Thresholds for Graph Properties and a Proof of the Kahn-Kalai Conjecture

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Overview

Discrete Probability Space

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A *discrete probability space* is a triple $(\Omega, \mathcal{A}, \mathbb{P})$ consisting of

1. the *sample space* Ω, which is the set of all possible outcomes of the random experiment,
2. an *algebra* $\mathcal{A}$ representing the allowable events, which is family of subset of Ω containing the empty set and closed under complement and finite union,
3. and a probability function $\mathbb{P} : \mathcal{A} \to [0, 1]$.

[Mitzenmacher, M. and E. Upfal (2005)].

Tail Bounds

Markov's Inequality

Let $Y \geq 0$ be a random variable on a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\alpha > 0$. Then $\mathbb{P}(Y \geq \alpha) \leq \frac{\mathbb{E}(Y)}{\alpha}$. In particular, $\mathbb{P}(Y \geq 1) \leq \mathbb{E}(Y)$

Chebyshev's Inequality

Let Y be a random variable on a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(Y) < \infty$ and $Var(Y) < \infty$. Then for $\alpha > 0$,
$\mathbb{P}(|Y - \mathbb{E}(Y)| \geq \alpha) \leq \frac{Var(Y)}{\alpha^2}$.

[Mitzenmacher, M. and E. Upfal (2005)].

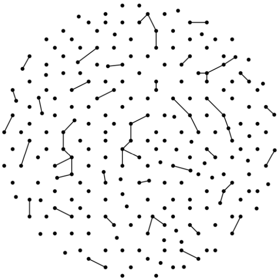


Figure: A Random Graph $G_{n,p}$ under the Erdős-Rényi Model. [Park, J. (2023)]

Consider the graph $G_{n,p}$ where each edge ω is drawn uniformly from the set of edges X with probability p. This is known as the Erdős-Rényi Model.

Random Graph on 3 vertices

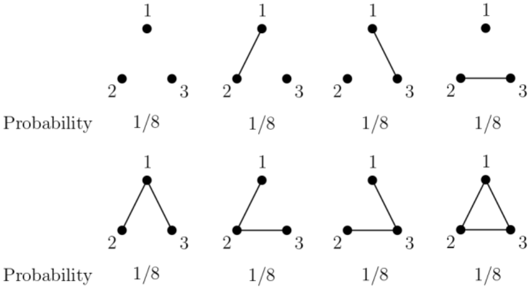


Figure: Probability Distribution of $G_{3,1/2}$. [Park, J. (2023)]

Each of the eight possible graph is is equally likely as each edge is drawn with probability $1/2$. However, the probability distribution of $G_{3,p}$ will look different as we change p.

Random Graph on 3 vertices -ctd

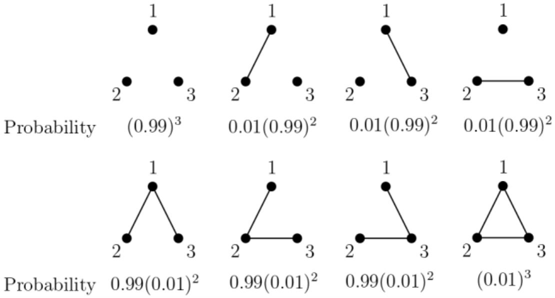


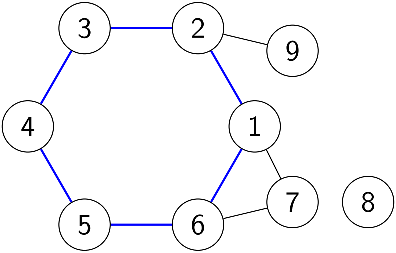
Figure: Probability Distribution of $G_{3,0.01}$ [Park, J. (2023)].

Natural question: What p should be chosen so that $G_{n,p}$ admits a property almost surely?

Properties in Random Graphs

Natural question: What p should be chosen so that $G_{n,p}$ admits a property almost surely?

- $G_{n,p}$ contains a triangle almost surely? $\rightarrow p \geq O(1/n)$ [Frieze, A. M. and Karoński (2015)]
- $G_{n,p}$ contains a Hamiltonian Cycle almost surely? $\rightarrow p \geq O(\log n/n)$ [Pósa, L. (1976)]



6-cycle

Figure: 6-cycle in $G(9, p)$

Example:

- Each cycle is a fixed set of k edges C_k in $G_{n,p}$. Denote $G_{n,p} \in \mathcal{F}$ the event 'for some k, the random graph $G_{n,p}$ contains a k-cycle on the potential set of edges C_k'.
- Define $Y_k^{(n)}$ to be the number of k-cycles cycles in the obtained graph, with $3 \leq k \leq n$.

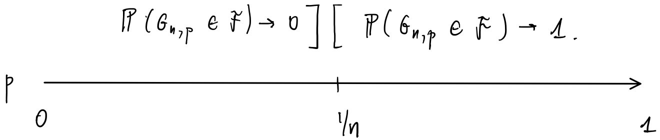


Figure: Location of Threshold $p^*(\mathcal{F})$.

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{F}) = \begin{cases} 0, & \text{if } p(n) \ll 1/n \\ 1, & \text{if } p(n) \gg 1/n \end{cases}$$

Notation:

- $p(n) \ll q(n)$ if $p(n)/q(n) \to 0$ as $n \to \infty$
- $p(n) \gg q(n)$ if $p(n)/q(n) \to \infty$ as $n \to \infty$

k-Cycles in Random-Graphs

- $Y^{(n)}$ the number of all possible sets of k edges $Y_k^{(n)} = \sum_{C_k} 1_{C_k \in \mathcal{F}}$ is obtained by marginalizing over k.

$$Y^n = \sum_{k=3} Y_k^{(n)} = \sum_{k=3}^{n} \sum_{C_k} 1_{C_k \in \mathcal{F}}$$

- Denote $(n)_k := n(n-1)...(n-k+1)$.
- There are $\frac{(n)_k}{2k} = \binom{n}{k}\frac{(k-1)!}{2}$ possible k-cycles.
- For a given $G_{n,p}$, $1_{C_k \in \mathcal{F}}$ is Bernoulli with parameter p^k.

$$\mathbb{E}(Y^{(n)}) = \sum_{k=3}^{n} \mathbb{E}(Y_k^{(n)}) = \sum_{k=3}^{n} \frac{(n)_k}{2k} p^k < (np)^3 \sum_{k=0}^{n-3} (np)^k = \frac{(np)^n - (np)}{np - 1}$$

By linearity of expectation, and noting $\binom{n}{k}(k-1)! < \binom{n}{k}k! < n^k$.

'First Moment Method': Markov's Inequality:

$$\lim_{n\to\infty} \mathbb{P}(Y^{(n)} \geq 1) \leq \lim_{n\to\infty} \mathbb{E}(Y^{(n)}) \leq \lim_{n\to\infty} \frac{(np)^n - (np)^3}{np - 1}$$

$$1 - \lim_{n\to\infty} \mathbb{P}(Y^{(n)} \geq 1) \geq 1 - \lim_{n\to\infty} \mathbb{E}(Y^{(n)}) \geq 1 - \lim_{n\to\infty} \frac{(np)^n - (np)^3}{np - 1}$$

- Using Markov's inequality, we can show that $G_{n,p} \notin \mathcal{F}$ if p is such that $\lim_{n\to\infty} np = 0$.
- Hence if p is such that $\lim_{n\to\infty} \frac{(np)^n}{1-np} = 0$, then $\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{F}) = 0$, which is the case if $p \leq (1/n)$.

'Second Moment Method': Chebyshev's Inequality: We would like to show that for $p \gg 1/n$ $G_{n,p} \in \mathcal{F}$. In particular, let $p = \frac{10}{n}$.

- A sufficient condition for $G_{n,p} \in \mathcal{F}$ is that $G_{n,p}$ contains n edges.
- The number of edges $Z^{(n)} = \sum_{i=1}^{n} E_i$ is distributed as $bin(\binom{n}{2}, p)$.
- $\mathbb{E}(Z^{(n)}) = \binom{n}{2}\frac{10}{n} = \frac{10}{n}\left(1 - \frac{1}{n}\right)$, $Var(Z^{(n)}) = \binom{n}{2}\frac{10}{n}$.

Using Chebyshev's inequality with $\alpha = \frac{10}{n}$,

$$\mathbb{P}(|\mathbb{E}(Z^{(n)}) - Z^{(n)}| \geq \frac{n}{10}) \leq \frac{Var(Z^{(n)})}{(n/10)^2} = \frac{10^3}{2n} \cdot (1 - \frac{1}{n}) \cdot (1 - \frac{10}{n}) \to 0.$$

$Z^{(n)} > \mathbb{E}(Z^{(n)}) - \frac{n}{10}$ a.s. implies $Z_n > \frac{49}{10}n - 5$, and it can easily be shown by induction that with $n \geq 3$, we have $Z^{(n)} \geq n$.

Monotone Increasing Property

Increasing Property

An *increasing property* $\mathcal{F} \subseteq 2^X$ is a family of sets such that if $B \supseteq A \in \mathcal{F}$, then $B \in \mathcal{F}$. A property is *non-trivial* if $\mathcal{F} \neq \emptyset$.

The event $X_p \in \mathcal{F}$

X_p is a uniformly drawn subset of X, where each element is taken with probability p. Denote by $X_p \in \mathcal{F}$ the event ' $\exists A \in \mathcal{F}$ s.t. $X_p \subseteq A$'.

$$\mathbb{P}(X_p \in \mathcal{F}) = \sum_{A \in \mathcal{F}} \mathbb{P}(A) = \sum_{A \in \mathcal{F}} p^{|A|}(1-p)^{|X \setminus A|}.$$

Threshold Functions

Threshold

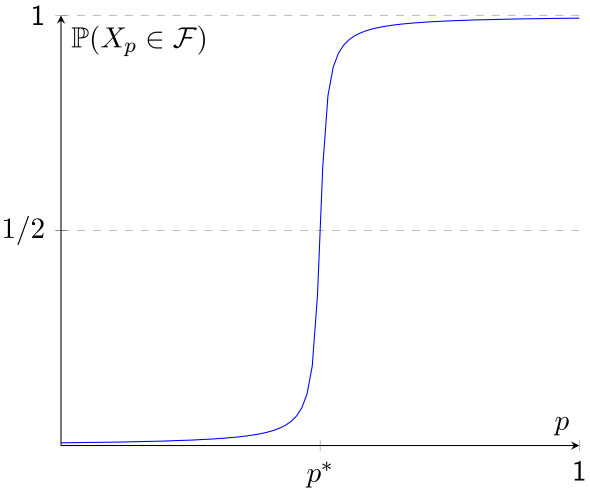
A function $p^* = p^*(n)$ is a *threshold* for a monotone increasing property $\mathcal{F} \subseteq 2^X$ is the unique p^* such that $\mathbb{P}(X_{p^*} \in \mathcal{F}) = 1/2$ a.s. .

Bollobás-Thomason Theorem

Let $\mathcal{F}$ be non-trivial increasing. For every $0 < \epsilon \leq 1/2$, there exists a unique p^* and some constants $K_\epsilon^1, K_\epsilon^2$ such that

$$\mathbb{P}(X_{p^*} \in \mathcal{F}) = 1/2,\ \mathbb{P}(X_{k_\epsilon^2 p^*} \notin \mathcal{F}) \leq \epsilon \text{ and } \mathbb{P}(X_{K_\epsilon^2 p^*} \in \mathcal{F}) \geq 1 - \epsilon.$$

Threshold Functions-ctd



Hamiltonian Cycles

Threshold for Hamiltonian Cycle is non-trivial.

$$\lim_{n\to\infty} \mathbb{P}(X_p \in \mathcal{F}) \begin{cases} = 0, & \text{if } p^*(n) \gg p(n) \\ \neq 1, & \text{if } p(n) \gg p^*(n) \end{cases}$$

- Y_n the number of Hamiltonian Cycles in $G_{n,p}$
- $\mathbb{E}(Y_n) = p^n(n-1)/2$
- Using Markov's Inequality,

$$\mathbb{P}(G_{n,p} \notin \mathcal{F}) = 1 - \lim_{n\to\infty} \mathbb{P}(Y_n \geq 1) \geq 1 - \lim_{n\to\infty} p^n(n-1)/2$$

- Markov's inequality implies $G_{n,p} \notin \mathcal{F}$ if $p \ll (n^{-1/n})$.
- However, it is not the case that $G_{n,p} \in \mathcal{F}$ if $p \gg (n^{-1/n})$!

[Pósa, L. (1976)] proved that for $\mathcal{F}$ given by '$G_{n,p}$' admits a Hamiltonian cycle, $p^*(\mathcal{F}) = \frac{\log n}{n}$.

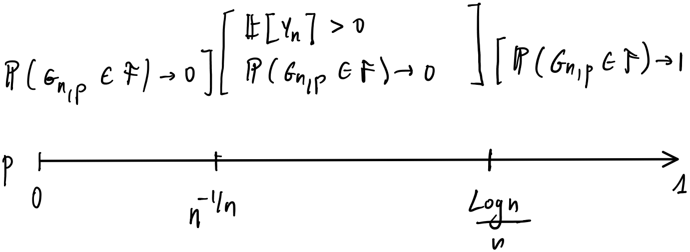


Figure: Location of $q(\mathcal{F})$ and $p^*(\mathcal{F})$.

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{F}) = \begin{cases} 0, & \text{if } p(n) \ll \frac{\log n}{n} \\ 1, & \text{if } p(n) \gg \frac{\log n}{n} \end{cases}$$

Finding Thresholds

- Given $\mathcal{F}$, we are interested in finding $p^*(\mathcal{F})$. This is generally a difficult task. While it ensures existence of p^*, the Bollobás-Thomason Theorem does not provide information on its location.
- One intuitive starting point to find these threshold function is the 'Expectation Threshold'.
- A key concept behind the expectation threshold is that of a p-*small cover* .

Minimal Element

A *minimal* element G of a set $\mathcal{G}$ is such that $\forall G' \in \mathcal{G}$, then $G \subsetneq G'$.

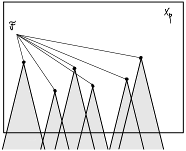


Figure: Increasing Property $\mathcal{F}$ and its Minimal Elements

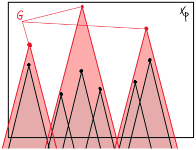


Figure: Minimal elements for a Cover $\mathcal{G}$ of $\mathcal{F}$

- The black dots represent the sets in $\mathcal{F}_{min} := F \in \mathcal{F} : F$ minimal.
- The points contained inside the cones is $\mathcal{F} = \langle \mathcal{F}_{min} \rangle$.
- $\mathcal{F} \subseteq \langle G \rangle$, but, crucially, the sets in G have less overlap.

p-Small Cover

p-Small Cover

We say that the property $\mathcal{F}$ is *p-small* if there exists $\mathcal{G} \subseteq 2^X$ such that both of the following hold.

$$\mathcal{F} \overset{(1)}{\subseteq} \langle \mathcal{G} \rangle := \bigcup_{S \in \mathcal{G}} \left\{ T \subseteq 2^X : S \subseteq T \right\} \text{ and } \sum_{S \in \mathcal{G}} p^{|S|} \overset{(2)}{\leq} \frac{1}{2}$$

If (1) holds, we say $\mathcal{G}$ is a *cover* of $\mathcal{F}$.

Remark

$\sum_{S \in \mathcal{G}} p^{|S|} = \mathbb{E}[|\{S \in \mathcal{G} : S \subseteq X_p\}|]$

- We are interested in the size of $\mathcal{F} := F \in \mathcal{F} : F \subseteq X_p$. The idea is to approximate the set $\mathcal{F}$ with one of lower expected 'size'.

Expectation Threshold

Expectation Threshold

The *expectation threshold* $q(\mathcal{F})$ of $\mathcal{F}$ is the maximum p such that $\mathcal{F}$ is p-small.

- One immediately notices that the expectation threshold provides a trivial lower bound on the threshold, as shown below.

$$\mathbb{P}(X_p \in \mathcal{F}) \leq \mathbb{P}(X_p \in \langle G \rangle) \leq \sum_{S \in \mathcal{G}} p^{|S|},$$

- For an arbitrary $p < q(\mathcal{F})$, $\mathbb{P}(X_p \in \mathcal{F}) < \frac{1}{2}$, preventing p from being the threshold p^*.

Location of p^*

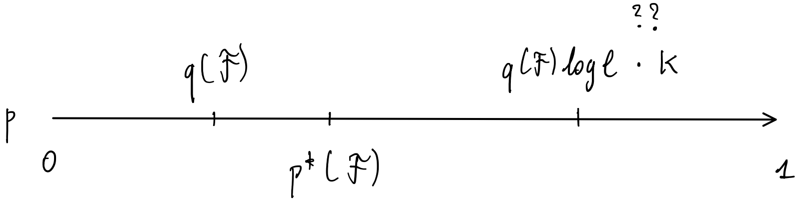


Figure: Location of $q(\mathcal{F})$ and $p^*(\mathcal{F})$.

[Kahn, J. and G. Kalai (2007)] conjectured that $p^*(\mathcal{F})$ is in fact bounded above by $q(\mathcal{F})$ multilied by some small constant, and stated that it would 'be more sensible to conjecture that it is not true'.

The Kahn-Kalai Conjecture

Kahn-Kalai Conjecture

For every finite set X and non-trivial increasing property $\mathcal{F} \subseteq 2^X$ with largest element of size l, there exists a universal constant K such that

$$p^*(\mathcal{F}) \leq Kq(\mathcal{F}) \log l(\mathcal{F}).$$

.[Kahn, J. and G. Kalai (2007)]

Reformulation

Let $l \geq 2$. There exists a universal constant L such that for any nonempty, l-bounded hypergraph $\mathcal{H}$ on X which is not p-small, a uniformly random $(Lp \log l)|X|$-element subset of X belongs to $\langle \mathcal{H} \rangle$ with probability $1 - o(1)$. [Park, J. and H. T. Pham (2022)]

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Remark: The upper bound provided by the Kahn-Kalai Conjecture is in fact the threshold for Hamiltonian Cycles!

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{F}) = \begin{cases} 0, & \text{if } p(n) < \frac{\log n}{n} - \epsilon \\ 1, & \text{if } p(n) > \frac{\log n}{n} + \epsilon \end{cases}$$

Proof: Strategy

Strategy: iteratively construct two sets of elements: $\bigcup_i \mathcal{U}_i$ and $W = \bigcup_i W_i$ through a randomized iterative process. This algorithmic construction has two outputs.

1. $\bigcup_i \mathcal{U}_i$ is a p-small cover of $\mathcal{H}$.
2. W satisfied the conclusion of the reformulated conjecture: it is a uniformly random $(Lp \log l)|X|$-element subset of X belongs to $\langle \mathcal{H} \rangle$ w.h.p.

Proof: Minimum Fragments

- Say we have $W \in \binom{X}{w}$, and some hypergraph $\mathcal{H} \subseteq 2^X$.
- Given any element S of $\mathcal{H}$, then $T(S, W)$ is a *minimum (S, W)-fragment if $T(S, W) = S' \setminus W$ for some $S' \in \mathcal{H}$ such that $S' \subseteq S \cup W$ and $|S' \setminus W|$ is smallest.*
- *We denote the size of a* minimum fragment$|T(S, W)|$ *by $t(S, W)$.*

Algorithm 1: ConstructCover

Input : l-bounded hypergraph $\mathcal{H}$ which is not p-small, finite set X
Output: A family $\mathcal{U}$, a $m = (L_i p \log l)|X|$-sized random $W \in \binom{X}{m}$ subset of X.

$i := 0$, $X_0 := X$, $\mathcal{H}_0 := \mathcal{H}$, $\mathcal{U}_0 := \emptyset$, $\mathcal{G}_0 := \emptyset$;
while $\mathcal{H}_i \neq \{\emptyset\}$ *or* $\mathcal{H}_i \neq \emptyset$ **do**
 $W_i =$ uniformly drawn from $\binom{X_i}{m}$;
 $X_{i+1} \leftarrow X_i \setminus W_i$;
 $\mathcal{G}_i \leftarrow \{S_{i-1} \in \mathcal{H}_{i-1} : t(S_{i-1}, W_i) \geq 0.9 l_i\}$;
 $\mathcal{U}_i \leftarrow \{T(S_{i-1}, W_i) : S_{i-1} \in \mathcal{G}_i\}$;
 $\mathcal{H}_i \leftarrow \{T(S_{i-1}, W_i) : S_{i-1} \in \mathcal{H}_{i-1} \setminus \mathcal{G}_i\}$;
end
return $\bigcup_i \mathcal{U}_i$, $W = \bigcup_i W_i$

Squares of Hamiltonian Cycles

Power of Hamilton Cycle

The k^{th} power C^k of C is the cycle graph with vertex set $[n]$ and edge set $E(H) = uv : d_H(u, v) \leq k$, where $d_H(u, v) \leq k$ is the length of a shortest u, v-path in C if one exists. [Montgomery, R. (2018)]

- [Riordan, O. (2000)] showed that for $k \geq 3$, if $p = O(n^{-1/k})$, then $G_{n,p}$ contains a k-cycle C_n^k a.s. using an argument based on the 'Second Moment' method.
- [Kühn, D. and D., Osthus (2012)] conjectured that the threshold is $n^{-1/2}$
- The best bounds before 2020 were of $(\log n)^3 n^{-1/2}$ by [Fischer, M. and N. and S. Škorić (2022)] and of $(\log n)^2 n^{-1/2}$ by [Friedgut, E. (2005)].

Squares of Hamiltonian Cycles

- Let $\mathcal{H}_{C_n^2}$ be the hypergraph of all graphs on n vertices containing C^2.
- [Frieze, A. M. and Karoński (2015)] show that $\mathcal{H}_{C_n^2}$ is not $(e^{-1}n^{1/2})^{-1}$ – small.
- Noting that the largest element of $\mathcal{H}_{C_n^2}$ has size at most $l(\mathcal{H}_{C_n^2}) := n + \lfloor n/2 \rfloor = O(n)$.
- The Kahn-Kalai Conjecture (Theorem) implies there exists a universal constant K such that

$$n^{-1/2} \leq Kn^{-1/2} \log l(\mathcal{H}_{C_n^2}) = O(n^{-1/2} \log n).$$

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