Supplementary Materials for "Speeding Up EM-type Algorithms via Surrogation for MLE of the Generalized Hyperbolic Distribution"

Chenyu Gao and Ziping Zhao

APPENDIX A
PROOF OF THE VARIATIONAL REPRESENTATIONS

Proof: We first prove the following inequality

$$-\log K_\lambda(\underline{\delta\gamma}) \geq \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\underline{\lambda}^2}\lambda^2 + \text{const.,} \tag{1}$$

where the equality holds for $\lambda = \underline{\lambda}$. To prove the inequality, a result is introduced.

Fact 1. *Denote functions $f_1(x)$ and $f_2(x)$. A sufficient condition for*

$$f_1(x) \geq f_2(x), \ \text{with} \ f_1(\underline{x}) = f_2(\underline{x}) \ \text{and} \ f_1'(\underline{x}) = f_2'(\underline{x}),$$

is given by

$$\begin{cases} g(x) > 0 & x > \underline{x}, \\ g(x) < 0 & x < \underline{x}. \end{cases} \tag{2}$$

where $g(x) = f_1'(x) - f_2'(x)$. Besides, if $g(x)$ is monotonically increasing, the condition (2) holds true.

For inequality (1), we compute the difference of the first derivative of the both sides with respect to λ as

$$\begin{aligned} g_1(\lambda) &= \frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial \lambda} - 2\frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\underline{\lambda}^2}\lambda \\ &= \frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial \lambda} - \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial \underline{\lambda}}\frac{\lambda}{\underline{\lambda}} \\ &= \lambda\left(\frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial \lambda}\frac{1}{\lambda} - \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial \underline{\lambda}}\frac{1}{\underline{\lambda}}\right) \\ &= 2\lambda\left(\frac{\partial \log K_\lambda(\underline{\delta\gamma})}{\partial \lambda^2} - \frac{\partial \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial \underline{\lambda}^2}\right), \end{aligned}$$

with $g_1(\underline{\lambda}) = 0$.

Consider the case where $\lambda > 0$ and $\underline{\lambda} > 0$. According to [1, Theorem 1], the function $-\log K_\lambda(x)$ is convex to λ^2 on $\lambda > 0$. Hence, $g_1(\lambda)$ is monotonically increasing. Based on Lemma 1, the inequality (1) is valid for $\lambda > 0$ and $\underline{\lambda} > 0$. Since the functions on both sides of (1) are even, the upperbound is also valid for all $\lambda \neq 0$ and $\underline{\lambda} \neq 0$.

Now we consider proving the other inequality.

$$-\log K_{\underline{\lambda}}(\delta\gamma) \leq \frac{1}{2}\frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial \log(\underline{\delta\gamma})}\log(\delta^2\gamma^2) + \text{const.,} \tag{3}$$

where the equality holds for both $\delta = \underline{\delta}$ and $\gamma = \underline{\gamma}$.

For inequality (3), the difference of the first derivative of the both sides with respect to $\gamma\delta$ is given by[1]

$$\begin{aligned} g_2(\delta\gamma) &= -\frac{K_{\underline{\lambda}}'(\delta\gamma)}{K_{\underline{\lambda}}(\delta\gamma)} + \frac{\nu}{\delta\gamma} \\ &= -\frac{K_{\underline{\lambda}}'(\delta\gamma)}{K_{\underline{\lambda}}(\delta\gamma)} + \frac{K_{\underline{\lambda}}'(\underline{\delta\gamma})}{K_{\underline{\lambda}}(\underline{\delta\gamma})}\frac{\underline{\delta\gamma}}{\delta\gamma} \\ &= -\frac{1}{\delta\gamma}\left(\gamma\delta\frac{K_{\underline{\lambda}}'(\delta\gamma)}{K_{\underline{\lambda}}(\delta\gamma)} - \underline{\delta\gamma}\frac{K_{\underline{\lambda}}'(\underline{\delta\gamma})}{K_{\underline{\lambda}}(\underline{\delta\gamma})}\right). \end{aligned}$$

Since the function $x\frac{K_{\underline{\lambda}}'(x)}{K_{\underline{\lambda}}(x)}$ is strictly decreasing on $x > 0$, by Lemma 1, the inequality (3) is valid. ∎

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[1]Throughout this paper, $\frac{\partial}{\partial x}K_\lambda(x)$ is denoted as $K_\lambda'(x)$.

APPENDIX B
PROOF OF LEMMA 1

Proof: Since both $f(x)$ and $f_2(x)$ are convex and $f_1(x)$ is concave, we have these two surrogate functions as follows:

$$f(x) \geq f_1(\underline{x}) + f_1'(\underline{x})(x - \underline{x}) + f_3(x|\underline{x}),$$

and

$$f(x) \geq f_1(x) + f_3(x|\underline{x}).$$

Denote the two surrogate functions on the right-hand side of the inequalities as $g_1(x)$ and $g_2(x)$. Now we will compare which surrogate function is tighter, i.e., the positivity of the following expression

$$g_1(x) - g_2(x) = f_1(\underline{x}) - f_1(x) + f_1'(\underline{x})(x - \underline{x}).$$

Since $f_1(x)$ is concave, we have

$$f_1(x) \leq f_1(\underline{x}) + f_1(\underline{x})(x - \underline{x}).$$

Hence, we have $g_1(x) - g_2(x) \leq 0$ and the proposition is valid. ∎

APPENDIX C
PROOF OF PROPOSITION 3

Proof: Without loss of generality, in this proof, we assume $n = 1$. For the function

$$\phi(\delta, \gamma) = \underline{\lambda} \log \frac{\gamma}{\delta} - \log K_{\underline{\lambda}}(\delta\gamma),$$

we define that $\phi(\delta) = \phi(\delta, \gamma = \underline{\gamma})$ and $\phi(\gamma) = \phi(\delta = \underline{\delta}, \gamma)$. The second derivative for $\phi(\delta)$ can be computed asFor $\underline{\lambda} > \frac{1}{2}$, the second derivative of $f(\delta)$ is given by

$$\phi''(\delta) = \frac{\underline{\lambda}}{\delta^2} - \frac{K_{\underline{\lambda}}''\left(\gamma\delta\right)}{K_{\underline{\lambda}}\left(\gamma\delta\right)}\underline{\gamma}^2 + \left(\frac{K_{\underline{\lambda}}'\left(\gamma\delta\right)}{K_{\underline{\lambda}}\left(\gamma\delta\right)}\gamma\right)^2.$$

Since the second derivative of Bessel function satisfies

$$K_{\lambda}''(x) = (1 + \frac{\lambda^2}{x^2})K_{\lambda}(x) - \frac{K_{\lambda}'(x)}{x}, \tag{4}$$

we have

$$\phi''(\delta) = -\frac{\underline{\lambda} + \delta^2\gamma^2 + \underline{\lambda}^2}{\delta^2} + \frac{K_{\underline{\lambda}}'\left(\delta\underline{\gamma}\right)}{K_{\underline{\lambda}}\left(\delta\underline{\gamma}\right)}\frac{\delta}{\gamma} + \left(\frac{K_{\underline{\lambda}}'\left(\delta\underline{\gamma}\right)}{K_{\underline{\lambda}}\left(\delta\underline{\gamma}\right)}\gamma\right)^2.$$

For the function $\frac{K_{\lambda}'(x)}{K_{\lambda}(x)}$, we have the bounds on $|\lambda| > \frac{1}{2}$ [2] as

$$-\frac{1 + \sqrt{(|\lambda| - 1)^2 + x^2}}{x} \leq \frac{K_{\lambda}'(x)}{K_{\lambda}(x)} \leq -\frac{\frac{1}{2} + \sqrt{(|\lambda| - \frac{1}{2})^2 + x^2}}{x}. \tag{5}$$

By the above bounds, we can obtain

$$\phi''(\delta) > -\frac{-\underline{\lambda} + \delta^2\underline{\gamma}^2 + \underline{\lambda}^2}{\delta^2} - \frac{\gamma}{\delta}\frac{1 + \sqrt{(|\underline{\lambda}| - 1)^2 + \delta^2\underline{\gamma}^2}}{\delta\gamma}$$

$$+ \left(-\frac{\frac{1}{2} + \sqrt{(|\underline{\lambda}| - \frac{1}{2})^2 + \delta^2\underline{\gamma}^2}}{\delta\underline{\gamma}}\underline{\gamma}\right)^2$$

$$= -\frac{-2\underline{\lambda} + \frac{1}{2} + h(\delta, \gamma)}{\delta^2},$$

where

$$h(\delta, \gamma) = \sqrt{(|\underline{\lambda}| - 1)^2 + \delta^2\gamma^2} - \sqrt{(|\underline{\lambda}| - \frac{1}{2})^2 + \delta^2\gamma^2}$$

$$= \frac{-|\underline{\lambda}| + \frac{3}{4}}{\sqrt{(|\underline{\lambda}| - 1)^2 + \delta^2\gamma^2} + \sqrt{(|\underline{\lambda}| - \frac{1}{2})^2 + \delta^2\gamma^2}}. \tag{6}$$

Hence, we can obtain that $\phi(\delta)$ is a convex function by the following two cases:

1) When $\underline{\lambda} > \frac{3}{4}$, we have $h(\delta, \underline{\gamma}) < 0$ so that $\phi''(\delta) > 0$.

2) When $\frac{1}{2} < \underline{\lambda} < \frac{3}{4}$, we have $h(\delta, \gamma)$ is strictly decreasing on both δ and γ, i.e., $0 < h(\delta, \underline{\gamma}) < h(0, \underline{\gamma}) = -2|\underline{\lambda}| + \frac{3}{2}$. Hence, we have

$$\phi''(\delta) > -\frac{-2\underline{\lambda} + \frac{1}{2} + h(\delta, \underline{\gamma})}{\delta^2} > -\frac{-4\underline{\lambda} + 2}{\delta^2} > 0.$$

Then for $\underline{\lambda} < -\frac{1}{2}$, the second derivative for $\phi(\gamma)$ can be computed as

$$\phi''(\gamma) = -\frac{\underline{\lambda}}{\gamma^2} - \frac{K''_{\underline{\lambda}}\left(\delta\gamma\right)}{K_{\underline{\lambda}}\left(\delta\gamma\right)}\delta^2 + \left(\frac{K'_{\underline{\lambda}}\left(\delta\gamma\right)}{K_{\underline{\lambda}}\left(\delta\gamma\right)}\delta\right)^2$$
$$> -\frac{2\underline{\lambda} + \frac{1}{2} + h(\underline{\delta}, \gamma)}{\gamma^2}.$$

Therefore, we can similarly obtain that $\phi''(\gamma)$ is convex on $\gamma > 0$ and $\underline{\lambda} > \frac{1}{2}$. ∎

References

[1] Á. Baricz and S. Ponnusamy, "On Turán type inequalities for modified Bessel functions," *Proceedings of the American Mathematical Society*, vol. 141, no. 2, pp. 523–532, 2012.

[2] J. Segura, "Simple bounds with best possible accuracy for ratios of modified Bessel functions," *Journal of Mathematical Analysis and Applications*, vol. 526, no. 1, p. 127211, 2023.