Anti-de Sitter supersymmetry and hyperKähler geometry

Sergei M. Kuzenko

School of Physics, The University of Western Australia

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Based on:
SMK & G. Tartaglino-Mazzucchelli, arXiv:1109.0496
SMK, U. Lindström & G. Tartaglino-Mazzucchelli, arXiv:1205.4622
D. Butter, SMK & G. Tartaglino-Mazzucchelli, arXiv:1210.5906

What this talk is about

This talk is mainly about implications of the geometry of spacetime on the geometry of rigid supersymmetric sigma models.

Of considerable interest are nonlinear sigma models with:

- eight supercharges in four and five dimensions;
- six and eight supercharges in three dimensions.

It turns out that the 4D and 5D stories are special cases of what happens in 3D. It suffices to discuss the latter case in some detail.

As toy examples, I first consider sigma models with four supercharges in four and three dimensions.

Bosonic nonlinear sigma-model is a field theory over a space-time $\mathbb{S}$ in which the fields take values in a Riemannian manifold $(\mathcal{M}, g)$ (target space). If $\mathbb{S}$ is D-dimensional Minkowski space, the σ-model action is

$$S = -\frac{1}{2}\int \mathrm{d}^D x \, g_{\mu\nu}(\varphi)\partial^a\varphi^\mu\partial_a\varphi^\nu \ ,$$

where $\varphi^\mu(x)$ are scalar fields on $\mathbb{S}$ and local coordinates on $\mathcal{M}$.

A supersymmetric nonlinear σ-model consists of a bosonic σ-model coupled to fermions in such a way that the combined action functional is supersymmetric.

If the number of supercharges is at least four, supersymmetry imposes nontrivial conditions on the geometry of target space.

4D $\mathcal{N} = 1$ Poincaré supersymmetry

Target spaces of 4D $\mathcal{N} = 1$ supersymmetric nonlinear sigma models are constrained to be arbitrary Kähler spaces.
One-to-one correspondence between massless supersymmetric σ-models and Kähler manifolds.

Zumino (79)

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, K(\phi, \bar{\phi})$$

$K(\phi^a, \bar{\phi}^{\bar{b}})$ Kähler potential defined modulo Kähler transformations

$$K(\phi, \bar{\phi}) \to K(\phi, \bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi})$$

Massive extension

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, K(\phi, \bar{\phi}) + \int \mathrm{d}^4x\mathrm{d}^2\theta\, W(\phi) + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{W}(\bar{\phi})$$

$W(\phi^a)$ superpotential, invariant under the Kähler transformations.

Kähler geometry

A Riemannian manifold $(\mathcal{M}_{2n}, g)$ is called Kähler if

- $\mathcal{M}_{2n}$ is endowed with an almost complex structure $J^\mu{}_\nu(\varphi)$

$$J^2 = -\mathbb{1}_{2n} ;$$

- the metric $g_{\mu\nu}(\varphi)$ is Hermitian with respect to $J^\mu{}_\nu(\varphi)$,

$$g_{\lambda\rho} J^\lambda{}_\mu J^\rho{}_\nu = g_{\mu\nu} \quad \Longleftrightarrow \quad J_{\mu\nu} := g_{\mu\lambda} J^\lambda{}_\nu = -J_{\nu\mu}$$

- $J^\mu{}_\nu(\varphi)$ is covariantly constant,

$$\nabla_\lambda J^\mu{}_\nu = 0 \quad \Longleftrightarrow \quad \mathrm{d}\Omega = 0 ,$$

with $\Omega := \frac{1}{2} J_{\mu\nu}(\varphi) \mathrm{d}\varphi^\mu \wedge \mathrm{d}\varphi^\nu$ the Kähler two-form which is closed. The latter requirement implies that $J^\mu{}_\nu$ is a complex structure.

Kähler geometry II

Choose local complex coordinates for $\mathcal{M}_{2n}$,

$$\varphi^\mu = (\phi^a, \bar{\phi}^{\bar{b}}) ,$$

such that the complex structure and the metric take the form

$$J^\mu{}_\nu = \begin{pmatrix} \mathrm{i}\,\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix} , \qquad g_{\mu\nu} = \begin{pmatrix} 0 & g_{a\bar{b}} \\ g_{\bar{a}b} & 0 \end{pmatrix} .$$

In these coordinates, the Kähler two-form becomes

$$\Omega = -\mathrm{i} g_{a\bar{b}} \mathrm{d}\phi^a \wedge \bar{\phi}^{\bar{b}} , \qquad \mathrm{d}\Omega = 0 \quad \Longrightarrow \quad g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(\phi, \bar{\phi}) .$$

The Kähler potential, $K(\phi, \bar{\phi})$, is defined modulo Kähler transformations

$$K(\phi, \bar{\phi}) \to K(\phi, \bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}) .$$

4D $\mathcal{N} = 1$ anti-de Sitter supersymmetry

For the most general $\mathcal{N} = 1$ supersymmetric σ-model in AdS_4, its target space must be a Kähler manifold $\mathcal{M}$ with an exact Kähler form

$$\Omega = -\mathrm{i}\, g_{a\bar{b}}\, \mathrm{d}\phi^a \wedge \mathrm{d}\bar{\phi}^{\bar{b}} \ , \qquad g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$$

and hence $\mathcal{M}$ is necessarily non-compact.

Adams, Jockers, Kumar & Lapan, arXiv:1104.3155
Festuccia & Seiberg, arXiv:1105.0689
Butter & SMK, arXiv:1104.2153, arXiv:1105.3111

Most general $\mathcal{N} = 1$ nonlinear σ-model in AdS

$$S = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, \mathcal{K}(\phi^a, \bar{\phi}^{\bar{b}}) \ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}} \phi^a = 0$$

Unlike in the Minkowski case, the action does not possess Kähler invariance. Because of

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, E\, F(\phi) = \mu \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{E}\, F(\phi) \neq 0$$

the Lagrangian $\mathcal{K}(\phi, \bar{\phi})$ has to be a globally defined function on $\mathcal{M}$.

In principle, Kähler invariance naturally emerges if we represent the Lagrangian as

$$\mathcal{K}(\phi,\bar\phi) = K(\phi,\bar\phi) + \frac{1}{\mu} W(\phi) + \frac{1}{\bar\mu}\bar W(\bar\phi)~,$$

for some Kähler potential K and superpotential W. Under a Kähler transformation, these transform as

$$K(\phi,\bar\phi) \to K(\phi,\bar\phi) + F(\phi) + \bar F(\bar\phi), \qquad W(\phi) \to W(\phi) - \mu F(\phi)$$

The Kähler metric

$$g_{a\bar b} := \partial_a \partial_{\bar b} \mathcal{K} = \partial_a \partial_{\bar b} K$$

is invariant under the Kähler transformations.
However, unlike the Minkowski case, only $\mathcal{K}$ is unambiguously defined.

3D $\mathcal{N} = 2$ SUSY versus 4D $\mathcal{N} = 1$ SUSY

What happens when we go down to three dimensions and keep four unbroken supercharges (3D $\mathcal{N} = 2$ supersymmetry) ?

- In the Minkowski case, the story is the same as in 4D $\mathcal{N} = 1$ SUSY.
- The AdS story gets a twist. The point is that there are two types of $\mathcal{N} = 2$ supersymmetry in AdS_3:
  (2,0) AdS supersymmetry;
  (1,1) AdS supersymmetry.
  They support different target-space geometries for σ-models.

AdS_3 supersymmetry and target space geometry: $\mathcal{N} = 2$

- (1,1) AdS SUSY: Any σ-model target space must be a Kähler manifolds with exact Kähler form, and therefore necessarily non-compact.
  (1,1) AdS SUSY is completely analogous to 4D $\mathcal{N} = 1$ AdS SUSY.
- (2,0) AdS SUSY: If a superpotential $W(\phi)$is present, any σ-model target space must possess a U(1) isometry group.

$$\delta\phi^a = \xi^a(\phi) \ , \qquad \xi^a W_a = -2W$$

Without superpotential, arbitrary Kähler manifolds may originate as σ-model target spaces, with ϕ^a being neutral under the $\mathrm{U}(1)_R$.

Izquierdo &Townsend (1995)
Deger, Kaya, Sezgin & Sundell (2000)
SMK & Tartaglino-Mazzucchelli (2011)

Specific features of (super) AdS in three dimensions

In three dimensions, the anti-de Sitter (AdS) group is reducible,

$$\mathrm{SO}_0(2,2) \cong \Big(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\Big)/\mathbb{Z}_2$$

and so are its supersymmetric extensions,

$$\mathrm{OSp}(p|2;\mathbb{R}) \times \mathrm{OSp}(q|2;\mathbb{R})$$

As a result, $\mathcal{N}$-extended AdS supergravity exists in several incarnations

Achúcarro &Townsend (1986)

and so are its maximally symmetric solutions – (p,q) AdS superspaces

SMK, Lindström & Tartaglino-Mazzucchelli (2012)

Different choices of p and q, $p \geq q$, for fixed $\mathcal{N} = p + q$, lead to supersymmetric field theories with drastically different properties.

3D $\mathcal{N}$-extended AdS superspaces I

$\mathcal{N}$-extended superspace parametrized by real bosonic (x^m) and real fermionic ($\theta^\mu_{\rm I}$) coordinates,

$$z^M = (x^m, \theta^\mu_{\rm I}) \ , \qquad m = 0, 1, 2 \ , \quad \mu = 1, 2 \ , \quad {\rm I} = \mathbf{1}, \cdots, \boldsymbol{\mathcal{N}}$$

Structure group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(\mathcal{N})$.
The superspace covariant derivatives (A tangent space index)

$$\mathcal{D}_A \equiv (\mathcal{D}_a, \mathcal{D}^I_\alpha) = E_A + \Omega_A + \Phi_A$$

$a = 0, 1, 2$ (vector), $\alpha = 1, 2$ (spinor), $I = \mathbf{1}, \cdots, \boldsymbol{\mathcal{N}}$ (R-symmetry).
$E_A = E_A{}^M \partial_M$ is the supervielbein, $\partial_M = \partial / \partial z^M$;

$$\Omega_A = \frac{1}{2} \Omega_A{}^{bc} \mathcal{M}_{bc} = \frac{1}{2} \Omega_A{}^{\beta\gamma} \mathcal{M}_{\beta\gamma} \ , \quad \mathcal{M}_{ab} = -\mathcal{M}_{ba} \ , \quad \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$$

is the Lorentz connection; and

$$\Phi_A = \frac{1}{2} \Phi_A{}^{KL} \mathcal{N}_{KL} \ , \qquad \mathcal{N}_{KL} = -\mathcal{N}_{LK} \qquad \text{SO}(\mathcal{N})\text{-connection}$$

$\mathcal{N}$-extended AdS superspaces II

AdS$_3$ supergeometry: Special backgrounds originating within the supergeometry for 3D $\mathcal{N}$-extended conformal supergravity

$$\begin{aligned}
\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} &= 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}S^{IJ}\mathcal{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\Big(X^{IJKL} - 4S^{K[I}\delta^{J]L}\Big)\mathcal{N}_{KL} \ , \\
[\mathcal{D}_a, \mathcal{D}^J_\beta] &= S^J{}_K(\gamma_a)_\beta{}^\gamma\mathcal{D}^K_\gamma \ , \\
[\mathcal{D}_a, \mathcal{D}_b] &= 4\,S^2\,\varepsilon_{abc}\mathcal{M}^c = -4\,S^2\,\mathcal{M}_{ab} \ .
\end{aligned}$$

The structure-group indices are raised and lowered using δ^{IJ} and δ_{IJ}.

$$X^{IJKL} = X^{[IJKL]} \ , \qquad S^{IJ} = S^{(IJ)} \ .$$

The torsion and curvature tensors are covariantly constant.

$$\mathcal{D}^I_\alpha S^{JK} = \mathcal{D}_a S^{JK} = 0 \ , \quad \mathcal{D}^I_\alpha X^{JKLM} = \mathcal{D}_a X^{JKLM} = 0 \ .$$

Integrability conditions I

Integrability conditions

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} S^{KL} = 0 \ , \qquad \{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} X^{KLMN} = 0$$

are equivalent to the following algebraic constraints:

$$\begin{aligned}
0 &= X^{IJN(K} S_N{}^{L)} - S^{IM} S_M{}^{(K} \delta^{L)J} + S^{JM} S_M{}^{(K} \delta^{L)I} \ , \\
0 &= X^{IJN[K} X_N{}^{LPQ]} + 2 S^{M[I} \delta^{J]N} \delta^{[K}_M X_N{}^{LPQ]} - 2 S^{M[I} \delta^{J]N} \delta^{[K}_N X_M{}^{LPQ]} \ .
\end{aligned}$$

Bianchi identities

$$\sum_{[ABC)} [\mathcal{D}_A, [\mathcal{D}_B, \mathcal{D}_C\}\} = 0$$

imply

$$\hat{S}^2 = S^2 \mathbb{1} \ , \qquad \hat{S} := (S^{IJ}) = \hat{S}^{\mathrm{T}} \ , \qquad S^2 := \frac{1}{\mathcal{N}} \mathrm{tr}(\hat{S}^2) \geq 0 \ .$$

Thus $\hat{S}$ is a nonsingular symmetric $\mathcal{N} \times \mathcal{N}$ matrix, and hence $\hat{S}/S$ is an orthogonal matrix, in the case $S^2 > 0$.

Integrability conditions II

Making use of $\hat{S}^2 = S^2\mathbb{1}$ simplifies the integrability conditions

$$
\begin{aligned}
(*) \qquad 0 &= S^{(K}{}_N X^{L)IJN} , \\
(**) \qquad 0 &= X_N{}^{IJ[K} X^{LPQ]N} + S^{I[K} X^{LPQ]J} - S^{J[K} X^{LPQ]I} \\
&\quad - S^{IM} X_M{}^{[LPQ} \delta^{K]J} + S^{JM} X_M{}^{[LPQ} \delta^{K]I} .
\end{aligned}
$$

A local SO($\mathcal{N}$) transformation can be performed to diagonalise $\hat{S}$. Without loss of generality, the general solution of $\hat{S}^2 = S^2\mathbb{1}$ is

$$
S^{IJ} = S\,\mathrm{diag}(\overbrace{+1, \cdots, +1}^{p}, \overbrace{-1, \cdots, -1}^{q=\mathcal{N}-p}) , \quad S > 0
$$

(p, q) AdS superspace

Constraint $(*)$ implies that

$$
q > 0 \quad \longrightarrow \quad X^{IJKL} = 0 .
$$

In the $(n, 0)$ case, constraint $(**)$ reduces to

$$
X_N{}^{IJ[K} X^{LPQ]N} = 0
$$

(Deformed) Minkowski superspaces

$$S^2 = 0 \quad \longleftrightarrow \quad S^{IJ} = 0$$

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}X^{IJKL}\mathcal{N}_{KL} \ ,$$
$$[\mathcal{D}_a, \mathcal{D}^J_\beta] = 0 \ , \qquad [\mathcal{D}_a, \mathcal{D}_b] = 0 \ .$$

This superspace is of Minkowski type for $\mathcal{N} = 1, 2, 3$.
In the case $\mathcal{N} \geq 4$, there may exist a non-zero constant antisymmetric tensor X^{IJKL} constrained by

$$X_N{}^{IJ[K}X^{LPQ]N} = 0 \ ,$$

so that the resulting superspace is a deformation of $\mathcal{N}$–extended Minkowski superspace.

$X^{IJKL} = \varepsilon^{IJKL}X$ for $\mathcal{N} = 4$.

AdS supersymmetry and target space geometry

3D AdS supersymmetry imposes severe restrictions on the target space geometry of supersymmetric sigma models, as compared with the super-Poincare case.

Important special cases
$\mathcal{N}=3$ or $\mathcal{N}=4$ Poincaré supersymmetry
σ-model target spaces are arbitrary hyperkähler manifolds.
$\mathcal{N}=3$ or $\mathcal{N}=4$ AdS supersymmetry
σ-model target spaces are hyperkähler manifolds of restricted type.

We classified all possible types of hyperkähler target space geometry for the cases $\mathcal{N}=3$ and $\mathcal{N}=4$ in AdS by developing two different realizations for the most general (p,q) supersymmetric sigma models:
(i) an off-shell formulation in terms of $\mathcal{N}=3$ and $\mathcal{N}=4$ projective supermultiplets; and
(ii) an on-shell formulation in terms of covariantly chiral superfields in (2,0) AdS superspace.

AdS supersymmetry and target space geometry: $\mathcal{N} = 3$

- (3,0) AdS SUSY: For any supersymmetric sigma model, its target space must be a hyperkähler cone.
  Hyperkähler cones are the target spaces of 3D $\mathcal{N} = 3$ and $\mathcal{N} = 4$, 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ superconformal sigma models.
  All hyperkähler cones are non-compact.
- (2,1) AdS SUSY: Target space must must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations of the two-sphere of complex structures.

SMK, U. Lindström & G. Tartaglino-Mazzucchelli, arXiv:1205.4622

Target spaces of (2,1) supersymmetric sigma models in AdS_3
is the same as those of $\mathcal{N} = 2$ supersymmetric sigma models in AdS_4

Butter & SMK arXiv:1105.3111

and $\mathcal{N} = 1$ supersymmetric sigma models in AdS_5

Bagger & Xiong, arXiv:1105.4852

HyperKähler geometry

A Riemannian manifold $(\mathcal{M}_{4n}, g)$ is called hyperKähler if it is endowed with three complex structure $(J_A)^\mu{}_\nu$, where $A = 1, 2, 3$, such that

- they obey the quaternion algebra

$$J_A J_B = -\delta_{AB} \mathbb{1} + \varepsilon_{ABC} J_C \ ;$$

- $\mathcal{M}_{4n}$ is Kähler with respect to each of them.

Unlike Kähler metrics, hyperKähler metrics are difficult to construct explicitly. The points is that all hyperKähler manifolds are Ricci flat

$$R_{\mu\nu} = 0$$

and therefore they are Einstein spaces.
Powerful techniques to generate hyperKähler metrics come from physics: Extended supersymmetric nonlinear σ-models.

HyperKähler geometry II

Introduce local complex coordinates, ϕ^a, with respect to one of the complex structures on $\mathcal{M}_{4n}$,

$$J_3 = \begin{pmatrix} \mathrm{i}\,\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar a}{}_{\bar b} \end{pmatrix} .$$

Two other complex structures can be chosen to look like

$$J_1 = \begin{pmatrix} 0 & g^{a\bar c}\omega_{\bar c\bar b} \\ g^{\bar a c}\omega_{cb} & 0 \end{pmatrix} , \qquad J_2 = \begin{pmatrix} 0 & \mathrm{i}\,g^{a\bar c}\omega_{\bar c\bar b} \\ -\mathrm{i}\,g^{\bar a c}\omega_{cb} & 0 \end{pmatrix} ,$$

where $\omega_{ab}(\phi)$ and $\omega_{\bar a\bar b}(\bar\phi)$ are covariantly constant (2,0) and (0,2) two-forms on $\mathcal{M}_{4n}$ such that

$$\omega^{ac}\omega_{cb} = -\delta^a{}_b \ , \qquad \omega^{ab}(\phi) := g^{a\bar c}(\phi,\bar\phi) g^{b\bar d}(\phi,\bar\phi)\,\omega_{\bar c\bar d}(\bar\phi) \ .$$

Kähler cones

A Kähler manifold $(\mathcal{M}, g_{a\bar{b}})$ parametrized by local complex coordinates ϕ^a is called a Kähler cone if it has a homothetic conformal Killing vector or infinitesimal dilatation

$$\chi = \chi^a \frac{\partial}{\partial \phi^a} + \bar{\chi}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \equiv \chi^\mu \frac{\partial}{\partial \varphi^\mu}$$

with the property

$$\nabla_\nu \chi^\mu = \delta_\nu{}^\mu \quad \Longleftrightarrow \quad \nabla_b \chi^a = \delta_b{}^a \ , \qquad \nabla_{\bar{b}} \chi^a = \partial_{\bar{b}} \chi^a = 0 \ .$$

In particular, χ is holomorphic. The properties of χ include the following:

$$\chi_a := g_{a\bar{b}}\, \bar{\chi}^{\bar{b}} = \partial_a K \quad \Longrightarrow \quad \chi^a K_a = K \ ,$$

where $K := g_{a\bar{b}}\, \chi^a \bar{\chi}^{\bar{b}}$ can be used as global Kähler potential, $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$. Complex coordinates for $\mathcal{M}$ can be chosen such that

$$\chi = \phi^a \frac{\partial}{\partial \phi^a} + \bar{\phi}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \quad \longrightarrow \quad \phi^a K_a(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) \ .$$

Hyperkähler cones

A hyperkähler cone is simply a hyperkähler manifold $(\mathcal{M}, g_{\mu\nu}, J_A{}^\mu{}_\nu)$ admitting an infinitesimal dilatation χ. Here $J_A{}^\mu{}_\nu$ are the three integrable complex structures obeying the quaternion algebra

$$J_A J_B = -\delta_{AB}\mathbb{I} + \varepsilon_{ABC} J_C \ .$$

Associated with the conformal Killing vector field χ are three Killing vectors $X_A{}^\mu := J_A{}^\mu{}_\nu \chi^\nu$, which leave the Kähler potential invariant, $X_A{}^\mu \partial_\mu \mathcal{K} = 0$. These obey the $\mathrm{SU}(2)$ algebra

$$[X_A, X_B] = -2\varepsilon_{ABC} X_C \ .$$

Hyperkähler spaces with U(1) isometry group rotating the complex structures

Let V^μ be the Killing vector generating the group U(1). Without loss of generality, V^μ is holomorphic w.r.t. J_3

$$\mathcal{L}_V J_1 = -J_2 \ , \quad \mathcal{L}_V J_2 = +J_1 \ , \quad \mathcal{L}_V J_3 = 0 \ .$$

The three closed Kähler two-forms are

$$\Omega_A = \frac{1}{2}(\Omega_A)_{\mu\nu}\,\mathrm{d}\phi^\mu \wedge \mathrm{d}\phi^\nu \ , \qquad (\Omega_A)_{\mu\nu} = g_{\mu\rho}(J_A)^\rho{}_\nu \ .$$

From Ω_1 and Ω_2 we construct the complex $(2,0)$ and $(0,2)$ forms with respect to $\mathcal{J}_3$

$$\Omega_\pm = \frac{1}{2}\Omega_1 \pm \frac{\mathrm{i}}{2}\Omega_2 \ , \qquad \mathcal{L}_V \Omega_\pm = \pm\mathrm{i}\,\Omega_\pm. \tag{1.1}$$

Ω_+ is holomorphic with respect to J_3.
Ω_+ and Ω_- prove to be exact. Indeed, $\rho_+ := -\mathrm{i}\,\imath_V\Omega_+$ is a holomorphic (1,0) form with respect to $\mathcal{J}_3$. Using $\mathcal{L}_V = \mathrm{d}\imath_V + \imath_V\mathrm{d}$ gives $\mathrm{d}\rho_+ = \Omega_+$.
Because some of the Kähler two-forms are exact, $\mathcal{M}$ must be a non-compact manifold.

AdS supersymmetry and target space geometry: $\mathcal{N} = 4$

Target spaces of 3D $\mathcal{N} = 4$ sigma models in AdS are decomposable

$$\mathcal{M}_L \times \mathcal{M}_R$$

where $\mathcal{M}_L$ and $\mathcal{M}_R$ are certain hyperkähler manifolds.

- (3,1) AdS SUSY: For any supersymmetric sigma model, its left and right target spaces must be hyperkähler cones.
- (2,2) AdS SUSY: Left and right target spaces must must be non-compact hyperkähler manifolds endowed with a Killing vector field which generates an SO(2) group of rotations of the two-sphere of complex structures.

The story is much more interesting in the (4,0) case.

(4,0) AdS superspace

Geometry

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}\, S\, \delta^{IJ}\mathcal{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\Big(X\varepsilon^{IJKL}\mathcal{N}_{KL} - 4\, S\,\mathcal{N}^{IJ}\Big) ,$$
$$[\mathcal{D}_a, \mathcal{D}^J_\beta] = S\,(\gamma_a)_\beta{}^\gamma\mathcal{D}^J_\gamma , \qquad [\mathcal{D}_a, \mathcal{D}_b] = -4\, S^2\,\mathcal{M}_{ab} .$$

X is a free parameter that does not affect the bosonic AdS.
The algebra simplifies if we switch from SO(4) isovector indices to pairs of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ isospinor indices making use of the isomorphism $\mathrm{SO}(4) \cong \big(\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R\big)/\mathbb{Z}_2$.

$$\{\mathcal{D}^{i\bar{i}}_\alpha, \mathcal{D}^{j\bar{j}}_\beta\} = 2\mathrm{i}\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}}(2S+X)\mathbf{L}^{ij} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}(2S-X)\mathbf{R}^{\bar{i}\bar{j}}$$
$$-4\mathrm{i}\, S\,\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{M}_{\alpha\beta} ,$$
$$[\mathcal{D}_a, \mathcal{D}^{j\bar{j}}_\beta] = S\,(\gamma_a)_\beta{}^\gamma\mathcal{D}^{j\bar{j}}_\gamma , \qquad [\mathcal{D}_a, \mathcal{D}_b] = -4\, S^2\,\mathcal{M}_{ab} .$$

Critical case: $X = \pm 2S$
Different isometry groups depending on the choice of X.

AdS supersymmetry and target space geometry: $\mathcal{N} = 4$

- (4,0) AdS SUSY with $X = 0$: For any supersymmetric sigma model, its left and right target spaces must be hyperkähler cones. The sigma model is superconformal.
- (4,0) AdS SUSY with $X \neq \pm 2S$: For any supersymmetric sigma model, its left and right target spaces must be hyperkähler cones. The sigma model is not superconformal. The presence of X leads to non-trivial scalar potentials in both sectors.
- (4,0) AdS SUSY with $X = \pm 2S$: One of the two target spaces, left or right, must be a hyperkähler cone (nontrivial scalar potential). The other target space is an arbitrary hyperkähler manifold; in particular, it may be compact.

Deformed Minkowski superspace

$$\{\mathcal{D}_\alpha^{i\bar{i}}, \mathcal{D}_\beta^{j\bar{j}}\} = 2\mathrm{i}\varepsilon^{ij}\varepsilon^{\bar{i}\bar{j}}\mathcal{D}_{\alpha\beta} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{\bar{i}\bar{j}} X \mathbf{L}^{ij} - 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij} X \mathbf{R}^{\bar{i}\bar{j}} ,$$
$$[\mathcal{D}_a, \mathcal{D}_\beta^{j\bar{j}}] = 0 , \qquad [\mathcal{D}_a, \mathcal{D}_b] = 0$$

The presence of X leads to the appearance of nontrivial potentials in both left and right sectors.

New mechanism to generate massive sigma models:
The potential is generated by a non-zero expectation value of the supersymmetric Cotton tensor.

$$V = \frac{1}{4} X^2 g_{a\bar{b}} \chi^a \bar{\chi}^{\bar{b}}$$

V^a homothetic conformal Killing vector field.