Examples paper 1 notes

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1 Brief summary of the four subspaces

A *vector subspace* is first of all a *vector space*. Let's recall what that is. A vector space is defined in your notes, and obeys the following:

- $\mathbf{0}$ is always in the space.
- If $\mathbf{a}$ and $\mathbf{b}$ are in the space, then the *linear combination* $\alpha\mathbf{a} + \beta\mathbf{b}$ is also in the space with α, β being scalars.

For example, one vector space is the set of all vectors with three real number components [1]. We can call that $\mathbb{R}^3$.

Now, every vector space can be represented by a *basis*. This is a small set of vectors that belong to the vector space that should represent the vector space. That is, we can construct all other vectors in the vector space as (unique) linear combinations of these vectors. One basis for $\mathbb{R}^3$ is

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{1}$$

because every vector in $\mathbb{R}^3$ can be represented uniquely as a linear combination of these three linearly independent vectors (please revise the definition and intuition behind linear independence if you are unfamiliar; how can we test for linear independence?). Another basis can be

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}. \tag{2}$$

This also contains three linearly independent vectors. Can we have more? No, because another 3-vector cannot possibly be linearly independent from the three already present. Can we have fewer? No, because then some vectors in $\mathbb{R}^3$ cannot be represented. In fact, the number of vectors in the basis is fixed for a vector space; this is called the *dimension* of the vector space. As a quick exercise, think about the following space and explain why it has dimension 2 (the dimension has nothing to do with the number of elements of the vectors contained in the space):

$$\left\{ \begin{bmatrix}a\\b\\0\end{bmatrix} \text{ for real numbers } a, b \right\}. \tag{3}$$

Usually, we want to work with *orthogonal* bases, meaning that the vectors are not only linearly independent but also orthogonal to each other (pairwise inner products are all zero).

[1] Let's just work with real numbers and vectors in this section. Generalisation to complex numbers does not require much more machinery.

Now, a vector subspace is a subset of a vector space, but itself also a vector space. For example, a subspace of $\mathbb{R}^3$ is the vector space spanned by

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}. \tag{4}$$

Another subspace is the vector space spanned by

$$\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{5}$$

In fact, these two subspace can be "added"[2] up to form $\mathbb{R}^3$ (think of it as a set union). Thus, one can say that one subspace is in fact the *complement* of the other. Furthermore, any vector from the first subspace is going to be orthogonal to the other (Why?). Because of this, we call these subspaces *orthogonal complements* of each other.

Matrices are associated with four fundamental subspaces: the column space, the left null space, the row space and the null space. First, consider the matrix

$$\boldsymbol{A} = \begin{bmatrix}1 & 2\\10 & 6\\1 & 3\\3 & 12\end{bmatrix}. \tag{6}$$

The column space is the vector space spanned by the columns. That is,

$$\text{colspan}(\boldsymbol{A}) = \text{span}\left\{\begin{bmatrix}1\\10\\1\\3\end{bmatrix}, \begin{bmatrix}2\\6\\3\\12\end{bmatrix}\right\}. \tag{7}$$

These two vectors are linearly independent and 4-dimensional, so this is a 2-dimensional subspace of $\mathbb{R}^4$. Can you find an orthogonal basis for this space? (Hint: Gram-Schmidt process or QR factorisation)

The left null space is the space of vectors

$$\text{leftnull}(\boldsymbol{A}) = \{\mathbf{x} \text{ such that } \mathbf{x}^T\boldsymbol{A} = 0\}. \tag{8}$$

Is this a vector space? Check that it satisfies the properties!

This space is in fact the orthogonal complement of the column space. Why? Consider $\mathbf{z}$ from the column space and $\mathbf{y}$ from the left null space. Compute the dot product,

$$\mathbf{y}^T\mathbf{z} = \mathbf{y}^T\boldsymbol{A}\mathbf{x} = 0. \tag{9}$$

This is because $\mathbf{z}$ is in the column space, so it can be expressed as $\boldsymbol{A}\mathbf{x}$ for some $\mathbf{x}$. Furthermore, $\mathbf{y}$ satisfies $\mathbf{y}^T\boldsymbol{A} = 0$ because it is from the left null space. This implies that any vector from the column space is orthogonal to any vector in the left null space, and vice versa. Some more food for thought: Why does this mean that $\mathbf{y} \neq \mathbf{z}$ except when both are zero (no non-zero vector can exist in both the column space and left null space!)?

The row space is the space spanned by the rows. For $\boldsymbol{A}$, we have four rows of length 2. They are also all pairwise linearly independent. We only need two linearly independent 2-vectors to span the $\mathbb{R}^2$ and this is more than enough, so the row space is the entirety of $\mathbb{R}^2$, dimension 2.

The null space is

$$\text{null}(\boldsymbol{A}) = \{\mathbf{x} \text{ such that } \boldsymbol{A}\mathbf{x} = 0\}. \tag{10}$$

Exercise for the reader: Go through the same logic as above and conclude that the null space is the orthogonal complement of the row space for any matrix. For $\boldsymbol{A}$, what is the dimension of the null space? List all of its elements.

[2] This is called the *direct sum*.

2 Determinants of transpose and conjugates

Satisfy yourself that

$$\begin{aligned} \det(\boldsymbol{A}) &= \det(\boldsymbol{A}^T) \\ \det(\overline{\boldsymbol{A}}) &= \overline{\det(\boldsymbol{A})}. \end{aligned} \tag{11}$$

3 On question 9

4 (From linear algebra) More on pseudoinverses

In the notes, we have defined $\boldsymbol{A}^+$ as follows:

$$\boldsymbol{A}^+ = \begin{cases} (\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H, & \text{if } \boldsymbol{A} \text{ is tall} \\ \boldsymbol{A}^H(\boldsymbol{A}\boldsymbol{A}^H)^{-1}, & \text{if } \boldsymbol{A} \text{ is fat} \end{cases} \tag{12}$$

Is this well-defined (i.e. do the matrix products make sense in terms of sizes? Are the matrices invertible whenever we indicate them?) Consider $(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H$ for a fat matrix. Can you explain why this would not work?