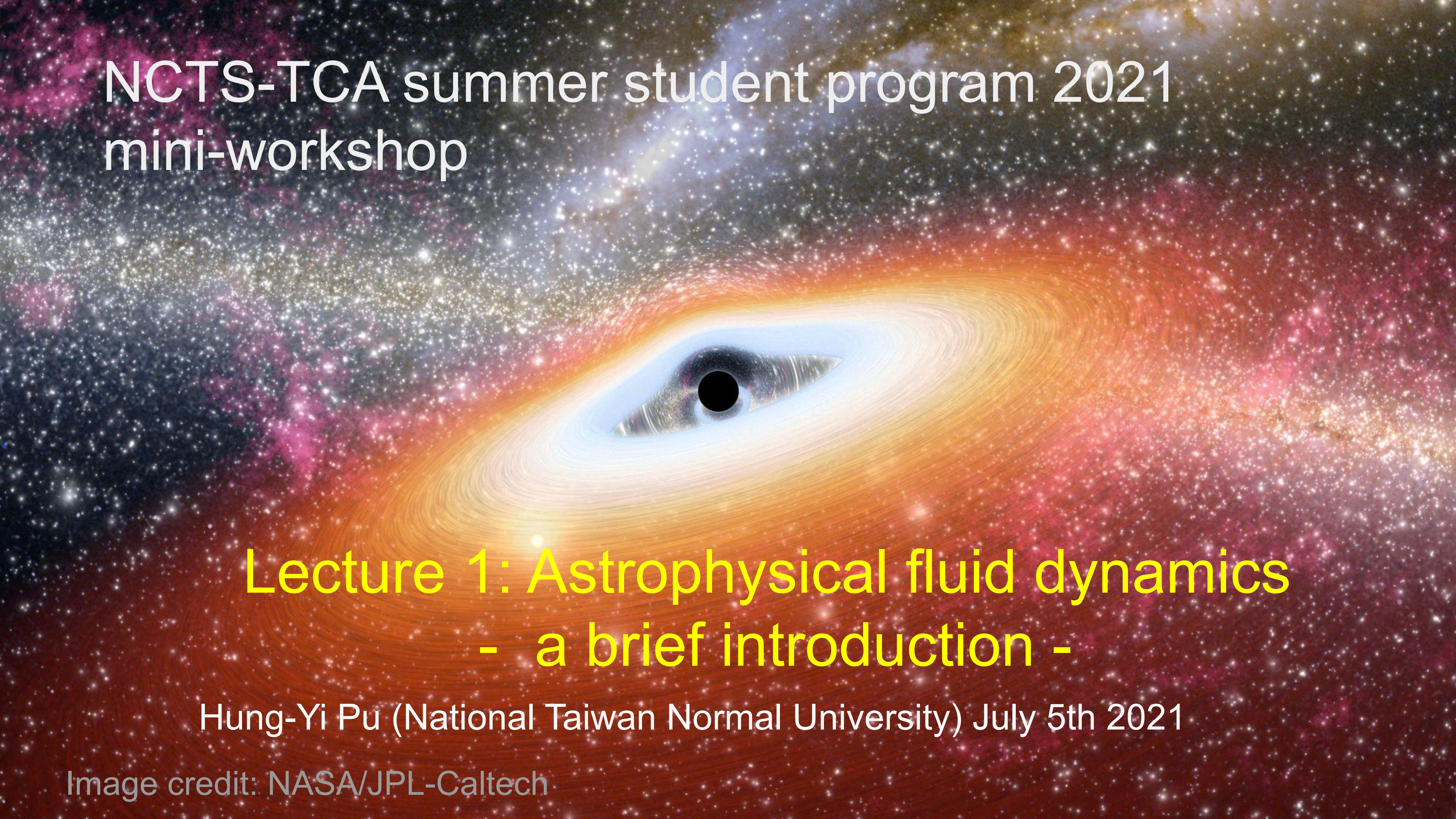


NCTS-TCA summer student program 2021 mini-workshop

Lecture 1: Astrophysical fluid dynamics - a brief introduction -

Hung-Yi Pu (National Taiwan Normal University) July 5th 2021

Image credit: NASA/JPL-Caltech

reference

- Principle of astropphysical fluid dynamics by Clarke & Carswell
- The physics of plasmas by Boyd & Sanderson
- The physics of fluids and plasmas by Choudhuri
- The physics of astrophysics volume II: gas dynamics by Shu
- Fluid Mechanics by Frank M. White
- MIT open course: Fluid Dynamics (National Committee for Fluid Mechanics Films), see also this YouTube playlist and the notes

“when I meet God, I am going to ask him two questions: Why relativity? and why trubulence?

I really belive he will have an answer for the first.”

W. Heisenberg (1907-1976)

Millennium Prize Problems (千禧年大獎難題)

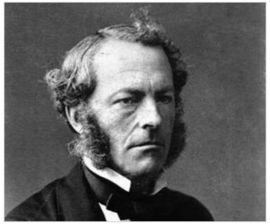
seven unsolved problems in mathematics that were stated by the Clay Mathematics Institute on May 24, 2000.

A correct solution to any of the problems results in a **US$1 million** prize being awarded by the institute to the discoverer(s).

CMI

ABOUT PROGRAMS PEOPLE MILLENNIUM PROBLEMS PUBLICATIONS EVENTS NEWS

Navier–Stokes Equation



Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Image: Sir George Gabriel Stokes (13 August 1819–1 February 1903). Public Domain

This problem is: Unsolved

Rules:

Rules for the Millennium Prizes

Related Documents:

Official Problem Description

Related Links:

Lecture by Luis Cafarelli

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

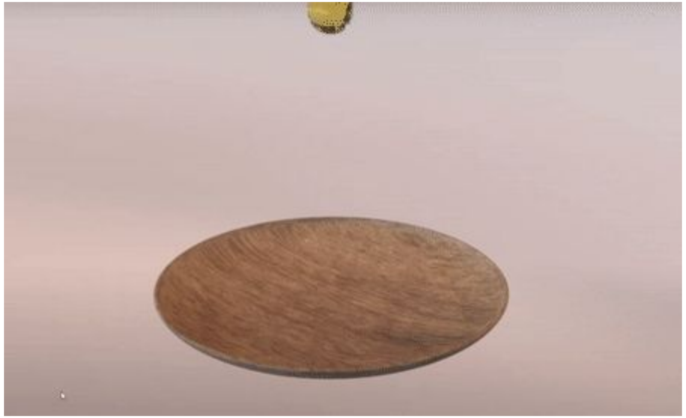
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

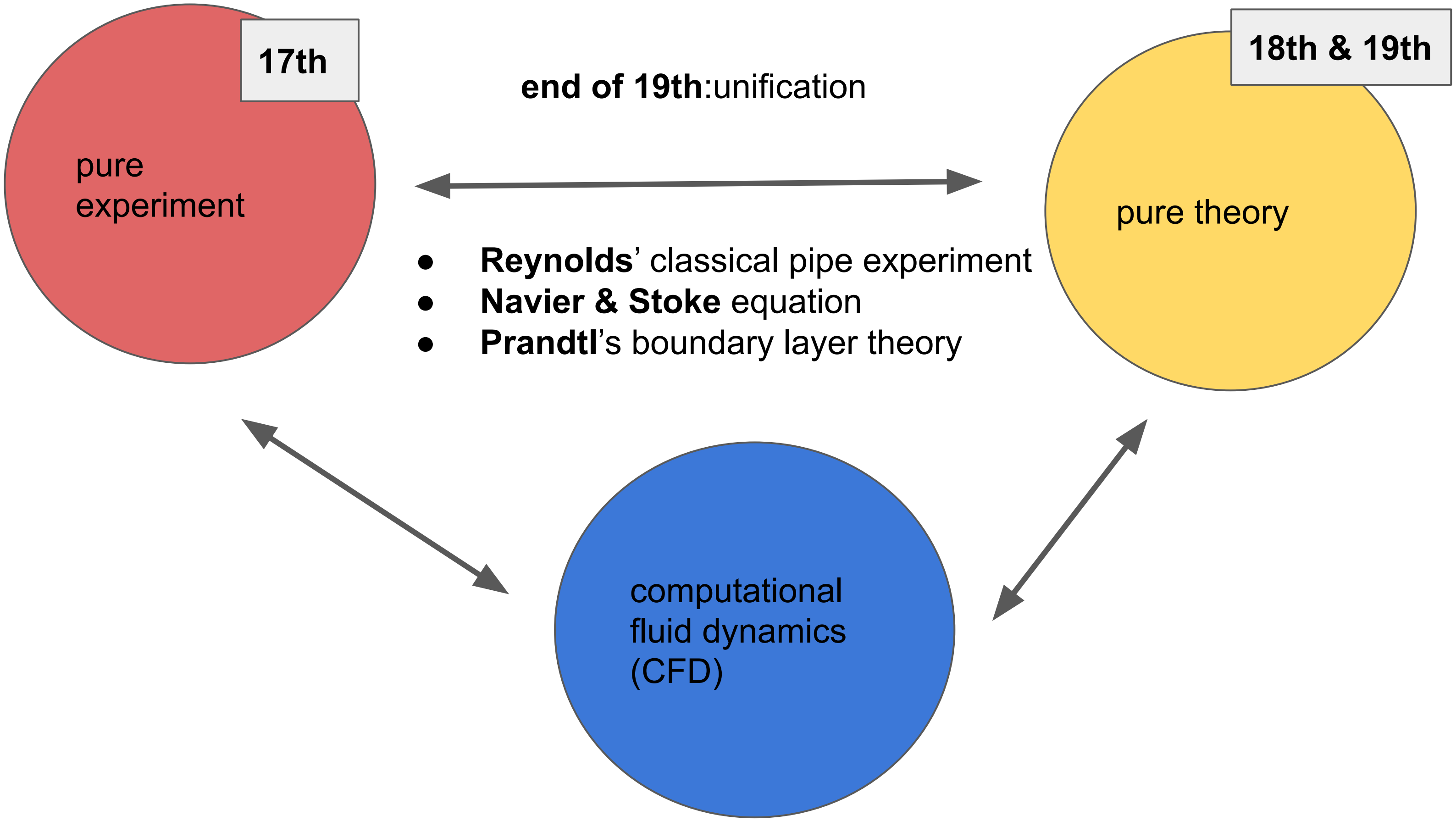
why fluid dynamics is hard?



simluation or real honey?
credit: https://www.youtube.com/watch?v=I3c4m29coB4

simluation
credit: wiki

using one equation (Navier-Stoke equation) to descibe all personality of different fluids!


17th
pure
experiment
end of 19th:unification
18th & 19th
pure theory
Reynolds' classical pipe experiment
Navier & Stoke equation
Prandtl's boundary layer theory
computational
fluid dynamics
(CFD)

setting up the stage -- outline

- background:
  - important concepts in fluid mechanics
- astrophysical fluid:
  - plama and magnetohydrodynamics (MHD)
- basic flow analysis techniques
  - integral analysis
  - differential analysis
  - dimensional analysis

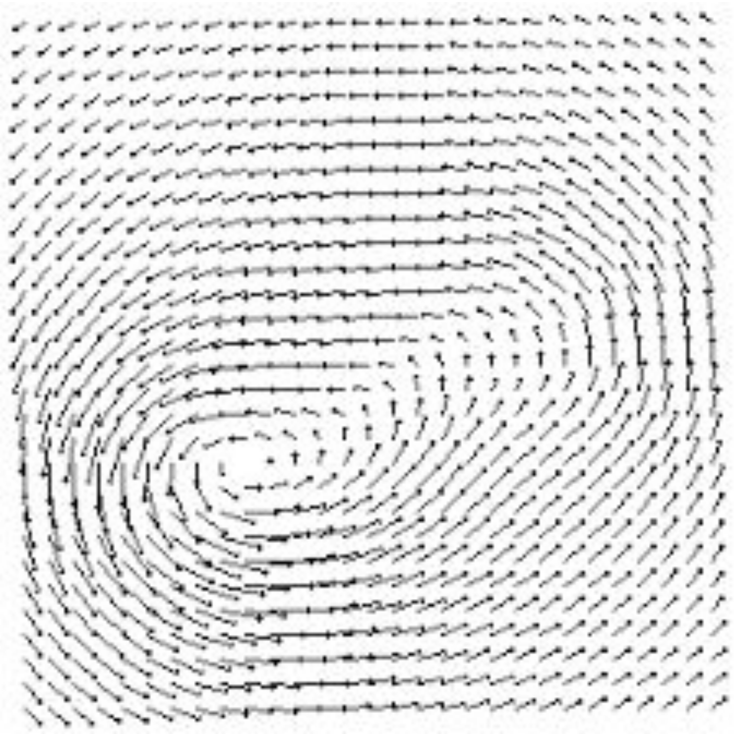
notation

- cartesian coordinate
- for 3D flow

$$(V_x, V_y, V_z) = (u, v, w)$$

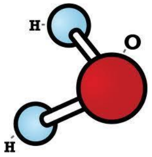
- for 2D flow

$$(V_x, V_y) = (u, v)$$



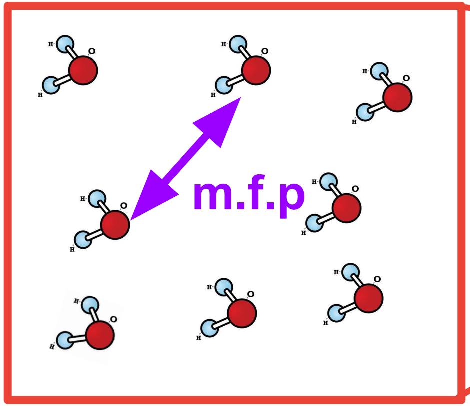
fluid: a macroscopic approach

different levels: particle → distribution funciton→ continuum model (**L** >> **m.f.p**)

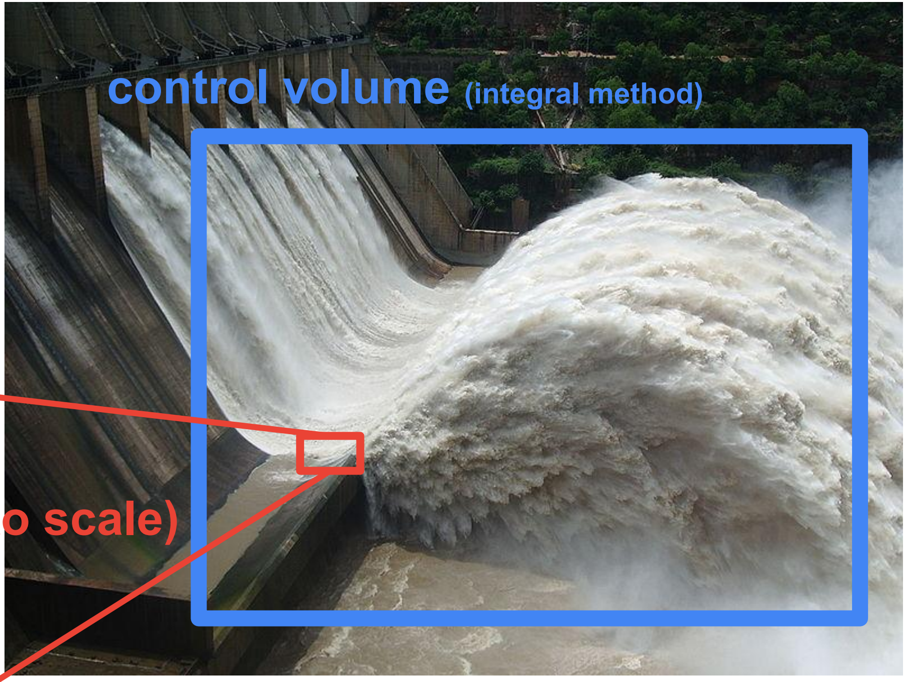


L

fluid element (differential method)



(not to scale)



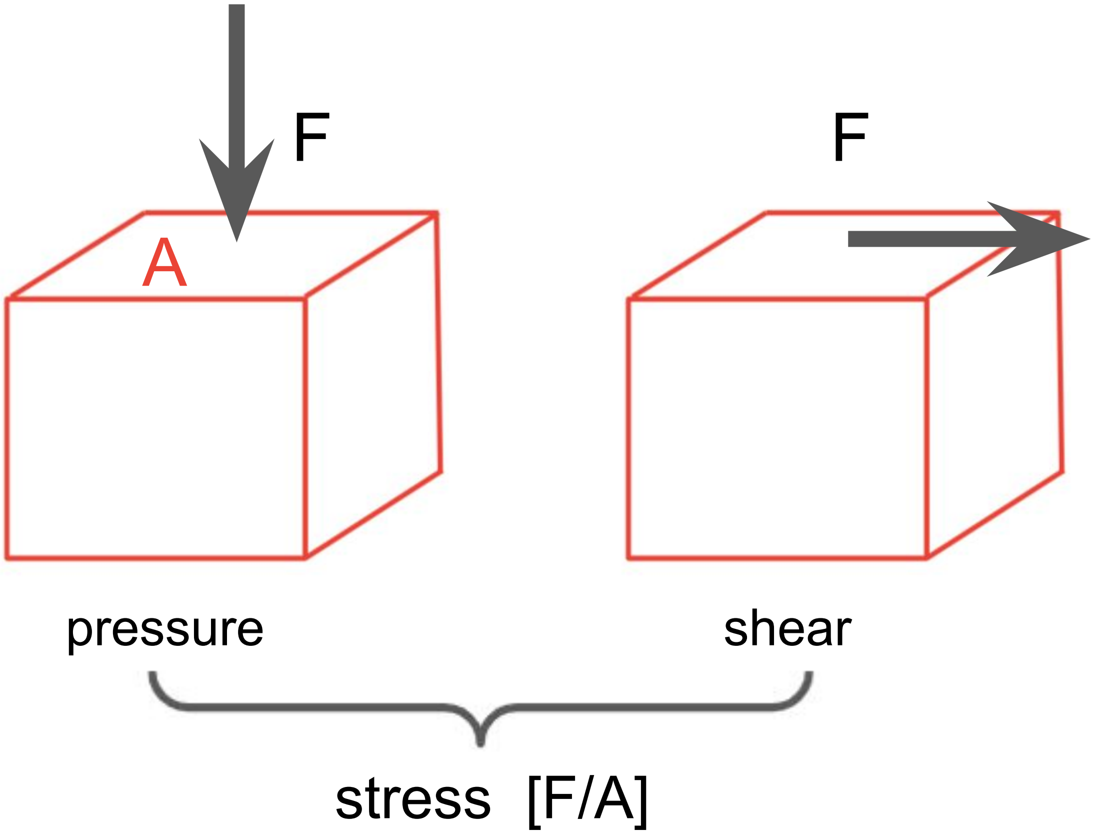
mean free path (m.f.p): average distance a particle travels before it colldes with another particle

from water molecular to river

stress and shear

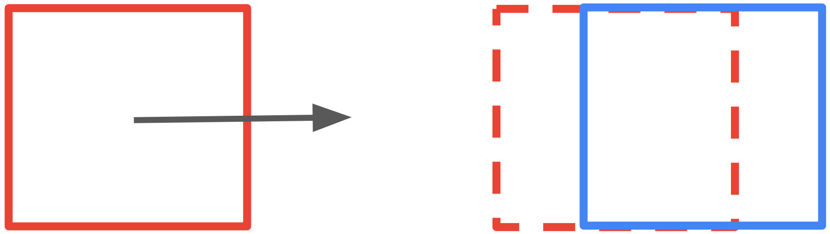
typical definition of fluid:

can move under the action of a **shear stress**, no matter how samll that stress may be

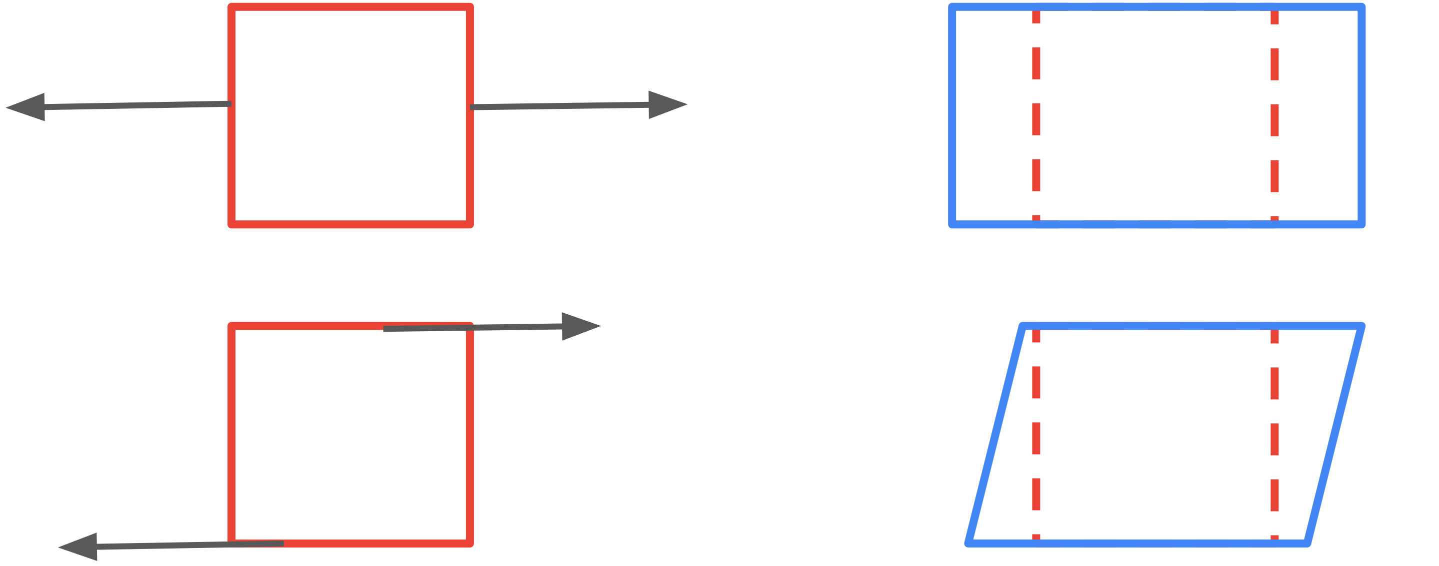


shear stress is a surface force

body force (does not require contact of the element)

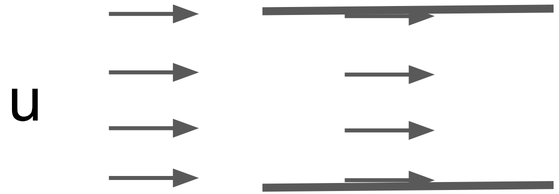


surface force (requires contact of the element)



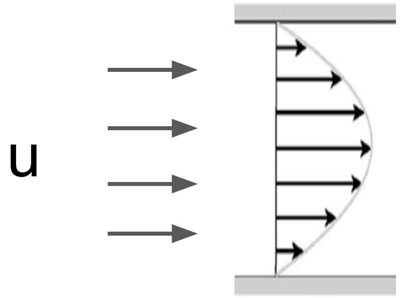
viscous and invicid flow

invicid flow



does not exist!

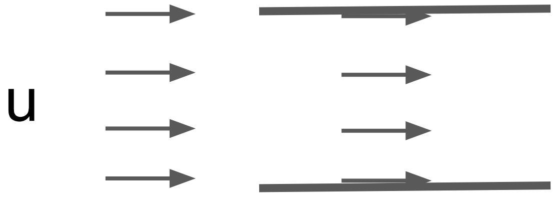
viscous flow



Poiseuille flow

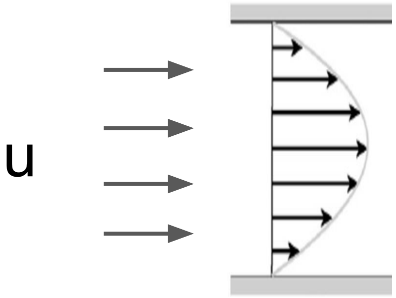
viscous and invicid flow

invicid flow

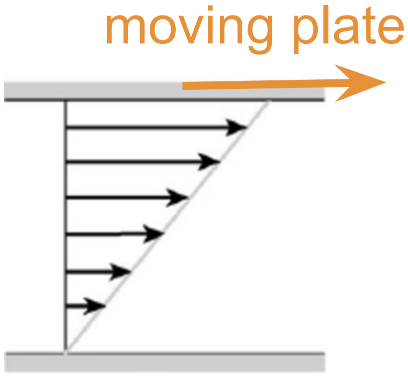


does not exist!

viscous flow



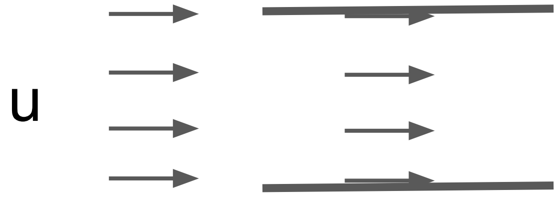
Poiseuille flow



Couette flow

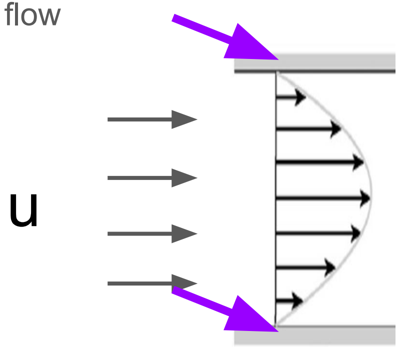
viscous and invicid flow

invicid flow

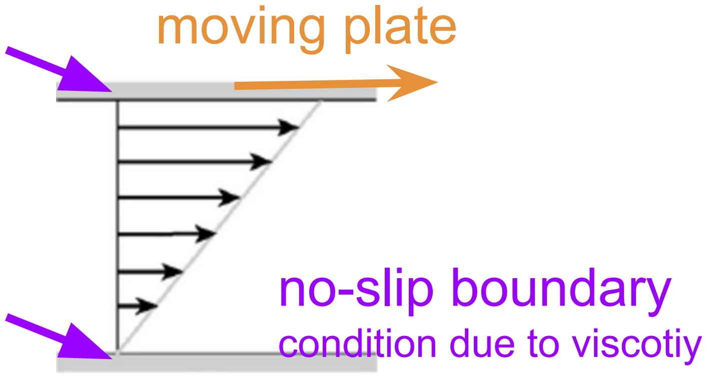


does not exist!

viscous flow



Poiseuille flow



Couette flow

viscosity and shear stress

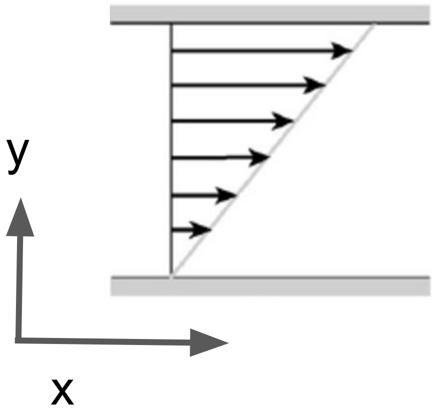
meassure of the resistence of a fluid to gradual deformations by shear stress

dynamic viscosity

$$\tau = \mu \frac{\partial u}{\partial y}$$

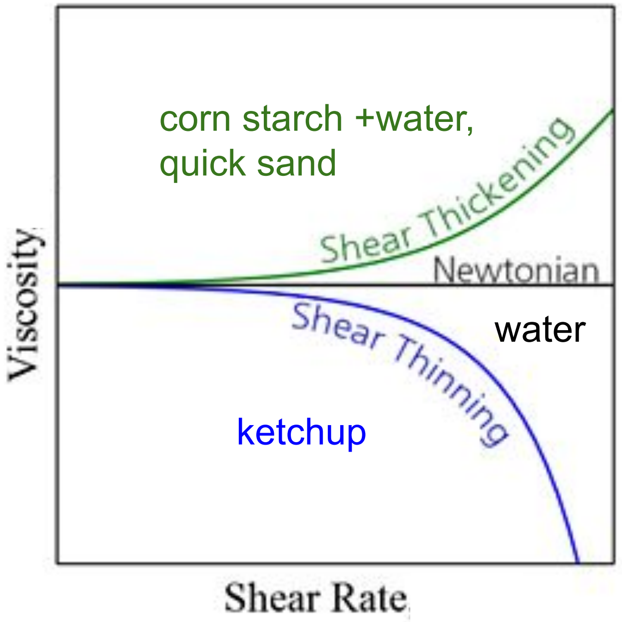
shear stress [F/A]

shear rate [1/s]



$$\nu = \frac{\mu}{\rho}$$

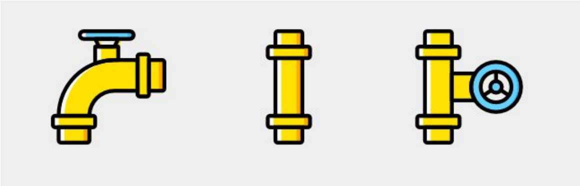
kinetic viscosity [VL]



movie credit: 國立台中教育大學 NTCU
科學教育與應用學系

classification of fluid

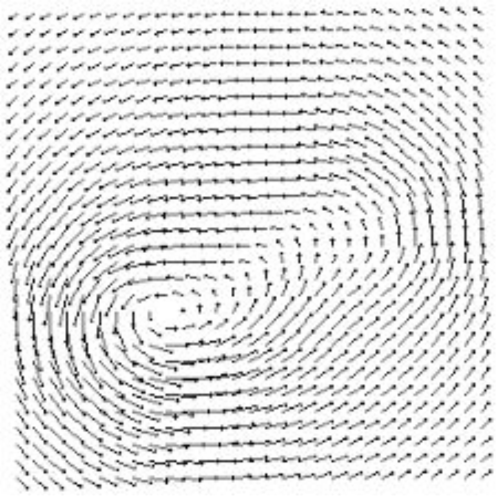
materials
- solid
- fluid
  - liquid
  - gas
  - (plasma)
    - statics (fluid statics/ hydrostatics)
    - in motion
      - internal flows



      - external flows

Newtonian		non-Newtonian
incompressible		compressible
inviscid	vs.	viscous
laminar		trubulent
steady		unsteady
one phase		multi phase

velocity field: V(u,v,w,t)

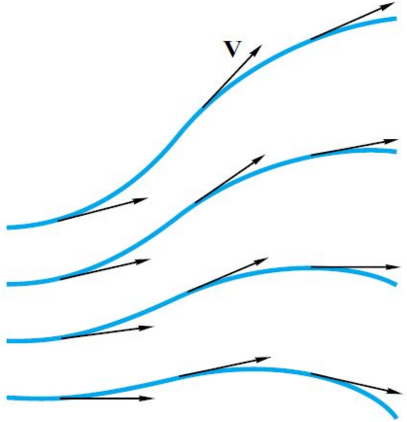


$$\nabla \cdot \vec{V}$$ imcompressible if $\nabla \cdot \vec{V} = 0$

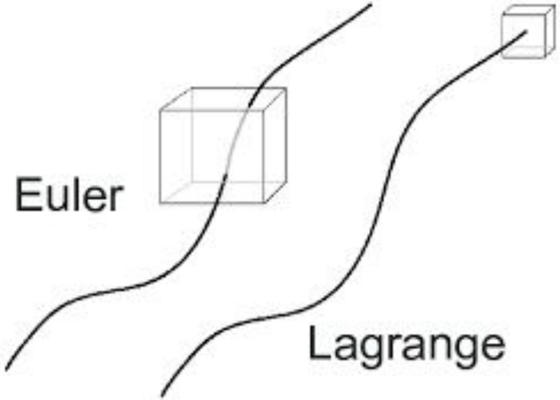
$$\nabla \times \vec{V}$$ vorticity: measure of local rotation

streamline (at constant t):

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}$$



a tale of two views



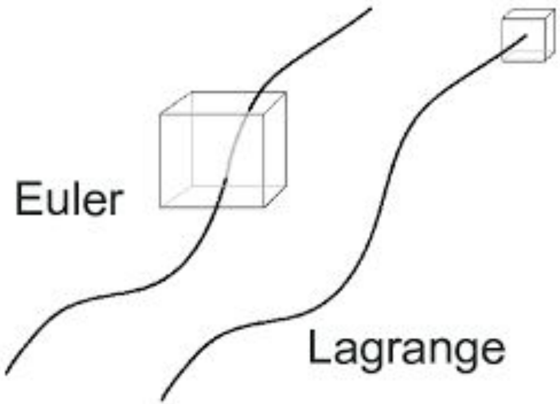
substantial/material derivative

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

LHS: Largragian point of view
(ride on the particles)

RHS: Eulerian point of view
(stay at the fixed grid)

a tale of two views



substantial/material derivative

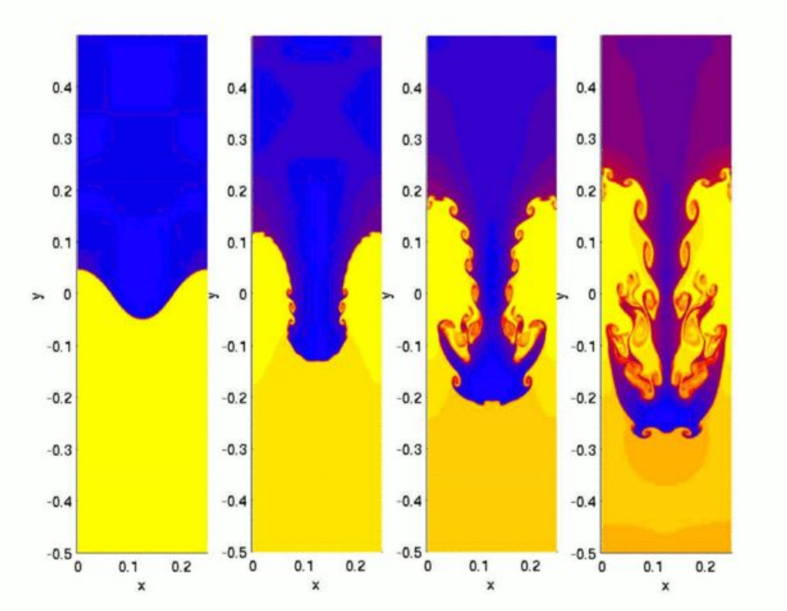
$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

<u>proof</u>

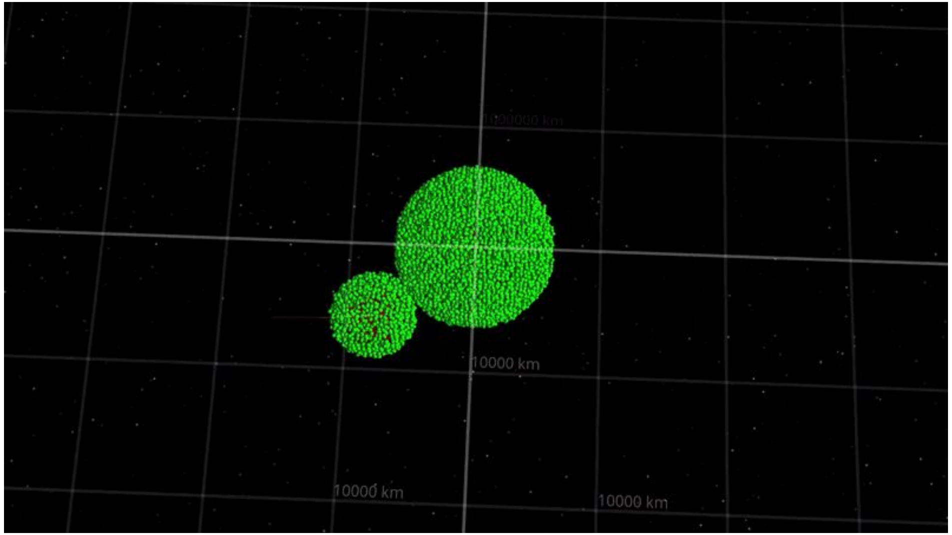
$$d\rho(t, x, y, z) = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz$$

$$\frac{d}{dt}\rho = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt})\rho$$

grid-based simulation



particle-based simulation

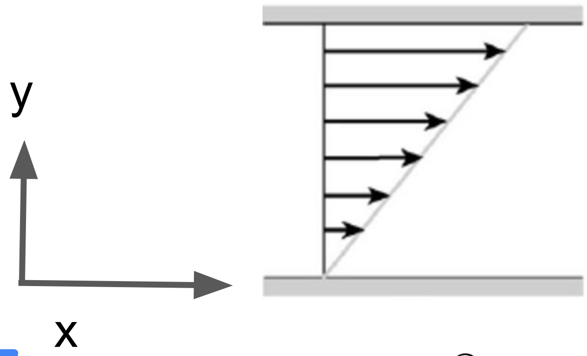


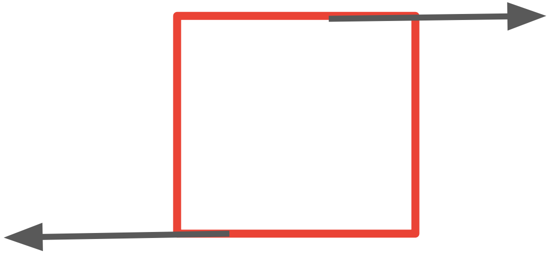
shear tensor

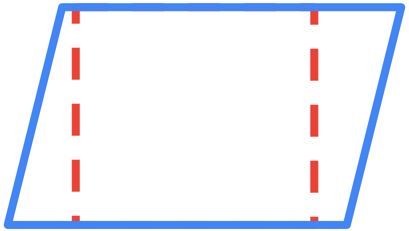
$$\tau_{ij}$$

surface

direction





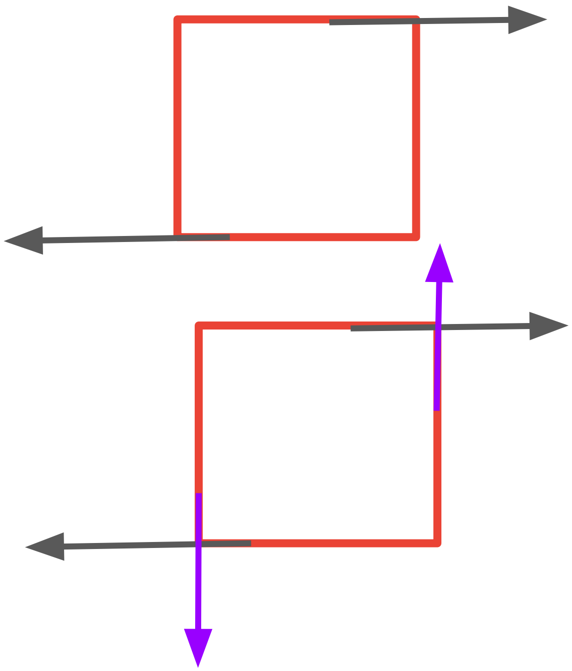


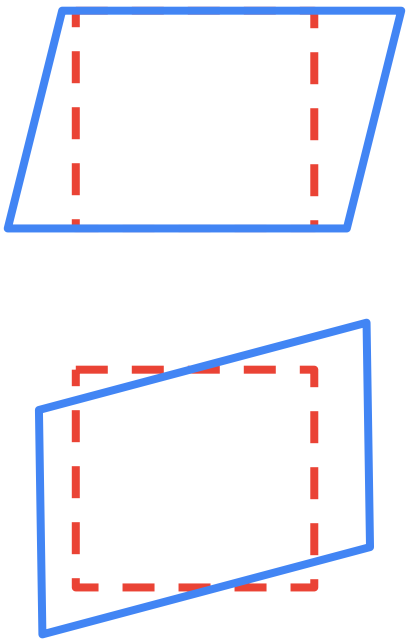
$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

shear tensor τ_{ij}

surface

direction





NO! $\tau_{yx} = \mu \dfrac{\partial u}{\partial y}$

$$\tau_{yx}$$
$$= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$= \tau_{xy}$$

shear tensor

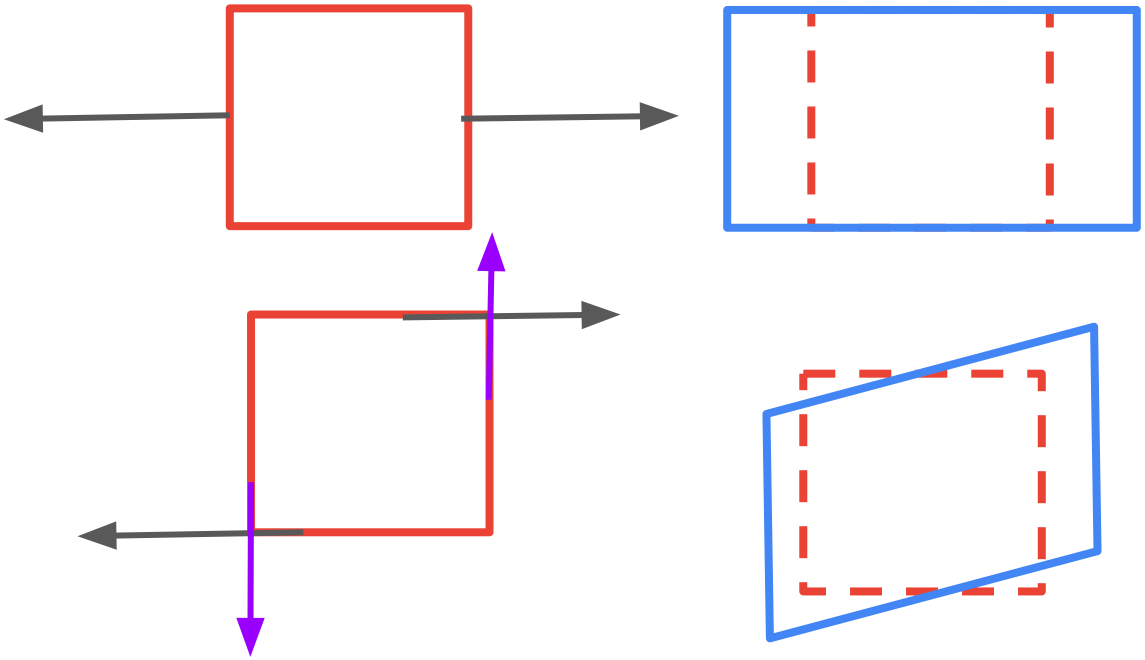
τ_{ij}

surface

direction

net florce
per unit
volumn

$$\nabla \cdot \bar{\bar{\tau}} = \Sigma(\nabla_i \tau_{ij})$$



$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \vec{V})$$

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

ma = F

body force

surface force

gravity

EM

pressure

viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

normal

shear

ma = F

body force

gravity

EM

surface force

pressure

viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

normal

shear

viscous term

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \nabla P + \nabla \cdot \bar{\bar{\tau}} = \rho \vec{g} - \nabla P + \mu \nabla^2 \vec{V}$$

for costant viscosity,
imcompressible fluid

Navior-Stoke equation

$$\boxed{(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}}$$

convection diffusion equation

$$\boxed{\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}}$$

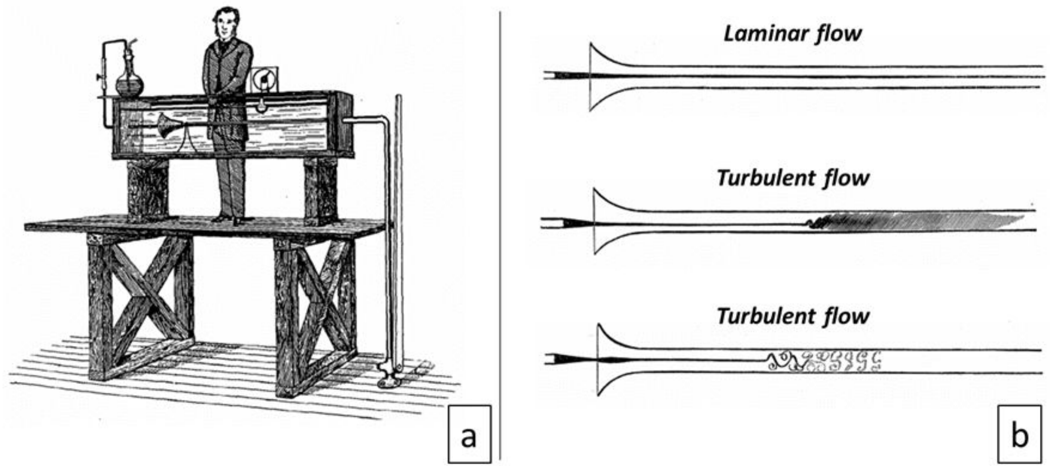
- linear
  - U= constant
- non-linear
  - U=f(x,t): Burger's equation

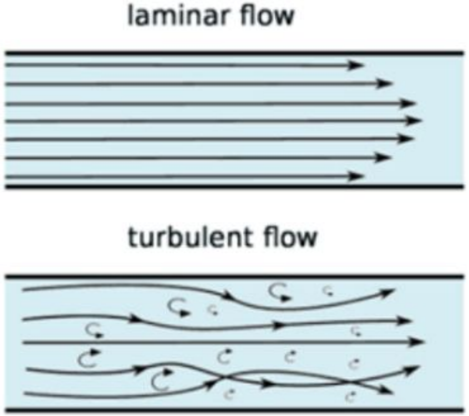
Navior-Stoke equation

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}$$

$$Re = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho \,.\, V \,.\, D}{\mu}$$

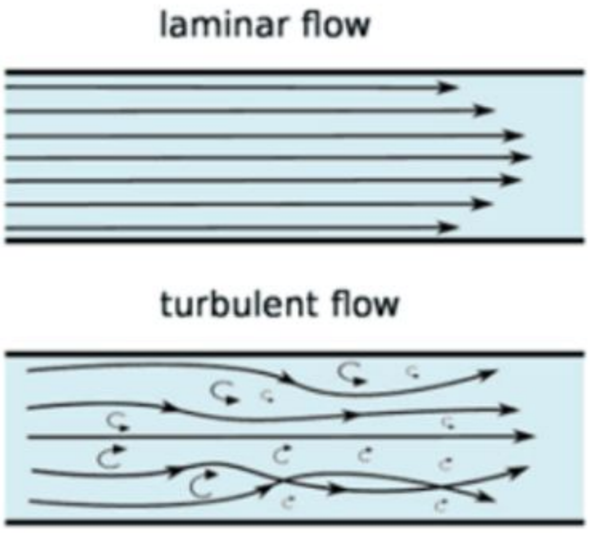
Rynolds' pipe experiment

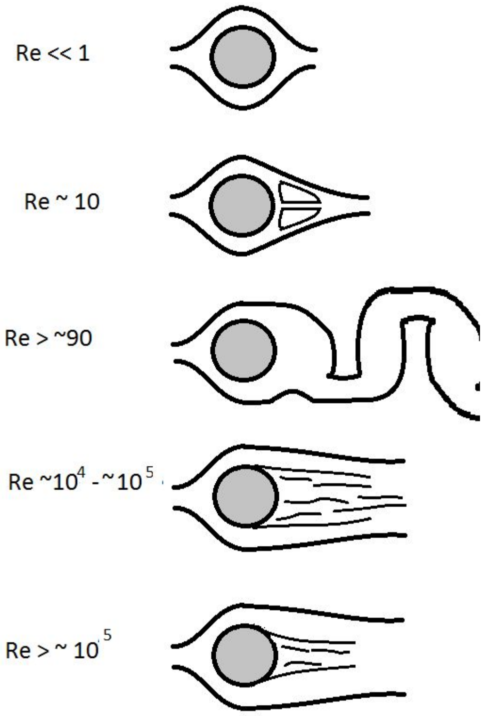




what causes turbulece? inertial or viscosity?

turbulence appears when Reynolds number is high enough (~10^5)





$$Re = \frac{\text{inertia forces}}{\text{viscous forces}} = \frac{\rho \, . \, V \, . \, D}{\mu}$$

inertial !

Bernoulli equation

$$(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2}\vec{V}$$

stationary, invicid

imcompressible

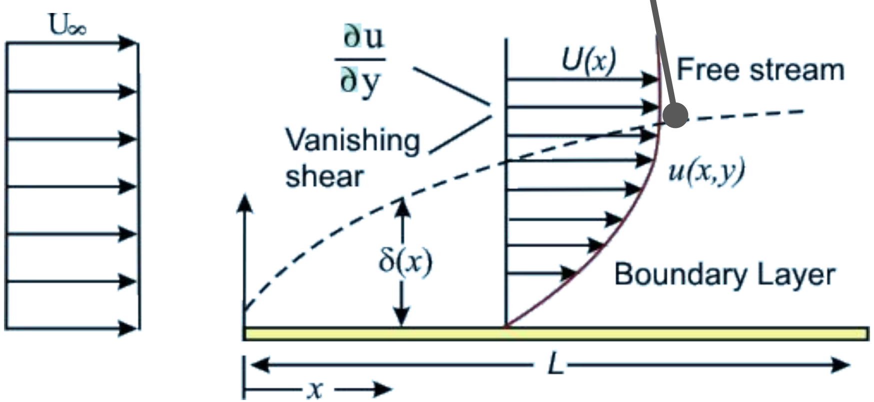
integral assiming $d\rho = 0$

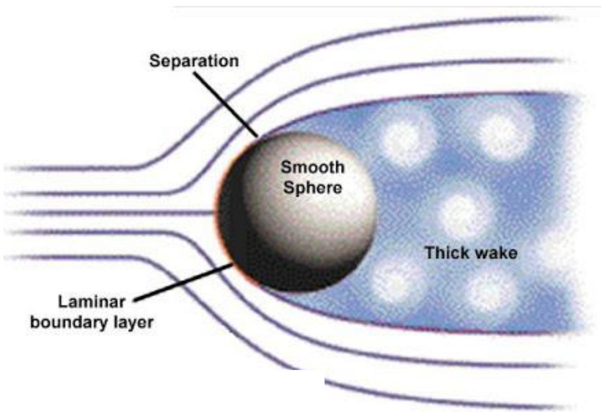
$$\boxed{\mathcal{B} = \frac{|\vec{V}|^2}{2} + \frac{P}{\rho} + gz}$$

(along a streamline)

$$\mathcal{B}(x1, y1, z1) = \mathcal{B}(x2, y2, z2)$$

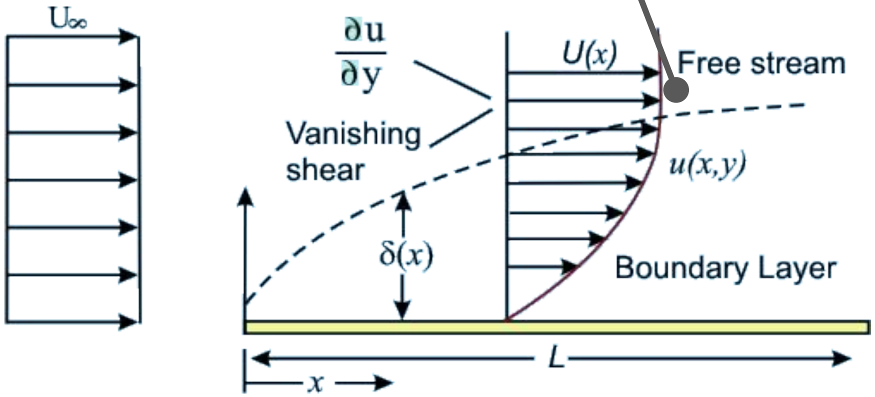
$$U(x, y = \delta) = 0.99U_\infty$$



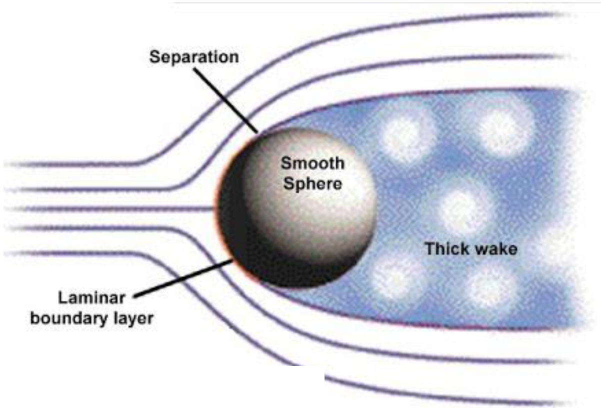


“**Prandtl**’s boundary layer theory”

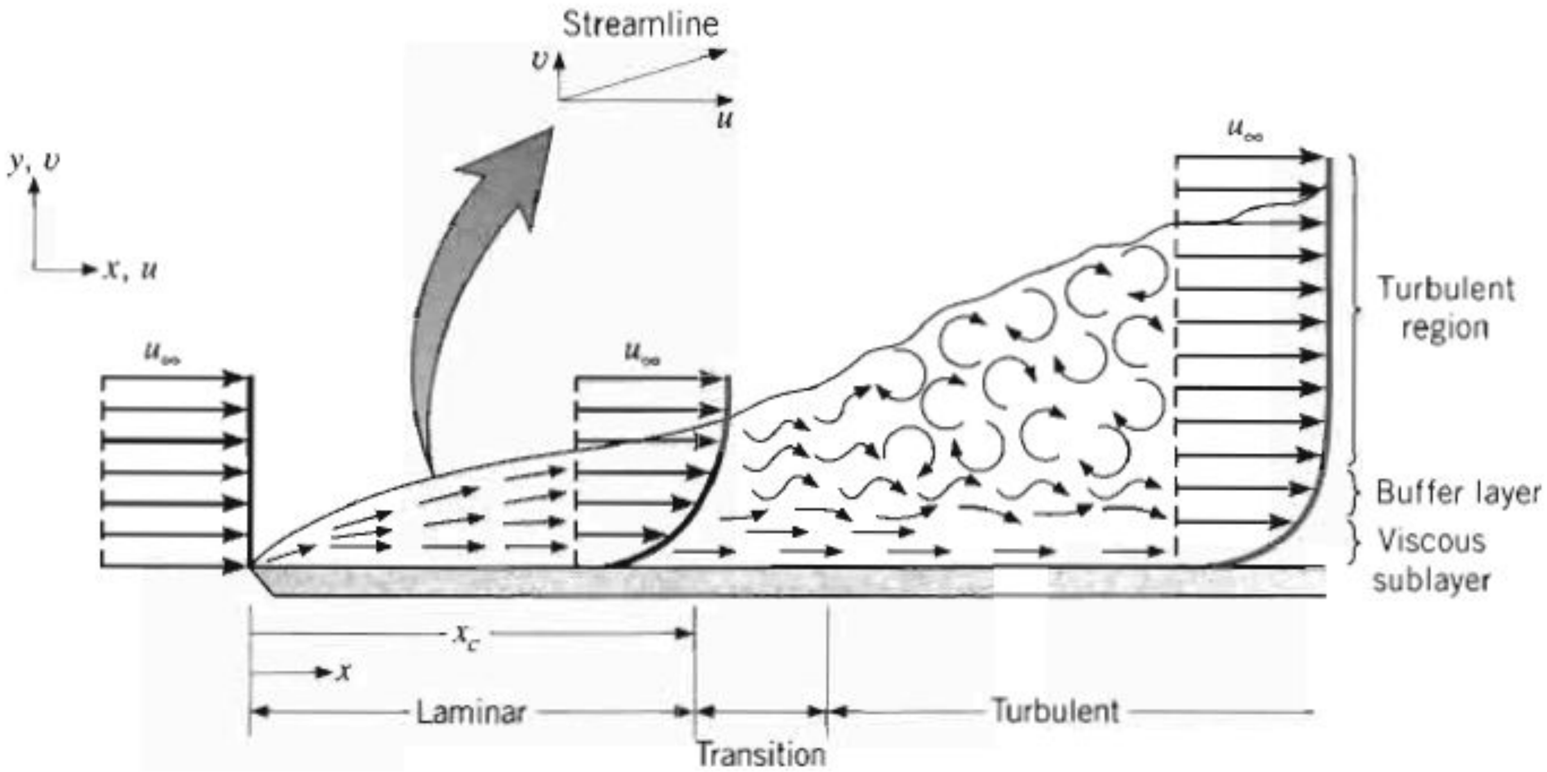
$$U(x, y = \delta) = 0.99 U_\infty$$



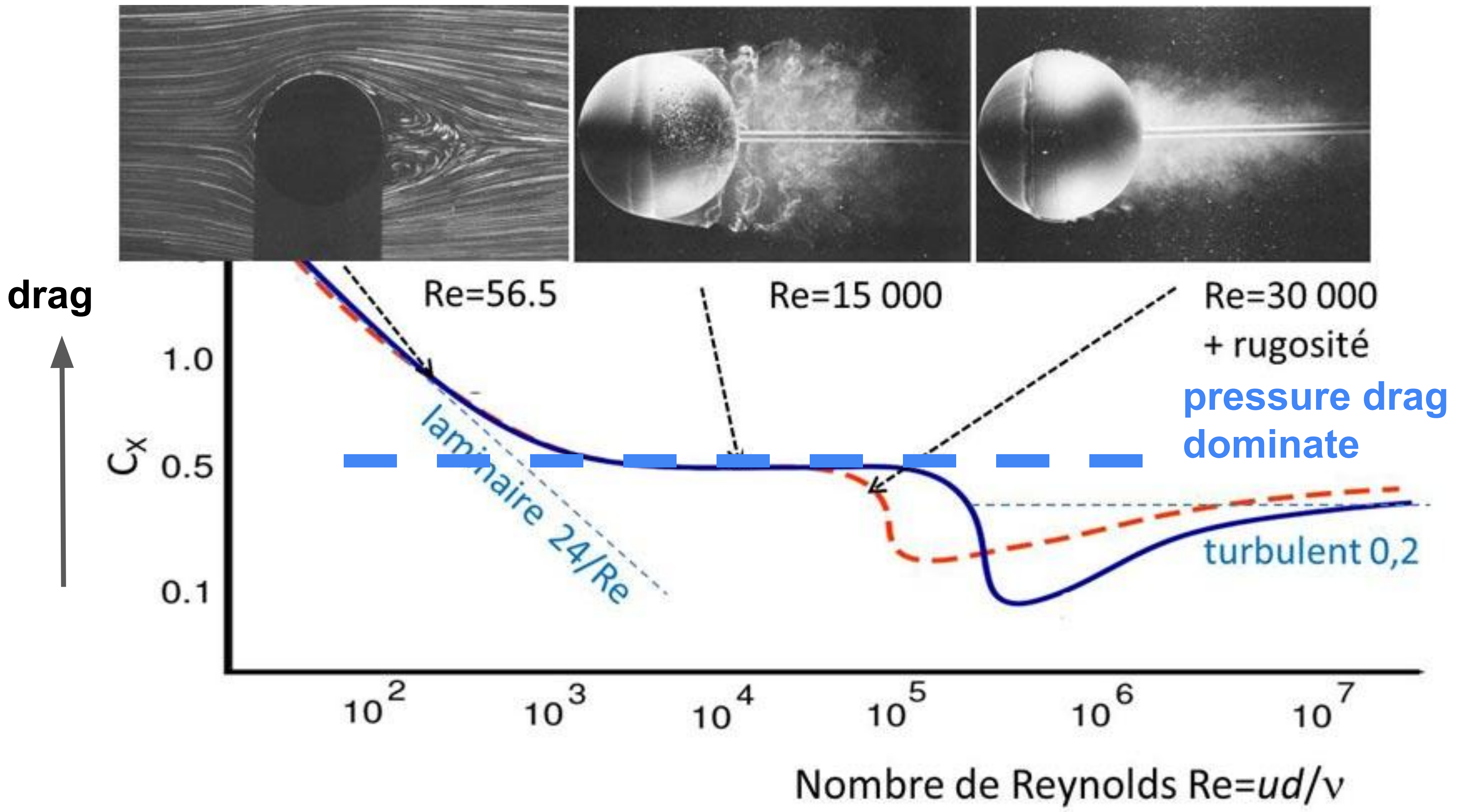
friction drag



pressure drag

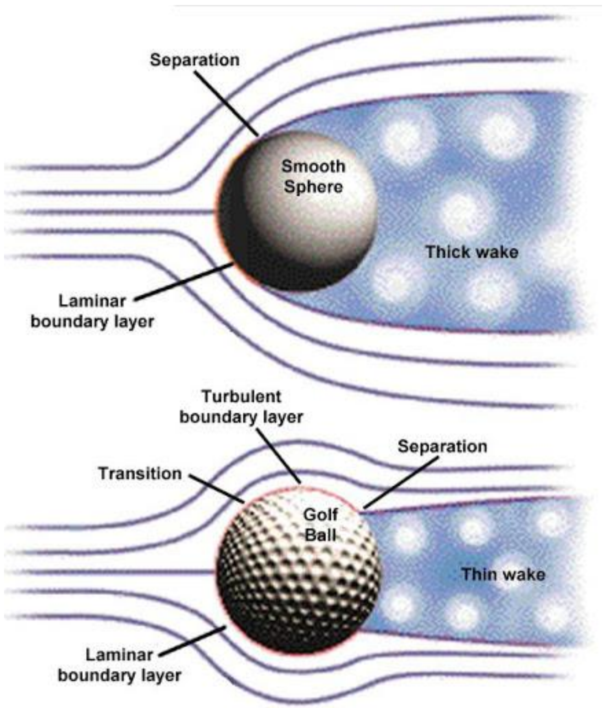


Streamline
v
u
y, v
x, u
u_∞
u_∞
u_∞
Turbulent region
Buffer layer
Viscous sublayer
x_c
x
Laminar
Transition
Turbulent



drag
Re=56.5
Re=15 000
Re=30 000
+ rugosité
pressure drag
dominate
1.0
0.5
0.1
C_x
laminaire 24/Re
turbulent 0,2
10^2
10^3
10^4
10^5
10^6
10^7
Nombre de Reynolds Re=ud/ν

trubulence can be helpful



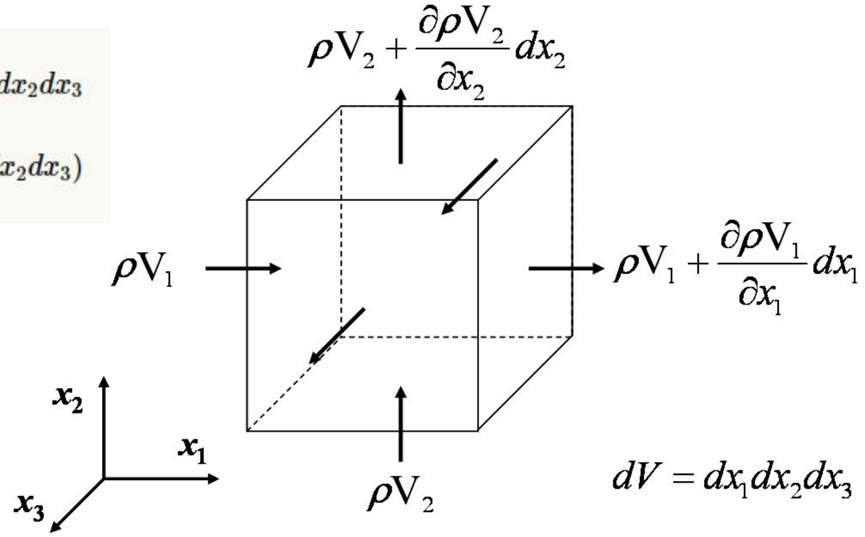
Bill Abbott

737-800

conservation of mass (continity equation)

$$\rho v_1(dx_2dx_3) + \rho v_2(dx_1dx_3) + \rho v_3(dx_1dx_2) - \left(\rho v_1 + \frac{\partial(\rho v_1)}{\partial x_1}dx_1\right)dx_2dx_3$$
$$-\left(\rho v_2 + \frac{\partial(\rho v_2)}{\partial x_2}dx_2\right)dx_1dx_3 - \left(\rho v_3 + \frac{\partial(\rho v_3)}{\partial x_3}dx_3\right)dx_1dx_2 = \frac{\partial}{\partial t}(\rho dx_1dx_2dx_3)$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$$

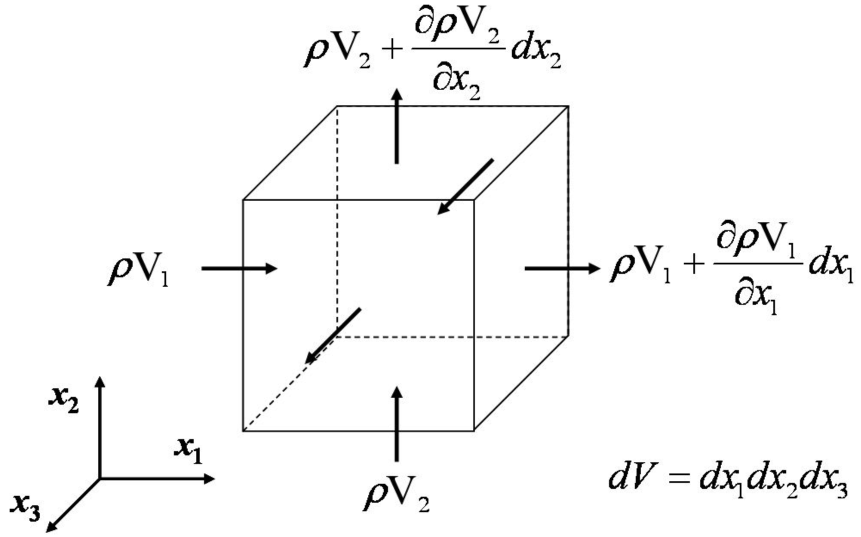


conservation of mass, continity equation

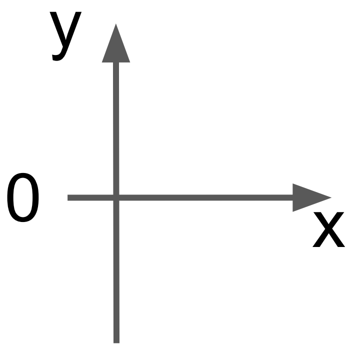
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

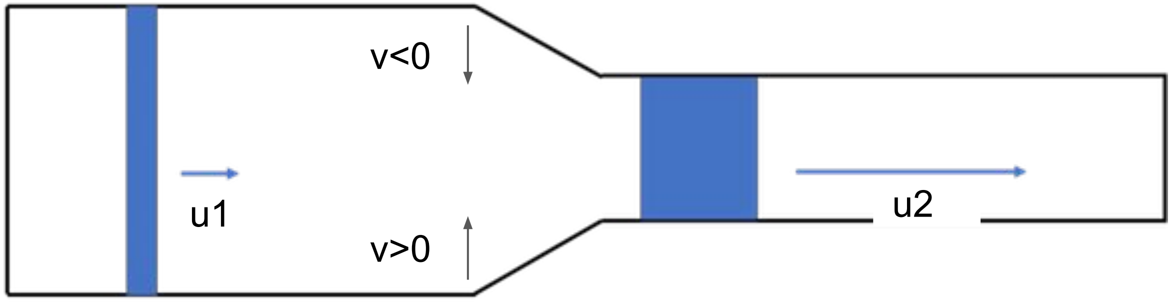
$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

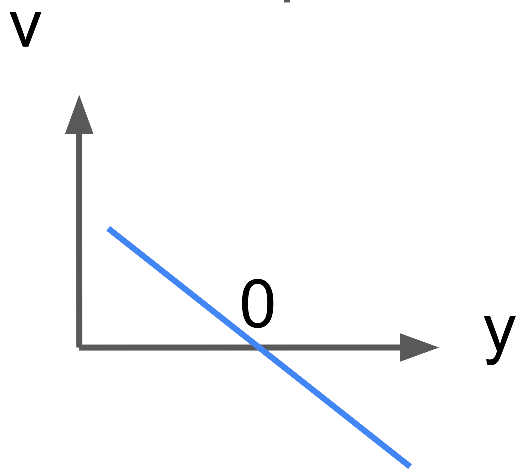
$$\rho u A = \text{constant}$$



an example







$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

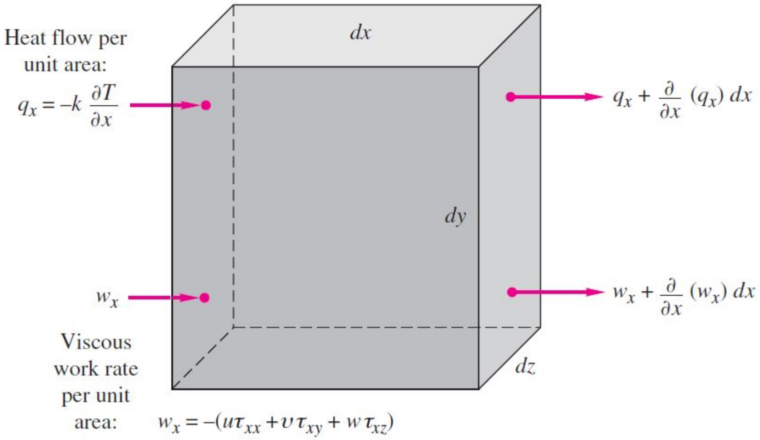
conservation of energy (1st law of thermodynamics)

$$e = \hat{u} + \frac{|\vec{V}|^2}{2} + gz$$

$$\rho \frac{De}{Dt} = -\nabla \cdot p\vec{V} - \nabla \cdot (\vec{V} \cdot \bar{\bar{\tau}}) + \nabla \cdot (kT)$$

momentum eqn.

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \vec{V} = \Phi + \nabla \cdot (kT)$$



$$\Phi = \mu[2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial w}{\partial z})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2]$$

conservative forms

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = J$$

variations

$$d\hat{u} = C_{\mathrm{V}} dT$$

$$dh = d(\hat{u} + \frac{p}{\rho}) = C_{\mathrm{p}} dT$$

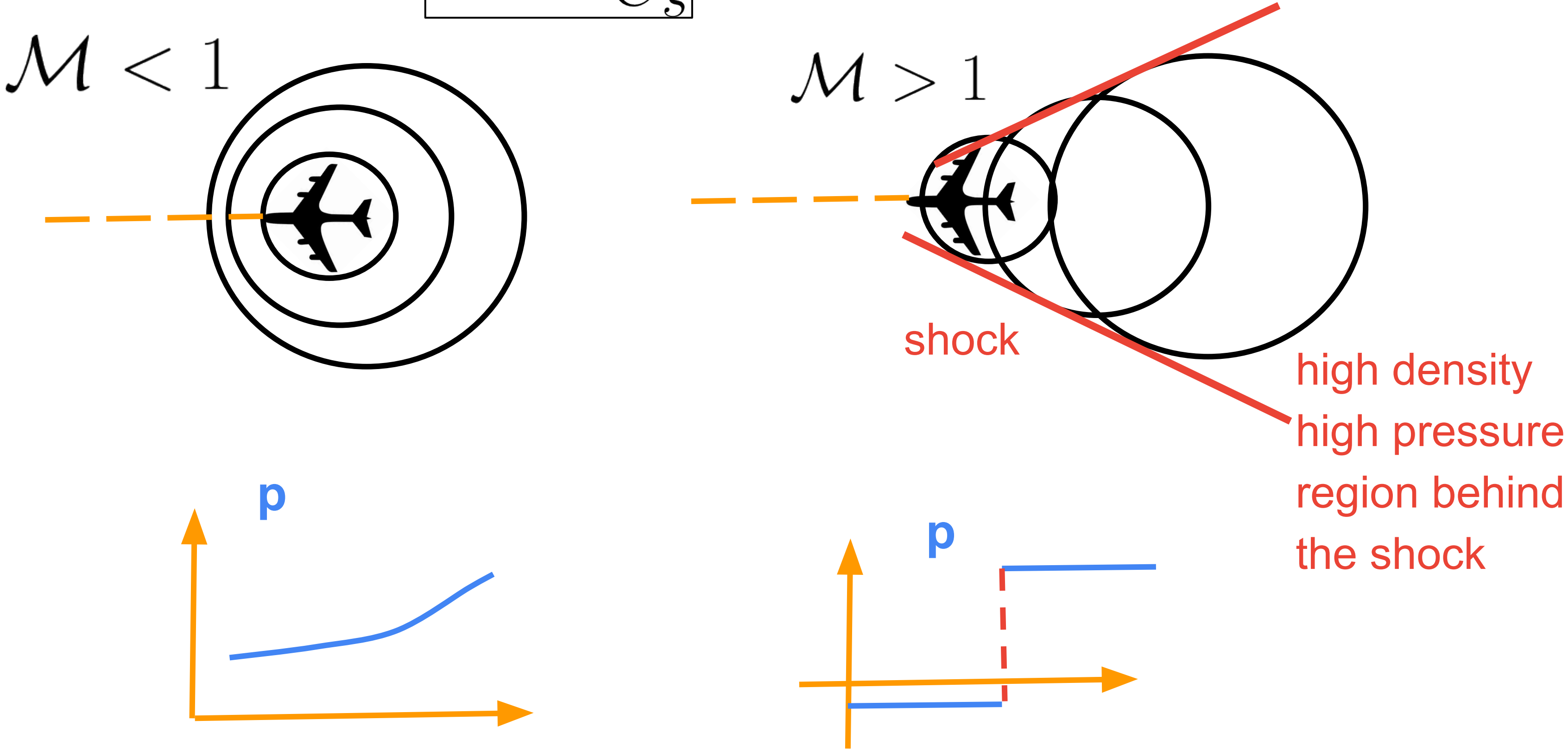
isothermal $P \propto \rho$

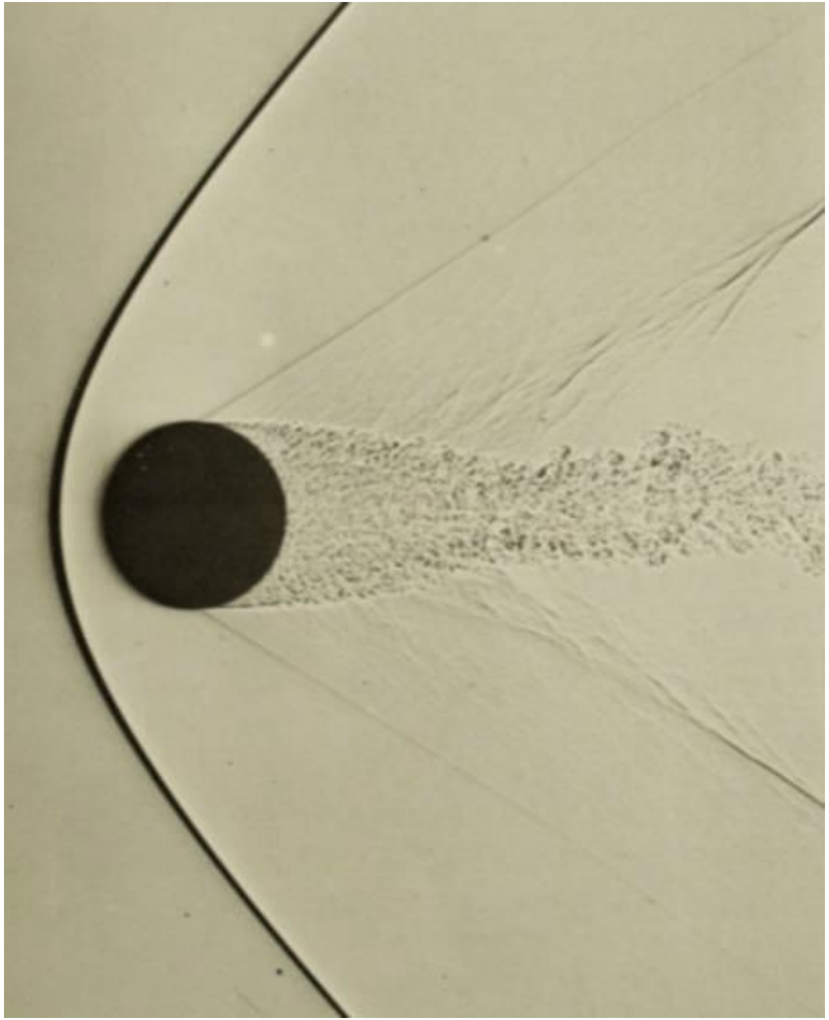
$$C_s = \frac{d\rho}{dp} = \frac{\rho}{p}$$

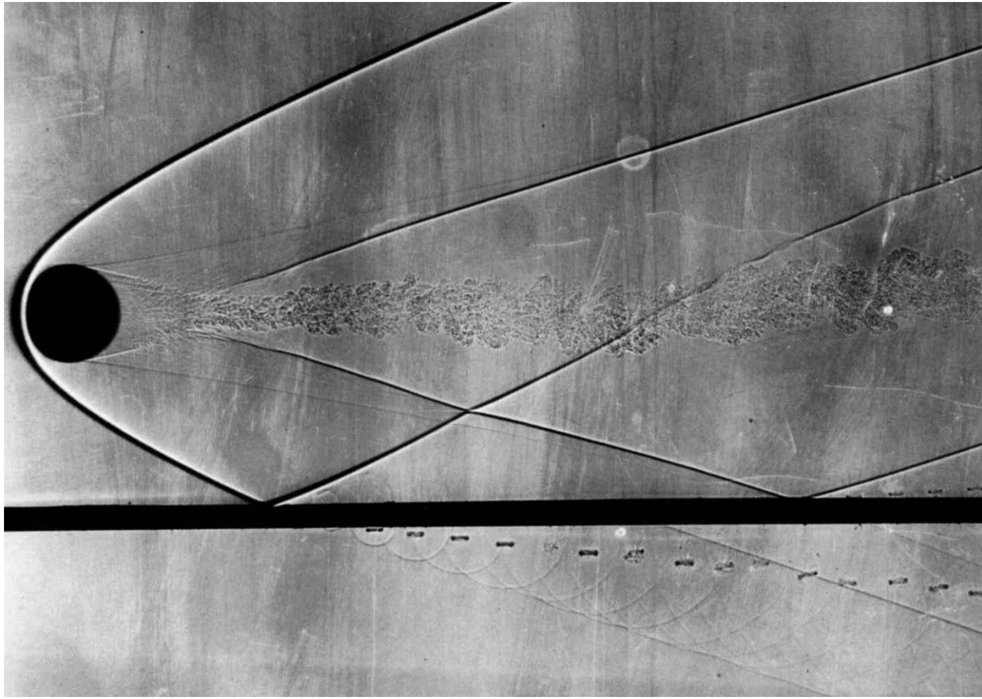
adiabatic $P \propto \rho^{\gamma}$

$$C_s = \frac{d\rho}{dp} = \gamma \frac{\rho}{p}$$

$$\boxed{\mathcal{M} \equiv \frac{V}{C_s}}$$







normal shock and Rankine-Hugoniot relations (adiabatic)

$$\rho_1 < \rho_2$$
$$p_1 < p_2$$
$$u_1 > u_2$$

shock

$$\rho_1 u_1 = \rho_2 u_2$$

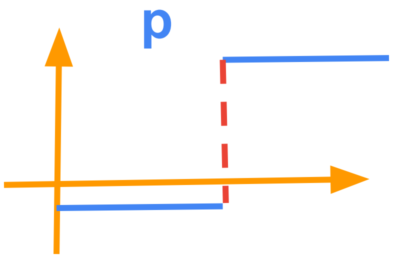
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

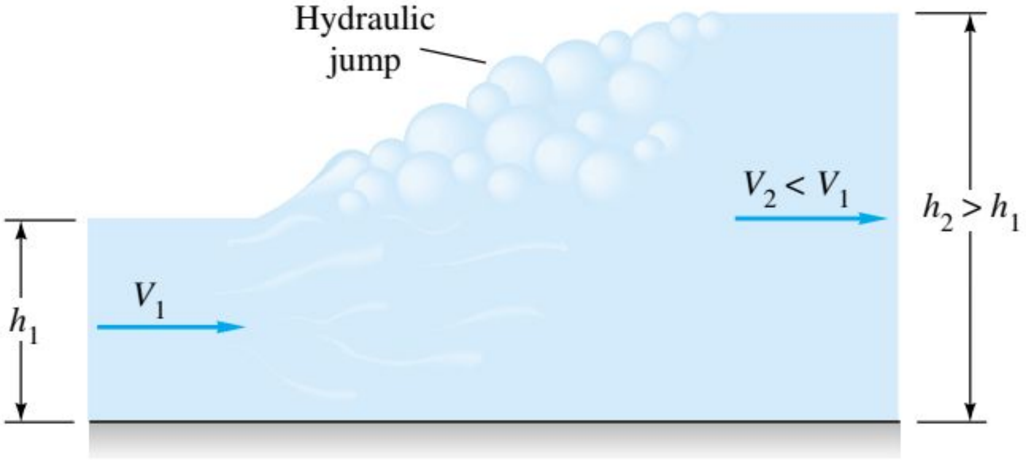
$$\frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2}$$

$$\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma + 1}{\gamma - 1}$$

*for isothermal or radiative shock, density contrast can reach arbitrarily high values


p


Hydraulic
jump
V_1
$V_2 < V_1$
h_1
$h_2 > h_1$

$$(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla)\vec{V} = \cancel{\vec{g}} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2\vec{V}}$$

$$udu + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

$$\rho u A = \mathrm{constant}$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$\boxed{(1 - \mathcal{M}^2)\frac{d\rho}{\rho} + \frac{du}{u} = -\frac{dA}{A}}$$

$$\left(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla\right)\vec{V} = \cancel{\vec{g}} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2\vec{V}}$$

$$\rho u A = \text{constant}$$

$$u\,du + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

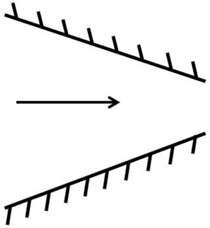
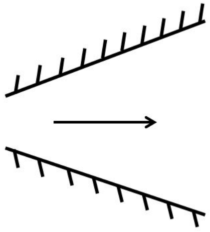
$$\boxed{(1 - \mathcal{M}^2)\frac{d\rho}{\rho} + \frac{du}{u} = -\frac{dA}{A}}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

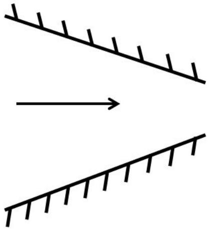
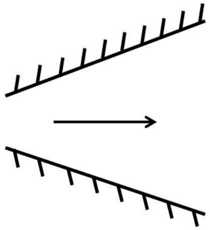
<1% 0.3 10%

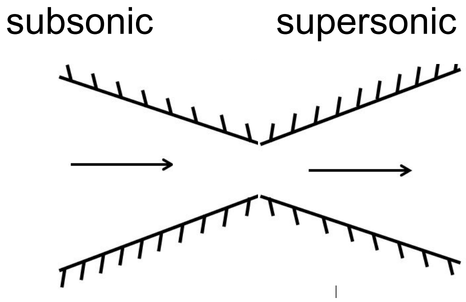
if $\mathcal{M} < 0.3 \rightarrow$ impcompressible fluid

$$\boxed{(1-\mathcal{M}^2)\frac{d\rho}{\rho}+\frac{du}{u}=-\frac{dA}{A}}$$

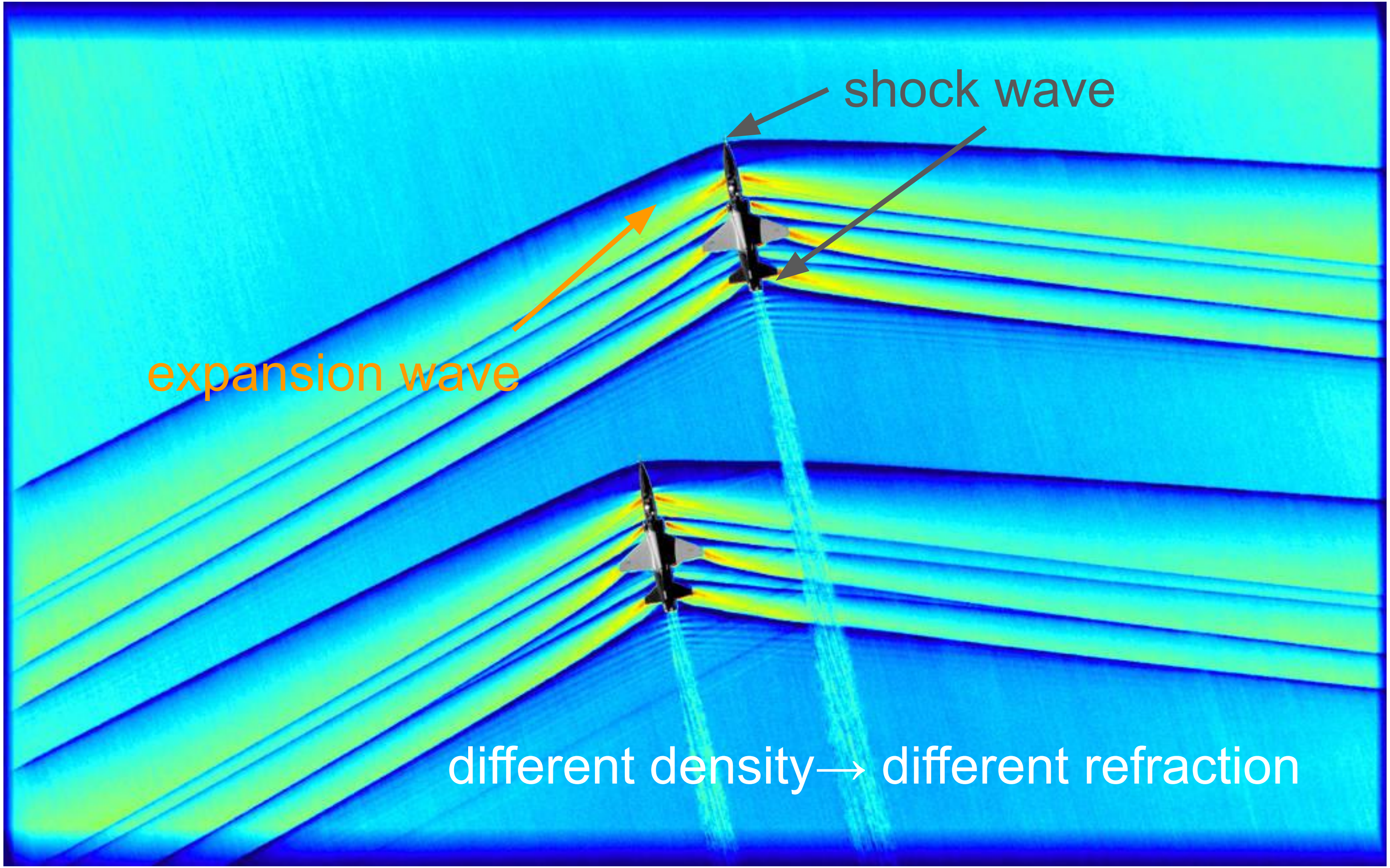
		subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$		$du>0$	$du<0$
$dA>0$		$du<0$	$du>0$

$$\boxed{(1-\mathcal{M}^2)\frac{d\rho}{\rho}+\frac{du}{u}=-\frac{dA}{A}}$$

		subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$		$du>0$	$du<0$
$dA>0$		$du<0$	$du>0$



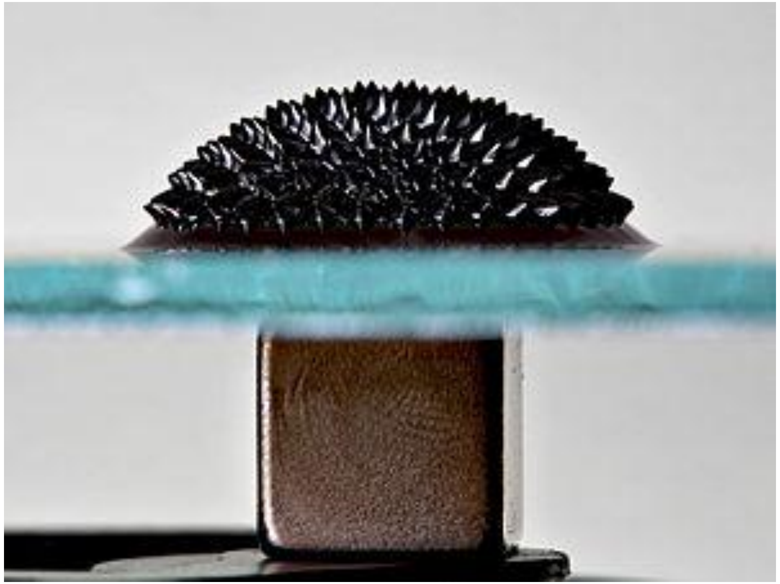
If the sonic transition does no occur in the nozzle flow, the fluid speed reaches an extremum (du=0) when dA=0


shock wave
expansion wave
different density→ different refraction

MagnetoHydroDynamics (磁流體力學)

we are not talking about

ferrofluid (鐵磁流體)



credit: wiki

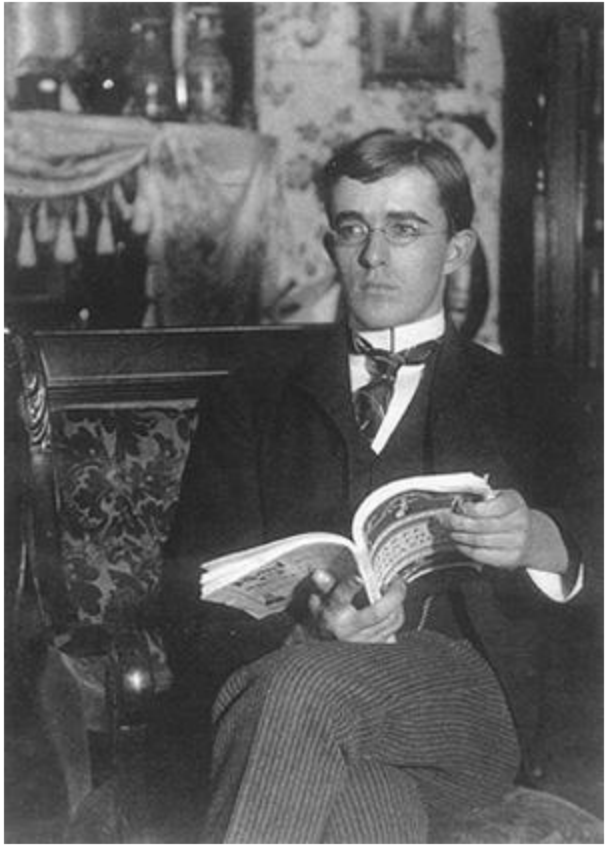
a quasi-neutral gas of charged (ionized) and neutral particles which exhibits collective behaviors

e.g. Sun, neon light etc

key: ionization

In practice **quite modest degrees of ionization are sufficient for a gas to exhibit electromagnetic properties.**

“Principles of Plasma Physics for Engineers and Scientists” by Inan et al.: a gas achieves an electrical conductivity of about half its possible maximum at about 0.1% ionization and has a conductivity nearly equal to that of a fully ionized gas at 1% ionization.



The word plasma was first used by Langmuir in 1928 to describe the ionized regions in gas discharges.

general ideas

when fluid is composed of charged particles, their behaviour is modified by **EM fields**

magnetic fields are important in many astrophysical situations (sun, pulsars, radio falaxies)

in general there will be an interplay between the magnetic field and the fluid of charged particles

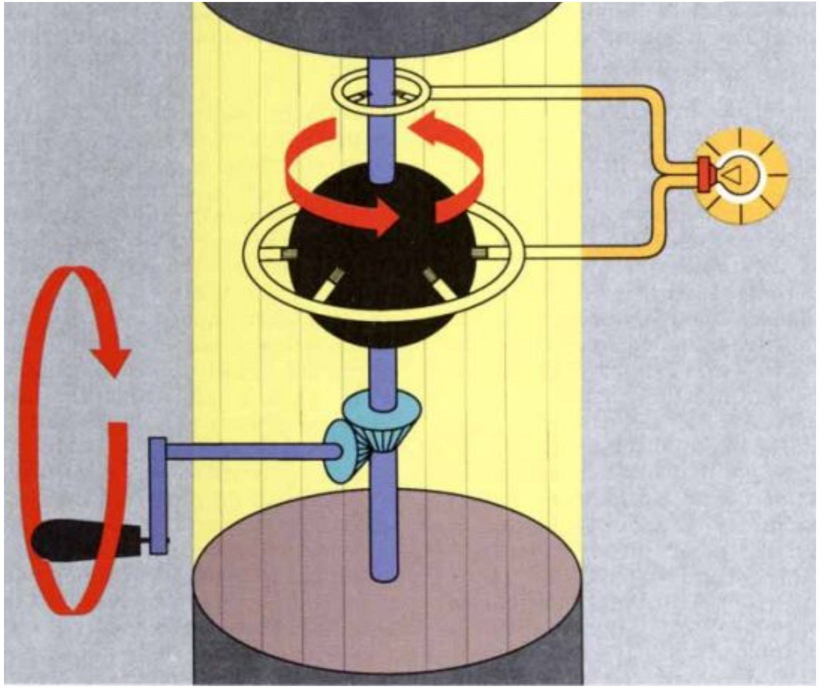
the magnetic field <=====> fluid motion

fluid approach ⇒ magnetohdrodynamics

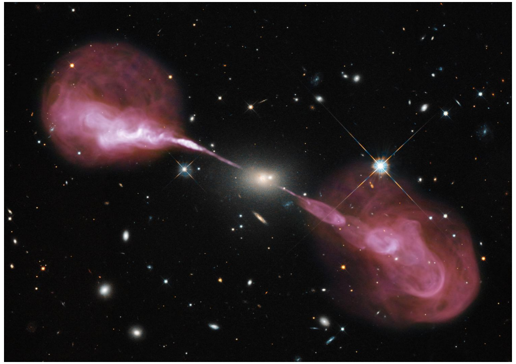
we can define different mean properties for the particle species with different charges

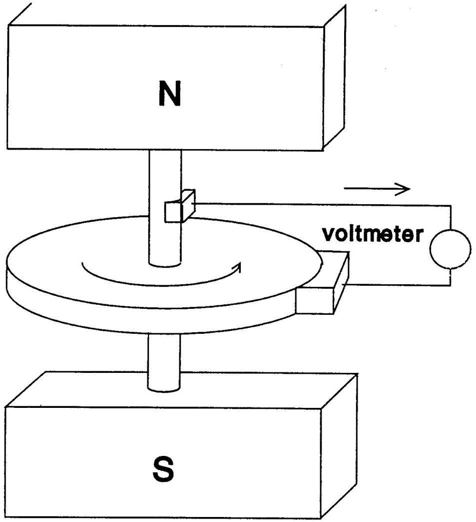
The Membrane Paradigm for Black Holes

How can one picture the interaction of a hole in spacetime with the matter and fields of its environment? It is fruitful to conceive of the black hole as an electrically conducting, spheroidal membrane



credit: Price and Thorne (Scientific American 1988)





Astrophysical proverbs:

If we don’t understand it, invoke magnetic fields.

before MHD: paritcle orbits

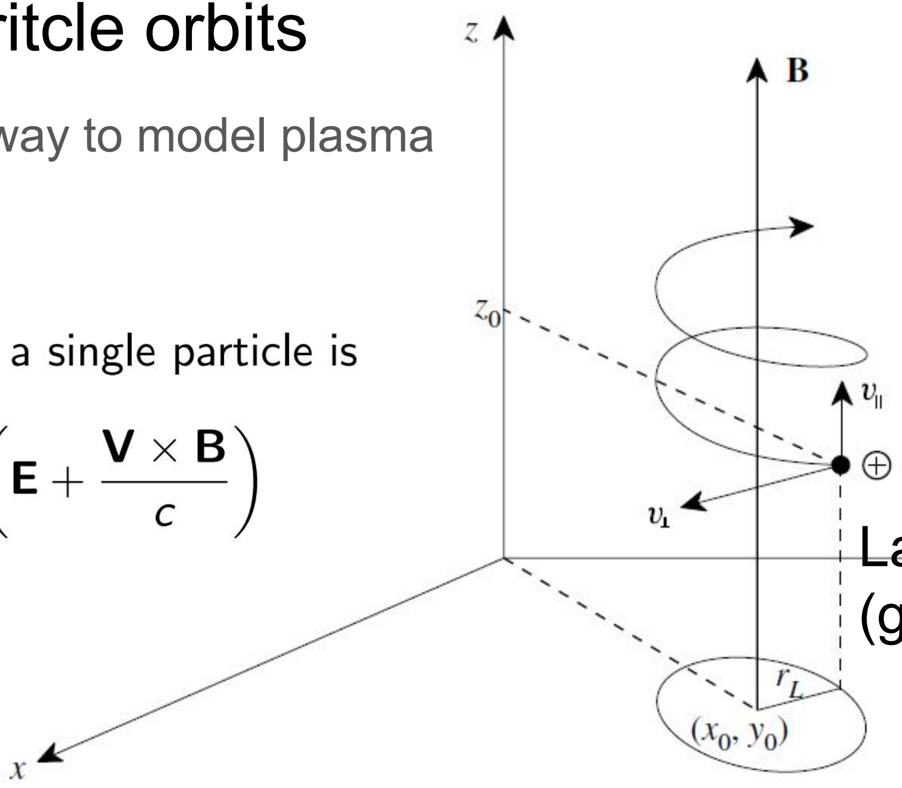
the most fundamental way to model plasma

for low density plasma

The Lorentz force acting on a single particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

in cgs unit



Larmor radius
(gyrofrequency)

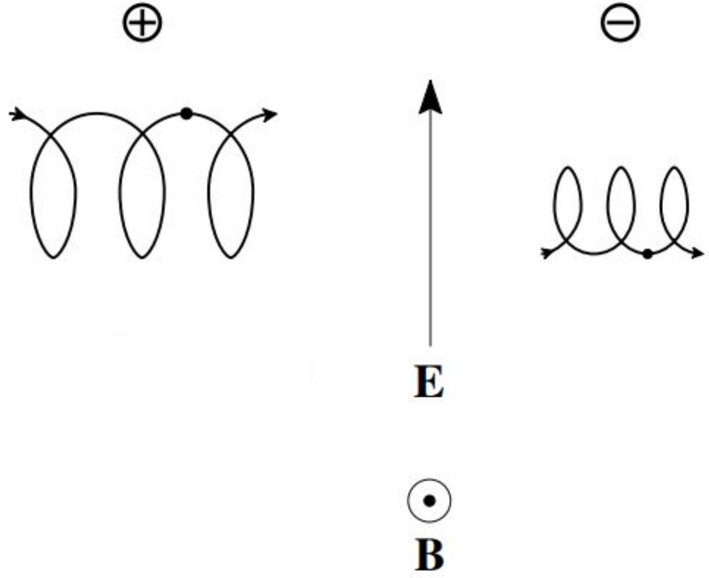
$$= \mathrm{m}v_\perp/\mathrm{q}\boldsymbol{B}.$$

Fig. 2.1. Orbit of a positively charged particle in a uniform magnetic field.

paritcle orbits: (ExB) drift

drift velocity:

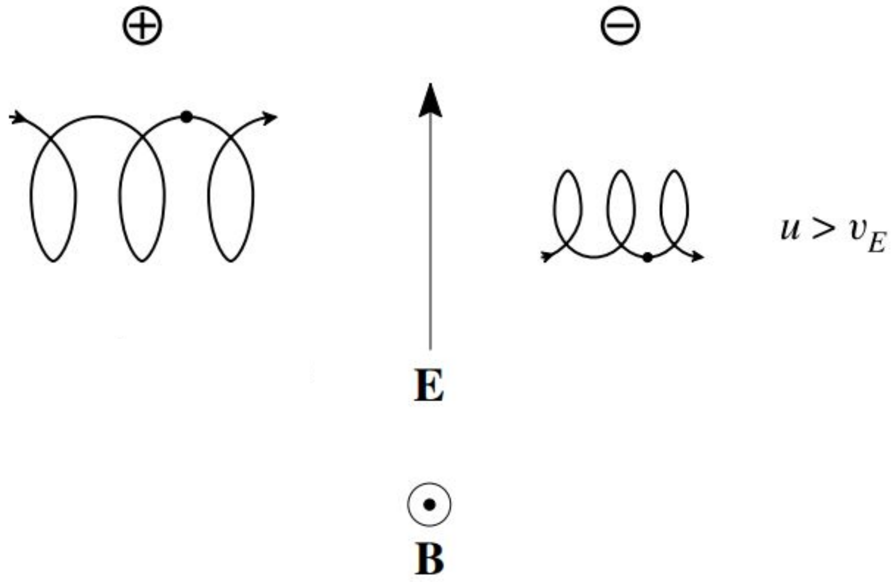
$$\mathbf{v}_{\mathrm{E}} = (\mathbf{E} \times \mathbf{B})/B^2$$



paritcle orbits: (ExB) drift

drift velocity:

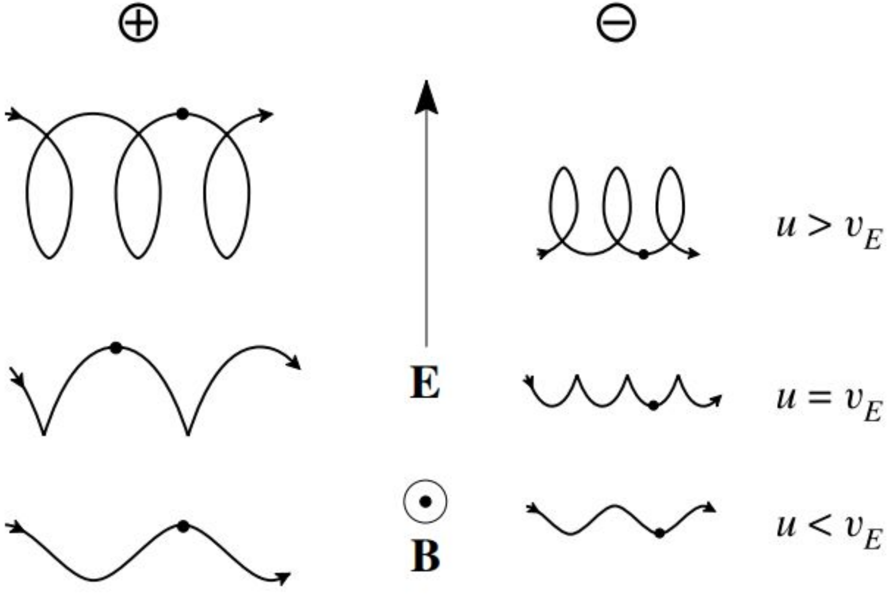
$$\mathbf{v}_{\mathrm{E}} = (\mathbf{E} \times \mathbf{B})/B^2$$



paritcle orbits: (ExB) drift

drift velocity:

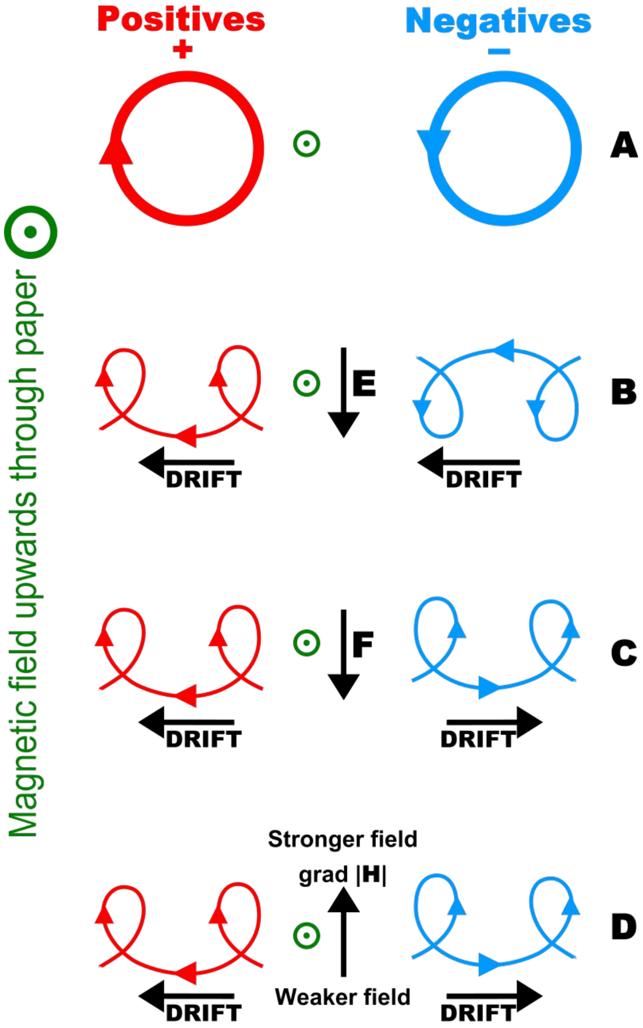
$$\mathbf{v}_{\mathrm{E}} = (\mathbf{E} \times \mathbf{B})/B^2$$



increase of E

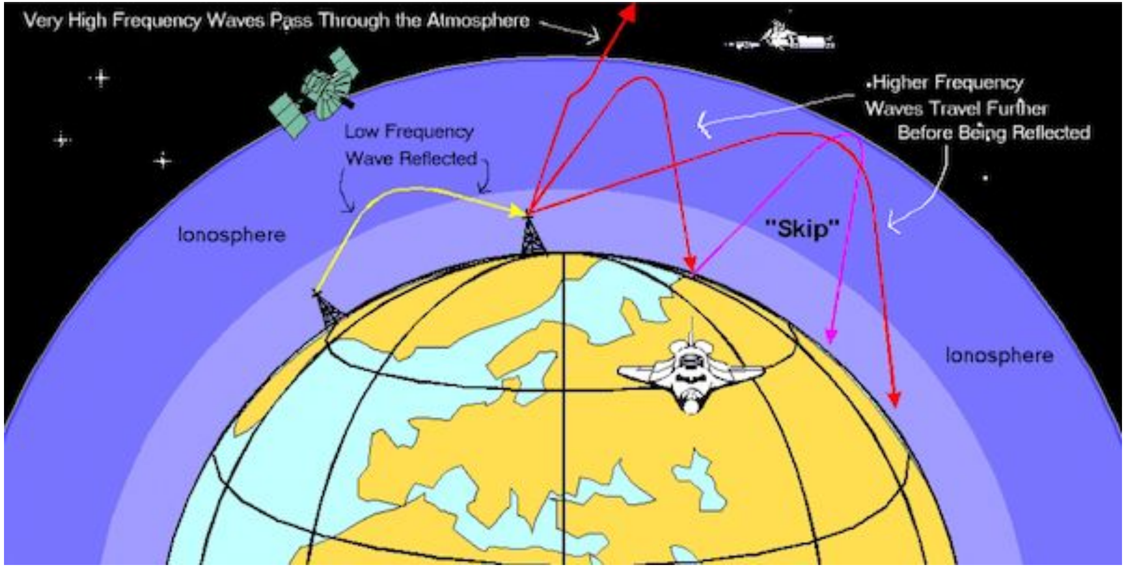
paritcle orbits: other drifts

gyrofrequency
~1.76 x 10^7 Hz
(for electron)



plasma frequency (~9 $n^{1/2}$Hz for electron)

- the frequency at which the electrons in the plasma naturally oscillate relative to the ions
- For the ionosphere, plasma frquency ~1010^{7}Hz
- f<10^7Hz: reflected by ionosphere



Credit: NASA/GSFC

Debye length and Debye shielding

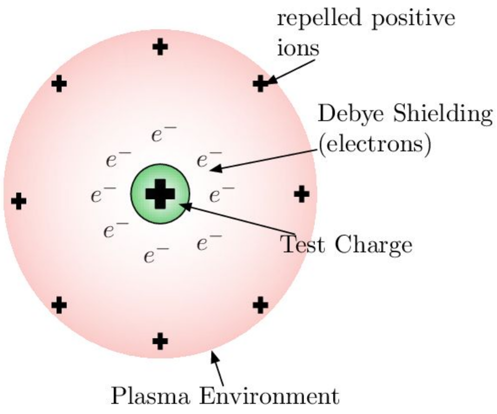
$$\lambda_D = \sqrt{\frac{K_B T}{4\pi n q^2}}$$

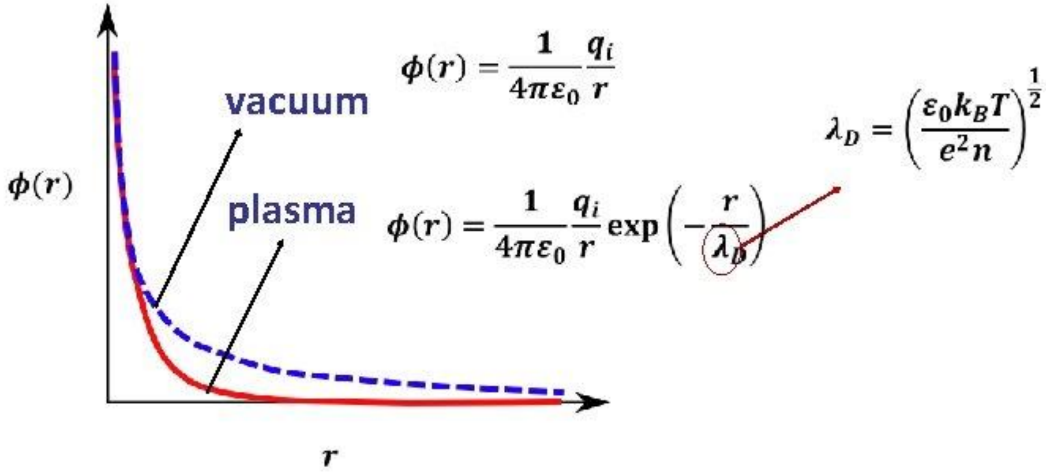
(Debye length)

any charge imbalance osillates with the plasma frequency

Debye length x plasma frequency= thermal velocity

Debye length as an effective shielding length (thermal motions “iron out” plasma oscillations)





MHD:

describes the “slow” evolution of an electrically conducting fluid,

and a region >> Debye length, Larmor radius

some initial guess

adding Lorentz force (qE +J xB) to momentum equation

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} = \frac{\mathbf{J}\times\mathbf{B}}{c} - \nabla p$$

adding Ohm's law (J) to close the set of equations

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

conductivity

some initial guess

SI unit

- In lab: apply presence of E and/or B
- astrophysical: generated by the motion and distribution of the charged particles

relation to related E B to the charge and current ⇒ Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

some initial guess

from (3): E/B~L/T~u

from (4):

RHS 2nd term/LHS ~ u^2/c^2

nonrelativistic flow (u<<c): ignored terms related to E!

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}}$$

SI unit

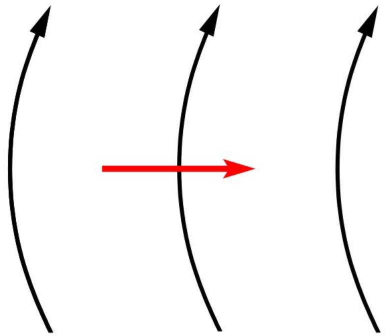
$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} + \frac{\partial B}{\partial t} &= 0 \\
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} &= \mu_0 \vec{J}
\end{aligned}$$

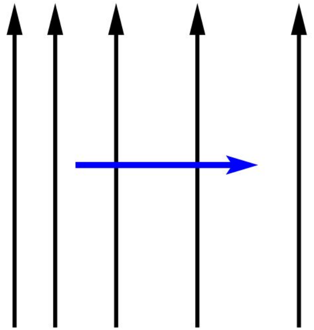
cgs unit

$$\begin{aligned}
\vec{\nabla} \cdot \vec{B} &= 0 \\
\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} &= 0 \\
\vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\
\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}.
\end{aligned}$$

$$\underbrace{\frac{\mathbf{J} \times \mathbf{B}}{c}}_{\text{Lorentz force}} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}$$

$$= \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{\sim\text{magnetic tension}} - \underbrace{\nabla \left(\frac{B^2}{8\pi} \right)}_{\sim\text{magnetic pressure}} \quad (20)$$





$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

finite conductivity

$$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$$

magnetic diffusivity

$$\boxed{\frac{\partial B}{\partial t} = \nabla\times(v\times B) + \eta\nabla^2 B}$$

(induction equation)

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B$$

$$\mathrm{R_m} = \frac{UL}{\eta} \sim \frac{\text{induction}}{\text{diffusion}}$$

L - Typical length scale of the flow **U - Typical Velocity scale of the flow**

R_m- Reynolds Magnetic Number **η - Magnetic Diffusivity**

usually >>1 in astrophysics

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}\right)$$

ideal MHD:
infinite/perfect conductivity

$$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0 \text{ and } \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})}$$

flux freezing! - Alfven’s forzen-in theorem

$$\Psi = \int_S \mathbf{B}\cdot d\mathbf{S}.$$

$$\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}.$$

$$\frac{d\Psi}{dt} = -\int_S \nabla\times(\mathbf{E}+\mathbf{V}\times\mathbf{B})\cdot d\mathbf{S}.$$

$= 0$

$$\int_C \mathbf{B}\cdot\mathbf{V}\times d\mathbf{l} = \int_C \mathbf{B}\times\mathbf{V}\cdot d\mathbf{l}.$$

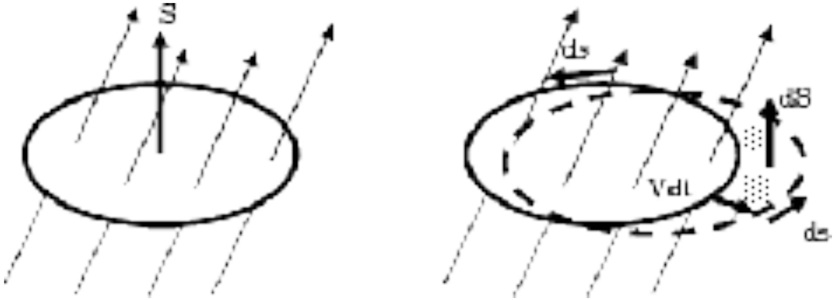


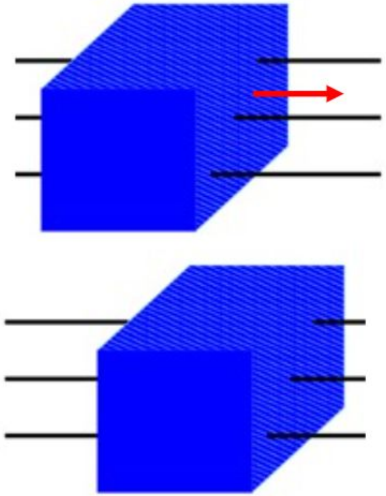
image credit:Kohji Tomisaka

\+ the help of Stoke thm:

$$\iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,d\Sigma = \int_C \mathbf{F}\cdot d\mathbf{r}$$

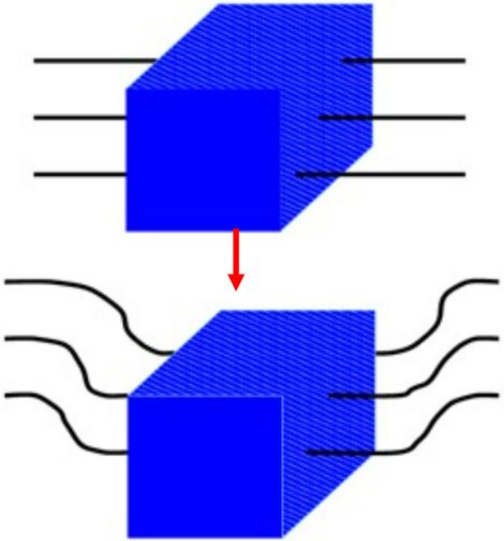
Ideal MHD: flux freezing

<u>Strong field:</u> matter can only move along given field lines (beads on a string):



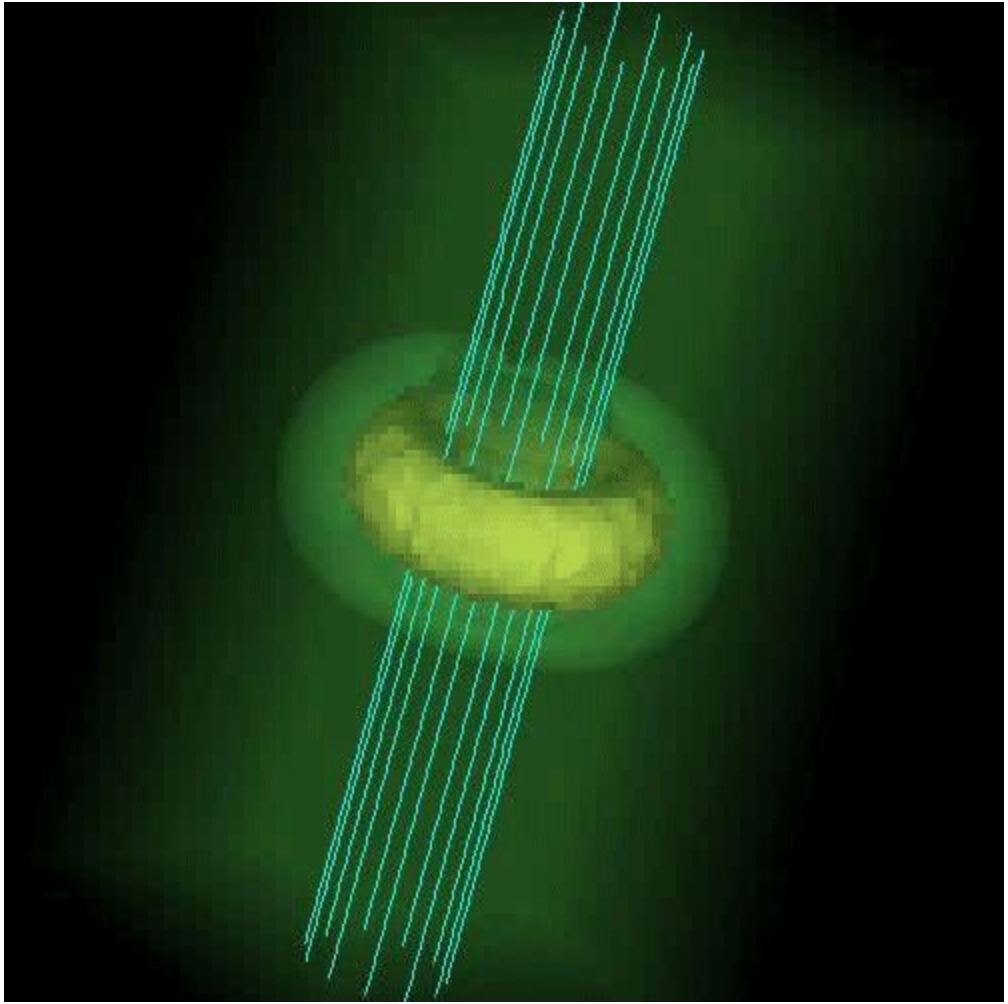
$$\frac{|B|^2}{8\pi} >> P_{\mathrm{gas}} + \rho|\mathbf{v}|^2$$

<u>Weak field:</u> field lines are forced to move along with the gas:

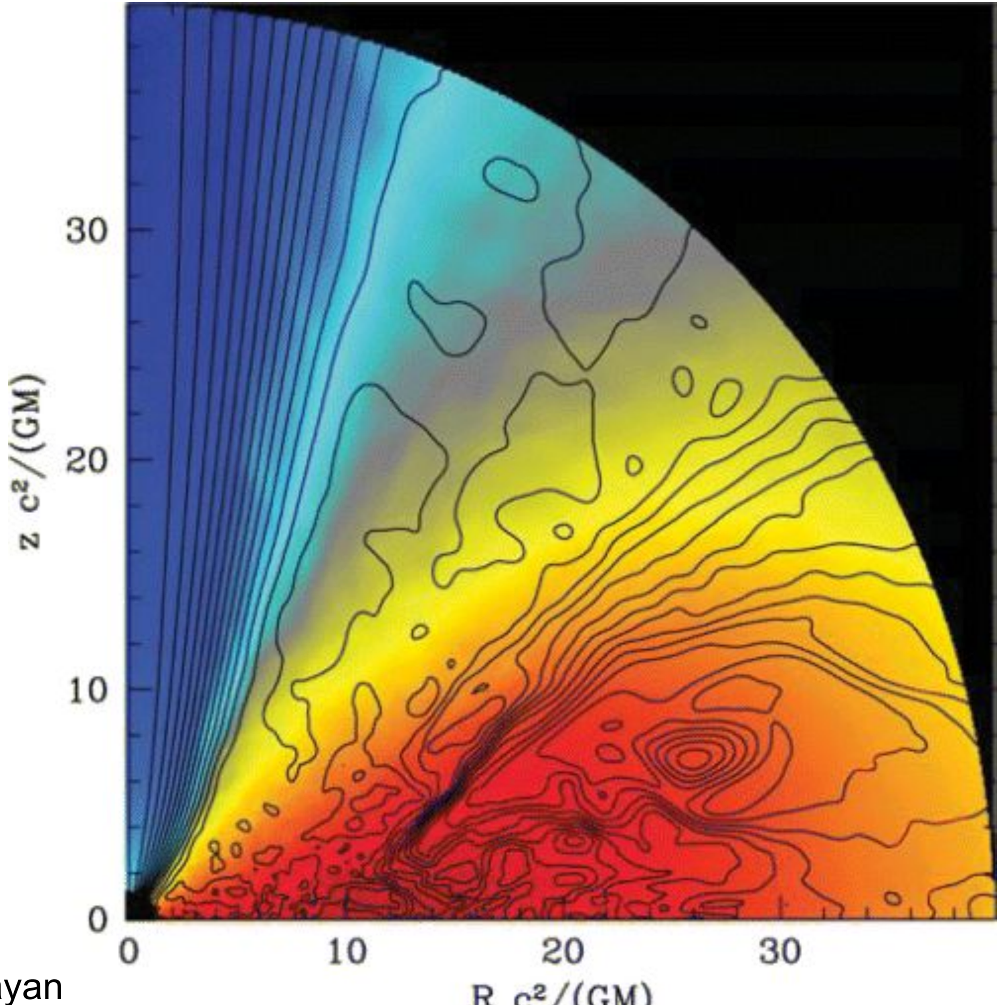


$$\frac{|B|^2}{8\pi} << P_{\mathrm{gas}} + \rho|\mathbf{v}|^2$$

slide credit: C. P. Dullemond



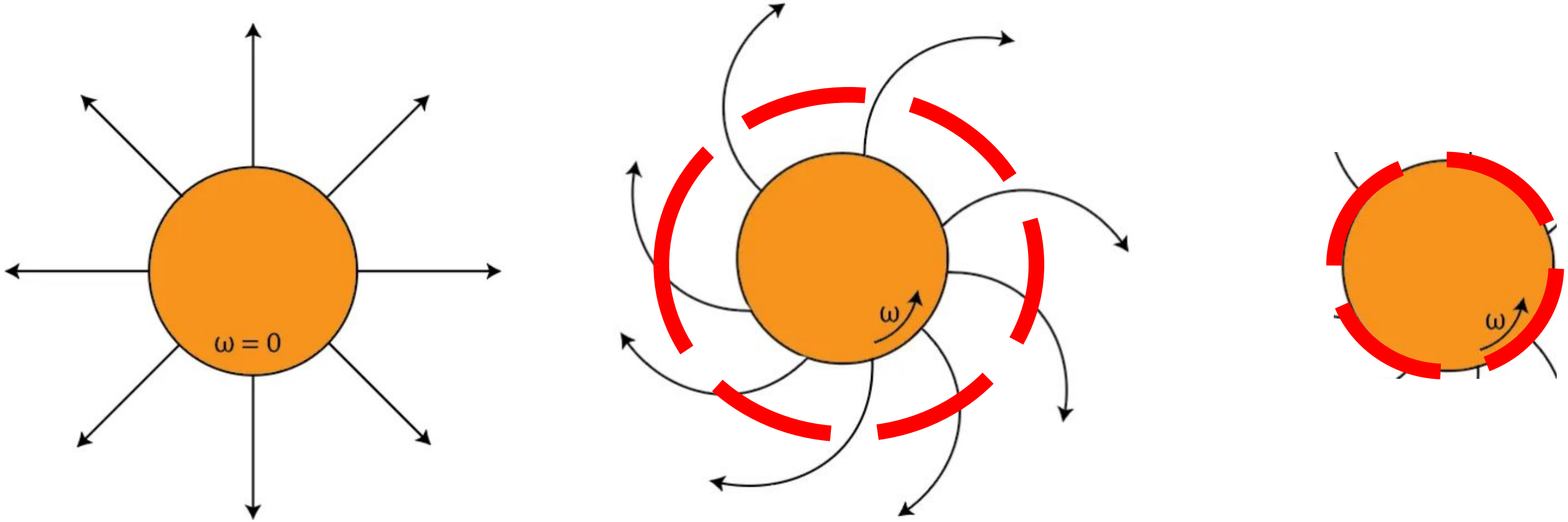
credit: Kuwabara

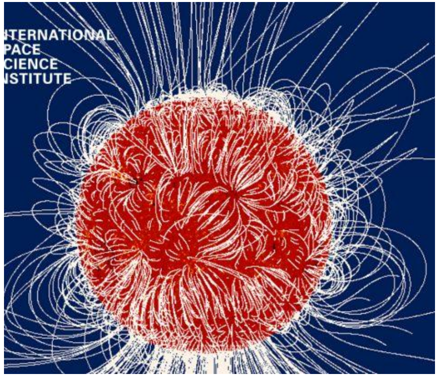


credit: McKinney & Narayan

magnetized wind

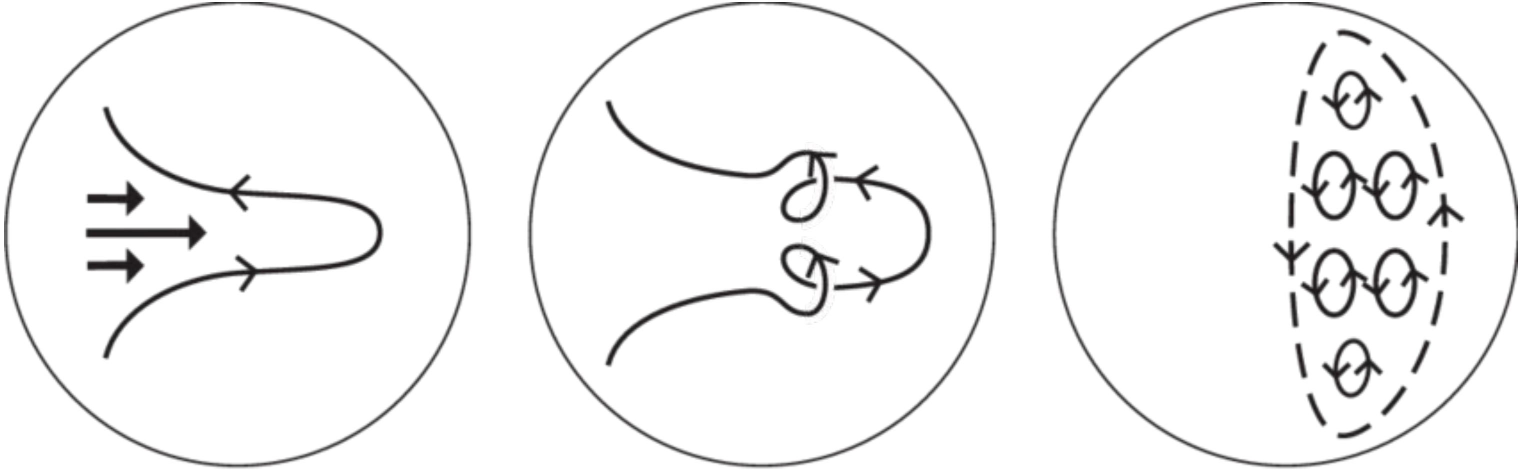
the plasma rotates approximately like a solid body out to the **Alfven radius**, at where the magnetic energy equals the kinetic energy





Ω-effect

α-effect



(a) (b) (c)

ideal MHD equations (cgs unit)

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$	**(note: compressible)**
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$	
Ampere's law	$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$	
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$	
Ideal Ohm's law	$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0$	
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$	
Adiabatic Energy Equation	$\frac{DP}{Dt} = -\gamma P \boldsymbol{\nabla} \cdot \mathbf{V}$	

Definitions: **B**, magnetic field; **V**, plasma velocity; **J**, current density; **E**, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

Fast mode has field and gas compression in phase
Slow mode has field and gas compression out of phase

three MHD waves

slow magnetosonic

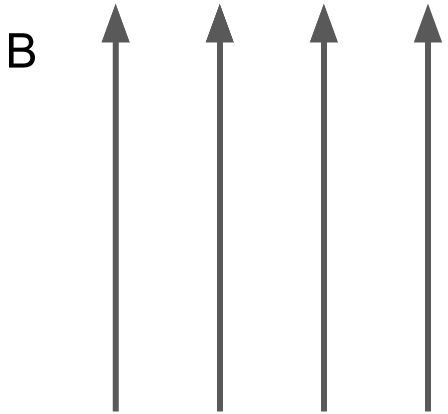
Alfven (resoroing force= magnetic tension)

$$V_A = \frac{B}{\sqrt{4\pi\rho}}$$

fast magnetosonic

apply perturbation to:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} &= \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}) \\
\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) p &= -\gamma P \boldsymbol{\nabla} \cdot \mathbf{V}
\end{aligned}$$



results depends on theta, Alfven speed, sound speed

“when I meet God, I am going to ask him two questions: Why relativity? and why trubulence?

I really belive he will have an answer for the first.”

W. Heisenberg (1907-1976)

Final remarks

- astrophysical fluid
  - large scale
  - gravity is important
  - radiation cooling
  - MHD, most of the time ideal MHD
  - multi-phase

- fluid mechanics as a physical problem:
  - govering equations + e.o.s + boundary conditions