An FEM Analysis of Risk-Sensitive Real Business Cycles

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Abstract

In this project, we will try to apply Finite Element Method to the model used in Tallarini (2000).

1 Model

The model is the usual business cycle model except for consumers preferences. Consumer's preferences are given by a subclass of preferences defined in Epstein and Zin (1989). The inter-temporal elasticity of substitution is 1 while the coefficient for relative risk-aversion can be varied freely. Hence the non-expected utility of an agent at date t is defined recursively as follows:

$$U_t = \log c_t + \beta \frac{1}{(1-\beta)(1-\chi)} \log(E_t[\exp\{(1-\beta)(1-\chi)U_{t+1}\}]) \tag{1}$$

Furthermore, if $\sigma = 2(1-\beta)(1-\chi)$, then:

$$U_t = \log c_t + \beta(2/\sigma)\log(E_t[\exp\{(\sigma/2)U_{t+1}\}])$$

Allowing for elastic labor supply, preferences will be in the following form:

$$U_t = \log C_t + \theta \log L_t + \beta(2/\sigma)\log(E_t[\exp\{(\sigma/2)U_{t+1}\}]) \tag{2}$$

where L_t is leisure. Tallarini (2000) uses a random walk with drift for the stochastic process governing productivity. He then divides everything by productivity to make problem stationary. While this is a valid approach, we will use the usual approach of having a constant productivity growth and using a stationary process for TFP.[1] Therefore, the production

[1] Random walk is not a stationary process and therefore we cannot discretize the state space of shocks. So we will assume, instead, a mean-reverting process for TFP in order to apply methods described in Tauchen (1986).

side of the economy is given by

$$
\begin{aligned}
Y_t &= K_t^{\alpha}(N_t(1+g)^t X_t)^{1-\alpha} \\
\log X_{t+1} &= \rho \log X_t + \epsilon_{t+1} \\
K_{t+1} &= (1-\delta)K_t + Y_t - C_t
\end{aligned}
$$

We define small letters to be variable divided by growth rate of TFP: $v_t = V_t/(1+g)^t$. Then, the Pareto problem becomes the following:

$$
\begin{aligned}
\max_{C_t, L_t, K_t, Y_t, U_t} \quad & U_0 \\
\text{s.t.} \quad & C_t + K_{t+1} = (1-\delta)K_t + Y_t \\
& Y_t = K_t^{\alpha}(N_t(1+g)^t X_t)^{1-\alpha} \\
& U_t = \log C_t + \theta \log(1-N_t) + \beta(2/\sigma)\log(E_t[\exp\{(\sigma/2)U_{t+1}\}])
\end{aligned}
$$

We define $u_t = U_t - t\frac{\log(1+g)}{1-\beta} - \frac{\beta}{(1-\beta)^2}\log(1+g)$. Then one can show that u_t satisfies the following equation:

$$
u_t = \log c_t + \theta \log(1-n_t) + \beta(2/\sigma)\log(E_t[\exp\{(\sigma/2)u_{t+1}\}])
$$

Therefore the Pareto problem is equivalent to the following problem:

$$
\begin{aligned}
\max_{c_t, n_t, k_t, y_t, u_t} \quad & u_0 \\
\text{s.t.} \quad & c_t + (1+g)k_{t+1} = (1-\delta)k_t + k_t^{\alpha}(n_t x_t)^{1-\alpha} && (3) \\
& u_t = \log c_t + \theta \log(1-n_t) + \beta(2/\sigma)\log(E_t[\exp\{(\sigma/2)u_{t+1}\}]) && (4) \\
& \log x_{t+1} = \rho \log x_t + \epsilon_{t+1}
\end{aligned}
$$

The FOCs for this problem are the following:

$$
\begin{aligned}
\frac{\theta}{1-n_t} &= \frac{1}{c_t}(1-\alpha)k_t^{\alpha}(n_t x_t)^{-\alpha} x_t && (5) \\
1 &= \frac{\beta}{1+g} E_t\left[\frac{c_t}{c_{t+1}} \frac{\exp((\sigma/2)u_{t+1})}{E_t[\exp((\sigma/2)u_{t+1})]}\left\{1-\delta+\alpha k_{t+1}^{\alpha-1}(n_{t+1}x_{t+1})^{1-\alpha}\right\}\right] && (6)
\end{aligned}
$$

Therefore the policy functions that we have to solve for are $k'(k,x), c(k,x), n(k,x), u(k,x)$ using equations (3), (4), (5) and (6).

The function to be solved for in McGrattan (1996) is $c(k,s)$. Here we will look for the following function:

$$
\begin{pmatrix} c(k,x) \\ u(k,x) \end{pmatrix} = \sum_{j=1}^{m} \boldsymbol{\alpha}_j^x \Phi_j(k) = \sum_{j=1}^{m} \begin{pmatrix} \alpha_{j,c}^x \\ \alpha_{j,u}^x \end{pmatrix} \Phi_j(k) \tag{7}
$$

where $\Phi_j(k)$ is a one-dimensional basis function and $\boldsymbol{\alpha}_j^x \in \mathbb{R}^2$.

The residual function, $R(k,x)$, in this case, would be

$$
R(k,x;\boldsymbol{\alpha}) = \begin{pmatrix} \log c(k,x) + \theta\log(1-n(k,x)) + \beta(2/\sigma)\log(E_t\{\exp((\sigma/2)u(\tilde{k},x'))\}) - u(k,x) \\ \frac{\beta}{1+g}E_t[\frac{c(k,x)}{c(\tilde{k},x')}\frac{\exp((\sigma/2)u(\tilde{k},x'))}{E_t[\exp((\sigma/2)u(\tilde{k},x'))]}\left(1-\delta+\alpha k_{t+1}^{\alpha-1}(n_{t+1}x_{t+1})^{1-\alpha}\right)] - 1 \end{pmatrix} \tag{8}
$$

where $n(k,x)$ is the solution to the following equation:

$$(1-\alpha)(1-n)(nx)^{-\alpha}xk^{\alpha}-\theta c(k,x)=0$$

and $k'=\frac{1}{1+g}\left((nx)^{1-\alpha}k^{\alpha}+(1-\delta)k-c(k,x)\right)$.

In order to differentiate (8), we define it as a function of $z=(c,c',k',n,n',u,u')$. We can calculate the derivative of R w.r.t z. Then, we can calculate $dz/du, dz/dc$ and using the chain rule, we can calculate R_{α}. We have:

$$R_1=\log c+\theta\log(1-n)+\beta(2/\sigma)\log(\sum_{x'}\pi_{x',x}\exp((2/\sigma)u'(x')))-u \tag{9}$$

Therefore,

$$\begin{aligned}
R_{1c} &= \frac{1}{c}\\
R_{1n} &= -\frac{\theta}{1-n}\\
R_{1u} &= -1\\
R_{1u'(x')} &= \beta\frac{\pi_{x',x}\exp((2/\sigma)u'(x'))}{\sum_{x'}\pi_{x',x}\exp((2/\sigma)u'(x'))}
\end{aligned}$$

Moreover,

$$\begin{aligned}
R_2 &= \frac{\beta}{1+g}\sum_{x'}\pi_{x',x}\frac{c}{c(x')}\frac{\exp((\sigma/2)u'(x'))}{\sum\limits_{x'}\pi_{x',x}\exp((\sigma/2)u'(x'))}(1-\delta+\alpha k'^{\alpha-1}(n'(x')x')^{1-\alpha})-1\\
&= \frac{\beta}{1+g}\frac{\sum\limits_{x'}\pi_{x',x}\frac{c}{c'(x')}\exp((\sigma/2)u'(x'))[1-\delta+\alpha k'^{\alpha-1}(n'(x')x')^{1-\alpha}]}{\sum\limits_{x'}\pi_{x',x}\exp((\sigma/2)u'(x'))}-1
\end{aligned} \tag{10}$$

For less numerical complexity, we redefine R_2 to be the following

$$R_2=\frac{\beta}{1+g}\sum_{x'}\pi_{x',x}\frac{c}{c'(x')}\exp((\sigma/2)u'(x'))[1-\delta+\alpha k'^{\alpha-1}(n'(x')x')^{1-\alpha}]-\sum_{x'}\pi_{x',x}\exp((\sigma/2)u'(x')) \tag{11}$$

Hence, the derivatives are

$$\begin{aligned}
R_{2c} &= \frac{R_2+\sum\limits_{x'}\pi_{x',x}\exp((\sigma/2)u'(x'))}{c}\\
R_{2c'(x')} &= -\frac{\beta}{1+g}\pi_{x',x}\frac{c}{c'(x')^2}\exp((\sigma/2)u'(x'))[1-\delta+\alpha k'^{\alpha-1}(n'(x')x')^{1-\alpha}]\\
R_{2k'} &= \frac{\beta}{1+g}\sum_{x'}\pi_{x',x}\frac{c}{c'(x')}\exp((\sigma/2)u'(x'))[\alpha(\alpha-1)k'^{\alpha-2}(n'(x')x')^{1-\alpha}]\\
R_{2n'(x')} &= \frac{\beta}{1+g}\pi_{x',x}\frac{c}{c'(x')}\exp((\sigma/2)u'(x'))[\alpha(1-\alpha)k'^{\alpha-1}(n'(x')x')^{-\alpha}x']\\
R_{2u'(x')} &= \frac{\sigma\beta}{2(1+g)}\pi_{x',x}\frac{c}{c'(x')}\exp((\sigma/2)u'(x'))[1-\delta+\alpha k'^{\alpha-1}(n'(x')x')^{1-\alpha}]\\
&- \frac{\sigma}{2}\pi_{x',x}\exp((\sigma/2)u'(x'))
\end{aligned}$$

We also have:

$$\left[-\alpha n^{-\alpha-1} - (1-\alpha)n^{-\alpha}\right]\frac{\partial n}{\partial c} = \frac{\theta}{(1-\alpha)k^\alpha x^{1-\alpha}} \Rightarrow \frac{\partial n}{\partial c} = \frac{\theta n^\alpha}{(1-\alpha)k^\alpha x^{1-\alpha}(-(1-\alpha)-\alpha n^{-1})}$$

$$\frac{\partial k'}{\partial c} = \frac{(1-\alpha)k^\alpha (nx)^{-\alpha} x}{1+g}\frac{\partial n}{\partial c} - \frac{1}{1+g}$$

$$\frac{\partial n'(x')}{c'(x')} = \frac{\theta n'(x')^\alpha}{(1-\alpha)k'^\alpha x'^{1-\alpha}(-(1-\alpha)-\alpha n'(x')^{-1})}$$

$$\frac{\partial n'(x')}{k'} = -\frac{\alpha\theta n'(x')^\alpha c'(x')}{(1-\alpha)k'^{\alpha-1} x'^{1-\alpha}(-(1-\alpha)-\alpha n'(x')^{-1})}$$

Therefore, taking derivative with respect to $\alpha^{\bar{x}}_{j,c}$, we have:

Contemporary derivatives

$$\frac{\partial c(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \Phi_j(k)\mathbf{1}[\bar{x}=x] \tag{12}$$

$$\frac{\partial n(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \frac{\partial n}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] \tag{13}$$

$$\frac{\partial u(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = 0 \tag{14}$$

Future Derivatives

$$\frac{\partial k'(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \frac{\partial k'}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] \tag{15}$$

$$\begin{aligned}\frac{\partial c'(x')}{\partial \alpha^{\bar{x}}_{j,c}} &= \frac{\partial c(k'(k,x;\boldsymbol{\alpha}),x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \frac{\partial c}{\partial k}|_{k'(k,x;\boldsymbol{\alpha})}\frac{\partial k'}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] + \mathbf{1}[\bar{x}=x']\Phi_j(k') \\ &= \left(\sum_{i=1}^m \alpha^{x'}_{i,c}\Phi'_i(k')\right)\frac{\partial k'}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] + \mathbf{1}[\bar{x}=x']\Phi_j(k') \end{aligned}\tag{16}$$

$$\frac{\partial n'(k',x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \frac{\partial n'(x')}{\partial c'(x')}\frac{\partial c'(x')}{\partial \alpha^{\bar{x}}_{j,c}} + \frac{\partial n'(x')}{\partial k'}\frac{\partial k'}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] \tag{17}$$

$$\frac{\partial u'(x')}{\partial \alpha^{\bar{x}}_{j,c}} = \frac{\partial u(k',x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = \left(\sum_{i=1}^m \alpha^{x'}_{i,u}\Phi'_i(k')\right)\frac{\partial k'}{\partial c}\Phi_j(k)\mathbf{1}[\bar{x}=x] \tag{18}$$

Applying chain rule:

$$\frac{\partial R_1(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = R_{1c}\frac{\partial c(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} + R_{1n}\frac{\partial n(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} + R_{1u}\times 0 \tag{19}$$

$$+ \sum_{x'} R_{1u'(x')}\frac{\partial u'(k',x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} \tag{20}$$

$$\frac{\partial R_2(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} = R_{2c}\frac{\partial c(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} + R_{2k'}\frac{\partial k'(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,c}} \tag{21}$$

$$+ \sum_{x'}\left\{R_{2c'(x')}\frac{\partial c'(x')}{\partial \alpha^{\bar{x}}_{j,c}} + R_{2n'(x')}\frac{\partial n'(x')}{\partial \alpha^{\bar{x}}_{j,c}} + R_{2u'(x')}\frac{\partial u'(x')}{\partial \alpha^{\bar{x}}_{j,c}}\right\} \tag{22}$$

For derivative w.r.t. $\alpha^{\bar{x}}_{j,u}$, we have the following:

$$
\begin{aligned}
&\text{Contemporary derivatives}\\
&\frac{\partial c(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = 0 && (23)\\
&\frac{\partial n(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = 0 && (24)\\
&\frac{\partial u(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = \mathbf{1}[\bar{x}=x]\Phi_j(k) && (25)\\
&\text{Future Derivatives}\\
&\frac{\partial k'(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = 0 && (26)\\
&\frac{\partial c'(x')}{\partial \alpha^{\bar{x}}_{j,u}} = \frac{\partial c(k'(k,x;\boldsymbol{\alpha}),x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = 0 && (27)\\
&\frac{\partial n'(k',x';\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} = 0 && (28)\\
&\frac{\partial u'(x')}{\partial \alpha^{\bar{x}}_{j,u}} = \mathbf{1}[\bar{x}=x']\Phi_j(k') && (29)
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\frac{\partial R_1(k,x;\boldsymbol{\alpha})}{\partial \alpha^{\bar{x}}_{j,u}} &= -\Phi_j(k)\mathbf{1}[\bar{x}=x] + R_{1u'(\bar{x})}\Phi_j(k') && (30)\\
\frac{\partial R_2(k,x;\boldsymbol{\alpha})}{\partial \alpha^{x}_{j,u}} &= R_{2u'(\bar{x})}\Phi_j(k') && (31)
\end{aligned}
$$

1.1 Calculating Utility When $\sigma = 0$

In this section, I describe how to calculate utility in a one-sector growth model with expected utility, i.e. $\sigma = 0$, and when we know the consumption function, $c(k,s)$. Suppose that our grid points for k are $\{k_j\}_{j=1}^{n_x}$ and $\{s_i\}_{i=1}^{n_s}$ for the stochastic state s. A consumption function is comprised of $n_x \times n_s$ numbers, $c(k_j,s_i)$. What we are looking for is $n_x \times n_s$ numbers $u(k_j,s_i)$ given the fact that utility $u(k,s)$ is a linear combination of basis function over our grid. Since we know $c(k_j,s_i)$, we can calculate $n(k_j,s_i)$ using the Intra-temporal Euler Equation. Having found hours, we can calculate next period's capital $k'(k_j,s_i)$. Therefore, we have the following relation for all i,j:

$$
u(k_j,s_i) = \log(c(k_j,s_i)) + \theta\log(1-n(k_j,s_i)) + \beta\sum_{s'}\pi_{s,s'}u(k'(k_j,s_i),s') \tag{32}
$$

Suppose, we find that $k'(k_j,s_i) \in [k_{l(j,i)}, k_{l(j,i)+1}]$, then substituting in (32), we will have:

$$
\begin{aligned}
u(k_j,s_i) &= \log(c(k_j,s_i)) + \theta\log(1-n(k_j,s_i)) && (33)\\
&+ \beta\sum_{s'}\pi_{s,s'}\{\frac{k_{l(j,i)+1}-k'(k_j,s_i)}{k_{l(j,i)+1}-k_{l(j,i)}}u(k_{l(j,i)},s') + \frac{k'(k_j,s_i)-k_{l(j,i)}}{k_{l(j,i)+1}-k_{l(j,i)}}u(k_{l(j,i)+1},s')\}
\end{aligned}
$$

Now, define $\mathbf{u}$ to be the vector of $\{u(k_j,s_i)\}_{j,i}$ and define $\mathcal{P}_{ij}$ to be the vector operation equivalent of second line in (33). Then, equation (33) is equivalent to the following equation:

$$
(I-\beta\mathcal{P})\,\mathbf{u} = \log(\mathbf{c}) + \theta\log(1-\mathbf{n}) \tag{34}
$$

This method closely follows, Ljungqvist and Sargent (2004) section 4.5. Notice that we cannot do this when $\sigma \neq 0$ since equation (32) becomes nonlinear and we have to use finite element method to estimate $\mathbf{u}$.

2 Computer Code

2.1 List of Variables

$$
\begin{array}{cl}
\text{Capital :} & \texttt{k} : k_t; \quad \texttt{kp} : k_{t+1}; \quad \text{grid : } \texttt{ka} \\
\text{Consumption :} & \texttt{c} : c_t; \quad \texttt{cp} : c_{t+1} \\
\text{Hours :} & \texttt{hr} : n_t; \quad \texttt{hrp} : n_{t+1} \\
\text{Utility :} & \texttt{u} : u_t; \quad \texttt{up} : u_{t+1} \\
\text{Basis :} & \texttt{basis1, basis2} : \Phi_j(k); \quad \texttt{basis1p, basis2p} : \Phi_j(k') \\
\text{Basis Derivative :} & \texttt{dbasis1, dbasis2} : \Phi_j'(k); \quad \texttt{dbasis1p, dbasis2p} : \Phi_i'(k') \\
\text{Derivatives :} & \texttt{dndc :} \ \dfrac{\partial n}{\partial c}; \quad \texttt{dkpdc :} \ \dfrac{\partial k'}{\partial c}; \quad \texttt{dnpdcp} : \dfrac{\partial n'(x')}{\partial c'(x')} \\
& \texttt{dnpdkp} : \dfrac{\partial n'(x')}{\partial k'}
\end{array}
$$

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