Optimal Control: The Maximum Principle

• Presented for leading example: Problem of a social planner or representative household

Maximize $U = \int_0^\infty \{e^{-\rho t} u[C(t)]L(t)\} dt$ s.t. $\frac{dK}{dt} = F(K, AL) - CL - \delta K$

- Concise set of necessary conditions known as Maximum Principle. Here focus on application. For general analysis see Barro/Sala-i-Martin, App.A3; Acemoglu, ch.7, or math textbooks.

1. Setup: Define choice variables, state variables, and **costate** variables
   - Here: Choice = C. State = K. For each state variable define a costate variable (here: λ for K)

2. Define the **Hamiltonian**: (Time-t Objective) + (Costate variables) * (RHS of the constraints)
   - Here: $H(C, K, \lambda, t) = e^{-\rho t} u(C) L + \lambda \cdot [F(K, AL) - CL - \delta K]$
   - Intuition: Co-state is the shadow value of a marginal increase in the state variable.

3. Apply the **Maximum Principle**: Three first order conditions, which involve derivatives of the Hamiltonian with respect to choice variables, state variables, and costate variables:

   (i) $\frac{\partial H}{\partial C} = 0$; (ii) $\frac{\partial H}{\partial K} = -\frac{d\lambda}{dt}$; (iii) $\frac{\partial H}{\partial \lambda} = \frac{dK}{dt}$. [Note: negative sign in (ii), not in (iii)]

4. Impose suitable boundary conditions: here initial condition that K(0) is given; terminal condition to formalize the intuition that no resources should be "left over" at the end.

Motivation for the Maximum Principle

- Problem: max $U = \int_0^\infty \{e^{-\rho t} u[C(t)]L(t)\}dt$ s.t. $\frac{dK}{dt} = F(K, AL) - CL - \delta K$
- General format: $U = \int_0^T h[c(t), x(t), t]dt + V[x(T)]$ s.t. $\frac{dx(t)}{dt} = g[c(t), x(t), t]$

  with control function c(t), state x(t), and time $t \in [0,T]$, and with $T \to \infty$ as limiting case.
  - For finite T, include terminal value V(). Initial conditions $x(0) = x_0$.
  - Assume h, g, and V are continuously differentiable. Time-dependence in h() and g() allows for discounting and known functions of time such as A(t), L(t).
  - Controls and state may be scalars or vectors; use scalar notation here.
  - Note that changing c(t) at isolated points would not change U. Motivates restricting attention to piecewise continuous functions c(t).
- Objective: Find necessary conditions for a control $\hat{c}(t)$ to be optimal within the space of piecewise continuous functions on [0,T].

- Apply Lagrangian idea: if $g[c(t), x(t), t] - \frac{dx(t)}{dt} = 0$ for all t, then

  $\int_0^T \lambda(t)\{g[c(t), x(t), t] - \frac{dx(t)}{dt}\}dt = 0$ for any function $\lambda(t)$. Write

  $$U = L \equiv \int_0^T h[c(t), x(t), t]dt + \int_0^T \lambda(t)\{g[c(t), x(t), t] - \frac{dx(t)}{dt}\}dt + V[x(T)].$$

- Rearrange and integrate by parts:

$$L = \int_0^T \{h[c(t),x(t),t] + \lambda(t)\cdot g[c(t),x(t),t]\}dt - \int_0^T \lambda(t)\frac{dx(t)}{dt}\,dt + V[x(T)]$$

$$= \int_0^T \{h[c(t),x(t),t] + \lambda(t)\cdot g[c(t),x(t),t]\}dt + \int_0^T \frac{d\lambda(t)}{dt}x(t)dt + \lambda(0)x(0) - \lambda(T)x(T) + V[x(T)].$$

$$= \int_0^T \left\{H[c(t),x(t),\lambda(t),t] + \frac{d\lambda(t)}{dt}x(t)\right\}dt + \lambda(0)x_0 - \lambda(T)x(T) + V[x(T)]$$

written in terms of the **Hamiltonian** $H[c(t),x(t),\lambda(t),t] = h[c(t),x(t),t] + \lambda(t)\cdot g[c(t),x(t),t]$

Result: for interior times $t \in (0,T)$, c(t) and x(t) influence L only through the $\{\ldots\}$-term

- Suppose $\hat{c}(t)$ is candidate for optimal control and $\hat{\hat{c}}(t)$ an arbitrary alternative
  - Define a parametric family of alternatives ("variations") by

$$c(t,\varepsilon) = \hat{c}(t) + \varepsilon\cdot\eta(t), \text{ where } \eta(t) = \hat{c}(t) - \hat{\hat{c}}(t).$$

  - Note: $c(t,\varepsilon)$ is piecewise continuous for any ε because $\hat{c}(t)$ and $\hat{\hat{c}}(t)$ are.
  - For given ε, let $x(t,\varepsilon)$ denote the solution to $\frac{dx(t,\varepsilon)}{dt} = g[c(t,\varepsilon),x(t,\varepsilon),t]$ with $x(0,\varepsilon) = x_0$.
  - Define:

$$L(\varepsilon) = \int_0^T \left\{H[c(t,\varepsilon),x(t,\varepsilon),\lambda(t),t] + \frac{d\lambda(t)}{dt}x(t,\varepsilon)\right\}dt + \lambda(0)x_0 - \lambda(T)x(T,\varepsilon) + V[x(T,\varepsilon)]$$

  - Necessary condition for optimal $\hat{c}(t)$ is that $dL(0)/d\varepsilon = 0$.

• Differentiate:

$$\frac{dL(\varepsilon)}{d\varepsilon} = \int_0^T \left\{ H_c \frac{dc(t,\varepsilon)}{d\varepsilon} + \{H_x + \frac{d\lambda(t)}{dt}\} \frac{dx(t,\varepsilon)}{d\varepsilon} \right\} dt - \{V_x - \lambda(T)\} \frac{dx(T,\varepsilon)}{d\varepsilon}, \text{ where } \frac{dc(t,\varepsilon)}{d\varepsilon} = \eta.$$

- Condition $dL(0)/d\varepsilon = 0$ must hold for any functions $\lambda(t)$ and $\eta(t)$.
- Convenient choice for $\lambda(t)$ to pick the solution to $\frac{d\lambda(t)}{dt} = -H_x[\hat{c}(t), x(t,0), \lambda(t), t]$ with boundary condition $\lambda(T) = V_x(x(T,0))$. Then

$$\frac{dL(0)}{d\varepsilon} = \int_0^T \left\{ H_c[\hat{c}(t), x(t,0), \lambda(t), t] \cdot \eta(t) \right\} dt$$

• Suppose for contradiction that $H_c[\hat{c}(t), x(t,0), \lambda(t), t] \neq 0$ for any $t \in (0,T)$. Then continuity implies that $H_c \neq 0$ on a surrounding interval and $\frac{dL(0)}{d\varepsilon} \neq 0$ if on picks $\eta(t) \neq 0$ on this interval and zero elsewhere. Thus $H_c[\hat{c}(t), x(t,0), \lambda(t), t] = 0$ for all $t \in (0,T)$

• Summarize:

$H_c[\hat{c}(t), x(t,0), \lambda(t), t] = 0$ for all $t \in (0,T)$ is necessary for optimality, if $\lambda(t)$ is picked to solve $\frac{d\lambda(t)}{dt} = -H_x[\hat{c}(t), x(t), \lambda(t), t]$ and $\frac{dx(t)}{dt} = g[\hat{c}(t), x(t), t] = H_\lambda[\hat{c}(t), x(t), \lambda(t), t]$

• Observation: The same system of differential equation can be obtained quickly by defining H, taking partial derivatives, and imposing: $H_c = 0$, $H_x = -\frac{d\lambda(t)}{dt}$ and $H_\lambda = \frac{dx(t)}{dt}$

- Intuition: recall that L depends on c(t) and x(t) through $H[c(t), x(t), \lambda(t), t] + \frac{d\lambda(t)}{dt} x(t)$: maximizing point-wise would yield first order conditions $H_c = 0$ and $H_x + \frac{d\lambda(t)}{dt} = 0$.
- Cannot literally choose (c,x) pointwise, but answers are correct if $\lambda(t)$ is set correctly.

Application of the Maximum Principle

• Hamiltonian: $H(C,K,\lambda,t) = e^{-\rho t}u(C)L + \lambda \cdot [F(K,AL) - CL - \delta K]$

i. "Maximize the Hamiltonian w.r.t. each choice variable."

- Apply to consumption:

$$\frac{\partial H}{\partial C} = e^{-\rho t}u'(C) \cdot L - \lambda L = 0 \quad => \quad \lambda = e^{-\rho t}u'(C)$$

- Starting point for characterizing optimal consumption (rigorous derivation of the Euler equation).

ii. "For each state variable, equate -∂H/∂(state) to d(costate)/dt."

- Apply to capital:

$$-\frac{\partial H}{\partial K} = \frac{d\lambda}{dt} \quad <=> \quad \frac{d\lambda}{dt} = -\lambda \cdot [F_K(K,AL) - \delta]$$

- Starting point for characterizing the optimal dynamics of the capital stock.

iii. "For each costate variable, equate ∂H/∂(costate) to d(state)/dt."

- Apply to the costate variable for capital:

$$\frac{\partial H}{\partial \lambda} = \frac{dK}{dt} \quad <=> \quad \frac{dK}{dt} = F(K,AL) - CL - \delta K$$

- Formal way of recovering the constraints. Note the positive sign here, vs. the negative sign in Step (ii).

• Provides three equations for three variables (C,K,λ). Two are differential equations.

Interpretation (I): Costate = Shadow Value of Capital

- Claim: The shadow value of capital declines over time at the rate of interest.
- Proof:
  - Step (ii) of the maximum principle implies

$$-\dot{\lambda}/\lambda = F_K(K, AL) - \delta = f'(k) - \delta = r$$

  - Conclude: $\lambda(t)$ is a decreasing function of time if and only if $r > 0$.
  - Linear differential equation solved by: $\lambda(T) = \lambda(0)e^{-\int_0^T r(t)dt}$

- General lesson: In dynamic problems, part (ii) of the Maximum Principle implies that future resources (like assets or capital) are discounted at an appropriate rate of interest.

- Intuition: high (or low) return means: easy (or difficult) to shift current resources into the future => Future resources are discounted deeply (or not much)

- Intuition based on the Lagrangian: recall that L includes the term $\lambda(0)x_0$. Suggests that $dL/dx_0 = dU/dx_0 = \lambda(0)$ is the marginal value of varying initial conditions.

Interpretation (II): Optimal Consumption Growth

- Claim: The Maximum Principle implies the Euler equation $\dot{C}/C = \frac{1}{\theta(C)}(r-\rho)$.
  - Idea: Express optimality in terms of observables (C,K) => eliminate λ and dλ/dt.
  - Proof: From step (i): $\lambda(t) = e^{-\rho t}u'(C(t))$

    Differentiate: $\frac{d\lambda}{dt} = -\rho e^{-\rho t}\cdot u'(C(t)) + e^{-\rho t}\cdot u''(C(t))\frac{dC}{dt}$

    Divide: $\frac{d\lambda}{dt}/\lambda = -\rho + \frac{u''(C(t))}{u'(C(t))}\frac{dC}{dt} = -\rho + \frac{u''(C(t))C}{u'(C(t))}(\frac{1}{C}\frac{dC}{dt}) = -\rho - \theta(C)(\dot{C}/C)$

    From step (ii): $\frac{d\lambda}{dt}/\lambda = -[F_K(K,AL) - \delta] = -r,$

    Combine: $\rho + \theta(C)(\dot{C}/C) = r \quad \Rightarrow \dot{C}/C = \frac{1}{\theta(C)}(r-\rho).$

- Result: Same Euler equation as in Romer's household problem:
  1. Per-capita consumption growth is proportional to the interest rate minus rate of time preference.
  2. Responsiveness to interest rate changes is 1/θ = the elasticity of intertemporal substitution.

- Related definition: Current-value Hamiltonian (See Acemoglu ch.7.5; not required here)

  $$\tilde{H}(C,K,\lambda,t) = u(C)L + \tilde{\lambda}\cdot[F(K,AL) - CL - \delta K]$$

  - Modified form of Maximum principle:

    (i) $\partial\tilde{H}/\partial C = 0$ and (ii) $\partial\tilde{H}/\partial K = -d\tilde{\lambda}/dt + \rho\tilde{\lambda}$

    Step (i) simplifies: $\partial\tilde{H}/\partial C = u'(C)L + \tilde{\lambda}L = 0 \Rightarrow \tilde{\lambda} = u'(C)$

    Step (ii): $\partial\tilde{H}/\partial K = \tilde{\lambda}[F_K(K,AL) - \delta] = -d\tilde{\lambda}/dt + \rho\tilde{\lambda}$ => same Euler equation.

Transformation to Effective Units

- Differential equations in "natural" units:

$$\tfrac{dK}{dt} = F(K, AL) - CL - \delta K \qquad \text{and} \qquad \dot{C}/C = \tfrac{1}{\theta(C)}(r - \rho)$$

- Transformation to effective units is convenient for steady state analysis:
  - Interest rate & return to capital: $\quad r = f'(k) - \delta$
  - Consumption: $c = C/A \;\Rightarrow\; \dot{c}/c = \dot{C}/C - g$

  $\Rightarrow$ Euler equation: $\quad \dot{c}/c = 1/\theta(C)\,[f'(k) - \delta - \rho] - g$

  - Capital: $k = K/AL \Rightarrow \dot{k}/k = \dot{K}/K - n - g \Rightarrow \dot{k} = \tfrac{1}{AL}\dot{K} - (n+g)k$

  $\Rightarrow$ Dynamics of capital: $\quad \dot{k} = f(k) - c - (n + g + \delta)k$

- Left as exercise: Consider problem with (c, k) as choice and state variables

  Maximize $U = \int_0^\infty \{e^{-\rho t} u[c(t)\cdot A(t)]L(t)\}dt$ s.t. $\dot{k} = f(k) - c - (n + g + \delta)k$

  Or maximize $\tfrac{1}{H}U = \int_0^\infty \{e^{-\rho t} u[c(t)\cdot A(t)]\tfrac{L(t)}{H}\}dt$ for household with population L/H.

  Show that the solutions imply the same optimality conditions as above.

Another Example of Optimal Control

- Apply optimal control approach to Romer's household problem (population L/H):

  Maximize $\qquad U = \int_0^T [e^{-\rho t} u(C(t)) \frac{L(t)}{H}] dt$, with finite horizon T.

  Subject to $\qquad \dot{a}(t) = r(t) \cdot a(t) + W(t)^{L(t)}/_H - C(t)^{L(t)}/_H$

- Hamiltonian: $\qquad H(C, a, \lambda, t) = e^{-\rho t} u(C) \frac{L}{H} + \lambda \cdot [r \cdot a + W \cdot {}^L/_H - C \cdot {}^L/_H]$

  where now λ = shadow value of household assets

- Maximum Principle:

  i. $\qquad \frac{\partial H}{\partial C} = 0 \quad \Rightarrow \quad e^{-\rho t} u'(C) \cdot \frac{L}{H} - \lambda \cdot \frac{L}{H} = 0 \Rightarrow \qquad \lambda = e^{-\rho t} u'(C)$

  ii. $\qquad \frac{\partial H}{\partial a} = -\frac{d\lambda}{dt} = \lambda r \quad \Rightarrow \quad {}^{\dot{\lambda}}/_{\lambda} = -r$

  iii. $\qquad \frac{\partial H}{\partial \lambda} = \frac{da}{dt} = r \cdot a + W \cdot {}^L/_H - C \cdot {}^L/_H$

- Left as exercise: Show that ${}^{\dot{C}}/_C = \frac{1}{\theta(C)}(r - \rho)$ holds.
- Conclude: Maximum Principle yields the same differential equations as Romer's solution.

  => A systematic and effective way of deriving necessary conditions.

Results and Open Questions

- Result: Key differential equations

    1. Euler equation: $\dot{C}/C = \frac{1}{\theta(C)}(f'(k) - \delta - \rho)$

        or $\dot{c}/c = 1/\theta(C)[f'(k) - \delta - \rho] - g$

    2. Dynamics of capital: $\dot{k} = f(k) - c - (n + g + \delta)k$

  - Starting point for graphical analysis (phase diagrams).

- Math fact: Solving a pair of differential equations requires two boundary conditions.

    1. Initial capital K(0) is given. 2. Open question => *Not a complete solution.*

- Open Questions:

    1. What is the second boundary condition?

       Claim (to prove): A suitable terminal condition.

    2. Is optimal growth consistent with a steady state? With balanced growth?

       Claim (to prove): Not in general. Requires restrictions on preferences.

    3. How do we solve or characterize the optimal solution (the differential equations)?

       Several approaches. Here: Phase diagrams; linearization around a steady state.

The Terminal Condition (I): Concepts

• Setting: Maximize utility subject to resource constraints—on capital and/or financial assets.

- More dependent on context than the Maximum Principle. Hence consider specific cases.

• Key concepts and arguments:

1. Transversality condition: Don't leave valuable resources unused.

   - Finite horizon problems: Rules out a strictly *positive* value at the terminal date.

   - Infinite horizon problems: Rules out a strictly *positive* present value in the limit.

   - Necessary condition for optimality. [Otherwise one could raise utility by spending the resource.]

2. No-Ponzi condition: Incentive to borrow if repayment is not required—must be prevented. [Named for Charles Ponzi, inventor of the chain letter.]

   - Finite horizon problems: Rules out positions with strictly *negative* value at the terminal date.

   - Infinite horizon problems: Rules out positions with strictly *negative* present value in the limit.

   - Property of the equilibrium: Must be justified in each application.

     - Common argument: An optimizing lender will not lend unless repayment is credibly promised.

       => Borrowers must satisfy a No-Ponzi condition if all lenders satisfy transversality conditions.

     - Counter examples: Government credit. Speculative bubbles. Overlapping generations of lenders.

   - Ponzi problem does not arise if the resource is naturally non-negative, e.g., for real capital.

The Terminal Condition (II): Motivation

• Recall the Lagrangian for variation $c(t,\varepsilon) = \hat{c}(t) + \varepsilon \cdot \eta(t)$

$$L(\varepsilon) = \int_0^T \left\{ H[c(t,\varepsilon), x(t,\varepsilon), \lambda(t), t] + \tfrac{d\lambda(t)}{dt} x(t,\varepsilon) \right\} dt + \lambda(0)x_0 - \lambda(T)x(T,\varepsilon) + V[x(T,\varepsilon)]$$

- For finite T, find variation with $\frac{dL(0)}{d\varepsilon} \neq 0$ unless $V_x[x(T)] - \lambda(T) = 0$

  Not useful in economic application with V=0 because $\lambda(T) = 0$ => $\lambda(t) = 0$ for all t

1. Economic problems with endpoint **constraint** $x(T) \geq 0$ (e.g., $a(T) \geq 0$)
   - Impose $x(T) \geq 0$ with Kuhn-Tucker multiplier $\mu \geq 0$

   $$L(\varepsilon) = \int_0^T \{\ldots\} dt + \lambda(0)x_0 - \lambda(T)x(T,\varepsilon) + \mu \cdot x(t,\varepsilon)$$

   - Pick λ(t) with boundary condition $\lambda(T) = \mu \geq 0$. Then Kuhn-Tucker conditions require $x(T) = 0$ for $\mu > 0$ or $\mu = 0$ for $x(T) > 0$. Often one can rule out $\mu = 0$.

2. Economic problems with **bounded domain** $x(t) \geq 0$ (e.g. $k(t) \geq 0$)
   - Often imposing $x(T) \geq 0$ is enough to obtain solutions that satisfy $x(t) \geq 0$ for all t.

• Intuition for limiting cases $T \to \infty$:

- Note that $\lambda(T)x(T)$ enters negatively into L and that $\lambda(T)x(T) \geq 0$.
- Suggests that candidate solutions with $\lim_{T\to\infty} \lambda(T)x(T) > 0$ can be improved on the margin.

The Terminal Condition (III): Results

1. Finite horizon problems with terminal date T:

   - Conditions usually reduce to $k(T) = 0$ or $a(T) = 0$.

2. Limit conditions for infinite horizons:

   - Conditions are: $\lambda(T) \cdot k(T) \to 0$ or $\lambda(T) \cdot a(T) \to 0$ as $T \to \infty$.
   - Usually find that $\lambda(T) \propto e^{-xt} \to 0$ for some limiting discount rate x>0.

   => Condition limits the *growth rate* of asset positions (must be less than x).

• Note: Limit condition does **not** require zero assets/capital at any finite date.

   - Positive limit $k(T) \to k^* > 0$ is fine, provided $\lambda(T) \to 0$. [Common misperception!]
   - Perpetual growth, say at rate x/2, would also work, even though $k(T) \propto e^{(x/2) \cdot t} \to \infty$.

• Reconsider the Intertemporal budget constraint.

   - Recall the Maximum Principle in the Romer problem: $\dot{\lambda}/\lambda = -r(t)$. Solve: $\lambda(T) = \lambda(0) \cdot e^{-\int_0^T r(v)dv}$.
   - Romer's IBC derivation assumed $a(T) \cdot e^{-\int_0^T r(v)dv} \to 0$. Equivalent to $\lambda(T)a(T) \to 0$.

• Insight: The IBC implicitly relies on the transversality and No-Ponzi conditions.

Conditions for Balanced Growth

- Claim: Balanced growth requires homothetic preferences: Power or logarithmic.
  - Proof: Balanced growth means convergence to a steady state in efficiency units.

$$k(t) \to k^*, \quad c(t) \to c^*, \quad r(t) \to r^* = f'(k^*) - \delta$$

  - Dynamics: $\dot{k} = f(k) - c - (n+g+\delta)k$ and $\dot{c}/c = \dot{C}/C - g = 1/\theta(C)[f'(k) - \delta - \rho] - g$.
  - Two differential equations => Two steady state conditions: $\dot{k} = 0$ and $\dot{c} = 0$.

$$\dot{k} = 0 \iff \quad f(k^*) - (n+g+\delta)k^* = c^*$$

$$\dot{c} = 0 \iff \quad g \cdot \theta(C(t)) = f'(k^*) - \delta - \rho, \text{ where } \theta(C(t)) = \theta[c^* \cdot A(t)].$$

  - If A(t) grows, $\theta(C(t))$ varies unless EIS is constant.

  => Steady state requires preferences with constant $\theta(C(t)) = \theta(c^* \cdot A(t)) = \theta$.

- Fact from micro: Constant EIS requires power utility or log-utility (for time-separable preferences)
  1. Power utility: $u(C) = \frac{C^{1-\theta}}{1-\theta}, u'(C) = C^{-\theta}, \theta \geq 0, \theta \neq 1.$
  2. Logarithmic: $u(C) = \ln(C), u'(C) = 1/C$ Limiting case of $\theta \to 1$.

  - Sloppy language: *Marginal* utility must be homothetic of degree (-θ) for some θ>0. Called homothetic utility.
  - Proof: Define $z(\ln(C) = \ln(u'(e^{\ln C}))$. Constant $z' = u''c/u' = -\theta$ implies $z = \ln u'(C) = z_0 - \theta \ln C$ => $u'(C) = e^{z_0} C^{-\theta}$

- Empirical observation: Long-run macro data are roughly consistent with balanced growth.

  => Makes sense to assume homothetic utility. *Motivates Romer's assumption.*

  - Note: No restrictions needed in models without growth. Then one may consider arbitrary utility.

Preferences in Effective Units

- Claim: If one assumes power utility, all relevant problems can be stated in efficiency units.

  => Convenient to transform preferences and constraints into efficiency units at the outset.

  - Argument for constraints: Routine – as practiced in the Solow model.

- Argument for Preferences:

  - Impose $u(C) = \frac{1}{1-\theta} C^{1-\theta}$ and invoke $C(t) = c(t) \cdot A(t)$:

$$U = \int_0^\infty e^{-\rho t} \left(\frac{C(t)^{1-\theta}}{1-\theta} \right) \frac{L(t)}{H} dt = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta}}{1-\theta} \right) A(t)^{1-\theta} \frac{L(t)}{H} dt$$

  - Invoke $L(t) = L(0) \cdot e^{nt}$, and $A(t) = A(0) \cdot e^{gt}$:

$$U = \int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\theta}}{1-\theta} \right) \left(A(0) e^{gt} \right)^{1-\theta} \frac{L(0) e^{nt}}{H} dt = \frac{L(0) A(0)^{1-\theta}}{H} \cdot \int_0^\infty e^{-[\rho - n - g \cdot (1-\theta)] t} \left(\frac{c(t)^{1-\theta}}{1-\theta} \right) dt$$

  - Scale factor is irrelevant: Normalize $A(0)^{1-\theta} L(0) / H = 1$. Define $\beta = \rho - n - (1-\theta) g$.

- Result: Preferences in efficiency units: $U = \int_0^\infty e^{-\beta t} \frac{1}{1-\theta} c(t)^{1-\theta} dt$

  with growth-adjusted rate of time preference $\beta = \rho - n - (1-\theta) g$.

Conditions for Finite Utility

- Observation #1: Finite utility along a balanced growth path requires β>0.
  - Proof: If consumption converges to a steady state $c(t) \to c^*$, then
    $U = \int_0^\infty e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$ is finite if and only if β>0. Assumed in the following.

- Observation #2: The Euler equation can be written as $\dot{c}/c = 1/\theta\,(r - n - g - \beta)$.
  - Proof: $\dot{c}/c = 1/\theta\,(r-\rho) - g = 1/\theta\,[r - \{\rho + \theta \cdot g\}]$, where $\rho = \beta + n + (1-\theta)g$

    => $\dot{c}/c = 1/\theta\,[r - \{\beta + n + (1-\theta)g + \theta \cdot g\}] = 1/\theta\,[r - \{\beta + n + g\}]$

  => Steady state condition $\dot{c} = 0$ implies $r^* = f'(k^*) - \delta = \beta + n + g$.

- Recall from the Solow model: Dynamic efficiency <=> $r^* > n + g$ <=> $f'(k)^* > n + g + \delta$

- Conclusion: Assumption β>0 ensures dynamic efficiency.
  - Implies capital stock strictly less than Golden Rule level.