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Spectral Properties of Quaternionic Unit Gain Cycles

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SPECTRAL PROPERTIES OF QUATERNIONIC UNIT GAIN CYCLES

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Abstract

The quaternions are a non-commutative division ring that extends the complex numbers. A gain graph is a simple graph together with a gain function that assigns a value from an arbitrary group to each edge of the graph. We can define certain concepts on these graphs such as adjacency and Laplacian matrices, gains of paths, and more. If we restrict ourselves to the unit norm quaternions, we can define quaternionic unit gain graphs, or $U(\mathbb{H})$-gain graphs, as gain graphs where the domain of the gain function is the unit quaternions. Traditional methods from spectral graph theory are not directly extended to quaternionic unit gain graphs due to non-commutativity. In this thesis, we extend a previous result from complex unit gain graphs, so that the right eigenvalues of the adjacency matrix for a $U(\mathbb{H})$-gain cycle can be written explicitly from the gain of the cycle. A thorough treatment of quaternions, quaternionic linear algebra, $\mathbb{T}$-gain graphs, and $U(\mathbb{H})$-gain cycles is given. At the end, the right eigenvalues are calculated for a particular $U(\mathbb{H})$-gain cycle, and the results are compared to those obtained from a MATLAB method for approximating them.

Contents

Chapter 1

Quaternions

1.1 Introduction

A **quaternion**, is a vector

$$q = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$$

with coefficients $q_i \in \mathbb{R}$. Note that we will typically refer to the basis element $\mathbf{1}$ as simply the real number 1. The space of quaternions, denoted $\mathbb{H} = \mathbb{C}^2 = \mathbb{R}^4$, is a division ring, or skew-field. Addition and multiplication on $\mathbb{H}$ follow the typical distributive laws, but multiplication, in particular, is non-commutative. Multiplication of the basis elements $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is given by the following:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Note that this definition implies

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}$$

The **conjugate** of q is another quaternion given by

$$q^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$

We then note

$$q^*q = qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2 = |q|^2$$

With this, we can define the **norm** of q as

$$|q| = \sqrt{q^*q}$$

which is equivalent to the definition of the Euclidean norm on $\mathbb{R}^4$

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

The inverse of $q \in \mathbb{H}\backslash\{0\}$ is then given by

$$q^{-1} = \frac{q^*}{|q|^2} = \frac{q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}}{q_0^2 + q_1^2 + q_2^2 + q_3^2} \in \mathbb{H}$$

It is trivial to check that this definition of q^{-1} is in fact the quaternionic inverse of q.

We define the following functions. The **real part** and **imaginary part** of a quaternion q are given by

$$Re(q) : \mathbb{H} \to \mathbb{R} \equiv q \mapsto q_0$$

and

$$Im(q) : \mathbb{H} \to \mathbb{R}^3 \equiv q \mapsto (q_1, q_2, q_3) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

We note that

$$|q|^2 = |Re(q)|^2 + |Im(q)|^2 = (q_0^2) + (q_1^2 + q_2^2 + q_3^2)$$

This fact will be used later in Section 1.3, and will be important for the later results in Chapter 3.

Some useful properties of quaternions are given in the following theorem.

Theorem 1.1. *[14] Let $q, w, z \in \mathbb{H}$. The following are true*

1. $|\cdot|$ *is a norm on* $\mathbb{H}$*, namely*

$$|q| = 0 \Longleftrightarrow q = 0$$
$$|q + w| \leq |q| + |w|$$
$$|qw| = |wq| = |w||q|$$

2. $\boldsymbol{j}c = \bar{c}\boldsymbol{j}$ *or* $\boldsymbol{j}c\boldsymbol{j}^* = \bar{c} \qquad \forall c \in \mathbb{C}$

3. $(qw)^* = w^*q^*$

4. $(qw)z = q(wz)$

5. $q = q^* \iff q \in \mathbb{R}$

6. $aq = qa \quad \forall q \in \mathbb{H} \iff a \in \mathbb{R}$

7. $|q^{-1}| = \frac{1}{|q|}$

8. $\exists$ *unique* $c_1, c_2 \in \mathbb{C} \ni q = c_1\boldsymbol{1} + c_2\boldsymbol{j}$

Proof. See [14] $\square$

There are numerous ways to represent the product of two quaternions. Since $\mathbb{H} \cong \mathbb{R}^4 \cong \mathbb{R} \bigoplus \mathbb{R}^3$, we can write a quaternion q as $q = q_0 + \vec{q}$, where $q_0 = Re(q) \in \mathbb{R}$ and $\vec{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. From this, we can define the product $qw \in \mathbb{H}$ as

$$qw = (q_0 + \vec{q})(w_0 + \vec{w}) = (q_0 w_0 - \vec{q} \cdot \vec{w}) + (q_0 \vec{w} + w_0 \vec{q} + \vec{q} \times \vec{w})$$

This definition is equivalent to typical component-wise multiplication, and the proof of this is left as an exercise to the reader. The following can be obtained from the above definition of multiplication.

Lemma 1.2. *Let* $q = q_0 + \vec{q}$ *and* $w = w_0 + \vec{w} \in \mathbb{H}$. *The following are true.*

1. $Re(qw) = Re(wq)$

2. $Im(qw) = Im(wq) - 2\vec{w} \times \vec{q}$

Proof. 1.

$$Re(qw) = q_0 w_0 - \vec{q} \cdot \vec{w} = w_0 q_0 - \vec{w} \cdot \vec{q} = Re(wq)$$

2.

$$\begin{aligned} Im(qw) &= q_0 \vec{w} + w_0 \vec{q} + \vec{q} \times \vec{w} \\ &= w_0 \vec{q} + q_0 \vec{w} - \vec{w} \times \vec{q} \\ &= w_0 \vec{q} + q_0 \vec{w} + \vec{w} \times \vec{q} - 2\vec{w} \times \vec{q} \\ &= Im(wq) - 2\vec{w} \times \vec{q} \end{aligned}$$

$\square$

There is one more representation for quaternions which can be useful. Since $\mathbf{ij} = \mathbf{k}$, we can write a quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ as

$$\begin{aligned} q &= q_0 + q_1\mathbf{i} + (q_2 + q_3\mathbf{i})\mathbf{j} \in \mathbb{H} \\ &= z_1 + z_2\mathbf{j} \qquad \text{where } z_1, z_2 \in \mathbb{C} \end{aligned}$$

This form will be used briefly in 1.4. Note that this form does not always have nice properties. For instance

$$q^* = z_1^* - z_2\mathbf{j} \neq z_1^* + z_2^*\mathbf{j} \neq z_1 - z_2\mathbf{j}$$

The two not-equal forms are pleasing to look at, but are unfortunately *not* correct. Nonetheless, this representation for q can be useful.

1.2 Equivalence Classes

There is a useful equivalence relation that we can define on $\mathbb{H}$. We call two quaternions, q and w, **similar** if there exists another quaternion, $u \in \mathbb{H}\backslash\{0\}$, such that

$$q = u^{-1}wu$$

We will denote this by $x \sim y$.

Lemma 1.3. *$\sim$ is an equivalence relation on $\mathbb{H}$*

Proof. Trivial. □

This equivalence relation allows us to define the *equivalence class* of $q \in \mathbb{H}$ as

$$\begin{aligned} [q] &= \{w \in \mathbb{H} \;\mid\; w \sim q\} \\ \equiv [q] &= \{h^{-1}qh \;\mid\; h \in \mathbb{H}\backslash\{0\}\} \end{aligned}$$

The use of this definition of similarity will become apparent in Sections 1.3 and 1.4, and later in Section 3.2.

The following lemma is given in several papers [3, 12].

Lemma 1.4. *If q and w are two quaternions, then $Re(q) = Re(w)$ and $|q| = |w|$ is equivalent to $q \sim w$.*

The following is immediate

Corollary 1.5. *$Re(q) = Re(w)$ and $|Im(q)| = |Im(w)|$ if and only if $q \sim w$.*

Proof. By Theorem 1.4 $\Rightarrow$

$$\begin{aligned} q_0^2 + q_1^2 + q_2^2 + q_3^2 &= w_0^2 + w_1^2 + w_2^2 + w_3^2 \\ q_0^2 + q_1^2 + q_2^2 + q_3^2 &= q_0^2 + w_1^2 + w_2^2 + w_3^2 \\ q_1^2 + q_2^2 + q_3^2 &= w_1^2 + w_2^2 + w_3^2 \end{aligned}$$

$\square$

Lemma 1.6. $\forall q \in \mathbb{H} \Rightarrow q \sim q^*$

Proof. $\because Re(q) = Re(q^*) \wedge |q| = |q^*| \Rightarrow q \sim q^*$ $\square$

The following states that the equivalence class of every quaternion contains a complex number. In other words, there always exists a unitary transformation from $q \in \mathbb{H}$ to $\lambda \in \mathbb{C} \cap [q]$.

Lemma 1.7. *If $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ and $q_{\mathbb{C}} = q_0 + \sqrt{q_1^2 + q_2^2 + q_3^2}\,\boldsymbol{i}$, then $q \sim q_{\mathbb{C}} \sim q_{\mathbb{C}}^*$.*

Proof. Clearly, $Re(q) = Re(q_{\mathbb{C}}) = Re(q_{\mathbb{C}}^*)$ and $|Im(q)| = |Im(q_{\mathbb{C}})| = |Im(q_{\mathbb{C}}^*)|$. The similarity follows immediately from Corollary 1.2. $\square$

Typical proofs of the above lemma demonstrate the particular quaternion used for the similarity transformation. This proof is clear as to why the quaternion $q_{\mathbb{C}}$ is the similar complex number. By the above, it is clear that, in fact, $q_{\mathbb{C}}$ and $q_{\mathbb{C}}^*$ are the only two complex numbers that are similar to q. From the above results, we can obtain

1.3 Unit Quaternions

The set of **unit quaternions** is defined as follows:

$$U(\mathbb{H}) = \{q \in \mathbb{H} \mid |q| = 1\}$$

The unit quaternions represent a sphere in $\mathbb{R}^4$ and are useful in many applications (rotations, computer graphics, etc.). Unit length quaternions have several properties that make them more useful than typical quaternions. For instance, if $q \in U(\mathbb{H})$ then $|q^{-1}| = 1$ and so $q^{-1} = q^*$.

Lemma 1.8. *If $q, w \in \mathbb{H} \ni q \sim w \Rightarrow \exists \nu \in U(\mathbb{H}) \ni q = \nu^* w \nu$*

Proof. $\exists u \in \mathbb{H} \ni q = u^{-1} w u$. Let $\nu = \frac{u}{|u|} \in U(\mathbb{H})$. Then $\nu^* w \nu = q$ $\square$

Corollary 1.9. *If $q \in \mathbb{H} \setminus \mathbb{R}$, then there exists two unique complex numbers $z, z^* \in [q] \cap \mathbb{C}$ where $|z| = |z^*| = 1$.*

In particular, we can denote the equivalence class, $[q]$, by one of its **complex identifiers**, $[z] = [q]$. The following lemma is also useful for unit quaternions.

Lemma 1.10. *For every $q, w \in U(\mathbb{H}) \Rightarrow qw \sim wq$*

Proof. By Lemma 1.2 $\Rightarrow Re(qw) = Re(wq)$. Since $|qw| = |wq|$, by Lemma 1.4 $\Rightarrow qw \sim wq$ $\square$

1.4 Quaternionic Linear Algebra

For the results in this thesis, we will require the notion of quaternionic matrices. The set $M_{m \times n}(\mathbb{H})$ is the set of all $m \times n$ matrices with elements in $\mathbb{H}$. Matrix addition and multiplication is defined in the usual way; however, since quaternions are non-commutative, we must define scalar multiplication on a particular side. For $A \in M_{m \times n}(\mathbb{H})$ and $q \in \mathbb{H}$ left scalar multiplication will be defined as

$$qA = qa_{ij} \quad \forall i, j$$

and right scalar multiplication as

$$A = a_{ij} q \quad \forall i, j$$

or left/right multiplication of all elements. It is trivial to check that all operations are associative. The following theorem comes from [14].

Theorem 1.11. $\forall A \in M_{m \times n}(\mathbb{H})$ *and* $B \in M_{n \times p}(\mathbb{H})$

1. $(\overline{A})^T = \overline{(A^T)} = A^*$

2. $(AB)^* = B^*A^*$

3. $\overline{AB} \neq \overline{A}\,\overline{B}$

4. $(AB)^T \neq B^TA^T$

5. $(AB)^{-1} = A^{-1}B^{-1}$

6. $(A^{-1})^* = (A^*)^{-1}$

7. $(\overline{A})^{-1} \neq \overline{(A^{-1})}$

8. $(A^{-1})^T \neq (A^T)^{-1}$

Where there are matrices, there are eigenvalues. Due to non-commutativity, we must define two very different eigenvalue equations. Given $A \in M_{n\times n}(\mathbb{H}) = M_n(\mathbb{H})$ and $\mathbf{x} \in \mathbb{H}^n$ we say λ is a **left eigenvalue** of A if

$$A\mathbf{x} = \lambda\mathbf{x}$$

or λ is a **right eigenvalue** of A if

$$A\mathbf{x} = \mathbf{x}\lambda$$

Upon first examination, these two equations may appear to be similar, but they in fact *very* different. Right eigenvalues are particularly well understood, whereas left eigenvalue remain largely untouched. This division occurs because the right eigenvalue equation is a linear equation; the linearity lends itself nicely to proofs. The non-linearity of the right eigenvalue equation causes many issues. For instance, a direct algebraic proof showing the existence of right eigenvalues exists, but only a topological proof of the existence of left eigenvalues has been found to date [2]. Left eigenvalues will no longer be mentioned in this thesis (see for more). From this point on, all mentions of eigenvalues will assume right eigenvalues.

We will denote the **spectrum** of A as $\sigma(A)$, which consists of all of the [right] eigenvalues of A. Quaternionic eigenvalue problems have several nice properties. Let $A \in M_n(\mathbb{H})$.

Lemma 1.12. *If $\lambda \in \sigma(A)$, then $[\lambda] \subset \sigma(A)$.*

Corollary 1.13. *If $\lambda \in \sigma(A)$, then $q^{-1}\lambda q \in \sigma(A)$ for all $q \in \mathbb{H}$.*

Lemma 1.14. *$\sigma(A)$ consists of* n *distinct equivalence classes.*

Lemma 1.15. *If A is Hermitian (i.e. $A^* = A$) then $\sigma(A) \subset \mathbb{R}$.*

Lemma 1.16. *If $U \in M_n(\mathbb{H})$ is unitary (i.e. $U^* = U^{-1}$) then $\sigma(A) = \sigma(U^*AU)$.*

Proof. Suppose $U \in M_n(\mathbb{H})$ is unitary.
$\Rightarrow (U^*AU)(U^*\mathbf{x}) = U^*A\mathbf{x} = (U^*\mathbf{x})\lambda$ □

Given $A \in M_n(\mathbb{H})$, we define the **complex representative** of A as

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$$

where $A = A_1 + A_2\mathbf{j}$ (recall the representation at the end of Section 1.1), and $A_1, A_2 \in M_n(\mathbb{C})$. Note again that

$$\overline{A} \neq \overline{A_1} + \overline{A_2}\mathbf{j}, \quad A^* \neq A_1^* + A_2^*\mathbf{j}$$

From this point on, the right eigenvalue theory for quaternionic matrices becomes very simple. The following theorem is found in many papers.

Theorem 1.17. *Suppose $A \in M_n(\mathbb{H})$. $\sigma(\chi_A) = \sigma(A) \cap \mathbb{C}$.*

Essentially, we can find all of the complex eigenvalues of A by finding the eigenvalues of its complex representative. Moreover, since $q \in \sigma(\chi_A) \Rightarrow q^* \in \sigma(\chi_A)$, by looking at the n distinct equivalence classes of the elements of $\sigma(\chi_A)$, we have found all of the right eigenvalues of A!

For the results in Chapter 3, we don't actually need the above theorem. However, we can use MATLAB to find the eigenvalues of χ_A and check the results that we find in Section 3.4 (which we will do).

The theory of quaternionic linear algebra is rapidly growing, with many papers being published [4, 6, 14, 12, 3, 2]. [4] also includes some possible applications of quaternionic linear algebra, particularly in physics. Outside of algebra, quaternionic analysis and differential equations are also being developed [1, 5, 8]. These are far beyond the scope of this paper, however.

Chapter 2

Graphs

2.1 Introduction & Gain Graphs

A **graph** is a pair $G = (V, E)$ where V is the set of **vertices** and E is the set of **edges**. In particular, E is a set of 2-element sets with elements in V, i.e. $\{v_1, v_2\} \in E$, abbreviated as e_{12}, is an edge from v_1 to v_2. The **degree** of a vertex is the number of incident edges on that degree. A **simple graph** is a graph where we require

1. $e_{ii} \notin E$ for all i

2. $e_{ij} = e_{ji} \in E$

That is, we do not allow self-edges, and there are no directions to the edges. This is of course different from the case of **directed graphs**, where we relax this symmetric condition. A **symmetric directed graph**, in particular, is a directed graph where the second condition is changed to $e_{ij} \in E \iff e_{ji} \in E$. At first glance this may appear equivalent to the definition of a simple graph. The difference is that the edges e_{ij} and e_{ji} are distinct from one another. This will be important for gain graphs.

The last basic type of graph we will mention here is **signed graphs**. A signed graph is a pair $\Phi = (G, \varphi)$ where G is a graph, and $\varphi : E \to \{\pm 1\}$ is a function that assigns a value of 1 or -1 to each edge of the graph. An example of this is shown in Figure 2.1. We will not look at signed graphs here, but their notion is helpful in building gain graphs.

A $\mathbb{G}$**-gain graph** is a triple $\Phi = (G, \mathbb{G}, \varphi)$ where G is a directed graph, $\mathbb{G}$ is

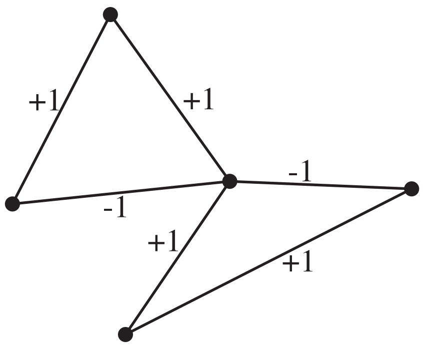


Figure 2.1: Example of a signed graph.

a group, and $\varphi : E \to \mathbb{G}$ is a function assigning values from the group $\mathbb{G}$ to each edge of the directed graph. Furthermore, we require the following of φ

$$\varphi(e_{ji}) = \varphi(e_{ij})^{-1}$$

or that the value of the reverse direction of an edge is equal to the inverse of that value. We denote the value associated with an edge as the **gain of that edge**. Note that this requirement on φ implies that Φ is, in fact, a symmetric directed graph. An example of a $(\mathbb{Z}_3, + \bmod 3)$-gain graph is shown in Figure 2.2. Gain graphs generalize the notion of many other graphs. Namely, a symmetric directed graph is the case of a $(\{1\}, \cdot)$-gain graph, and a signed graph is a $(\{\pm 1\}, \cdot)$-gain graph.

Associating a group to a graph allows us to look at various algebraic properties of these graphs. Before we can do that, there are still a few definitions we will need to know.

A **walk** in a graph is a finite sequence of edges which joins together vertices of the graph. A **path** is a walk with the additional requirement that each vertex and edge is only visited once. We will adopt a shorthand notation for a path, $p = e_1 e_2 \ldots e_n$ where $e_i = e_{jk}$ for some vertices v_j, v_k, and $e_{i+1} = e_{kl}$ for the same vertex v_k and a new vertex v_l. If Φ is a gain graph, we define the **gain of a path** to be the product of all of the gains of the edges in the path. Or rather, if $p = e_1 e_2 \ldots e_n$ is a path in Φ, then $\varphi(p) = \prod_{i=1}^{n} \varphi(e_i)$.

A **cycle** in a graph is a path that starts and ends at the same vertex, i.e. $c = e_1 e_2 \ldots e_n$ where $e_1 = e_{ij}$ and $e_n = e_{mi}$. The **gain of a cycle** is defined in

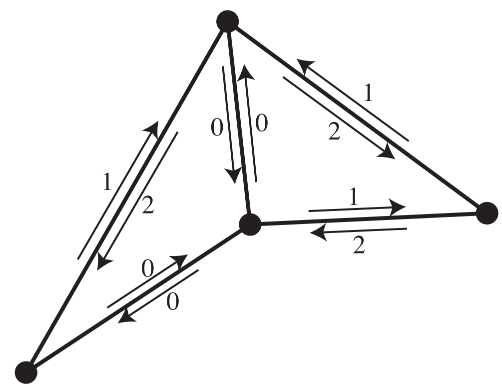


Figure 2.2: Example of $(Z_3, + \bmod 3)$-gain graph. Note that the backward direction will be the additive inverse $\pmod 3$. i.e. for this group, the addition of the forward and backward gains will equal 0 in the group.

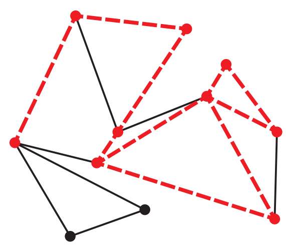
exactly the same way as the gain of a path. Examples of a walk, path, and cycle in an arbitrary graph are given in Figure 2.3.

The specific type of graph we focus on in this thesis is a cycle graph. A **cycle graph** is graph C_n with n vertices which consists of a single cycle. C_n can be drawn as a regular n-gon, and the degree of each vertex is 2. We will, of course, assume that C_n is a symmetric directed graph. Lastly, we define a $\mathbb{G}$-cycle graph as a gain graph $\mathcal{C}_n = (C_n, \mathbb{G}, \varphi)$. The **gain of a cycle graph**, $\varphi(\mathcal{C}_n)$, is naturally defined as the gain of its cycle.

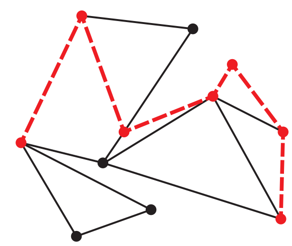
As with quaternions, gain graphs themselves have been increasingly studied within the last few years [11, 7], complex unit gain graphs (which we will talk about later), in particular[10, 9, 13]. Many of these results may be extendable to the quaternionic unit gain graph case, but to date very little has been done in this regard. In this thesis, a result from [10] is extended for complex unit cycle graphs.

2.2 Representations

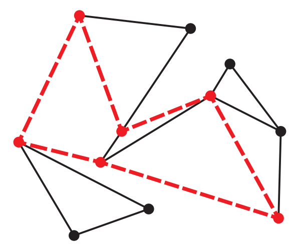
There are many ways of representing a graph. We will look at several of the algebraic ways to do so here. For the following, let $\Phi = (G, \mathbb{G}, \varphi)$ be a $\mathbb{G}$-gain



(a) Walk



(b) Path



(c) Cycle

Figure 2.3: Examples of a walk, path, and cycle within the same graph.

graph with n vertices.

The **adjacency matrix** $A(\Phi)$ of Φ is an $n \times n$ matrix with elements $(a_{ij}) \in \mathbb{G} \cup \{0\}$ defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } e_{ij} \in E \\ 0 & \text{otherwise;} \end{cases}$$

i.e. the (i, j)-th element of A corresponds to the gain of the edge e_{ij}.

The **degree matrix** $D(G)$ of the underlying graph G is an $n \times n$ matrix with elements $(d_{ij}) \in \mathbb{Z}_{\geq 0}$ defined by

$$d_{ij} = \begin{cases} degree(v_i) & \text{if } i = j \\ 0 & \text{if} i \neq j; \end{cases}$$

i.e. D is a diagonal matrix with elements corresponding to the degree of each vertex.

The **incidence matrix** $H(\Phi)$ of Φ is an $n \times |E|$ matrix with elements $(h_{v_i e} \in \mathbb{G} \cup \{0\}$ defined by

$$h_{v_i e} = \begin{cases} -\eta_{v_j e}\varphi(e_{ij}) & \text{if } e = e_{ij} \in E \\ 0 & \text{otherwise;} \end{cases}$$

Lastly, the **Laplacian matrix** $L(\Phi)$ of Φ is define as $L(\Phi) = D(G) - A(\Phi)$.

Just looking at $\mathbb{G}$-gain graphs would be a difficult task. To look for more interesting algebraic results, we need to define what group we are actually looking at.

2.3 Complex Unit Gain Graphs

A **complex unit gain graph** is a gain graph $\Phi = (G, U(\mathbb{C}), \varphi)$ where the underlying group is the set of complex numbers with unit length $U(\mathbb{C}) = \mathbb{T}$. The adjacency, degree, incidence, and Laplacian matrices are defined exactly as before. From [10] we have the following

Lemma 2.1. *Let Φ be a $\mathbb{T}$-gain graph. Then $L(\Phi) = D(G) - A(\Phi) = H(\Phi)H(\Phi)^*$, where * denotes the complex adjoint.*

Many of the results from the theory of $\mathbb{T}$-gain graphs are unnecessary for the purposes of this paper, and will be omitted. For further information see [10, 13, 9].

2.3.1 Switching

Switching is a way to change the graph we are looking at, while keeping several properties of the graphs intact. The properties we are mainly interested in are the spectral properties of the representative matrices. That is, we are interested in finding $\sigma(A(\Phi))$ and $\sigma(L(\Phi))$. Gain functions have the property $\varphi(e_{ji}) = \varphi(e_{ij})^{-1}$. For $z \in \mathbb{T} \Rightarrow z^{-1} = z^*$, so it is obvious that the adjacency and Laplacian matrices of a $\mathbb{T}$-unit gain graph are Hermitian, which of course implies that their eigenvalues will be real. In general, finding these eigenvalues is a difficult task. For certain cases, such as cycle graphs, finding the eigenvalues can be greatly simplified.

Let $\Phi = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph. A switching function $\zeta : V \to \mathbb{T}$ assigns a complex unit to each vertex of Φ. We can generate a new graph $\Phi^\zeta = (G, \mathbb{T}, \varphi^\zeta)$ where $\varphi^\zeta : E \to \mathbb{T}$ is defined by

$$\varphi^\zeta(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j) = \zeta(v_i)^*\varphi(e_{ij})\zeta(v_j).$$

Note that since all $z \in \mathbb{T}$ have unit length, $\Rightarrow z^{-1} = z^*$. Two gain graphs Φ_1, Φ_2 are said to be **switching equivalent** if there exists a switching function ζ such that $\Phi_1 = \Phi_2^\zeta$. This definition is, in fact, an equivalence relation which we will denote $\sim_\zeta$. We therefore define the **switching class** of Φ as

$$[\Phi]_\zeta = \{\Gamma \mid \Gamma \sim_\zeta \Phi\}$$

We define the **switching matrix** of ζ, Z_ζ, by $(z_{ij} = diag(\zeta))$. That is

$$z_{ij} = \begin{cases} \zeta(v_i) & \text{if } i = j \\ 0 & \text{if } i \neq j; \end{cases}$$

The following lemma shows that this definition of a switching matrix corresponds to the correct action on a gain graph.

Lemma 2.2. *Let Φ be a $\mathbb{T}$-gain graph and ζ be a switching function. The following are true:*

- $A(\Phi^\zeta) = Z_\zeta^* A(\Phi) Z_\zeta$
- $L(\Phi^\zeta) = Z_\zeta^* L(\Phi) Z_\zeta$

The following is also true of the switching matrix.

Lemma 2.3. *Let ζ be a switching function on an n-vertex gain graph. Z_ζ is unitary. i.e. $Z_\zeta^* Z_\zeta = I_n$*

The proof of the above is by direct computation, and its result is extremely important for the following:

Theorem 2.4. *For all $\Gamma \in [\Phi]_\zeta$ the following are true*

- $\sigma(A(\Gamma)) = \sigma(A(\Phi))$
- $\sigma(L(\Gamma)) = \sigma(L(\Phi))$

Proof. Since $\Gamma \sim_\zeta \Phi$, there exists a switching function ζ such that $\Gamma = \Phi^\zeta$. Since Z_ζ is unitary, we have the following

- $\sigma(A(\Gamma)) = \sigma(Z_\zeta^* A(\Phi) Z_\zeta) = \sigma(A(\Phi))$
- $\sigma(L(\Gamma)) = \sigma(Z_\zeta^* L(\Phi) Z_\zeta) = \sigma(L(\Phi))$

□

In other words, the eigenvalues for a gain graph's adjacency and Laplacian matrices are **switching invariant**. This will lead to a nice form for these eigenvalues for cycle graphs.

2.3.2 Eigenvalues for Cycles

A **complex unit gain cycle** is a gain graph $\mathcal{C}_n = (C_n, \mathbb{T}, \varphi)$ where C_n is an n-vertex cycle graph, and the underlying gain group is the group of all unit norm complex numbers, $\mathbb{T}$. The **gain of a $\mathbb{T}$-gain cycle graph** is defined as the gain of its only cycle.

Theorem 2.5. *For every $\mathbb{T}$-gain cycle, $\mathcal{C}_n = (C_n, \mathbb{T}, \varphi)$, there exists another $\mathbb{T}$-gain cycle $\Xi = (C_n, \mathbb{T}, \varphi')$ such that $\Xi \in [\mathcal{C}_n]_\zeta$ and $\varphi(\mathcal{C}_n) = \varphi'(\Xi)$ where $\varphi' : E \to \mathbb{T}$ is given by*

$$\varphi'(e_{n,1}) = \varphi(\mathcal{C}_n)$$

and

$$\varphi'(e_{i,i+1}) = 1 \qquad \textit{for } 0 \leq i < n$$

Proof. Let $\mathcal{C}_n$ be a $\mathbb{T}$-gain cycle. We define a switching function $\xi : V \to \mathbb{T}$ as follows:

$$\xi(v_i) = \begin{cases} 1 & i = 1 \\ \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} & i \neq 1 \end{cases}$$

In other words, the switch for the starting vertex is 1, and the switch for each vertex, i, after is the inverse of the gain of the path from vertices 1 to i. From ξ, we generate φ^ξ as:

$$\varphi^\xi(e_{i,i+1}) = \xi(v_i)^{-1}\varphi(e_{i,i+1})\xi(v_{i+1})$$

From the construction of ξ, this reduces to

$$\varphi^\xi(e_{12}) = \varphi(e_{12})\xi(v_2) = \varphi(e_{12})\varphi(e_{12})^{-1} = 1$$

for $i = 1$,

$$\begin{aligned}
\varphi^\xi(e_{i,i+1}) &= \xi(v_i)^{-1}\varphi(e_{i,i+1})\xi(v_{i+1}) \\
&= \left[\left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1}\right]^{-1} \varphi(e_{i,i+1}) \left(\prod_{j=1}^{i} \varphi(e_{j,j+1})\right)^{-1} \\
&= \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right) \varphi(e_{i,i+1})\varphi(e_{i,i+1})^{-1} \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} \\
&= \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)\left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} = 1
\end{aligned}$$

for $1 < i < n$, and

$$\begin{aligned}\varphi^{\xi}(e_{n,1}) &= \xi(n)^{-1}\varphi(e_{n,1}) \\ &= \Big[\Big(\prod_{j=1}^{n-1}\varphi(e_{j,j+1})\Big)^{-1}\Big]^{-1}\varphi(e_{n,1}) \\ &= \Big(\prod_{j=1}^{n-1}\varphi(e_{j,j+1})\Big)\varphi(e_{n,1}) \\ &= \varphi(e_{12})\varphi(e_{23})\dots\varphi(e_{n-1,n})\varphi(e_{n,1}) = \varphi(\mathcal{C}_n)\end{aligned}$$

for $i = n$.

Lastly, we have

$$\begin{aligned}\varphi^{\xi}(\Xi) &= \varphi^{\xi}(e_{12})\varphi^{\xi}(e_{23})\dots\varphi^{\xi}(e_{n,n-1})\varphi^{\xi}(e_{n,1}) \\ &= 1\cdot 1\cdot \dots \cdot 1\cdot\varphi(\mathcal{C}_n) \\ &= \varphi(\mathcal{C}_n)\end{aligned}$$

φ^{ξ} satisfies all results of the theorem. $\square$

This result tells us that given any $\mathbb{T}$-gain cycle, we can switch the cycle to an equivalent cycle with only one non-neutral edge ($\varphi(e) \neq 1$). This edge, in fact, contains the gain of the original cycle. Figure 2.4 shows a general $\mathbb{T}$-gain cycle with 7 vertices, along with the switching equivalent cycle based on this construction. The adjacency matrices for these graphs are

$$A(\mathcal{C}_7) = \begin{bmatrix} 0 & \varphi(e_{12}) & 0 & 0 & 0 & 0 & \varphi(e_{17}) \\ \varphi(e_{12})^* & 0 & \varphi(e_{23}) & 0 & 0 & 0 & 0 \\ 0 & \varphi(e_{23})^* & 0 & \varphi(e_{34}) & 0 & 0 & 0 \\ 0 & 0 & \varphi(e_{34})^* & 0 & \varphi(e_{45}) & 0 & 0 \\ 0 & 0 & 0 & \varphi(e_{45})^* & 0 & \varphi(e_{56}) & 0 \\ 0 & 0 & 0 & 0 & \varphi(e_{56})^* & 0 & \varphi(e_{67}) \\ \varphi(e_{17})^* & 0 & 0 & 0 & 0 & \varphi(e_{67})^* & 0 \end{bmatrix}$$

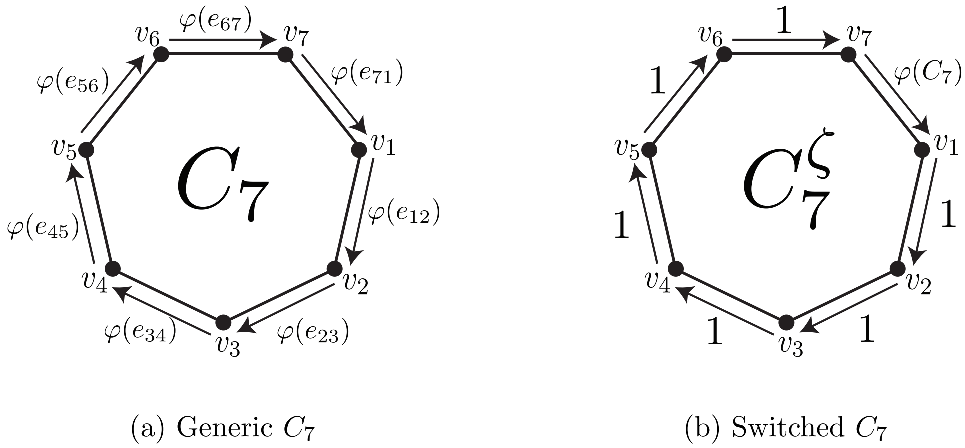


Figure 2.4: Examples of C_7, generic and switched. Note that the switched version has only one non-neutral edge, and that $\varphi(C_7) = \varphi(C_7^\zeta)$.

and

$$A(\mathcal{C}_7^\zeta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \varphi(\mathcal{C}_7)^* \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \varphi(\mathcal{C}_7) & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The adjacency matrix for a switched cycle can be decomposed into a useful form. Given a $\mathbb{T}$-unit gain cycle $\mathcal{C}_n$, we will define a new matrix, $P(\mathcal{C}_n) = P \in M_n(\mathbb{C})$, with elements $(p_{ij}) \in \mathbb{T} \cup \{0\}$ as follows

$$p_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ \varphi(\Phi) & \text{if } i = n, j = 1 \\ 0 & \text{otherwise;} \end{cases}$$

This matrix will look like

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varphi(\Phi) & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that if we switch $\mathcal{C}_n$ to $\mathcal{C}_n^\xi$, then $A(\mathcal{C}_n^\xi) = P + P^*$. It can easily be shown that P is unitary, implying that $A(\mathcal{C}_n^\xi) = P + P^{-1}$.

The following theorem is from [10], and its proof will be reproduced here.

Theorem 2.6. *Let $\mathcal{C}_n = (C_n, \mathbb{T}, \varphi)$ with $\varphi(\mathcal{C}_n) = e^{i\theta}$. Then*

$$\sigma(A(\mathcal{C}_n)) = \left\{2\cos\left(\frac{\theta + 2\pi j}{n}\right) \;\middle|\; j \in \{0, 1, \ldots, n-1\}\right\}$$

and

$$\sigma(L(\mathcal{C}_n)) = \left\{2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right) \;\middle|\; j \in \{0, 1, \ldots, n-1\}\right\}$$

Proof. Switch $\mathcal{C}_n$ to $\mathcal{C}_n^\xi$ with $A(\mathcal{C}_n^\xi) = P + P^{-1}$. Since switching is spectrum preserving, we have $\sigma(A(\mathcal{C}_n)) = \sigma(A(\mathcal{C}_n^\xi)) = \sigma(P + P^{-1})$. This implies that $\lambda \in A(\mathcal{C}_n)$ have the form $\lambda(A) = \lambda(P) + \lambda(P)^{-1}$. We will now find the eigenvalues of P.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ be an eigenvector of P with corresponding eigenvalue λ. Examine the eigenvalue equation

$$P\mathbf{x} = \lambda\mathbf{x}$$

Doing this multiplication out

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ e^{i\theta} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ e^{i\theta}x_1 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n \end{bmatrix}$$

This vector equality leads to the following n equations.

$$x_2 = \lambda x_1$$

$$x_3 = \lambda x_2$$

$$\vdots = \vdots$$
$$x_n = \lambda x_{n-1}$$
$$e^{i\theta} x_1 = \lambda x_n$$

With substitutions, these equations yield

$$x_2 = \lambda x_1$$
$$x_3 = \lambda^2 x_1$$
$$\vdots = \vdots$$
$$x_n = \lambda^{n-1} x_1$$
$$e^{i\theta} x_1 = \lambda^n x_1$$

We find that $\lambda \in \sigma(P)$ must satisfy $e^{i\theta} x_1 = \lambda^n x_1$. We end up with

$$e^{i\theta} x_1 = \lambda^n x_1 \Longleftrightarrow e^{i\theta} = \lambda^n \Longleftrightarrow \lambda = \left\{ e^{i(\frac{\theta + 2\pi j}{n})} \mid j = \{0, 1, \ldots, n-1\} \right\}$$

So that the eigenvalues of A have the form $\lambda(A) = \lambda(P) + \lambda(P)^1 = e^{i(\frac{\theta+2\pi j}{n})} + e^{-i(\frac{\theta+2\pi j}{n})}$. From Euler's equation, we find that the complex components cancel and yield $\lambda(A) = \left\{ 2\cos\left(\frac{\theta+2\pi j}{n}\right) \mid j \in \{0, 1, \ldots, n-1\} \right\}$, the result we were looking for.

Lastly, since each vertex in $\mathcal{C}_n$ has degree 2, we know that $L(\mathcal{C}_n) = 2I_n - A(\mathcal{C}_n)$. This implies that, in fact, $\lambda(A) = \left\{ 2 - 2\cos\left(\frac{\theta+2\pi j}{n}\right) \mid j \in \{0, 1, \ldots, n-1\} \right\}$. $\square$

The outcome of this theorem is the following: given an arbitrary $\mathbb{T}$-gain graph with gain $z \in \mathbb{T}$, we can find the eigenvalues of its adjacency and Laplacian matrices simply by taking the real part of the n-th roots of z. We have found all of the eigenvalues by knowing only the gain of the cycle. As we will see in the next chapter, this result can easily be generalized to gain graphs where the underlying group is $U(\mathbb{H})$, the unit quaternions that we discussed in Chapter 1.

Chapter 3

Quaternionic Unit Gain Graphs

3.1 Introduction

A **quaternionic unit gain graph** is a gain graph $\Phi = (G, U(\mathbb{H}), \varphi)$ where the underlying group is the set of quaterions with unit length $U(\mathbb{H})$. The adjacency, degree, incidence, and Laplacian matrices are defined exactly as in Section 2.2. Due to the non-commutative nature of quaternions, one would assume that many of the results from complex unit gain graphs would not be true. Generally speaking, this is actually not the case. Few (if any) of the results from Chapter 2 actually rely on commutativity. In fact, only Theorem 2.6 relied directly on the commutativity of complex numbers. That downfall, however, does not affect the results in the following sections.

Despite all of the similarities, there are two important differences that must be mentioned. First, since only the right eigenvalue equations are unitarily invariant, we can typically only find the adjacency and Laplacian right eigenvalues. Second, we must take care in dealing with the gain of a cycle. If $c = e_1 e_2 \ldots e_n$ where $e_1 = e_{ij}$ and $e_n = e_{mi}$ is a cycle in $\Phi = (G, U(\mathbb{H}), \varphi)$, then the **gain of the cycle starting at vertex** v_i is defined as

$$\varphi_i(c) = \prod_{\alpha=1\ldots n} \varphi(e_\alpha)$$

Because of non-commutativity the gain will change depending on what vertex we start at, so we *must* define a starting vertex. Despite this, we do have the following result.

Lemma 3.1. *Let $\Phi = (G, U(\mathbb{H}), \varphi)$. Let $c = e_1 e_2 \dots e_n$ where $e_1 = e_{ij}$ and $e_n = e_{mi}$ be a cycle in Φ starting at vertex i. Suppose there is another cycle which visits the exact same vertices as c, but starts instead at vertex k. We can denote this as $c = e_p e_{p+1} \dots e_n e_1 \dots e_{p-1}$ where $e_p = e_{kl}$ and $e_{p-1} = e_{jk}$. Then*

$$\varphi_i(c) \sim \varphi_k(c).$$

Proof. We have

$$\varphi_i(c) = \prod_{\alpha=1\dots n} \varphi(e_\alpha)$$

and

$$\varphi_k(c) = \prod_{\alpha=p,p+1,\dots,n,1,\dots,p-1} \varphi(e_\alpha)$$

We can split these two products up in two particular places. For $\varphi_i(c)$ we will split the product between positions $p-1$ and p, and for $\varphi_k(c)$ we will split the product between positions n and 1. This yields

$$\varphi_i(c) = \Big(\prod_{\alpha=1\dots p-1} \varphi(e_\alpha)\Big) \cdot \Big(\prod_{\alpha=p\dots n} \varphi(e_\alpha)\Big) = Q \cdot W \qquad \text{for some } Q, W \in U(\mathbb{H})$$

and

$$\varphi_k(c) = \Big(\prod_{\alpha=p,\dots,n} \varphi(e_\alpha)\Big) \cdot \Big(\prod_{\alpha=1,\dots,p-1} \varphi(e_\alpha)\Big) = W \cdot Q \qquad \text{for some } Q, W \in U(\mathbb{H})$$

By Lemma 1.10, $\because Q, W \in U(\mathbb{H})$ and $QW \sim WQ \Rightarrow \varphi_i(c) \sim \varphi_k(c)$. $\square$

This Lemma allows us to revisit our definition for the gain for a cycle. The **gain of a cycle c in a $U(\mathbb{H})$-gain graph** is the set $\varphi(c) = [\varphi_i(c)]$, where v_i is any vertex in the cycle. i.e. the gain of a cycle is the quaternionic equivalence class of the gain starting at any vertex v_i of the cycle. From Lemma 3.1 this definition is consistent. From Lemma 1.9, this equivalence class can uniquely be identified by one of its complex members.

3.2 Switching

All of the definitions and proofs from Section 2.3.1 are valid for quaternions. Of course, a switching function will be defined as $\zeta : E \to U(\mathbb{H})$. If Φ is a $U(\mathbb{H})$-gain graph, the result from Theorem 2.4 is changed very slightly.

Lemma 3.2. *For all $\Gamma \in [\Phi]$ the following are true*

- $\sigma_r(A(\Gamma)) = \sigma_r(A(\Phi))$
- $\sigma_r(L(\Gamma)) = \sigma_r(L(\Phi))$

Proof. Since the right spectrum of quaternionic matrices is unitarily invariant, the result is immediate. $\square$

3.3 Eigenvalues for Cycles

A **quaternionic unit gain cycle** is a gain graph $\mathcal{C}_n = (C_n, U(\mathbb{H}), \varphi)$ where C_n is an n-vertex cycle graph, and the underlying gain group is the group of all unit norm quaternions, $U(\mathbb{H})$. From the definition in Section 3.1, we will define the **gain of a cycle graph** $\mathcal{C}_n$ as the set $\varphi(\mathcal{C}_n) = [\varphi_1(\mathcal{C}_n)]$, where v_1 is the first vertex in the cycle. In particular, we will refer to this set as $\varphi(\mathcal{C}_n) = [e^{i\theta}]$ where $\varphi_1(\mathcal{C}_n) \sim e^{i\theta} \in \mathbb{C}$.

The following lemma is useful, but not entirely necessary. Nonetheless, it is a nice result.

Lemma 3.3. *Let $\Phi = (C_n, U(\mathbb{H}), \varphi)$ and $\Psi = (C_n, U(\mathbb{H}), \psi)$ be two $U(\mathbb{H})$-gain cycles of the same size. Then $\Phi \sim_\zeta \Psi$ if and only if $\varphi_1(\Phi) \sim \psi_1(\Psi)$, in the sense of quaternions.*

Proof. ($\Rightarrow$)
The forward direction is easier, so we begin there. We denote the edges of C_n, $e_{12}, e_{23}, \ldots, e_{n,1}$ as $e_i = e_{i,i+1}$ when $1 \leq i < n$ and $e_n = e_{n,1}$ (when $i = n$).
Since $\Phi \sim_\zeta \Psi \rightarrow$ there exists $\zeta : V \rightarrow U(\mathbb{H})$ such that

$$\begin{aligned} \psi(e_i) &= \zeta(i)^{-1}\varphi(e_i)\zeta(i+1) \qquad\qquad \text{when } 1 \leq i < n \\ \text{and } \psi(e_n) &= \zeta(n)^{-1}\varphi(e_i)\zeta(1) \end{aligned}$$

So, when we calculate the gain of Ψ starting at vertex v_1, we find

$$\begin{aligned} \psi_1(\Psi) &= \prod_{i=1}^{n} \psi(e_i) \\ &= \Big(\prod_{i=1}^{n-1} \zeta(i)^{-1}\varphi(e_i)\zeta(i+1)\Big)\zeta(n)^{-1}\varphi(e_i)\zeta(1) \end{aligned}$$

$$
\begin{aligned}
&= \zeta(1)^{-1}\varphi(e_1)\zeta(2)\zeta(2)^{-1}\dots\zeta(n-1)^{-1}\varphi(e_{n-1})\zeta(n)\zeta(n)^{-1}\varphi(e_n)\zeta(1) \\
&= \zeta(1)^{-1}\varphi(e_1)\varphi(e_2)\dots\varphi(e_n)\zeta(1) \\
&= \zeta(1)^{-1}\Big(\prod_{i=1}^{n}\varphi(e_i)\Big)\zeta(1) \\
\psi_1(\Psi) &= \zeta(1)^{-1}\varphi_1(\Phi)\zeta(1)
\end{aligned}
$$

($\Leftarrow$)

Since $\varphi_1(\Phi) \sim \psi_1(\Psi) \Rightarrow$there exists a $\nu \in U(\mathbb{H})$ such that

$$
\begin{aligned}
\psi_1(\Psi) &= \nu\varphi_1(\Phi)\nu^{-1} \\
\prod_{i=1}^{n}\psi(e_i) &= \nu\Big(\prod_{i=1}^{n}\varphi(e_i)\Big)\nu^{-1}
\end{aligned}
$$

We define a switching function $\zeta : V \to U(\mathbb{H})$ as

$$
\begin{aligned}
\zeta(1) &= \nu^{-1} \\
\zeta(i) &= \Big[\Big(\prod_{j=i}^{n}\psi(e_j)\Big)\cdot\nu\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)^{-1}\Big]^{-1} \qquad \text{when } 1 < i \leq n
\end{aligned}
$$

We will now show that $\Phi^\zeta = \Psi$, thereby proving the switching equivalence of the two. We will do this in three parts. Namely, for e_1, e_i when $1 < i < n$, and e_n.

1. $i = 1$

$$
\begin{aligned}
\varphi(e_1)^\zeta &= \zeta(1)^{-1}\varphi(e_1)\zeta(2) \\
&= \nu\varphi(e_1)\Big[\Big(\prod_{j=2}^{n}\psi(e_j)\Big)\cdot\nu\Big(\prod_{j=2}^{n}\varphi(e_j)\Big)^{-1}\Big]^{-1} \\
&= \nu\varphi(e_1)\Big[\Big(\prod_{j=2}^{n}\varphi(e_j)\Big)\nu^{-1}\Big(\prod_{j=2}^{n}\psi(e_j)\Big)^{-1}\Big] \\
&= \left(\nu\varphi(e_1)\Big(\prod_{j=2}^{n}\varphi(e_j)\Big)\nu^{-1}\right)\Big(\prod_{j=2}^{n}\psi(e_j)\Big)^{-1} \\
&= \left(\nu\Big(\prod_{j=1}^{n}\varphi(e_j)\Big)\nu^{-1}\right)\Big(\prod_{j=2}^{n}\psi(e_j)\Big)^{-1} \\
\text{Since } \varphi_1(\Phi) \sim \psi_1(\Psi) \Rightarrow &= \Big(\prod_{j=1}^{n}\psi(e_j)\Big)\Big(\prod_{j=2}^{n}\psi(e_j)\Big)^{-1}
\end{aligned}
$$

$$= \psi(e_1)\Big(\prod_{j=2}^{n}\psi(e_j)\Big)\Big(\prod_{j=2}^{n}\psi(e_j)\Big)^{-1}$$
$$\varphi(e_1)^\zeta = \psi(e_1)$$

2. $1 < i < n$

$$\varphi(e_i)^\zeta = \zeta(i)^{-1}\varphi(e_i)\zeta(i+1)$$

$$\begin{aligned}
&= \Big[\Big(\prod_{j=i}^{n}\psi(e_j)\Big)\nu\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)^{-1}\Big]\varphi(e_i)\Big[\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)\nu\Big(\prod_{j=i+1}^{n}\varphi(e_j)\Big)^{-1}\Big]^{-1}\\
&= \Big[\Big(\prod_{j=i}^{n}\psi(e_j)\Big)\nu\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)^{-1}\Big]\varphi(e_i)\Big(\prod_{j=i+1}^{n}\varphi(e_j)\Big)\nu^{-1}\Big(\prod_{j=i}^{n}\psi(e_j)\Big)^{-1}\\
&= \Big[\Big(\prod_{j=i}^{n}\psi(e_j)\Big)\nu\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)^{-1}\Big]\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)\nu^{-1}\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)^{-1}\\
&= \Big[\Big(\prod_{j=i}^{n}\psi(e_j)\Big)\nu\Big]\Big[\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)^{-1}\Big(\prod_{j=i}^{n}\varphi(e_j)\Big)\Big]\nu^{-1}\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)^{-1}\\
&= \Big(\prod_{j=i}^{n}\psi(e_j)\Big)\nu\nu^{-1}\Big(\prod_{j=i}^{n}\psi(e_j)\Big)^{-1}\\
&= \Big(\prod_{j=i}^{n}\psi(e_j)\Big)\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)^{-1}\\
&= \psi(e_i)\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)\Big(\prod_{j=i+1}^{n}\psi(e_j)\Big)^{-1}
\end{aligned}$$

$$\varphi^\zeta(e_i) = \psi(e_i)$$

3. $i = n$

$$\begin{aligned}
\varphi^\zeta(e_n) &= \zeta(n)^{-1}\varphi(e_n)\zeta(1)\\
&= \Big(\psi(e_n)\nu\varphi(e_n)^{-1}\Big)\varphi(e_n)\nu^{-1}\\
&= \psi(e_n)(\nu\varphi(e_n)^{-1}\varphi(e_n)\nu)\\
\varphi^\zeta(e_n) &= \psi(e_n)
\end{aligned}$$

In all cases the switching results with $\varphi^\zeta(e_i) = \psi(e_i)$, so $\Phi \sim_\zeta \Psi$. □

We immediately obtain the following.

Corollary 3.4. *Let $\Phi = (C_n, U(\mathbb{H}), \varphi)$ and $\Psi = (C_n, U(\mathbb{H}), \psi)$ be two $U(\mathbb{H})$-gain cycles of the same size. Then $\Phi \sim_\zeta \Psi$ if and only if $\varphi(\Phi) = \psi(\Psi)$.*

That is to say, given two $U(\mathbb{H})$-gain cycles, their gains are equal if and only if they are switching equivalent.

The following theorem is the quaternionic version of Theorem 2.5, and its result is just as powerful. We prove it two ways, one is direct and the other makes use of Lemma 3.3.

Theorem 3.5. *For every $U(\mathbb{H})$-gain cycle, $\mathcal{C}_n = (C_n, U(\mathbb{H}), \varphi)$, there exists a $\mathbb{T}$-unit gain cycle $\Xi = (C_n, \mathbb{T}, \varphi')$ such that $\Xi \in [\mathcal{C}_n]_\zeta$ and $\varphi(\mathcal{C}_n) = [\varphi'(\Xi)]$ where $\varphi'(\Xi) = e^{i\theta}$ and $\varphi' : E \to \mathbb{T}$ is defined by*

$$\varphi'(e_{n,1}) = \nu^* \varphi_1(\mathcal{C}_n)\nu$$

where $\nu \in U(\mathbb{H})$ such that $\nu^*\varphi_1(\mathcal{C}_n)\nu = e^{i\theta} = \varphi'(\Xi) \in \mathbb{C}$ for $v_1 \in C_n$ and

$$\varphi'(e_{i,i+1}) = 1 \qquad \text{for } 0 \leq i < n$$

Proof. (Direct proof) Let $\mathcal{C}_n$ be a $U(\mathbb{H})$-gain cycle. We define a switching function $\xi : V \to U(\mathbb{H})$ as follows:

$$\xi(v_i) = \begin{cases} 1 & i = 1 \\ \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} & i \neq 1 \end{cases}$$

This is the exact same switching function as from Theorem 2.5, and it produces a new $U(\mathbb{H})$-gain graph Φ^ξ. Following the same steps from Theorem 2.5, we will find that

$$\varphi^\xi(e_{i,i+1}) = 1 \qquad \text{for } 1 \leq i < n$$
$$\text{and } \varphi^\xi(e_{n,1}) = \varphi_1(\mathcal{C}_n)$$

We define $\varphi_1(\mathcal{C}_n)_\mathbb{C} = Re(\varphi_1(\mathcal{C}_n)) + |Im(\varphi_1(\mathcal{C}_n))|\mathbf{i}$. By Lemma 1.7 $\varphi_1(\mathcal{C}_n)_\mathbb{C} \sim \varphi_1(\mathcal{C}_n)$, so there exists $\nu \in U(\mathbb{H})$ such that $\varphi_1(\mathcal{C}_n)_\mathbb{C} = \nu^*\varphi(\mathcal{C}_n)\nu$. We define a second switching function, ν, by

$$\nu(v_i) = \nu \qquad \text{for all } v_i.$$

Note that the switching matrix for this function is $Z_\nu = \nu I_n$ (which was why we simply named it ν). The result of this switching function is a gain graph $\Phi^{\xi\nu}$, with gain function

$$\begin{aligned} \varphi^{\xi\nu}(e_{i,i+1}) &= \nu^*\nu = 1 \qquad\qquad \text{for } 1 \le i < n \\ \varphi^{\xi\nu}(e_{n,1}) &= \nu^*\varphi_1(\mathfrak{C}_n)\nu \in \mathbb{C} \end{aligned}$$

Since we chose ν with the particular property $\nu^*\varphi_1(\mathfrak{C}_n)\nu = \varphi_1(\mathfrak{C}_n)_{\mathbb{C}} \in \mathbb{C}$, we have reduced the $U(\mathbb{H})$-gain graph to a $\mathbb{T}$-gain graph. Additionally, $\varphi^{\xi\nu}(\Phi^{\xi\nu}) = \varphi^{\xi\nu}(e_{12})\varphi^{\xi\nu}(e_{23})\dots\varphi^{\xi\nu}(e_{n-1,n})\varphi^{\xi\nu}(e_{n,1}) = 1\cdot 1\cdot\dots\cdot 1\cdot\nu^*\varphi_1(\mathfrak{C}_n)\nu = \nu^*\varphi_1(\mathfrak{C}_n)\nu$. The switched gain function $\varphi^{\xi\nu}$ satisfies all results of the theorem.

(Alternate Proof) Let $\mathfrak{C}_n$ be a $U(\mathbb{H})$-gain cycle. There exists $e^{i\theta} \in \mathbb{T}$ such that $\varphi_1(\mathfrak{C}_n) \sim e^{i\theta}$. We construct a $\mathbb{T}$-gain graph $\Xi = (C_n, \mathbb{T}, \varphi')$, where φ' is defined as

$$\begin{aligned} \varphi'(e_{i,i+1}) &= 1 \qquad\qquad\qquad\qquad \text{for } 0 \le i < n \\ \varphi'(e_{n,1}) &= \nu^*\varphi_1(\mathfrak{C}_n)\nu = e^{i\theta} \end{aligned}$$

By construction, $\varphi'(\Xi) = e^{i\theta} = \nu^*\varphi_1(\mathfrak{C}_n)\nu$, which implies $\varphi'(\Xi) \sim \varphi_1(\mathfrak{C}_n)$. By Lemma 3.3 $\Rightarrow \Xi \sim_\zeta \mathfrak{C}_n$. □

This result shows us that any arbitrary $U(\mathbb{H})$-gain graph can be switched to $\mathbb{T}$-gain graph, with gains being similar in the sense of quaternions. The next result is the quaternionic version of Theorem 2.6.

Theorem 3.6. *Let $\mathfrak{C}_n = (C_n, U(\mathbb{H}), \varphi)$ and let $\varphi(\mathfrak{C}_n) = [e^{i\theta}]$. Then*

$$\sigma_r(A(\mathfrak{C}_n)) = \left\{2\cos\left(\frac{\theta + 2\pi j}{n}\right) \;\middle|\; j \in \{0, 1, \dots, n-1\}\right\}$$

and

$$\sigma_r(L(\mathfrak{C}_n)) = \left\{2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right) \;\middle|\; j \in \{0, 1, \dots, n-1\}\right\}$$

Proof. By the previous lemma, we can switch $\mathfrak{C}_n$ to a $\mathbb{T}$-unit gain graph Ξ with $\varphi(\mathfrak{C}_n) = [\varphi'(\Xi)]$ where $\varphi'(\Xi) = e^{i\theta} \in \mathbb{C}$. Since this switch corresponds to a unitary matrix, we know that $\sigma_r(A(\mathfrak{C}_n)) = \sigma_r(A(\Xi))$ and $\sigma_r(L(\mathfrak{C}_n)) = \sigma_r(L(\Xi))$. By applying Theorem 2.6 to Ξ, the result is immediate. □

This result shows that given an arbitrary $U(\mathbb{H})$-gain cycle, we can determine the eigenvalues of its adjacency and Laplacian matrices simple by looking at its gain starting at vertex v_1. The procedure goes as follows. We will first determine the complex representative of the gain of the graph. Calculate the gain of the cycle, φ_1, starting from v_1. The complex representative will be equal to $Re(\varphi_1) + |Im(\varphi_1)|\mathbf{i} = e^{i\theta}$. The associated eigenvalues can be determined from the formulas in Theorem 3.6, using θ from the complex representative. This procedure, in fact, works starting from any vertex v_i, since they are all equivalent to the same complex numbers $e^{\pm i\theta}$.

3.4 Example of Finding Eigenvalues

In the following example, we go through this eigenvalue procedure for a simple 4-vertex $U(\mathbb{H})$-gain graph. For comparison, we also go through the typical procedure for finding eigenvalues using the complex representative of the quaternionic matrix. Figure 3.1 shows the cycle graph that we will be examining more closely, $\Phi = (C_4, U(\mathbb{H}), \varphi)$, along with its complex switched form $\Phi^{\xi\nu}$.

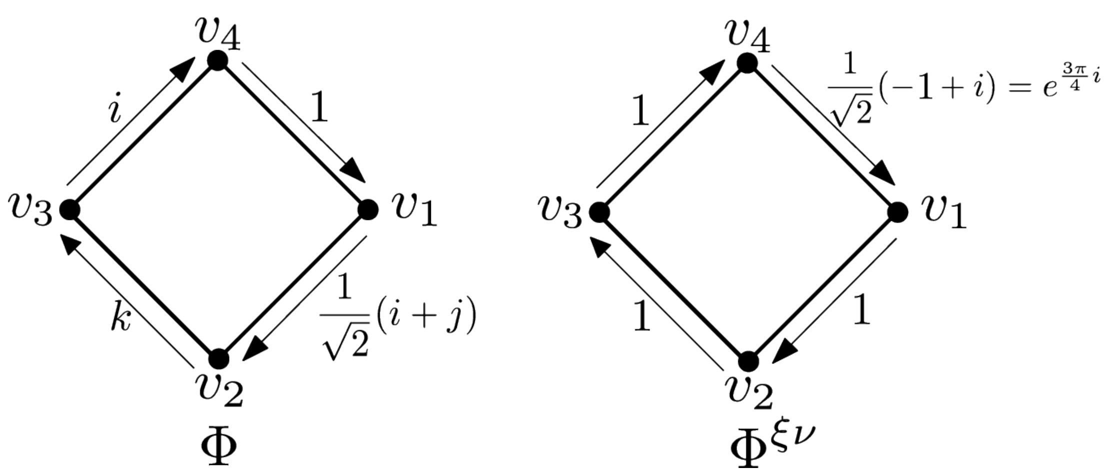


Figure 3.1: Sample $U(\mathbb{H})$-gain graphs. On the left is the original, and on the right is the switched complex version. Note that $\varphi(\Phi) = [e^{\frac{3\pi}{4}\mathbf{i}}]$.

The adjacency and Laplacian matrices are:

$$A(\Phi) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) & 0 & 1 \\ -\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) & 0 & \mathbf{k} & 0 \\ 0 & -\mathbf{k} & 0 & \mathbf{i} \\ 1 & 0 & -\mathbf{i} & 0 \end{bmatrix}$$

$$L(\Phi) = \begin{bmatrix} 2 & -\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) & 0 & -1 \\ \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) & 2 & -\mathbf{k} & 0 \\ 0 & \mathbf{k} & 2 & -\mathbf{i} \\ -1 & 0 & \mathbf{i} & 2 \end{bmatrix}$$

3.4.1 Method 1: Eigenvalues from the Gain of the Cycle

1. We begin by calculating the gain of the cycle starting at vertex v_1.

$$\begin{aligned} \varphi_1(c) &= \prod_{\alpha=1\ldots 4} \varphi(e_\alpha) \\ &= \varphi(e_{12}) \cdot \varphi(e_{23}) \cdot \varphi(e_{34}) \cdot \varphi(e_{41}) \\ &= \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \cdot \mathbf{k} \cdot \mathbf{i} \cdot 1 \\ &= \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \cdot \mathbf{j} \\ &= \frac{1}{\sqrt{2}}(\mathbf{k}-1) \\ \varphi_1(c) &= -\frac{1}{\sqrt{2}}(1-\mathbf{k}) \end{aligned}$$

So, $\varphi(\Phi) = [-\frac{1}{\sqrt{2}}(1-\mathbf{k})]$.

2. Next, we find the complex representative for the equivalence class $[e^{i\theta}] = [-\frac{1}{\sqrt{2}}(1-\mathbf{k})]$.

$$\begin{aligned} e^{i\theta} &= Re\Big(-\frac{1}{\sqrt{2}}(1-\mathbf{k})\Big) + \Big|Im\Big(-\frac{1}{\sqrt{2}}(1-\mathbf{k})\Big)\Big|\mathbf{i} \\ &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathbf{k} \\ e^{i\theta} &= -\frac{1}{\sqrt{2}}(1-\mathbf{i}) \end{aligned}$$

So $\theta = \arctan\left(\frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = \arctan(-1) = \frac{3\pi}{4}$.

3. We calculate the right eigenvalues for the adjacency matrix according to the formula in Theorem 3.6.

$$\begin{aligned}\sigma_r(A(\Phi)) &= \left\{2\cos\left(\frac{\frac{3\pi}{4}+2\pi j}{4}\right) \mid j \in \{0,1,\dots,n-1\}\right\}\\ &= \left\{2\cos\left(\frac{3\pi}{16}\right), 2\cos\left(\frac{11\pi}{16}\right), 2\cos\left(\frac{19\pi}{16}\right), 2\cos\left(\frac{27\pi}{16}\right)\right\}\\ &= \left\{\pm\sqrt{2+\sqrt{2-\sqrt{2}}}, \pm\sqrt{2-\sqrt{2-\sqrt{2}}}\right\}\\ &= \left\{\pm\sqrt{2\pm\sqrt{2-\sqrt{2}}}\right\}\end{aligned}$$

4. We calculate the right eigenvalues for the Laplacian matrix according to the formula in Theorem 3.6

$$\begin{aligned}\sigma_r(L(\Phi)) &= \left\{2-2\cos\left(\frac{\frac{3\pi}{4}+2\pi j}{4}\right) \mid j \in \{0,1,\dots,n-1\}\right\}\\ &= \left\{2\mp\sqrt{2\pm\sqrt{2-\sqrt{2}}}\right\}\end{aligned}$$

3.4.2 Method 2: Eigenvalues from the Complex Representative using MATLAB

The previous section calculated eigenvalues from the theorem proven earlier. Here we show how the same eigenvalues can be obtained from the **complex representatives** of the corresponding adjacency and Laplacian matrices. Note that since both matrices are Hermitian, we know their right eigenvalues will be real. The right eigenvalues of the complex representative will be all of the right eigenvalues in $\mathbb{C}$. Since they are all real, determining the eigenvalues of the complex representative will yield the real right eigenvalues of the quaternionic matrix with multiplicity two. A MATLAB code was written to expand the quaternionic matrix into its complex representative, which then found the eigenvalues of that matrix.

We begin by finding the **right eigenvalues of the adjacency matrix**.

1. We can decompose $A(\Phi)$ as

$$A(\Phi) = A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}\mathbf{i} & 0 & 1 \\ -\frac{1}{\sqrt{2}}\mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i} \\ 1 & 0 & -\mathbf{i} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{i} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{j} = A_1 + A_2\mathbf{j}$$

2. The complex representative of A will be

$$\chi_A = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}\mathbf{i} & 0 & 1 & 0 & \frac{1}{\sqrt{2}} & 0 & 1 \\ -\frac{1}{\sqrt{2}}\mathbf{i} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \mathbf{i} & 0 \\ 0 & 0 & 0 & \mathbf{i} & 0 & -\mathbf{i} & 0 & 0 \\ 1 & 0 & -\mathbf{i} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 & -\frac{1}{\sqrt{2}}\mathbf{i} & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & \mathbf{i} & 0 & \frac{1}{\sqrt{2}}\mathbf{i} & 0 & 0 & 0 \\ 0 & -\mathbf{i} & 0 & 0 & 0 & 0 & 0 & -\mathbf{i} \\ -1 & 0 & 0 & 0 & 1 & 0 & \mathbf{i} & 0 \end{bmatrix}$$

3. From MATLAB, the eigenvalues of χ_A will give us

$$\sigma(\chi_A) = \sigma_r(A(\Phi)) = \Big\{\pm 1.111, \pm 1.663\Big\}$$

which do, in fact, correspond to the eigenvalues obtained from the previous section.

Next, we will find the **right eigenvalues of the Laplacian matrix**.

1. We can decompose $L(\Phi)$ as

$$L(\Phi) = L = \begin{bmatrix} 2 & -\frac{1}{\sqrt{2}}\mathbf{i} & 0 & -1 \\ \frac{1}{\sqrt{2}}\mathbf{i} & 2 & 0 & 0 \\ 0 & 0 & 2 & -\mathbf{i} \\ -1 & 0 & \mathbf{i} & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}} & 0 & -\mathbf{i} & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\mathbf{j} = L_1 + L_2\mathbf{j}$$

2. The complex representative of L will be

$$\chi_L = \begin{bmatrix} L_1 & L_2 \\ -\overline{L_2} & \overline{L_1} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{\sqrt{2}}\mathbf{i} & 0 & -1 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}}\mathbf{i} & 2 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\mathbf{i} & 0 \\ 0 & 0 & 2 & -\mathbf{i} & 0 & \mathbf{i} & 0 & 0 \\ -1 & 0 & \mathbf{i} & 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 2 & \frac{1}{\sqrt{2}}\mathbf{i} & 0 & -1 \\ -\frac{1}{\sqrt{2}} & 0 & -\mathbf{i} & 0 & -\frac{1}{\sqrt{2}}\mathbf{i} & 2 & 0 & 0 \\ 0 & \mathbf{i} & 0 & 0 & 0 & 0 & 2 & \mathbf{i} \\ 1 & 0 & 0 & 0 & -1 & 0 & -\mathbf{i} & 2 \end{bmatrix}$$

3. From MATLAB, the eigenvalues of χ_L will give us

$$\sigma(\chi_L) = \sigma_r(L(\Phi)) = \Big\{0.337, 0.889, 3.111, 3.663\Big\}$$

   which do, in fact, correspond to the eigenvalues obtained from the previous section.

The right eigenvalues for these matrices can certainly be found using this method, implying that Theorem 3.6 is unnecessary. The following lists several pros to using the equations from Theorem 3.6.

1. The equations will yield exact results; the MATLAB code will only yield approximations. There are, of course, computational methods for finding exact eigenvalues of the complex representative, but they fail because of the following.

2. The computational difficulty of finding eigenvalues of the complex representative quickly outgrows the computational difficulty of using the equations. Given an n-vertex cycle graph, the equations require a product of n gains, whereas the complex representative will require finding the eigenvalues of a $2n \times 2n$ matrix, which is much more difficult.

3. Theorem 3.6 gives us an exact reasoning behind the values of the eigenvalues. The complex representative method does not.

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