Optimal Taxation with Heterogeneous Rates of Return
Model

1 Baseline Model: Immobile Capital, Constant Types

1.1 Discrete Types

I begin with a simple model in which exists a continuum of households, $i \in [0,1]$. Each household is endowed with initial wealth w_0, and owns the rights to its own production technology. Households are heterogeneous in their rates of return, given by $\theta \in \Theta = \{\theta_1, \dots, \theta_N\}$, where I denote $\Pr(\theta = \theta_i) = \pi_i$. Without loss of generality, I assume that $\theta_1 < \theta_2 < \dots < \theta_N$. If a household with productivity θ invests capital k in their production technology, their return is $y = \theta k$. The the households have utility over consumption, and discount the future at rate β. For exposition, I begin by studying a two-period model, with $t \in \{0, 1\}$. For additional simplicity, I assume that all uncertainty regarding output is resolved at $t = 0$; θ remains constant within a household across time.

A benevolent planner (here standing in for the government) allocates consumption $c_0(\theta), c_1(\theta)$ in order to maximize total utility, subject to its resource and incentive compatibility constraints:

$$\max_{c_0(\theta), c_1(\theta)} \sum_{i=1}^{N} u(c_0(\theta_i)) + \beta u(c_1(\theta)) \tag{1}$$

s.t.

$$\sum_{i=1}^{N} \left[c_0(\theta_i) + k(\theta_i)\right] \pi_i = w_0 \tag{2}$$

$$\sum_{i=1}^{N} c_1(\theta_i)\pi_i = \sum_{i=1}^{N} \theta_i k(\theta_i)\pi_i \tag{3}$$

$$u(c_0(\theta_i)) + \beta u(c_1(\theta_i)) \geq u(c_0(\theta_{i_r})) + \beta u(c_1(\theta_{i_r})) \; \forall i, i_r \in 1, \dots, N \tag{4}$$

Following Golosov *et al.* (2006), I denote i_r as the individual's reporting strategy; the incentive constraint (4) requires that truthful reporting be a dominant strategy. I solve this problem using Lagrangean methods, where λ_0 and λ_1 are the multipliers on the resource constraints (2) and (3), respectively, and $\psi(i, i_r)$ the multiplier on each of the N^2 incentive constraints (4). The first-order conditions for the planner's problem are as follows:

$$u'(c_0(\theta_i)) = \frac{\lambda_0 \pi_i}{\pi_i + \eta(i)} \tag{5}$$

$$\beta u'(c_1(\theta_i)) = \frac{\lambda_1 \pi_i}{\pi_i + \eta(i)} \tag{6}$$

$$\theta_i = \frac{\lambda_0}{\lambda_1} \tag{7}$$

where

$$\eta(i) = \sum_{i'} \psi(i', i) - \psi(i, i')$$

By assumption, no two values of θ are equal, and thus the FOC for $k(\theta_i)$ (7) can only hold for one i. In other words, there is only one i for which $k^*(\theta_i)$ is at an interior solution. Because the planner maximizes utility, it stands to reason that $k(\theta_N) > 0$, and $k(\theta_i) = 0$ for $i < N$. Thus, only the most productive agents are called upon to produce.

The Euler equation for the household problem is

$$u'(c_0) = \beta\theta u'(c_1)$$

and thus the intertemporal wedge is given by

$$\tau_k(i) = 1 - \frac{u'(c_0(\theta_i))}{\beta\theta_i u'(c_1(\theta_i))}$$

Combining the first-order conditions (5)-(7) yields

$$\tau_k(i) = 1 - \frac{\theta_N}{\theta_i}$$

Thus, for $i < N$, $\tau_k(i) < 0$; these agents are discouraged from saving. Agents for whom $i = N$, meanwhile, face a wedge of 0, and are on their Euler equations.

1.2 Continuous Types

The setup is the same as in section 1.1, but now I allow for a continuum of types. Households are now indexed by $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$. I assume that the distribution of θ has the CDF $F(\theta)$. The planning problem is now

$$\max_{c_0(\theta), c_1(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} u(c_0(\theta)) + \beta u(c_1(\theta)) dF(\theta) \tag{8}$$

s.t.

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[c_0(\theta) + k(\theta)\right] dF(\theta) = w_0 \tag{9}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} c_0(\theta) dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta k(\theta) dF(\theta) \tag{10}$$

$$u(c_0(\theta)) + \beta u(c_1(\theta)) \geq u(c_0(\hat{\theta})) + \beta u(c_1(\hat{\theta})) \; \forall \theta, \hat{\theta} \in \Theta \tag{11}$$

Following Golosov *et al.* (2006) and Kocherlakota (2010), I derive the inverse Euler equation in order to characterize the optimal wedges. As before, the Euler equation for a household of type θ is

$$u'(c_0(\theta)) = \beta\theta u'(c_1(\theta))$$

Thus, the intertemporal wedge is defined as in section 1.1:

$$\tau_k(\theta) = 1 - \frac{u'(c_0(\theta))}{\beta\theta u'(c_1(\theta))}$$

Fixing θ, I increase utility at $t = 1$ by Δ, and to compensate, decrease utility at $t = 0$ by $\beta\Delta$. Note that the period-0 cost of delivering consumption in period 1 is the aggregate rate of return, which is exogeneous. Denoting the aggregate return $\tilde{R}$, it is given by

$$\tilde{R} = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \theta k(\theta) dF(\theta)}{\int_{\underline{\theta}}^{\bar{\theta}} k(\theta) dF(\theta)} \tag{12}$$

However, assuming that $F(\theta)$ has no atoms, the contribution of each $k(\theta)$ is infinitesimally small, so changing $c_0(\theta)$ and thus $k(\theta)$ by a small Δ has no effect on $\tilde{R}$. Therefore, this perturbation to equilibrium allocations does not affect the objective function or the incentive constraints.

The total cost of these perturbations, then, is given by

$$u^{-1}\left(u'(c_0(\theta)) - \beta\Delta\right) + \frac{1}{\tilde{R}} u^{-1}\left(u(c_1(\theta)) + \Delta\right)$$

The first-order condition of the above minimization problem with respect to Δ, evaluated at $\Delta = 0$, gives the inverse Euler equation:

$$\frac{1}{u'(c_0(\theta)} = \frac{1}{B\tilde{R}u'(c_1(\theta))} \tag{13}$$

Because period-1 consumption is not stochastic, (13) can be rearranged to yield

$$u'(c_0(\theta) = \beta\tilde{R}u'(c_1(\theta))$$

Thus, the determinant of the optimal intertemporal wedge will be the relationship between θ and $\tilde{R}$: if $\theta > \tilde{R}$, $\tau_k(\theta) > 0$, while if $\theta < \tilde{R}$, $\tau_k(\theta) < 0$. This is consistent with the wedge in section 1.1. I am still in the process of deriving the optimal allocations in this environment, in order to determine which types face positive and negative wedges.

2 Immobile Capital, Idiosyncratic Shocks

2.1 Discrete Types

Here, I consider the model of section 1.2, but I allow for production at $t = 1$ to be subject to idiosyncratic shocks. A household investing k at $t = 0$ in its personal production technology now produces at $t = 1$ output $y = \theta k \varepsilon$, where the shocks ε are independent and identically distributed, with $\mathbb{E}[\varepsilon] = 1$ and the CDF $G(\varepsilon)$. For additional simplicity, I also assume that the shocks are independent of θ. The planner's problem is now

$$\max_{c_0(\theta), c_1(\theta,\varepsilon)} \int_{\underline{\theta}}^{\bar{\theta}} \left[u(c_0(\theta)) + \beta \int_{\mathbb{R}} u(c_1(\theta,\varepsilon)) dG(\varepsilon)\right] dF(\theta) \tag{14}$$

s.t.

$$\int_{\underline{\theta}}^{\bar{\theta}} \left[c_0(\theta) + k(\theta)\right] dF(\theta) = w_0 \tag{15}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\mathbb{R}} c_1(\theta,\varepsilon) dG(\varepsilon) dF(\theta) = \int_{\underline{\theta}}^{\bar{\theta}} \theta k(\theta) dF(\theta) \tag{16}$$

$$u(c_0(\theta)) + \beta u(c_1(\theta,\varepsilon)) \geq u(c_0(\hat{\theta})) + \beta u(c_1(\hat{\theta},\hat{\varepsilon}))\ \forall \theta, \hat{\theta} \in \Theta; \forall \varepsilon, \hat{\varepsilon} \in \mathbb{R} \tag{17}$$

Once again, I derive the inverse Euler equation. By the Law of Large Numbers, the aggregate interest rate is again given by (12). I again consider a small deviation from the equilibrium allocations: fixing θ, I increase utility in the second period by Δ for all realizations of ε, and to compensate, decrease utility in the initial period by $\beta\Delta$. The cost of doing so is

$$u^{-1}\left(u'(c_0(\theta)) - \beta\Delta\right) + \frac{1}{\tilde{R}} \int u^{-1}\left(u(c_1(\theta)) + \Delta\right) dG(\varepsilon)$$

As in section 1.2, the first-order condition of the above at $\Delta = 0$ gives the inverse Euler equation:

$$\frac{1}{u'(c_0(\theta))} = \frac{1}{\beta \tilde{R}} \int \frac{1}{u'(c_1(\theta, \varepsilon))} dG(\varepsilon) \tag{18}$$

Applying Jensen's inequality to the above yields

$$u'(c_0(\theta)) < \beta \tilde{R} \int u'(c_1(\theta, \varepsilon)) dG(\varepsilon)$$

As in section 1.2, the optimal intertemporal wedge is determined by the relationship between θ and $\tilde{R}$. The Euler equation derived from the household's problem, given type θ, is

$$u'(c_0(\theta)) = \beta\theta \int u'(c_1(\theta, \varepsilon)) dG(\varepsilon)$$

and thus the intertemporal savings wedge is

$$\tau_k(\theta) = 1 - \frac{u'(c_0(\theta))}{\beta\theta \int u'(c_1(\theta, \varepsilon)) dG(\varepsilon)}$$

Thus, for types with $\theta > \tilde{R}$, the optimal wedge is again positive.

References

Mikhail Golosov, Aleh Tsyvinski, Ivan Werning, Peter Diamond, and Kenneth L Judd. New dynamic public finance: A user's guide [with comments and discussion]. *NBER macroeconomics annual*, 21:317–387, 2006.

Narayana R Kocherlakota. *The new dynamic public finance.* Princeton University Press, 2010.