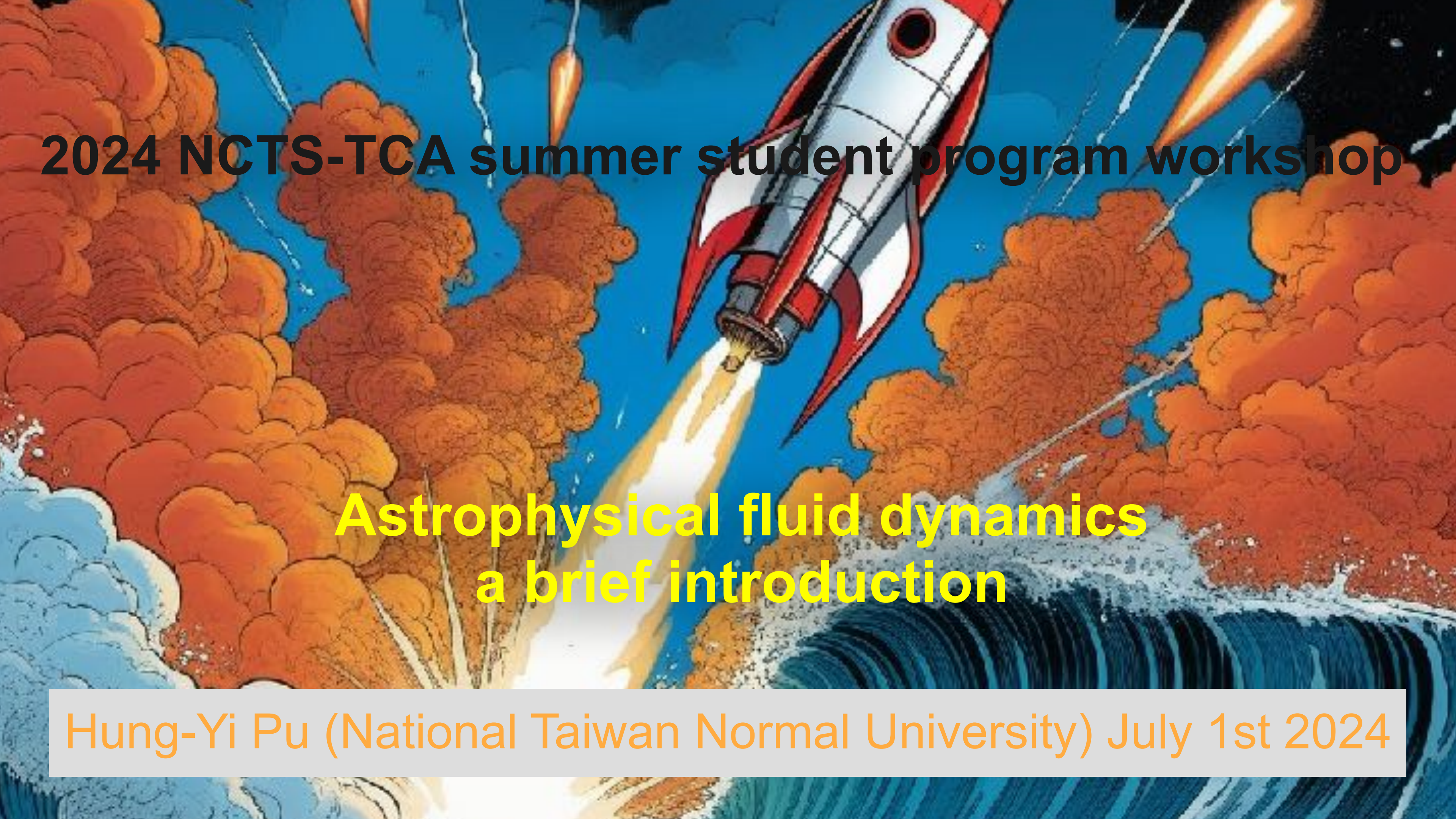
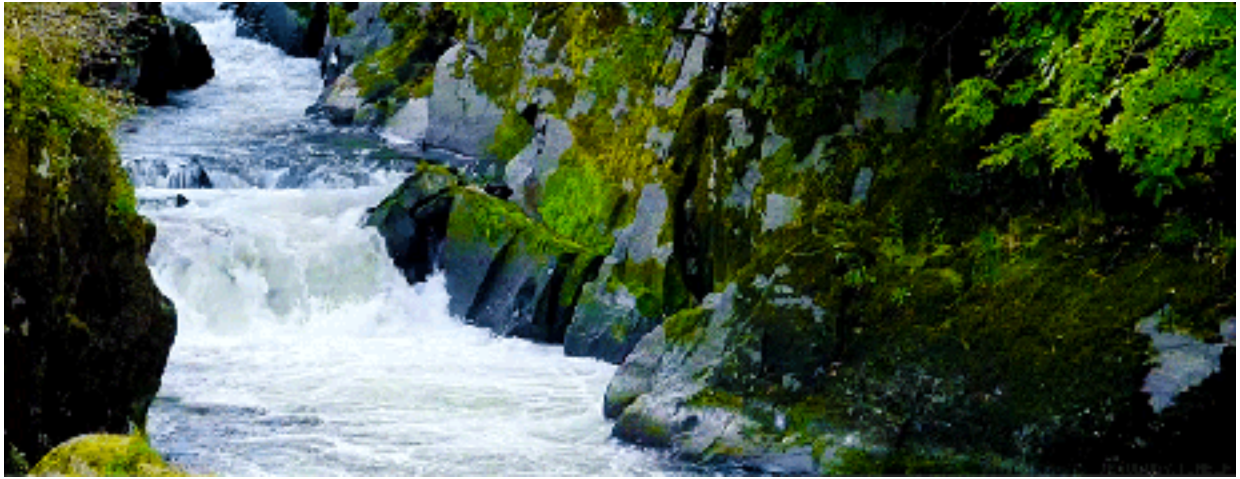
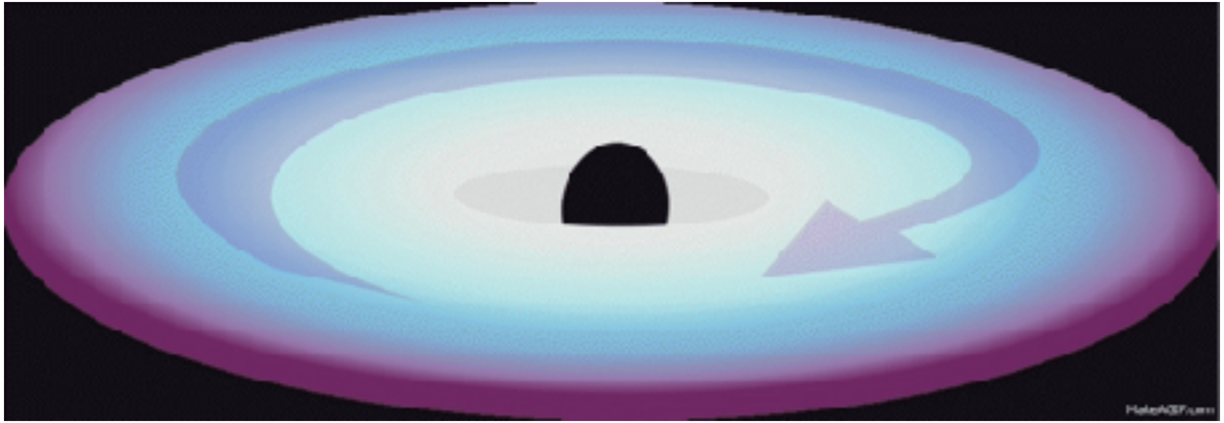


2024 NCTS-TCA summer student program workshop

Astrophysical fluid dynamics
a brief introduction

Hung-Yi Pu (National Taiwan Normal University) July 1st 2024





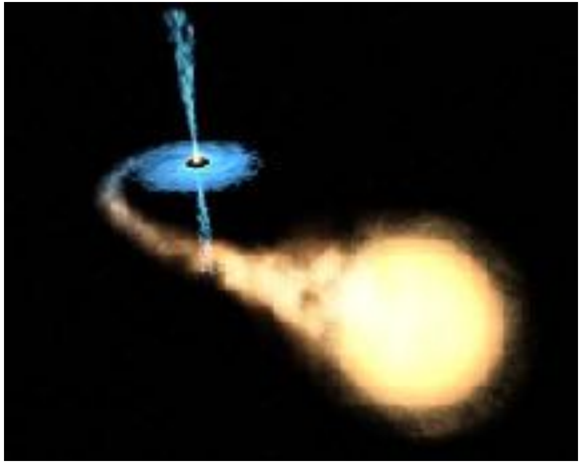
big pictures

- **liquid** and **gas** as fluid
  - as a collisional continuum
  - as a vector field
  - with thermodynamical properties
  - with viscosity
  - compressible, sound wave, and shock
- **plasma** as fluid
  - with conductivity
  - with EM interaction
- governing equations
  - conservation of mass, momentum, energy
  - continuity equation, F=ma, 1st law of thermodynamics

Astrophysical fluid dynamics?

Astrophysical fluid dynamics

- most of the baryonic matter (consists of three quarks) in the Universe can be treated as a fluid
- the liquid phase is less common
- **plasma**: magnetised fluid, interacting via EM interactions
- gravity is important (e.g. accretion)
- large scale

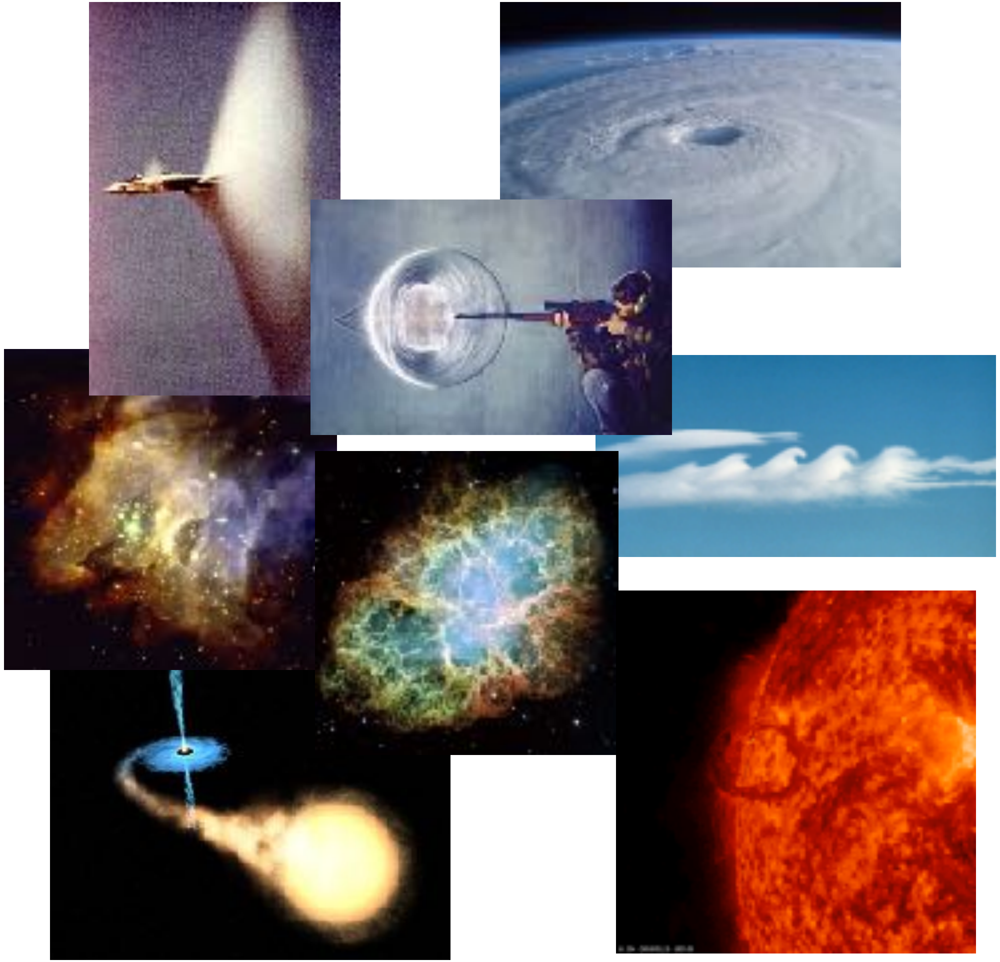


reference

- Astrophysical Flows by Pringle & King
- Principle of astrophysical fluid dynamics by Clarke & Carswell
- The physics of **plasmas** by Boyd & Sanderson
- The physics of fluids and **plasmas** by Choudhuri
- The physics of astrophysics volume II: **gas dynamics** by Shu
- Fluid Mechanics by Frank M. White

outline

- hydrodynamics (HD)
    - shear and viscosity
    - velocity field
    - governing equations
        - continuity
        - momentum
        - energy
    - turbulence and energy cascade
    - shock
- magnetohydrodynamics (MHD)
    - single fluid approach
    - plasma
    - ideal MHD
    - astrophysical applications



a macroscopic approach

“level”	neutral fluids		plasmas	
0	N quantum particles	Schrodinger’s eqn	N quantum particles	Schrodinger’s eqn
1	N classical particles	Newton’s law	N classical particles	Newton’s law
2	distribution function	Boltzmann eqn	distribution function	Vlassov eqn
				BBGKY hierarchy
3	continuum model	**HD** eqn	one-fluid model	**MHD** eqn

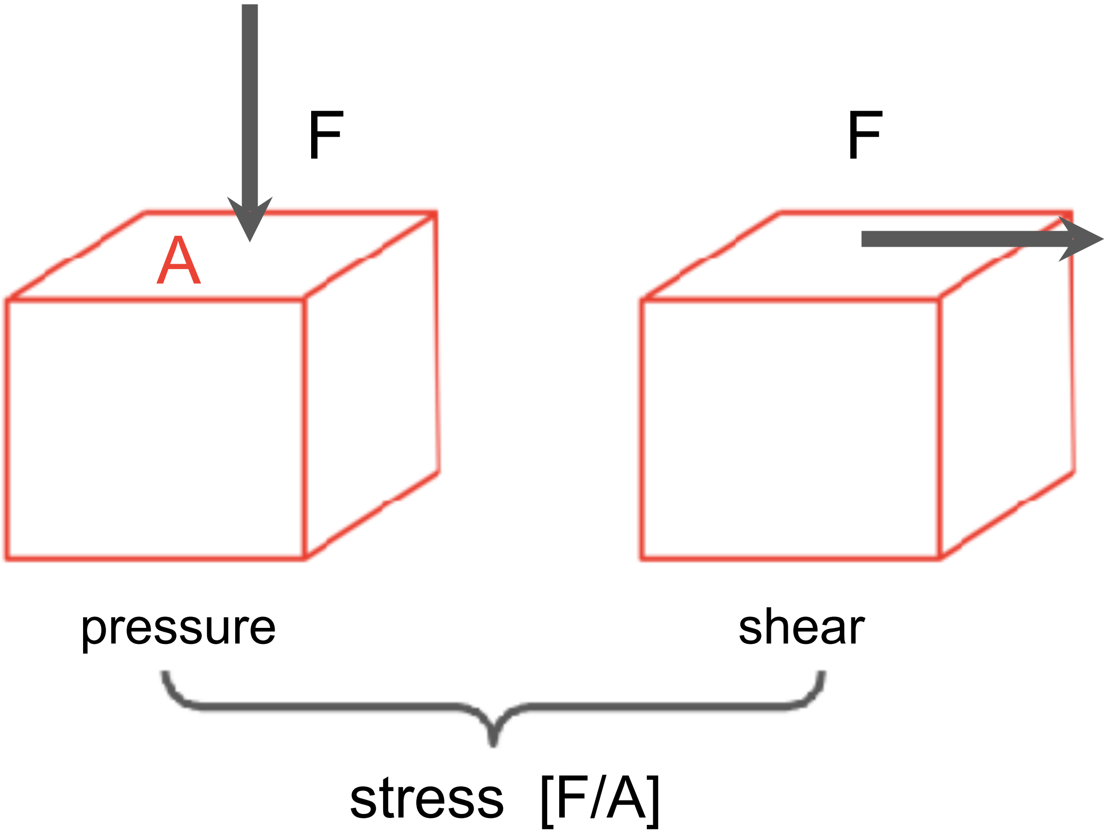
The physics of fluids and **plasmas** by Choudhuri today we can only cover some key concepts and conclusions hydrodynamics

stress and shear

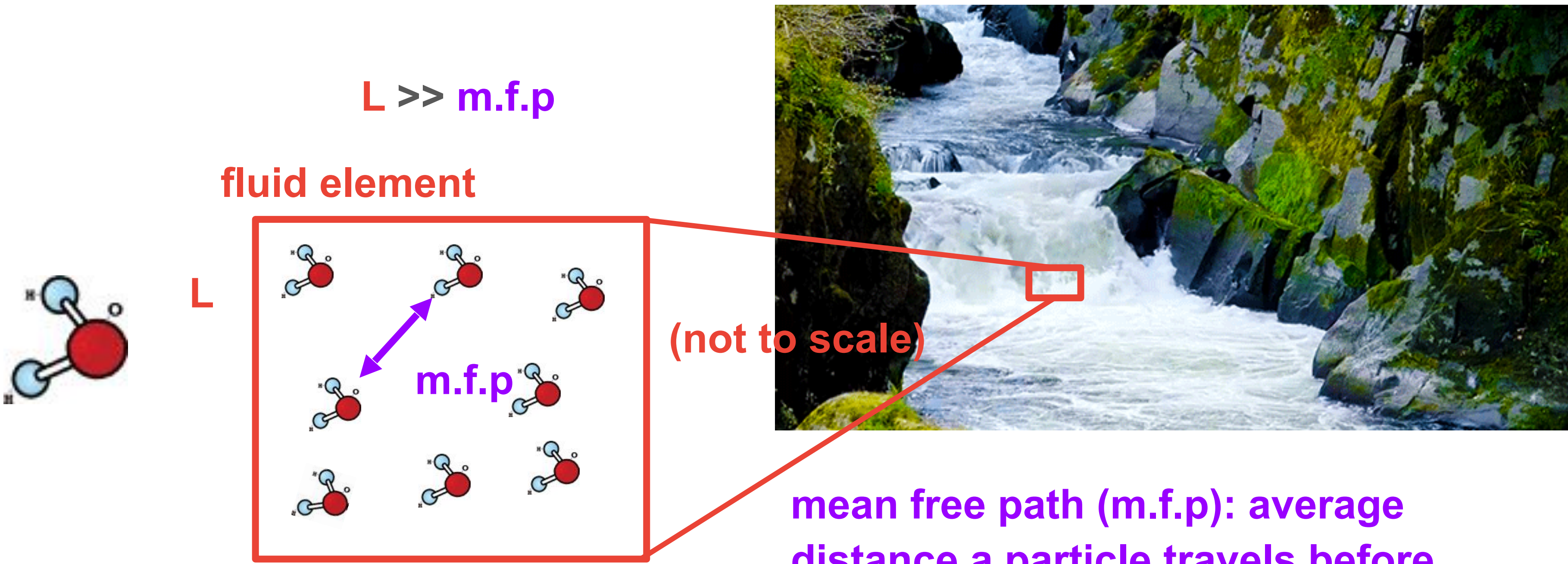
typical definition of fluid:

a fluid can move under the action of a **shear stress**, no matter how small that stress may be

- materials
  - solid
  - fluid
    - liquid
    - gas



fluid: a macroscopic approach



mean free path (m.f.p): average distance a particle travels before it collides with another particle

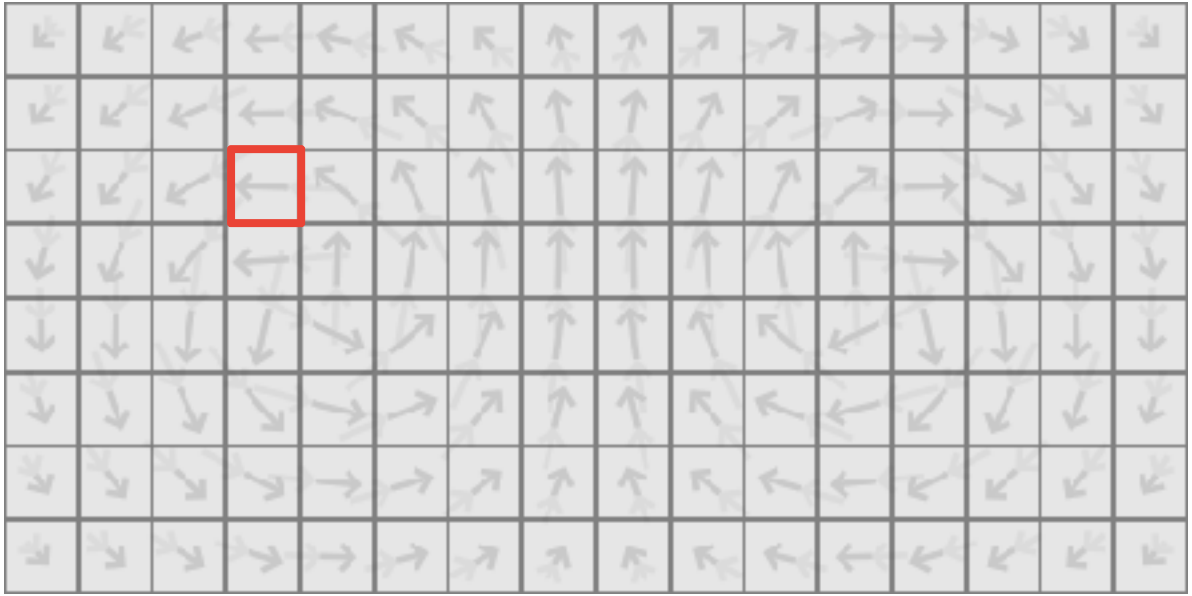
from water molecular to river

fluid dynamics as velocity field + fluid properties

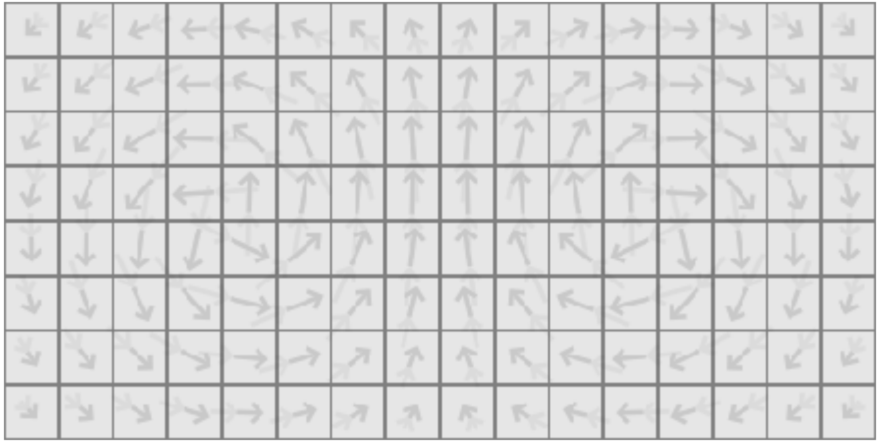
t1: velocity field

how?

t2: velocity field



velocity field



$\nabla \cdot \vec{V}$ relative change in volume per unit time if =0: incompressible

$\nabla \times \vec{V}$ **vorticity**: measure of local rotation if >0: counter-clockwise locally

common notation

- for 3D flow in cartesian coordinate

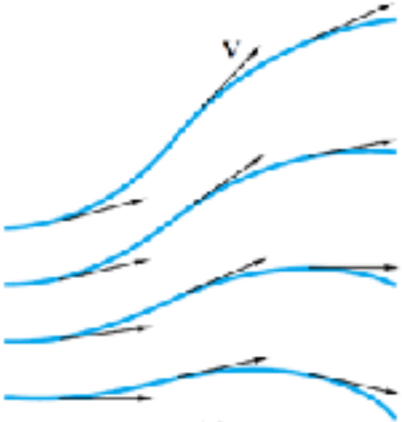
$$(V_x, V_y, V_z) = (u, v, w)$$

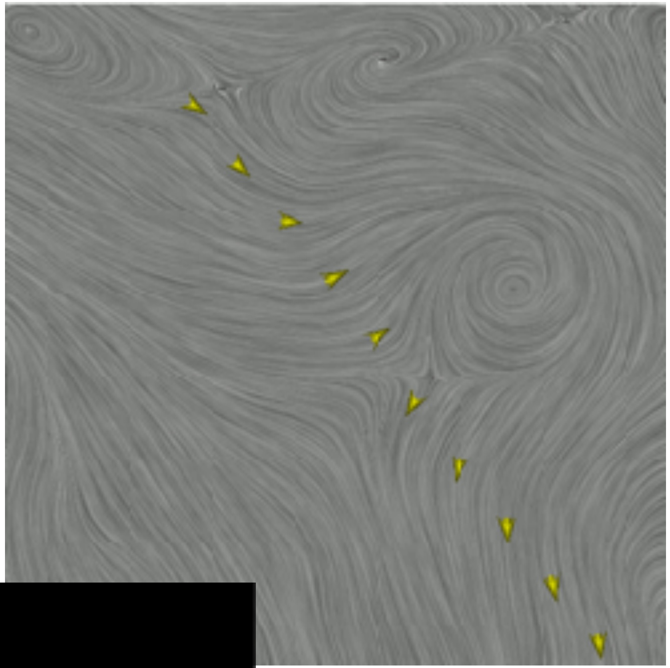
- for 2D flow

$$(V_x, V_y) = (u, v)$$

streamline (at the instant t):

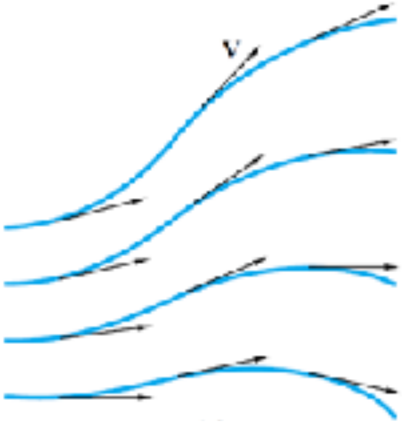
$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}}$$

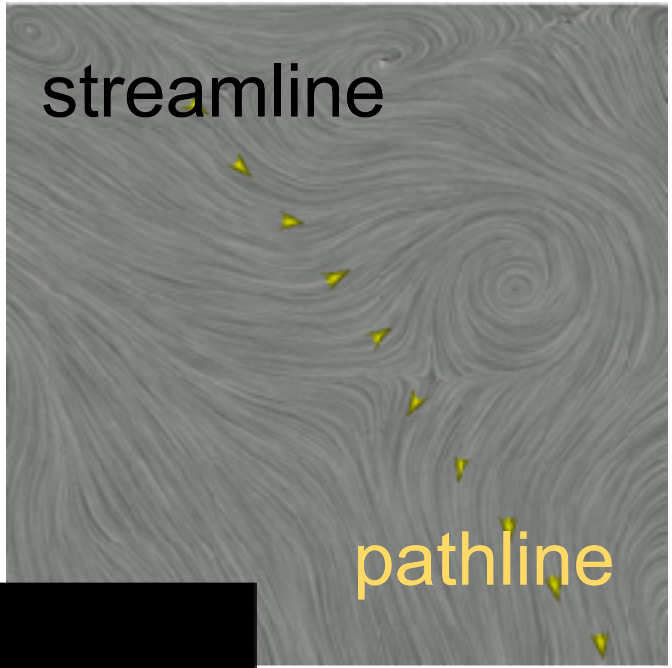


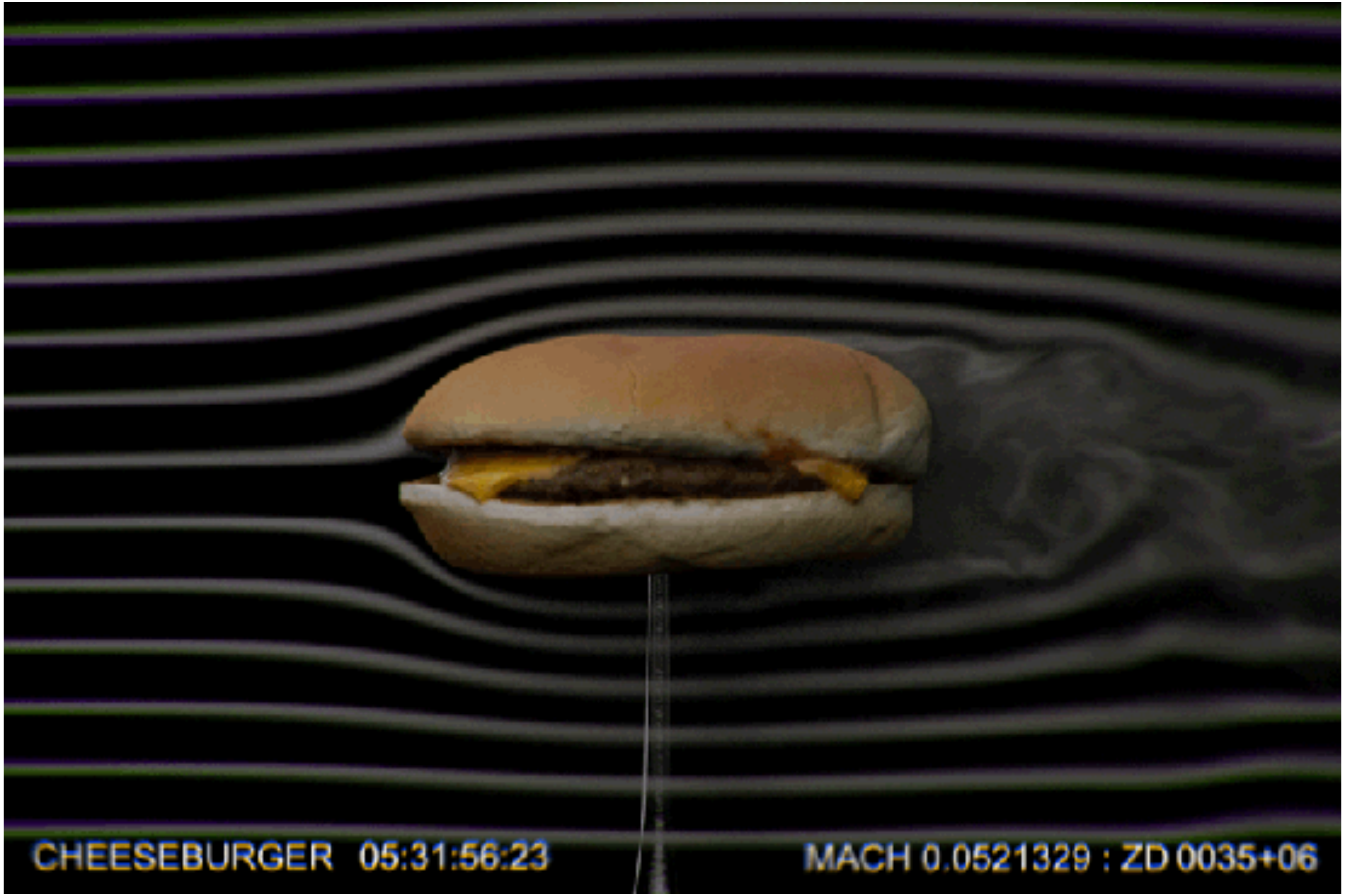


streamline (at the instant t):

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V}$$






CHEESEBURGER 05:31:56:23
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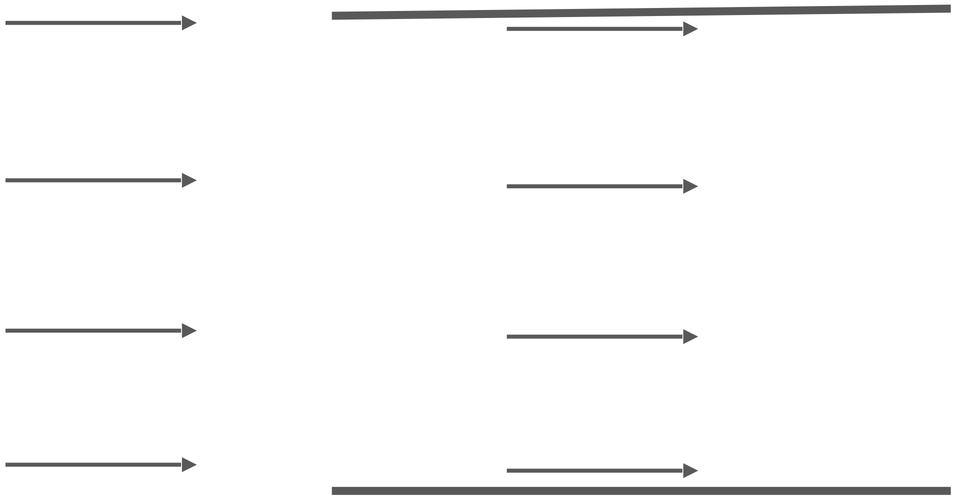
When steady $(\partial/\partial t = 0)$,
all three lines are the same

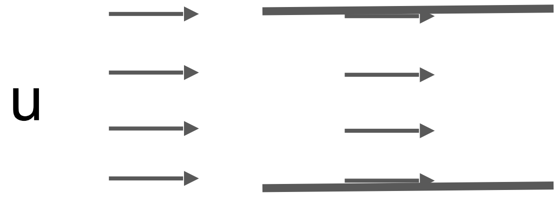
example: velocity field of a steady flow in a pipe

u



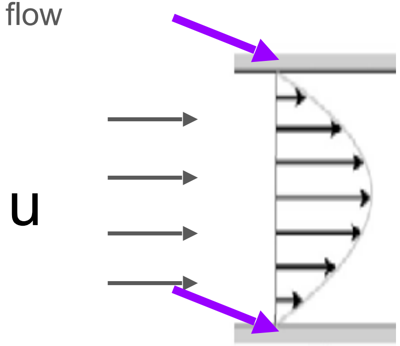
viscous and invicid (steady) flow

invicid flow



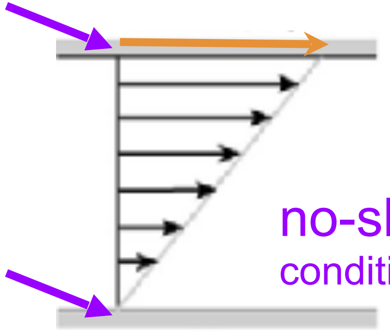
does not exist!

viscous flow



Poiseuille flow

moving plate



no-slip boundary
condition due to viscotiy

Couette flow

(shear) stress causes strain (via viscosity)

measure of the resistance of a fluid to gradual deformations by shear stress

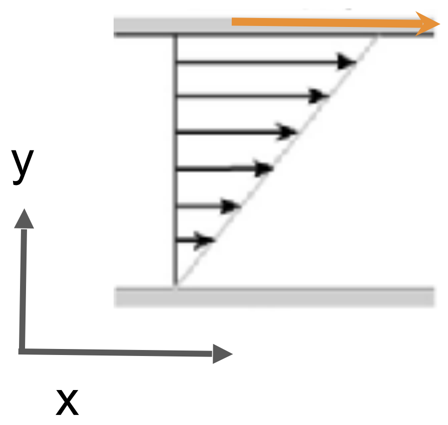
dynamic viscosity

$$\tau = \mu \frac{\partial u}{\partial y}$$

shear stress [F/A]

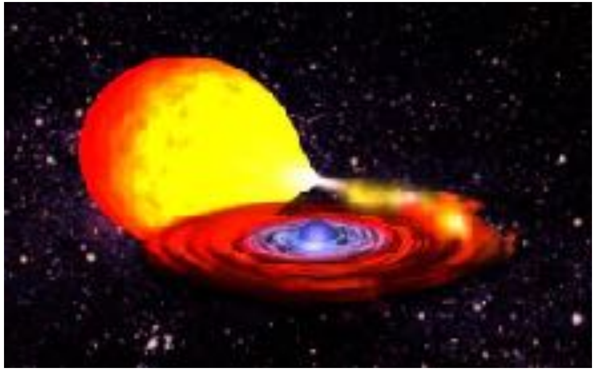
shear/strain rate [1/s]

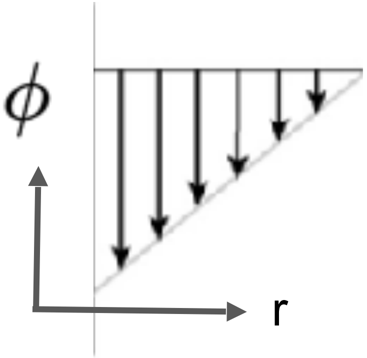
$\boxed{\text{stress = viscosity x strain}}$



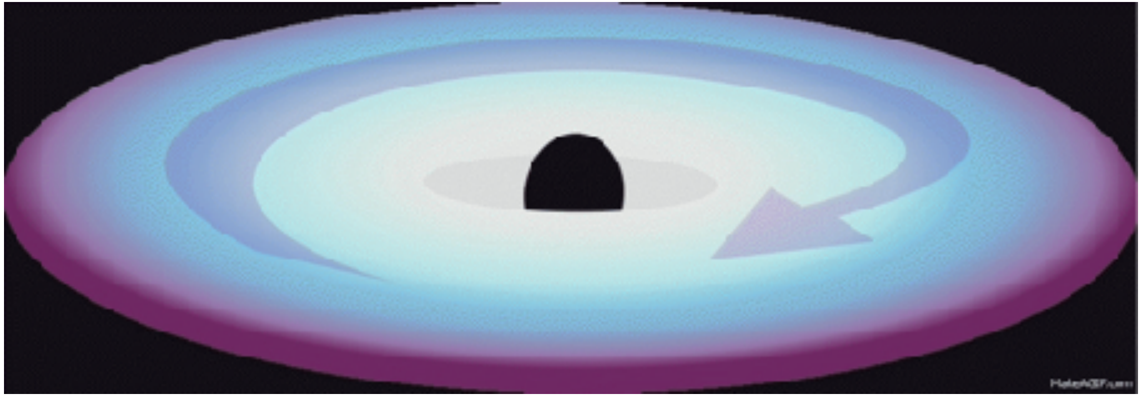
$$\nu = \frac{\mu}{\rho}$$

kinetic viscosity [VL]





$$\omega_{\rm Kepler} = (\frac{GM}{r^3})^{1/2}$$



$\alpha - \text{disk}:$

$$\boxed{\nu = \alpha H C_s}$$

kinetic viscosity [VL]

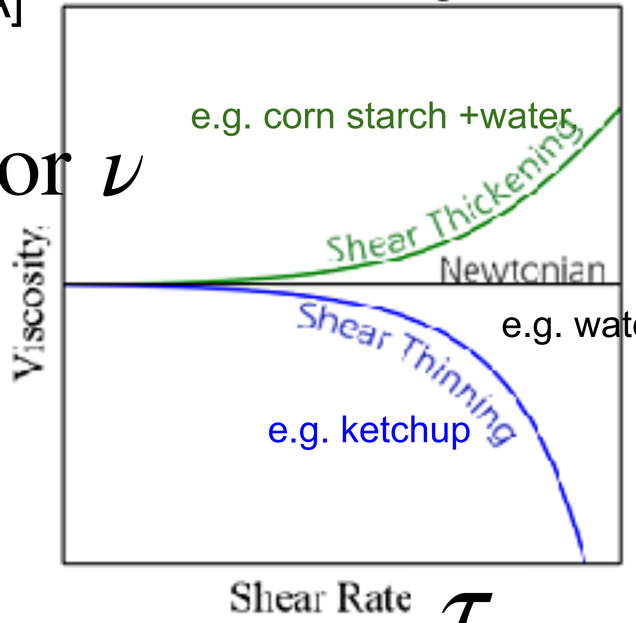
dynamic viscosity

shear/strain rate [1/s]

$$\tau = \mu \frac{\partial u}{\partial y}$$

shear stress [F/A]

μ or ν



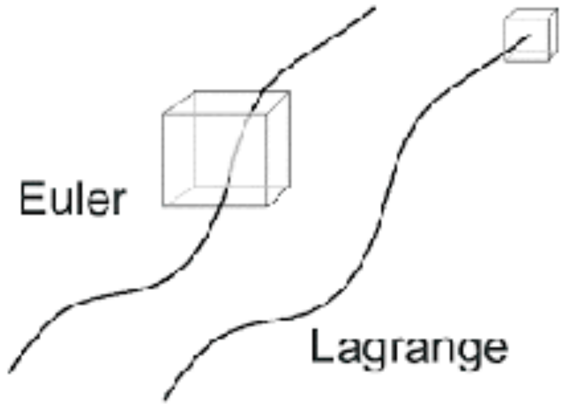
τ

movie credit: 國立台中教育大學 NTCU

shearing thickening!

Ketchup: shearing thinning!

a tale of two views (for everything!)



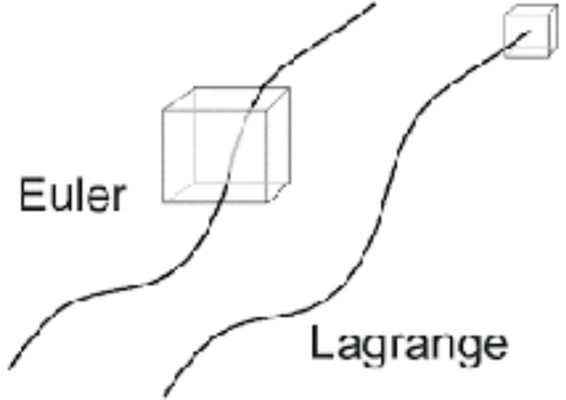
substantial/material derivative

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

LHS: Largragian point of view
(**ride on the particles**)

RHS: Eulerian point of view
(**stay at the fixed grid**)

a tale of two views



substantial/**material derivative**

$$\frac{D}{Dt} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \nabla)$$

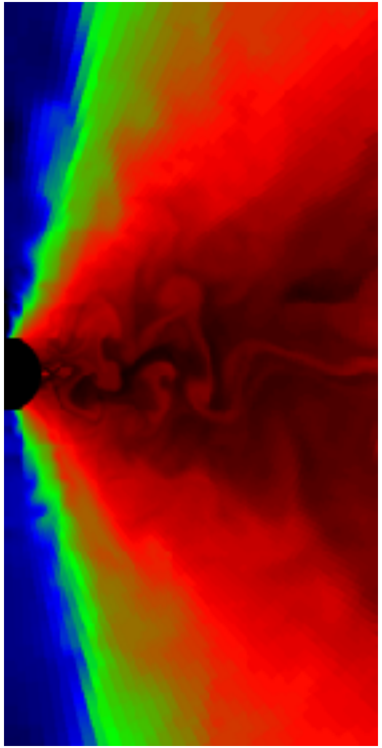
proof

$$d\rho(t,x,y,z) = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz$$

$$\frac{d}{dt}\rho = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt} + \frac{\partial}{\partial z}\frac{dz}{dt}\right)\rho$$

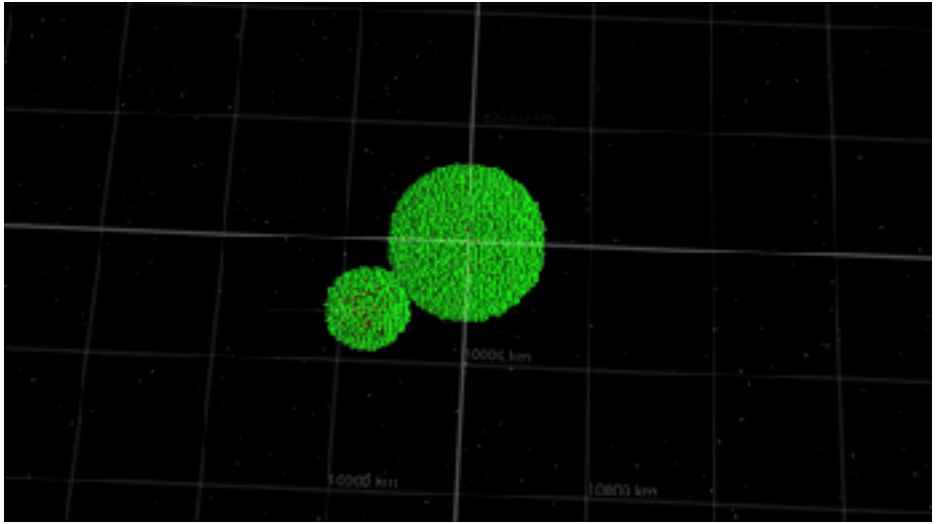

Eulerian
Lagrangian

grid-based simulation



movie credit: Stone

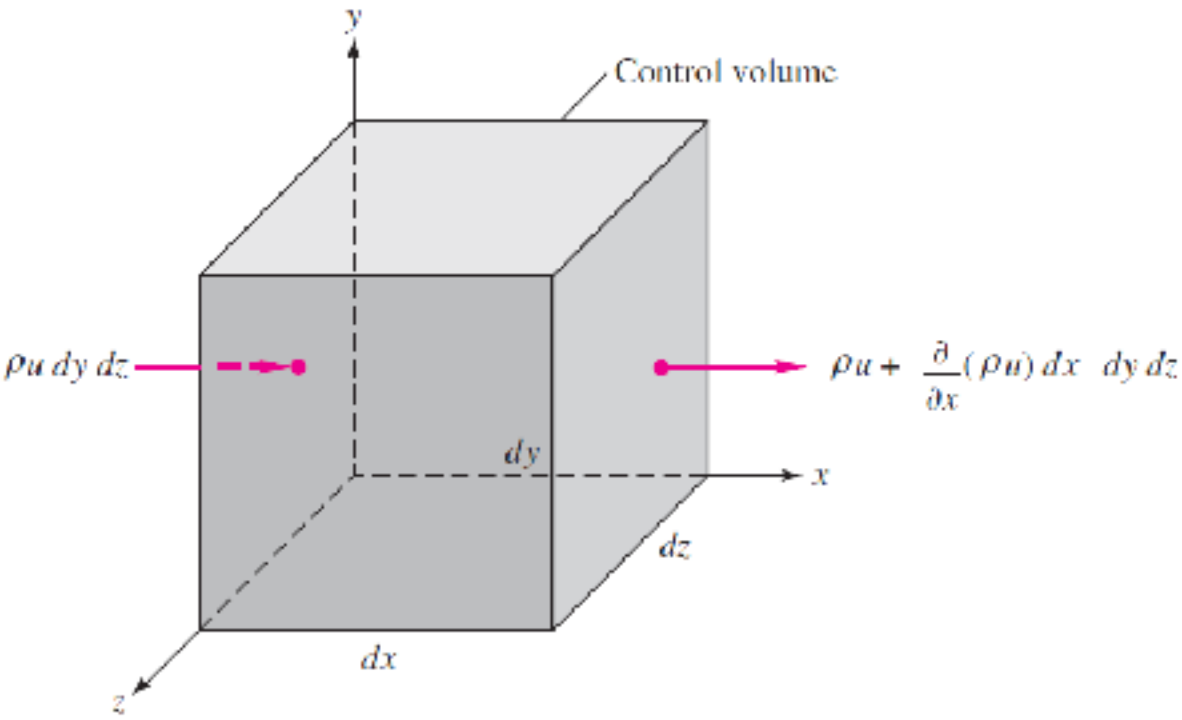
particle-based simulation



governing equation I:

continuity equation (mass conservation)

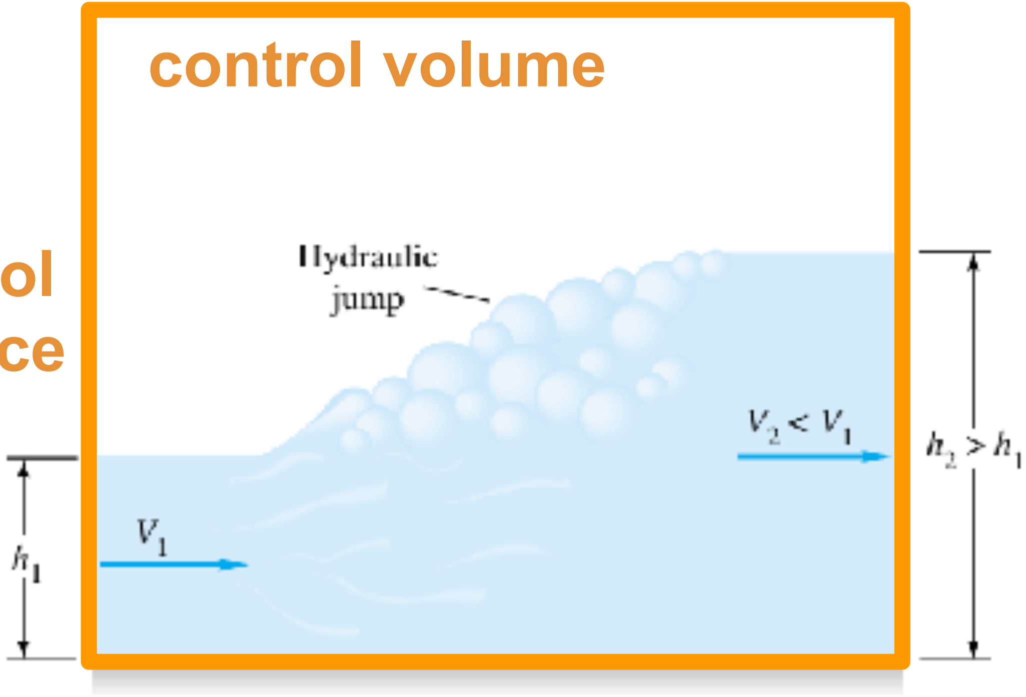
mass conservation



side note: integral form and differential form

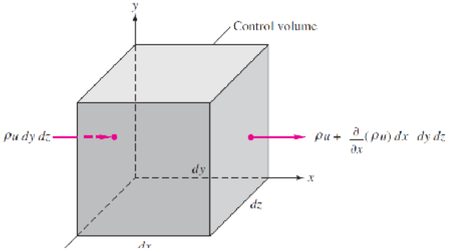
hydraulic jump (水躍)

control surface



control volume → 0 : differential form

mass conservation



Face	Inlet mass flw	Outlet mass flw
x	$\rho u \, dy \, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy \, dz$
y	$\rho v \, dx \, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx \, dz$
z	$\rho w \, dx \, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx \, dy$

conservation of mass (continuity equation): differential form

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0} \longleftrightarrow \boxed{\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0}$$

$$(\mathbf{v} \cdot \nabla)\rho + \rho(\nabla \cdot \mathbf{v})$$

if incompressible $(\frac{D\rho}{Dt} = 0)$:

$$\boxed{\nabla \cdot \mathbf{v} = 0}$$

$$\nabla \cdot \mathbf{v} = \frac{\frac{d(\delta V)}{dt}}{\delta V}$$ **relative change in volume per unit time**

not necessarily zero!

if:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$$

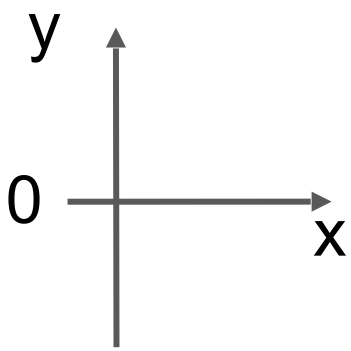
$$\frac{D\rho}{Dt} = 0 \longleftrightarrow \boxed{\nabla \cdot \mathbf{v} = 0}$$

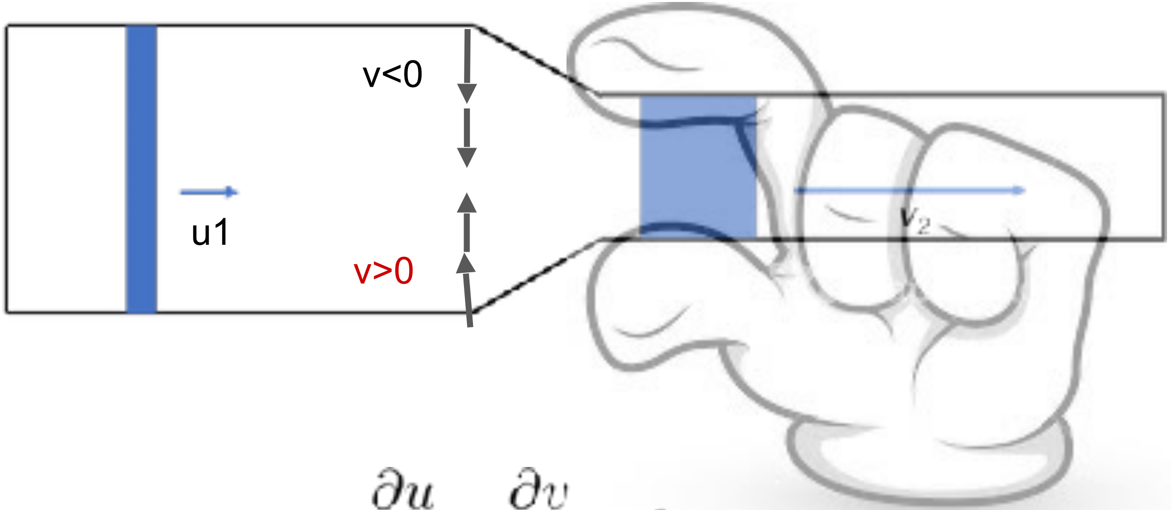
1D, steady, incompressible fluid is trivial!

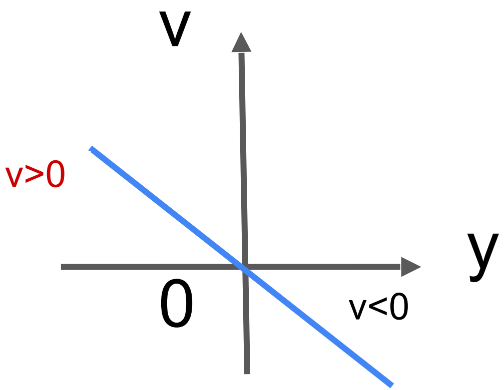
an example: 2D incompressible flow

what will you do?

an example: 2D incompressible flow







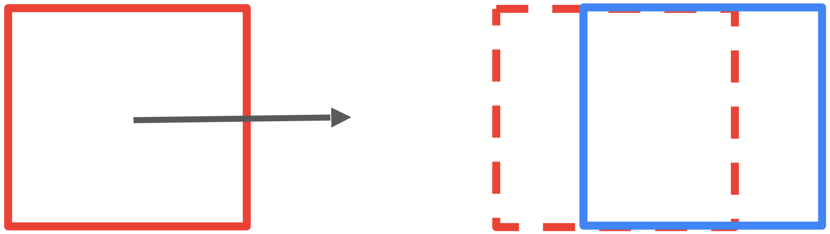
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

>0 $\quad$ <0

governing equation II:

momentum equation (Newton's 2nd law)

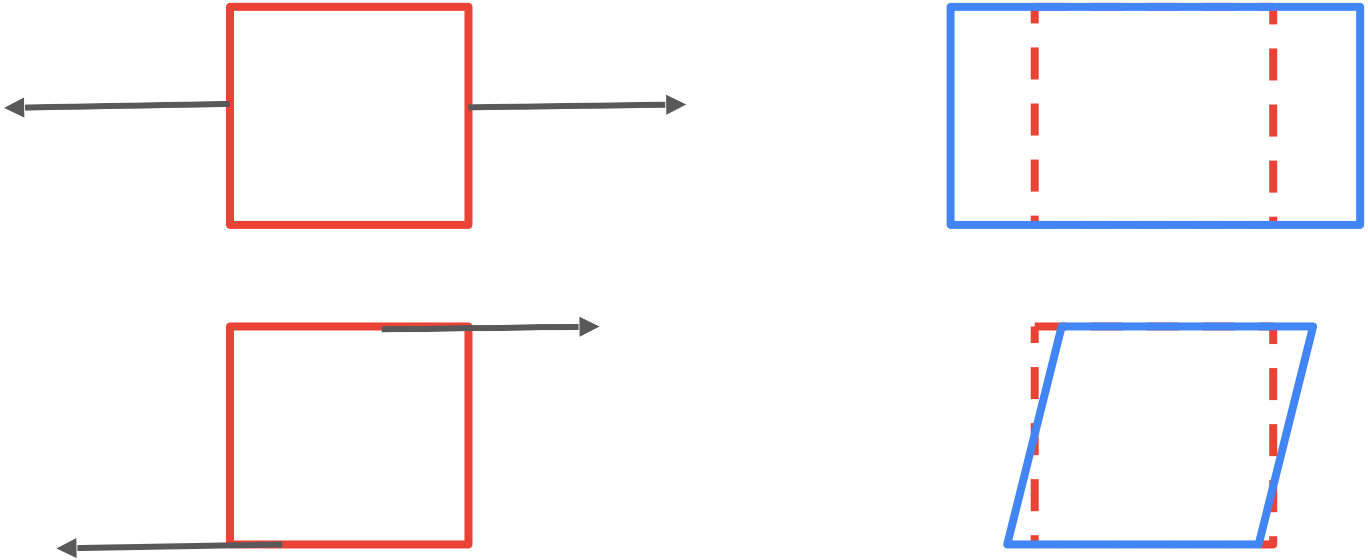
body force (acting on mass; does not require contact of the element)



stress as a “surface force”

surface force

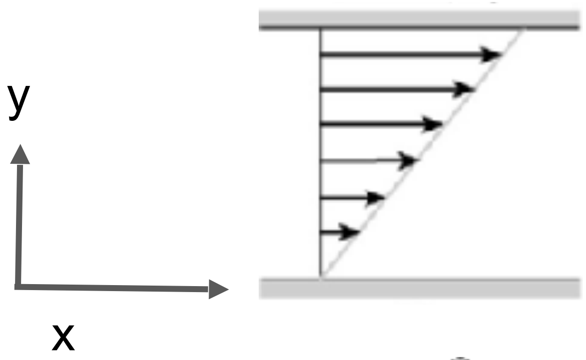
(acting on surface; requires contact of the element)

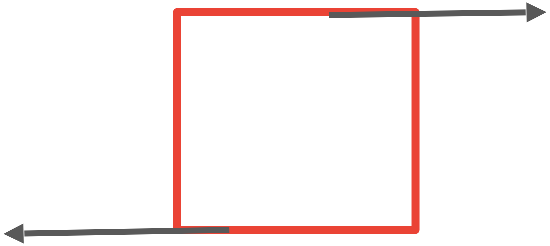


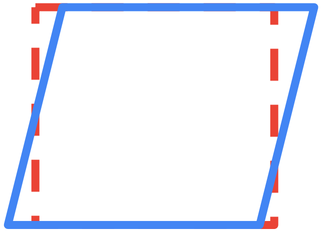
shear tensor τ_{ij}

surface

direction





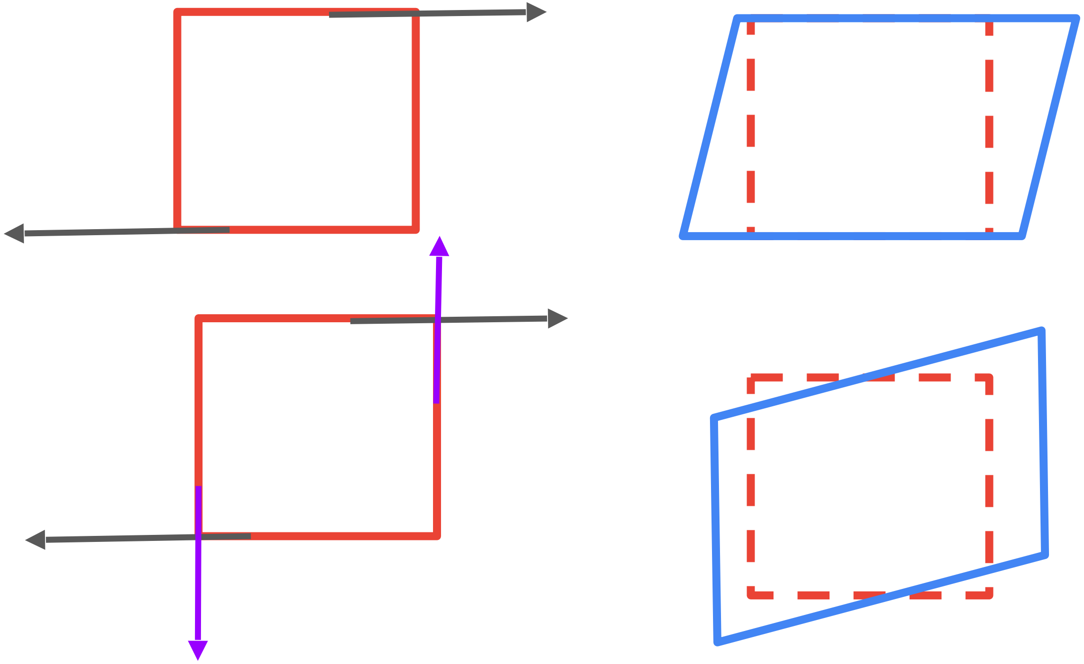


$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \quad ?$$

shear tensor τ_{ij}

surface

direction



NO! $\tau_{yx} = \mu \frac{\partial u}{\partial y}$?

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

(otherwise the element would rotate like crazy) shear tensor

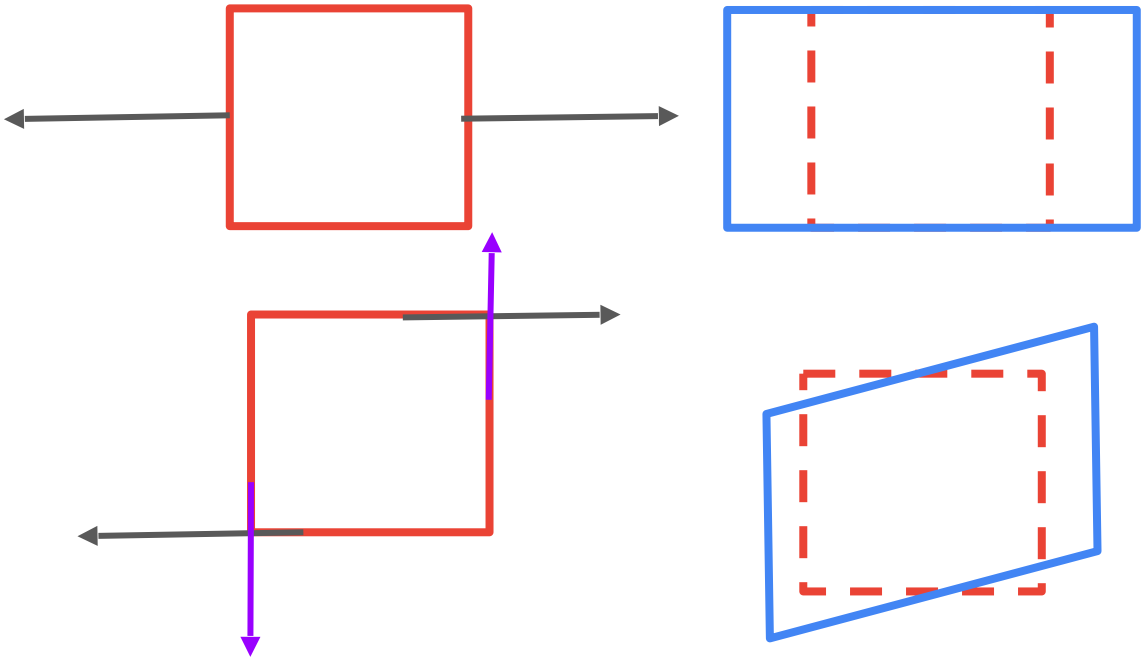
τ_{ij}

surface

direction

net florce per unit volumn

$$\nabla \cdot \bar{\bar{\tau}} = \Sigma(\nabla_i \tau_{ij})$$



$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla \cdot \vec{V})$$

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}$$

ma = F

body force

surface force

gravity

EM

pressure

viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

normal

shear

Navier-Stokes equations

ma = F

body force

surface force

gravity

EM

pressure

viscous

$$\begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

normal

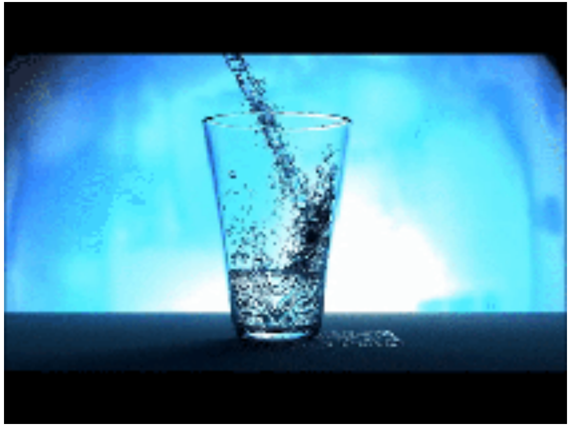
shear

viscous term

$$\rho \frac{d\vec{V}}{dt} = \rho\vec{g} - \nabla P + \nabla \cdot \bar{\bar{\tau}} = \rho\vec{g} - \nabla P + \mu\nabla^2\vec{V}$$

if constant viscosity, incompressible fluid

* divergence/laplacian of a vector



$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Navier-Stokes equations

$$\boxed{\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}}$$

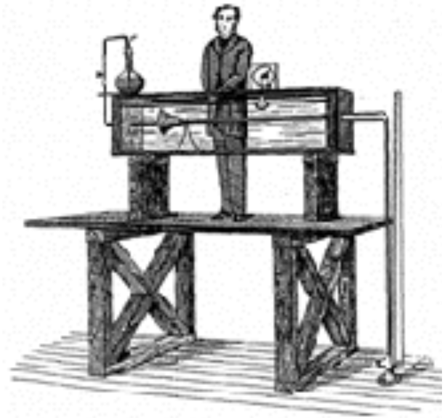
$$\mathrm{Re} = \frac{\text{inertia forces}}{\text{viscous forces}}$$

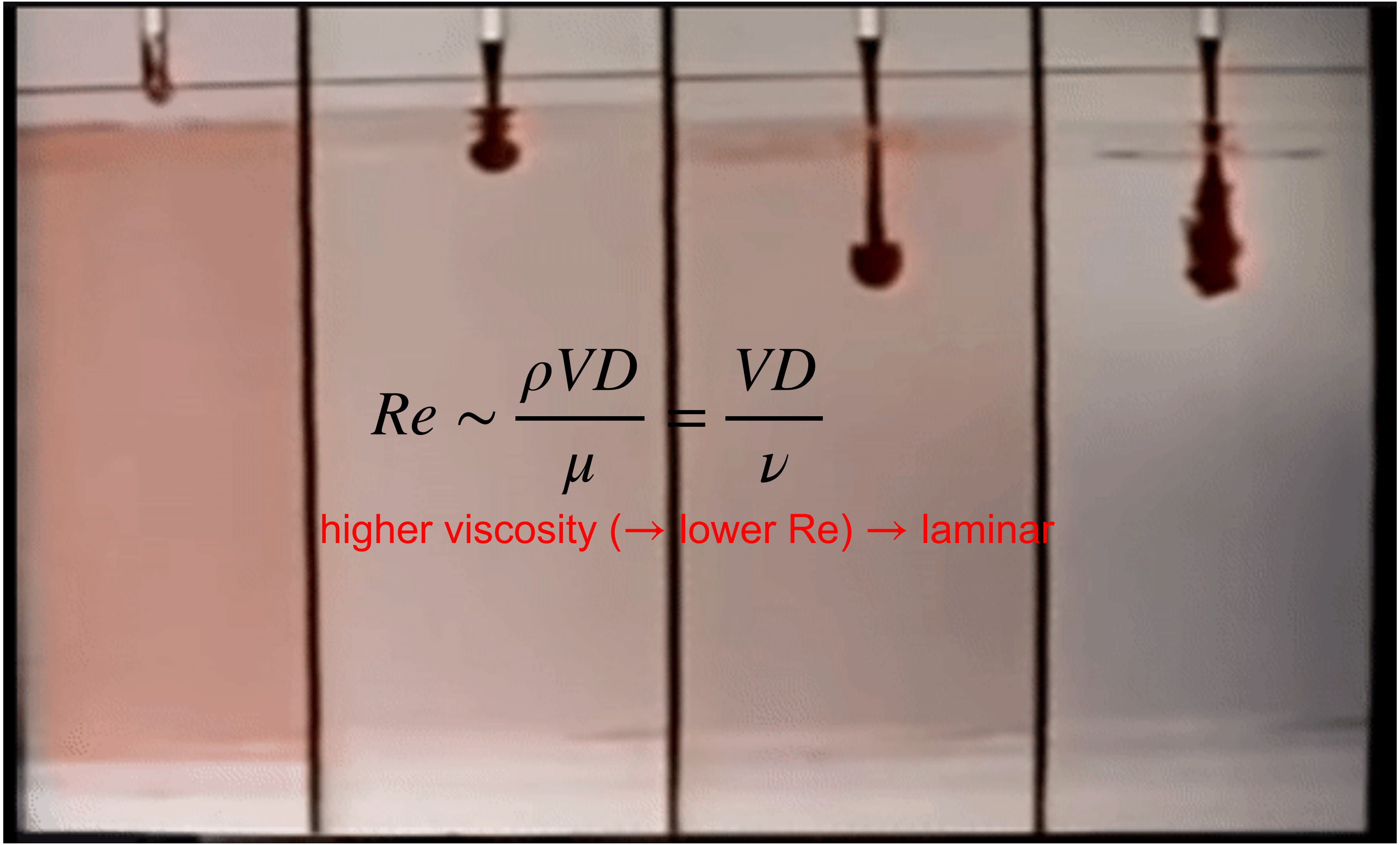
$$Re \sim \frac{\rho V D}{\mu} = \frac{VD}{\nu}$$

turbulence appears when Reynolds number is high enough!

$$Re \sim \frac{\rho V D}{\mu} = \frac{V D}{\nu}$$

Reynolds’ pipe experiment





$$Re \sim \frac{\rho V D}{\mu} = \frac{VD}{\nu}$$

higher viscosity (→ lower Re) → laminar

diffusion of “momentum”

Navier-Stokes equation

$$\boxed{(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu}{\rho}\nabla^2\vec{V}}$$

convection diffusion equation

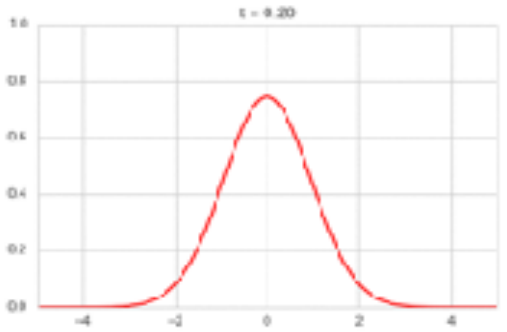
$$\boxed{\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}}$$

- linear
  - U= constant
- non-linear
  - U=f(x,t): Burger’s equation

the good guy: diffusion/conduction

$$\boxed{\frac{\partial f}{\partial t} + \cancel{U\frac{\partial f}{\partial x}} = D\frac{\partial^2 f}{\partial x^2}}$$

D= constant

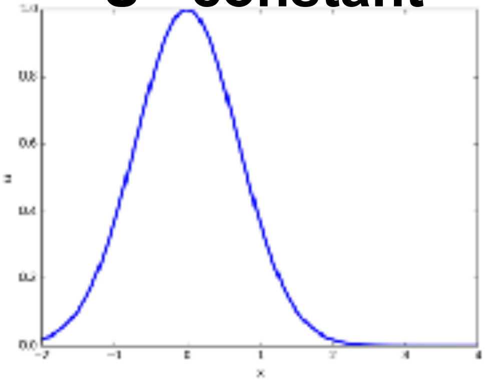


the “bad” guy who cause turbulence:

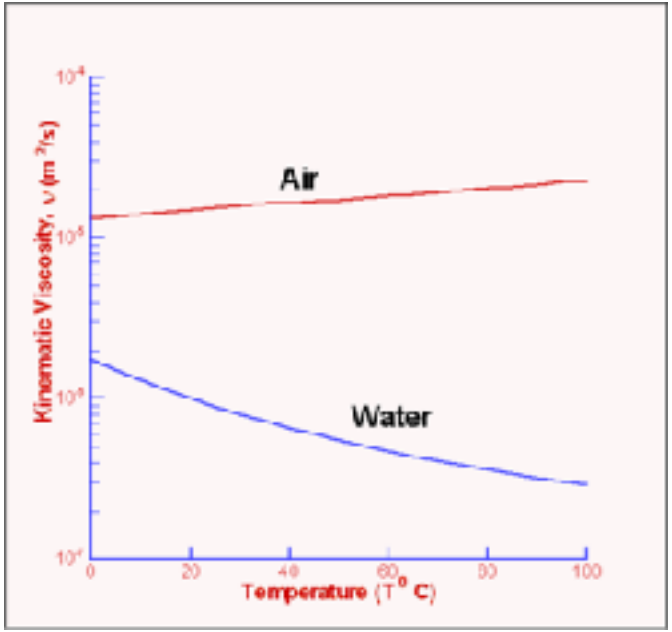
convection/advection

$$\boxed{\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \cancel{D\frac{\partial^2 f}{\partial x^2}}}$$

U= constant



the ubiquitous nature of turbulence in our (human) daily lives



acceleration $\Rightarrow$ larger Re $\Rightarrow$ Turbulent

$$Re \sim \frac{\rho V D}{\mu} = \frac{VD}{\nu}$$

hydrostatic: Euler equation

$$\left(\cancel{\frac{\partial}{\partial t}} + \cancel{\vec{V}} \cdot \nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2}\vec{V}$$

v=0, stationary, invicid

a fish tank with special designs

Bernoulli equation for incompressible fluid*

$$\left(\cancel{\frac{\partial}{\partial t}} + \vec{V}\cdot\nabla\right)\vec{V} = \vec{g} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2\vec{V}}$$

stationary, invicid

integration assumption $d\rho = 0$

(incompressible)

$$\boxed{\mathcal{B} = \frac{|\vec{V}|^2}{2} + \frac{P}{\rho} + gz}$$

(along a streamline)

$$\mathcal{B}(x1, y1, z1) = \mathcal{B}(x2, y2, z2)$$

*there is Bernoulli equation for compressible fluid too! (by taking into account the change of internal energy) governing equation III:

energy equation (1st law of thermodynamics) conservation of energy (1st law of thermodynamics)

viscous work

internal energy

$$\frac{D\hat{u}}{Dt} = \frac{dQ}{dt} - \frac{dW}{dt}$$

pressure work

recall:

$$\boxed{\frac{D\rho}{Dt} + \rho\,(\nabla \cdot \mathbf{v}) = 0}$$

$$p\frac{dV}{dt} = p\frac{d(\frac{1}{\rho})}{dt} = -\frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\rho\frac{d\hat{u}}{dt} + p(\nabla \cdot \vec{v}) = -\dot{Q}_{cool} + \Phi$$

Viscous dissipation function

Viscous dissipation function

$$\Phi = \mu[2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial w}{\partial z})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2] > 0$$

→ always increase the internal energy (irreversible)

example: the importance of energy equation

$$\dot{Q}^{+} \approx \dot{Q}^{-} (\gg \dot{Q}^{adv})$$

(radiative efficient)

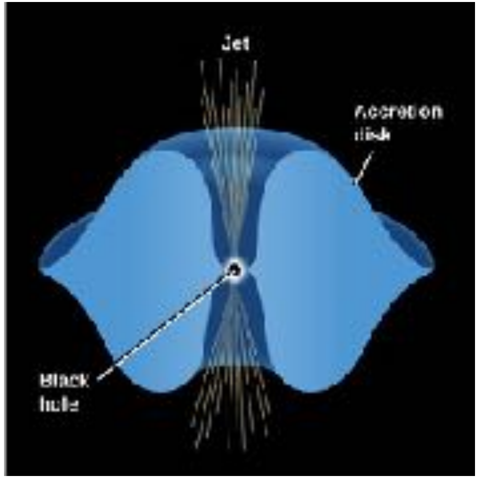
$$TdS = dQ$$

$$\rho T \frac{ds}{dt} = \dot{Q}^{+} - \dot{Q}^{-}$$

$$\boxed{\rho v_r T \frac{ds}{dr} \equiv \dot{Q}^{adv} = \dot{Q}^{+} - \dot{Q}^{-}}$$

(radiative inefficient)

$$\dot{Q}^{+} \approx \dot{Q}^{adv} (\gg \dot{Q}^{-})$$



conservative form

$$\boxed{\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=J}$$

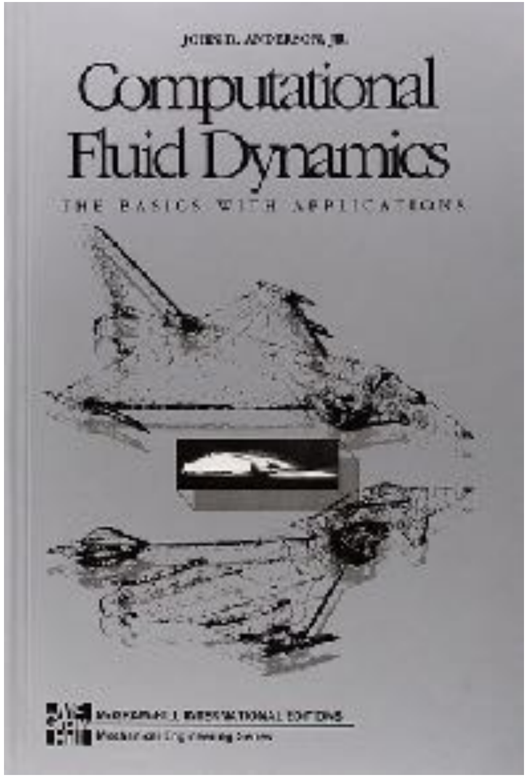
$$U=\begin{Bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho\left(e+\frac{V^2}{2}\right)\end{Bmatrix}$$

$$F=\begin{Bmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho vu-\tau_{xy}\\ \rho wu-\tau_{xz}\\ \rho\left(e+\frac{V^2}{2}\right)u+pu-k\frac{\partial T}{\partial x}-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}\end{Bmatrix}$$

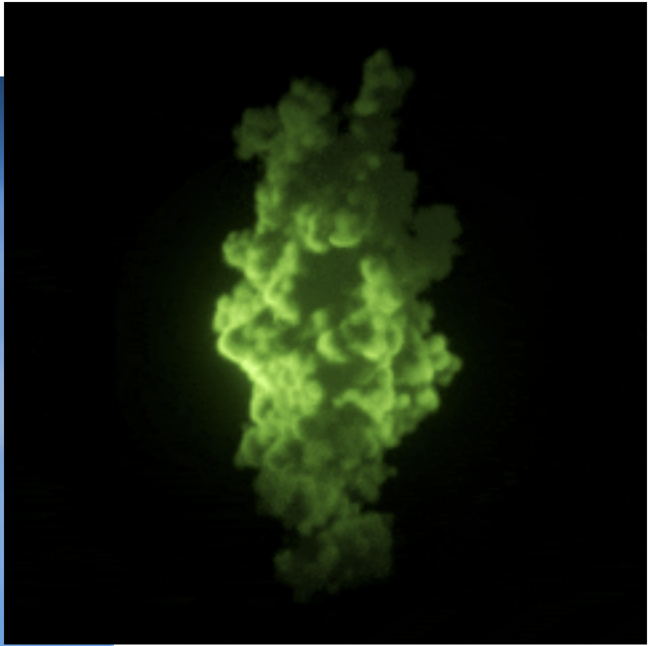
$$G=\begin{Bmatrix}\rho v\\ \rho uv-\tau_{yx}\\ \rho v^2+p-\tau_{yy}\\ \rho wv-\tau_{yz}\\ \rho\left(e+\frac{V^2}{2}\right)v+pv-k\frac{\partial T}{\partial y}-u\tau_{yx}-v\tau_{yy}-w\tau_{yz}\end{Bmatrix}$$

$$H=\begin{Bmatrix}\rho w\\ \rho uw-\tau_{zx}\\ \rho vw-\tau_{zy}\\ \rho w^2+p-\tau_{zz}\\ \rho\left(e+\frac{V^2}{2}\right)w+pw-k\frac{\partial T}{\partial z}-u\tau_{zx}-v\tau_{zy}-w\tau_{zz}\end{Bmatrix}$$

$$J=\begin{Bmatrix}0\\ \rho f_x\\ \rho f_y\\ \rho f_z\\ \rho(uf_x+vf_y+wf_z)+\rho\dot{q}\end{Bmatrix}$$

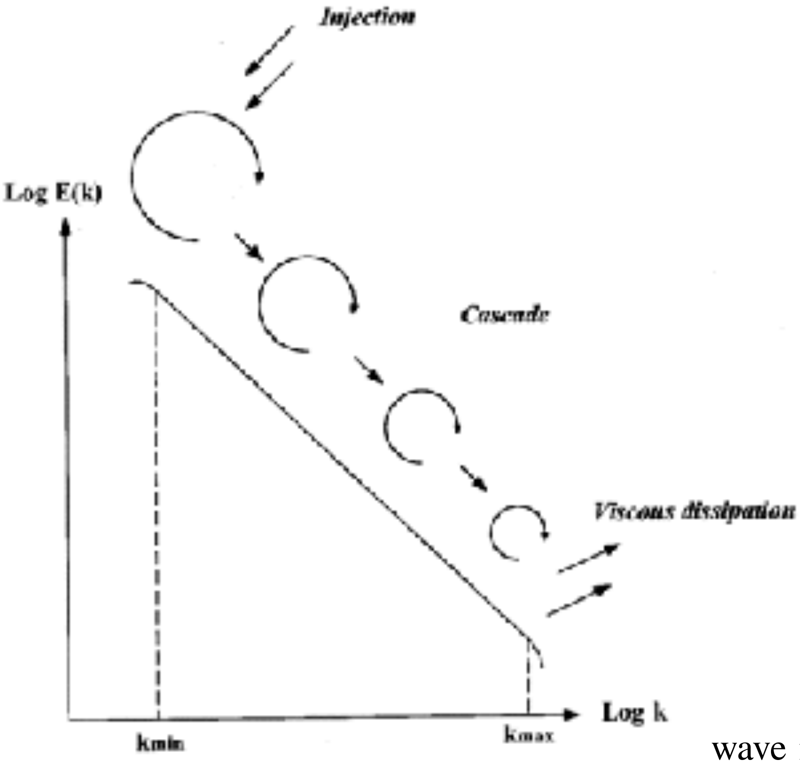


understanding turbulent flows….



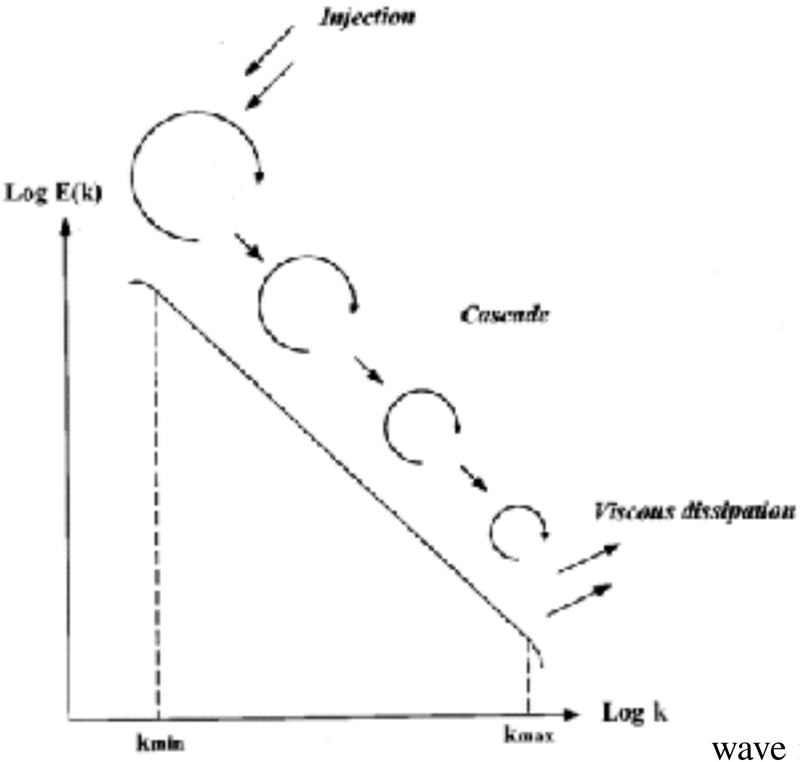
self-similar

key word: eddies, energy cascade



wave number k $= 2\pi/\lambda$

Komogrolov’s law for homogenous and isotropic turbulence



roughly no loss of energy

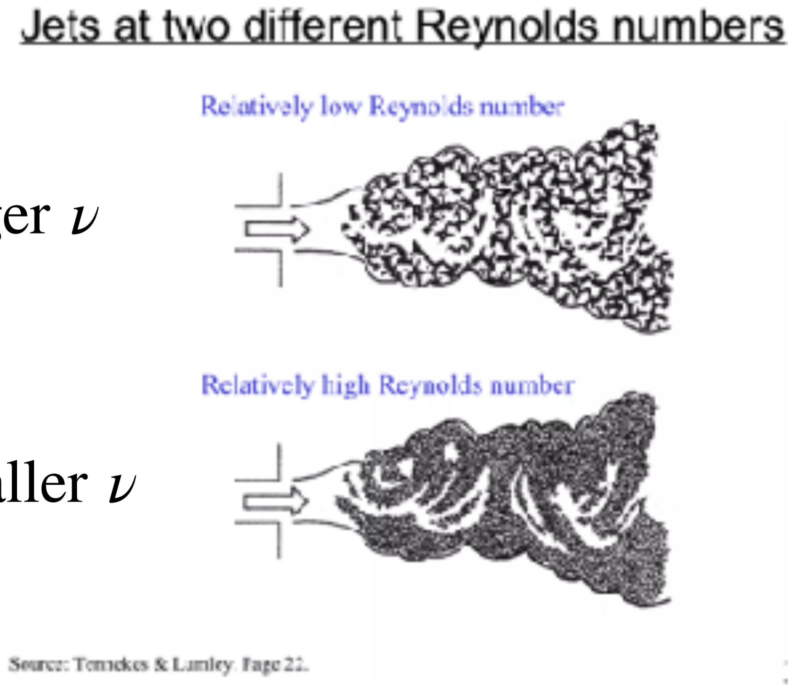
$$E(k) \propto k^{-5/3}$$

energy finally lost due to viscous dissipation

$$\Longrightarrow \lambda_0 \sim R_e^{-3/4} L$$

wave number $k = 2\pi/\lambda$

Komogrolov’s law for homogenous and isotropic turbulence



larger ν

roughly no loss of energy

$$E(k) \propto k^{-5/3}$$

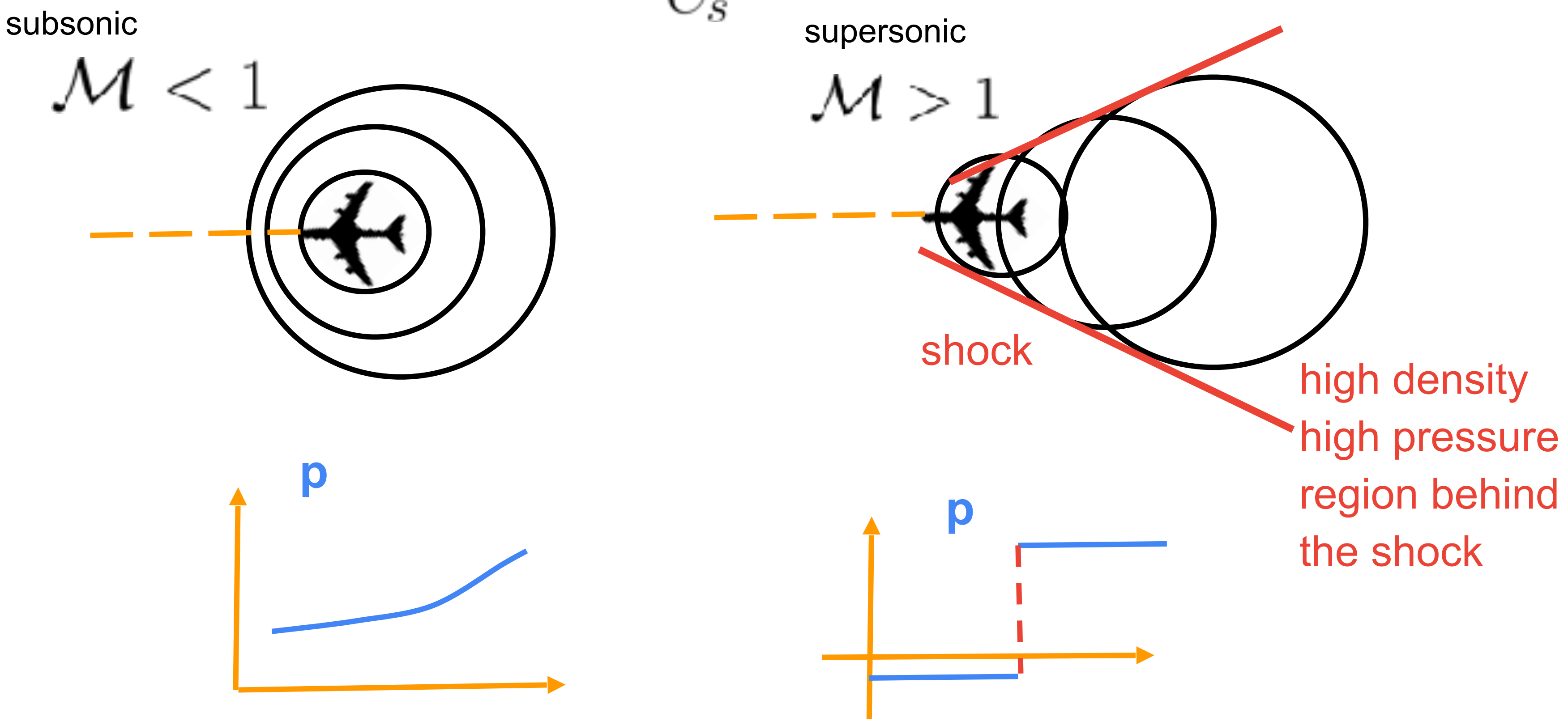
smaller ν

energy finally lost due to viscous dissipation

$$\Longrightarrow \lambda_0 \sim R_e^{-3/4} L$$

towards to compressible gas

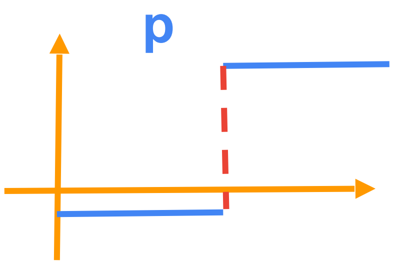
Mach number: $\mathcal{M} \equiv \dfrac{V}{C_s}$

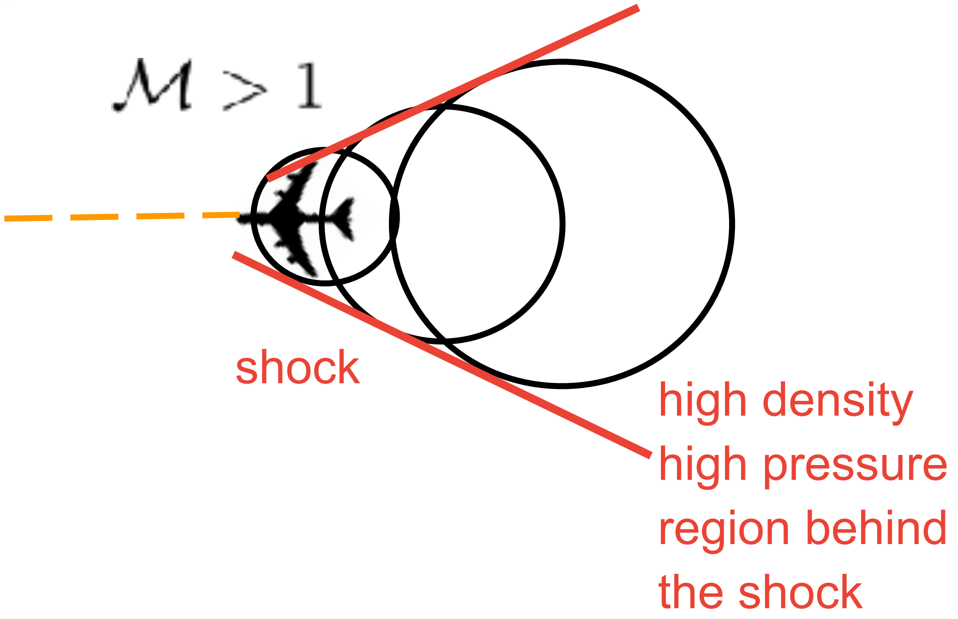


(at the flight-nose frame)

Mach number: $\mathcal{M} \equiv \dfrac{V}{C_s}$

(at the flight-nose frame)





upstream

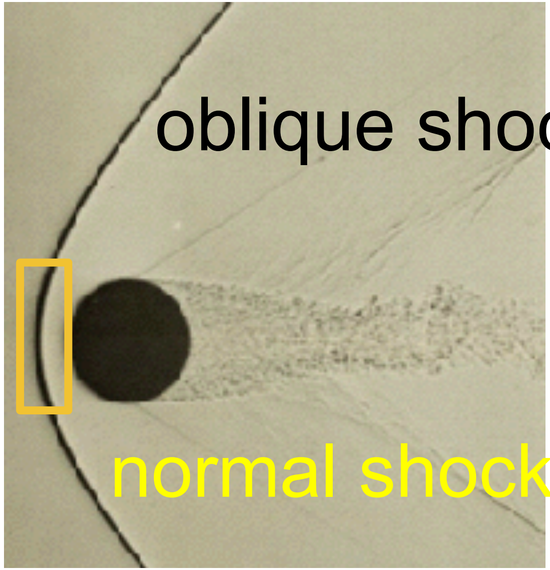
downstream
(post-shock)

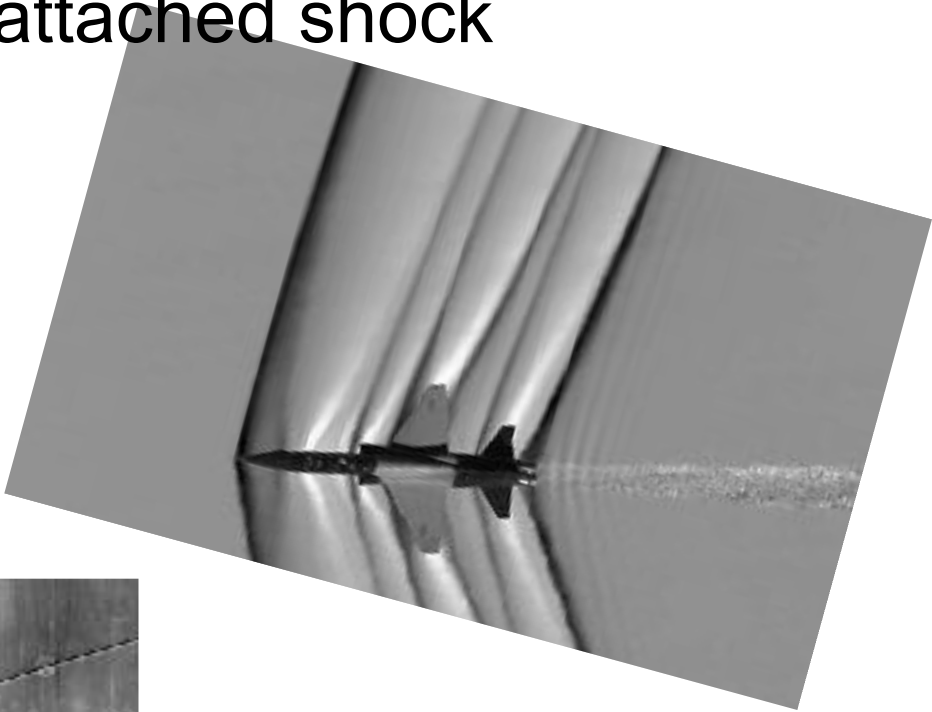
region 1

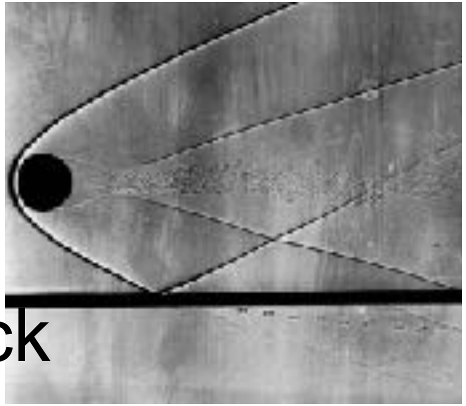
region 2

detached shock

attached shock





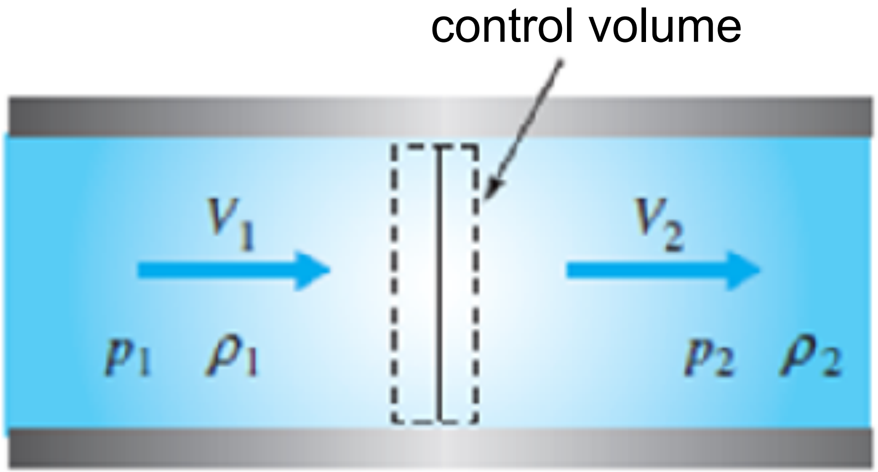


reflection of shock

avoiding bow shock (and therefore drag)

normal shock: must be a strong shock (M_2<1)

upstream



downstream (the after-shock region must be subsonic)

equation of state (EOS)

barotropic $\boldsymbol{p(\rho)}$

isothermal $\boldsymbol{p \propto \rho}$ $C_s^2 = \frac{dp}{d\rho} = \frac{p}{\rho}$

adiabatic $\boldsymbol{p \propto \rho^{\gamma}}$ $C_s^2 = \frac{dp}{d\rho} = \gamma\frac{p}{\rho}$

*for air, γ =1.4

$\mathcal{M}_1 > 1$ $\mathcal{M}_2 < 1$

$\rho_1 < \rho_2$

$p_1 < p_2$

$u_1 > u_2$

shock

(adiabatic) **normal shock** and Rankine-Hugoniot relations
*governing equations are written down in integral forms

RH relation at the shock frame $\left(\frac{\partial}{\partial t} = 0\right)$

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\ \frac{1}{2}u_1^2 + \mathcal{E}_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \mathcal{E}_2 + \frac{p_2}{\rho_2} \end{cases}$$

a useful relation for the internal energy of adiabatic flow:

$$\boxed{\mathcal{E} = \frac{1}{\gamma - 1}\frac{p}{\rho}}$$

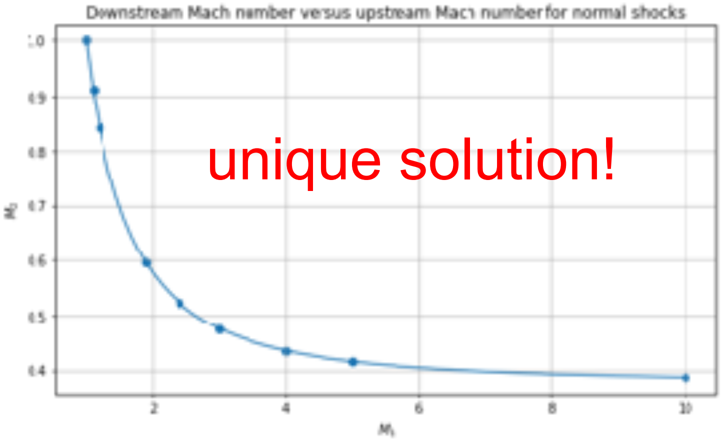
if $\mathcal{M}_1 \to \infty$

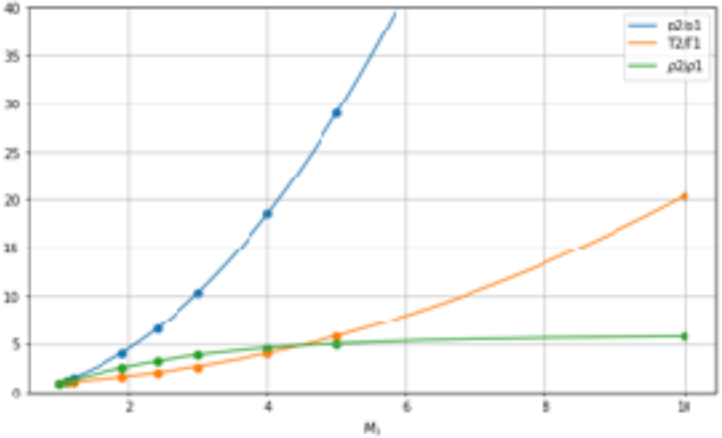
$$\frac{\rho_2}{\rho_1} \to \frac{\gamma + 1}{\gamma - 1}$$

*for adiabatic shock, density contrast is finite

The (adiabatic) normal shock tables (for air;r γ = 1.4)

M_1	M_2	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$
1.00	1.00000	1.0000	1.0000	1.0000
1.10	0.91177	1.2450	1.1691	1.0649
1.20	0.84217	1.5133	1.3416	1.1280
1.30	0.78596	1.8050	1.5157	1.1909
1.40	0.73971	2.1200	1.6896	1.2547
1.50	0.70109	2.4583	1.8621	1.3202
1.60	0.66844	2.8200	2.0317	1.3880
1.70	0.64055	3.2050	2.1977	1.4583
1.80	0.61650	3.6133	2.3592	1.5316
1.90	0.59562	4.0450	2.5157	1.6079
2.00	0.57735	4.5000	2.6666	1.6875
2.10	0.56128	4.9784	2.8119	1.7704
2.20	0.54706	5.4800	2.9512	1.8569
2.30	0.53441	6.0050	3.0846	1.9468
2.40	0.52312	6.5533	3.2119	2.0403
2.50	0.51299	7.1250	3.3333	2.1375
2.60	0.50387	7.7200	3.4489	2.2383
2.70	0.49563	8.3383	3.5590	2.3429
2.80	0.48817	8.9800	3.6636	2.4512
2.90	0.48138	9.6450	3.7629	2.5632
3.00	0.47519	10.333	3.8571	2.6790
4.00	0.43496	18.500	4.5714	4.0469
5.00	0.41523	29.000	5.0000	5.8000
10.00	0.38758	116.50	5.7143	20.388
∞	0.37796	∞	6.000	∞





$$(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla)\vec{V} = \cancel{\vec{g}} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2\vec{V}}$$

$$\rho u A = \text{constant}$$

$$u\,du + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

$$\boxed{(1 - \mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

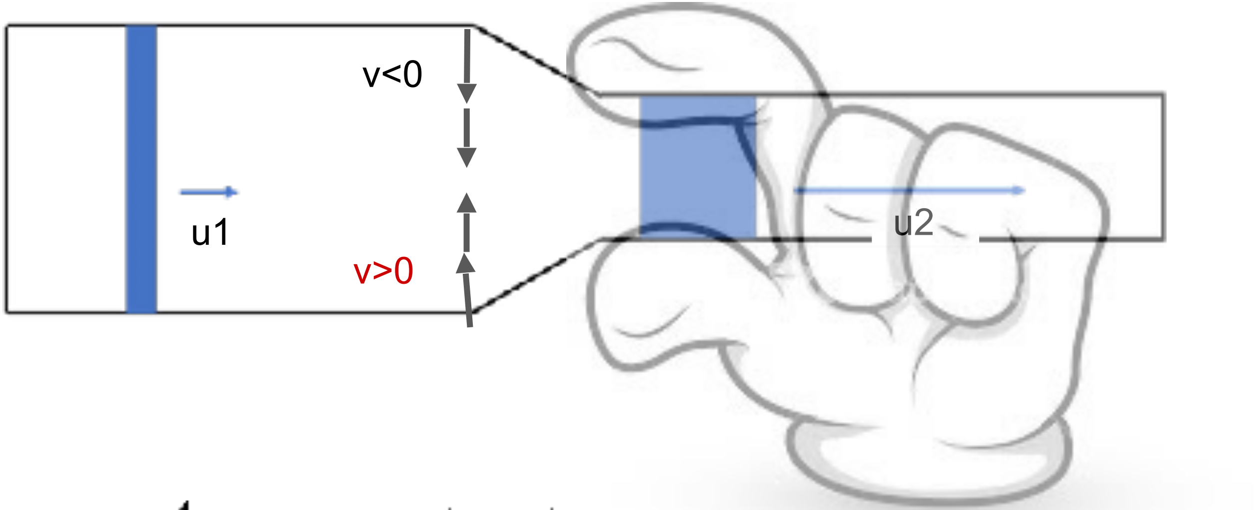
conservation of mass (continuity equation): integral form

from Reynolds transport theorem: $\boxed{\rho u A = \text{constant}}$

e.g. Fluid Mechanics by Frank M. White

if incompressible: $uA = \text{constant}$

an example: 2D incompressible flow



$$uA = \text{constant}$$

$$\left(\cancel{\frac{\partial}{\partial t}} + \vec{V} \cdot \nabla\right)\vec{V} = \cancel{\vec{g}} - \frac{\nabla P}{\rho} + \cancel{\frac{\mu}{\rho}\nabla^2\vec{V}}$$

$$\longrightarrow \quad u\,du + \frac{dp}{d\rho}\frac{d\rho}{\rho} = 0$$

$$u^2\frac{du}{u} = C_s^2\frac{d\rho}{\rho}$$

$$\frac{d\rho}{\rho} = -\mathcal{M}^2\frac{du}{u}$$

<1%　　0.3　　10%

$$\rho u A = \text{constant}$$

$$\longrightarrow \quad \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

$$\boxed{(1-\mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

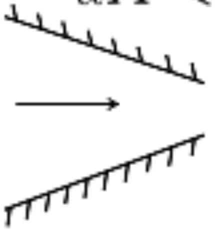
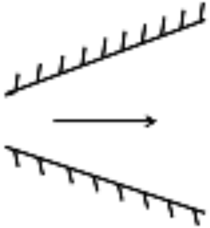
if $\mathcal{M} < 0.3$ $\rightarrow$ incompressible fluid incompressible:

$$\boxed{\nabla \cdot \mathbf{v} = 0}$$

or

$$\boxed{\mathcal{M} < 0.3}$$

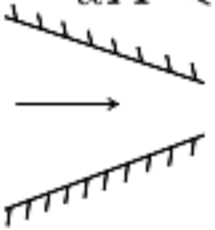
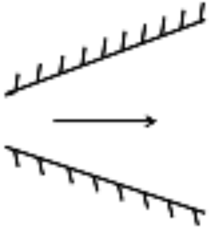
$$\boxed{(1-\mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

	subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$ 	$du>0$	$du<0$
$dA>0$ 	$du<0$	$du>0$

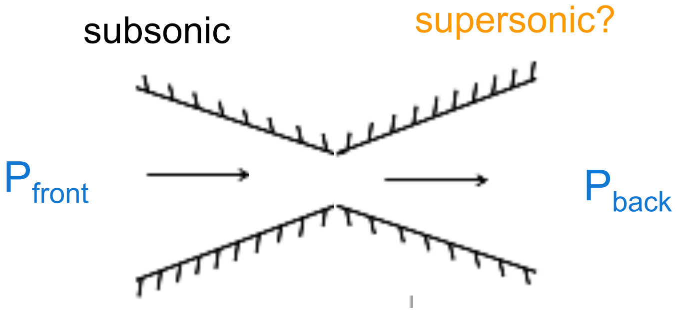
$$\boxed{(1-\mathcal{M}^2)\frac{du}{u} = -\frac{dA}{A}}$$

why?

$$\boxed{\rho u A = \text{constant}}$$

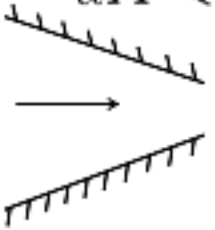
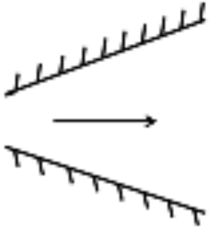
	subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$ 	$du>0$	$du<0$
$dA>0$ 	$du<0$	$\boxed{du>0}$

(convergence-divergence) nozzle flow



- A supersonic flow can be produced when the back pressure (P_{back}) is low enough

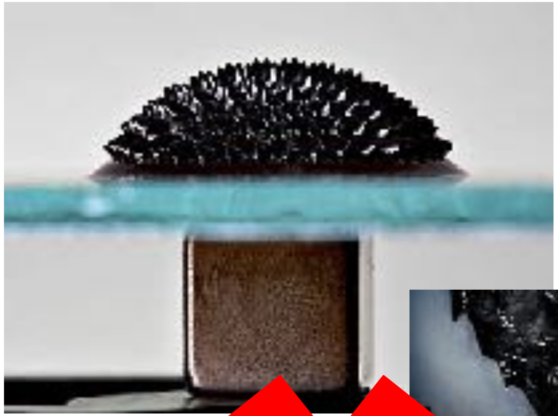
- If the sonic transition does not occur in the nozzle flow, the fluid speed reaches an extremum (du=0) when dA=0

	subsonic $1-\mathcal{M}^2>0$	supersonic $1-\mathcal{M}^2<0$
$dA<0$ 	$\boxed{du>0}$	$du<0$
$dA>0$ 	$du<0$	$\boxed{du>0}$

magnetohydrodynamics

MagnetoHydroDynamics（磁流體力學)

credit: wiki



we are NOT talking about

ferrofluid (鐵磁流體)

single and many charged particles

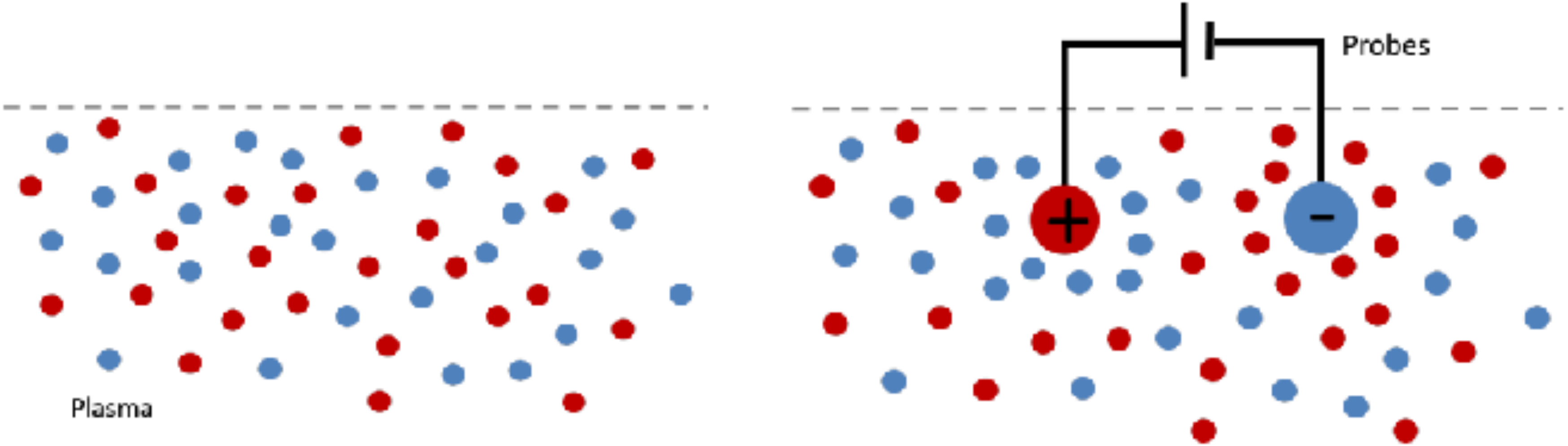
single particle acceleration: related to energetic particle in the Universe (e.g. cosmic ray)

Here we focus on plasma:

A plasma is a quasi-neutral gas consisting of positive and negative charged particles (usually ions & electrons)

*A stricter definition of a plasma is a gas where there are enough freed electrons and ions that they act collectively.

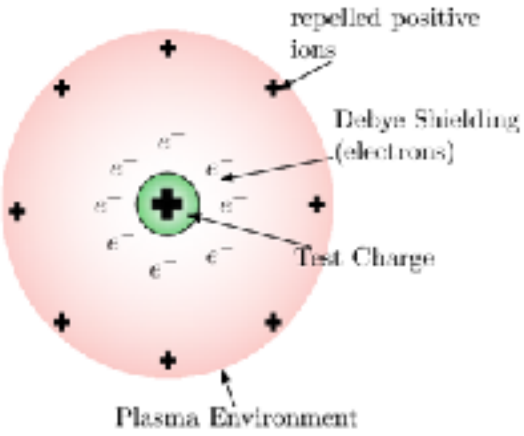
charge imbalances may exist only over a distance (**Debye length**) and a period of time (inverse of **plasma frequency**)

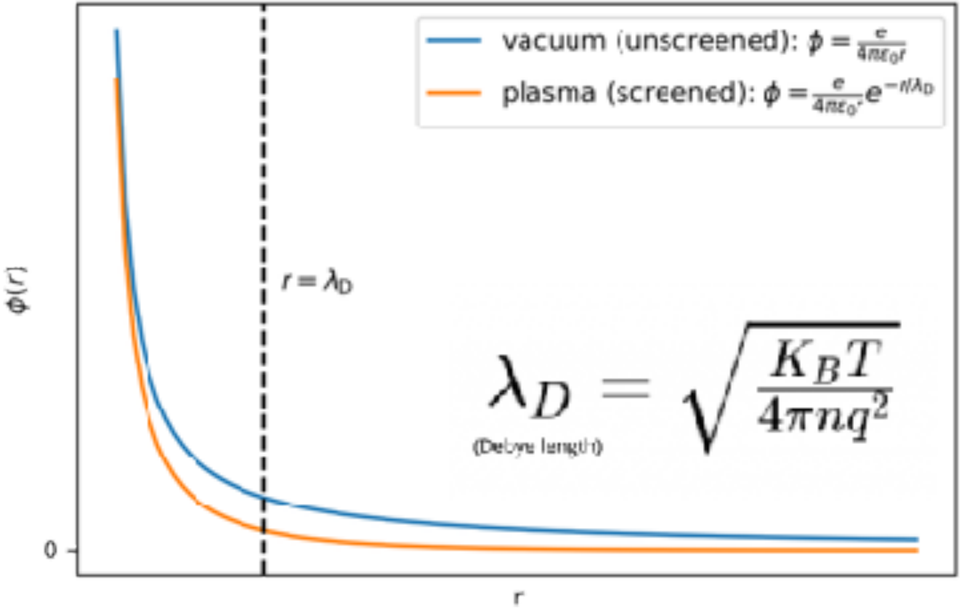


Two important parameters in plasma physics (I)

λ_D *electron Debye length*

a measure of the distance over which the electric potential of a point charge is significantly influenced by the surrounding charges.





$$\lambda_D = \sqrt{\frac{K_B T}{4\pi n q^2}}$$

Two important parameters in plasma physics (II)

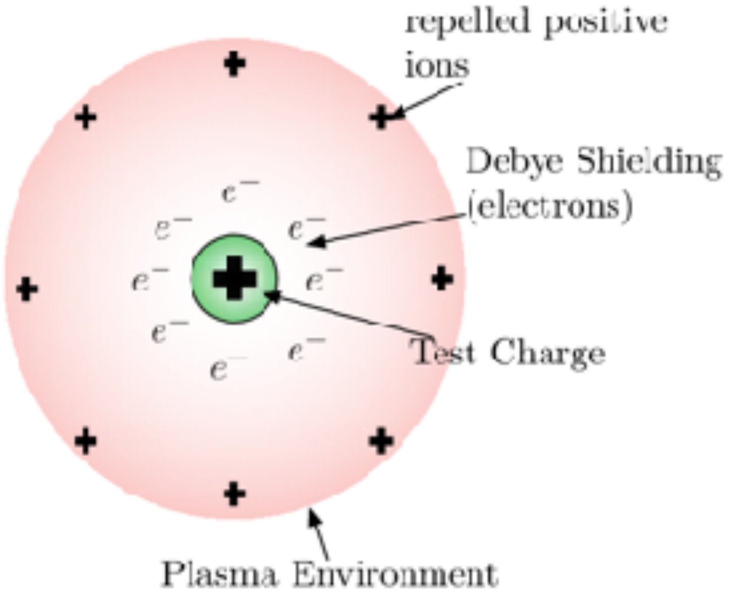
number of electrons in a “Debye cube" or “Debye sphere”:

$$N_D(\sim n\lambda_D^3)$$

The condition for an ionized gas to be considered a plasma is

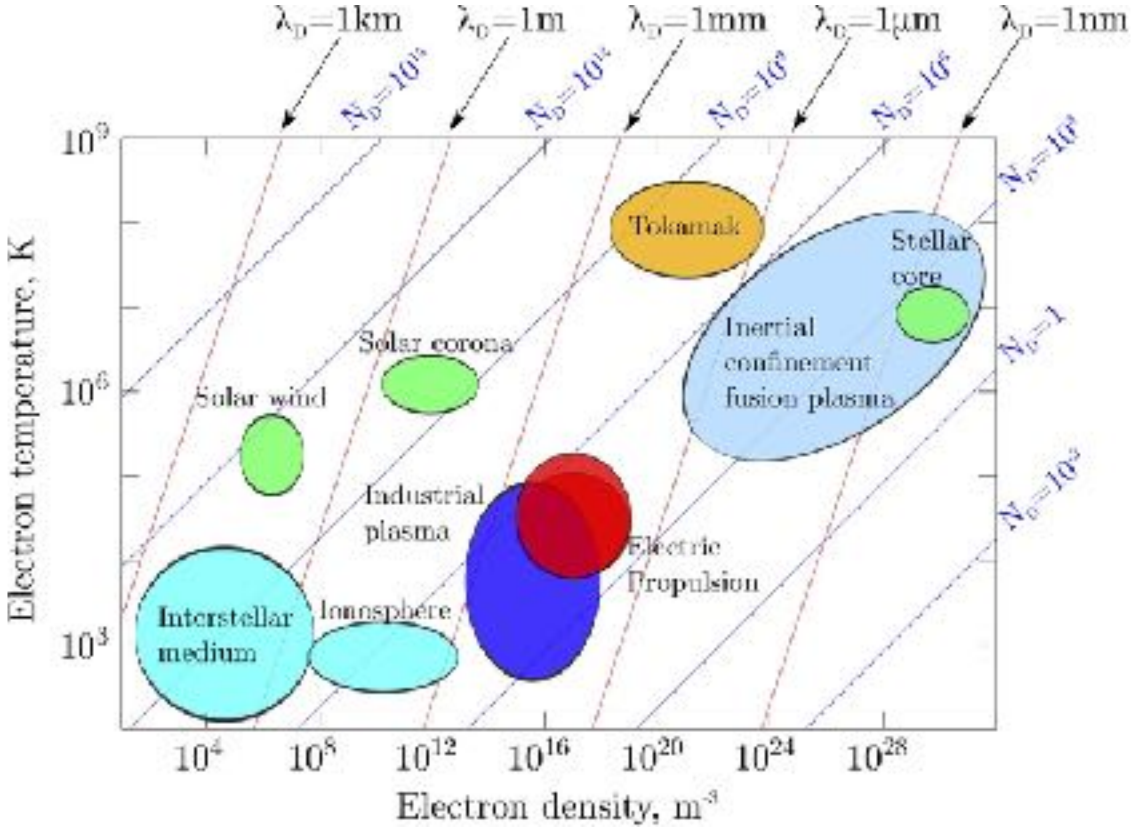
$$N_D >> 1$$

many charged particles within a Debye cube.



diffuse

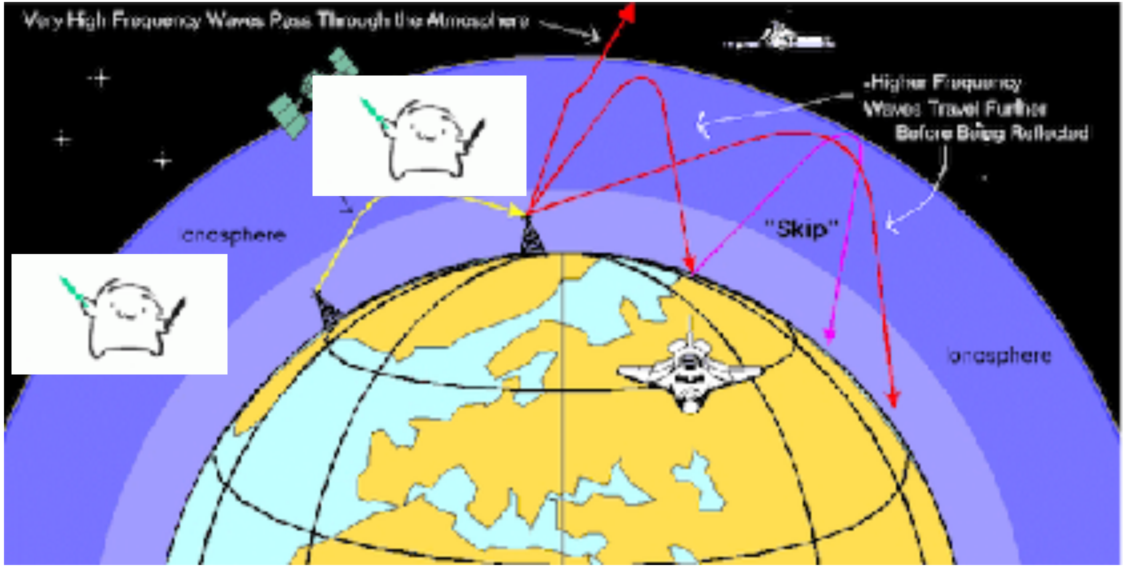
hot



Plasma exist **wide range of number densities and temperatures**

plasma frequency (~8980 x $n^{1/2}$Hz for electron)

- the frequency at which the electrons in the plasma naturally oscillate relative to the ions
- For the ionosphere, plasma frequency ~10^7Hz
- f<10^7Hz: reflected by ionosphere



Credit: NASA/GSFC

MHD describes:

the “slow*” evolution of an electrically conducting fluid,
and a region >> Debye length, Larmor radius

*compare to plasma frequency

collective behavior of fluids composted of charged particles (but electrically neutral!)

Lorentz force

magnetic field

motion of
charged particle

Maxwell equation

*magnetic forces on the particles in the fluid are not isotropic

SI unit | cgs unit

$$\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E}+\frac{\partial B}{\partial t} &= 0\\ \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0}\\ \vec{\nabla}\times\vec{B}-\frac{1}{c^2}\frac{\partial E}{\partial t} &= \mu_0\vec{J}\end{aligned}$$

$\longrightarrow$

$$\begin{aligned}\vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial B}{\partial t} &= 0\\ \vec{\nabla}\cdot\vec{E} &= 4\pi\rho\\ \vec{\nabla}\times\vec{B}-\frac{1}{c}\frac{\partial E}{\partial t} &= \frac{4\pi}{c}\vec{J}.\end{aligned}$$

SI to gaussian:

$$\epsilon_0 \to \frac{1}{4\pi} \qquad B \to B/c$$

$$\mu_0 \to \frac{4\pi}{c^2} \qquad E \to E$$

MHD: single fluid approach

$$\rho = m^+ n^+ + m^- n^-$$

$$v = \frac{m^+ n^+ v^+ + m^- n^- v^-}{m^+ n^+ + m^- n^-}$$

some initial guess

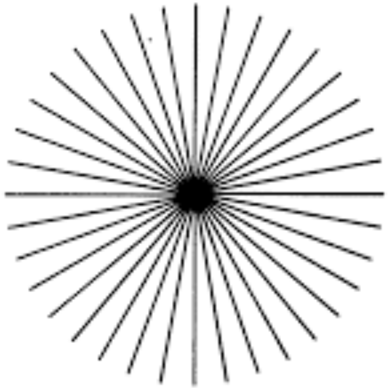
adding Lorentz force (as body force) to momentum equation

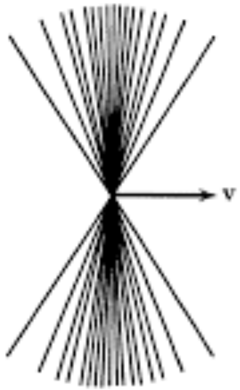
$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$$

adding Ohm's law (J) to close the set of equations

Lab frame: $$\mathbf{j} = \sigma \left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right)$$

conductivity





$$\mathbf{E}_{\parallel}{}' = \mathbf{E}_{\parallel}$$

$$\mathbf{B}_{\parallel}{}' = \mathbf{B}_{\parallel}$$

$$\mathbf{E}_{\perp}{}' = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}\right)$$

$$\mathbf{B}_{\perp}{}' = \gamma\left(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

some initial guess

E/B~L/T~u

RHS 2nd term/LHS ~ u^2/c^2

non-relativistic flow (u<<c): ignored terms related to E!

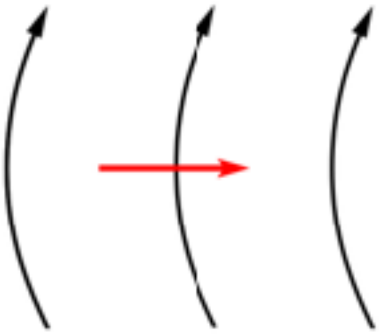
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

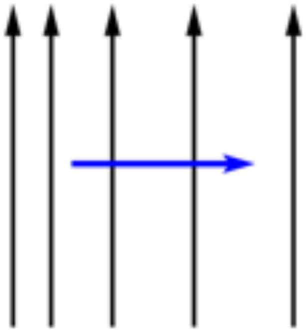
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \cancel{\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}}$$

$$
\begin{aligned}
\underbrace{\frac{\mathbf{J}\times\mathbf{B}}{c}}_{\text{Lorentz force}} &= \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} \\
&= \underbrace{\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}}_{\sim\text{magnetic tension}} - \underbrace{\nabla\left(\frac{B^2}{8\pi}\right)}_{\sim\text{magnetic pressure}} \qquad (20)
\end{aligned}
$$





Lab frame:

finite

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right)$$

$\boxed{\text{plasma frame: E'=0}}$ ⇔ ideal MHD:
infinite/perfect conductivity

$$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0 \text{ and } \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})}$$

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c}\right)$$

finite conductivity

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

magnetic diffusivity

$$\boxed{\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B}$$

(induction equation)

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B$$

$$\mathrm{R_m} = \frac{UL}{\eta} \sim \frac{\text{induction}}{\text{diffusion}}$$

L - Typical length scale of the flow   U - Typical Velocity scale of the flow

R_m - Reynolds Magnetic Number   η - Magnetic Diffusivity

usually >>1 in astrophysics
(ideal MHD is a good approximation)

magnetic tension, magnetic pressure

Lorentz force

magnetic field

motion of
charged particle

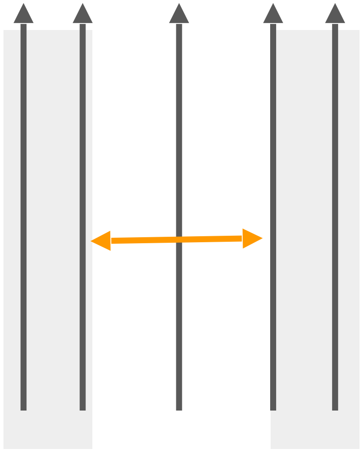
Maxwell equation + Ohm's law

$$\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c} = 0 \qquad \text{(ideal MHD)}$$

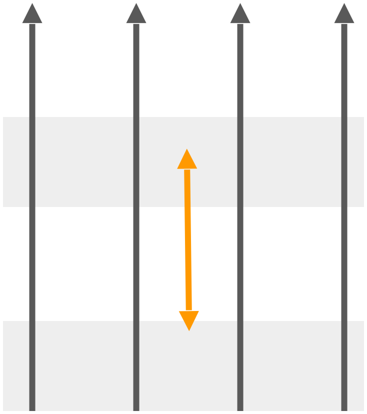
three MHD waves

fast magnetosonic wave

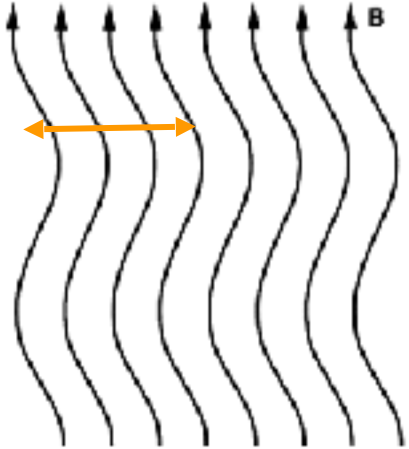
B



slow magnetosonic wave

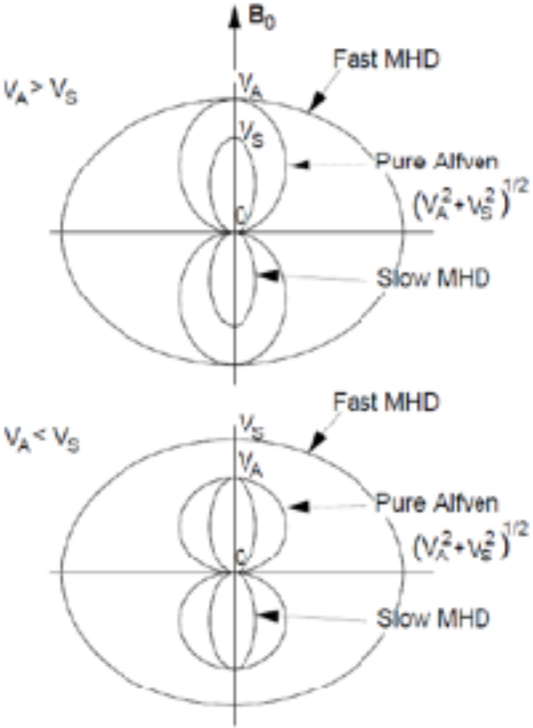


Aflven wave

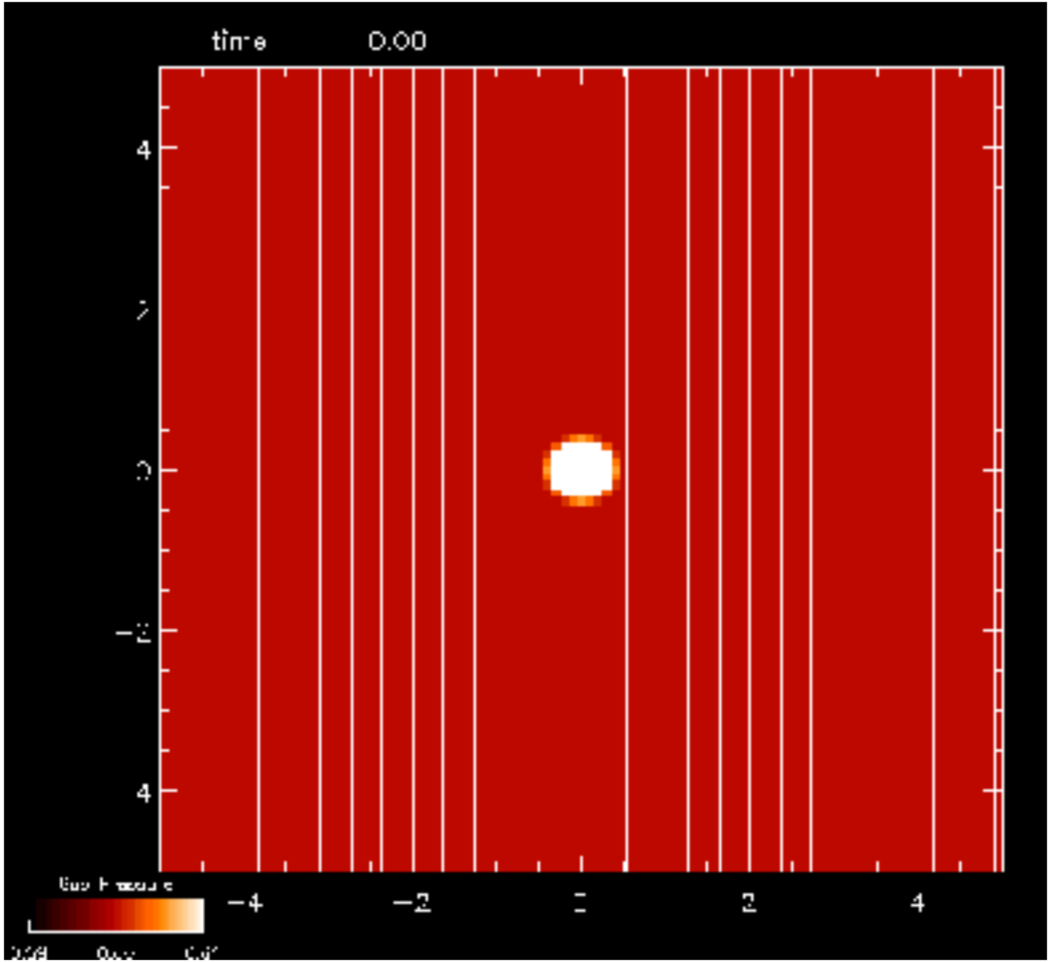


density enhancement

*from "Astrophysical flows" by Pringle and King



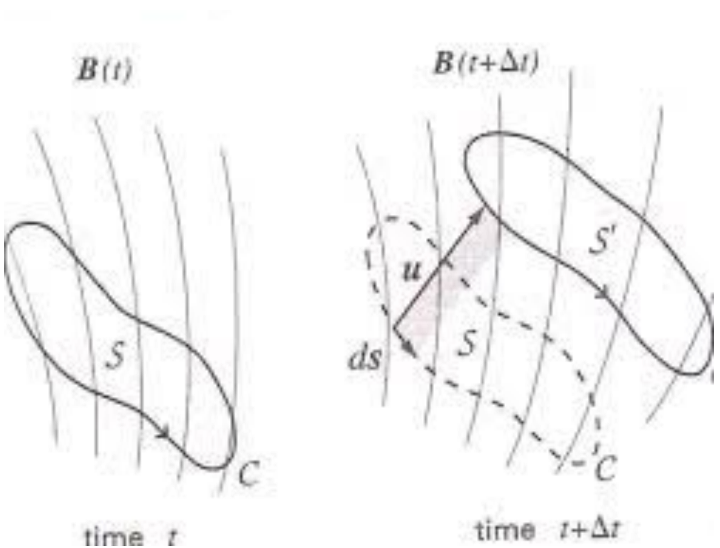
B_0
$V_A > V_S$
Fast MHD
V_A
V_S
Pure Alfven
$(V_A^2+V_S^2)^{1/2}$
0
Slow MHD
$V_A < V_S$
Fast MHD
V_S
V_A
Pure Alfven
$(V_A^2+V_S^2)^{1/2}$
0
Slow MHD



time 0.00
4
4
−4
−2
2
4

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$
Ampere's law	$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0$
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^{\gamma}} \right) = 0$

ideal MHD: flux freezing!



$$\Psi = \int_{\mathbf{S}} \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

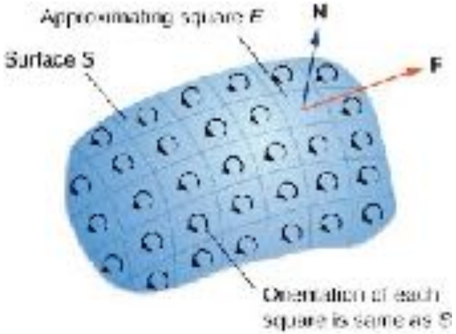
$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\int_{\mathbf{S}} \nabla \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{S}.$$

$= 0$

by the help of Stoke thm:

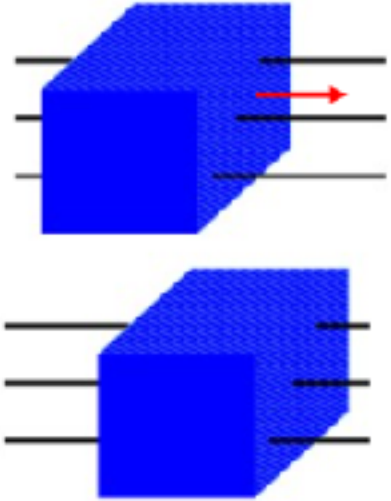
$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, d\Sigma$$

$$= \int_C \mathbf{F} \cdot d\mathbf{r}$$



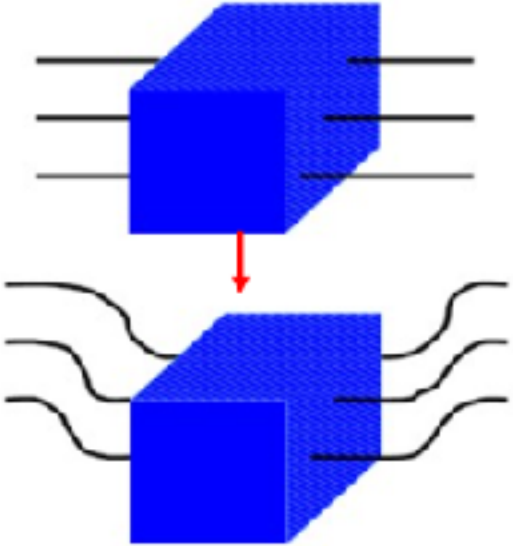
(apply F= v x B)

Strong field: matter can only move along given field lines (beads on a string):



$$\frac{|B|^2}{8\pi} >> P_{gas} + \rho|\mathbf{v}|^2$$

Weak field: field lines are forced to move along with the gas:

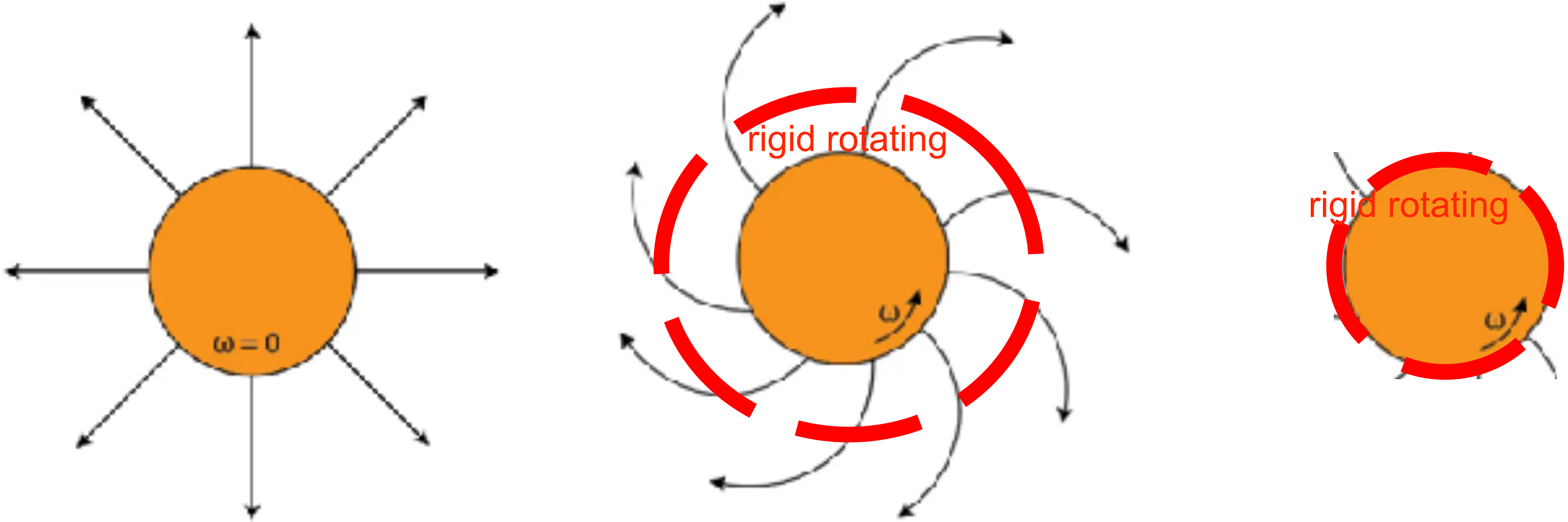


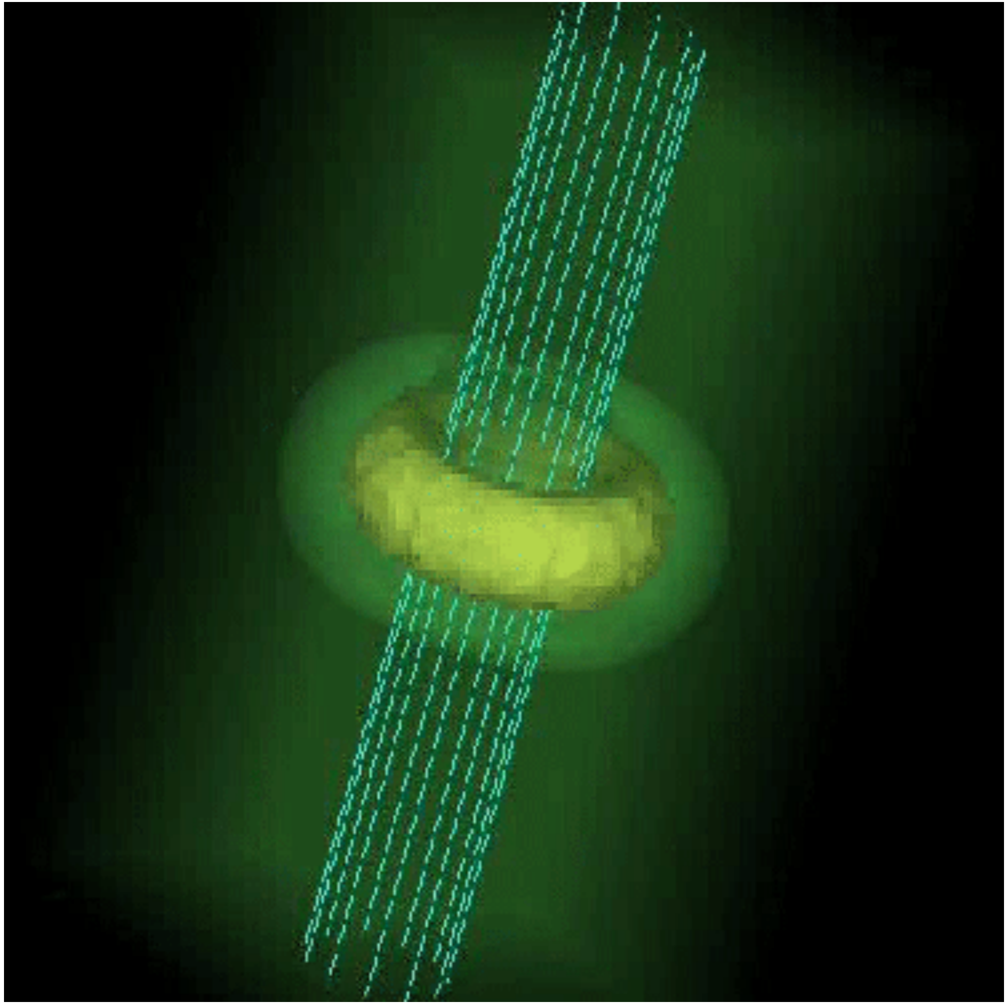
$$\frac{|B|^2}{8\pi} << P_{gas} + \rho|\mathbf{v}|^2$$

slide credit: C. P. Dullemond

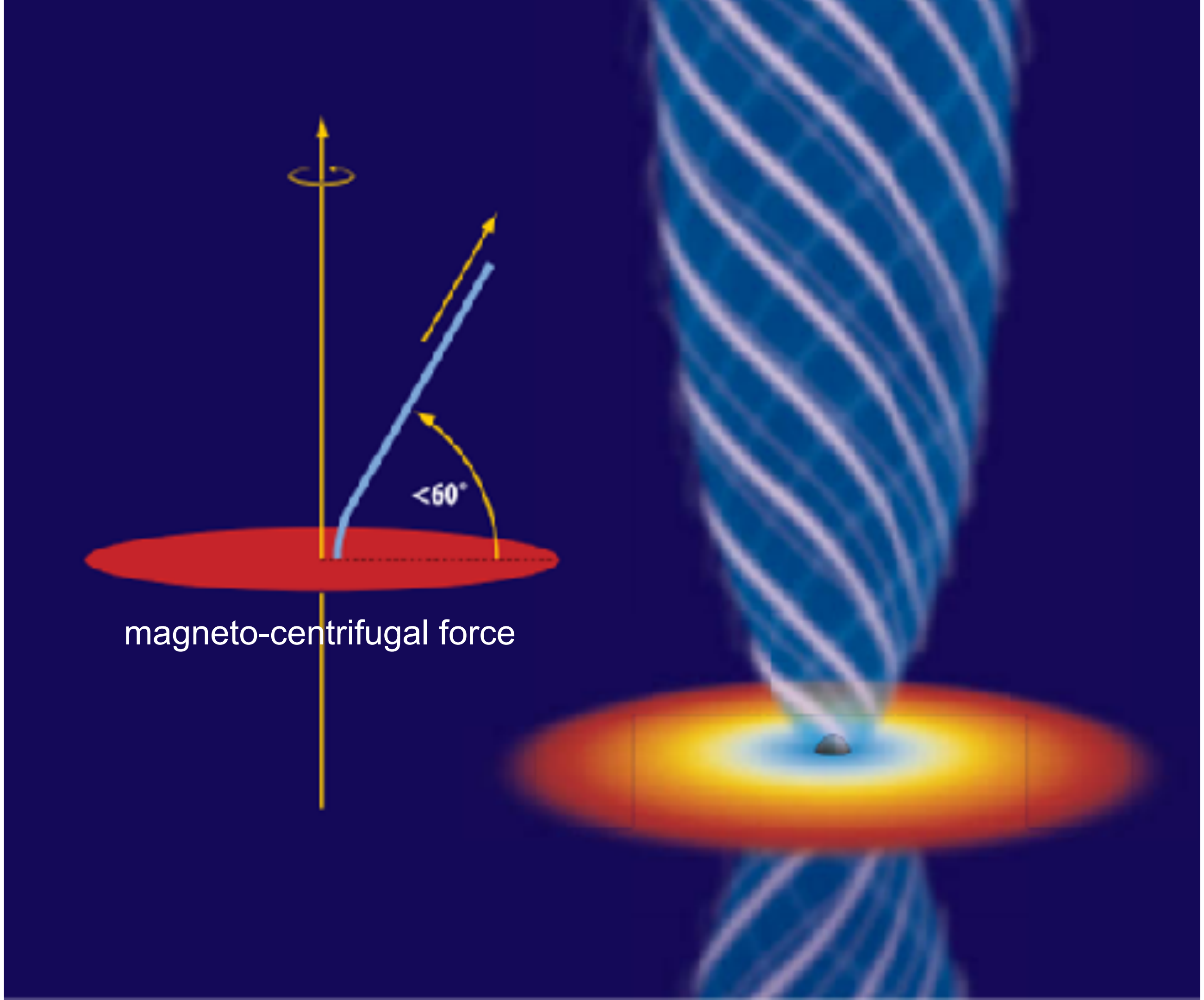
magnetized wind

the plasma rotates approximately like a solid body out to a radius, at where the magnetic energy equals the kinetic energy



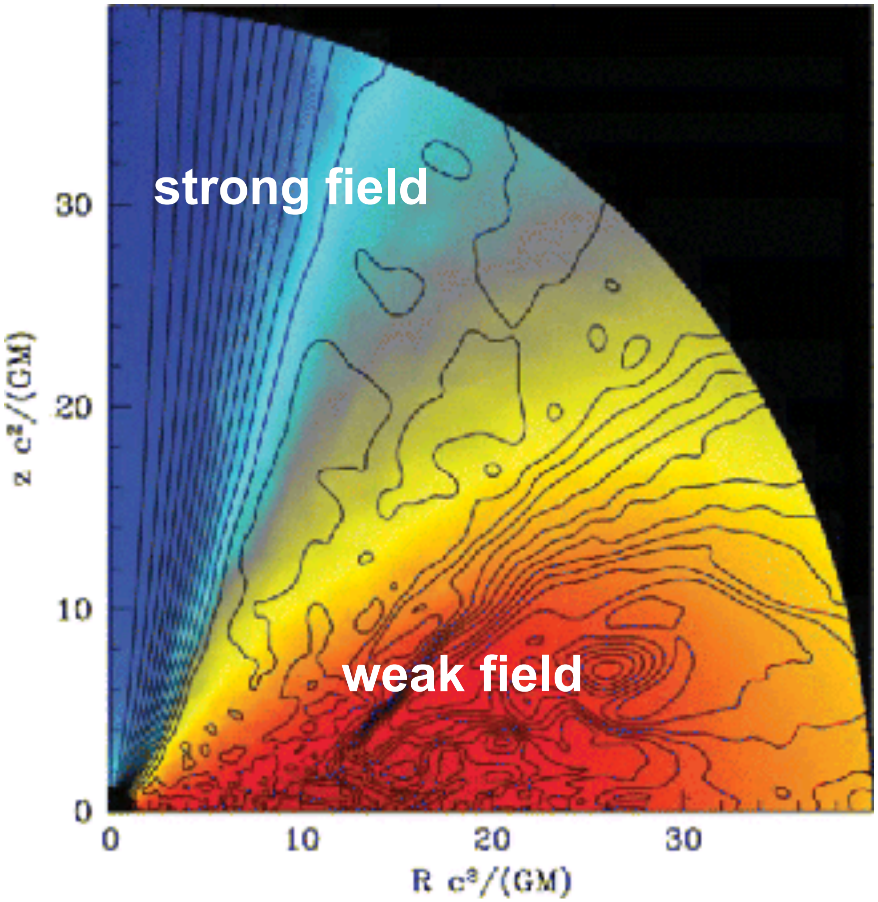


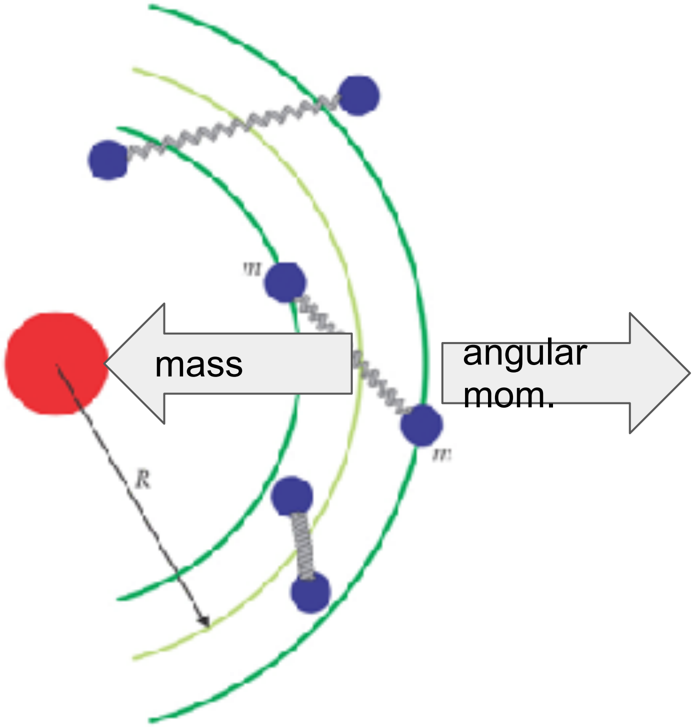
credit: Kuwabara



<60°
magneto-centrifugal force

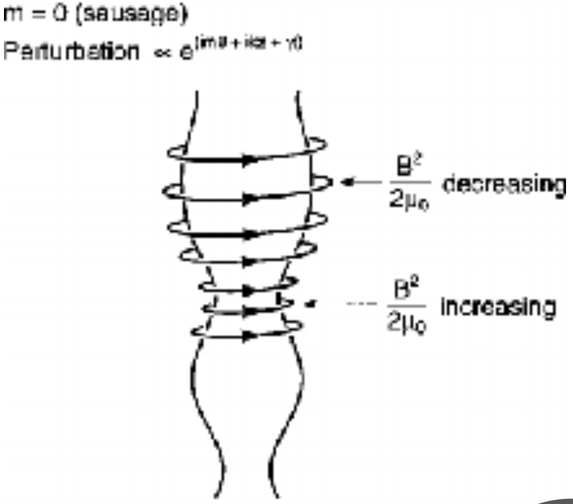
magnetorotational instability

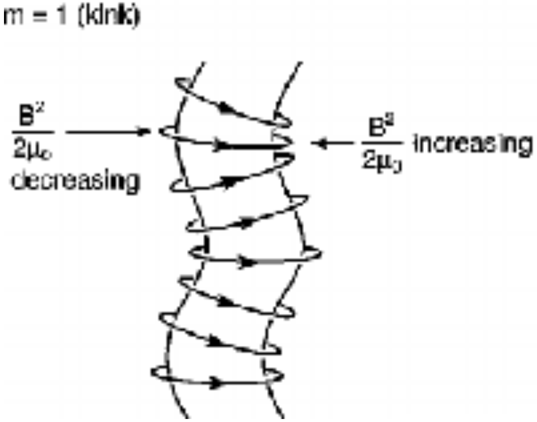




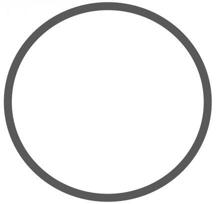
credit: McKinney & Narayan

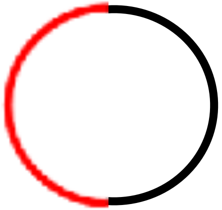
Sausage and Kink instability





m: axial wave number





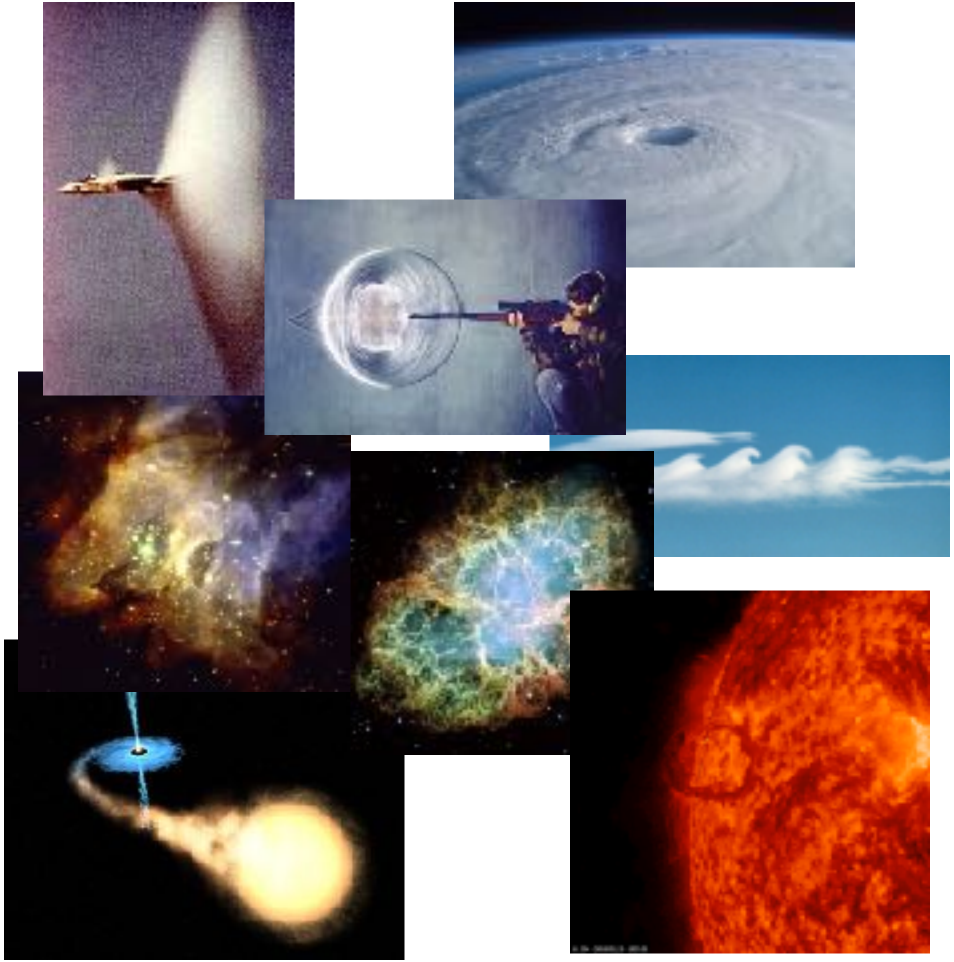
Final remarks

- astrophysical fluid
  - large scale
  - gravity is important
  - radiation cooling
  - MHD, most of the time: ideal MHD
  - multi-phase

- fluid mechanics as a physical problem:
  - governing equations + e.o.s + boundary conditions

summary

- hydrodynamics (HD)
    - shear and viscosity
    - velocity field
    - governing equations
        - continuity
        - momentum
        - energy
    - turbulence and energy cascade
    - shock
- magnetohydrodynamics (MHD)
    - plasma
    - ideal MHD
    - astrophysical applications



“when I meet God, I am going to ask him two questions: Why relativity? and why turbulence?

I really beleve he will have an answer for the first.”

W. Heisenberg (1907-1976)