994), the borrowing constraint together with stochastic incomes assures a lower bound to wealth acting as a reflecting barrier.[9] We analytically show that enough idiosyncratic capital income risk induces an ergodic stationary wealth distribution which is fat tailed, more precisely, a Pareto distribution in the right tail. Furthermore, we show that the consumption function under borrowing constraints is strictly concave at lower wealth levels, consistent with, e.g. Saez and Zucman (2014)'s evidence of substantial saving rate differentials across wealth levels. In this environment, therefore, the rich can get richer through savings, while the poor may not save enough to become rich. Such non-ergodicity however would imply no social mobility between rich and poor, which seems incompatible with observed levels of social mobility in income over time and across generations; see for example Chetty, Hendren, Kline, and Saez (2014). In our analysis it is capital income risk that induces the necessary mobility across wealth levels to generate an ergodic stationary wealth distribution.[10]

Piketty (some with co-authors), starting with Saez and Piketty (2003) on the U.S. We refer to Atkinson, Saez, and Piketty (2011) for a survey and to the excellent website of the database they have collected (with Facundo Alvaredo), The World Top Incomes Database. However, most empirical studies of labor earnings find some form of stationarity of the earnings process; see Guvenen (2007) and e.g., the discussion of Primiceri and van Rens (2006) by Heathcote (2008). Persistent income shocks are often postulated to explain the cross-sectional distribution of consumption but seem hardly enough to produce fat tailed distributions of wealth; see e.g., Storesletten, Telmer, Yaron (2004); see also Cagetti and De Nardi (2008) for a survey.

[7]Capital income risk has been introduced by Angeletos and Calvet (2005) and Angeletos (2007) and further studied by Panousi (2008) and by ourselves (Benhabib, Bisin, and Zhu, 2011 and 2013). Quadrini (1999, 2000) and Cagetti and De Nardi (2006) study entrepreneurial risk, one of the leading examples of capital income risk, explicitly. Jones and Kim (2014) study entrepreneurs in a growth context under risk introduced by creative destruction. Relatedly, Krusell and Smith (1998) introduce heterogeneous discount rates to numerically produce some skewness in the distribution of wealth. We refer to these papers and our previous papers, as well as to Benhabib and Bisin (2006) and Benhabib and Zhu (2008), for more general evidence on the macroeconomic relevance of capital income risk.

[8]The work by Levhari and Srinivasan (1969), Schectman (1976), Schectman and Escudero (1977), Chamberlain-Wilson (2000), Huggett (1993), Rabault (2002), Carroll and Kimball (2005) has been instrumental to provide several incremental pieces to our characterization of the solution of (various specifications of) the Income Fluctuation problem; see Ljungqvist and Sargent (2004), Ch. 16, as well as Rios-Rull (1995) and Krusell-Smith (2006), for a review of results regarding the standard Income Fluctuation problem.

[9]See also Achdou, Lasry, Lions and Moll (2014) for a continuous time model with stochastic returns and borrowing constraints, exploring the interaction of aggregate shocks and inequality on the transition dynamics of the macroeconomy.

[10]This complements the results in our previous papers (Benhabib, Bisin, and Zhu, 2011 and 2013), which focus on overlapping generation economies. An alternative approach to generate fat tails without stochastic returns is to introduce a model with bequests, where the probability of death (and/or

The rest of the paper is organized as follows. We present the basic setup of our economy in Section 2. In Section 3 we obtain the characterization of the income fluctuation problem with idiosyncratic capital income risk. In Section 4 we show that the wealth accumulation process has a unique stationary distribution and the stationary distribution displays a fat right tail. In Section 5 we introduce a model of entrepreneurship which is embedded in our analysis of the wealth distribution induced by the income fluctuation problem.[11] In Section 6 we extend our analysis of Bewley economies to allow for a market for loans. In Section 7 we briefly conclude.

2 The economy

Consider an infinite horizon economy with a continuum of agents uniformly distributed with measure 1.[12] Let $\{c_t\}_{t=0}^{\infty}$ denote an agent consumption process. Let $\{y_t\}_{t=0}^{\infty}$ represent the agent's labor earnings process and $\{R_{t+1}\}_{t=0}^{\infty}$ his/her idiosyncratic rate of return on wealth process, that is, capital income risk.

The agent's budget constraint at time t is then

$$w_{t+1} = R_{t+1}\left(w_t + y_t - c_t\right),$$

where $\{w_{t+1}\}_{t=0}^{\infty}$ is wealth before earnings. In the economy, each agent faces a no-borrowing constraint at each time t:

$$w_{t+1} \geq 0.$$

It is convenient however for our purposes to work with the process of wealth after earnings, that is $a_t = w_t + y_t$. In this case, the agent's budget constraint and his/her borrowing constraint take respectively the following form:

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}$$

$$c_t \leq a_t$$

Each agent in the economy then solves the *Income Fluctuation (IF) problem* which is obtained under Constant Relative Risk Aversion (CRRA) preferences,

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma \geq 1,$$

retirement) is independent of age. In these models, the stochastic component is not stochastic returns but the length of life. For models that embody such features, see Wold and Whittle (1957), Castaneda, Gimenez and Rios-Rull (2003), and Benhabib and Bisin (2006). On the other hand, sidestepping the income fluctuation problem by assuming a constant savings rate, Nirei and Aoki (2015) shows that thick tails are a direct consequence of the linearity of the wealth equation.

[11] The NBER W.P. version of this paper, Benhabib, Bisin, and Zhu (2014), also contains some simulation results regarding the stationary wealth distribution and the social mobility of the wealth accumulation process.

[12] We avoid introducing notation to index agents in the paper.

constant discounting $\beta < 1$, and *capital income risk and earnings* processes, $\{R_{t+1}\}_{t=0}^{\infty}$ and $\{y_t\}_{t=0}^{\infty}$:

$$\max_{\{c_t\}_{t=0}^{\infty},\{a_{t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma} \tag{IF}$$

$$\text{s.t. } a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}$$

$$c_t \leq a_t$$

$$a_0 \text{ given.}$$

The following assumptions characterize formally the stochastic properties of the economic environment:

Assumption 1 *R_t and y_t are stochastic processes, independent and identically distributed (i.i.d.) over time and across agents: y_t has probability density function $f(y)$ on bounded support $[\underline{y}, \bar{y}]$, with $\underline{y}>0$ and R_t has probability density function $g(R)$ with closed support $[\underline{R}, \bar{R}]$.*[13] *R_t and y_t are independent. Furthermore, y_t satisfies* ***i)*** $(\bar{y})^{-\gamma} < \beta E\left[R_t\left(y_t\right)^{-\gamma}\right]$, *while R_t satisfies:* ***ii)*** $\bar{R}>\underline{R}>0$ *large enough,* ***iii)*** $\beta E R_t^{1-\gamma} < 1$; ***iv)*** $\left(\beta E R_t^{1-\gamma}\right)^{\frac{1}{\gamma}} E R_t < 1$; *and* ***v)*** $\Pr(\beta R_t > 1) > 0$ *and any finite moment of R_t exists.*

To induce a limit stationary distribution of wealth, these assumptions guarantee that the contractive and expansive components of the rate of return process $\{R_t\}_{t=0}^{\infty}$ tend to balance and the earnings process $\{y_t\}_{t=0}^{\infty}$ act as a reflecting barrier on wealth. The assumption that these processes are i.i.d over time is restrictive as a positive correlation in earnings and returns would capture economic environments with limited social mobility (for example, environments in which returns economic opportunities are in part transmitted across generations); but it could possibly be relaxed.[14]

2.1 Outline

It is useful to briefly outline the role of our assumptions and our strategy to obtain the main results in the paper. Assumptions 1.i) and 1.ii) guarantee that an agent with zero wealth at some time t will not consume all his/her income at time $t+1$ for high enough realizations of earnings and rates of return; as a consequence, the lower bound of the

[13] Note however we can allow the support of R to be the real numbers over the half-line $[\bar{R}, \infty)$, which is closed in the real numbers. While $\bar{R} = \infty$ is allowed for, a finite $\bar{R}$, as derived in the proof of Theorem 4, is sufficient for all our results. In the case R takes discrete values in state space $\tilde{R}$, we also assume the elements of $\tilde{R}$ are not all integral multiples of each other; see Saporta (2005), Theorem 1. This non-aritmeticity assumption is immediately satisfied if the support of R contains an interval of real numbers; it assures that the discrete stochastic process for wealth results a distribution with a continuous power tail without holes.

[14] See the next section for a detailed discussion of Assumptions 1.i)-1.v).

wealth space is a reflecting barrier, i.e., the wealth accumulation process is not trapped in the lower part of the wealth space in which savings of the agent are zero (see Lemma 7 in the Appendix) The stochastic process for wealth is then ergodic.

Assumptions 1.iii) and 1.iv) guarantee that the wealth accumulation process is stationary. In particular, Assumption 1.iii) guarantees that the aggregate economy displays no unbounded growth in consumption and wealth.[15] Assumption 1.iv) implies that

$$\beta E R_t < 1.$$

This is enough to guarantee that the economy contracts, giving rise to a stationary distribution of wealth. However, since we cannot obtain explicit solutions for consumption or savings policies, we have to explicitly show that under suitable assumptions there are no disjoint invariant sets or cyclic sets in wealth, so that agents do not get trapped in subsets of the support of the wealth distribution. In other words we have to show that the stochastic process for wealth is ergodic, and that a unique stationary distribution exists. We show this in Lemmata 6 and 8 in the Appendix.

We then have to show that idiosyncratic capital income risk can give rise to a fat-tailed wealth distribution. Since in our economic environment policy functions are not linear and explicit solutions are not available even under CRRA preferences, we cannot use the results of Kesten (1973), for example as in Benhabib, Bisin and Zhu (2011). We are nonetheless able to show that consumption and savings policies are asymptotically linear; a result which, under appropriate assumptions, in particular i.i.d. processes for R_{t+1} and y_t, allow us to apply Mirek (2011)'s generalization of Kesten (1973).[16] We do this in Propositions 3, 4 and 5. The fat right tail of the stationary distribution of wealth, obtained in Theorem 3, exploits crucially that $\Pr(\beta R_t > 1) > 0$, that is, Assumption 1.v).

3 The income fluctuation problem with idiosyncratic capital income risk

In this section we show several technical results about the consumption function $c(a)$ which solves the (IF) problem, as a build-up for its characterization of the wealth distribution in the next section. All proofs are in the Appendix.

[15] We can allow for exogenous growth $g > 1$ in earnings, as in Aiyagari and McGrattan (1998). To this end, we need to deflate the variables by the growth rate and let the borrowing constraint grow at growth rate. (In our context, since we allow for no borrowing, no modification of the constraint is needed. However, Assumption 1.2.iii) would have to be modified so that $\Pr(\frac{\beta R_t}{g^\gamma} > 1) > 0$.)

[16] We conjecture that the analysis could be extended to serially correlated earnings and returns processes along the lines of Benhabib, Bisin, and Zhu (2001), though this would require extending Saporta (2004, 2005) and Roitershtein (2007) main theorems to asymptotic Kesten processes.

Theorem 1 *A consumption function $c(a)$ which satisfies the constraints of the (IF) problem and furthermore satifies*

i) *the Euler equation*

$$u'(c(a)) \geq \beta E R_{t+1} u'(c\left[R_{t+1}(a - c(a)) + y\right]) \text{ with equality if } c(a) < a, \tag{1}$$

and

ii) *the transversality condition*

$$\lim_{t\to\infty} E\beta^t u'(c_t) a_t = 0, \tag{2}$$

represents a solution of the (IF) problem.

By strict concavity of $u(c)$, there exists a unique $c(a)$ which solves the (IF) problem.

The study of $c(a)$ requires studying two auxiliary problems. The first is a version the IF problem in which the stochastic process for earnings $\{y_t\}_0^\infty$ is turned off, that is, $y_t = 0$, for any $t \geq 0$. The second is a finite horizon version of the IF problem. In both cases we naturally maintain the relevant specification and assumptions imposed on our main IF problem.

3.1 The IF problem with no earnings

The formal IF problem with no earnings is:

$$\max_{\{c_t\}_{t=0}^\infty, \{a_{t+1}\}_{t=0}^\infty} E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \tag{IF with no earnings}$$
$$s.t.\ a_{t+1} = R_{t+1}(a_t - c_t)$$
$$c_t \leq a_t$$
$$a_0 \text{ given.}$$

This problem can indeed be solved in closed form, following Levhari and Srinivasan (1969). Note that for this problem the borrowing constraint is never binding because Inada conditions are satisfied for CRRA utility.

Proposition 1 *The unique solution to the (IF with no earnings) problem is*

$$c^{no}(a) = \phi a, \text{ for } 0 < \phi = 1 - \left(\beta E \left(R_{t+1}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}} < 1. \tag{3}$$

3.2 The finite IF problem

For any $\tau \in \mathbb{Z}$, $T > 0$, let the finite IF problem be:

$$\max_{\{c_t\}_{t=\tau}^{T},\{a_{t+1}\}_{t=\tau}^{T-1}} E\sum_{t=\tau}^{T}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma} \tag{finite IF}$$
$$s.t.\ a_{t+1}=R_{t+1}(a_t-c_t)+y_{t+1},\quad \text{for } \tau\leq t\leq T-1$$
$$c_t\leq a_t,\quad \text{for } \tau\leq t\leq T$$
$$a_\tau \text{ given.}$$

Proposition 2 *The unique solution to the (finite IF) problem is a consumption function $c_{t,\tau}(a)$ which is continuous and increasing in a. Furthermore, let $s_{t,\tau}(a)$ denote the induced savings function,*

$$s_{t,\tau}(a)=a-c_{t,\tau}(a).$$

Then $s_{t,\tau}(a)$ is also continuous and increasing in a.

3.3 Characterization of $c(a)$

We can now derive a relation between $c_{t,\tau}(a)$, $c^{no}(a)$ and $c(a)$. The following Lemma is a straightforward extension of Proposition 2.3 and Proposition 2.4 in Rabault (2002).

Lemma 1 $\lim_{t,\tau\to-\infty}c_{t,\tau}(a)$ *exists, it is continuous, and satisfies the Euler equation. Furthermore,*

$$\lim_{t,\tau\to-\infty}c_{t,\tau}(a)\geq c^{no}(a).$$

The main result of this section follows:

Theorem 2 *The unique solution to the (IF) problem is the consumption function $c(a)$ which satisfies:*

$$c(a)=\lim_{t,\tau\to-\infty}c_{t,\tau}(a).$$

Let the induced savings function $s(a)$ be

$$s(a)=a-c(a).$$

Proposition 3 *The consumption and savings functions $c(a)$ and $s(a)$ are Lipschitz continuous and increasing in a.*

Carroll and Kimball (2005) show that $c_{t,\tau}(a)$ is concave.[17] But Lemma 2 guarantees that $c(a)=\lim_{t,\tau\to-\infty}c_{t,\tau}(a)$ and thus $c(a)$ is also a concave function of a.

[17]See also Carroll, Slacalek, and Tokuoka (2014).

Proposition 4 *The consumption function $c(a)$ is a concave function of a.*

The most important result of this section is that the optimal consumption function $c(a)$, in the limit for $a \to \infty$, is linear and has the same slope as the optimal consumption function of the income fluctuation problem with no earnings, ϕ.

Proposition 5 *The consumption function $c(a)$ satisfies $\lim_{a\to\infty} \frac{c(a)}{a} = \phi$.*

The proof, in the Appendix, is non-trivial.

4 The stationary distribution

In this section we study the distribution of wealth in the economy. The wealth accumulation equation of the (IF) problem is

$$a_{t+1} = R_{t+1}(a_t - c(a_t)) + y_{t+1}. \tag{4}$$

It is useful to compare it with the IF with no earnings. Using Lemma 1 we have:

$$\begin{aligned} a_{t+1} &= R_{t+1}(a_t - c(a_t)) + y_{t+1} \\ &\leq R_{t+1}(a_t - c^{no}(a_t)) + y_{t+1} \\ &= R_{t+1}(1-\phi)a_t + y_{t+1}. \end{aligned}$$

Let

$$\mu = 1 - \phi = \left(\beta E R^{1-\gamma}\right)^{\frac{1}{\gamma}}.$$

Thus $\mu < 1$ by Assumption 1.iii). We have

$$a_{t+1} \leq \mu R_{t+1} a_t + y_{t+1}.$$

The main results in this section are the following two theorems.[18]

Theorem 3 *The process $\{a_{t+1}\}_{t=0}^{\infty}$ is ergodic and hence there exists a unique stationary distribution for a_{t+1} which satisfies the stochastic wealth accumulation equation (4).*

The proof, in the Appendix, requires several steps. First we show that the wealth accumulation process $\{a_{t+1}\}_{t=0}^{\infty}$ induced by equation (4) above is $\varphi-$ irreducible, *i.e.,* there exists a non trivial measure φ on [y,∞) such that if $\varphi(A) > 0$, the probability that the process enters the set A in finite time is strictly positive for any initial condition (see Chapter 4 of Meyn and Tweedie (2009)). We also show that $a = \underline{y}$ represents a reflecting barrier for the process. To show that there exists a unique stationary wealth distribution

[18]The result in Theorem 3 can also be obtained as an application of Theorem 2 in Kamihigashi and Stachurski (2014) under slightly weaker assumptions. We thank a referee for pointing this out to us.

we exploit the results in Meyn and Tweedie (2009) and show that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ergodic.

Finally, in the next theorem we show that the wealth accumulation process $\{a_{t+1}\}_{t=0}^{\infty}$ has a fat tail. More precisely,

Definition 1 *A distribution X is said to have a right fat tail if there exists $\alpha > 0$ such that*

$$\lim_{x\to+\infty}\inf \frac{\Pr(X > x)}{x^{-\alpha}} \geq C,$$

where C is a positive constant.[19]

We use the characterization of $c(a)$ and $s(a)$ in Section 3.3, and in particular the fact that $\frac{s(a)}{a}$ is increasing in a and $\frac{s(a)}{a}$ approaches μ as a goes to infinity (see the discussion after the proof of Lemma 3 in the Appendix); this allows us to apply some results by Mirek (2011) regarding conditions for asymptotically Pareto stationary distributions for processes induced by non-linear stochastic difference equations.

Theorem 4 *The unique stationary distribution for a_{t+1} which satisfies the stochastic wealth accumulation equation (4) has a fat tail.*

Proof. We use a comparison method to show the result. Firstly, we construct an auxiliary process, $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$. Then we show that the tail of the stationary distribution for $\tilde{a}_{t+1}$ is asymptotic to a Pareto law. Finally, we show that the stationary distribution for a_t, which satisfies the stochastic wealth accumulation equation (4) has a fat tail, through comparing processes $\{a_{t+1}\}_{t=0}^{\infty}$ and $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$.

Construction of $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$. Since $\frac{s(a)}{a}$ is increasing in a and $\frac{s(a)}{a}$ approaches μ as a goes to infinity (see the discussion after the proof of Lemma 3 in the Appendix), there exist an $\epsilon > 0$ arbitrarily small such that we can pick a large a^{ϵ} to satisfy

$$\mu - \frac{s(a^{\epsilon})}{a^{\epsilon}} < \epsilon.$$

[19] A simple definition of a power law, or fat tailed, distribution is as follows. Define a regularly varying function with index $\alpha \in (0,\infty)$ as

$$\lim_{z\to\infty} \frac{L(tx)}{L(x)} = t^{-a}, \forall t > 0$$

Then, a distribution with a differentiable cumulative distribution function (cdf) $F(x)$ and counter-cdf $1-F(x)$ is defined as a power-law with tail index α if $1-F(z)$ is regularly varying with index $\alpha > 0$. If $\lim_{z\to\infty} \frac{1-F(tx)}{1-F(x)} = 1, \forall t > 0$, this is a slowly-varying function. If $\lim_{z\to\infty} \frac{1-F(tx)}{1-F(x)} = \infty, \forall t > 0$, then the function is neither a slowly-varying function nor a power-law. For example, the counter-cdf of the Cauchy distribution is slowly varying (i.e., $\alpha = 0$ above), while for the lognormal and normal distributions, the limit is infinite. Intuitively, α captures the number of moments: $\alpha = 0$ means the Cauchy has no moments, while $\alpha = \infty$ means the distribution has all the moments; see Soulier (2009).

Let

$$\mu^\epsilon = \frac{s(a^\epsilon)}{a^\epsilon}.$$

Thus $\mu - \epsilon < \mu^\epsilon \leq \mu$.

Let

$$l(a) = \begin{cases} s(a), & a \leq a^\epsilon \\ \mu^\epsilon a, & a \geq a^\epsilon \end{cases}. \tag{5}$$

Note that $l(a) \leq s(a)$ for $\forall a \in [\underline{y}, \infty)$, since $\frac{s(a)}{a}$ is increasing in a; furthermore, the function $l(a)$ in (5) is Lipschitz continuous, since $s(a)$ is Lipschitz continuous.

Let $\theta = (R,\ y)$ and

$$\psi_\theta(a) = Rl(a) + y. \tag{6}$$

The stochastic process $\{\tilde{a}_{t+1}\}_{t=0}^\infty$ is induced by $\tilde{a}_{t+1} = \psi_\theta(\tilde{a}_t)$. Now we apply Theorem 1.8 of Mirek (2011) to show that $\{\tilde{a}_{t+1}\}_{t=0}^\infty$ has a unique stationary distribution and that the tail of the stationary distribution for $\tilde{a}_{t+1}$ is asymptotic to a Pareto law, *i.e.*

$$\lim_{a \to \infty} \frac{\Pr(\tilde{a}_\infty > a)}{a^{-\alpha}} = C,$$

where C is a positive constant.

In order to apply Theorem 1.8 of Mirek (2011), we need to verify Assumption 1.6 and Assumption 1.7 of Mirek (2011). Assumption 1.6 essentially guarantees that $\psi_\theta(\cdot)$ is asymptotically linear. Assumption 1.7 instead is the standard assumption which induces fat tails in the stationary distribution of a Kesten (linear) process.

Verification of Assumption 1.6 of Mirek (2011). For every $z > 0$, let

$$\psi_{\theta,z}(a) = z\psi_\theta\left(\frac{1}{z}a\right).$$

$\psi_{\theta,z}$ are called dilatations of ψ_θ. Let

$$\bar{\psi}_\theta(a) = \lim_{z \to 0} \psi_{\theta,z}(a).$$

By the definition of $\psi_\theta(\cdot)$ we have

$$\bar{\psi}_\theta(a) = \lim_{z \to 0} \psi_{\theta,z}(a) = \lim_{z \to 0}\left[z\psi_\theta\left(\frac{1}{z}a\right)\right] = \mu^\epsilon Ra, \quad \text{for } \forall a \in [\underline{y}, \infty).$$

Let

$$M^\epsilon = \mu^\epsilon R.$$

Thus

$$\bar{\psi}_\theta(a) = M^\epsilon a.$$

Let

$$N_\theta = \Omega R + y$$

where

$$\Omega = \max_{a \in [\underline{y}, a^\epsilon]} |s(a) - \mu^\epsilon a|.$$

It is easy to verify that

$$|\psi_\theta(a) - M^\epsilon a| < N_\theta, \quad \text{for } \forall a \in [\underline{y}, \infty).$$

and hence that Assumption 1.6 (Shape of the mappings) in Mirek (2011) is satisfied.

Verification of Assumption 1.7 of Mirek (2011) As for Assumption 1.7 in Mirek (2011), condition (H3) is satisfied since $M^\epsilon = \mu^\epsilon R$, R_t is *i.i.d.* over time and the support of R_t is closed. The conditional law of $log M$ is non-arithmetic by Assumption 1 (see footnote 13) so H(4) in Assumption 1.7 of Mirek is satisfied. Let $h(d) = \log E\,(M^\epsilon)^d$. By Assumption 1.iv) we have $E\,(\mu R_t) < 1$. Thus $h(1) = \log E\,(M^\epsilon) \le \log E\,(\mu R) < 0$. We now show that Assumption 1.iv) and Assumption 1.v) imply that there exists $\kappa > 1$ such that $\mu^\kappa E(R_t)^\kappa > 1$. By Jensen's inequality we have $E(R_t)^{1-\gamma} \ge (ER_t)^{1-\gamma}$. Also, Assumption 1.iv) implies that $\beta E R_t < 1$. Thus

$$\mu = \left(\beta E(R_t)^{1-\gamma}\right)^{\frac{1}{\gamma}} \ge \left[\beta\,(ER_t)^{1-\gamma}\right]^{\frac{1}{\gamma}} \ge \left[\beta\left(\frac{1}{\beta}\right)^{1-\gamma}\right]^{\frac{1}{\gamma}} = \beta.$$

Thus

$$E\,(\mu R_t)^\kappa \ge E\,(\beta R_t)^\kappa \ge \int_{\{\beta R_t > 1\}} (\beta R_t)^\kappa\,.$$

By Assumption 1.v), $\Pr(\beta R_t > 1) > 0$. Thus there exists $\kappa > 1$ such that $\mu^\kappa E(R_t)^\kappa > 1$. We could pick μ^ϵ such that $(\mu^\epsilon)^\kappa E(R_t)^\kappa > 1$. Thus $h(\kappa) = \log E\,(M^\epsilon)^\kappa > 0$. By Assumption 1.v), any finite moment of R_t exists. Thus $h(d)$ is a continuous function of d. Thus there exists $\alpha > 1$ such that $h(\alpha) = 0$, *i.e.* $E\,(M^\epsilon)^\alpha = 1$. Also we know that $h(d)$ is a convex function of d. Thus there is a unique $\alpha > 0$, such that $E\,(M^\epsilon)^\alpha = 1$.

Moreover, $E\left[(M^\epsilon)^\alpha\,|\log M^\epsilon|\right] < \infty$, since M^ϵ has a lower bound, and, by Assumption 1.v), any finite moment of R exists.

We also know that $E(N_\theta)^\alpha < \infty$ since y has bounded support and, by Assumption 1.v), any finite moment of R exists.

Thus M^ϵ and N satisfy Assumption 1.7 (Moments condition for the heavy tail) of Mirek (2011).

The comparison method. Applying Theorem 1.8 of Mirek (2011), we find that the stationary distribution of $\{\tilde{a}_{t+1}\}_{t=0}^\infty$, $\tilde{a}_\infty$, has an asymptotic Pareto tail. Finally, we show that the stationary distribution of $\{a_{t+1}\}_{t=0}^\infty$, a_∞, has a fat tail.

Pick $a_0 = \tilde{a}_0$. The stochastic process $\{a_{t+1}\}_{t=0}^{\infty}$ is induced by

$$a_{t+1} = R_{t+1}s(a_t) + y_{t+1}.$$

And the stochastic process $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$ is induced by

$$\tilde{a}_{t+1} = R_{t+1}l(\tilde{a}_t) + y_{t+1}.$$

For a path of $\{(R_{t+1}, y_{t+1})\}_{t=0}^{\infty}$, we have $a_t \geq \tilde{a}_t$, $\forall t \geq 0$. Thus for $\forall a > \underline{y}$, we have

$$\Pr(a_t > a) \geq \Pr(\tilde{a}_t > a), \quad \text{for } \forall t \geq 0.$$

This implies that

$$\Pr(a_\infty > a) \geq \Pr(\tilde{a}_\infty > a),$$

since stochastic processes $\{a_{t+1}\}_{t=0}^{\infty}$ and $\{\tilde{a}_{t+1}\}_{t=0}^{\infty}$ are ergodic. Thus

$$\liminf_{a\to\infty} \frac{\Pr(a_\infty > a)}{a^{-\alpha}} \geq \liminf_{a\to\infty} \frac{\Pr(\tilde{a}_\infty > a)}{a^{-\alpha}} = \lim_{a\to\infty} \frac{\Pr(\tilde{a}_\infty > a)}{a^{-\alpha}} = C. \blacksquare$$

5 Investment risk and entrepreneurship

In this section we discuss how to embed the analysis of the distribution of wealth induced by the (IF) problem in an equilibrium economy of entrepreneurship, one of the leading examples of investment risk. Following Angeletos (2007) we assume that each agent acts as entrepreneur of his own individual firm. Each firm has a constant returns to scale neo-classical production function

$$F(k, n, A)$$

where k, n are, respectively, capital and labor, and A is an idiosyncratic productivity shock. Agents can only use their own savings as capital in their own firm. In each period $t+1$, each agent observes his/her firm's productivity shock A_{t+1} and decides how much labor to hire in a competitive labor market, n_{t+1}. Therefore, each firm faces the same market wage rate w_{t+1}. The capital he/she invests is instead predetermined, but the agent can decide not to engage in production, in which case $n_{t+1} = 0$ and the capital invested is carried over (with no return nor depreciation) to the next period. The firm's profits in period $t+1$ are denoted π_{t+1}:

$$\pi_{t+1} = \max\left\{F(k_{t+1}, n_{t+1}, A_{t+1}) - wn_{t+1} + (1-\delta)k_{t+1},\ k_{t+1}\right\}. \tag{7}$$

Letting each agent's earnings in period $t+1$ are denoted $w_{t+1}e_{t+1}$, where e_{t+1} is his/her idiosyncratic (exogenous) labor supply, we have

$$a_{t+1} = \pi_{t+1} + w_{t+1}e_{t+1}.$$

Furthermore,

$$k_{t+1} = a_t - c_t.$$

Given a sequence $\{w_t\}_{t=0}^{\infty}$, each agent solves the following modified (IF) problem:

$$\max_{\{c_t,n_t\}_{t=0}^{\infty},\{k_{t+1},a_{t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}}{1-\gamma} \qquad \text{(IF with entrepreneurship)}$$

$$s.t.\ a_{t+1} = \pi_{t+1} + w_{t+1}e_{t+1} \text{ where } \pi_{t+1} \text{ is defined in (7)}$$

$$k_{t+1} = a_t - c_t$$

$$c_t \leq a_t$$

$$k_0 \text{ given.}$$

A stationary equilibrium in our economy consists of a constant wage rate w, sequences $\{c_t, n_t\}_{t=0}^{\infty}, \{k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$ which constitute a solution to the (IF with entrepreneurship) problem under $w_t = w$ for any $t \geq 0$, and a distribution $v(a_{t+1})$, such that the following conditions hold:

(i) labour markets clear: $En_t = Ee_t$;[20]

(ii) v is a stationary distribution of a_{t+1}.

We can now illustrate how such an equilibrium can be constructed, inducing a stationary distribution of wealth $v(a_{t+1})$ with the same properties, notably the fat tail, as the one characterized in the previous section under appropriate assumptions for the stochastic processes $\{A_{t+1}\}_{t=0}^{\infty}$ and $\{e_t\}_{t=0}^{\infty}$. The first order conditions of each agent firm's labor choice requires

$$\frac{\partial F}{\partial n}(k_{t+1}, n_{t+1}, A_{t+1}) = w_{t+1};$$

which, under constant returns to scale implies,

$$\frac{\partial F}{\partial n}\left(1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1}\right) = w_{t+1}. \tag{8}$$

Equation (8) can be solved to give

$$\frac{n_{t+1}}{k_{t+1}} = n\left(w_{t+1}, A_{t+1}\right);\ \text{ or } n_{t+1} = g\left(w_{t+1}, A_{t+1}\right)k_{t+1}.$$

The market clearing condition i) in Definition 2 is then satisfied by a constant wage rate w such that

[20]The usual abuse of the Law of Large Number guarantees that the market clearing condition as stated holds in the cross-section of agents.

$$En_{t+1} = E\left(g\left(w, A_{t+1}\right)\right) Ek_{t+1},$$

as long as the process $\{A_{t+1}\}_{t=0}^{\infty}$ is i.i.d. over time and in the cross-section and Ek_{t+1} is constant over time.

From the constant returns to scale assumption, once again, we can write profits π_{t+1} as:

$$\pi_{t+1} = R_{t+1}k_{t+1}$$

where $\{R_{t+1}\}_{t=0}^{\infty}$ is induced by the process $\{A_{t+1}\}_{t=0}^{\infty}$ as follows:

$$R_{t+1} = \max\left\{\frac{\partial F}{\partial k}\left(1, \frac{n_{t+1}}{k_{t+1}}, A_{t+1}\right) + 1 - \delta,\ 1\right\}.$$

Let $y_{t+1} = we_{t+1}$. Then the dynamic equation for wealth can be written as

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}.$$

We conclude that the solution to (IF with entrepreneurship) induces a stochastic process $\{a_{t+1}\}_{t=0}^{\infty}$ which has the same properties as the one induced by the IF problem as long as i) Ek_{t+1} is constant and ii) the process $\{R_{t+1}\}_{t=0}^{\infty}$ induced by $\{A_{t+1}\}_{t=0}^{\infty}$ and the process$\{y_t\}_{t=0}^{\infty}$ induced by $\{e_t\}_{t=0}^{\infty}$ satisfy Assumption 1.[21] In particular, in this case, $\{a_{t+1}\}_{t=0}^{\infty}$ has a unique stationary distribution. The stationary distribution of $\{a_{t+1}\}_{t=0}^{\infty}$ induces in turn a stationary distribution of k_{t+1}. The aggregate capital Ek_{t+1} is the first moment of the stationary distribution of k_{t+1} and is therefore constant. As a consequence, the labor market indeed clear with a constant wage w as postulated. It is verified then that at a stationary general equilibrium, as long as ii) above is satisfied, the stochastic process $\{a_{t+1}\}_{t=0}^{\infty}$ has the same properties as the one induced by the IF problem; it displays, in particular, a fat tail.

6 Market for loans

Our analysis of Bewley economies is constructed on the assumption that the agent's borrowing is restricted as in the (IF) problem. More specifically, the agent at t can only invest in a risky asset with idiosyncratic return R_{t+1} and no market for loans is active in the economy. In this section we show how to extend the analysis to relax this assumption.

Let b_{t+1} denote the agent's holdings of the riskless asset at time $t+1$, while k_{t+1} denotes his/her risky asset holdings. Let then a_{t+1} denote the total wealth after earnings:

[21] General conditions on $\{A_{t+1}\}_{t=0}^{\infty}$ that induce a process $\{R_{t+1}\}_{t=0}^{\infty}$ that satisfies Assumption 1 are hard to characterize. Simulations might have to be used to have a better sense of the range of parameters which induces a stationary distribution of wealth with a fat right tail.

$a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1}$, where R_f is the rate of return of the riskless asset and R_{t+1} is the rate of return of the risky asset, as in Section 2.[22] We maintain Assumption 1 and we impose a negative borrowing limit on bond holdings: $b_{t+1} \geq -L$, where $0 \leq L$.[23].

The IF problem, after allowing for an active market for loans to complement the risky asset, takes the following form:

$$E \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma} \tag{IF with loan market}$$

$$\begin{gathered} s.t. \quad b_{t+1} + k_{t+1} = a_t - c_t \\ a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1} \\ k_{t+1} \geq 0 \\ b_{t+1} \geq -L \\ a_0 \text{ given.} \end{gathered}$$

We can now illustrate how the solution of the (IF with loan market) problem induces a stochastic process $\{a_{t+1}\}_{t=0}^{\infty}$ which has the same properties as the one induced by the IF problem. Indeed, the key to this result is that, as a_t becomes large, the solution to the (IF with loan market) problem is characterized by asymptotically constant portfolio shares and as a consequence by an asymptotically linear consumption function, as in the (IF) problem (Proposition 5).[24]

More specifically, the policy functions of the (IF with loan market) problem can be written as

$$c_t = c\left(a_t\right),\ k_{t+1} = k\left(a_t\right),\ b_{t+1} = b\left(a_t\right).$$

Most importantly, they satisfy

$$\lim_{a \to \infty} \frac{k(a)}{a} = \omega(1 - \tilde{\phi}), \quad \lim_{a \to \infty} \frac{b(a)}{a} = (1 - \omega)(1 - \tilde{\phi})$$

and hence

$$\lim_{a \to \infty} \frac{c(a)}{a} = \tilde{\phi}$$

for some $0 < \tilde{\phi}, \omega < 1$.

[22]We assume R_f is constant and exogenous; though a constant R_f can be endogenously obtained at the stationary distribution by imposing market clearing in the market for loans.

[23]To guarantee that the constraints are binding and induce a reflecting barrier, it is enough for instance to assume that $L < \frac{\mathrm{y}}{R_f - 1}$

[24]Achdou, Lasry, Lions, and Moll (2014) have an elegant analysis of a related problem in continuous time, using viscosity solutions.

In fact we can easily solve for $\tilde{\phi}$ and ω. For large a_t, the first order conditions for the problem are

$$c_t^{-\gamma} = \beta R_f E c_{t+1}^{-\gamma} \tag{9}$$

and

$$E c_{t+1}^{-\gamma} \left(R_f - R_{t+1}\right) = 0. \tag{10}$$

Equation (10) implies

$$E \left[R_f \left(1-\omega\right) + R_{t+1}\omega\right]^{-\gamma} \left(R_f - R_{t+1}\right) = 0, \tag{11}$$

which determines ω; and in turn equation (9) implies

$$\left(\frac{c_t}{a_t}\right)^{-\gamma} = \beta R_f E \left(\frac{a_{t+1}}{a_t}\right)^{-\gamma} \left(\frac{c_{t+1}}{a_{t+1}}\right)^{-\gamma}.$$

and thus

$$\tilde{\phi} = 1 - \left(\beta R_f E \left[R_f \left(1-\omega\right) + R_{t+1}\omega\right]^{-\gamma}\right)^{\frac{1}{\gamma}}. \tag{12}$$

Note that the equation for $\tilde{\phi}$ is analogous to equation (3) for ϕ, the asymptotic slope of the consumption function in the (IF) problem we obtained in Proposition 5:

$$\phi = 1 - \left(\beta E \left(R_{t+1}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}},$$

once the rate of return on the risky asset R_{t+1} is substituted by the rate of return on the agent's portfolio, $R_f \left(1-\omega\right) + R_{t+1}\omega$.

Assuming the upper bound on labor earnings, $\bar{y}$, is large enough, we obtain a reflecting barrier at the lower bound of the wealth accumulation process

$$a_{t+1} = R_f b_{t+1} + R_{t+1} k_{t+1} + y_{t+1},$$

as in the benchmark model with only the risky asset. It is straighforward now to proceed as in benchmark to construct a stationary wealth distributions with fat tails.

7 Conclusion

In this paper we construct an equilibrium model with idiosyncratic capital income risk in a Bewley economy and analytically demonstrate that the resulting wealth distribution has a fat right tail under well defined and natural conditions on the parameters and stochastic structure of the economy.

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Appendix

Proof of Theorem 1. A feasible policy $c(a)$ is said to overtake another feasible policy $\hat{c}(a)$ if starting from the same initial wealth a_0, the policies $c(a)$ and $\hat{c}(a)$ yield stochastic consumption processes (c_t) and $(\hat{c}_t)$ that satisfy

$$E\left[\sum_{t=0}^{T} \beta^t \left(u(c_t) - u(\hat{c}_t)\right)\right] > 0 \qquad \text{for all } T > \text{some } T_0.$$

Also, a feasible policy is said to be optimal if it overtakes all other feasible policies.

Proof: For an a_0, the stochastic consumption process (c_t) is induced by the policy $c(a)$. Let $(\hat{c}_t)$ be an alternative stochastic consumption process, starting from the same initial wealth a_0. By the strict concavity of $u(\cdot)$, we have

$$E\left[\sum_{t=0}^{T} \beta^t \left(u(c_t) - u(\hat{c}_t)\right)\right] \geq E\left[\sum_{t=0}^{T} \beta^t u'(c_t)(c_t - \hat{c}_t)\right].$$

From the budget constraint we have

$$a_{t+1} = R_{t+1}(a_t - c_t) + y_{t+1}$$

and

$$\hat{a}_{t+1} = R_{t+1}(\hat{a}_t - \hat{c}_t) + y_{t+1}.$$

For a path of (R_t, y_t), we have

$$\frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} = a_t - c_t - (\hat{a}_t - \hat{c}_t) \tag{13}$$

and

$$c_t - \hat{c}_t = a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}}.$$

Therefore we have

$$\sum_{t=0}^{T} \beta^t u'(c_t)(c_t - \hat{c}_t) = \sum_{t=0}^{T} \beta^t u'(c_t) \left(a_t - \hat{a}_t - \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}}\right).$$

Using $a_0 = \hat{a}_0$ and rearranging terms, we have

$$\sum_{t=0}^{T} \beta^t u'(c_t)(c_t - \hat{c}_t) = -\sum_{t=0}^{T} \beta^t [u'(c_t) - \beta R_{t+1} u'(c_{t+1})] \frac{a_{t+1} - \hat{a}_{t+1}}{R_{t+1}} - \beta^T u'(c_T) \frac{a_{T+1} - \hat{a}_{T+1}}{R_{T+1}}.$$

Using equation (13) we have

$$\begin{aligned}\sum_{t=0}^{T}\beta^t u'(c_t)(c_t-\hat{c}_t) &= -\sum_{t=0}^{T}\beta^t[u'(c_t)-\beta R_{t+1}u'(c_{t+1})]\{a_t-c_t-(\hat{a}_t-\hat{c}_t)\}\\ &\quad -\beta^T u'(c_T)[a_T-c_T-(\hat{a}_T-\hat{c}_T)]\\ &\geq -\sum_{t=0}^{T}\beta^t[u'(c_t)-\beta R_{t+1}u'(c_{t+1})]\{a_t-c_t-(\hat{a}_t-\hat{c}_t)\}-\beta^T u'(c_T)a_T.\end{aligned}$$

Thus we have

$$\begin{aligned}E\left[\sum_{t=0}^{T}\beta^t u'(c_t)(c_t-\hat{c}_t)\right] &\geq -E\left(\sum_{t=0}^{T}\beta^t[u'(c_t)-\beta ER_{t+1}u'(c_{t+1})]\{a_t-c_t-(\hat{a}_t-\hat{c}_t)\}\right)\\ &\quad -E\beta^T u'(c_T)a_T. \end{aligned}\tag{14}$$

By the Euler equation (1) we have $u'(c_t)-\beta ER_{t+1}u'(c_{t+1})\geq 0$. If $c_t<a_t$, then $u'(c_t)=\beta ER_{t+1}u'(c_{t+1})$. If $c_t=a_t$, then $a_t-c_t-(\hat{a}_t-\hat{c}_t)=-(\hat{a}_t-\hat{c}_t)\leq 0$. Thus

$$-E\left(\sum_{t=0}^{T}\beta^t[u'(c_t)-\beta ER_{t+1}u'(c_{t+1})]\{a_t-c_t-(\hat{a}_t-\hat{c}_t)\}\right)\geq 0. \tag{15}$$

Combining equations (14) and (15) we have

$$E\left[\sum_{t=0}^{T}\beta^t u'(c_t)(c_t-\hat{c}_t)\right]\geq -E\beta^T u'(c_T)a_T.$$

By the transversality condition (2) we know that for large T,

$$E\left[\sum_{t=0}^{T}\beta^t\left(u(c_t)-u(\hat{c}_t)\right)\right]\geq 0. \quad \blacksquare$$

Proof of Proposition 1. The Euler equation of this problem is

$$c_t^{-\gamma}=\beta ER_{t+1}c_{t+1}^{-\gamma}. \tag{16}$$

Guess $c_t=\phi a_t$. From the Euler equation (16) we have

$$\phi=1-\left(\beta ER^{1-\gamma}\right)^{\frac{1}{\gamma}},$$

which is >0 by Assumption 1.iii).

It is easy to verify the transversality condition,

$$\lim_{t\to\infty} E\left(\beta^t c_t^{-\gamma} a_t\right) = 0. \quad \blacksquare$$

In the finite IF problem, let $V_t(a)$ be the optimal value function of an agent who has wealth a in period t. Thus we have

$$V_t(a) = \max_{c\leq a}\left\{u(c) + \beta E V_{t+1}\left(R(a-c)+y\right)\right\} \quad \text{for } t \geq \tau$$

and

$$V_T(a) = \max_{c\leq a} u(c).$$

We have then the Euler equation for this problem, for $t > 1$:

$$u'(c_t(a)) \geq \beta E[Ru'(c_{t+1}(R(a-c_t(a))+y))] \text{ with equality if } c_t(a) < a.$$

Proof of Proposition 2. Continuity is a consequence of the Theorem of the Maximum and mathematical induction. The proof that $c_{t,\tau}(a)$ and $s_{t,\tau}(a)$ are increasing can be easily adapted from the proof of Theorem 1.5 of Schechtman (1976); it makes use of the fact that $c_{t,\tau}(a) > 0$, a consequence of Inada conditions which hold for CRRA utility functions. $\blacksquare$

Proof of Theorem 2. By Lemma 1 we know that $c(a)$ satisfies the Euler equation. Now we verify that $c(a)$ satisfies the transversality condition (2).

By Lemma 1 and Theorem 2 we have

$$c_t \geq \phi a_t.$$

Note that $a_t \geq \underline{y}$ for $t \geq 1$. We have

$$u'(c_t)a_t \leq \phi^{-\gamma}\left(\underline{y}\right)^{1-\gamma} \quad \text{for } t \geq 1.$$

Thus

$$\lim_{t\to\infty} E\beta^t u'(c_t)a_t = 0. \quad \blacksquare$$

Proof of Proposition 3. By Lemma 1, $c(a)$ is continuous. Thus $s(a)$ is continuous since $s(a) = a - c(a)$.

Also, by Theorem 2, $\lim_{t,\tau\to-\infty} s_{t,\tau}(a) = s(a)$, since $\lim_{t,\tau\to-\infty} c_{t,\tau}(a) = c(a)$, $s_{t,\tau}(a) = a - c_{t,\tau}(a)$, and $s(a) = a - c(a)$. The conclusion that $c(a)$ and $s(a)$ are increasing in a follows from Proposition 2.

For $\tilde{a}$, $\hat{a} > 0$, without loss of generality, we assume that $\tilde{a} < \hat{a}$. We have $c(\tilde{a}) \leq c(\hat{a})$ and $s(\tilde{a}) \leq s(\hat{a})$. Also $c(\tilde{a}) + s(\tilde{a}) = \tilde{a}$ and $c(\hat{a}) + s(\hat{a}) = \hat{a}$. Thus

$$c(\hat{a}) - c(\tilde{a}) + s(\hat{a}) - s(\tilde{a}) = \hat{a} - \tilde{a}.$$

Thus we have

$$0 \leq c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}$$

and

$$0 \leq s(\hat{a}) - s(\tilde{a}) \leq \hat{a} - \tilde{a}.$$

Thus

$$|c(\hat{a}) - c(\tilde{a})| \leq |\hat{a} - \tilde{a}|$$

and

$$|s(\hat{a}) - s(\tilde{a})| \leq |\hat{a} - \tilde{a}|.$$

Therefore, $c(a)$ and $s(a)$ are Lipschitz continuous. ■

Proof of Proposition 5. The proof involves several steps, producing a characterization of $\frac{c(a)}{a}$.

Lemma 2 *$\exists \zeta > \underline{y}$, such that $s(a) = 0$, $\forall a \in (0, \zeta]$.*

Proof. Suppose that $s(a) > 0$ for $a > \underline{y}$. Pick $a_0 > \underline{y}$. For any finite $t \geq 0$, we have $a_t > \underline{y}$ and $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$. Thus

$$u'(c_0) = \beta^t E R_1 R_2 \cdots R_{t-1} R_t u'(c_t). \tag{17}$$

By Lemma 1 and Theorem 2 we have

$$c_t \geq \phi a_t > \phi \underline{y}.$$

Thus equation (17) implies that

$$u'(c_0) \leq \left(\phi \underline{y}\right)^{-\gamma} (\beta E R)^t. \tag{18}$$

Thus the right hand side of equation (18) approaches 0 as t goes to infinity. A contradiction. Thus $s(\zeta) = 0$ for some $\zeta > \underline{y}$. By the monotonicity of $s(a)$, we know that $s(a) = 0$, $\forall a \in (0, \zeta]$. ■

We can now show the following:

Lemma 3 *$\frac{c(a)}{a}$ is decreasing in a.*

Proof. By Lemma 2 we know that $c(\underline{y}) = \underline{y}$. For $\forall a > \underline{y}$, $\frac{c(a)}{a} \leq 1 = \frac{c(\underline{y})}{\underline{y}}$. Note that $-c(a)$ is a convex function of a, since $c(a)$ is a concave function of a. For $\hat{a} > \tilde{a} > \underline{y}$, we have[25]

$$\frac{c(\hat{a}) - c(\underline{y})}{\hat{a} - \underline{y}} \leq \frac{c(\tilde{a}) - c(\underline{y})}{\tilde{a} - \underline{y}}.$$

This implies that

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a} - [\hat{a} - \tilde{a} - (c(\hat{a}) - c(\tilde{a}))]\,\underline{y}. \tag{19}$$

From the Proof of Proposition 3 we know that

$$c(\hat{a}) - c(\tilde{a}) \leq \hat{a} - \tilde{a}. \tag{20}$$

Combining inequalities (19) and (20) we have

$$c(\hat{a})\tilde{a} \leq c(\tilde{a})\hat{a},$$

i.e.

$$\frac{c(\hat{a})}{\hat{a}} \leq \frac{c(\tilde{a})}{\tilde{a}}.$$

■

By Lemma 1 and Proposition 1 we know that $\frac{c(a)}{a} \geq \phi$.we know that $\frac{c(a)}{a} \geq \phi$. Thus we have

$$\lim_{a \to \infty} \frac{c(a)}{a} \text{ exists.}$$

Let

$$\lambda = \lim_{a \to \infty} \frac{c(a)}{a}. \tag{21}$$

Note that $\lambda \leq 1$ since $c(a) \leq a$. This furthermore implies that $\frac{s(a)}{a}$ is increasing and converges to a limit as a goes to infinity.

The Euler equation of this problem is

$$c_t^{-\gamma} \geq \beta E R_{t+1} c_{t+1}^{-\gamma} \text{ with equality if } c_t < a_t. \tag{22}$$

Lemma 4 $\lambda \in [\phi, 1)$.

[25]See Lemma 16 on page 113 of Royden (1988).

Proof. Suppose that $\lambda = 1$. Thus

$$\liminf_{a_t \to \infty} \frac{c(a_t)}{a_t} = \lim_{a_t \to \infty} \frac{c(a_t)}{a_t} = 1.$$

From the Euler equation (22) we have

$$c_t^{-\gamma} \geq \beta E R_{t+1} c_{t+1}^{-\gamma} \geq \beta E R_{t+1} a_{t+1}^{-\gamma}$$

since $c_{t+1} \leq a_{t+1}$ and $\gamma \geq 1$.

Thus

$$\left(\frac{c(a_t)}{a_t}\right)^{-\gamma} \geq \beta E R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma}.$$

By Fatou's lemma we have

$$\begin{aligned} & \liminf_{a_t \to \infty} E R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma} \\ \geq \; & E \liminf_{a_t \to \infty} \left[R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma}\right]. \end{aligned}$$

Thus

$$\begin{aligned} 1 &= \lim_{a_t \to \infty} \left(\frac{c(a_t)}{a_t}\right)^{-\gamma} \\ &\geq \beta \lim_{a_t \to \infty} E R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma} \\ &= \beta \liminf_{a_t \to \infty} E R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma} \\ &\geq \beta E \liminf_{a_t \to \infty} \left[R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma}\right] \\ &= \beta E \lim_{a_t \to \infty} \left[R_{t+1} \left(R_{t+1}\left(1 - \frac{c(a_t)}{a_t}\right) + \frac{y_{t+1}}{a_t}\right)^{-\gamma}\right] \\ &= \infty. \end{aligned}$$

A contradiction. ■

From Lemma 4 we know that $c_t < a_t$ when a_t is large enough. Thus the equality of the Euler equation holds

$$c_t^{-\gamma} = \beta E R_{t+1} c_{t+1}^{-\gamma}.$$

Thus

$$\left(\frac{c_t}{a_t}\right)^{-\gamma} = \beta E R_{t+1} \left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}. \tag{23}$$

Taking limits on both sides of equation (23) we have

$$\lim_{a_t \to \infty} \left(\frac{c_t}{a_t}\right)^{-\gamma} = \beta \lim_{a_t \to \infty} E R_{t+1} \left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}.$$

Thus

$$\lambda^{-\gamma} = \beta \lim_{a_t \to \infty} E R_{t+1} \left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}. \tag{24}$$

We turn to the computation of $\lim_{a_t \to \infty} E R_{t+1} \left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}$.

In order to compute $\lim_{a_t \to \infty} E R_{t+1} \left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}$, we first show a lemma.

Lemma 5 *For $\forall H > 0$, $\exists J > 0$, such that $a_{t+1} > H$ for $a_t > J$. Here J does not depend on realizations of R_{t+1} and y_{t+1}.*

Proof. Note that

$$\frac{a_{t+1}}{a_t} = \frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t} \geq R_{t+1}\left(1 - \frac{c_t}{a_t}\right).$$

From equation (21) we know that for some $\varepsilon > 0$, $\exists J_1 > 0$, such that

$$\frac{c_t}{a_t} < \lambda + \varepsilon$$

for $a_t > J_1$. Thus

$$\frac{a_{t+1}}{a_t} \geq R_{t+1}\left(1 - \frac{c_t}{a_t}\right) \geq R_{t+1}(1 - \lambda - \varepsilon). \tag{25}$$

And

$$\frac{a_{t+1}}{a_t} \geq R_{t+1}(1 - \lambda - \varepsilon) \geq \underline{R}(1 - \lambda - \varepsilon).$$

We pick $J > J_1$ such that $\underline{R}(1 - \lambda - \varepsilon) \geq \frac{H}{J}$. Thus for $a_t > J$, we have

$$\frac{a_{t+1}}{a_t} \geq \frac{H}{J}.$$

This implies that

$$a_{t+1} \geq \frac{H}{J} a_t > H.$$

■

From equation (21) we know that for some $\eta > 0$, $\exists H > 0$, such that

$$\frac{c_{t+1}}{a_{t+1}} > \lambda - \eta \tag{26}$$

for $a_{t+1} > H$.

From Lemma 5 and equations (25) and (26) we have

$$R_{t+1}\left(\frac{c_{t+1}}{a_t}\right)^{-\gamma} = R_{t+1}\left(\frac{c_{t+1}}{a_{t+1}}\frac{a_{t+1}}{a_t}\right)^{-\gamma} \leq (\lambda - \eta)^{-\gamma}(1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$$

for $a_t > J$. And

$$(\lambda - \eta)^{-\gamma}(1 - \lambda - \varepsilon)^{-\gamma} E R_{t+1}^{1-\gamma} < \infty$$

since $\gamma \geq 1$. Thus when a_t is large enough, $(\lambda - \eta)^{-\gamma}(1 - \lambda - \varepsilon)^{-\gamma} R_{t+1}^{1-\gamma}$ is a dominant function of $R_{t+1}\left(\frac{c_{t+1}}{a_t}\right)^{-\gamma}$.

Note that

$$\lim_{a_t \to \infty} \frac{c_{t+1}}{a_{t+1}} = \lim_{a_t \to \infty} \frac{c(a_{t+1})}{a_{t+1}} = \lambda \quad a.s.$$

by Lemma 5 and equation (21). And

$$\lim_{a_t \to \infty} \frac{a_{t+1}}{a_t} = \lim_{a_t \to \infty}\left(\frac{R_{t+1}(a_t - c_t) + y_{t+1}}{a_t}\right) = R_{t+1}(1 - \lambda) \quad a.s.$$

since $y_{t+1} \in [\underline{y}, \bar{y}]$. Thus

$$\lim_{a_t \to \infty} \frac{c_{t+1}}{a_t} = \lim_{a_t \to \infty} \frac{c_{t+1}}{a_{t+1}}\frac{a_{t+1}}{a_t} = \lambda(1 - \lambda)R_{t+1} \quad a.s.$$

Thus by the Dominated Convergence Theorem, we have

$$\lim_{a_t \to \infty} E R_{t+1}\left(\frac{c_{t+1}}{a_t}\right)^{-\gamma} = E R_{t+1}\left(\lim_{a_t \to \infty}\frac{c_{t+1}}{a_t}\right)^{-\gamma} = \lambda^{-\gamma}(1 - \lambda)^{-\gamma} E R_{t+1}^{1-\gamma}. \tag{27}$$

Combining equations (24) and (27) we have

$$\lambda^{-\gamma} = \beta\lambda^{-\gamma}(1 - \lambda)^{-\gamma} E R_{t+1}^{1-\gamma}. \tag{28}$$

By Lemma 4 we know that $\lambda \geq \phi > 0$. Thus we find λ from equation (28)

$$\lambda = 1 - \left(\beta E R^{1-\gamma}\right)^{\frac{1}{\gamma}}.$$

Thus $\lambda = \phi$.

Proof of Theorem 3. The proof requires several steps.

Lemma 6 *The wealth accumulation process* $\{a_{t+1}\}_{t=0}^{\infty}$ *is* $\psi-$*irreducible.*

Proof. First we show that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is $\varphi-$irreducible. We construct a measure φ on $[\underline{y},\infty)$ such that

$$\varphi(A) = \int_A f(y)dy.$$

where $f(y)$ is the density of labor earnings y_t. Note that the borrowing constraint binds in finite time with a positive probability for $\forall a_0 \in [\underline{y},\infty)$. Suppose not. For any finite $t \geq 0$, we have $a_t > \underline{y}$ and $u'(c_t) = \beta E R_{t+1} u'(c_{t+1})$. Following the same procedure as in the proof of Lemma 2, we obtain a contradiction. If the borrowing constraint binds at period t, then $a_{t+1} = y_{t+1}$. Thus any set A such that $\int_A f(y)dy > 0$ can be reached in finite time with a positive probability. The process $\{a_{t+1}\}_{t=0}^{\infty}$ is $\varphi-$irreducible.

By Proposition 4.2.2 in Meyn and Tweedie (2009), there exists a probability measure ψ on $[\underline{y},\infty)$ such that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is $\psi-$irreducible, since it is $\varphi-$irreducible. ■

Lemma 7 $a = \underline{y}$ *is a reflecting barrier of the process* $\{a_{t+1}\}_{t=0}^{\infty}$.

Proof. If $a_t = \underline{y}$, then there exists $\hat{y}$ close to $\bar{y}$ such that $\Pr(a_{t+1} \in [\hat{y}, \bar{y}] | a_t = \underline{y}) = \Pr(y_{t+1} \in [\hat{y}, \bar{y}]) > 0$, since $s(\underline{y}) = 0$. To show that a_{t+2} can be greater than $\bar{y}$ with a positive probability, it is sufficient to show that $s(\bar{y}) > 0$. Suppose that $s(\bar{y}) = 0$. Thus $s(a) = 0$ for $a \in [\underline{y}, \bar{y}]$. Thus by the Euler equation we have

$$(\bar{y})^{-\gamma} \geq \beta E \left[R_t (y_t)^{-\gamma} \right].$$

This is impossible under Assumption 1.i). Thus $s(\bar{y}) > 0$ and $a = \underline{y}$ is a reflecting barrier of the process $\{a_{t+1}\}_{t=0}^{\infty}$. ■

To show that there exists a unique stationary wealth distribution, we have to show that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is ergodic. Actually, we can show that it is geometrically ergodic.

Lemma 8 *The process* $\{a_{t+1}\}_{t=0}^{\infty}$ *is geometrically ergodic.*

Proof. To show that the process $\{a_{t+1}\}_{t=0}^{\infty}$ is geometrically ergodic, we use part (iii) of Theorem 15.0.1 of Meyn and Tweedie (2009). We need to verify that

a the process $\{a_{t+1}\}_{t=0}^{\infty}$ is $\psi-$irreducible;

b the process $\{a_{t+1}\}_{t=0}^{\infty}$ is aperiodic;[26] and

[26]For the definition of aperiodic, see page 114 of Meyn and Tweedie (2009).

c there exists a petite set C,[27] constants $b < \infty$, $\rho > 0$ and a function $V \geq 1$ finite at some point in [$\underline{y}$,∞) satisfying

$$EV(a_{t+1}) - V(a_t) \leq -\rho V(a_t) + bI_C(a_t), \quad \forall a_t \in [\underline{y}, \infty).$$

By Lemma 6, the process $\{a_{t+1}\}_{t=0}^{\infty}$ is $\psi-$irreducible.

For a $\varphi-$irreducible Markov process, when there exists a v_1-small set A with $v_1(A) > 0$,[28] then the stochastic process is called strongly aperiodic; see Meyn and Tweedie (2009, p. 114). We construct a measure v_1 on [$\underline{y}$,∞) such that

$$v_1(A) = \int_A f(y)dy.$$

By Lemma 2, we know that $s(a) = 0$, $\forall a \in [\underline{y}, \zeta]$. Thus $[\underline{y}, \zeta]$ is v_1-small and $v_1([\underline{y}, \zeta]) = \int_{\underline{y}}^{\zeta} f(y)dy > 0$. The process $\{a_{t+1}\}_{t=0}^{\infty}$ is strongly aperiodic.

We now show that an interval [$\underline{y}$,B] is a petite set for $\forall B >\underline{y}$. To show this, we first show that $\underline{R}s(a)+\underline{y}< a$ for $a \in (\underline{y}, \infty)$. For $s(a) = 0$, this is obviously true. For $s(a) > 0$, suppose that $\underline{R}s(a)+\underline{y}\geq a$, we have

$$u'(c(a)) = \beta ER_t u'(c(R_t s(a) + y)) \leq \beta ER_t u'(c(a)).$$

We obtain a contradiction since Assumption 1.iv) implies that $\beta ER_t < 1$. Also by Lemma 2, there exists an interval $[\underline{y}, \zeta]$, such that $s(a) = 0$, $\forall a \in [\underline{y}, \zeta]$. For an interval [$\underline{y}$,$B$], $\forall a_0 \in$ [$\underline{y}$,B], there exists a common t such that the borrowing constraint binds at period t with a positive probability. Then for any set $A \subset [\underline{y}, \bar{y}]$, $\Pr(a_{t+1} \in A | s(a_t) = 0) = \int_A f(y)dy$. Note that a $t-$step probability transition kernel is the probability transition kernel of a specific sampled chain. Thus we construct a measure v_a on [$\underline{y}$,∞) such that v_a has a positive measure on $[\underline{y}, \bar{y}]$ and $v_a((\bar{y}, \infty)) = 0$. The $t-$step probability transition kernel of a process starting from $\forall a_0 \in$ [$\underline{y}$,B] is greater than the measure v_a. An interval [$\underline{y}$,B] is a petite set for $\forall B >\underline{y}$.

We pick a function $V(a) = a + 1$, $\forall a \in$ [$\underline{y}$,∞). Thus $V(a) > 1$ for $a \in$ [$\underline{y}$,∞). Pick $0 < q < 1 - \mu ER_t$. Let $\rho = 1 - \mu ER_t - q > 0$ and $b = 1 - \mu ER_t + Ey$. Pick $B > \underline{y}$, such that $B + 1 \geq \frac{b}{q}$. Let $C =$ [$\underline{y}$,B]. Thus C is a petite set. Therefore, for $\forall a_t \in$ [$\underline{y}$,∞), we have

$$\begin{aligned} EV(a_{t+1}) - V(a_t) &= E\left(a_{t+1}\right) - a_t \\ &\leq -\left(1 - \mu ER_t\right) V(a_t) + 1 - \mu ER_t + Ey \\ &\leq -\rho V(a_t) + bI_C(a_t) \end{aligned}$$

where $I_C(\cdot)$ is an indicator function.

By Theorem 15.0.1 of Meyn and Tweedie (2009) the process $\{a_{t+1}\}_{t=0}^{\infty}$ is geometrically ergodic. ■

[27]For the definition of petite sets, see page 117 of Meyn and Tweedie (2009).

[28]For the definition of small sets, see page 102 of Meyn and Tweedie (2009).