Existence of Parameter-dependent Lyapunov Functions Assuring Robust Stability via SOS

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Notations

an HPPD Lyapunov function P_g of degree g

$$\sum_{k \in \mathcal{K}(g)} P_k \alpha^k$$

with Pólya notation $(\sum_{i=1}^{N} \alpha_i)^d$ of degree d.

$$\begin{aligned}
k &= k_1 k_2 \cdots k_N \\
\alpha^k &= \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N} \\
e_i &= 0 \cdots \underbrace{1}_{i^{th}} \cdots 0 \\
k - e_i &= k_1 k_2 \cdots (k_i - 1) \cdots k_N \\
\pi(k) &= (k_1!)(k_2!) \cdots (k_N!).
\end{aligned}$$

Pólya Theorem

Let $F(\alpha) < 0$ be a matrix polynomial function for all α on a unit simplex.
Then for a sufficiently large integer $d > 0$, the product $(\sum_{i=1}^{N} \alpha_i)^d F(\alpha)$ has all its matrix coefficients strictly negative-definite

Polytopic Robust Model

$$\sigma\zeta(t) = A(\alpha)\zeta(t) \tag{1}$$

where $\zeta(t) \in R^n$ and $\alpha \in R^N$ belongs to the unit simplex Δ displayed below

$$\Delta = \left\{ \alpha \in R^N | \sum_{i=1}^{N} \alpha_i = 1 \text{ where } \alpha_i \geq 0 \right\}$$

and

$$\Omega = \left\{ \ \mathcal{A} : \exists \ \alpha \in \Delta, \mathcal{A} = \textstyle\sum_{i=1}^{N} \alpha_i A_i \ \right\}.$$

Hurwitz Stability [6,7]

The system (1) is Hurwitz stable if and only if there exist matrices P_g, $k \in \mathcal{K}(g)$ and a sufficiently large $d > 0$ such that

$$\sum_{k \in \mathcal{K}(g+d+1)} R_k \alpha^k < 0 \text{ and } \sum_{k \in \mathcal{K}(g+d)} T_k \alpha^k > 0 \qquad (2)$$

where

$$R_k = \sum_{k' \in \mathcal{K}(d)} \sum_{i=1}^{N} \frac{d!}{\pi(k')} (A_i^T P_{k-k'-e_i} + P_{k-k'-e_i} A_i)$$

$$T_k = \sum_{k' \in \mathcal{K}(d)} \sum_{i=1}^{N} \frac{d!}{\pi(k')} P_{k-k'}$$

Schur Stability [6,7]

The system (1) is Schur stable if and only if there exist matrices $P_g > 0$ and a sufficiently large $d > 0$ such that

$$\sum_{k \in \mathcal{K}(g+d+1)} R_k \alpha^k < 0 \tag{3}$$

where

$$R_k = \sum_{k' \in \mathcal{K}(d)} \sum_{i=1}^{N} \frac{d!}{\pi(k')} \begin{bmatrix} -P_{k-k'-e_i} & P_{k-k'-e_i} A_i \\ A_i^T P_{k-k'-e_i} & -P_{k-k'-e_i} \end{bmatrix}$$

SOS Relaxation Theorem

The system (1) is Hurwitz/Schur stable if for each $k \in \mathcal{K}(g+d+1)$ and a sufficiently large $d > 0$ there exist matrices $P_k = P_k'$ such that

$$-v_1' \left(\sum_{k \in \mathcal{K}(d+g+1)} (R_k + \phi_1) x^{2k} \right) v_1 \text{ is SOS}$$

$$v_2' \left(\sum_{k \in \mathcal{K}(d+g)} (T_k - \phi_2) x^{2k} \right) v_2 \text{ is SOS}$$

ϕ_1 and ϕ_2 are sufficiently small positive constants and v_1, v_2 are vectors independent of x.

Example 1

Consider a continuous-time uncertain system $(n = 3, N = 2)$[6]

$$A_1 = \begin{bmatrix} -0.1938 & 0.3961 & -0.7104 \\ 0.0374 & 0.0988 & -0.9082 \\ 0.4803 & -0.2257 & -0.4496 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.6343 & 0.1343 & -0.9079 \\ -0.7179 & -0.6443 & -0.2978 \\ 0.3733 & 0.4191 & 0.3495 \end{bmatrix}$$

Example 1

Table I Maximum eigenvalues for $\lambda_{max}(R_k)$

k	d=0	d=1	d=2	d=3	d=4	d=5
1	-0.0021	-0.0021	-0.0021	-0.0020	-0.0020	-0.0020
2	2.4813	1.1241	0.5429	0.1783	0.0031	-0.1052
3	-0.0061	1.4027	1.5341	0.8847	0.1708	-0.5401
4	NA	-0.0065	0.9194	1.4237	0.9559	-0.5183
5	NA	NA	-0.0065	0.6203	1.2860	0.7054
6	NA	NA	NA	-0.0066	0.4564	1.0963
7	NA	NA	NA	NA	-0.0064	0.3397
8	NA	NA	NA	NA	NA	-0.0062
r9	-0.7765	-1.7208	-3.5535	-7.1931	-14.6145	-28.8897
r10	-0.0011	-0.0008	-0.0008	-0.0008	-0.0008	-0.0008

r 9: $\lambda_{\mathrm{max}}(\sum_k R_k)$ r10: $\lambda_{\mathrm{max}}(-Q)$

Example 2

Consider a discrete-time uncertain system $(n = 2, N = 4)$ [6]:

$$A_1 = \begin{bmatrix} -.468 & .845 \\ .272 & -.423 \end{bmatrix}, A_2 = \begin{bmatrix} .825 & .427 \\ .299 & -.346 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -.744 & .214 \\ 1.242 & .545 \end{bmatrix}, A_4 = \begin{bmatrix} .330 & -1.140 \\ -.322 & .309 \end{bmatrix}$$

Example 2

For $g = 1, d = 0$, feasibility test yields

$$P_{1000} = \begin{bmatrix} 0.5446 & 0.0323 \\ 0.0323 & 1.0881 \end{bmatrix}$$

$$P_{0100} = \begin{bmatrix} 0.8525 & 0.0944 \\ 0.0944 & 0.8262 \end{bmatrix}$$

$$P_{0010} = \begin{bmatrix} 1.3856 & 0.3344 \\ 0.3344 & 0.6349 \end{bmatrix}$$

$$P_{0001} = \begin{bmatrix} 0.4385 & 0.0122 \\ 0.0122 & 1.1798 \end{bmatrix}.$$

Example 2

For $g = 1, d = 1$, we have

$$P_{1000} = \begin{bmatrix} 0.3790 & -0.0347 \\ -0.0347 & 0.7671 \end{bmatrix}$$

$$P_{0100} = \begin{bmatrix} 0.6173 & 0.0653 \\ 0.0653 & 0.5679 \end{bmatrix}$$

$$P_{0010} = \begin{bmatrix} 1.0297 & 0.2389 \\ 0.2389 & 0.4384 \end{bmatrix}$$

$$P_{0001} = \begin{bmatrix} 0.3157 & -0.0213 \\ -0.0213 & 0.8588 \end{bmatrix}.$$

The $\lambda_{\max}(\sum_k R_k)$ for each case is -7.8807 and -23.3413 respectively and $\lambda_{\max}(-Q)$ are -0.0079 and -0.0075, respectively. Thus convergence is obtained for $d = 0$.

Example 3

Table II Summary of total decision variables required

Cases in [9]	Chesi's	The present
Ex 1 $q = 2, n = 2,$		
$m = 0$	4	3
$m = 1$	13	6
Ex 2 $q = 2, n = 3$		
$m = 0$	9	6
$m = 1$	30	12
$m = 2$	69	18
Ex 3 $q = 3, n = 4$		
$m = 0$	28	10
$m = 1$	198	30
$m = 2$	750	60

Conclusion

- An SOS relaxation method.
- Non-quadratic HPPD Lyapunov function.
- Pólya Theorem.
- Numerical Comparison.

Lastly

Thank you for your attentions