# 1D ADCIRC Derivation

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## 1 Continuity Equation

Start with the vertically integrated continuity equation:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH) = 0 \tag{1}$$

where

$$\begin{aligned}
H &\equiv \zeta + h \\
\zeta &= \text{free surface departure from the geoid} \\
h &= \text{bathymetric depth (distance from the geoid to the bottom)} \\
u &= \text{vertically varying velocity in the x-direction} \\
U &= \frac{1}{H}\int_{-h}^{\zeta} u \mathrm{d}z = \text{depth-averaged velocity in the x-direction}
\end{aligned}$$

Take $\partial/\partial t$ of (1):

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) = 0 \tag{2}$$

Add (2) to (1) multiplied by the parameter $\tau_0$, which may be variable in space:

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + \tau_0\left(\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH)\right) = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + \tau_0\frac{\partial H}{\partial t} + \tau_0\frac{\partial}{\partial x}(UH) = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \tau_0\frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) = 0 \tag{3}$$

Now, define $\tilde{J}_x$:

$$\tilde{J}_x \equiv \frac{\partial}{\partial t}(UH) + \tau_0 UH \tag{4}$$

$$= \frac{\partial Q}{\partial t} + \tau_0 Q \tag{5}$$

where

$$Q = UH$$

Recall that $\tau_0$, $U$, and $H$ are all variable in $x$ and take $\partial/\partial x$ of (5), noting the use of the product rule:

$$\frac{\partial \tilde{J}_x}{\partial x} = \frac{\partial}{\partial x}\left[\tau_0 Q + \frac{\partial Q}{\partial t}\right]$$

$$= \frac{\partial}{\partial x}(\tau_0 Q) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q$$

$$= Q\frac{\partial \tau_0}{\partial x} + \tau_0\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q$$

$$= \tau_0\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q + Q\frac{\partial \tau_0}{\partial x}$$

$$= \tau_0\frac{\partial (UH)}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + UH\frac{\partial \tau_0}{\partial x} \tag{6}$$

Now, returning to equation (3), let's add zero to it in the form of:

$$UH\frac{\partial \tau_0}{\partial x} - UH\frac{\partial \tau_0}{\partial x} = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \underbrace{\tau_0\frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + UH\frac{\partial \tau_0}{\partial x}}_{\text{Note that this is equivalent to (6)}} - UH\frac{\partial \tau_0}{\partial x} = 0$$

and substituting (6) in gives us:

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \frac{\partial \tilde{J}_x}{\partial x} - UH\frac{\partial \tau_0}{\partial x} = 0 \tag{7}$$

If we assume that bathymetric depth is constant, then

$$\frac{\partial H}{\partial t} = \frac{\partial \zeta}{\partial t}$$

$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 \zeta}{\partial t^2}$$

and (7) can be rewritten as

$$\frac{\partial^2 \zeta}{\partial t^2} + \tau_0 \frac{\partial \zeta}{\partial t} + \frac{\partial \tilde{J}_x}{\partial x} - UH \frac{\partial \tau_0}{\partial x} = 0 \tag{8}$$

## 1.1 Apply the weighted residual method to arrive at the weak form

First, we'll define the inner product notation $\langle A, B \rangle$ as the integral over the domain $\Omega$ of $A$ and $B$ multiplied together.

$$\langle A, B \rangle \equiv \int\limits_{\Omega} AB d\Omega$$

We apply the weighted residual method to (8) by multiplying each term by a weighting function $\phi_j$ and integrating over the horizontal computational domain $\Omega$. Written using the inner product notation, we have

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_0 \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle + \left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle - \left\langle UH \frac{\partial \tau_0}{\partial x}, \phi_j \right\rangle = 0 \tag{9}$$

The third term, which involves $\tilde{J}_x$, can be integrated using integration by parts. Recall that integration by parts is defined as

$$\int u dv = uv - \int v du$$

So looking at the third term from (9),

$$\left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle = \int \phi_j \frac{\partial \tilde{J}_x}{\partial x} dx$$

we see that if

$$u = \phi_j \qquad\qquad v = \tilde{J}_x$$

$$\frac{du}{dx} = \frac{d\phi_j}{dx} \qquad\qquad \frac{dv}{dx} = \frac{d\tilde{J}_x}{dx}$$

$$du = \frac{d\phi_j}{dx} dx \qquad\qquad dv = \frac{d\tilde{J}_x}{dx} dx$$

then we can use integration by parts, leaving us with

$$\begin{aligned}
\left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle &= \int \phi_j \frac{\partial \tilde{J}_x}{\partial x} dx \\
&= \phi_j \tilde{J}_x - \int \tilde{J}_x \frac{d\phi_j}{dx} dx \\
&= \phi_j \tilde{J}_x - \left\langle \tilde{J}_x, \frac{d\phi_j}{dx} \right\rangle
\end{aligned} \tag{10}$$

Substituting this back in to (8) gives us the weak form

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_0 \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle - \left\langle \tilde{J}_x, \frac{d\phi_j}{dx} \right\rangle - \left\langle UH \frac{\partial \tau_0}{\partial x}, \phi_j \right\rangle + \phi_j \tilde{J}_x = 0 \tag{11}$$

## 1.2 Complete the GWCE derivation

The GWCE derivation is completed by substituting the vertically-integrated momentum equations, in either conservative or non-conservative forms, into the weak form of the continuity equation (11).

We'll start with the non-conservative form of the vertically-integrated momentum equation. Note that this is the same equation we'll be using in Section 2 to determine depth-averaged velocities.

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = -g \frac{\partial [\zeta + P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{H\rho_0} + \frac{\tau_{bx}}{H\rho_0} \tag{12}$$

This can be substitued in to equation (11) in the term that involves $\tilde{J}_x$. Let's recall the definition of $\tilde{J}_x$, as shown in (4), and expand the derivative term using the chain rule:

$$\begin{aligned}
\tilde{J}_x &\equiv \frac{\partial}{\partial t}(UH) + \tau_0 UH \\
&= H \frac{\partial U}{\partial t} + U \frac{\partial \zeta}{\partial t} + \tau_0 UH
\end{aligned} \tag{13}$$

Now, lets rearrange (12) to isolate the first term.

$$\frac{\partial U}{\partial t} = -U \frac{\partial U}{\partial x} - g \frac{\partial [\zeta + P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{H\rho_0} + \frac{\tau_{bx}}{H\rho_0}$$

This can now be directly substituted in to (13) and simplified.

$$\tilde{J}_x = H\left(-U\frac{\partial U}{\partial x} - g\frac{\partial[\zeta + P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{H\rho_0} + \frac{\tau_{bx}}{H\rho_0}\right) + U\frac{\partial \zeta}{\partial t} + \tau_0 UH$$

$$= -UH\frac{\partial U}{\partial x} - gH\frac{\partial[\zeta + P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{\rho_0} + \frac{\tau_{bx}}{\rho_0} + U\frac{\partial \zeta}{\partial t} + \tau_0 UH$$

$$= -Q_x\frac{\partial U}{\partial x} \underbrace{-gH\frac{\partial \zeta}{\partial x}}_{*} -gH\frac{\partial[P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{\rho_0} + \frac{\tau_{bx}}{\rho_0} + U\frac{\partial \zeta}{\partial t} + \tau_0 Q_x \tag{14}$$

Now take a look at the starred term in (14), recalling that $H = h + \zeta$.

$$-gH\frac{\partial \zeta}{\partial x} = -g(h+\zeta)\frac{\partial \zeta}{\partial x}$$

$$= -gh\frac{\partial \zeta}{\partial x} - g\zeta\frac{\partial \zeta}{\partial x}$$

$$= -gh\frac{\partial \zeta}{\partial x} - \frac{g}{2}\frac{\partial \zeta^2}{\partial x} \tag{15}$$

Note that to arrive at (15) from the previous step, the chain rule was used:

$$\frac{du^2}{dx} = u\frac{du}{dx} + u\frac{du}{dx}$$

$$= 2u\frac{du}{dx}$$

So now if we plug (15) back in to (14), we get:

$$\tilde{J}_x = -Q_x\frac{\partial U}{\partial x} - gh\frac{\partial \zeta}{\partial x} - \frac{g}{2}\frac{\partial \zeta^2}{\partial x} - gH\frac{\partial[P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{\rho_0} + \frac{\tau_{bx}}{\rho_0} + U\frac{\partial \zeta}{\partial t} + \tau_0 Q_x \tag{16}$$

We isolate the linear free surface gravity wave term to arrive at

$$\tilde{J}_x = J_x - gh\frac{\partial \zeta}{\partial x} \tag{17}$$

where

$$J_x = -Q_x\frac{\partial U}{\partial x} - \frac{g}{2}\frac{\partial \zeta^2}{\partial x} - gH\frac{\partial[P_s/g\rho_0 + \alpha\eta_0]}{\partial x} + \frac{\tau_{sx}}{\rho_0} + \frac{\tau_{bx}}{\rho_0} + U\frac{\partial \zeta}{\partial t} + \tau_0 Q_x$$

Finally, we substitute (17) in to (11), leaving us with:

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_0 \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle + \left\langle gh \frac{\partial \zeta}{\partial x}, \frac{\partial \phi_j}{\partial x} \right\rangle = \left\langle J_x, \frac{d\phi_j}{dx} \right\rangle + \left\langle Q_x \frac{\partial \tau_0}{\partial x}, \phi_j \right\rangle - \phi_j \tilde{J}_x$$

# 2 Momentum Equation