First polynomial optimisation exercises

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1 Questions

1. **AM-GM inequality** Consider the following polynomial problem without constraints:

$$\min_{x,y\in\mathbb{R}} \quad (x+y)^2 - 4xy \tag{1}$$

   (a) Write down the level 1 moment matrix for this problem.

   (b) Use this moment matrix to prove that

$$(x+y)^2 - 4xy \geq 0 \tag{2}$$

   Hence you have a polynomial optimisation proof of the arithmetic-geometric mean inequality $\sqrt{xy} \leq \frac{x+y}{2}$.

   (c) You can also prove the AM-GM inequality using the dual approach of sum-of-squares. I'll illustrate the idea very briefly here for this problem. Let w be a vector of monomials up to degree 1, i.e.

$$w = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \tag{3}$$

   Now take any positive semidefinite matrix M. As M is positive semidefinite, we can write it as $M = N^T N$ for some matrix N (for example the square root of M would work). Then we have that

$$w^T M w = w^T N^T N w = (Nw)^T (Nw) = \sum_i r_i(x,y)^2 \tag{4}$$

   where we notice that the elements of the vector Nw are just polynomials in the variables (x, y). Thus, for any positive semidefinite matrix we get a sum-of-squares polynomial. So if we want to prove that the polynomial $(x+y)^2 - 4xy$ is nonnegative, we could prove $(x+y)^2 - 4xy$ it is a sum of squares, or by the above it is sufficient to prove that there exists a positive semidefinite matrix M such that

$$(x+y)^2 - 4xy = w^T M w. \tag{5}$$

   By equating coefficients of the monomials on both sides, we get **linear** constraints on the elements of M.

   Use this idea to systematically find a sum of squares decomposition of $(x+y)^2 - 4xy$ and hence give a second proof of the AM-GM inequality.

2. **A commutative problem**

   Consider the following polynomial optimisation problem

$$p^* = \begin{array}{rl} \max\limits_{x_1,x_2\in\mathbb{R}} & 2x_0x_1 \\ \text{s.t.} & x_0^2 - x_0 = 0 \\ & -x_1^2 + x_1 + 1/4 \geq 0 \end{array} \tag{6}$$

(a) Write down a level 1 and level 2 SDP relaxations of the problem.

(b) Using an SDP solver[1], find the solutions to the 2 SDP relaxations.

(c) Use your computations to prove that the optimal value of the original problem is $p^* = 1 - \sqrt{2}$.

3. **A noncommutative generalisation**

Consider the following noncommutative polynomial optimisation problem

$$p^*_{NC} = \begin{aligned} \max_{\substack{X_1, X_2 \in B(H) \\ v \in H}} \quad & v^\dagger (X_0 X_1 + X_1 X_0) v \\ \text{s.t.} \quad & X_0^2 - X_0 = 0 \\ & -X_1^2 + X_1 + 1/4 \geq 0 \\ & X_i^\dagger = X_i \\ & v^\dagger v = 1 \end{aligned} \tag{7}$$

(a) Write down a level 1 SDP relaxation of the problem.

(b) Solve the SDP relaxation (perhaps you can use the previous question?)

(c) Find a Hilbert space, unit vector and $X_1, X_2 \in B(H)$ that are feasible and achieve the same objective value as the SDP relaxation. Use this to argue about the optimal value of p^*_{NC}. [Hint: the Hilbert space required for the optimal value might be very simple.]

[1] I'd recommend using PICOS to solve SDPs using python. You can find the package here https://picos-api.gitlab.io/picos/