Chapter 4
AC Circuits

Me

Mechanical Engineering
National Central University

Feb 24, 2012–Apr 02, 2019

Outlines

$$
\begin{aligned}
v(t) &= V_{\max}\cos(\omega t+\theta) = \sqrt{2}V_{rms}\cos(\omega t+\theta) \\
i(t) &= I_{\max}\cos(\omega t+\phi) = \sqrt{2}I_{rms}\cos(\omega t+\phi)
\end{aligned}
$$

Euler's(/oiler/) formula $e^{jx} = \cos x + j\sin x$, leads to

$$
\begin{aligned}
v(t) &= Re[V_{\max}\cos(\omega t+\theta) + jV_{\max}\sin(\omega t+\theta)] \\
&= Re[V_{\max}e^{j(\omega t+\theta)}] = Re[\sqrt{2}V_{rms}e^{j(\omega t+\theta)}] \\
&= Re[v_e(t)] \\
i(t) &= Re[I_{\max}\cos(\omega t+\phi) + jI_{\max}\sin(\omega t+\phi)] \\
&= Re[I_{\max}e^{j(\omega t+\phi)}] = Re[\sqrt{2}I_{rms}e^{j(\omega t+\phi)}] \\
&= Re[i_e(t)]
\end{aligned}
$$

- $v(t) = Re[v_e(t)], \quad i(t) = Re[i_e(t)]$
- $jwe^{jwt} = \frac{de^{jwt}}{dt}, \quad \frac{1}{jw}e^{jwt} = \int_{-\infty}^{t} e^{jwt}dt.$

Impedance $v_e = Zi_e$

- Resistors (Ohm's law) $v_e(t) = i_e(t)R = Z_R i_e$
- Inductors (Henry's law)

$$v_e(t) = L\frac{di_e(t)}{dt} = L\frac{d}{dt}(I_{max}e^{j(\omega t+\phi)})$$

$$= (j\omega L)I_{max}e^{j(\omega t+\phi)} = \underline{j\omega L}i_e = Z_L i_e$$

- Capacitors (Faraday's law)

$$v_e(t) = \frac{1}{C}\int_{-\infty}^{t} i_e(\tau)d\tau = \frac{1}{C}\int_{-\infty}^{t} I_{max}e^{j(\omega\tau+\phi)}d\tau$$

$$= (\frac{1}{j\omega C})I_{max}e^{j(\omega t+\phi)} = \underline{\frac{1}{j\omega C}}i_e = Z_C i_e$$

Summary

$$
\begin{array}{c|c}
V_{\max}\cos(\omega t+\theta)=v(t) & Re[v_e(t)]=Re[V_{\max}e^{j(\omega t+\theta)}] \\
I_{\max}\cos(\omega t+\phi)=i(t) & Re[i_e(t)]=Re[I_{\max}e^{j(\omega t+\phi)}]
\end{array}
$$

Only magnitudes and angles are kept

$$\bar{V} = V_{rms}e^{j\theta} = \frac{V_{max}}{\sqrt{2}}e^{j\theta} =$$

$$\bar{I} = I_{rms}e^{j\phi} = \frac{V_{max}}{\sqrt{2}}e^{j\phi} =$$

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{V_{rms}e^{j\theta}}{I_{rms}e^{j\phi}} = \frac{V_{rms}\angle\theta}{I_{rms}\angle\phi} = \frac{V_{rms}}{I_{rms}}\angle\theta-\phi = Z\angle\theta_z$$

Summary

$V_{\max}\cos(\omega t+\theta)=v(t)$	$Re[v_e(t)]=Re[V_{\max}e^{j(\omega t+\theta)}]$
$I_{\max}\cos(\omega t+\phi)=i(t)$	$Re[i_e(t)]=Re[I_{\max}e^{j(\omega t+\phi)}]$
$\frac{d\cos wt}{dt}=-w\sin wt$	$\frac{e^{jwt}}{dt}=jwe^{jwt}$

Only magnitudes and angles are kept

$$\bar{V} = V_{rms}e^{j\theta} = \frac{V_{max}}{\sqrt{2}}e^{j\theta} =$$

$$\bar{I} = I_{rms}e^{j\phi} = \frac{V_{max}}{\sqrt{2}}e^{j\phi} =$$

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{V_{rms}e^{j\theta}}{I_{rms}e^{j\phi}} = \frac{V_{rms}\angle\theta}{I_{rms}\angle\phi} = \frac{V_{rms}}{I_{rms}}\angle\theta-\phi = Z\angle\theta_z$$

Summary

$$
\begin{array}{c|c}
V_{\max}\cos(\omega t+\theta)=v(t) & Re[v_e(t)]=Re[V_{\max}e^{j(\omega t+\theta)}] \\
I_{\max}\cos(\omega t+\phi)=i(t) & Re[i_e(t)]=Re[I_{\max}e^{j(\omega t+\phi)}] \\
\frac{d\cos wt}{dt}=-w\sin wt & \frac{e^{jwt}}{dt}=jwe^{jwt} \\
\int\cos wtdt=\frac{\sin wt}{w} & \int e^{jwt}dt=\frac{e^{jwt}}{jw}
\end{array}
$$

Only magnitudes and angles are kept

$$\bar{V} = V_{rms}e^{j\theta} = \frac{V_{max}}{\sqrt{2}}e^{j\theta} =$$

$$\bar{I} = I_{rms}e^{j\phi} = \frac{V_{max}}{\sqrt{2}}e^{j\phi} =$$

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{V_{rms}e^{j\theta}}{I_{rms}e^{j\phi}} = \frac{V_{rms}\angle\theta}{I_{rms}\angle\phi} = \frac{V_{rms}}{I_{rms}}\angle\theta-\phi = Z\angle\theta_z$$

Summary

$V_{\max}\cos(\omega t+\theta)=v(t)$	$Re[v_e(t)]=Re[V_{\max}e^{j(\omega t+\theta)}]$
$I_{\max}\cos(\omega t+\phi)=i(t)$	$Re[i_e(t)]=Re[I_{\max}e^{j(\omega t+\phi)}]$
$\frac{d\cos wt}{dt}=-w\sin wt$	$\frac{e^{jwt}}{dt}=jwe^{jwt}$
$\int\cos wtdt=\frac{\sin wt}{w}$	$\int e^{jwt}dt=\frac{e^{jwt}}{jw}$
Computationally hard	Algebraic Ohm's law $(v_e=Zi_e)$

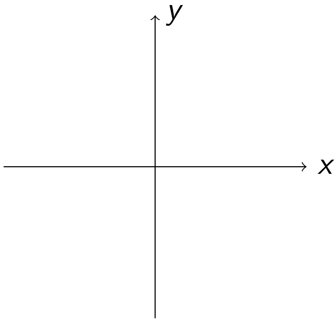
Only magnitudes and angles are kept

$$\bar{V} = V_{rms}e^{j\theta}=\frac{V_{max}}{\sqrt{2}}e^{j\theta}=$$

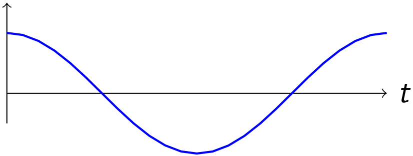
$$\bar{I} = I_{rms}e^{j\phi}=\frac{V_{max}}{\sqrt{2}}e^{j\phi}=$$

$$\bar{Z}=\frac{\bar{V}}{\bar{I}}=\frac{V_{rms}e^{j\theta}}{I_{rms}e^{j\phi}}=\frac{V_{rms}\angle\theta}{I_{rms}\angle\phi}=\frac{V_{rms}}{I_{rms}}\angle\theta-\phi=Z\angle\theta_z$$

Time domain and Phasor domain



$$v(t) = \cos wt$$



$$\cos(wt + \theta) = Re[e^{j(wt+\theta)}]$$

A $\cos wt$ function is converted into a rotating vector by Euler

Example 4.2

$$
\begin{aligned}
v(t) &= 170\cos(377t - 40^\circ)V \longleftrightarrow \bar{V} = 170\angle -40^\circ \\
i(t) &= 10\sin(1000t + 20^\circ)A \longleftrightarrow \bar{I} = 10\angle -70^\circ \\
\bar{V} &= 86.3\angle 26^\circ V \longleftrightarrow v(t) = 86.3\cos(\omega t + 26^\circ)V
\end{aligned}
$$

where $\sin(z) = \cos(z - 90^\circ)$. And more examples to practice.

Summary

	$R, jwL, \frac{1}{jwC}$	
$\Uparrow$ Phasor $M\angle\theta$	$R, jwL, \frac{1}{jwC}$	$\Downarrow$ $e^{j(wt+\theta)}$
$\Uparrow$ Euler $e^{j(wt+\theta)}$		$\Downarrow$ $Re(e^{j(wt+\theta)})$

Example 4.2

$$
\begin{aligned}
v(t) &= 170\cos(377t - 40^\circ)V \longleftrightarrow \bar{V} = 170\angle -40^\circ \\
i(t) &= 10\sin(1000t + 20^\circ)A \longleftrightarrow \bar{I} = 10\angle -70^\circ \\
\bar{V} &= 86.3\angle 26^\circ V \longleftrightarrow v(t) = 86.3\cos(\omega t + 26^\circ)V
\end{aligned}
$$

where $\sin(z) = \cos(z - 90^\circ)$. And more examples to practice.

Summary

	$R, jwL, \frac{1}{jwC}$	
$\Uparrow$ Phasor $M\angle\theta$	$R, jwL, \frac{1}{jwC}$	$\Downarrow$ $e^{j(wt+\theta)}$
$\Uparrow$ Euler $e^{j(wt+\theta)}$		$\Downarrow$ $Re(e^{j(wt+\theta)})$
Sinusoidal Input	$\rightarrow$ Differential eqn. $\rightarrow$	Sinusoidal Output

Example 4.2

$$
\begin{aligned}
v(t) &= 170\cos(377t - 40^\circ)V \longleftrightarrow \bar{V} = 170\angle -40^\circ \\
i(t) &= 10\sin(1000t + 20^\circ)A \longleftrightarrow \bar{I} = 10\angle -70^\circ \\
\bar{V} &= 86.3\angle 26^\circ V \longleftrightarrow v(t) = 86.3\cos(\omega t + 26^\circ)V
\end{aligned}
$$

where $\sin(z) = \cos(z - 90^\circ)$. And more examples to practice.

Summary

$$
\begin{array}{lcl}
 & R, jwL, \frac{1}{jwC} & \\
 & & \\
\Uparrow \text{Phasor } M\angle\theta & R, jwL, \frac{1}{jwC} & \Downarrow e^{j(wt+\theta)} \\
\text{Exponential Input} & \rightarrow \bar{Z}, \text{ Algebraic eqn.} \quad \rightarrow & \text{Exponential Output} \\
\Uparrow \text{Euler } e^{j(wt+\theta)} & & \Downarrow Re(e^{j(wt+\theta)}) \\
\text{Sinusoidal Input} & \rightarrow \text{Differential eqn.} \quad \rightarrow & \text{Sinusoidal Output}
\end{array}
$$

Example 4.2

$$
\begin{aligned}
v(t) &= 170\cos(377t - 40^\circ)V \longleftrightarrow \bar{V} = 170\angle -40^\circ \\
i(t) &= 10\sin(1000t + 20^\circ)A \longleftrightarrow \bar{I} = 10\angle -70^\circ \\
\bar{V} &= 86.3\angle 26^\circ V \longleftrightarrow v(t) = 86.3\cos(\omega t + 26^\circ)V
\end{aligned}
$$

where $\sin(z) = \cos(z - 90^\circ)$. And more examples to practice.

Summary

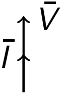
	$R, jwL, \frac{1}{jwC}$		
Complex input	$\rightarrow \bar{Z}$, Algebraic eqn.	$\rightarrow$	Complex output
$\Uparrow$ Phasor $M\angle\theta$	$R, jwL, \frac{1}{jwC}$		$\Downarrow e^{j(wt+\theta)}$
Exponential Input	$\rightarrow \bar{Z}$, Algebraic eqn.	$\rightarrow$	Exponential Output
$\Uparrow$ Euler $e^{j(wt+\theta)}$			$\Downarrow Re(e^{j(wt+\theta)})$
Sinusoidal Input	$\rightarrow$ Differential eqn.	$\rightarrow$	Sinusoidal Output

Resistor, $\bar{I}_R$ and $\bar{V}_R$ are in phase

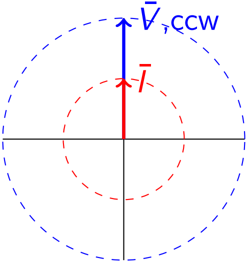
Assumed Form: $v(t) = V_{\max}\cos(\omega t + \theta) = \sqrt{2}V_{rms}\cos(\omega t + \theta)$

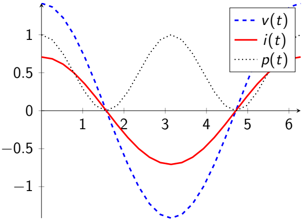
Assume $\theta = 90^\circ$ for voltage. Could be any angle

$v = iR$, $i(t) = \frac{V_{\max}}{R}\cos(\omega t + \theta) = I_{\max}\cos(wt + 90^\circ)$



$p(t) = v(t)i(t) = V_{\max}I_{\max}cos^2(wt)$



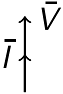


Resistor, $\bar{I}_R$ and $\bar{V}_R$ are in phase

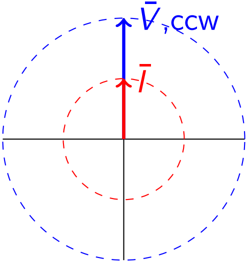
Assumed Form: $v(t) = V_{\max}\cos(\omega t + \theta) = \sqrt{2}V_{rms}\cos(\omega t + \theta)$

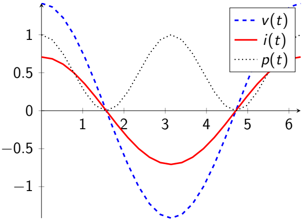
Assume $\theta = 90^\circ$ for voltage. Could be any angle

$v = iR$, $i(t) = \frac{V_{\max}}{R}\cos(\omega t + \theta) = I_{\max}\cos(wt + 90^\circ)$



$p(t) = v(t)i(t) = V_{\max}I_{\max}cos^2(wt)$



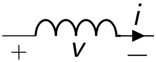


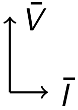
Inductor, $\bar{I}_L$ lags $\bar{V}_L$ by 90°

Assumed Form: $v(t) = V_{\max}\cos(\omega t + \theta) = \sqrt{2}V_{rms}\cos(\omega t + \theta)$

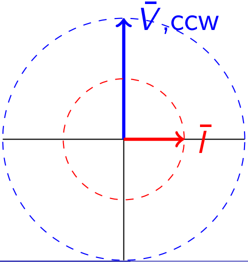
Assume $\theta = 90^\circ$ for voltage. Could be any angle

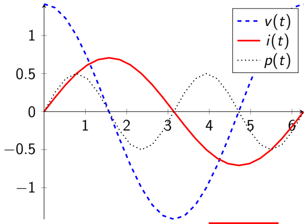
$v = i(jwL), i(t) = \frac{1}{jwL}V_{\max}\cos(\omega t + 90^\circ) = I_{max}\sin(\omega t)$





$p(t) = v(t)i(t) = V_{\max}I_{\max}\cos(wt)\sin(wt) = \frac{V_{\max}I_{\max}}{2}\sin(2wt)$



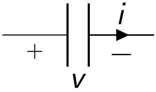


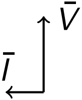
Capacitor, $\bar{I}_C$ leads $\bar{V}_C$ by 90°

Assumed Form: $v(t) = V_{\max}\cos(\omega t + \theta) = \sqrt{2}V_{rms}\cos(\omega t + \theta)$

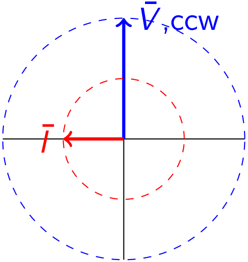
Assume $\theta = 90^\circ$ for voltage. Could be any angle

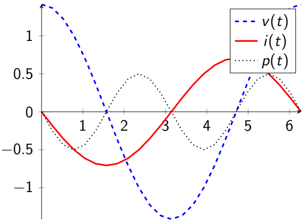
$v = i(\frac{1}{jwC}), i(t) = jwCV_{\max}\cos(\omega t + 90^\circ) = -I_{max}\sin(\omega t)$





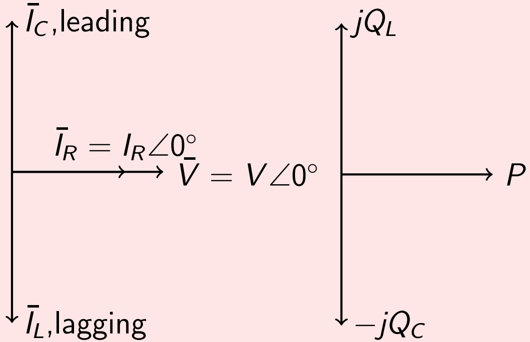
$p(t) = v(t)i(t) = -V_{\max}I_{\max}\cos(wt)\sin(wt) = -\frac{V_{\max}I_{\max}}{2}\sin(2wt)$





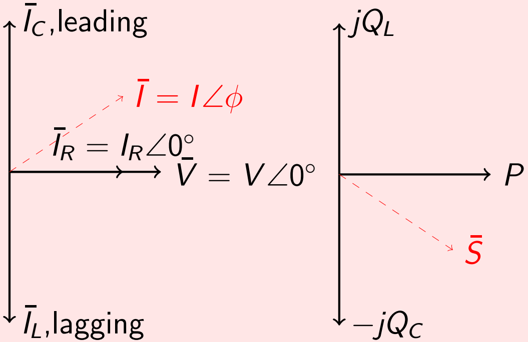
back

Summary



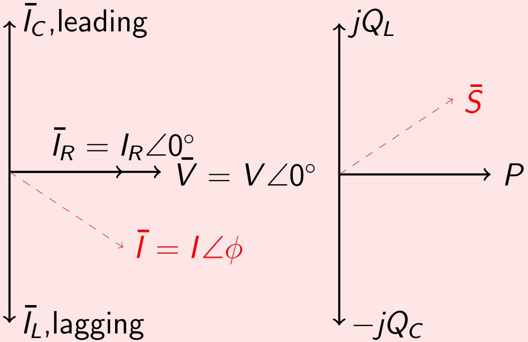
back

Summary



back

Summary



We have discussed the special cases for R, L and C. How about a more general case? Say, combination of RLC.

Instantaneous power

$$p(t) = V_{\max} \cos(wt + \theta) I_{\max} \cos(wt + \phi)$$
$$= V_{rms} I_{rms} \cos(\theta - \phi)\{1 + \cos(2wt + 2\phi)\}$$
$$+ V_{rms} I_{rms} \sin(\theta - \phi) \sin(2wt + 2\phi)$$

Average real/reactive power

$$P_{ave} = \frac{1}{2\pi} \int_0^{2\pi} p(\tau)\tau = V_{rms} I_{rms} \cos(\theta - \phi) \ W$$

$$Q_{ave} = \frac{1}{2\pi} \int_0^{2\pi} p(\tau)\tau = 0 \ VAR$$

Complex power

$$p(t) = V_{rms}I_{rms}\cos(\theta-\phi)\{1+\cos(2wt+2\phi)\} + V_{rms}I_{rms}\sin(\theta-\phi)\sin(2wt+2\phi)$$

$$\begin{aligned}\bar{S} &= P + jQ = \bar{V}\bar{I}^* = (V_{rms}\angle\theta)(I_{rms}\angle\phi)^* = V_{rms}I_{rms}\angle(\theta-\phi) \\ &= V_{rms}I_{rms}\cos(\theta-\phi) + jV_{rms}I_{rms}\sin(\theta-\phi)\end{aligned}$$

Since $\bar{V} = \bar{I}\bar{Z}$ (Ohm's law in AC), we have

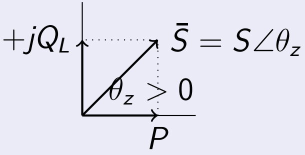
$$\begin{aligned}\bar{S} &= P + jQ = \bar{V}\bar{I}^* = (\bar{I}\bar{Z})\bar{I}^* = I_{rms}^2(R+jX) = I_{rms}^2R + jI_{rms}^2X \\ &= \bar{V}\left(\frac{\bar{V}}{\bar{Z}}\right)^\star = \frac{V_{rms}^2}{R-jX} = \frac{V_{rms}^2R}{Z^2} + j\frac{V_{rms}^2X}{Z^2}\end{aligned}$$

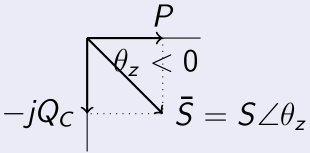
If $X = 0$, it becomes a DC formula.

Apparent power

$$S = V_{rms} I_{rms} = \frac{V_{max}}{\sqrt{2}} \frac{I_{max}}{\sqrt{2}} = \frac{1}{2} V_{max} I_{max}$$

Power triangle





$$P = V_{rms} I_{rms} \cos(\theta - \phi) = I_{rms}^2 R = V_{rms}^2 R/Z^2$$

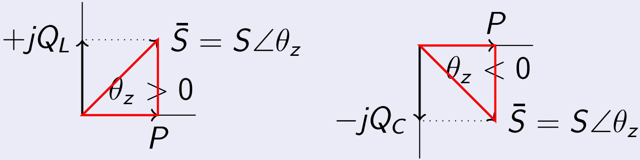
$$Q = V_{rms} I_{rms} \sin(\theta - \phi) = I_{rms}^2 X = V_{rms}^2 X/Z^2$$

$$\theta_z = \theta - \phi = cos^{-1} pf$$

Apparent power

$$S = V_{rms} I_{rms} = \frac{V_{max}}{\sqrt{2}} \frac{I_{max}}{\sqrt{2}} = \frac{1}{2} V_{max} I_{max}$$

Power triangle



$$P = V_{rms} I_{rms} \cos(\theta - \phi) = I_{rms}^2 R = V_{rms}^2 R / Z^2$$

$$Q = V_{rms} I_{rms} \sin(\theta - \phi) = I_{rms}^2 X = V_{rms}^2 X / Z^2$$

$$\theta_z = \theta - \phi = cos^{-1} pf$$

pf

Leading/Lagging power factor

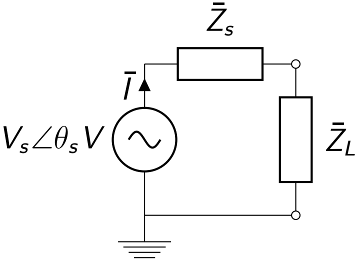
Based on the analysis in the previous page, it is clear that the power factor is the cosine value of the impedance angle θ_z.

$\bar{Z} = R + jX = R + j(wL - \frac{1}{wC}) = \sqrt{R^2 + X^2} \angle tan^{-1} \frac{wL - \frac{1}{wC}}{R}$. For a pure resistor network: $X = 0$, $\theta_z = 0$, $pf = \cos\theta_z = 1$, pointing to the right.

For a pure inductor network: $X = L$, $\theta_z > 0$, $pf = \cos\theta_z < 1$, pointing upward.

For a pure capacitor network: $X = C$, $\theta_z < 0$, $pf = \cos\theta_z < 1$, pointing downward.

To maximize the power received at the load end.



$$\bar{I} = \frac{\bar{V}_s}{\bar{Z}_s + \bar{Z}_L} = \frac{V_s\angle\theta_s}{(R_s + R_L) + j(X_s + X_L)}$$

$$I = \frac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

$$P_L = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

which implies that

$$\frac{\partial P_L}{\partial X_L} = 0 \Rightarrow X_L = -X_s$$

$$\frac{\partial P_L}{\partial R_L} = 0 \Rightarrow R_L = R_s$$

$$\bar{Z}_L = R_L + jX_L = R_s - jX_s = \bar{Z}_s^*$$

$$P_L = I^2 R_L = \frac{V_{s,rms}^2}{4R_s} = \frac{V_{s,max}^2}{8R_s}$$