Exercise 2

Problem

$$\max_{x1,x2\in\mathbb{R}} 2x_0x_1$$
$$\text{s.t.} \begin{cases} x_0^2 - x_0 = 0 \\ -x_1^2 + x_1 + \frac{1}{4} \geq 0 \end{cases}$$

Solution

We have $x_0^2 - x_0 = 0$. So $x_0 \in \{0, 1\}$.

Case 1: $x_0 = 0$

The objective function becomes 0.

Case 2: $x_0 = 1$

The objective function becomes $2x_1$.

The condition is $-x_1^2 + x_1 + \frac{1}{4} \geq 0$.
And yet, $-x_1^2 + x_1 + \frac{1}{4} = \frac{1}{2} - \left(x_1 - \frac{1}{2}\right)^2$.
So we need to have $\frac{1}{2} - \left(x_1 - \frac{1}{2}\right)^2 \geq 0$.
So we need to have $\frac{1}{2} \geq \left(x_1 - \frac{1}{2}\right)^2$.
So we need to have $x_1 - \frac{1}{2} \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.
So we need to have $x_1 \in \left[\frac{1}{2} - \frac{1}{\sqrt{2}}, \frac{1}{2} + \frac{1}{\sqrt{2}}\right]$.

So we have $2x_1 \in \left[1 - \sqrt{2}, 1 + \sqrt{2}\right]$.
So the maximum is $1 + \sqrt{2}$, for $x_1 = \frac{1}{2} + \frac{1}{\sqrt{2}}$.

Conclusion

The maximum is :

$$1 + \sqrt{2}$$

for

$$(x_0, x_1) = \left(1, \frac{1}{2} + \frac{1}{\sqrt{2}}\right)$$