Exercise 1

We want to show that $(x+y)^2 - 4xy \geq 0$

Question 1

The moment matrix is :

$$M = \begin{bmatrix} 1 & x & y \\ x & x^2 & xy \\ y & yx & y^2 \end{bmatrix}$$

Question 2

M semi-positive definite
So $\forall X, X^\top MX \geq 0$

For $X = (0, -1, 1)$
We get $(x+y)^2 - 4xy \geq 0$

Question 3

$w = (1, x, y)^\top$

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \; N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have $M = N^\top N$
$w^\top Mw = w^\top N^\top Nw = (Nw)^\top Nw = ||Nw||^2 \geq 0$
So $(x+y)^2 - 4xy \geq 0$