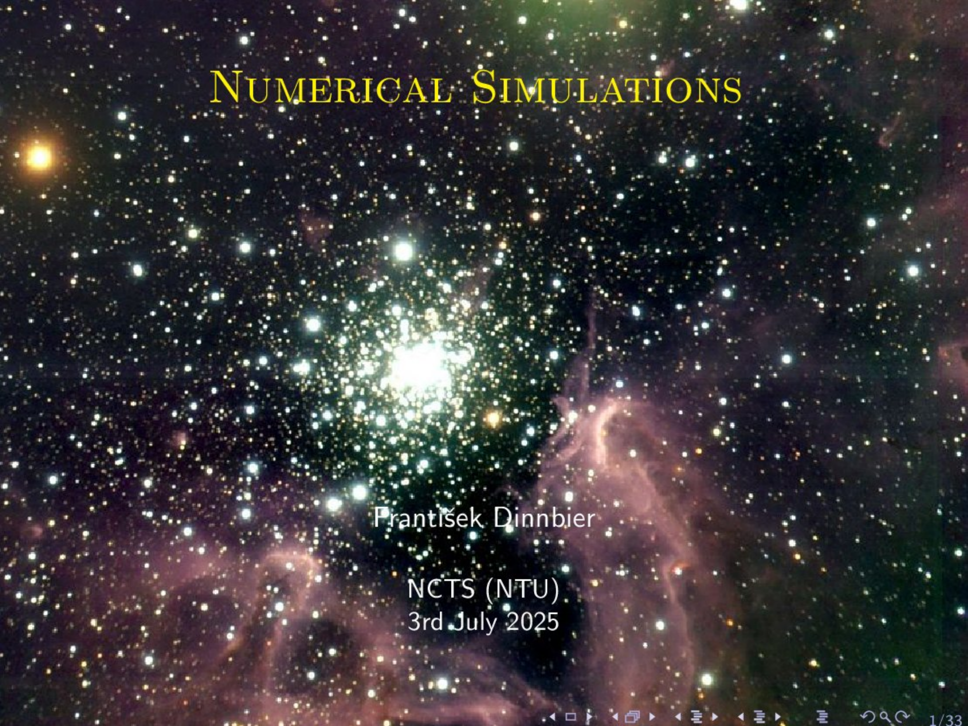

Numerical Simulations

František Dinnbier

NCTS (NTU)

3rd July 2025

Motivation for N-body dynamics

The problem is called N-body problem.

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Lets' start with $N = 2$.

Motivation for the N-body problem

- Let's start with two bodies
- The law of universal gravitation

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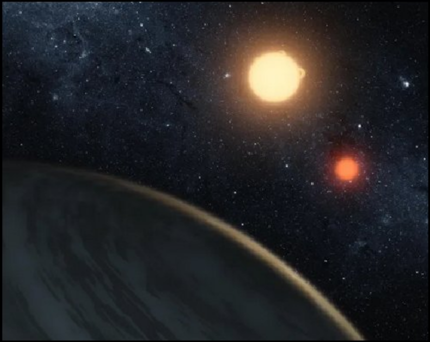

Two body problem - analytical solution

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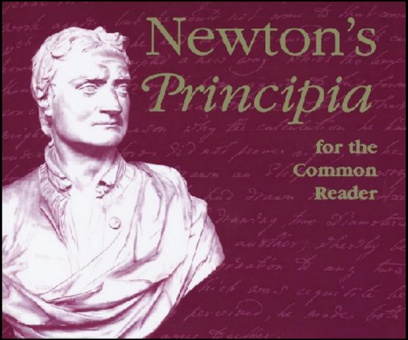

Two body problem - analytical solution

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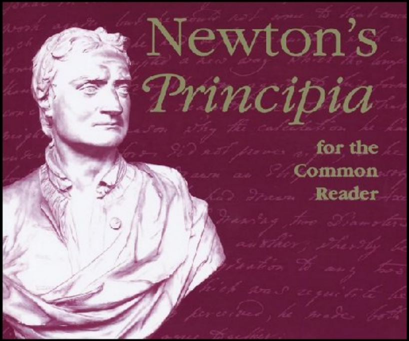
- $$r = \frac{a(1 - e^2)}{1 + e\cos\phi}$$



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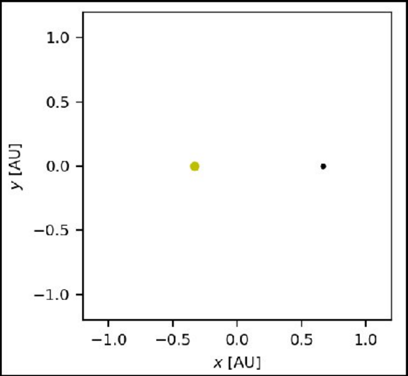

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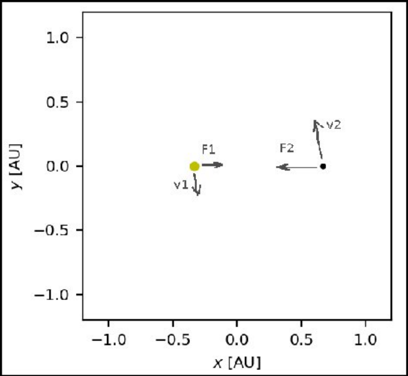

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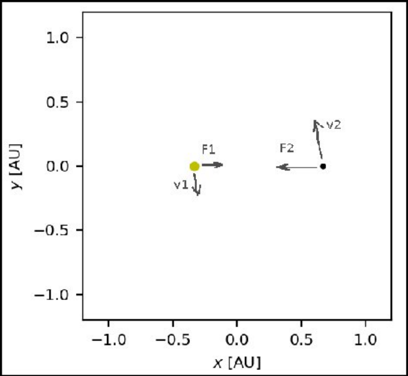
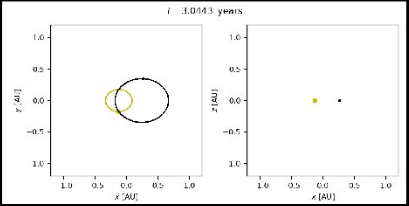


- Two stars of masses $1\,\mathrm{M}_\odot$ and $0.5\,\mathrm{M}_\odot$ on a mildly eccentric orbit with semi-major axis $a = 0.64$ AU.

Two body problem - numerical solution



The transition to complexity

Two body problem is well understood.

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What happens if $N = 3$.

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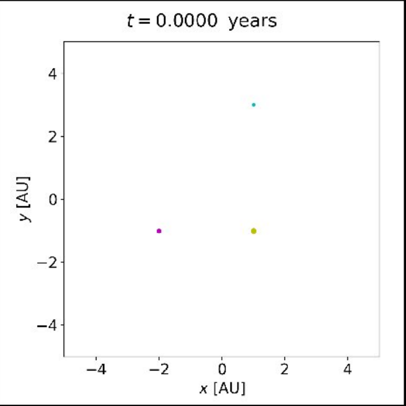
What happens if $N = 3$.

Is there also an elegant analytic solution as for $N = 2$?

Introduction to three body problem

- Lets have three stars of masses $3\,\mathrm{M}_{\odot}$, $4\,\mathrm{M}_{\odot}$ and $5\,\mathrm{M}_{\odot}$ at rest at the beginning.
- The stars move only as the influence of its mutual gravity (i.e. star 2 is attracted only by star 1 and 3).



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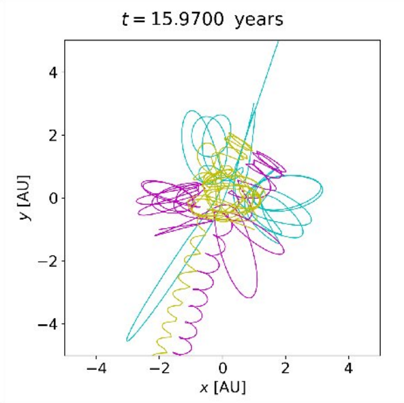

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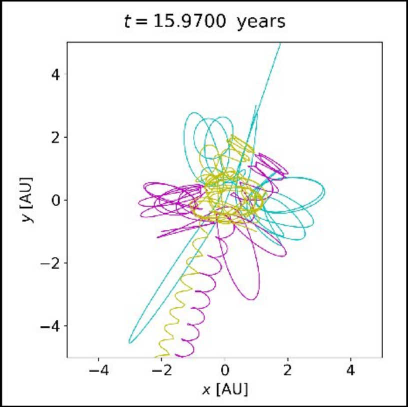
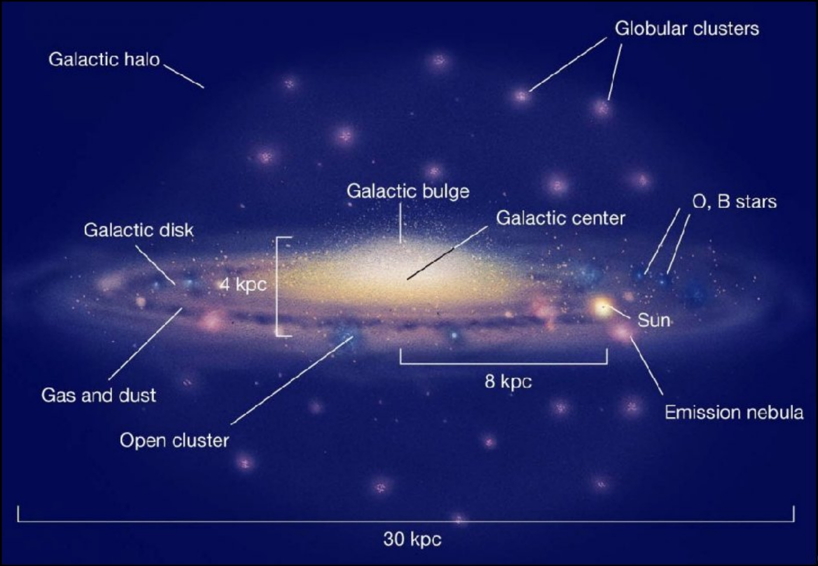
- It is difficult to imagine that the complex trajectories have analytic solution.



The transition to complexity

What are the typical values of N in the nature?

The ecosystem of a spiral galaxy



Systems with larger N: open star clusters

Open star clusters contain up to several thousand of stars ($N = 30$ to $\approx 10^4$ (10^6); $\log_{10}(N) = 1.5$ to 4).

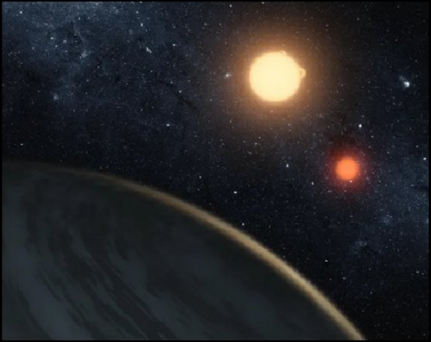
Systems with larger N: open star clusters

There is a large diversity in open star clusters (they differ in their sizes, ages, mass, . . .)

Systems with large N: globular star clusters

Globular star clusters contain up to several million of stars ($N \approx 10^3$ to $\approx 10^7$; $\log_{10}(N) = 3$ to 7).

Stellar binaries and multiples



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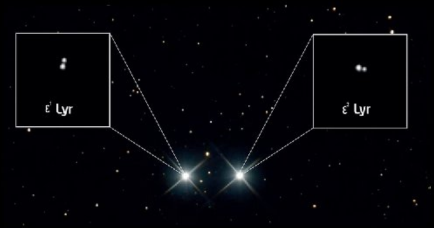

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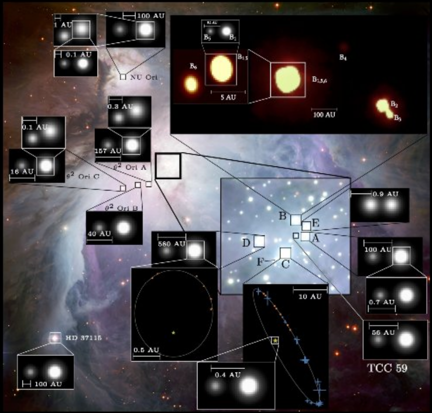

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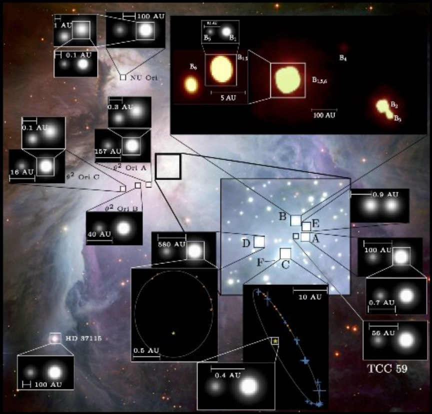

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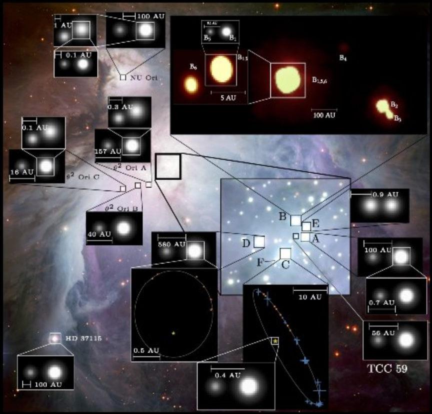

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- The Sun has no companion.

Systems with huge N: galaxies

Milky Way galaxy contains $\approx 5.10^{10}$ stars ($\log_{10}(N) = 10.7$).

Zooming-in on the Andromeda galaxy (M 31)

Panchromatic Hubble Andromeda Treasury (PHAT) program

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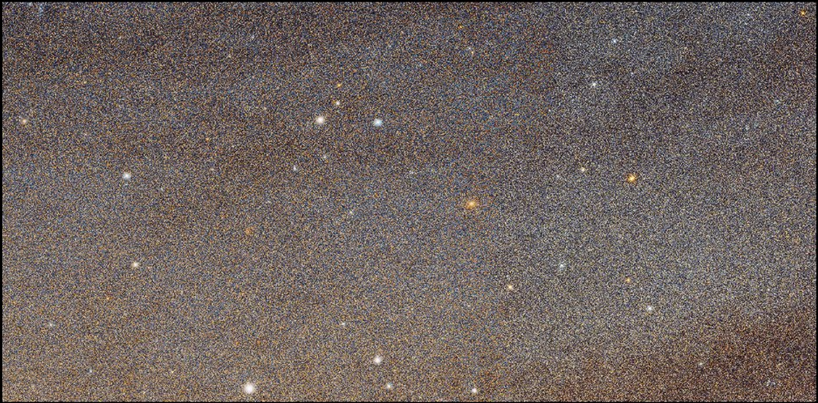

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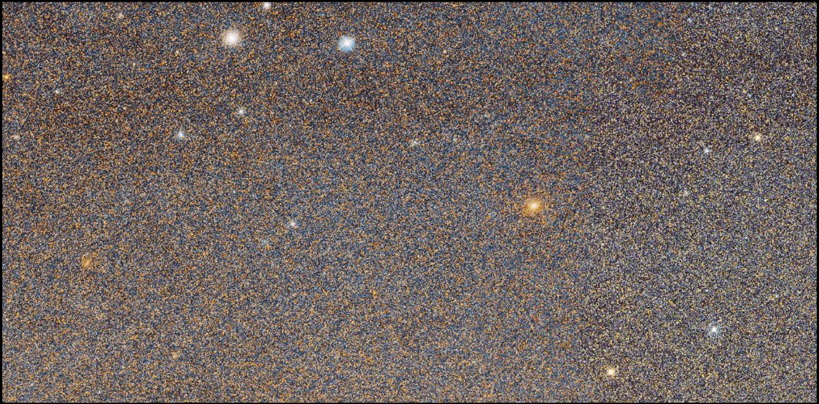

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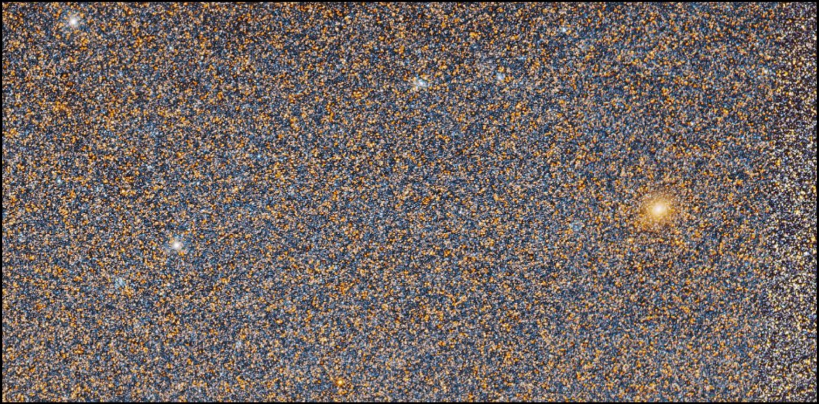

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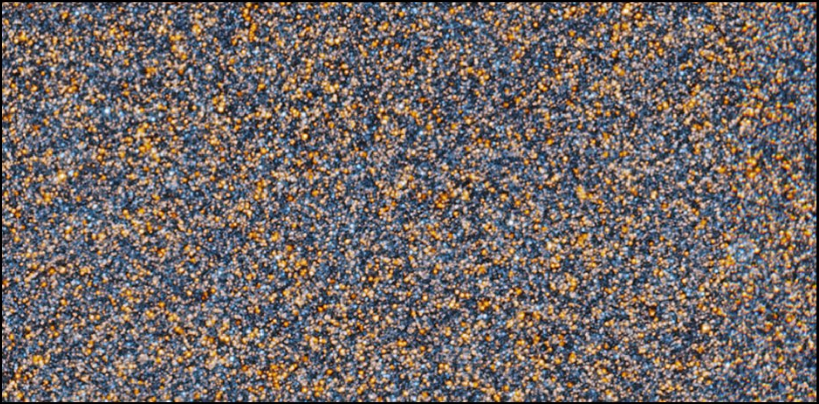

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N-body integrators

- Each star is represented by one particle.

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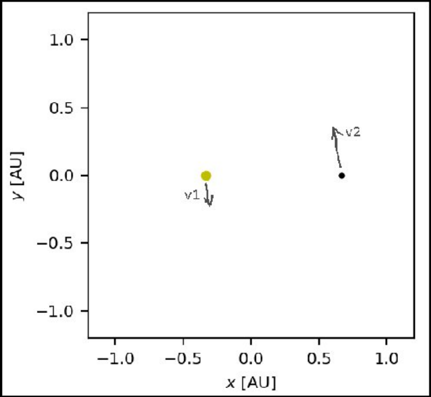
- Tidal field of the galaxy, molecular clouds, . . .
- $\rightarrow$ here we assume that all stars are point-mass (the integrating scheme is the most difficult and complex part of the numerical scheme).

The initial conditions

- For each star, we know its position $\vec{r}$, velocity $\vec{v}$, and mass m.



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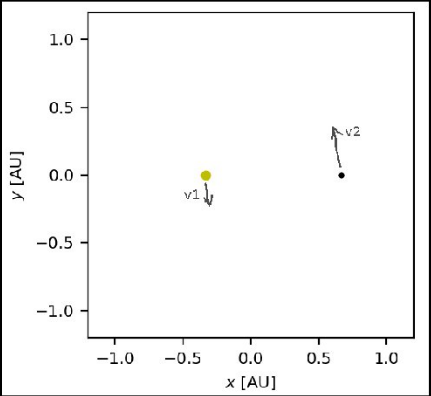

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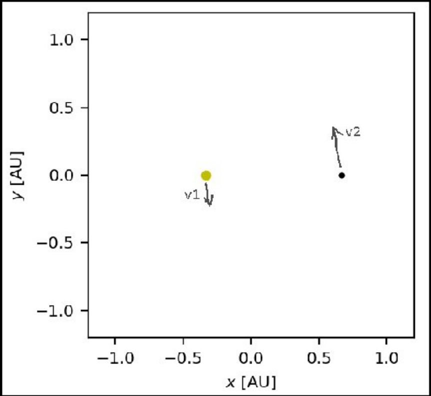
- Each integration time-step, the position and velocity of the stars is updated.



The types of N-body integration schemes

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- Leap frog (symplectic, the energy error does not increase with time).

Overview of the predictor-corrector scheme

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$$\vec{a}_1 = -\frac{GM_2\vec{r}_{1,2}}{r_{1,2}^3}$$

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- From these formulae, we can predict the positions at time $\mathrm{d}t$

$$\vec{r}_p = \vec{r}_0 + \vec{v}_0 \mathrm{d}t + 1/2 \vec{a}_0 (\mathrm{d}t)^2 + 1/6 \dot{\vec{a}}_0$$

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- Note that positions and velocities $\vec{r}_p$ and $\vec{v}_p$ are only predicted values; their accuracy can be further increased by correcting.

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- We evaluate acceleration ($\vec{a}_1$) and its time derivative ($\dot{\vec{a}}_1$) at the predicted positions at new time $t_1 = t_0 + \mathrm{d}t$.

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- We evaluate acceleration ($\vec{a}_1$) and its time derivative ($\dot{\vec{a}}_1$) at the predicted positions at new time $t_1 = t_0 + \mathrm{d}t$.
- It is reasonable to assume that the velocity and its derivatives change smoothly (lets drop the vector symbols for simplicity), i.e.
  $v(t_0) = v_0$
  $\dot{v}(t_0) = a_0$
  $\ddot{v}(t_0) = \dot{a}_0$
  $\dot{v}(t_1) = a_1$
  $\ddot{v}(t_1) = \dot{a}_1$

Overview of the predictor-corrector scheme

- Correcting step is achieved by fitting by Hermite polynomials:
  $r(t_1) = r_p + 3/20(-a_0 + a_1)(\mathrm{d}t)^2 - 1/60(7\dot{a}_0 + 2\dot{a}_1)(\mathrm{d}t)^3$
  $v(t_1) = v_p + 1/2(-a_0 + a_1)\mathrm{d}t - 1/12(5\dot{a}_0 + \dot{a}_1)(\mathrm{d}t)^2$

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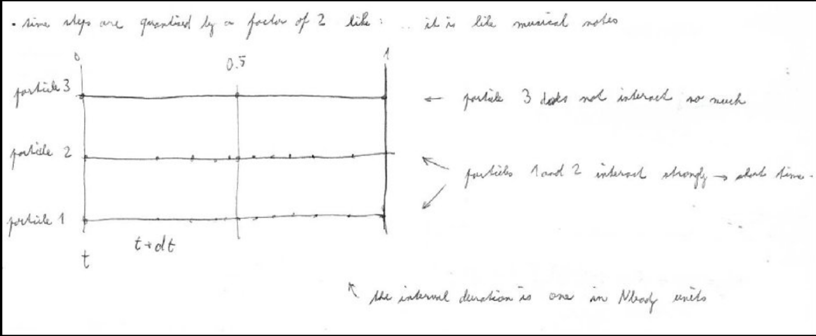
- The best condition turned out to be (Aarseth 1985)

$$\mathrm{d}t = \sqrt{\eta \frac{|F||\ddot{F}| + |\dot{F}|^2}{|\dot{F}||\dddot{F}| + |\ddot{F}|^2}}$$

The individual time-steps



Individual time-steps as the main challenge in star cluster modelling

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- Close binary stars in the cluster can have orbital periods as short as several hours $\rightarrow$ the necessary time-step is several minutes.
- This would mean $\approx 10^{14}$ individual integration time-steps for a single star.
- Smart algorithms usually wins over brute force.

Checking accuracy of N-body models

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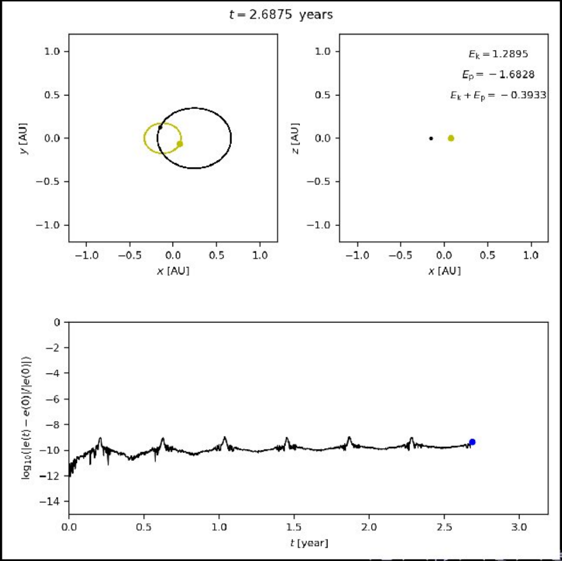
- Consider evolution of a system of N-point masses under its mutual gravity.
- Is there a way how to check that the simulation was correctly calculated?
- Conservation of mechanical energy $E_{\mathrm{tot}} = E_k + E_p$.
- Relative energy error:

$$\mathrm{d}E = |E_{\mathrm{tot}} - E_{\mathrm{tot},0}|/|E_{\mathrm{tot},0}|$$

Checking accuracy of N-body models



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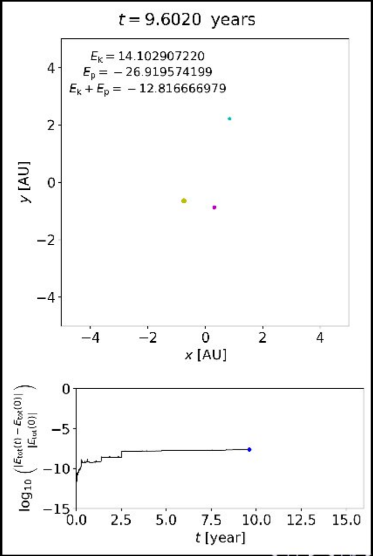
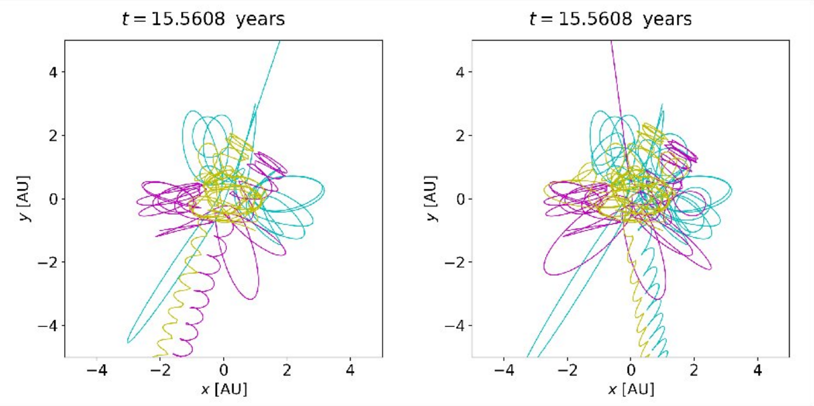


The sensitivity on initial conditions



Both simulations initially have the same initial conditions with the difference that the 3 $\mathrm{M}_\odot$ star in the model on the right panel has initial speed of $10\mathrm{cm/s}$.

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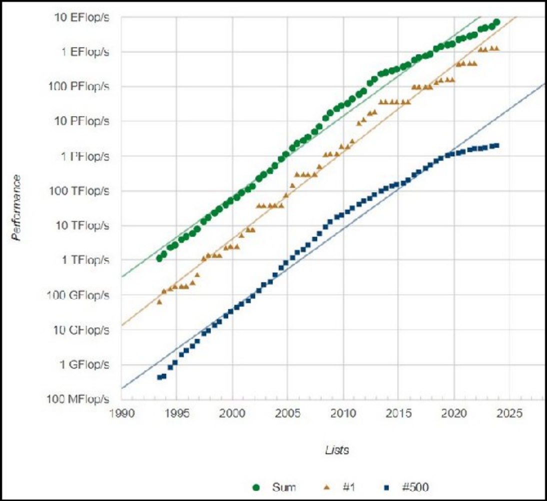
- Divergence of close trajectories $\rightarrow$ typical for chaotic systems.
- Initially nearby trajectories diverge as

$$x_p - x_r \propto e^{\lambda t},$$

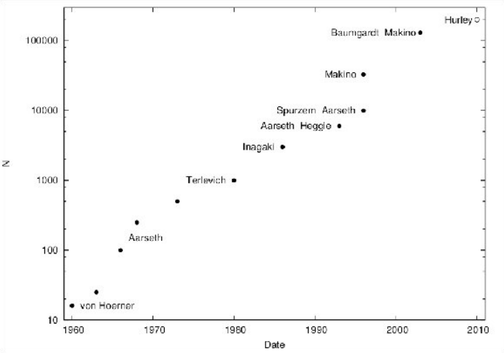
where λ is a positive number.

Performance of supercomputers



Speed of computers doubled every 18 months

The past and future of N-body calculations



- Speed of computers doubles every 18 months (Moore's law).
- In 10 years, it is possible to calculate only 10 times more massive clusters.
- Modelling of globular star clusters is currently possible.

A general note on performing numerical simulations

- Alongside observations and theory, numerical modelling is a powerful tool in science.
- Numerical algorithms are often very efficient on the task which they were developed.
- Numerical algorithms can become inefficient or provide wrong results when incorrectly understood and setup.
- $\rightarrow$ avoid using a code as a black box (for doing real science, experiments are OK); it is better to understand at least the basics of the numerical scheme, its strengths and weaknesses.

Thank you for your attention