1D ADCIRC Derivation

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1 Continuity Equation

Start with the vertically integrated continuity equation:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH) = 0 \tag{1}$$

where

$$\begin{aligned} H &\equiv \zeta + h \\ \zeta &= \text{free surface departure from the geoid} \\ h &= \text{bathymetric depth (distance from the geoid to the bottom)} \\ u &= \text{vertically varying velocity in the x-direction} \\ U &= \frac{1}{H}\int_{-h}^{\zeta} u \mathrm{d}z = \text{depth-averaged velocity in the x-direction} \end{aligned}$$

Take $\partial/\partial t$ of (2):

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) = 0 \tag{2}$$

Add (2) to (1) multiplied by the parameter τ_0, which may be variable in space:

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + \tau_0\left(\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(UH)\right) = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + \tau_0\frac{\partial H}{\partial t} + \tau_0\frac{\partial}{\partial x}(UH) = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \tau_0\frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) = 0 \tag{3}$$

Now, define $\tilde{J}_x$:

$$\tilde{J}_x \equiv \tau_0(UH) + \frac{\partial}{\partial t}(UH) \tag{4}$$

$$\tilde{J}_x = \tau_0 Q + \frac{\partial Q}{\partial t} \tag{5}$$

where

$$Q = UH$$

Recall that τ_0, U, and H are all variable in x and take $\partial/\partial x$ of (5), noting the use of the product rule:

$$\begin{aligned}
\frac{\partial \tilde{J}_x}{\partial x} &= \frac{\partial}{\partial x}\left[\tau_0 Q + \frac{\partial Q}{\partial t}\right] \\
&= \frac{\partial}{\partial x}(\tau_0 Q) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q \\
&= Q\frac{\partial \tau_0}{\partial x} + \tau_0\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q \\
&= \tau_0\frac{\partial Q}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}Q + Q\frac{\partial \tau_0}{\partial x} \\
&= \tau_0\frac{\partial (UH)}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + UH\frac{\partial \tau_0}{\partial x} \qquad (6)
\end{aligned}$$

Now, returning to equation (3), let's add zero to it in the form of:

$$UH\frac{\partial \tau_0}{\partial x} - UH\frac{\partial \tau_0}{\partial x} = 0$$

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \underbrace{\tau_0\frac{\partial}{\partial x}(UH) + \frac{\partial}{\partial x}\frac{\partial}{\partial t}(UH) + UH\frac{\partial \tau_0}{\partial x}}_{\text{Note that this is equivalent to (6)}} - UH\frac{\partial \tau_0}{\partial x} = 0$$

and substituting (6) in gives us:

$$\frac{\partial^2 H}{\partial t^2} + \tau_0\frac{\partial H}{\partial t} + \frac{\partial \tilde{J}_x}{\partial x} - UH\frac{\partial \tau_0}{\partial x} = 0 \tag{7}$$

If we assume that bathymetric depth is constant, then

$$\frac{\partial H}{\partial t} = \frac{\partial \zeta}{\partial t}$$

$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial^2 \zeta}{\partial t^2}$$

and (7) can be rewritten as

$$\frac{\partial^2 \zeta}{\partial t^2} + \tau_0 \frac{\partial \zeta}{\partial t} + \frac{\partial \tilde{J}_x}{\partial x} - UH\frac{\partial \tau_0}{\partial x} = 0 \tag{8}$$

1.1 Apply the weighted residual method

First, we'll define the inner product notation $\langle A, B\rangle$ as the integral over the domain Ω of A and B multiplied together.

$$\langle A, B\rangle \equiv \int\limits_{\Omega} AB d\Omega$$

We apply the weighted residual method to (8) by multiplying each term by a weighting function ϕ_j and integrating over the horizontal computational domain Ω. Written using the inner product notation, we have

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_0 \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle + \left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle - \left\langle UH\frac{\partial \tau_0}{\partial x}, \phi_j \right\rangle = 0 \tag{9}$$

The third term, which involves $\tilde{J}_x$, can be integrated using integration by parts. Recall that integration by parts is defined as

$$\int u dv = uv - \int v du$$

So looking at the third term from (9),

$$\left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle = \int \phi_j \frac{\partial \tilde{J}_x}{\partial x} dx$$

we see that if

$$u = \phi_j \qquad\qquad v = \tilde{J}_x$$

$$\frac{du}{dx} = \frac{d\phi_j}{dx} \qquad\qquad \frac{dv}{dx} = \frac{d\tilde{J}_x}{dx}$$

$$du = \frac{d\phi_j}{dx}dx \qquad\qquad dv = \frac{d\tilde{J}_x}{dx}dx$$

then we can use integration by parts, leaving us with

$$\begin{aligned}
\left\langle \frac{\partial \tilde{J}_x}{\partial x}, \phi_j \right\rangle &= \int \phi_j \frac{\partial \tilde{J}_x}{\partial x} dx \\
&= \phi_j \tilde{J}_x - \int \tilde{J}_x \frac{d\phi_j}{dx} dx \\
&= \phi_j \tilde{J}_x - \left\langle \tilde{J}_x, \frac{d\phi_j}{dx} \right\rangle
\end{aligned} \tag{10}$$

Substituting this back in to (8) gives us

$$\left\langle \frac{\partial^2 \zeta}{\partial t^2}, \phi_j \right\rangle + \left\langle \tau_0 \frac{\partial \zeta}{\partial t}, \phi_j \right\rangle - \left\langle \tilde{J}_x, \frac{d\phi_j}{dx} \right\rangle - \left\langle UH \frac{\partial \tau_0}{\partial x}, \phi_j \right\rangle + \phi_j \tilde{J}_x = 0$$