The cyclic bandwidth problem

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THE CYCLIC BANDWIDTH PROBLEM*

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Abstract. The cyclic bandwidth problem for a graph G is to label its n vertices by the elements of the additive group $(Z_n, \oplus)$ of integers modulo n so that the quantity $\max\{d(f(u), f(v)) : (u,v) \in E(G)\}$ is minimized, where $f(v)$ is the label of $v \in V(G)$ and $d(x,y) = \min\{x \oplus (n-y), y \oplus (n-x)\}$ represents the distance of $x, y \in Z_n$. This paper describes the background of this labelling problem, some basic properties, and the computational complexity. In particular, a local density lower bound for trees and several exact results for special graphs are presented.

Key words. Graph labelling, circuit layout, cyclic bandwidth.

1. Introduction

The bandwidth minimization problem for graphs arises from a wide application area including sparse matrix computation, code theory, data structure and the circuit layout of VLSI designs. Especially, in the circuit layout models the following general framework is well acceptable[1,3]. First, we are given a graph H, called the host graph, which represents a fixed layout structure. Then, let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. A labelling f of G in the host graph H is a one-to-one mapping from $V(G)$ to $V(H)$. This can be viewed as a layout (embedding) of G into H. The bandwidth of a labelling f for G is defined by

$$B_H(G,f) = \max_{(u,v)\in E(G)} d_H(f(u), f(v))$$

where $d_H(x,y)$ denotes the distance between x and y in H. The bandwidth of G(relative to H) is

$$B_H(G) = \min_f B_H(G,f)$$

where the minimum is taken over all labellings f.

Up to now, most of the researchers have concentrated on the case where the host graph is a path[2-7]. In addition, the case where the host graph is a grid graph $P_m \times P_n$ has been investigated in connection with the rectilinear network layout designs[1]. F.R.K. Chung[3] proposed that circuits (cycles) and trees are among other good candidates for host graphs, which have applications in cyclic network designs and data structures. She also pointed out: "In taking a circuit as host graph, we deal with integers modulo n as labels".

In this paper, we shall study the bandwidth problem whose host graph is a cycle C_n; that is, to place the vertices of G into a cycle C_n(at the same time, embed the edges of G into paths of C_n) such that the maximum distance in C_n between adjacent vertices in G is minimized. We shall call it the cyclic bandwidth problem. For the sake of contrast, "the bandwidth problem" will mean that whose host graph is a path.

Our main results are the following: (1) some basic properties of cyclic labellings and cyclic bandwidths(in Section 2); (2) the NP-completeness of the problem(in Section 3); (3) the local density lower bound for trees and exact results for some special classes of trees(in Section 4).

Received November 9, 1992.
*This research is supported by the National Natural Science Foundation of China.

2. Definitions and Elementary Results

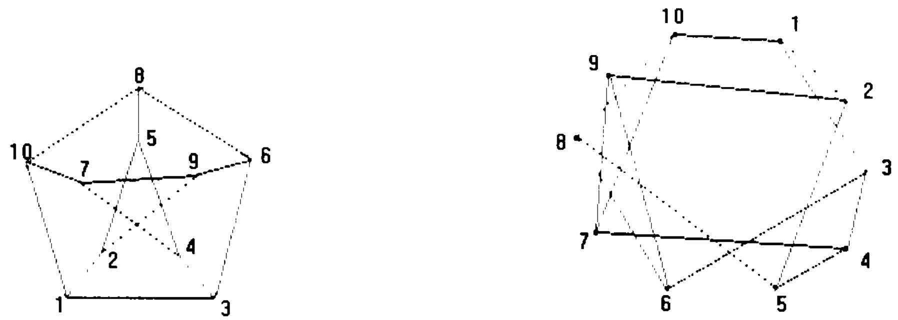
Let $(Z_n, \oplus)$ be the additive group of integers modulo n. For convenience, we assume $Z_n = \{1, 2, \cdots, n\}$ in which n is considered as the identity element, instead of 0. For $x, y \in Z_n (x \neq y)$, the distance between x and y is defined by

$$d(x, y) = \min\{x \oplus (-y),\ y \oplus (-x)\}$$

where $-x = n-x$ is the inverse of $x \in Z_n$. In fact, we may write $d(x, y) = \min\{|x-y|,\ n-|x-y|\}$ in the set Z of integers.

Let $G = (V(G), E(G))$ be a graph with n vertices. A bijection(one-to-one mapping) f: $V(G) \to Z_n$ will be called a cyclic labelling of G. when there is no confusion, we abbreviate it as "a labelling".

The cyclic bandwidth of a labelling f for G is defined by $B_c(G, f) = \max\limits_{(u,v)\in E(G)} d(f(u), f(v))$; and the cyclic bandwidth of G is $B_c(G) = \min\limits_{f} B_c(G, f)$. Moreover, a labelling attaining the above minimum value will be called an optimal labelling. For example, the cyclic bandwidth and an optimal labelling of the Petersen graph are shown in Figure 1.



(a) $B_c(G) = 3$ (b) an embedding

Fig. 1

For a given labelling f, all edges of G are classified into two sets: one consisting of those having distance $d(x, y) = |x-y|$; and the other consisting of those having $d(x, y) = n - |x-y| < |x-y|$. The edges in the former set are called normal edges; and the others are called overstep edges. In Figure 1, all edges, except (1, 10) and (2, 9), are normal; while (1, 10) and (2, 9) are overstep.

As mentioned before, a cyclic labelling can be viewed as an embedding of G into a cycle C_n in which each edge of G covers a path of C_n. For example, Figure 1 (b) represents the corresponding embedding of the labelling shown in Figure 1 (a). Note that the normal edge (2, 5) in G covers the path $(2, 3, 4, 5)$ in C_n. On the other hand, the overstep edge (2, 9) in G covers the path (9, 10, 1, 2), instead of $(2, 3, 4, 5, 6, 7, 8, 9)$.

In the sequel, the following basic property of cyclic labellings is useful.

Proposition 2.1. *For two labellings f and f', if there is a $k \in Z_n$ such that $f'(v) = f(v) \oplus k$ for all $v \in V(G)$, then $B_c(G, f) = B_c(G, f')$.*

This means that we can rotate the labels in any distance k without changing the value of bandwidth. Therefore, the labellings f and f' under consideration may be said to be equivalent.

By the definitions, we have a basic criterion for the cyclic bandwidth as follows.

Theorem 2.2. *A graph G on n vertices has cyclic bandwidth k if k is the smallest integer that G can be embedded in C_n^k(the k-th power of a cycle), that is,*

$$B_c(G) = \min\{k \mid G \subseteq C_n^k\}.$$

This is similar to the well-known result of bandwidth k: G can be embedded in P_n^k[2]. In general, the smallest integer k is not easy to compute; but from this theoretic criterion many useful results can be deduced.

Corollary 2.3. *If a graph G has maximum degree $\triangle(G)$ and bandwidth $B(G)$, then*

$$\left\lceil \frac{\triangle(G)}{2} \right\rceil \leq B_c(G) \leq \min\left\{B(G), \left\lfloor \frac{|V(G)|}{2} \right\rfloor\right\}.$$

Here, the lower and upper bounds are both achievable by the following examples respectively.

Corollary 2.4.

$$B_c(P_n) = B_c(C_n) = 1, \qquad B_c(K_{1,n}) = \left\lceil \tfrac{n}{2} \right\rceil,$$
$$B_c(K_n) = \left\lceil \tfrac{n-1}{2} \right\rceil, \qquad B_c(P_n^k) = B_c(C_n^k) = k \quad (k \leq \left\lfloor \tfrac{n}{2} \right\rfloor).$$

The following is a little complicated.

Theorem 2.5. $$B_c(K_{m,n}) = \left\lceil \tfrac{m}{2} \right\rceil + \left\lceil \tfrac{n}{2} \right\rceil - 1.$$

Proof. Consider a complete bipartite graph $G = (X, Y; E)$ where $|X| = m$, $|Y| = n$, $E = X \times Y$. Let f be a cyclic labelling of G. Denote by $u_i = f^{-1}(i)$, $1 \leq i \leq m+n$, the vertex of label i. There are three cases as follows:

Case 1. m, n are both even. By Proposition 2.1, we may assume that u_1 and u_{m+n} are adjacent(otherwise, we may appropriately rotate the labels). Further assume that $u_{m+n} \in X$, $u_1 \in Y$. Take an integer $k = (m+n)/2$. If $u_k \in X$ then $(u_1, u_k) \in E$ and thus $d(f(u_1), f(u_k)) = (m+n)/2 - 1$; If $u_k \in Y$ then $(u_{m+n}, u_k) \in E$ and thus $d(f(u_{m+n}), f(u_k)) = (m+n)/2$. In any possibility we have

$$B_c(G, f) = \max_{(u,v)\in E} d(f(u), f(v)) \geq \frac{m+n}{2} - 1 = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 1.$$

Case 2. m is odd and n is even. Without loss of generality, we may assume $u_1 \in X$. Take an integer $k = (m+n-1)/2$. Consider the sequence $u_1, u_{1\oplus k}, u_{1\oplus k\oplus k}, \cdots$, which can eventually pass through all vertices of G. We can assert that there must be a $u_i \in X$ such that $u_{i\oplus k} \in Y$. So $(u_i, u_{i\oplus k}) \in E$ and thus

$$B_c(G, f) \geq d(f(u_i), f(u_{i\oplus k})) = \frac{m+n-1}{2} = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 1.$$

Case 3. m, n are both odd. Take $k = (m+n)/2$. For each i, if u_i and $u_{i\oplus k}$ both belong to X, we will match them. Since $|X| = m$ is odd, it is impossible to form a perfect matching of X in this way. Hence, there must be a $u_i \in X$ such that $u_{i\oplus k} \in Y$. Thus $(u_i, u_{i\oplus k}) \in E$ and

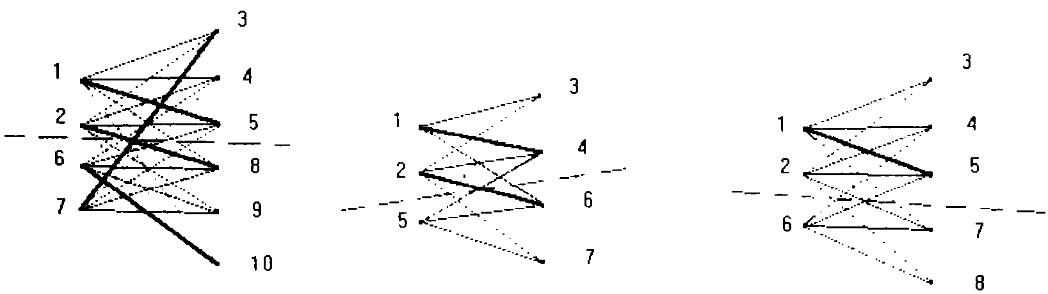
$$B_c(G, f) \geq d(f(u_i), f(u_{i\oplus k})) = \frac{m+n}{2} = \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil - 1.$$

To sum up the foregoing cases, we obtain the lower bound of $B_c(G,f)$. On the other hand, it is a routine to check that the following labelling attains the lower bound:

$$f(x_i)=\begin{cases} i, & \text{if } 1\le i\le \left\lceil \frac{m}{2}\right\rceil \\ \left\lceil \frac{n}{2}\right\rceil+i, & \text{if } \left\lceil \frac{m}{2}\right\rceil+1\le i\le m \end{cases}$$

$$f(y_j)=\begin{cases} \left\lceil \frac{m}{2}\right\rceil+j, & \text{if } 1\le j\le \left\lceil \frac{n}{2}\right\rceil \\ m+j, & \text{if } \left\lceil \frac{n}{2}\right\rceil+1\le j\le n \end{cases}$$

(as shown in Figure 2). Therefore we show what we want to prove. ∎



(a) $B_c(G)=4$ (b) $B_c(G)=3$ (c) $B_c(G)=4$

Fig. 2

3. Computational Complexity

In this section we consider the recognition(decision) versions of the bandwidth problem and the cyclic bandwidth problem as follows:

The Bandwidth Problem (BP). Given a graph G on n vertices and an integer k, does there exist a labelling f such that $B(G,f)\le k(1\le k\le n-1)$?

The Cyclic Bandwidth Problem (CBP). Given a graph G on n vertices and an integer k, does there exist a cyclic labelling f such that $B_c(G,f)\le k(1\le k\le \lfloor n/2\rfloor)$?

BP is known to be NP-complete[7]. As will be seen, CBP is as hard as BP.

Theorem 3.1. *The cyclic bandwidth problem* (CBP) *is* NP-*complete.*

Proof. It is obvious that CBP is in the class NP. Hence it suffices to show that a known NP-complete problem(e.g. BP) can polynomially transform to CBP. Now, given an instance of BP, i.e. a graph G on n vertices and integer $k(1\le k\le n-1)$. We construct an instance of CBP by taking $H=2nG(2n$ copies of G, i.e. a disconnected graph with $2n$ components each of which is isomorphic to $G)$ and the same integer k. It remains to show the following

Claim. The instance of BP has a solution if and only if the instance of CBP has a solution.

Suppose that the instance of BP has a solution, i.e. a labelling f such that $B(G,f)\le k$. Denote the vertices of G by $u_j=f^{-1}(j)$, $j=1,2,\cdots,n$. In the instance of CBP, let $G^{(i)}$ be the i-th copy of G, $i=1,2,\cdots,2n$. Then the vertex in $G^{(i)}$ that corresponds to u_j in G may be denoted by $u_j^{(i)}$, $1 \le j \le n$. We construct a cyclic labelling f^* for the instance of CBP as follows:

$$f^*(u_j^{(i)}) = (i-1)n + j, \qquad i = 1, 2, \cdots, 2n; \quad j = 1, 2, \cdots, n.$$

It is easy to see that $B_c(H, f^*) = B(G, f) \le k$. Namely, the instance of CBP has a solution.

Conversely, suppose that there is a cyclic labelling f^* such that $B_c(H, f^*) \le k$. By Proposition 2.1 we may assume that the labels of $G^{(1)}$ are included in the interval $[1, kn]$(if G is disconnected, then each component of $G^{(1)}$ can be so). Note that $kn \le (n-1)n \le \frac{1}{2}|V(H)|$. Therefore

$$d(f^*(u), f^*(v)) = |f^*(u) - f^*(v)| \quad \text{for } (u, v) \in E(G^{(1)}).$$

So we can define a labelling f in $G^{(1)}$ by $f(u) < f(v)$ if and only if $f^*(u) < f^*(v)$ for all $u, v \in V(G^{(1)})$. It follows that

$$|f(u) - f(v)| \le |f^*(u) - f^*(v)| = d(f^*(u), f^*(v)) \le k$$

for all $(u, v) \in E(G^{(1)})$. Hence $B(G^{(1)}, f) \le k$. That is to say, the instance of BP has a solution. This completes the proof. ∎

It is known in the literature that the bandwidth problem remains NP-complete even when restricted to trees with maximum degree three[4], or to caterpillars with hairs of length at most three[6]. So the cyclic bandwidth problem also remains NP-complete when all components of G are restricted to these classes of trees. We believe that CBP remains NP-complete for connected graphs.

4. The Density Lower Bound

The following lower bound for bandwidth is due to V.Chvátal[2,3,5].

Theorem 4.1. *Let $D(G)$ denote the diameter of G. Then*

$$B(G) \ge \left\lceil \frac{|V(G)| - 1}{D(G)} \right\rceil. \tag{4.1}$$

The right hand side of (4.1) is called the density of G, denoted by $ds(G)$. This is an important lower bound in the study of bandwidth. However, in the case of cyclic bandwidth, it is no longer true. For example, when G is a cycle C_n, then $B_c(G) = 1$, but $ds(G) = 2$. As another counterexample, we may take the complete bipartite graph $K_{m,n}$ where both m, n are even. By Theorem 2.5, we have $B_c(K_{m,n}) = (m+n)/2 - 1$, but $ds(K_{m,n}) = \lceil (m+n-1)/2 \rceil = (m+n)/2$. Hence, for a general graph G, the cyclic bandwidth does not have the lower bound of (4.1). Nevertheless, we shall fortunately see that this density lower bound remains true for trees.

Theorem 4.2. *If G is a tree, then*

$$B_c(G) \ge \left\lceil \frac{|V(G)| - 1}{D(G)} \right\rceil. \tag{4.2}$$

Proof. Let f be a cyclic labelling of G, and let $u_i = f^{-1}(i)$, $1 \le i \le n$, be the vertex of label i. Namely, we embed G into a cycle $(u_1, \cdots, u_n, u_1)$. Since G is a tree, there must be some pairs $(u_i, u_{i+1}) \notin E(G)$, which will be called gaps. When considering a gap, we may appropriately rotate the labels(recall Proposition 2.1) so that this gap becomes (u_n, u_1), which will be called the standard gap.

Now, we consider the standard gap (u_n, u_1). On the one hand, the path from u_1 to u_n on the cycle C_n is represented by $[1, n]$, the label interval. On the other hand, the unique path from u_1 to u_n in G is denoted by

P: $u_{i_0}(=u_1), u_{i_1}, \cdots, u_{i_k}(=u_n)$

which will be called the path relative to the gap (u_n, u_1). For example, in Figure 3 (a) or (b), the path P is depicted by heavy lines.

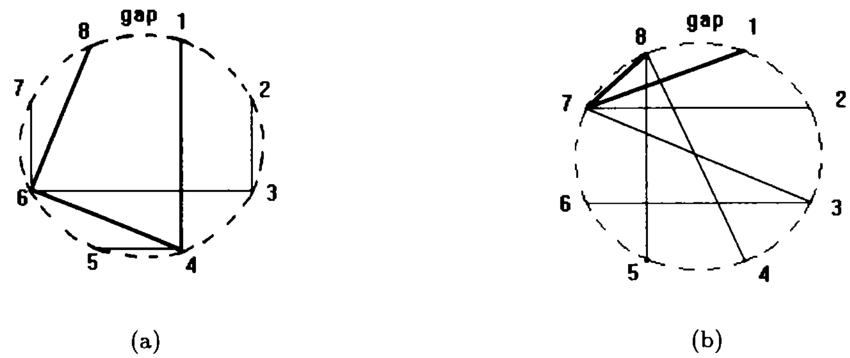


Fig. 3

Note that in Figure 3 (a) the set of all edges in P covers the label interval [1, n], since these edges are all normal. In this situation we will say in brief that the path P covers the label interval. On the contrary, in Figure 3 (b), the path P can not cover the label interval, since there is an overstep edge (1, 7). In the latter situation, however, we can find another gap, e.g. (6, 7), which has the expected property that the relative path covers its label interval. In general, we have the following

Claim. For any cyclic labelling f(any embedding of G into a cycle), there is a gap whose relative path covers the label interval.

We shall prove this claim by induction on the number of gaps. When there is only one gap (u_n, u_1), the path relative to this gap exactly coincides with the label interval [1, n]. Assume that the claim holds for the case where the number of gaps is less than k. Now we consider an embedding of G which has k gaps. First, we take a gap $e = (u_i, u_{i+1}) \notin E(G)$. If all edges in the path P relative to e are normal, then P covers the label interval(relative to e). Otherwise, there is an overstep edge e' in P. Note that $C = P + e$ forms a basic cycle of the tree G, and $e' \in C$. So we can make an elementary tree transformation: $G' = G + e - e'$. Then G' is a tree with $k-1$ gaps. By the assumption of induction, there is a gap e^* for G' whose relative path P' covers the label interval of e^*. If P' does not contain e, then P' is included in G, that is what we want to show. If P' indeed contains e, then $P^* = P' \triangle C$ is the relative path of e^* in G(where $\triangle$ denotes the symmetric difference operation for edge sets). It can be obviously seen that P^* also covers the label interval of e^*. This completes the proof of the claim.

Now, suppose that the standard gap (u_n, u_1) satisfies the claim. By observing the labels along the relative path $P = (u_{i_0}, u_{i_1}, \cdots, u_{i_k})$, it follows that

$$B_c(G,f) = \max_{(u,v)\in E(G)} d(f(u), f(v)) \geq \max_{0\leq r\leq k-1} d(f(u_{i_r}), f(u_{i_{r+1}}))$$
$$\geq \frac{1}{k}\sum_{r=0}^{k-1} d(i_r, i_{r+1}) \geq \frac{n-1}{d_G(u_1, u_n)} \geq \frac{|V(G)|-1}{D(G)}.$$

By the arbitrariness of f, we have the inequality (4.2). ∎

In the bandwidth problem, the local density of G, i.e., the maximum density of all subgraphs of G(see (4.3)), can sometimes provide better lower bounds[3]. In the case of cyclic bandwidth, we have a similar result:

Theorem 4.3. *If G is a tree, then*

$$B_c(G) \geq \max_{S \subseteq V(G)} \left\lceil \frac{|S|-1}{D(S)} \right\rceil, \tag{4.3}$$

where $D(S)$ denotes the diameter of S with respect to the distance in G.

The proof of this generalization is similar to that of Theorem 4.2.

We have seen that $B_c(G) \leq B(G)$ in Corollary 2.3. Additionally, for many classes of trees, $B(G)$ is equal to the local density lower bound. So the corresponding cyclic bandwidth $B_c(G)$ has the same value. Therefore we have the following corollaries(for the results on $B(G)$, see [2, 3, 5]).

A caterpillar is a tree which yields a path, called the spine, when all its pendant vertices are removed.

Corollary 4.4. *Let T be a caterpillar with the spine $P=(v_1, v_2, \cdots, v_p)$. And T_{ij} denotes the subtree induced by $v_i, v_{i+1}, \cdots v_j$ and all vertices adjacent to them($i \leq j$). Then*

$$B_c(T) = B(T) = \max_{1 \leq i \leq j \leq p} \left\lceil \frac{|V(T_{ij})|-1}{j-i+2} \right\rceil.$$

Corollary 4.5. *Let T be a tree with diameter 4(two-layer star[5]). Then*

$$B_c(T) = B(T) = \max_{2 \leq k \leq 4} \max_{S \in \mathcal{S}_k} \left\lceil \frac{|S|-1}{k} \right\rceil$$

where $\mathcal{S}_2$ denotes the family of maximal stars(with diameter 2), $\mathcal{S}_3$ the family of maximal double-stars(with diameter 3), and $\mathcal{S}_4 = \{T\}$.

The reader can list more examples. However, a more interesting problem would be to characterize the graphs with $B_c(G) = B(G)$. In addition, the relations between $B_c(G)$ and other graph invariants are worth further studying.

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