Supplementary Materials for "Speeding Up EM-type Algorithms via Surrogation for MLE of the Generalized Hyperbolic Distribution"

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Appendix A
Proof of the variational representations

Proof: We first prove the following inequality

$$-\log K_\lambda(\underline{\delta\gamma}) \geq \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\lambda^2}\lambda^2 + \text{const.}, \tag{1}$$

where the equality holds for $\lambda = \underline{\lambda}$. To prove the inequality, a result is introduced.

Remark 1. Denote functions $f_1(x)$ and $f_2(x)$. A sufficient condition for

$$f_1(x) \geq f_2(x), \text{ with } f_1(\underline{x}) = f_2(\underline{x}) \text{ and } f_1'(\underline{x}) = f_2'(\underline{x}),$$

is given by

$$\begin{cases} g(x) > 0 & x > \underline{x}, \\ g(x) < 0 & x < \underline{x}. \end{cases} \tag{2}$$

where $g(x) = f_1'(x) - f_2'(x)$. Besides, if $g(x)$ is monotonically increasing, the condition (2) holds true.

For inequality (1), we compute the difference of the first derivative of the both sides with respect to λ as

$$\begin{aligned} g_1(\lambda) &= \frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial\lambda} - 2\frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\lambda^2}\lambda \\ &= \frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial\lambda} - \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial\lambda}\frac{\lambda}{\underline{\lambda}} \\ &= \lambda\left(\frac{\partial - \log K_\lambda(\underline{\delta\gamma})}{\partial\lambda}\frac{1}{\lambda} - \frac{\partial - \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial\underline{\lambda}}\frac{1}{\underline{\lambda}}\right) \\ &= 2\lambda\left(\frac{\partial \log K_\lambda(\underline{\delta\gamma})}{\partial\lambda^2} - \frac{\partial \log K_{\underline{\lambda}}(\underline{\delta\gamma})}{\partial\underline{\lambda}^2}\right), \end{aligned}$$

with $g_1(\underline{\lambda}) = 0$.

Consider the case where $\lambda > 0$ and $\underline{\lambda} > 0$. According to [1, Theorem 1], the function $-\log K_\lambda(x)$ is convex to λ^2 on $\lambda > 0$. Hence, $g_1(\lambda)$ is monotonically increasing. Based on Lemma 1, the inequality (1) is valid for $\lambda > 0$ and $\underline{\lambda} > 0$. Since the functions on both sides of (1) are even, the upperbound is also valid for all $\lambda \neq 0$ and $\underline{\lambda} \neq 0$.

Now we consider proving the other inequality.

$$-\log K_{\underline{\lambda}}\Big(\delta\gamma\Big) \leq \frac{1}{2}\frac{\partial - \log K_{\underline{\lambda}}\Big(\underline{\delta\gamma}\Big)}{\partial \log(\underline{\delta\gamma})}\log(\delta^2\gamma^2) + \text{const.}, \tag{3}$$

where the equality holds for both $\delta = \underline{\delta}$ and $\gamma = \underline{\gamma}$.

For inequality (3), the difference of the first derivative of the both sides with respect to $\gamma\delta$ is given by[1]

$$\begin{aligned} g_2(\delta\gamma) &= -\frac{K_{\underline{\lambda}}'\Big(\delta\gamma\Big)}{K_{\underline{\lambda}}\Big(\delta\gamma\Big)} + \frac{\nu}{\delta\gamma} \\ &= -\frac{K_{\underline{\lambda}}'\Big(\delta\gamma\Big)}{K_{\underline{\lambda}}\Big(\delta\gamma\Big)} + \frac{K_{\underline{\lambda}}'\Big(\underline{\delta\gamma}\Big)}{K_{\underline{\lambda}}\Big(\underline{\delta\gamma}\Big)}\frac{\underline{\delta\gamma}}{\delta\gamma} \\ &= -\frac{1}{\delta\gamma}\left(\gamma\delta\frac{K_{\underline{\lambda}}'\Big(\delta\gamma\Big)}{K_{\underline{\lambda}}\Big(\delta\gamma\Big)} - \underline{\delta\gamma}\frac{K_{\underline{\lambda}}'\Big(\underline{\delta\gamma}\Big)}{K_{\underline{\lambda}}\Big(\underline{\delta\gamma}\Big)}\right). \end{aligned}$$

Since the function $x\frac{K_{\underline{\lambda}}'(x)}{K_{\underline{\lambda}}(x)}$ is strictly decreasing on $x > 0$, by Lemma 1, the inequality (3) is valid. ■

[1]Throughout this paper, $\frac{\partial}{\partial x}K_\lambda(x)$ is denoted as $K_\lambda'(x)$.

APPENDIX B
PROOF OF LEMMA 1

Proof: Since both $f(x)$ and $f_2(x)$ are convex and $f_1(x)$ is concave, we have these two surrogate functions as follows:

$$f(x) \geq f_1(\underline{x}) + f_1'(\underline{x})(x - \underline{x}) + f_3(x|\underline{x}),$$

and

$$f(x) \geq f_1(x) + f_3(x|\underline{x}).$$

Denote the two surrogate functions on the right-hand side of the inequalities as $g_1(x)$ and $g_2(x)$. Now we will compare which surrogate function is tighter, i.e., the positivity of the following expression

$$g_1(x) - g_2(x) = f_1(\underline{x}) - f_1(x) + f_1'(\underline{x})(x - \underline{x}).$$

Since $f_1(x)$ is concave, we have

$$f_1(x) \leq f_1(\underline{x}) + f_1(\underline{x})(x - \underline{x}).$$

Hence, we have $g_1(x) - g_2(x) \leq 0$ and the proposition is valid. ■

APPENDIX C
PROOF OF PROPOSITION 3

Proof: Without loss of generity, in this proof, we assume $N = 1$. For the function

$$\phi(\delta, \gamma) = \underline{\lambda} \log \frac{\gamma}{\delta} - \log K_{\underline{\lambda}}(\delta\gamma),$$

we define that $\phi(\delta) = \phi(\delta, \gamma = \underline{\gamma})$ and $\phi(\gamma) = \phi(\delta = \underline{\delta}, \gamma)$. The second derivative for $\phi(\delta)$ can be computed asFor $\underline{\lambda} > \frac{1}{2}$, the second derivative of $f(\delta)$ is given by

$$\begin{aligned}\phi''(\delta) &= \frac{\underline{\lambda}}{\delta^2} - \frac{K''_{\underline{\lambda}}\Big(\gamma\delta\Big)}{K_{\underline{\lambda}}\Big(\gamma\delta\Big)}\underline{\gamma}^2 + \left(\frac{K'_{\underline{\lambda}}\Big(\gamma\delta\Big)}{K_{\underline{\lambda}}\Big(\gamma\delta\Big)}\gamma\right)^2 \\ &> -\frac{-2\underline{\lambda} + \frac{1}{2} + h(\underline{\gamma}, \delta)}{\delta^2},\end{aligned}$$

Since the second derivative of Bessel function satisfies

$$K''_{\lambda}(x) = (1 + \frac{\lambda^2}{x^2})K_{\lambda}(x) - \frac{K'_{\lambda}(x)}{x}, \tag{4}$$

we have

$$\phi''(\delta) = -\frac{\underline{\lambda} + \delta^2\underline{\gamma}^2 + \underline{\lambda}^2}{\delta^2} + \frac{K'_{\underline{\lambda}}\Big(\delta\underline{\gamma}\Big)}{K_{\underline{\lambda}}\Big(\delta\underline{\gamma}\Big)}\frac{\delta}{\gamma} + \left(\frac{K'_{\underline{\lambda}}\Big(\delta\underline{\gamma}\Big)}{K_{\underline{\lambda}}\Big(\delta\underline{\gamma}\Big)}\underline{\gamma}\right)^2.$$

For the function $\frac{K'_{\lambda}(x)}{K_{\lambda}(x)}$, we have the bounds on $|\lambda| > \frac{1}{2}$ [2] as

$$-\frac{1 + \sqrt{(|\lambda| - 1)^2 + x^2}}{x} \leq \frac{K'_{\lambda}(x)}{K_{\lambda}(x)} \leq -\frac{\frac{1}{2} + \sqrt{(|\lambda| - \frac{1}{2})^2 + x^2}}{x}. \tag{5}$$

By the above bounds, we can obtain

$$\begin{aligned}\phi''(\delta) &> -\frac{-\underline{\lambda} + \delta^2\underline{\gamma}^2 + \underline{\lambda}^2}{\delta^2} - \frac{\gamma}{\delta}\frac{1 + \sqrt{(|\underline{\lambda}| - 1)^2 + \delta^2\underline{\gamma}^2}}{\delta\underline{\gamma}} \\ &\quad + \left(-\frac{\frac{1}{2} + \sqrt{(|\underline{\lambda}| - \frac{1}{2})^2 + \delta^2\underline{\gamma}^2}}{\delta\underline{\gamma}}\underline{\gamma}\right)^2 \\ &= -\frac{-2\underline{\lambda} + \frac{1}{2} + h(\delta, \underline{\gamma})}{\delta^2},\end{aligned}$$

where

$$h(\delta,\gamma) = \sqrt{(|\underline{\lambda}|-1)^2+\delta^2\gamma^2} - \sqrt{(|\underline{\lambda}|-\frac{1}{2})^2+\delta^2\gamma^2} \\ = \frac{-|\underline{\lambda}|+\frac{3}{4}}{\sqrt{(|\underline{\lambda}|-1)^2+\delta^2\gamma^2}+\sqrt{(|\underline{\lambda}|-\frac{1}{2})^2+\delta^2\gamma^2}}. \tag{6}$$

Hence, we can obtain that $\phi(\delta)$ is a convex function by the following two cases:

1) When $\underline{\lambda} > \frac{3}{4}$, we have $h(\delta,\underline{\gamma}) < 0$ so that $\phi''(\delta) > 0$.
2) When $\frac{1}{2} < \underline{\lambda} < \frac{3}{4}$, we have $h(\delta,\gamma)$ is strictly decreasing on both δ and γ, i.e., $0 < h(\delta,\underline{\gamma}) < h(0,\underline{\gamma}) = -2|\underline{\lambda}| + \frac{3}{2}$. Hence, we have

$$\phi''(\delta) > -\frac{-2\underline{\lambda}+\frac{1}{2}+h(\delta,\underline{\gamma})}{\delta^2} > -\frac{-4\underline{\lambda}+2}{\delta^2} > 0.$$

Then for $\underline{\lambda} < -\frac{1}{2}$, the second derivative for $\phi(\gamma)$ can be computed as

$$\phi''(\gamma) = -\frac{\underline{\lambda}}{\gamma^2} - \frac{K''_{\underline{\lambda}}\Big(\underline{\delta}\gamma\Big)}{K_{\underline{\lambda}}\Big(\underline{\delta}\gamma\Big)}\underline{\delta}^2 + \left(\frac{K'_{\underline{\lambda}}\Big(\underline{\delta}\gamma\Big)}{K_{\underline{\lambda}}\Big(\underline{\delta}\gamma\Big)}\underline{\delta}\right)^2 \\ > -\frac{2\underline{\lambda}+\frac{1}{2}+h(\underline{\delta},\gamma)}{\gamma^2}.$$

Therefore, we can similarly obtain that $\phi''(\gamma)$ is convex on $\gamma > 0$ and $\underline{\lambda} > \frac{1}{2}$. ■

References

[1] Á. Baricz and S. Ponnusamy, "On Turán type inequalities for modified Bessel functions," *Proceedings of the American Mathematical Society*, vol. 141, no. 2, pp. 523–532, 2012.

[2] J. Segura, "Simple bounds with best possible accuracy for ratios of modified Bessel functions," *Journal of Mathematical Analysis and Applications*, vol. 526, no. 1, p. 127211, 2023.