Chapter 4 AC Circuits

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Outlines

- Complex Signals
 - Math form
- 2 Impedance Concepts
 - Impedance $v_e = Zi_e$
- Phasor Concepts
 - Rotating vector
 - An example
 - More examples
- Power in AC
 - Generalized expressions
 - Max. power transfer
 - Examples

$$v(t) = V_{\text{max}} \cos(\omega t + \theta) = \sqrt{2} V_{rms} \cos(\omega t + \theta)$$

$$i(t) = I_{\text{max}} \cos(\omega t + \phi) = \sqrt{2} I_{rms} \cos(\omega t + \phi)$$
Euler's(/oiler/) formula $e^{jx} = \cos x + j \sin x$, leads to
$$v(t) = Re[V_{\text{max}} \cos(\omega t + \theta) + j V_{\text{max}} \sin(\omega t + \theta)]$$

$$= Re[V_{\text{max}} e^{j(\omega t + \theta)}] = Re[\sqrt{2} V_{rms} e^{j(\omega t + \theta)}]$$

$$= Re[V_{\text{max}} \cos(\omega t + \phi) + j I_{\text{max}} \sin(\omega t + \phi)]$$

$$= Re[I_{\text{max}} \cos(\omega t + \phi) + j I_{\text{max}} \sin(\omega t + \phi)]$$

$$= Re[I_{\text{max}} e^{j(\omega t + \phi)}] = Re[\sqrt{2} I_{rms} e^{j(\omega t + \phi)}]$$

$$= Re[i_e(t)]$$

•
$$v(t) = Re[v_e(t)], \quad i(t) = Re[i_e(t)]$$

• $jwe^{jwt} = \frac{de^{jwt}}{dt}$, $\frac{1}{2}e^{jwt} = \int_{-\infty}^{t} e^{jwt} dt$.

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Impedance $v_e = Zi_e$

- Resistors (Ohm's law) $v_e(t) = i_e(t)R = Z_R i_e$
- Inductors (Henry's law)

$$v_e(t) = L \frac{di_e(t)}{dt} = L \frac{d}{dt} (I_{max} e^{j(\omega t + \phi)})$$

= $(j\omega L)I_{max} e^{j(\omega t + \phi)} = \underline{j\omega L} i_e = \underline{Z_L} i_e$

Capacitors (Faraday's law)

$$egin{aligned} v_e(t) &= rac{1}{C} \int_{-\infty}^t i_e(au) d au = rac{1}{C} \int_{-\infty}^t I_{max} e^{j(\omega au+\phi)} d au \ &= (rac{1}{j\omega C}) I_{max} e^{j(\omega t+\phi)} = rac{1}{j\omega C} i_e = Z_C i_e \end{aligned}$$

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$$egin{aligned} V_{\mathsf{max}}\cos(\omega t + heta) &= v(t) & Re[v_{\mathsf{e}}(t)] &= Re[V_{\mathsf{max}}e^{i(\omega t + heta)}] \ I_{\mathsf{max}}\cos(\omega t + \phi) &= i(t) & Re[i_{\mathsf{e}}(t)] &= Re[I_{\mathsf{max}}e^{j(\omega t + \phi)}] \end{aligned}$$

Only magnitudes and angles are kept

$$ar{V}$$
 = $V_{rms}e^{j heta}=rac{V_{max}}{\sqrt{2}}e^{j heta}=0$
 $ar{I}$ = $I_{rms}e^{j\phi}=rac{V_{max}}{\sqrt{2}}e^{j\phi}=0$
 $ar{Z}=rac{ar{V}}{I}=rac{V_{rms}e^{j\phi}}{I_{rms}e^{j\phi}}=rac{V_{rms} extstyle heta}{I_{rms}} = 0$
 $A_{rms}e^{j\phi}=\frac{V_{rms}e^{j\phi}}{\sqrt{2}}e^{j\phi}=0$

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$$V_{\max}\cos(\omega t + \theta) = v(t)$$
 $Re[v_e(t)] = Re[V_{\max}e^{j(\omega t + \theta)}]$ $Re[i_e(t)] = Re[I_{\max}e^{j(\omega t + \theta)}]$ $Re[i_e(t)] = Re[I_{\max}e^{j(\omega t + \phi)}]$ $\frac{d\cos wt}{dt} = -w\sin wt$ $\frac{e^{jwt}}{dt} = jwe^{jwt}$

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 $ar{Z}=rac{ar{V}}{I}=rac{V_{rms}e^{j\phi}}{I_{rms}e^{j\phi}}=rac{V_{rms}\angle\theta}{I_{rms}}=rac{V_{rms}}{I_{rms}}\angle\theta-\phi=Z\angle\theta_z$

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$$V_{\mathsf{max}} \cos(\omega t + \theta) = v(t)$$
 $Re[v_e(t)] = Re[V_{\mathsf{max}} e^{j(\omega t + \theta)}]$ $Re[i_e(t)] = Re[I_{\mathsf{max}} e^{j(\omega t + \theta)}]$ $Re[i_e(t)] = Re[I_{\mathsf{max}} e^{j(\omega t + \phi)}]$ $Re[i_e(t)] = Re[I_{\mathsf{max}} e^{j(\omega t + \phi)}]$ $\frac{e^{j\omega t}}{dt} = jwe^{jwt}$ $\int e^{jwt} dt = \frac{e^{jwt}}{jw}$

Only magnitudes and angles are kept

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$$V_{\mathsf{max}}\cos(\omega t + \theta) = v(t)$$
 $I_{\mathsf{max}}\cos(\omega t + \phi) = i(t)$
 $\frac{d\cos wt}{dt} = -w\sin wt$
 $\int \cos wt dt = \frac{\sin wt}{w}$
Computationally hard

$$egin{aligned} Re[v_e(t)] &= Re[V_{\mathsf{max}}e^{j(\omega t + heta)}] \ Re[i_e(t)] &= Re[I_{\mathsf{max}}e^{j(\omega t + \phi)}] \ rac{e^{jwt}}{dt} &= jwe^{jwt} \ \int e^{jwt}dt &= rac{e^{jwt}}{jw} \ Algebraic Ohm's law $(v_e = Zi_e) \end{aligned}$$$

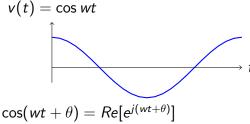
Only magnitudes and angles are kept

$$ar{V}$$
 =
$$ar{I}$$
 =
$$ar{Z} = \frac{ar{V}}{I} = \frac{V_{rms}e^{i\theta}}{I e^{i\phi}} = \frac{V_{rms}\angle\theta}{I e^{i\phi}} = \frac{V_{rms}}{I e^{i\phi}} \angle\theta - \phi = Z\angle\theta_z$$

$$V_{rms}e^{j heta}=rac{V_{max}}{\sqrt{2}}e^{j heta}= rac{V_{max}}{\sqrt{2}}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{max}}{\sqrt{2}e^{j\phi}= rac{V_{ma$$

$$I_{rms}e^{j\phi}=rac{V_{max}}{\sqrt{2}}e^{j\phi}=$$

Time domain and Phasor domain



A cos wt function is converted into a rotating vector by Euler

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$$v(t) = 170\cos(377t - 40^{\circ})V \longleftrightarrow \bar{V} = 170\angle - 40^{\circ}$$

 $i(t) = 10\sin(1000t + 20^{\circ})A \longleftrightarrow \bar{I} = 10\angle - 70^{\circ}$
 $\bar{V} = 86.3\angle 26^{\circ}V \longleftrightarrow v(t) = 86.3\cos(\omega t + 26^{\circ})V$

where $\sin(z) = \cos(z - 90^{\circ})$. And more examples to practice.

Summarv

	$R, jwL, \frac{1}{jwC}$	
\uparrow Phasor $M \angle \theta$	$R, jwL, \frac{1}{jwC}$	$\Downarrow e^{j(wt+ heta)}$
\uparrow Euler $e^{j(wt+\theta)}$		$\Downarrow Re(e^{j(wt+\theta)})$

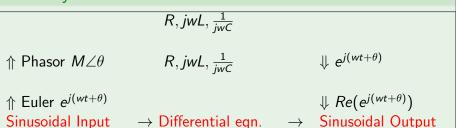
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where $sin(z) = cos(z - 90^{\circ})$. And more examples to practice.

Summary

	$R, jwL, \frac{1}{jwC}$		
Complex input	$ ightarrow ar{Z}$, Algebraic eqn.	\rightarrow	Complex output
\uparrow Phasor $M\angle\theta$	$R, jwL, \frac{1}{iwC}$		$\Downarrow e^{j(wt+\theta)}$
Exponential Input	$ ightarrow ar{Z}$, Algebraic eqn.	\rightarrow	Exponential Output
\Uparrow Euler $e^{j(wt+\theta)}$			$\Downarrow Re(e^{j(wt+\theta)})$
Sinusoidal Input	\rightarrow Differential eqn.	\rightarrow	Sinusoidal Output

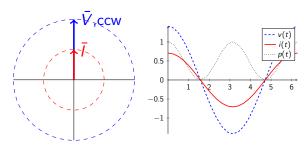
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Resistor, \bar{I}_R and \bar{V}_R are in phase

Assumed Form: $v(t) = V_{\text{max}} \cos(\omega t + \theta) = \sqrt{2} V_{rms} \cos(\omega t + \theta)$ Assume $\theta = 90^{\circ}$ for voltage. Could be any angle v = iR, $i(t) = \frac{V_{\text{max}}}{R} \cos(\omega t + \theta) = I_{\text{max}} \cos(wt + 90^{\circ})$

$$\frac{1}{+} \bigvee_{V} \bigvee_{-}$$

$$p(t) = v(t)i(t) = V_{\text{max}}I_{\text{max}}cos^{2}(wt)$$



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Resistor, \bar{I}_R and \bar{V}_R are in phase

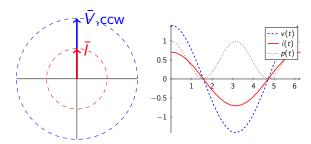
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$$-\sqrt{\sqrt{+}}$$

$$\bar{I} \uparrow^{\bar{V}}$$



$$p(t) = v(t)i(t) = V_{\text{max}}I_{\text{max}}cos^{2}(wt)$$



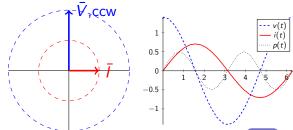
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Inductor, \bar{I}_L lags \bar{V}_L by 90°

Assumed Form: $v(t) = V_{\text{max}} \cos(\omega t + \theta) = \sqrt{2} V_{rms} \cos(\omega t + \theta)$ Assume $\theta = 90^{\circ}$ for voltage. Could be any angle $v = i(jwL), i(t) = \frac{1}{iwL} V_{\text{max}} \cos(\omega t + 90^{\circ}) = I_{max} \sin(\omega t)$

$$\bigcup_{j=1}^{\bar{V}} \bar{I}$$

$$p(t) = v(t)i(t) = V_{\max}I_{\max}\cos(wt)\sin(wt) = \frac{V_{\max}I_{\max}}{2}\sin(2wt)$$



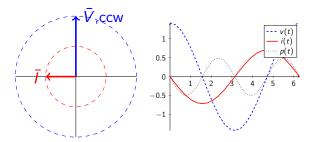
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Capacitor, \bar{I}_C leads \bar{V}_C by 90°

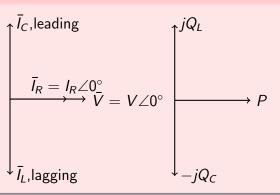
Assumed Form: $v(t) = V_{\text{max}} \cos(\omega t + \theta) = \sqrt{2} V_{rms} \cos(\omega t + \theta)$ Assume $\theta = 90^{\circ}$ for voltage. Could be any angle $v = i(\frac{1}{iwC}), i(t) = jwCV_{\text{max}} \cos(\omega t + 90^{\circ}) = -I_{\text{max}} \sin(\omega t)$

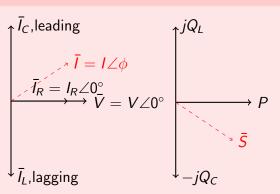
$$\bar{l}$$

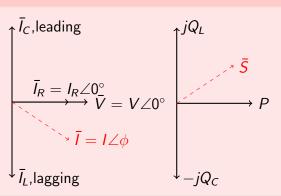
$$p(t) = v(t)i(t) = -V_{\text{max}}I_{\text{max}}\cos(wt)\sin(wt) = -\frac{V_{\text{max}}I_{\text{max}}}{2}\sin(2wt)$$



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We have discussed the special cases for R, L and C. How about a more general case? Say, combination of RLC.

Instantaneous power

$$p(t) = V_{\text{max}} \cos(wt + \theta) I_{\text{max}} \cos(wt + \phi)$$

$$= V_{\text{rms}} I_{\text{rms}} \cos(\theta - \phi) \{1 + \cos(2wt + 2\phi)\}$$

$$+ V_{\text{rms}} I_{\text{rms}} \sin(\theta - \phi) \sin(2wt + 2\phi)$$

Average real/reactive power

$$P_{ ext{ave}} = rac{1}{2\pi} \int_0^{2\pi} p(au) au = V_{ ext{rms}} I_{ ext{rms}} \cos(heta - \phi) \; W$$
 $Q_{ ext{ave}} = rac{1}{2\pi} \int_0^{2\pi} p(au) au = 0 \; VAR$

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Complex power

$$p(t) = \frac{V_{rms}I_{rms}}{V_{rms}I_{rms}}\cos(\theta - \phi)\{1 + \cos(2wt + 2\phi)\}$$
$$+ \frac{V_{rms}I_{rms}\sin(\theta - \phi)\sin(2wt + 2\phi)}{V_{rms}I_{rms}\sin(\theta - \phi)\sin(2wt + 2\phi)}$$

$$\bar{S} = P + jQ = \bar{V}\bar{I}^* = (V_{rms}\angle\theta)(I_{rms}\angle\phi)^* = V_{rms}I_{rms}\angle(\theta - \phi)$$
$$= V_{rms}I_{rms}\cos(\theta - \phi) + jV_{rms}I_{rms}\sin(\theta - \phi)$$

Since $\bar{V} = \bar{I}\bar{Z}$ (Ohm's law in AC), we have

$$\bar{S} = P + jQ = \bar{V}\bar{I}^* = (\bar{I}\bar{Z})\bar{I}^* = I_{rms}^2(R + jX) = I_{rms}^2R + jI_{rms}^2X
= \bar{V}\left(\frac{\bar{V}}{\bar{Z}}\right)^* = \frac{V_{rms}^2}{R - jX} = \frac{V_{rms}^2R}{Z^2} + j\frac{V_{rms}^2X}{Z^2}$$

If X = 0, it becomes a DC formula.

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Apparent power

$$S = V_{rms}I_{rms} = \frac{V_{max}}{\sqrt{2}}\frac{I_{max}}{\sqrt{2}} = \frac{1}{2}V_{max}I_{max}$$

Power triangle

$$\begin{array}{c|c}
+jQ_L & \overline{S} = S \angle \theta_z \\
\hline
P & -jQ_C & \overline{S} = S \angle \theta_z
\end{array}$$

$$P = V_{rms}I_{rms}\cos(\theta - \phi) = I_{rms}^2R = V_{rms}^2R/Z^2$$

$$Q = V_{rms}I_{rms}\sin(\theta - \phi) = I_{rms}^2X = V_{rms}^2X/Z^2$$

$$\theta_z = \theta - \phi = \cos^{-1}pf$$

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Apparent power

$$S = V_{rms}I_{rms} = \frac{V_{max}}{\sqrt{2}}\frac{I_{max}}{\sqrt{2}} = \frac{1}{2}V_{max}I_{max}$$

Power triangle

$$+jQ_{L}$$

$$\bar{S} = S \angle \theta_{z}$$

$$-jQ_{C}$$

$$\bar{S} = S \angle \theta_{z}$$

$$P = V_{rms}I_{rms}\cos(\theta - \phi) = I_{rms}^{2}R = V_{rms}^{2}R/Z^{2}$$

$$Q = V_{rms}I_{rms}\sin(\theta - \phi) = I_{rms}^{2}X = V_{rms}^{2}X/Z^{2}$$

$$\theta_{z} = \theta - \phi = \cos^{-1}pf$$

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Leading/Lagging power factor

Based on the analysis in the previous page, it is clear that the power factor is the cosine value of the impedance angle θ_z .

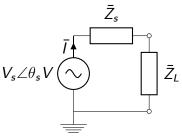
 $\bar{Z}=R+jX=R+j(wL-\frac{1}{wC})=\sqrt{R^2+X^2}\angle tan^{-1}\frac{wL-\frac{1}{wC}}{R}$. For a pure resistor network: X=0, $\theta_z=0$, $pf=\cos\theta_z=1$, pointing to the right.

For a pure inductor network: X = L, $\theta_z > 0$, $pf = \cos \theta_z < 1$, pointing upward.

For a pure capacitor network: X = C, $\theta_z < 0$, $pf = \cos \theta_z < 1$, pointing downward.

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To maximize the power received at the load end.



which implies that

$$\overline{Z_L} = R_L + jX_L = R_s - jX_s = \overline{Z_s^*}$$

$$P_L = I^2 R_L = \frac{V_{s,rms}^2}{4R_s} = \frac{V_{s,max}^2}{8R_s}$$

$$\bar{I} = \frac{\bar{V}_s}{\bar{Z}_s + \bar{Z}_L} = \frac{V_s \angle \theta_s}{(R_s + R_L) + \hat{J}(X_s + X_L)}$$

$$I = \frac{V_s}{\sqrt{(R_s + R_L)^2 + (X_s + X_L)^2}}$$

$$P_L = I^2 R_L = \frac{V_s^2 R_L}{(R_s + R_L)^2 + (X_s + X_L)^2}$$

