Introduction to Cryptography
Diophantine Equations

(1) Determine whether the following Diophantine equations have solutions. You do not need to solve the equations.
    (a) $10X + 15Y = 1$
    (b) $9X + 16Y = 1$
    (c) $24X + 56Y = 1$

(2) Find integers X and Y that solve the following Diophantine equations, or show that they do not have a solution.
    (a) $223X + 75Y = 1$
    (b) $35X + 49Y = 1$
    (c) $7X + 24Y = 1$

(3) Find an integer X that satisfy the following equations, or show that no such number exists.
    (a) $75X \equiv 1 \mod 223$
    (b) $35X \equiv 1 \mod 49$
    (c) $7X \equiv 1 \mod 24$

(4) (a) Find a solution to the Diophantine equation $13X + 8Y = 1$.
    (b) Find a solution to the Diophantine equation $21X + 13Y = 1$.
    (c) Based on your answers, make a conjecture for a solution to the following Diophantine equation $F_{n+1}X + F_nY = 1$.

(5) If $(\widetilde{X}, \widetilde{Y})$ is a solution to the Diophantine equation $AX + BY = 1$, what is a solution to the Diophantine equation $AX + BY = 2$? What about $AX + BY = n$ where n is an integer?

(6) You have a 7-liter bucket and a 24-liter bucket. Using these, you can add or remove exactly 7 or 24 liters of water at a time into or out of a large tank. If the large tank begins empty, explain how you can use the buckets to end with exactly 2 liters of water in the large tank.

(7) What if you instead have a 4-liter bucket and a 6-liter bucket. Can you use the buckets to end with exactly 1 liter of water in the large tank? What about 2 liters?

(8) If $(\widetilde{X}, \widetilde{Y})$ is a solution to the Diophantine equation $AX + BY = 1$, and if t is any integer, show that $(\widetilde{X} + Bt, \widetilde{Y} - At)$ is also a solution to this Diophantine equation.

(9) Use the result from the previous problem to find infinitely many solutions to the following Diophantine equations, if a solution exists.
    (a) $223X + 75Y = 1$
    (b) $35X + 49Y = 1$
    (c) $7X + 24Y = 1$

(10) (Optional Challenge Problem): When Dr. Benge cashed a check at his bank, the teller mistook the number of cents for the number of dollars and vice versa. Unaware of this, Dr. Benge spent 68 cents and then noticed to his surprise that he had twice the amount of the original check. Determine the smallest value for which the check could have been written.