

# Markov Chains

# Toolbox

- Search: uninformed/heuristic
- Adversarial search
- Probability
- Bayes nets
  - Naive Bayes classifiers
- Statistical inference

# Reasoning over time

- In a Bayes net, each random variable (node) takes on one specific value.
  - Good for modeling static situations.
- What if we need to model a situation that is changing over time?

# Example: Comcast

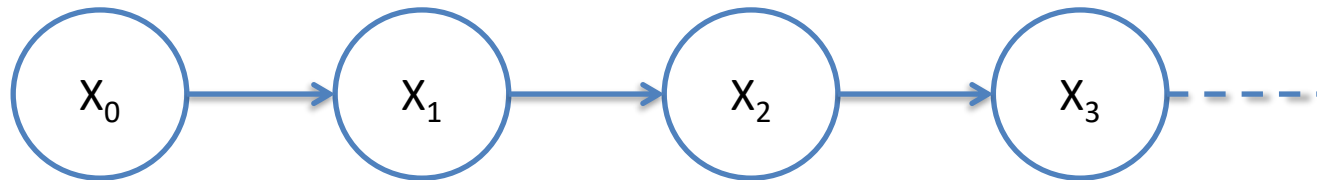
- In 2004 and 2007, Comcast had the worst customer satisfaction rating of any company or gov't agency, including the IRS.
- I have cable internet service from Comcast, and sometimes my router goes down. If the router is online, it will be online the next day with  $\text{prob}=0.8$ . If it's offline, it will be offline the next day with  $\text{prob}=0.4$ .
- How do we model the probability that my router will be online/offline tomorrow? In 2 days?

# Example: Waiting in line

- You go to the Apple Store to buy the latest iPhone. Every minute, the first person in line is served with  $\text{prob}=0.5$ .
- Every minute, a new person joins the line with probability
  - 1 if the line length=0
  - $2/3$  if the line length=1
  - $1/3$  if the line length=2
  - 0 if the line length=3
- How do we model what the line will look like in 1 minute? In 5 minutes?

# Markov Chains

- A Markov chain is a type of Bayes net with a potentially infinite number of variables (nodes).
- Each variable describes the state of the system at a given point in time ( $t$ ).



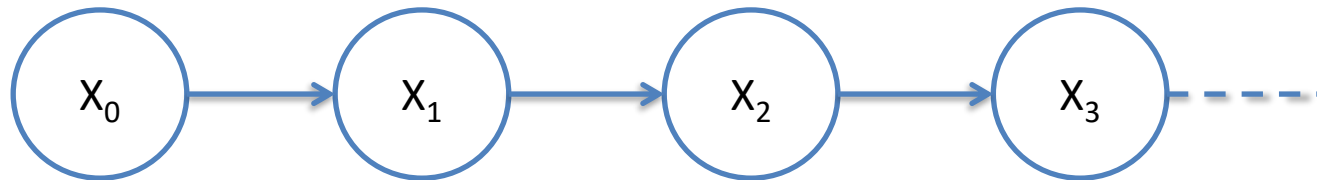
# Markov Chains

- Markov property:

$$P(X_t \mid X_{t-1}, X_{t-2}, X_{t-3}, \dots) = P(X_t \mid X_{t-1})$$

- Probabilities for each transition are identical:

$$P(X_t \mid X_{t-1}) = P(X_1 \mid X_0)$$



# Markov Chains

- Since these are just Bayes nets, we can use standard Bayes net ideas.
  - Shortcut notation:  $X_{i:j}$  will refer to all variables  $X_i$  through  $X_j$ , inclusive.
- Common questions:
  - What is the probability of a specific event happening in the future?
  - What is the probability of a specific sequence of events happening in the future?



# An alternate formulation

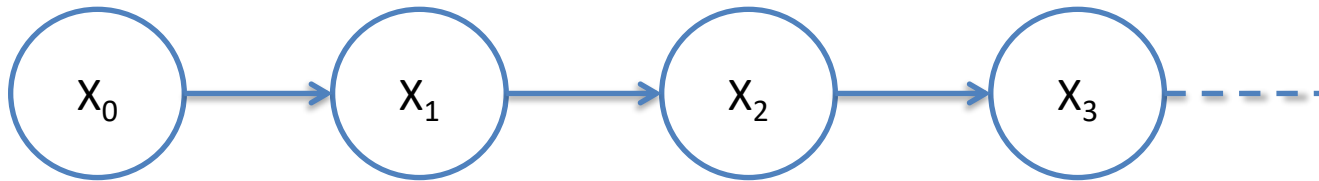
- We have a set of states,  $S$ .
- The Markov chain is always in *exactly one* state at any given time  $t$ .
- The chain transitions to a new state at each time  $t+1$  based only on the current state at time  $t$ .

$$p_{ij} = P(X_{t+1} = j \mid X_t = i)$$

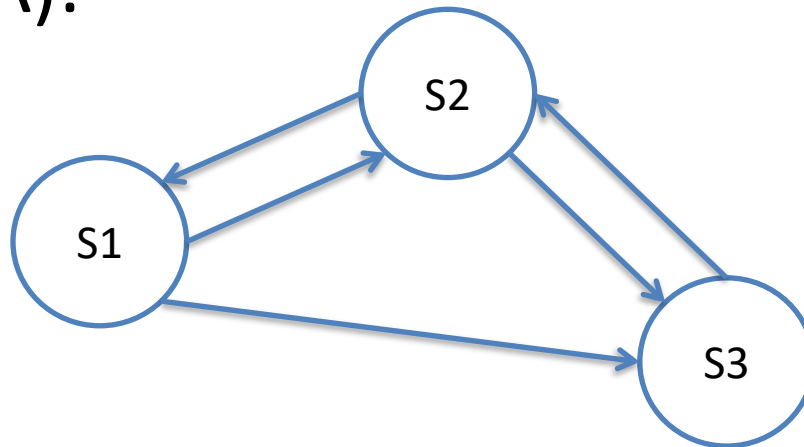
- Chain must specify  $p_{ij}$  for all  $i$  and  $j$ , and starting probabilities for  $P(X_0 = j)$  for all  $j$ .

# Two different representations

- As a Bayes net:



- As a state transition diagram (similar to a DFA/NFA):



# Formulate Comcast in both ways

- I have cable internet service from Comcast, and sometimes my router goes down. If the router is online, it will be online the next day with  $\text{prob}=0.8$ . If it's offline, it will be offline the next day with  $\text{prob}=0.4$ .
- Let's draw this situation in both ways.
- Assume on day 0, probability of router being down is 0.5.

# Comcast

- What is the probability my router is offline for 3 days in a row (days 0, 1, and 2)?
  - $P(X_2=\text{off}, X_1=\text{off}, X_0=\text{off})?$
  - $P(X_2=\text{off} | X_0=\text{off}, X_1=\text{off}) * P(X_0=\text{off}, X_1=\text{off})$  *[mult rule]*
  - $P(X_2=\text{off} | X_0=\text{off}, X_1=\text{off}) * P(X_1=\text{off} | X_0=\text{off}) * P(X_0=\text{off})$
  - $P(X_2=\text{off} | X_1=\text{off}) * P(X_1=\text{off} | X_0=\text{off}) * P(X_0=\text{off})$
  - $p_{\text{off,off}} * p_{\text{off,off}} * P(X_0=\text{off})$

$$P(x_{0:t}) = P(x_0) \prod_{i=1}^t P(x_i | x_{i-1})$$

# More Comcast

- Suppose I don't know if my router is online right now (day 0). What is the prob it is offline tomorrow?

- $P(X_1=\text{off})$

- $P(X_1=\text{off}) = P(X_1=\text{off}, X_0=\text{on}) + P(X_1=\text{off}, X_0=\text{off})$

- $P(X_1=\text{off}) = P(X_1=\text{off} | X_0=\text{on}) * P(X_0=\text{on})$   
 $+ P(X_1=\text{off} | X_0=\text{off}) * P(X_0=\text{off})$

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

# More Comcast

- Suppose I don't know if my router is online right now (day 0). What is the prob it is offline **the day after tomorrow?**

- $P(X_2=\text{off})$

- $P(X_2=\text{off}) = P(X_2=\text{off}, X_1=\text{on}) + P(X_2=\text{off}, X_1=\text{off})$

- $P(X_2=\text{off}) = P(X_2=\text{off} | X_1=\text{on}) * P(X_1=\text{on})$   
 $+ P(X_2=\text{off} | X_1=\text{off}) * P(X_1=\text{off})$

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

# Markov chains with matrices

- Define a transition matrix for the chain:

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

- Each row of the matrix represents the transition probabilities **leaving** a state; each column represents the **next** state.
- Let  $v_t$  = a row vector representing the probability that the chain is in each state at time  $t$ .
- $v_t = v_{t-1} * T$

# Formulate this matrix

- If the stock market is up one day, then it will be up the next day with  $\text{prob}=0.7$ .
- If it's down one day, it will be down the next day with  $\text{prob}=0.4$ .



# Mini-forward algorithm

- Suppose we are given the value of  $X_t$  or a probability distribution over  $X_t$  and we want to predict  $X_{t+1}, X_{t+2}, X_{t+3} \dots$
- Make row vector  $v_t = P(X_t)$ 
  - Note that  $v_t$  can be something like  $[1, 0]$  if you know the true value of  $X_t$ , or it can be a distribution over values.
- $v_{t+1} = v_t * T$
- $v_{t+2} = v_{t+1} * T = v_t * T * T = v_t * T^2$
- $v_{t+3} = v_t * T^3$
- $v_{t+n} = v_t * T^n$

# Back to the Apple Store...

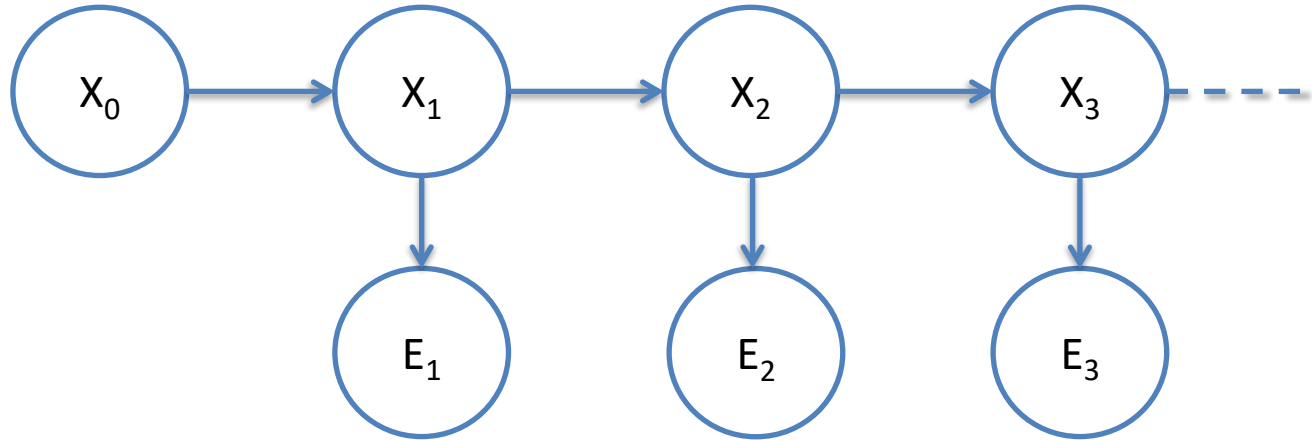
- You go to the Apple Store to buy the latest iPhone.
- Every minute, a new person joins the line with probability
  - 1 if the line length=0
  - $2/3$  if the line length=1
  - $1/3$  if the line length=2
  - 0 if the line length=3
- Immediately after (in the same minute), the first person is helped with prob = 0.5
- Model this as a Markov chain, assuming the line starts empty. Draw the state transition diagram.
- What is  $T$ ? What is  $v_0$ ?

- Markov chains are pretty easy!
- But sometimes they aren't realistic...
- What if we can't directly know the states of the model, but we can see some indirect evidence resulting from the states?

# Weather

- Regular Markov chain
  - Each day the weather is rainy or sunny.
  - $P(X_t = \text{rain} \mid X_{t-1} = \text{rain}) = 0.7$
  - $P(X_t = \text{sunny} \mid X_{t-1} = \text{sunny}) = 0.9$
- Twist:
  - Suppose you work in an office with no windows.  
All you can observe is whether your colleague brings their umbrella to work.

# Hidden Markov Models

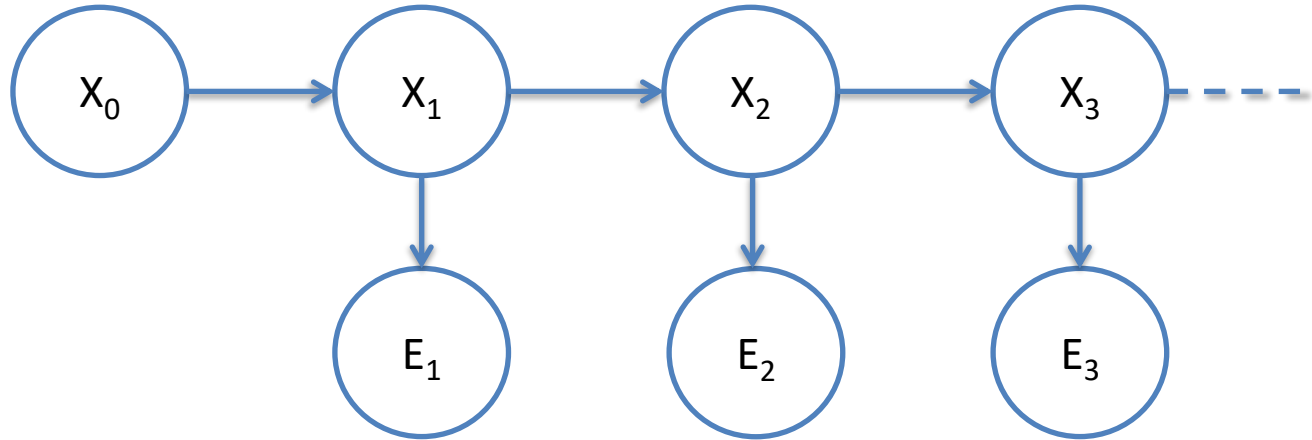


- The  $X$ 's are the state variables (never directly observed).
- The  $E$ 's are evidence variables.

# Common real-world uses

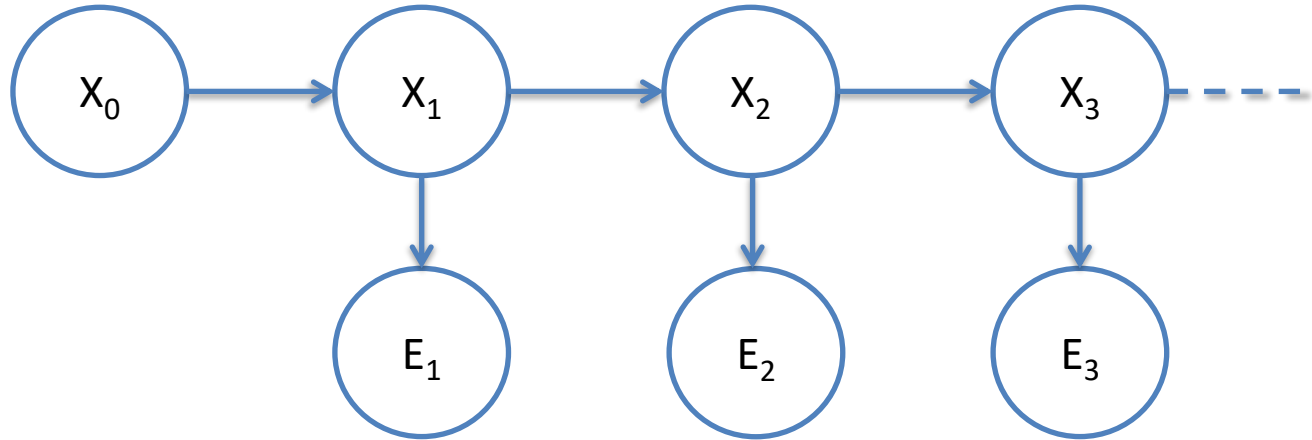
- Speech processing:
  - Observations are sounds, states are words or phonemes.
- Localization:
  - Observations are inputs from video cameras or microphones, state is the actual location.
- Video processing (example):
  - Extracting a human walking from each video frame. Observations are the frames, states are the positions of the legs.

# Hidden Markov Models



- $P(X_t \mid X_{t-1}, X_{t-2}, X_{t-3}, \dots) = P(X_t \mid X_{t-1})$
- $P(X_t \mid X_{t-1}) = P(X_1 \mid X_0)$
- $P(E_t \mid X_{0:t}, E_{0:t-1}) = P(E_t \mid X_t)$
- $P(E_t \mid X_t) = P(E_1 \mid X_1)$

# Hidden Markov Models



- What is  $P(X_{0:t}, E_{1:t})$ ?

$$P(X_0) \prod_{i=1}^t P(X_i \mid X_{i-1}) P(E_i \mid X_i)$$



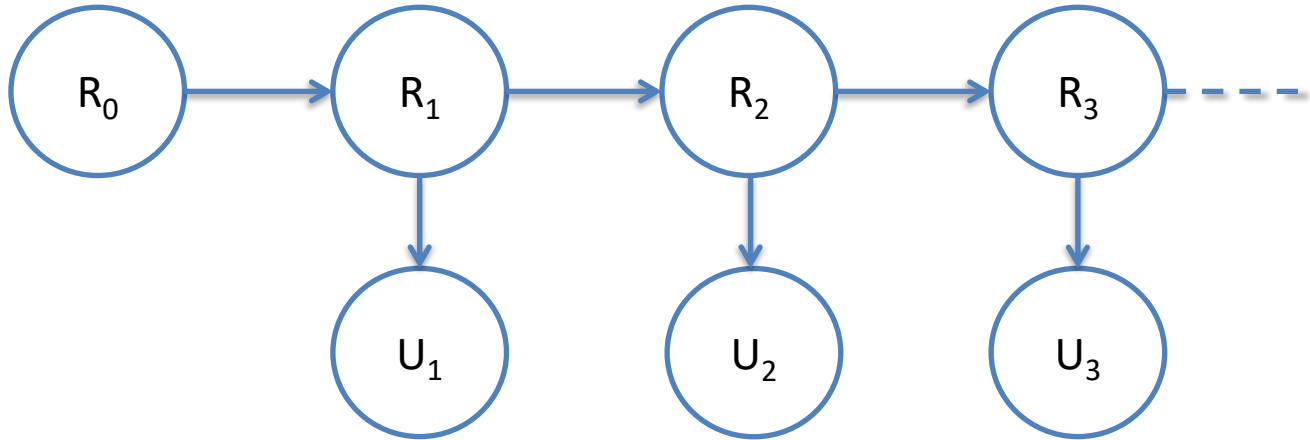
# Common questions

- **Filtering:** Given a sequence of observations, what is the most probable *current* state?
  - Compute  $P(X_t \mid e_{1:t})$
- **Prediction:** Given a sequence of observations, what is the most probable *future* state?
  - Compute  $P(X_{t+k} \mid e_{1:t})$  for some  $k > 0$
- **Smoothing:** Given a sequence of observations, what is the most probable *past* state?
  - Compute  $P(X_k \mid e_{1:t})$  for some  $k < t$

# Common questions

- **Most likely explanation:** Given a sequence of observations, what is the most probable sequence of states?
  - Compute  $\operatorname{argmax}_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
- **Learning:** How can we estimate the transition and sensor models from real-world data?  
(Future machine learning class?)

# Hidden Markov Models



- $P(R_t = \text{yes} \mid R_{t-1} = \text{yes}) = 0.7$   
 $P(R_t = \text{yes} \mid R_{t-1} = \text{no}) = 0.1$
- $P(U_t = \text{yes} \mid R_t = \text{yes}) = 0.9$   
 $P(U_t = \text{yes} \mid R_t = \text{no}) = 0.2$

# Filtering

- Filtering is concerned with finding the most probable "current" state from a sequence of evidence.
- Let's compute this.

# Recall the "mini-forward algorithm"

For Markov chains:

$$P(X_{t+1}) = \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t)$$

with matrices:  $v_{t+1} = v_t * T$ , with  $v_0 = P(X_0)$

For HMM's:

$$P(X_{t+1} \mid e_{1:t+1}) = \alpha P(e_{t+1} \mid X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})$$

# Forward algorithm

- Today is Day 2, and I've been pulling all-nighters for two days!
- My colleague brought their umbrella on days 1 and 2.
- What is the probability it is raining today?
  - that is, find  $P(X_t \mid e_{1:t})$  [*filtering*]
- Assume initial distribution of rain/not-rain for Day 0 is 50-50.

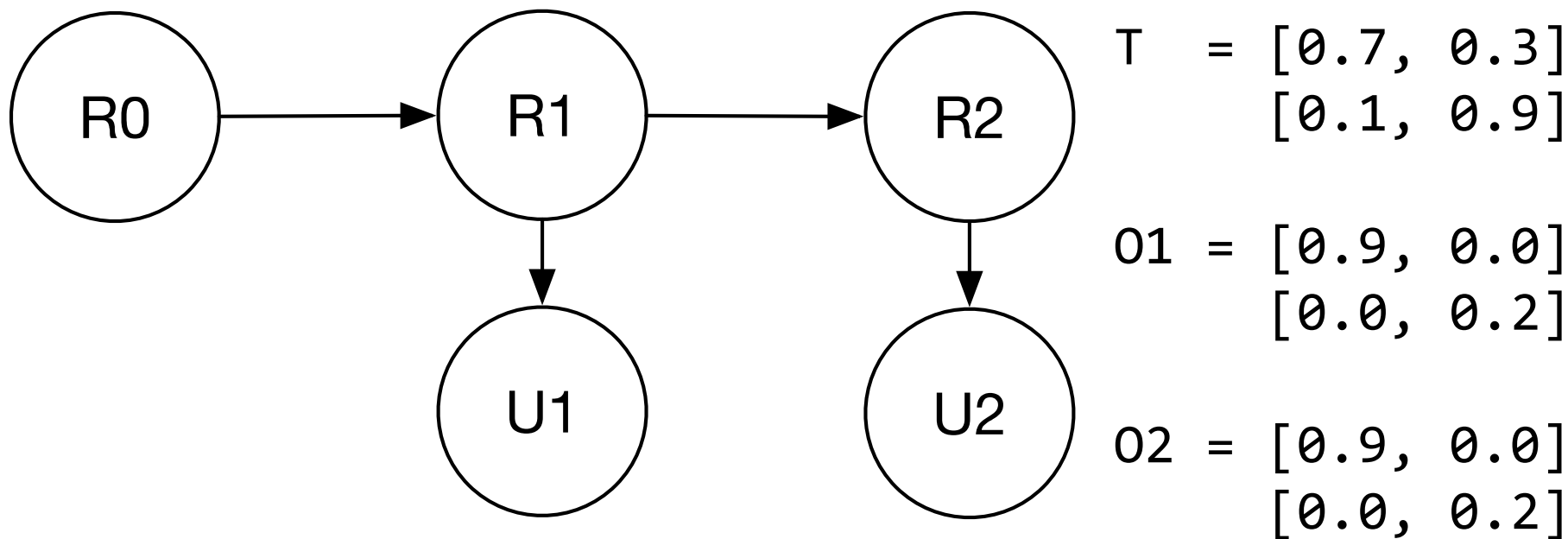
# Matrices to the rescue!

- Define a transition matrix  $T$  as normal.
- Define a sequence of observation matrices  $O_1$  through  $O_t$ .
- Each  $O$  matrix is a diagonal matrix with the entries corresponding to observation at time  $t$  given each state.

$$f_{1:t+1} = \alpha f_{1:t} \cdot T \cdot O_{t+1}$$

where each  $f$  is a row vector containing the probability distribution at timestep  $t$ .

$$f_{1:0}=[0.5, 0.5] \quad f_{1:1}=[0.75, 0.25] \quad f_{1:2}=[0.846, 0.154]$$



$$f_{1:0} = P(R_0) = [0.5, 0.5]$$

$$f_{1:1} = P(R_1 \mid u_1) = \alpha * f_{1:0} * T * O_1 = \alpha[0.36, 0.12] = [0.75, 0.25]$$

$$f_{1:2} = P(R_2 \mid u_1, u_2) = \alpha * f_{1:1} * T * O_2 = \alpha[0.495, 0.09] = [.846, .154]$$



# Forward algorithm

- Note that the forward algorithm only gives you the probability of  $X_t$  taking into account evidence at times 1 through  $t$ .
- In other words, say you calculate  $P(X_1 | e_1)$  using the forward algorithm, then you calculate  $P(X_2 | e_1, e_2)$ .
  - Knowing  $e_2$  changes your calculation of  $X_1$ .
  - That is,  $P(X_1 | e_1) \neq P(X_1 | e_1, e_2)$

# Backward algorithm

- Updates previous probabilities to take into account new evidence.
- Calculates  $P(X_k \mid e_{1:t})$  for  $k < t$ 
  - aka **smoothing**. (not the same kind of smoothing as in Naïve bayes)

# Backward algorithm

- Algorithm generates a *backward vector*  $b$  for every timestep  $t$ .
  - This vector is based on the observation at time  $k$  and the *next day's* backward vector.
- The initial backwards vector is for day  $t+1$  and is a column vector of all 1's.

$$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

$$b_{t+1:t} = [1; \cdots ; 1]$$

# Backwards algorithm

- Each backward vector is used to *scale* the previous day's forward vector.
- After normalization, this is the updated probability for day k.

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

- (Remember, that multiplication above is an item by item multiplication, not a matrix multiplication.)

# Backward matrices

- Main equations:

$$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

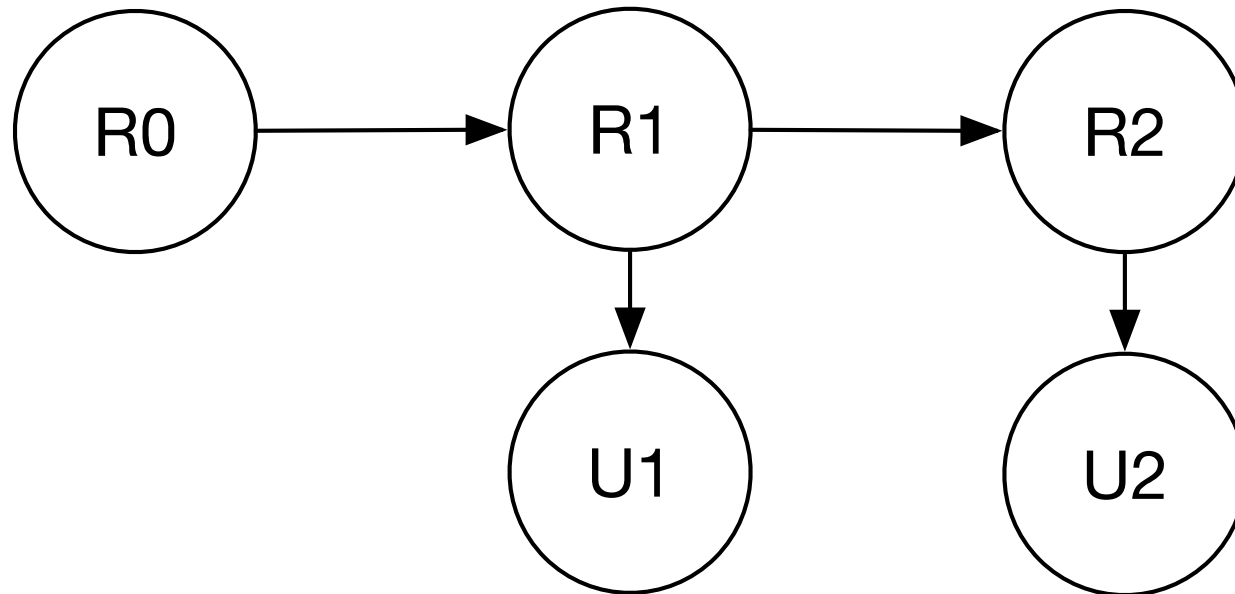
$$b_{t+1:t} = [1; \cdots ; 1] \quad (\text{column vector of 1s})$$

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

$f1:0=[0.5, 0.5]$        $f1:1=[0.75, 0.25]$        $f1:2=[0.846, 0.154]$

$b1:2=[0.4509, 0.1107]$     $b2:2=[0.69, 0.27]$        $b3:2=[1; 1]$

$mult=[0.803, 0.197]$        $mult=[0.885, 0.115]$



$T = \begin{bmatrix} 0.7, & 0.3 \\ 0.1, & 0.9 \end{bmatrix}$

$O1 = \begin{bmatrix} 0.9, & 0.0 \\ 0.0, & 0.2 \end{bmatrix}$

$O2 = \begin{bmatrix} 0.9, & 0.0 \\ 0.0, & 0.2 \end{bmatrix}$

$b3:2 = [1; 1]$

$b2:2 = T * O2 * b3:2 = [0.69, 0.27]$

$P(R1 \mid u1, u2) = \alpha f1:1 \times b2:2 = \alpha[0.5175, 0.0675] = [0.885, 0.115]$

$b1:2 = T * O1 * b2:2 = [0.4509, 0.1107]$

$P(R0 \mid u1, u2) = \alpha f1:0 \times b1:2 = \alpha[0.2255, 0.0554] = [0.803, 0.197]$

# Forward-backward algorithm

$$f_{1:0} = P(X_0)$$

$$f_{1:t+1} = \alpha f_{1:t} \cdot T \cdot O_{t+1}$$

Compute these forward from  $X_0$  to wherever you want to stop ( $X_t$ )

$$b_{t+1:t} = [1; \cdots ; 1]$$

$$b_{k:t} = T \cdot O_k \cdot b_{k+1:t}$$

$$P(X_k \mid e_{1:t}) = \alpha f_{1:k} \times b_{k+1:t}$$

Compute these backwards from  $X_{t+1}$  to  $X_0$ .