Thesis Proposal: Two-Sample Kernel Based Tests

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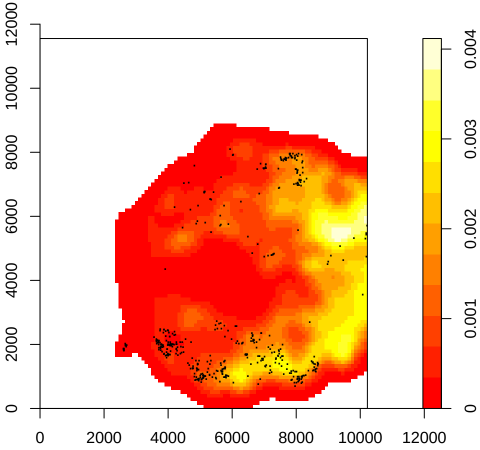
- Twitter example for text data
- Future work: theory for general case, heterogeneous data and combining kernels

Breast Cancer Data: Spatial



Breast Cancer Data: Survival

Pathology no.	Initial Diagnosis Date	Relapse or Disease Free	RDF (R=relapsed; F=DF)	Recurrence Date	Las
98_17969D	1997-08-25	Disease Free	F	Disease Free	
97_24046C8	1997-08-25	Disease Free	F	Disease Free	
98_8501C1	1998-04-03	Disease Free	F	Disease Free	
98_8501A1	1998-04-03	Disease Free	F	Disease Free	
98_9134D4	1998-04-09	Left in-situ BrCa in 1999 (2nd primary cancer, not a metastasis from the right BrCa in 1997)	F	Disease Free	
98_9134B	1998-04-09	Left in-situ BrCa in 1999 (2nd primary cancer, not a metastasis from the right BrCa in 1997)	F	Disease Free	
98_14783B1	1998-06-10	bone, brain, lymph nodes, pericardium, liver metastasis	R	2004-07-30	
98_14783A	1998-06-10	bone, brain, lymph nodes, pericardium, liver metastasis	R	2004-07-30	
98_16169C2	1998-06-24	Disease Free	F	Disease Free	
98_16169A	1998-06-24	Disease Free	F	Disease Free	
98_16169B	1998-06-24	Disease Free	F	Disease Free	
98_16253C1	1998-06-25	Disease Free	F	Disease Free	
60C1	1998-07-10	Disease Free	F	Disease Free	

Breast Cancer Data: Medical

Pathology no.	Age at time of diagnosis	Gender	SLN tumor status	Diagnosis	ER status	PR status	Her-2 overexpressio
98_17969D	68	F	+	Invasive ductal carcinoma (IDC)	-	-	-
97_24046C8	68	F	+	Invasive ductal carcinoma (IDC)	-	-	-
98_8501C1	51	F	+	IDC & DCIS	+	+	?
98_8501A1	51	F	+	IDC & DCIS	+	+	?
98_9134D4	70	F	+	IDC	+	+	n/a
98_9134B	70	F	+	IDC	+	+	n/a
98_14783B1	67	F	+	IDC & DCIS	+	+	+
98_14783A	67	F	+	IDC & DCIS	+	+	+
98_16169C2	79	F	+mic	IDC	+	+	+
98_16169A	79	F	+mic	IDC	+	+	+
98_16169B	79	F	+mic	IDC	+	+	+
98_16253C1	70	F	+mic	IDC & DCIS	+	-	-
60C1	51	F	- (rare keratin+ cells)	IDC & DCIS	+	+	+
				IDC; DCIS;			

Breast Cancer Study

- How do you deal with the data integration problem?

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- Are there any differences (spatial, medical) between women who relapse and those who remain disease free?
- Two-sample tests

Friedman's Two-Sample Test

$\{\mathbf{x}_i\}_1^N$ from $p(\mathbf{x})$ and $\{\mathbf{z}_i\}_1^M$ from $q(\mathbf{x})$ testing
$\mathcal{H}_A$: $p \neq q$ against $\mathcal{H}_0$: $p = q$

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2. Assign label $y_i = 1$ to the first group and $y_i = -1$ to the second group.
3. Apply a binary classification learning machine f to the training data to score the observations $\{s_i = f(\mathbf{u}_i)\}_1^{N+M}$.

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5. Determine the permutation null distribution of the above statistic to yield a p-value.

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Strategy: Analyze the simple case (univariate/linear) and attempt to generalize.

Other Work

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- Bentkus et al. [4], Shao [5] proved Berry-Esseen bounds for Student's t-statistic in independent (but not i.d.) case.

Let $\Phi(t)$ denote the standard normal CDF and T be a random variable that is distributed according to our permutation t null distribution. Can we get a bound on

$$\sup_{t \in \mathbb{R}} |P(T \leq t) - \Phi(t)|?$$

Stein's Method and the Randomization Distribution

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We are finishing up a proof using the method of exchangeable pairs where our bound is $O(N^{-1/4})$.

Proof Ideas

Chen et al. [6]:

Theorem

If T, T' are mean 0, variance 1 exchangeable random variables satisfying

$$\mathbb{E}[T - T' | T] = \lambda(T - R)$$

for some $\lambda \in (0,1)$ and some random variable R, then

$$\sup_{t \in \mathbb{R}} |P(T \leq t) - \Phi(t)| \leq B + (2\pi)^{-1/4} \sqrt{\frac{\mathbb{E}|T' - T|^3}{\lambda}} + \mathbb{E}|R|,$$

where $B \leq \frac{\Theta}{2\lambda}$ and $\Theta = \sqrt{\mathrm{var}(\mathbb{E}[(T' - T)^2 | T])}$.

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- General contraction property, or "approximate case," from Stein et al. [8] and Holmes [9]
- Computational and simulation aided proof (Borwein [10]) with efficient t-statistic updates similar to Diaconis et al. [11]

Exchangeable Pair

For simplicity, assume $M = N$. We have data $\{u_1, \ldots, u_N, u_{N+1}, \ldots, u_{2N}\}$. Take a uniformly random permutation π, and let

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Let (I, J) be a uniformly random transposition between groups: over the N^2 cases where $1 \leq I \leq N$ and $N+1 \leq J \leq 2N$. Then

$$T' = T\left(\{u_{\pi\circ(I,J)(i)}\}_{i=1}^{N}, \{u_{\pi\circ(I,J)(i)}\}_{i=N+1}^{2N}\right).$$

T and T' form an exchangeable pair.

Bound Calculations

$$\sup_{t \in \mathbb{R}} |P(T \leq t) - \Phi(t)| \leq \underbrace{\frac{\sqrt{\mathrm{var}(\mathbb{E}[(T' - T)^2 | T])}}{2\lambda}}_{1}$$
$$+ \underbrace{(2\pi)^{-1/4} \sqrt{\frac{\mathbb{E}|T' - T|^3}{\lambda}}}_{2}$$
$$+ \underbrace{\mathbb{E}| - \frac{1}{\lambda}\mathbb{E}[T - T' | T] + T|}_{3}$$

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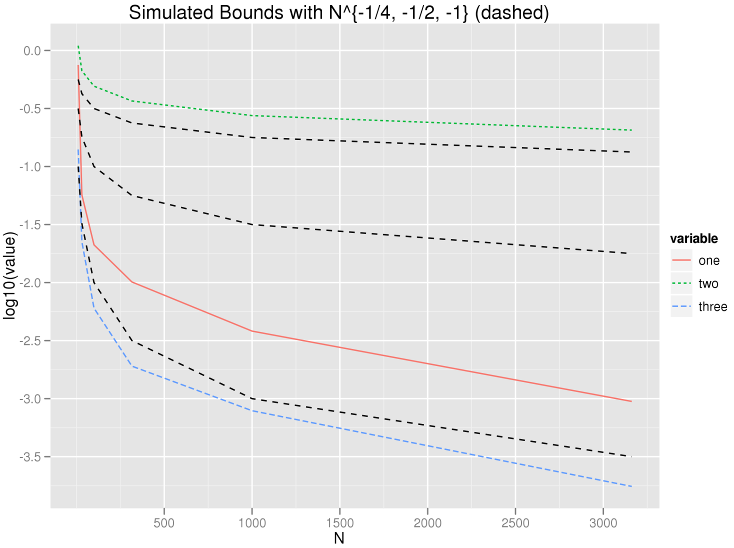
5. Calculate conditional expectations with respect to T and condition on T for the unconditional expectations.
6. Average over many values of T, and repeat for a sequence of N's.

Simulated Data

```
                 T      Tprime    N       lambda
1       -1.6646969 -1.4150824   10 0.2000000000
2       -1.6646969 -2.8302749   10 0.2000000000
3       -1.6646969 -1.5975851   10 0.2000000000
4       -1.6646969 -2.1813520   10 0.2000000000
5       -1.6646969 -2.5914846   10 0.2000000000
6       -1.6646969 -1.9817233   10 0.2000000000
...
88873283  0.2425782  0.3088987 3162 0.0006325111
88873284  0.2425782  0.2740881 3162 0.0006325111
88873285  0.2425782  0.2816923 3162 0.0006325111
88873286  0.2425782  0.2992468 3162 0.0006325111
88873287  0.2425782  0.2931195 3162 0.0006325111
88873288  0.2425782  0.2677967 3162 0.0006325111
```

Bounds Comparison



Twitter Example

Twitter Data

Raw:

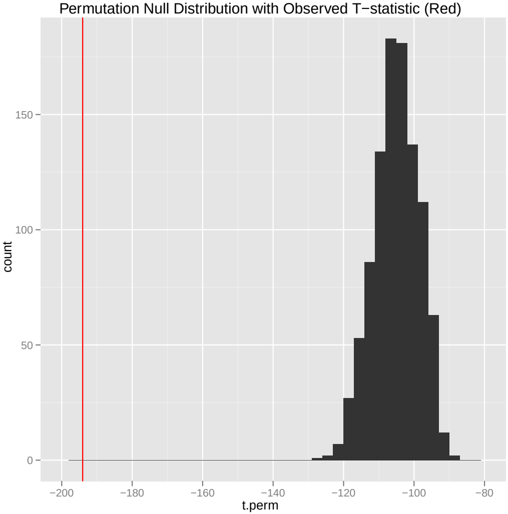
```
"BarackObama: We need to reward education reforms that are
driven not by Washington, but by principals and teachers and
parents. http://OFA.BO/6p2EMy"
"SarahPalinUSA: You betcha!! MT \"@AlaskaAces: Alaska Aces
are 2011 Kelly Cup Champs w/ 5-3 win over Kalamazoo Wings!
Aces win  ECHL Championship series 4-1\""
```

After pre-processing:

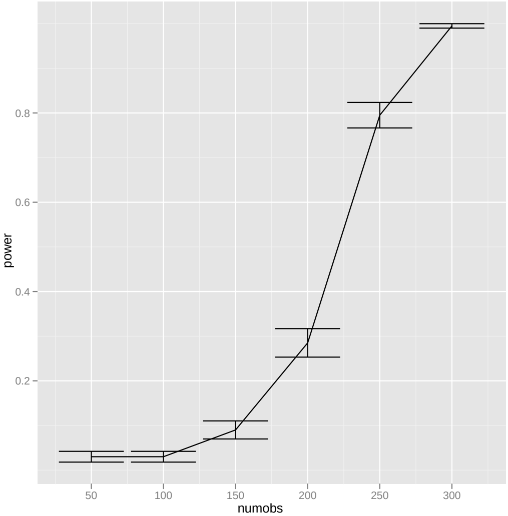
```
"we need to reward education reforms that are driven not by
washington but by principals and teachers and parents "
"you betcha mt alaskaaces alaska aces are  kelly cup champs
w  win over kalamazoo wings aces win  echl championship
series "
```

Twitter Example

$p < .001$:



Power Simulations at .05 Level



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References I

J. Friedman, "On Multivariate Goodness–of–Fit and Two–Sample Testing," *Proceedings of Phystat2003, http://www.slac.stanford.edu/econf/C*, vol. 30908, 2003.

R. Fisher, "The design of experiments.," 1935.

E. Lehmann, *Elements of large-sample theory*. Springer Verlag, 1999.

V. Bentkus and F. Götze, "The berry-esseen bound for student's statistic," *The Annals of Probability*, vol. 24, no. 1, pp. 491–503, 1996.

Q. Shao, "An explicit berry-esseen bound for students t-statistic via steins method," *Steins Method and Applications (AD Barbour and LHY Chen eds). Lecture Notes Series, Institute for Mathematical Sciences, NUS*, vol. 5, pp. 143–155, 2005.

References II

- L. Chen, L. Goldstein, and Q. Shao, *Normal Approximation by Stein's Method.* Springer Verlag, 2010.
- C. Stein, "Approximate computation of expectations," *Lecture Notes-Monograph Series*, vol. 7, 1986.
- C. Stein, P. Diaconis, S. Holmes, and G. Reinert, "Use of exchangeable pairs in the analysis of simulations," *Lecture Notes-Monograph Series*, pp. 1–26, 2004.
- S. Holmes, "Stein's method for birth and death chains," *Lecture Notes-Monograph Series*, pp. 45–67, 2004.
- J. Borwein and D. Bailey, *Mathematics by Experiment: Plausible reasoning in the 21st century.* AK Peters, 2004.

References III

P. Diaconis and S. Holmes, "Gray codes for randomization procedures," *Statistics and Computing*, vol. 4, no. 4, pp. 287–302, 1994.