Two-Sample Kernel Based Tests

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Outline

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1. The two-sample problem
2. Friedman's two-sample test [1]: leverage regression and classification techniques

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2. Friedman's two-sample test [1]: leverage regression and classification techniques
3. Univariate data and linear scoring functions: permutation t-test
4. Twitter example for text data
5. Image data: airplanes and cars / pigeons and roosters

The Two-Sample Problem

Problem

$\{\mathbf{x}_i\}_1^N$ from $p(\mathbf{x})$ and $\{\mathbf{z}_i\}_1^M$ from $q(\mathbf{z})$ testing $\mathcal{H}_A$: $p \neq q$ against $\mathcal{H}_0$: $p = q$

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5. Determine the permutation null distribution of the above statistic to yield a p-value.

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With multivariate data, the test is close to Hotelling's T^2-test.

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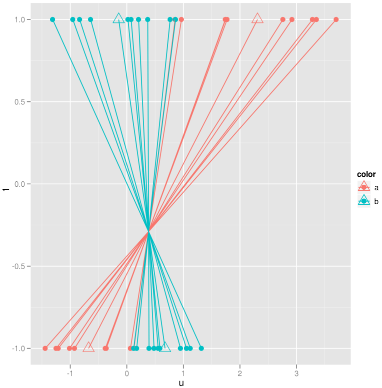
$$|T(\{u_i\}_1^N, \{u_i\}_{N+1}^{N+M})| = 6.12 = |T(\{s_i\}_1^N, \{s_i\}_{N+1}^{N+M})|$$



Twitter Example

Twitter Data

Raw:

```
"BarackObama: We need to reward education reforms that are
driven not by Washington, but by principals and teachers and
parents. http://OFA.BO/6p2EMy"
"SarahPalinUSA: You betcha!! MT \"@AlaskaAces: Alaska Aces
are 2011 Kelly Cup Champs w/ 5-3 win over Kalamazoo Wings!
Aces win  ECHL Championship series 4-1\""
```

After pre-processing:

```
"we need to reward education reforms that are driven not by
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- Define the feature map from $\mathcal{X}$ to $\mathbb{R}^{|\mathcal{A}|^k}$ by $\Phi_k(x) = (\phi_a(x))_{a \in \mathcal{A}^k}$ where $\phi_a(x)$ is the number of times a occurs in x: $\{\#aaa, \#aab, \#aac, \ldots, \}$.

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- $K_k(x, y) = \langle \Phi_k(x), \Phi_k(y) \rangle$.

Support Vector Machines for Regression

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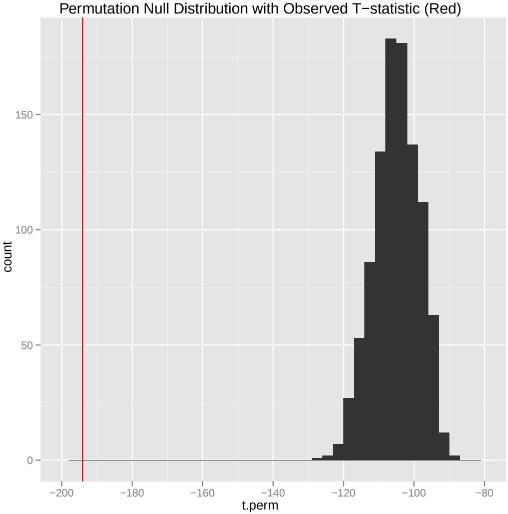
The solution has the form $\hat{f}(x) = \sum_{i=1}^{N} \hat{\alpha}_i K(x, x_i)$, where $K(x, y) = \langle h(x), h(y) \rangle$.

Twitter Example

$p < .001$:



Power Simulations at .05 Level

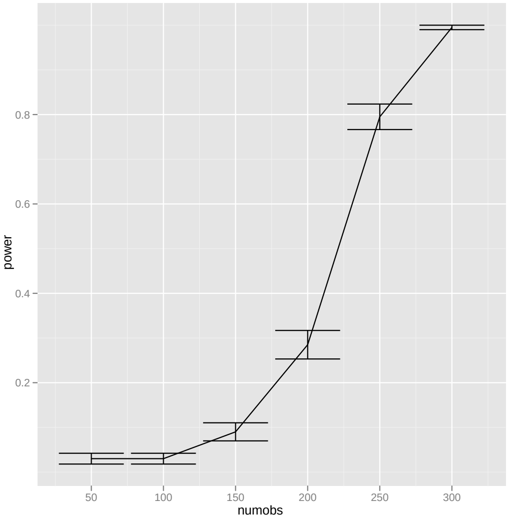


Image Data (Cars)

Caltech 101 Object Categories [2]
The cars are 300×197 grayscale.

Planes Before

The planes aren't.

Planes After

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The homogeneous polynomial kernel,

$$K(x, x') = \langle \Phi(x), \Phi(x') \rangle = \langle x, x' \rangle^d,$$

corresponds to the mapping

$\Phi(x) = [x_1^d, \ldots, x_p^d, x_1^{d-1} x_2, \ldots, x_p^{d-1} x_{p-1}]^T \in \mathbb{R}^{d'}$, where $d' = \binom{d+N-1}{d}$.

Standardization

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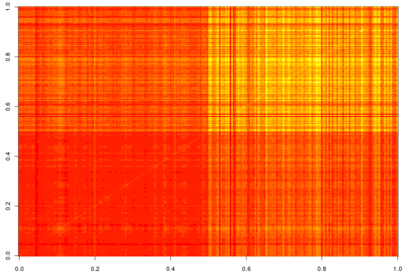
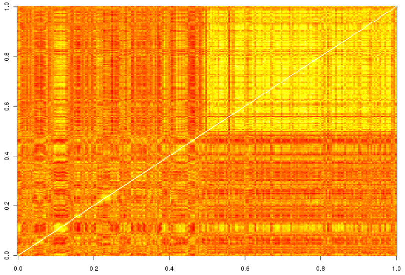
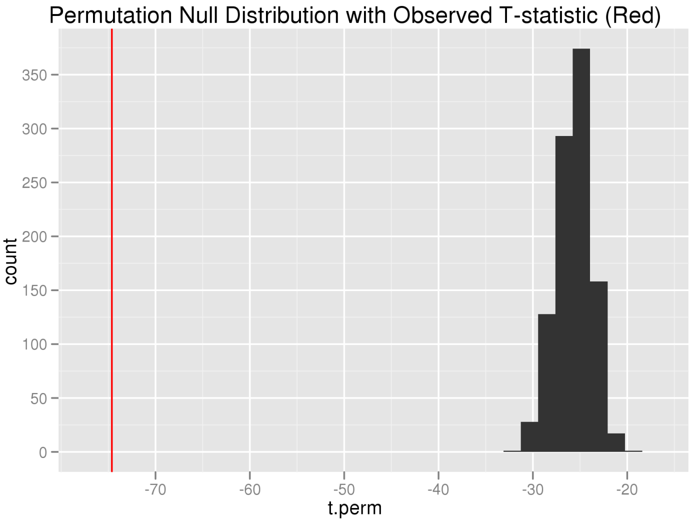
Unscaled linear kernel matrix, left; scaled, right

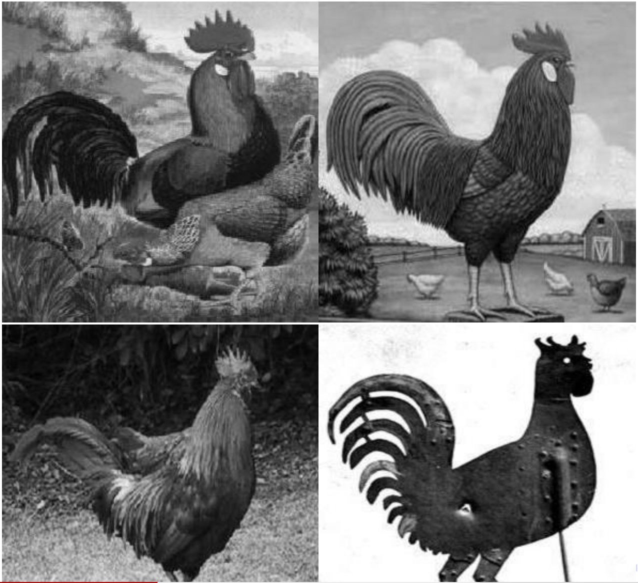




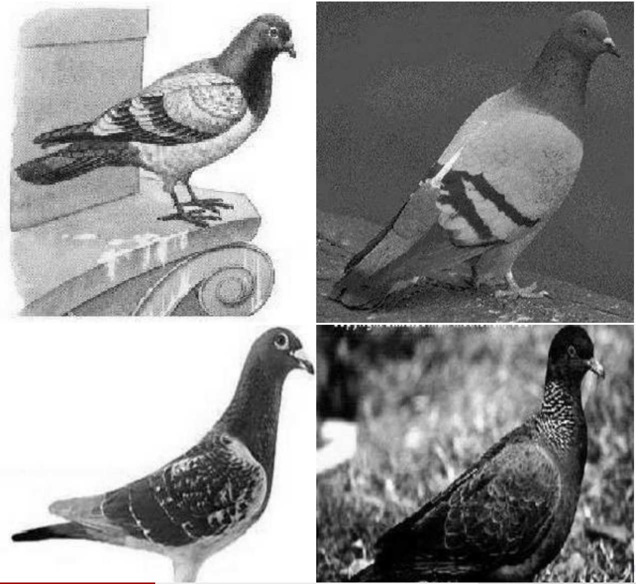
Car/Airplane Example (Linear Kernel)



Roosters

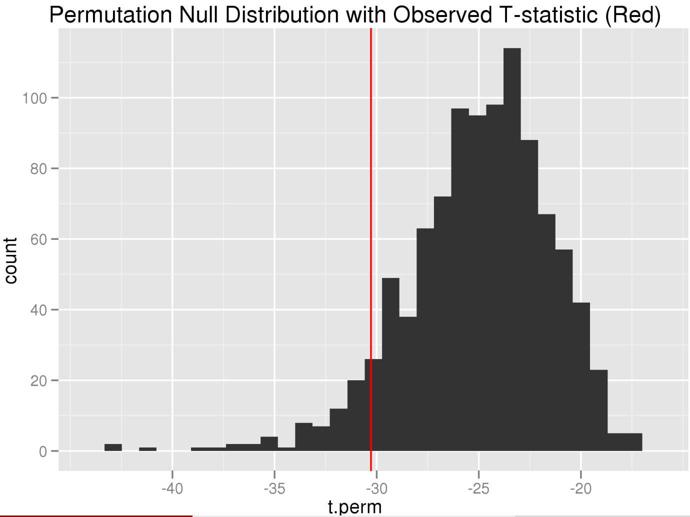


Pigeons



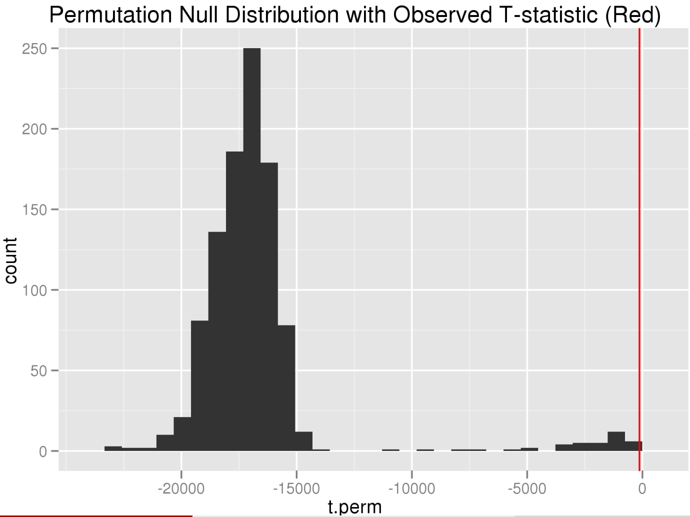
Rooster/Pigeon Example (Linear Kernel)

$p = .138$



Rooster/Pigeon Example (Inhomogeneous Degree 4)

$p < .001$



Future Work

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- Side Information:

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- Side Information: phylogenetic tree, Twitter post times
- Heterogeneous Data (Wikipedia pages): optimal combinations of kernels via SDPs, KL divergence

References I

J. Friedman, "On Multivariate Goodness–of–Fit and Two–Sample Testing," *Proceedings of Phystat2003, http://www.slac.stanford.edu/econf/C*, vol. 30908, 2003.

L. Fei-Fei, R. Fergus, and P. Perona, "Learning generative visual models from few training examples: An incremental bayesian approach tested on 101 object categories," *Computer Vision and Image Understanding*, vol. 106, no. 1, pp. 59–70, 2007.