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Trees and extensive forms

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Abstract

This paper addresses the question of what it takes to obtain a well-defined extensive form game. Without relying on simplifying finiteness or discreteness assumptions, we characterize the class of game trees for which all pure strategy combinations induce unique outcomes. The generality of the set-up covers "exotic" cases, like stochastic games or decision problems in continuous time (differential games). We find that the latter class, though a well-defined problem, fails this test.

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1. Introduction

Non-cooperative game theory is the theory of games with complete rules. Its hallmark is a thought experiment that leaves all decisions exclusively to the players. Unlike cooperative game theory, where axioms like efficiency or symmetry restrict the solutions, or competitive equilibrium, where an auctioneer provides the agents with prices, a non-cooperative game provides an idealized 'world of its own,' where the effects of individual decisions can be studied without external intervention—a purely individualistic "interactive decision theory" [9, p. 460].

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The formal device to verify complete rules is the *extensive form*, as introduced by von Neumann and Morgenstern [27] and generalized by Kuhn [21]. It employs a *tree* to model the order of decisions by players—who can do what when—and how those generate an outcome. A tree is usually taken to be a directed connected graph without loops and with a distinguished node, the "root," where the game starts. To focus on conceptual issues, much of the theoretical literature on extensive forms tacitly assumes finite trees. (A notable early exception is Aumann [8], who generalizes Kuhn's theorem to infinite extensive form games.)

Though a tree is an ingenious representation of sequential decisions, a graph is not a natural domain for received decision theory. In the theory of decisions under uncertainty a decision maker chooses between maps from states to consequences ("acts") once and for all. Thus, neither is the domain of preferences naturally a graph, nor is the traditional decision theory one about sequential decision making.

Expected utility theory offers, however, an attractive separability between probability assignments over states and a Bernoulli utility function on consequences. This separability lends itself quite naturally to a dynamic re-interpretation. Consecutive decision problems can be described by restricting probability assignments to the states that remain possible (Bayes' rule), without affecting the utility of consequences.

The framework of expected utility—in particular, the independence axiom—is even more powerful, though. By replacing consequences with *sets* of consequences, a richer class of dynamic decision problems can be formalized. This approach has been used to characterize preference orderings that allow for a preference for flexibility [18], temptation and self-control [13,16], or unforeseen contingencies [12].

In short, classical decision theory has proved capable of formalizing sequential decision making on a domain that considers maps from sets of states to sets of consequences. Contrary to how it may first appear, this notion is perfectly compatible with the tree model of dynamic decision making. It has been shown elsewhere [3] that every decision tree—in particular, Kuhn's [21] graph—can be represented as a collection of sets (of plays, consequences, or outcomes) that is ordered by set inclusion, without loss of generality. The advantage of such a set representation is that nodes become sets of plays/states, thus "events" in the sense of decision theory.

The concept of set-trees, moreover, is so general that it encompasses all examples from the literature, including exotic cases like "differential games" (decision problems in continuous time), repeated games, stochastic games [24], infinite bilateral bargaining [23], and long cheap-talk [10].[2] Here we take advantage of this generality to address a more fundamental issue: When do strategies generate well-defined outcomes, so that players can make up their minds as to which strategy to employ?

It was shown in [3], for instance, that differential games can be rigorously defined as extensive form games with the aid of the general concept of a set-tree. But are they such that players can indeed always decide on the basis of a well-defined outcome being associated with every strategy? This is important, because if a non-cooperative game is meant to leave all decisions to players, then players ought to be able to evaluate their strategies. Solution concepts like Nash equilibrium are also defined in terms of strategy combinations.[3] Hence, it is essential that they yield an object in the domain of the players' preferences—a play or an outcome. If there are

[2] An extended account of these classes of games is given in [3, Section 2.2].

[3] This is in contrast to cooperative solution concepts that are often defined in terms of utility allocations or "imputations."

strategy combinations that "evaporate" (induce no outcome at all) or yield multiple outcomes, they cannot be evaluated by the decision makers.

In this paper we characterize extensive forms that satisfy the following two desiderata: (A1) Every strategy combination induces some outcome/play. (A2) The outcome/play induced by a given strategy combination is unique.

These desiderata take a global perspective. Completeness of rules is a local criterion in the sense that at each "when" it is clear "who can do what." The differential game, for instance, has an extensive form representation and, thereby, complete rules. But, as shown in this paper, its rules are such that they allow for decisions that do not combine to a definite outcome.

The results have two virtues. First, because they are characterizations, they make explicit which assumptions have been implicitly used—and which have to be used necessarily—in applications that push the limits of extensive form analysis beyond finite games. Second, the main characterization is in terms of easily verifiable properties of the tree. This makes it more tractable to ascertain whether or not a given extensive form game can be analyzed by classical methods. This is illustrated by the insights about differential games.

The two desiderata are also necessary and sufficient for an extensive form game to have a normal form representation. Many non-cooperative solution concepts are defined directly in the normal form—on the implicit assumption that strategies always yield well-defined outcomes. Such an approach is meaningful if and only if the two desiderata hold. Therefore, our results provide the first *characterization* of the class of trees for which the traditional techniques of game theory are appropriate.

A further motivation of this work is that extensive form games are often solved by backward induction procedures. Those require that strategies induce outcomes after every history (roughly, in every subgame) even if those are unreached (counterfactuals). It will be shown that this is guaranteed for the class of games described by the main characterization. Thus, indeed the trees identified in this paper provide an adequate domain for sequential decision theory.

The paper is organized as follows. Section 2 introduces the general concept of a game tree and provides motivating examples. Nodes in a game tree are also classified in this section. Trees on which extensive decision problems can be *defined* are characterized in Section 3. In Section 4 the two desiderata (A1) and (A2) are defined and strategies are discussed. As for (A1), we characterize in Section 5 a slightly stronger criterion: that every strategy induces an outcome after every history. Since subgames correspond to histories, this forms the basis on which backward induction concepts can be built. At the end of Section 5 a converse to (A1), that every play can be reached by some strategy combination, is shown to hold. Section 6 provides a characterization of (A2). Finally, in Section 7 we characterize the class of trees on which extensive decision problems satisfy both desiderata. Conclusions are in Section 8. All proofs, except for the main theorems, are in Appendix A.

2. Preliminaries and preview

2.1. Game trees

In working with decision trees there is no loss of generality in assuming a game tree (for details see [3]). The following definition is based on the idea that nodes may be represented by the sets of plays passing through them.

Definition 2.1. A **game tree** $T = (N, \supseteq)$ is a collection of nonempty subsets (called nodes) $x \in N$ of a given set W partially ordered by set inclusion such that[4]

(TI) "Trivial Intersection": if $x \cap y \neq \emptyset$ then $x \subset y$ or $y \subseteq x$ for all $x, y \in N$,
(IR) "Irreducibility": if $w, w' \in W$ are such that $w \neq w'$ then there are $x, x' \in N$ such that $w \in x \setminus x'$ and $w' \in x' \setminus x$,
(BD) "Boundedness": for every nonempty chain[5] $h \in 2^N$ there is $w \in W$ such that $w \in \bigcap_{x \in h} x$.

Given a game tree $(N, \supseteq)$ and a node $x \in N$ define the *up-set* (or *order filter*) $\uparrow x$ and the *down-set* (or *order ideal*) $\downarrow x$ by

$$\uparrow x = \{y \in N \mid y \supseteq x\} \quad \text{and} \quad \downarrow x = \{y \in N \mid x \supseteq y\}. \tag{1}$$

By TI $\uparrow x$ is a chain for all $x \in N$. A game tree is *rooted* if $W \in N$, which is henceforth assumed. A chain h in N is a *play* if it is maximal in N, i.e. there is no $x \in N \setminus h$ such that $h \cup \{x\}$ is a chain. For any chain in N there is a play that contains it. This is the Hausdorff Maximality Principle, a version of the Axiom of Choice.[6]

In game trees the set of plays and the underlying set W can be identified [3, Theorem 3]. A node can then be identified with the set of plays passing through it. Equivalently, the underlying set W represents plays. Therefore, an element $w \in W$ can be viewed either as an outcome (element of some node) or as a play (maximal chain of nodes). Whenever a distinction is in order, we write w for the outcome and $\uparrow\{w\}$ for the play (chain of nodes), where $\uparrow\{w\} = \{x \in N \mid w \in x\}$ is the play formed by all nodes containing w. Theorem 3 of [3] implies that, if h is a play, there exists a unique outcome $w \in W$ such that $\bigcap_{x \in h} x = \{w\}$, or, equivalently, $\uparrow\{w\} = h$.

An additional advantage of a game tree is that the singleton sets from the underlying set W can be added (e.g. for repeated games) to the set of nodes—as "terminal nodes"—without changing any essential features of the tree [3, Proposition 10]. Define a *complete* game tree as one where $\{w\} \in N$ for all $w \in W$.

We introduce now three properties of a game tree that are critical for the results. This takes some terminology: For a game tree $T = (N, \supseteq)$ a *filter* is a chain h in N such that $\uparrow x \subseteq h$ for all $x \in h$. A *history* is a nonempty filter that is not maximal in T, i.e. that is not a play. For a history h in T a *continuation* is the complement of h in a play that contains h.

Definition 2.2. A game tree $T = (N, \supseteq)$ is

(a) **weakly up-discrete** if all maximal chains in $\downarrow x \setminus \{x\}$ have maxima, for all nodes $x \in N$ for which $\downarrow x \setminus \{x\} \neq \emptyset$;
(b) **coherent** if every history without minimum has at least one continuation with a maximum;
(c) **regular** if every history of the form $\uparrow x \setminus \{x\}$ for $x \in N$ has an infimum.

Intuitively, the first property allows inductive arguments to proceed from any non-terminal node by guaranteeing existence of all immediate successors. The second requires that, after an infinite chain of decisions, (transfinite) induction is still able to proceed. This would be prevented by a "hole" in the tree, which amounts to a history without minimum whose continuations have

[4] The symbols $\supseteq, \subseteq$ denote weak inclusion, while $\supset, \subset$ indicate proper inclusion.
[5] A chain is a subset of N that is *completely* ordered by set inclusion.
[6] This is also known as Kuratowski's Lemma; see e.g. [17].

no maxima. The third property makes sure that any node can be "singled out" by (transfinite) induction.

These properties are "local." That is, to verify them one considers a history and looks at what happens "after" it. If the history is the up-set of a node, then weak up-discreteness requires that all of its continuations have a maximum. If the history has no minimum, then coherence asks for at least one continuation with a maximum. If the history consists of the proper predecessors of a node, then regularity requires that it has an infimum. What may go wrong if the properties fail is illustrated next.

2.2. *Examples and preview*

Game trees (and extensive decision problems) as defined above encompass all classical examples of games, from finite to infinitely repeated and stochastic games, and even "Long Cheap Talk" [10], where the length of the plays is $\omega + 1$, with ω being the first infinite ordinal. Many effects that arise in the abstract can be illustrated with instances from the following family that encompasses the, in many respects, simplest trees of all.

Example 1 *(Centipedes).* Let W be any completely ordered set, i.e. there is an order relation $\geqslant$ defined on W such that either $w \geqslant w'$ or $w' \geqslant w$ for all $w, w' \in W$. Define $x_t = \{\tau \in W \mid \tau \geqslant t\}$ for all $t \in W$, and let $N = \{(\{t\})_{t \in W}, (x_t)_{t \in W}\}$.

The tree $(N, \supseteq)$ is irreducible. Since the play $v_\infty = \{x_t\}_{t \in W}$ may not have a lower bound (e.g. if W is the set of natural numbers), the tree may not be bounded. This is, however, a case in which the tree can be completed by the addition of an "infinite" element which does not affect the order-theoretic structure [3, Proposition 8]. Formally, if W has a maximum, $w_\infty \geqslant w$ for all $w \in W$, it follows that $w_\infty \in x$ for all $x \in v_\infty$. Then, $(N, \supseteq)$ is bounded, i.e. it is a game tree. Since all singleton sets are nodes, it is even a complete game tree. We refer to this tree as the W-*centipede*.[7]

Decision problems in continuous time are often formalized as "differential games." Those tend to be implicitly defined as normal form games, where strategies and payoffs are taken as the primitives, and a differential equation is used to compute the payoffs from strategy combinations [15, p. 784]. Thus, the set of strategies is taken small enough so that payoffs can be computed, e.g. a certain set of Lipschitz continuous functions. The general concept of a game tree allows us to treat decision problems in continuous time as non-cooperative games in extensive form.

Example 2 *(Differential game).* Let W be the set of functions $f : \mathbb{R}_+ \to A$, where A is some fixed set of "actions," with at least two elements, and let $N = \{x_t(g) \mid g \in W,\ t \in \mathbb{R}_+\}$, where $x_t(g) = \{f \in W \mid f(\tau) = g(\tau), \forall \tau \in [0, t)\}$, for any $g \in W$ and $t \in \mathbb{R}_+$. Intuitively, at each point in time $t \in \mathbb{R}_+$ a decision $a_t \in A$ is taken. The "history" of all decisions in the past (up to, but exclusive of, time t) is a function $f : [0, t) \to A$, i.e. $f(\tau) = a_\tau$ for all $\tau \in [0, t)$. A node at time t is the set of functions that coincide with f on $[0, t)$, all possibilities still open for their values thereafter.

[7] This family of examples shows that there are (complete) game trees with plays of arbitrary cardinality. Consider a set W' of the appropriate cardinality, endow it with a well order (hence, a total order) by applying Zermelo's well-order theorem (see e.g. [17]), and adjoin a "top" (maximum) to it. If the resulting set is called W, the corresponding W-centipede proves the claim.

It is shown in [3] that $(N, \supseteq)$ is a game tree. At each node $x_t(f)$, the decision that an agent has to take is merely her action at time t. Ultimately, a function $f \in W$ becomes a complete description of all decisions taken from the beginning to the end.

This tree is coherent. For, if h is a history without a minimum, let $f \in \bigcap_{x\in h} x$ and $\tau = \sup\{t \in \mathbb{R}_+ \mid x_t(f) \in h\}$. By hypothesis $x_\tau(f) \notin h$ and, therefore, $x_\tau(f)$ provides a maximum for the continuation $\{x_t(f) \in N \mid t \geqslant \tau\}$. The tree is also regular, because $x_t(f) = \inf \uparrow x_t(f) \setminus \{x_t(f)\}$ for all nodes. But weak up-discreteness fails. For, if $x_\tau(g) \in \downarrow x_t(f) \setminus \{x_t(f)\}$ were a maximum for a continuation of $\uparrow x_t(f)$, then $g|_{[0,t)} = f|_{[0,t)}$ and $\tau > t$ would imply the contradiction $x_t(f) \supset x_{(t+\tau)/2}(g) \supset x_\tau(g)$.

The formalization offered in Example 2 differs from Simon and Stinchcombe [25], who view continuous time as "discrete time, but with a grid that is infinitely fine" (p. 1171). Stinchcombe [26] determines strategies and outcomes simultaneously to find the maximal set of strategies that give well defined outcomes for such problems. By contrast, we start with an unrestricted set of outcomes and will define strategies from the associated extensive form. Therefore, we do not employ any additional restrictions on strategies, like measurability or continuity.

The problems, which arise from pushing the limits of extensive form games beyond the confines of finite games, are often difficult to see in the differential game, but are easy to identify in the following examples. (The following three examples also show that the three properties from Definition 2.2 are independent.)

Example 3 *(Hole in the middle).* Let $x_t = [(t-1)/(4t), (3t+1)/(4t)]$, $y_t = [1/4, (2t-1)/(4t)]$, and $y'_t = [(2t+1)/(4t), 3/4]$ for all $t = 1, 2, \ldots$ and $W = [0,1]$, and let $N = \{(x_t)_{t=1}^\infty, (y_t)_{t=1}^\infty, (y'_t)_{t=1}^\infty, [1/4, 3/4], (\{w\})_{w\in W}\}$. So, $\{x_t\}_{t=1}^\infty$ is a decreasing sequence of intervals starting with $W = [0,1]$ and converging to $[1/4, 3/4] \in N$; $\{y_t\}_{t=1}^\infty$ resp. $\{y'_t\}_{t=1}^\infty$ are increasing sequences of intervals that start at $\{1/4\} \in N$ resp. $\{3/4\} \in N$ and converge to $[1/4, 1/2)$ resp. $(1/2, 3/4]$ (which are not nodes). This is a complete game tree. Since $\{1/2\} \in N$ provides a maximum for a continuation of the history $\{x_t\}_{t=1}^\infty$, this tree is coherent. It is regular, because $[1/4, 3/4] \in N$ provides a minimum for $\{x_t\}_{t=1}^\infty$ and $\{1/4\} \in N$ resp. $\{3/4\} \in N$ provide minima for $\{y_t\}_{t=1}^\infty$ resp. $\{y'_t\}_{t=1}^\infty$. But it is not weakly up-discrete, since the chains $\{y_t\}_{t=1}^\infty$ and $\{y_t\}_{t=1}^\infty$ in $\downarrow[1/4, 3/4] \setminus \{[1/4, 3/4]\}$ have no maxima.[8]

Consider the following single-player perfect information extensive form. Assign all nodes except the root as choices to the personal player and add $[1/4, 1/2) \cup (1/2, 3/4] = (\bigcup_{t=1}^\infty y_t) \cup (\bigcup_{t=1}^\infty y'_t)$ as a further choice (but not as a node), so that at $[1/4, 3/4]$ the player chooses between $[1/4, 1/2) \cup (1/2, 3/4]$ and (the node) $\{1/2\}$. There exists a strategy that assigns to any x_t the non-singleton choice x_{t+1}, to every y_t resp. y'_t the corresponding singleton choice $\{(2t-1)/(4t)\}$ resp. $\{(2t+1)/(4t)\}$, and to $[1/4, 3/4]$ the choice $[1/4, 1/2) \cup (1/2, 3/4]$. This strategy "continues" along $\{x_t\}_{t=1}^\infty$, discards $\{1/2\}$, and "stops" at every y_t resp. y'_t. And it induces no outcome/play!

This example has a coherent and regular, but *not* weakly up-discrete tree, and a strategy that induces no outcome at all. A modification illustrates that with a regular and weakly up-discrete tree, that is *not* coherent, the same problem may appear.

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8 This game tree also satisfies that $\bigcap\{x \mid x \in h\} \in N$ for every history h. This condition is equivalent to regularity and coherence. We are grateful to a referee for alerting us to this observation.

Example 4. Remove the nodes $[1/4, 3/4]$ and $\{1/2\}$ from N and the element $1/2$ from the underlying set W. The resulting tree is regular and now weakly up-discrete (because $[1/4, 3/4] \notin N$), but not coherent anymore. Consider a strategy that assigns to each x_t the choice x_{t+1}, to each y_t resp. y'_t the (singleton) choice $\{(2t-1)/(4t)\}$ resp. $\{(2t+1)/(4t)\}$, i.e., a strategy that "continues" at all x_t, but "stops" at all y_t and y'_t, for all t. Once again, this strategy does not induce any outcome at all.

Reinserting the node $\{1/2\}$ to N and the element $1/2$ to W in the last example yields a tree that is weakly up-discrete (because still $[1/4, 3/4] \notin N$) and coherent (because $\{1/2\} \in N$ provides a maximum for a continuation of the history $\{x_t\}_{t=1}^{\infty}$), but not regular anymore, since $\uparrow\{1/2\} \setminus \{1/2\}$ has no infimum.[9] As a result, a strategy can be constructed that induces multiple outcomes.

Example 5. Let $N = \{(x_t)_{t=1}^{\infty}, (y_t)_{t=1}^{\infty}, (y'_t)_{t=1}^{\infty}, (\{w\})_{w\in W}\}$ and consider again the single-player extensive form with perfect information. The strategy that assigns to each x_t the choice x_{t+1}, to each y_{t+1} the choice y_t, and to each y'_{t+1} the choice y'_t for all $t = 1, 2, \ldots$ always "continues." It is intuitively clear that this strategy selects the outcomes/plays $1/4, 1/2, 3/4 \in W$. The reason for this multiplicity is that there is no move, where a decision between $\bigcup_{t=1}^{\infty} y_t = [1/4, 1/2)$ and $\bigcup_{t=1}^{\infty} y'_t = (1/2, 3/4]$ is taken, because a node of the form $[1/4, 3/4]$ is missing.

Examples 3 and 4 illustrate that a weakly up-discrete and coherent game tree is necessary for an arbitrary strategy to induce an outcome. Example 5 shows that a regular tree is necessary for a strategy to induce a unique outcome. Sufficiency is a different matter, though, and requires more work. Details aside, two of the (four) characterization results below will show the following:

- Every strategy induces an outcome/play *after every history* if and only if the game tree is weakly up-discrete and coherent (Theorem 2 below).
- Every strategy induces a *unique* outcome/play if and only if the game tree is weakly up-discrete, coherent, and regular (Theorem 6 below).

To make it precise when, for instance, a strategy induces an outcome/play requires further ingredients. A key one, that concerns purely the tree, is introduced next.

2.3. A classification of nodes

Nodes that are properly followed by other nodes are called *moves*. That is, $X = \{x \in N \mid \downarrow x \setminus \{x\} \neq \emptyset\}$ is the set of all moves. Nodes that are not properly followed by other nodes are called *terminal*.

Lemma 1. *For a game tree $(N, \supseteq)$ a node $x \in N$ is terminal if and only if there is $w \in W$ such that $x = \{w\}$.*

Thus, when $(N, \supseteq)$ is a game tree, $X = N \setminus \{\{w\}\}_{w\in W}$ can be taken as an alternative definition of (the set of) moves. Yet, this result does not imply that $\{w\} \in N$ for all $w \in W$, unless $(N, \supseteq)$ is

[9] The node $\{1/2\}$ is a lower bound for the chain $(x_t)_{t=1}^{\infty}$, as is any of the nodes y_t and y'_t. Since $y_t \cap y'_t = \emptyset$, the chain $\uparrow\{1/2\} \setminus \{\{1/2\}\} = (x_t)_{t=1}^{\infty}$ has no infimum.

a complete game tree. If all chains are finite, then $(N, \supseteq)$ is a game tree *if and only if* it satisfies TI and all singleton sets are nodes [3, Proposition 12]. In particular, this always holds if W is finite. Yet, even if W is infinite, certain nodes may have properties analogous to the finite case.

Let $(N, \supseteq)$ be a game tree and $x \in N \setminus \{W\}$. Say that x is *finite* if $\uparrow x \setminus \{x\}$ has a minimum, *infinite* if $x = \inf \uparrow x \setminus \{x\}$, and *strange* if $\uparrow x \setminus \{x\}$ has no infimum. Denote by $S(N)$ the set of strange nodes of N. The three possibilities in this definition are exhaustive, i.e., all nodes (other than the root) are either finite, infinite, or strange. For, if $\uparrow x \setminus \{x\}$ has an infimum z, it is either a minimum (and then x is finite), or $z \notin \uparrow x \setminus \{x\}$. In the latter case, it follows by definition of an infimum that $z = x$.

Lemma 2. *For a game tree $(N, \supseteq)$ a node $x \in N \setminus \{W\}$ is*

(a) *the infimum of a chain $h \in 2^N$ if and only if $x = \bigcap_{y \in h} y$;*
(b) *infinite if and only if $x = \bigcap_{y \in \uparrow x \setminus \{x\}} y$;*
(c) *strange if and only if $\uparrow x \setminus \{x\}$ has no minimum and $x \subset \bigcap_{y \in \uparrow x \setminus \{x\}} y$.*

By Lemma 2(b) infinite nodes can be reconstructed from the other nodes. Therefore, an infinite node has infinitely many predecessors, justifying its name. For, if $\uparrow x \setminus \{x\}$ were finite, it would have a minimum $z \in \uparrow x \setminus \{x\}$, implying that x was finite.

Example 6. If in Example 1 W is the set of natural numbers together with "infinity" ∞, $W = \{1, 2, \ldots, \infty\}$, an "infinite centipede" emerges. Terminal nodes are singletons $\{t\}$ (including $\{\infty\}$), moves are of the form $x_t = \{t, t+1, \ldots, \infty\}$, except for $x_\infty = \{\infty\}$ which is terminal. All nodes, except the terminal node $\{\infty\}$, are finite. Only $\{\infty\}$ is infinite, since $\bigcap\{x_t \mid 1 \leqslant t < \infty\} = \{\infty\}$, yet $\{\infty\} \notin \{x_t \in N \mid t = 1, 2, \ldots\} = \uparrow\{\infty\} \setminus \{\{\infty\}\}$. This represents a never-ending chain of decisions.

If $W = [0, 1]$, a "continuous centipede" emerges. This is again a game tree, as $[0, 1]$ has a maximum. Nodes are either singletons $\{t\}$ or of the form $x_t = [t, 1]$. All singletons $\{t\}$ are finite except the "last" node, $\{1\}$. For, $\bigcap\{y \mid y \in \uparrow\{t\} \setminus \{\{t\}\}\} = \bigcap\{x_\tau \mid \tau < t\} = \bigcap_{\tau < t}[\tau, 1] = [t, 1] = x_t$, for any $t \leqslant 1$. The latter is only identical to $\{t\}$ if $t = 1$. All moves, though, are infinite. For, $\bigcap\{y \mid y \in \uparrow x_t \setminus \{x_t\}\} = \bigcap\{x_\tau \mid \tau < t\} = [t, 1] = x_t$, provided $t > 0$ (otherwise the intersection is empty).

Trivially, a game tree is regular if and only if there are no strange nodes. All centipedes are regular. An example with strange nodes is the "hole in the middle" (Example 5 with $\{1/2\} \in N$), where $\{1/2\} \in S(N)$. If nodes $y = [1/4, 1/2)$ and $y' = (1/2, 3/4]$ were added, these would also be strange.[10] If, however, a node $x = [1/4, 3/4]$ were added, then none of the previous would be strange, but $[1/4, 3/4]$ would be infinite. But this later addition changes the tree by adding a decision.

3. Extensive decision problems

In this section the general concept of an "extensive decision problem" is introduced, and the class of trees is characterized on which such problems can be defined.

[10] If a node x is strange, then any maximal lower bound of $\uparrow x \setminus \{x\}$ is also strange.

3.1. Definition

For a game tree with set W of plays and a subset $a \subseteq W$ of plays (not necessarily a node), the *down-set* of a is $\downarrow a = \{x \in N \mid x \subseteq a\}$, the *up-set* of a is $\uparrow a = \{x \in N \mid a \subseteq x\}$, and the set of *immediate predecessors* of $a \in 2^W$ is

$$P(a) = \{x \in N \mid \exists y \in \downarrow a\colon \uparrow x = \uparrow y \setminus \downarrow a\} \tag{2}$$

Since nodes in a game tree are sets of plays, they too may, but need not have immediate predecessors. In fact, for a node $x \in N$ in a game tree, $P(x) \neq \emptyset$ holds if and only if x is finite; in this case, $P(x) = \{\min \uparrow x \setminus \{x\}\}$.[11]

The definition of an extensive decision problem from [3, Definition 7] is given below. It captures what is often called a "game form."

Definition 3.1. An **extensive decision problem** (EDP) with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a game tree with set of plays W and $C = (C_i)_{i \in I}$ is a system consisting of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of plays) for all $i \in I$, such that the properties below hold. Let $A_i(x) = \{c \in C_i \mid x \in P(c)\}$ be the choices available to $i \in I$ at $x \in X$, and $J(x) = \{i \in I \mid A_i(x) \neq \emptyset\}$ the set of decision makers at x, which is required to be nonempty for all $x \in X$. Then, (T, C) satisfies

- (EDP.i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;
- (EDP.ii) $x \cap [\bigcap_{i \in J(x)} c_i] \neq \emptyset$ for all $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ and for all $x \in X$;
- (EDP.iii) if $y, y' \in N$ with $y \cap y' = \emptyset$ then there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$;
- (EDP.iv) if $x \supset y \in N$, then there is $c \in A_i(x)$ such that $y \subseteq c$ for all $i \in J(x)$, for all $x \in X$.

Briefly, the interpretation of the conditions in Definition 3.1 is as follows [3, Section 5, for additional details]. (EDP.i) stands in for information sets. In words, whenever two choices are simultaneously available at a common move, then they are disjoint and their immediate predecessors coincide; that is, whenever one of them is available, so is the other.[12] (EDP.ii) requires that simultaneous decisions by different players at a common move do select some outcome. (EDP.iii) states that for any two disjoint nodes, there must be a player (possibly chance) who can eventually take a decision that selects among them. Finally, (EDP.iv) states that, if a player takes a decision at a given node, he must be able not to discard any given successor of the node. This requirement excludes absent-mindedness [3, Proposition 13].

Note, first, that the focal objects in the definition of an EDP are the choices, rather than information sets. The latter can be recovered, though, by taking the immediate predecessors. (Condition (EDP.i) ensures that players cannot tell from the menu of available choices at which move in their information set they are.) Second, Definition 3.1 allows several players to choose at the same move. Cascading information sets are not required to model simultaneous decisions.

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[11] If x is finite, $P(x) = \{\min \uparrow x \setminus \{x\}\}$ by uniqueness of the minimum. Conversely, if $x' \in P(x)$ for $x \in N$, then there is $y \in \downarrow x$ such that $\uparrow x' = \uparrow y \setminus \downarrow x$; since $\uparrow y \setminus \downarrow x = \uparrow x \setminus \{x\}$ for all $y \in \downarrow x$, that $\uparrow x' = \uparrow x \setminus \{x\}$ implies $x' = \min \uparrow x \setminus \{x\}$, and x is finite.

[12] The player set I may contain a "chance" player $i = 0$, whose behavior models what is not under the control of personal players. Property (EDP.i) is not necessary for such a player.

This is advantageous if, for instance, a one-shot game with a continuum of players is modeled. With the present formalism the corresponding tree can be taken to consist of the root and all terminal nodes, as all players decide simultaneously at the root. By contrast, with the cascading information sets of the traditional formalism plays of uncountable length are required (for details see Example 8 of [5]).

3.2. Perfect information choices

Whenever it is possible to define an EDP on a tree, it should be possible to define a single-player perfect information game on the same tree, where the single player takes all possible decisions under the best possible information. Also, the single-player perfect information game should be determined by the tree *alone*, in the sense that there should exist a unique form of defining it.

If the tree were discrete, choices available to the single player at $x \in X$ under perfect information would be the immediate successors of x. In the abstract framework used here a more general construction is required. For any move $x \in X$ and a play $w \in x$ define the *perfect information choice* $\gamma(x, w) \subseteq W$ as the set of plays

$$\gamma(x, w) = \bigcup \{z \mid w \in z \in \downarrow x \setminus \{x\}\}, \tag{3}$$

i.e. the union of all proper successors of x that contain w. Note that by TI these successors form a chain.

Example 7. A typical example, where this construction goes beyond immediate successors is the differential game (Example 2). Given $g \in x_t(f) \in X$, notice that $x_t(g) = x_t(f)$ and let $g(t) = a \in A$. Then,

$$\gamma\big(x_t(f), g\big) = \bigcup \{z \in N \mid g \in z \in \downarrow x_t(f) \setminus \{x_t(f)\}\} = \{h \in x_t(g) \mid h(t) = a\}$$

i.e., perfect information choices at $x_t(f) \in X$ are

$$c_t(f, a) = \{g \in x_t(f) \mid g(t) = a\} = \gamma\big(x_t(f), g\big) \subset x_t(f)$$

for any $g \in x_t(f)$ with $g(t) = a \in A$. A single-player game with perfect information choices $\gamma(x_t(f), g)$ is easily seen to be an EDP.

Note that this is a different specification as by Stinchcombe [26], who allows the players to change their action only at a set of time-points that is order isomorphic to an initial segment of the countable ordinals.

The properties of perfect information choices determine when they can indeed be considered choices in a well-defined EDP.

Proposition 1. *Let $(N, \supseteq)$ be a game tree, let $x \in X$, and $w \in x$. Then*:

(a) *if $w' \in x$ and $\gamma(x, w) \cap \gamma(x, w') \neq \emptyset$, then $\gamma(x, w) = \gamma(x, w')$*;
(b) *if $\gamma(x, w) \subset x$, then $P(\gamma(x, w)) = \{x\}$ and there exists at least one $w' \in x$ such that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$*;
(c) *if $\gamma(x, w) = x$, then the chain $\{y \in \downarrow x \setminus \{x\} \mid w \in y\}$ has no maximum, and no choice can be available at x, hence, no EDP can be defined on $(N, \supseteq)$.*

The last impossibility suggests that, in order to have an EDP defined on a tree, an extra condition is needed. Say that a game tree $(N, \supseteq)$ has *available choices* if $\gamma(x, w) \subset x$ for all $w \in x$ and all $x \in X$. The terminology is motivated by the fact that, by Proposition 1(c), if $\gamma(x, w) = x$ then there can be no choice available at x, i.e. $J(x) = \emptyset$. If, by contrast, $\gamma(x, w) \subset x$, then the sets $\gamma(x, w)$ can serve as choices—at least in a perfect information game. This may not always be the case.

Example 8 *(Inverse infinite centipedes).* Let $W = \{\dots, -2, -1\}$ be the set of negative integers with the natural order and consider the corresponding centipede (which is a game tree by Example 1, as -1 provides a maximum). The set of plays for $(N, \supseteq)$ consists of sets of the form $\{(x_{-\tau})_{\tau=t}^{\infty}, \{-t\}\}$ for all $t = 1, 2, \dots$ (since $x_{-1} = \{-1\}$, the play $\{(x_{-\tau})_{\tau=1}^{\infty}\}$ is included). Every play $\{(x_{-\tau})_{\tau=t}^{\infty}, \{-t\}\}$ can be represented by the negative integer $-t$, for all $t = 1, 2, \dots$. But none of the chains $\{x_{-\tau}\}_{\tau=t}^{\infty}$ has a maximum. Therefore, the perfect information choice $\gamma(W, -1)$ coincides with the root itself, i.e. $\gamma(W, -1) = \bigcup_{t=1}^{\infty} x_{-t} = W$. By Proposition 1(c) it is impossible to define an EDP on this tree.

Proposition 1(c) gives a sufficient condition for available choices: that all maximal chains in $\downarrow x \setminus \{x\}$ have a maximum, $\forall x \in X$, viz. weak up-discreteness.

Corollary 1. *If a game tree is weakly up-discrete, then it has available choices.*

3.3. Existence of EDPs

Denote the set of perfect information choices for a game tree $T = (N, \supseteq)$ by $\Gamma(T) = \{\gamma(x, w) \mid w \in x \in X\}$ and recall that $S(N)$ denotes the set of strange nodes. By Proposition 1(b), if a game tree has available choices, then all choices in $\Gamma(T)$ are available at some node. The following theorem shows that available choices indeed *characterize* game trees on which EDPs can be defined.

Theorem 1. *Let $T = (N, \supseteq)$ be a (rooted) game tree. The following are equivalent*:

(a) *Some EDP (T, C) can be defined on T*;
(b) *T has available choices*;
(c) *$\Pi(T) = (T, C_1)$ is a well-defined single-player EDP, where $C_1 = \Gamma(T) \cup S(N)$.*

Proof. (c) trivially implies (a), and (a) implies (b) by Proposition 1(c). To show that (b) implies (c), suppose that $T = (N, \supseteq)$ has available choices. Let $I = \{1\}$ and $C_1 = \Gamma(T) \cup S(N)$. Then choices are unions of nodes by construction. Proposition 1(b) implies that $A_1(x) \neq \emptyset$, hence, $J(x) = \{1\} \neq \emptyset$ for all $x \in X$.

It remains to verify (EDP.i)–(EDP.iv). Property (EDP.ii) follows trivially from $J(x) = \{1\}$ for all $x \in X$. (EDP.iv) is also simple: Consider $x \in X$ and $x \supset y \in N$. Choose $w \in y$ and $c = \gamma(x, w)$. Then $y \subseteq c$ and $c \in A_1(x)$ by Proposition 1(b).

Consider now (EDP.i). If $P(c) \cap P(c') \neq \emptyset$, then Proposition 1(b) implies that $P(c) = \{x\} = P(c')$ for some $x \in X$, where $c = \gamma(x, w)$ and $c' = \gamma(x, w')$ for some $w, w' \in x$. That $c \cap c' = \emptyset$ then follows from Proposition 1(a).

Turn to (EDP.iii). Strange nodes are choices in C_1 by construction. We claim that finite nodes are also in C_1. Let y be a finite node and $x = \min \uparrow y \setminus \{y\}$. Let $c = \gamma(x, w)$ for some $w \in y$. By definition $y \subseteq c$. If $y \subset c \subset x$ (the latter by (b)), then there exists $z \in \downarrow x \setminus \{x\}$ such that $w \in y \subset z$, in contradiction to $x = \min \uparrow y \setminus \{y\}$. Therefore, $y = c$. Hence, (EDP.iii) holds trivially for pairs of non-infinite nodes. Consider now an infinite node y. Hence, $y = \inf \uparrow y \setminus \{y\}$. Consider any other node $y' \in N$ (infinite or not) such that $y \cap y' = \emptyset$. It follows that $y' \notin \uparrow y$. Then, there exists $z \in \uparrow y \setminus \{y\}$ such that $z \cap y' = \emptyset$. For, if not, TI would imply that $y' \subseteq z$ for all $z \in \uparrow y \setminus \{y\}$; by the definition of the infimum, it would follow that $y \supseteq y'$, a contradiction. Take now $c = \gamma(z, w)$ for some $w \in y$. It follows that $y \subseteq c$ and $c \cap y' = \emptyset$. Hence, (EDP.iii) is verified if y is infinite and y' is not, since then $y' \in C_1$. If y, y' are infinite disjoint nodes, then, as above, there exist a choice $c = \gamma(z, w)$ with $z \in \uparrow y \setminus \{y\}$, $z \cap y' = \emptyset$, $y \subseteq c \subseteq z$ and $c \cap y' = \emptyset$. Repeating the argument with the (disjoint) nodes z, y' we obtain a choice c' such that $y' \subseteq c'$ and $c' \cap z = \emptyset$, implying that $c \cap c' = \emptyset$. This completes the verification of (EDP.iii). $\square$

The strange nodes that are choices in $C_1 = \Gamma(T) \cup S(N)$ are never available. This makes the interpretation of Theorem 1 odd. Obviously, if the tree is regular, the single-player game only requires the choices $C_1 = \Gamma(T)$ that are always available.

Say that an EDP (T, C) has *quasi-perfect information* if $P(c) \neq \emptyset$ implies that $P(c) = \{x\}$ for some $x \in X$, for all $c \in C_i$ and all $i \in I$. Quasi-perfect information differs from the traditional notion of "perfect information" in an extensive form in two respects. First, there may be choices that are never available at any move; second, if there are several players, they may choose at the same node. That is, unavailable choices aside, quasi-perfect information corresponds to the standard concept of perfect information, except that several players may decide at the same move.

Theorem 1(c) (and Proposition 1(b)) reveals that if an EDP (T, C) can be defined on the game tree T at all, then there exists one with quasi-perfect information defined on T, which is unique up to an assignment of decision points to players. In particular, it is unique if a single player is assumed. The associated single player decision problem with perfect information is henceforth denoted $\Pi(T) = (T, C_1)$, as in Theorem 1(c).

4. Strategies and the desiderata

4.1. Pure strategies

A key object derived from an EDP is a *pure strategy* for a player $i \in I$. This is a function $s_i : X_i = \{x \in X \mid i \in J(x)\} \to C_i$ such that

$$s_i^{-1}(c) = P(c) \quad \text{for all } c \in s_i(X_i), \tag{4}$$

where $s_i(X_i) \equiv \bigcup_{x \in X_i} s_i(x)$. That is, s_i assigns to every move $x \in X_i$ a choice $c \in C_i$ such that (a) choice c is *available* at x, i.e. $s_i(x) = c \Rightarrow x \in P(c)$ or $s_i^{-1}(c) \subseteq P(c)$, and (b) to every move x in an *information set* $\xi = P(c)$ the same choice gets assigned, i.e. $x \in P(c) \Rightarrow s_i(x) = c$ or $P(c) \subseteq s_i^{-1}(c)$, for all $c \in C_i$ that are chosen somewhere, viz. $c \in s_i(X_i)$.[13] Let S_i denote the set of all pure strategies for player $i \in I$. A *pure strategy combination* is an element $s = (s_i)_{i \in I} \in S \equiv \times_{i \in I} S_i$.

If an EDP captures complete and consistent rules, then there are a few desiderata that need to be satisfied, like that every strategy combination ought to induce an outcome. First, of course, it

[13] If chance, $i = 0$, is part of the EDP, it is treated symmetrically to personal players in Definition 3.1, so there is no problem with defining pure strategies for chance as well.

has to be clarified when a strategy combination "induces" a play. To this end define, for every $s \in S$, the correspondence $R_s : W \to W$ by

$$R_s(w) = \bigcap \{ s_i(x) \mid w \in x \in X,\ i \in J(x) \}. \tag{5}$$

Say that the strategy combination s *induces* the play w if $w \in R_s(w)$, i.e., if w is a fixed point of R_s. The following are the two key desiderata on the mapping R_s:

(A1) For every $s \in S$ there is some $w \in W$ such that $w \in R_s(w)$.
(A2) If for $s \in S$ there is $w \in W$ such that $w \in R_s(w)$, then R_s has no other fixed point and $R_s(w) = \{w\}$.

The desiderata (A1) and (A2), together with preference profiles for the players, are precisely what is needed for a normal form representation of the game. In *finite* games there is no problem with the transition from the extensive to the normal form: Pure strategies are derived from the extensive form and every pure strategy combination is associated with precisely one play, so that a utility representation of the players' preferences over plays generates a utility profile for each pure strategy combination. In general (A1) defines a nonempty-valued correspondence $\phi : S \to W$ from strategies to plays. Under (A2) the map ϕ is a function that, in turn, defines the normal form.

4.2. *Randomized strategies*

This also enables a treatment of *mixed* strategies, that is, probability distributions on pure strategies. For the function ϕ defined by (A1) and (A2) the preimage of a set $V \subseteq W$ of plays, $\phi^{-1}(V)$, is unambiguously defined. A mixed strategy combination gives a (product) probability distribution σ on pure strategy combinations $s \in S$. Assuming that ϕ is measurable, the measure that σ associates to the preimage $\phi^{-1}(V)$ of a set V of plays gives the probability $\sigma(\phi^{-1}(V))$ of the set V of plays under the mixed strategy combination. In this sense (A1) and (A2) for pure strategies implicitly also cover mixed strategies.

A similar comment applies to behavioral strategies. This is because by splitting players into agents—one agent per information set—behavioral strategies become the mixed strategies of the agents. Since the present framework imposes no restrictions on the player set I, the above construction works for behavioral strategies, too: Given a probability measure ρ on the pure strategy combinations (pure choices) of the agents, the probability mass assigned to a set V of plays is $\rho(\phi_a^{-1}(V))$, where ϕ_a is the function defined by (A1) and (A2) for the game played by the agents.

The caveat to this argument is that it ignores all measurability issues.[14] Consider a Stackelberg-duopoly with continuum action spaces. What the follower chooses is a function of the leader's choice. When both players randomize, the follower's strategy becomes a measure on a function space. As Aumann [7] observed, it is not clear which measurable structure on the function space is required so that a distribution on the functions and one on their inputs generates

[14] For instance, let a continuum of players decide simultaneously between two actions, 0 and 1. If all select i.i.d. randomizations, it is known that the relevant derived sets that will be needed in order to properly define payoffs, e.g. the set of plays where half the players choose 0 ex post, are not measurable in the standard Kolmogorov product measure space; see e.g. [14] or [1].

a distribution on the outputs.[15] Since all plays consist of three nodes only, this problem is not related to "long plays," but haunts any sequential game with large action spaces. Therefore, we view this as beyond the scope of the present paper.

5. When do strategies induce outcomes?

This section is devoted to (A1). The focus is on properties of the tree. This is because the crucial condition on choices is already part of the definition of an EDP: Clearly, property (EDP.ii) is necessary for (A1) to hold. For, if (EDP.ii) were not true, players could choose such that the game cannot continue from some move.

But the restriction on choices incorporated in (EDP.ii) is not enough to fulfill (A1). Below its is shown that weak up-discreteness and coherence characterize the class of trees for which every strategy induces outcomes *after every history*—a slightly stronger criterion than (A1). This, then, turns into a characterization of (A1) for the class of *regular* game trees.

5.1. Examples for non-existence

The following example provides a transparent illustration for what can go wrong with existence of outcomes.

Example 9 *(Augmented inverse infinite centipede).* No EDP can be defined on the inverse infinite centipede from Example 8. Construct, though, an *augmented* inverse centipede by adding a new element $-\infty$ to the underlying set W (which was previously just the negative integers), such that $-\infty < -t$ for any $t = 1, 2, \ldots$, and consider the corresponding W-centipede. Now the root has an "immediate successor," $\{-\infty\}$, and $\gamma(W, -1) \subset W$. This (regular) game tree has available choices and, hence, admits an EDP. The corresponding problem $\Pi(T)$ is easy to construct: $\gamma(x_{-t}, -\tau) = x_{1-t} \subset x_{-t}$ if $\tau < t$, and $\gamma(x_{-t}, -\tau) = \{-t\} \subset x_{-t}$ if $\tau = t$, for all $t = 1, 2, \ldots$. The interpretation of these choices as "continue" or "stop" is obvious.

Consider a strategy s which prescribes to continue at the beginning and to stop at every other move, i.e. $s(W) = \gamma(W, 1) = \{-1, -2, \ldots\}$ and $s(x_{-t}) = \gamma(x_{-t}, -t) = \{-t\}$ for all $t = 1, 2, \ldots$. There is no play that is consistent with this strategy. For, $R_s(-\infty) = s(W) = \{-1, -2, \ldots\}$ so that $-\infty \notin R_s(-\infty)$, and for any $t = 1, 2, \ldots$ one obtains $R_s(-t) = s(W) \cap [\bigcap_{\tau=t,t+1,\ldots} s(x_{-\tau})] = \bigcap_{\tau=t,t+1,\ldots} \{-\tau\} = \emptyset$. Thus, the strategy s induces no outcome at all.

A similar point can be made with the continuous centipede. Even though this problem is not peculiar to continuous time, it may well plague continuous-time decision problems. It is known [25,26] that for decision problems in continuous time the relation between outcomes/plays and strategies is subtle. The following example shows that in the differential game some strategies may not induce an outcome at all.

Example 10. Consider the differential game with a single player, perfect information, and $A = \{0, 1\}$ as in Examples 2 and 7. Specify a strategy $s \in S$ by $s(W) = c_0(h, 1)$, $s(x_t(f)) = c_t(f, 0)$ if $f(r) = 1$ for all $r < t$, and $s(x_t(f)) = c_t(f, 1)$ otherwise, for any $t > 0$ and any $f \in W$. Clearly,

[15] Aumann [8] proposes to circumvent (but not resolve) this difficulty by treating mixed strategies as jointly measurable mappings (into pure strategies) with respect to an extraneous probability space and decision points in X_i, rather than as distributions.

the constant function $\mathbf{1}$ (viz. $\mathbf{1}(t) = 1\ \forall t$) is not a fixed point of R_s, as $\mathbf{1}(s) = 1$ for all $s < t$ for any $t > 0$, so that by the construction of s it would follow that $\mathbf{1}(t) = 0$, a contradiction.

Suppose R_s has a fixed point f. It follows that $f(0) = 1$ but, since $f \neq \mathbf{1}$, there exists $t > 0$ such that $f(t) = 0$. Thus, the set of real numbers $\{t \geqslant 0 \mid f(t) = 0\}$ is nonempty and bounded below by 0. By the Supremum Axiom, this set has an infimum t^*. If $t^* > 0$, consider $t' = t^*/2$. Then, $f(t') = 1$, but also $f(r) = 1$ for all $r < t'$. By the definition of s we should have $f(t') = 0$, a contradiction.

It follows that $t^* = 0$. But then, consider any $t > 0$. By definition of infimum, there exists $0 < r < t$ such that $f(r) = 0$. By the definition of s we have that $f(t) = 1$. Since $t > 0$ was arbitrary, it follows that f must be identically 1, a contradiction.

5.2. Undiscarded nodes

While desideratum (A1), that strategies induce outcomes, is clearly appealing, it is not always sufficient. It may well be sufficient for pure one-shot decisions among strategies. But for truly sequential decision making it is necessary to evaluate counterfactuals, that is, ‘continuation’ strategies after arbitrary histories. This requires that strategies not only induce outcomes, but do so after every history. This is a prerequisite for any “backward induction” solution concept, like subgame perfection. But in the present framework such a criterion is pushed beyond subgames, as arbitrary (possibly infinite) histories need to be accounted for.

Define, for any history h in N, $W(h) = \bigcap_{x \in h} x$ as the set of outcomes that have still not been discarded after h. Clearly, $W(\{W\}) = W$ corresponds to the null history that consists only of the root. Let (T, C) be an EDP and s a pure strategy combination. Say that (a) *s induces outcomes* if there exists $w \in W$ such that $w \in R_s(w)$, where R_s is defined in (5), and that (b) *s induces outcomes after* history h if there exists $w \in W(h)$ such that $w \in R_s^h(w)$, where

$$R_s^h(w) = \bigcap \bigl\{ s_i(x) \bigm| w \in x \subseteq W(h),\ x \in X,\ i \in J(x) \bigr\}. \tag{6}$$

An EDP is *playable* if every strategy combination induces outcomes. It is *playable everywhere* if every strategy combination induces outcomes after every history.

That is, an EDP is playable (everywhere) if the mapping R_s (R_s^h) has a fixed point for every $s \in S$ (and every history h). Histories take the role of subgames in a general EDP, but only correspond to subgames (under quasi-perfect information) when they have infima. This has a useful characterization in terms of the sets $W(h) = \bigcap_{x \in h} x$.

Lemma 3. *Let $T = (N, \supseteq)$ be a game tree and h a history in T. Then*:

(a) $\emptyset \neq W(h) = \{w \in W \mid \uparrow\{w\} = h \cup g$ *for some continuation g of h*$\}$, *and*
(b) $W(h) \in N$ *if and only if h has an infimum.*

Fix a history h and a strategy combination $s \in S$. Define the set of *discarded nodes at h*, denoted $D^h(s)$, as the set of nodes $y \in N$ that are properly contained in $W(h)$ and for which there are $x \in \uparrow y \setminus \{y\}$, $i \in J(x)$, and $c \in A_i(x)$ such that $x \subseteq W(h)$ and $y \subseteq c \neq s_i(x)$. The set of *undiscarded nodes at h*, denoted $U^h(s)$, is the set of nodes contained in $W(h)$ that are not discarded. The sets of *discarded* and *undiscarded nodes* are defined as $D(s) = D^{\{W\}}(s) \subseteq N \setminus \{W\}$ and $U(s) = U^{\{W\}}(s) = N \setminus D(s)$ respectively. Clearly, $W \in U(s)$ by construction. Let $\downarrow W(h) = \{x \in N \mid x \subseteq W(h)\}$ be the set of nodes contained in $W(h)$.

Proposition 2. *Consider an EDP, a history h, and a strategy combination $s \in S$. Then, there exists $w \in W$ such that $w \in R_s^h(w)$ if and only if $U^h(s)$ contains a maximal chain in $\downarrow W(h)$.*

This result shows that existence of outcomes is equivalent to existence of plays consisting of undiscarded nodes. This is, of course, almost a tautology. But it illustrates what goes wrong in the "augmented inverse infinite centipede," Example 9. There, a strategy that "continues" at the root, but "stops" everywhere else, generates a set of undiscarded nodes that consists only of the root.

5.3. Perfect information and playability

It will now be shown that for playability it suffices to consider the perfect information case. The following concerns an arbitrary EDP (T, C) and the associated single-player perfect information problem $\Pi(T) = (T, C_1)$. Let $s \in S$ denote the strategy combinations in (T, C) and $s' \in S'$ the strategies in $\Pi(T)$. Whether or not every strategy combination induces outcomes in a given EDP is purely a matter of the tree and, therefore, independent of the choice (information) structure (granted (EDP.ii) holds). This is the essence of the following.

Proposition 3. *Fix a history h. If every strategy $s' \in S'$ for $\Pi(T)$ induces outcomes after h, then for any EDP (T, C) with the same tree every strategy combination $s \in S$ induces outcomes after h.*

Choosing $h = \{W\}$, for a fixed game tree, Proposition 3 implies the following.

Corollary 2. *If $\Pi(T)$ is playable* (*resp. playable everywhere*), *then any EDP (T, C) with the same tree is playable* (*resp. playable everywhere*).

Even though playability is a matter of the tree, it remains a surprisingly subtle problem. To clarify it, two issues need to be addressed. First, a history may or may not have a minimum. (E.g., the chain of proper predecessors of an infinite or strange node does not.) Second, a continuation of a history may or may not have a maximum. In "classical" games, all histories have minima and all continuations have maxima. Large games, e.g. in continuous time, provide examples, where this is not the case.

Recall that T is *weakly up-discrete* if for every move $x \in X$ all maximal chains in $\downarrow x \setminus \{x\}$ have a maximum (Definition 2.2). The next result gives two characterizations.

Lemma 4. *For a game tree $T = (N, \supseteq)$ the following statements are equivalent*:

(a) *T is weakly up-discrete*;
(b) *for every history with a minimum, every continuation has a maximum*;
(c) *$x \supset \gamma(x, w) \in N$ for all $w \in x$ and all $x \in X$.*

Weak up-discreteness implies available choices (Corollary 1), but Lemma 4(c) additionally states that perfect information choices are, in fact, nodes. This is not the case in the differential game (Example 2). Furthermore, it can be shown that for weakly up-discrete trees the converse of Proposition 3 holds too [4, Proposition 5]: If every strategy combination for an arbitrary EDP on a weakly up-discrete tree T induces outcomes after history h, then so does every strategy for the problem $\Pi(T)$.

This implies that if an EDP with a weakly up-discrete tree is playable resp. everywhere playable, then *every* EDP with the same tree is playable resp. everywhere playable. For, if (T, C) is an (everywhere) playable EDP with weakly up-discrete tree, then the single-player problem $\Pi(T)$ is (everywhere) playable; but then also *any other* EDP (T, C') is (everywhere) playable, by Proposition 3.

5.4. Everywhere playable EDPs

For sequential decision theory it is essential that decision makers can evaluate their 'continuation' strategies after arbitrary histories, in particular, if a game is to be solved by backward induction. Therefore, a domain appropriate for sequential decision theory has to be playable everywhere. Hence, we turn now to a characterization of the class of trees where this is fulfilled.

Theorem 2. *Let $T = (N, \supseteq)$ be a game tree with available choices. Then, any EDP (T, C) is playable everywhere if and only if T is coherent and weakly up-discrete.*

Proof. "If": By Proposition 3 it suffices to consider $\Pi(T)$. Fix a history h and a strategy s. We first show: (a) $D^h(s)$ is an ideal, i.e. $x \in D^h(s) \Rightarrow \downarrow x \subseteq D^h(s)$; (b) every maximal chain in $U^h(s)$ is a filter in $\downarrow W(h)$, i.e. $x \in U^h(s) \Rightarrow \uparrow x \cap \downarrow W(h) \subseteq U^h(s)$. Part (a) follows by definition. Then, every chain in $U^h(s)$ is a filter. For, if $x \in U^h(s)$ and $y \in D^h(s)$ with $y \supseteq x$, then $x \in D^h(s)$, a contradiction.

Next, $U^h(s)$ is nonempty: If h has an infimum, then $W(h) \in N$ by Lemma 3 and, by definition, $W(h) \in U^h(s)$. If h has no infimum (and hence no minimum), by coherence there exists a continuation g of h which has a maximum, $z = \max g$. By definition, $z \in U^h(s)$, because there exists no node $x \in \uparrow z \setminus \{z\}$ such that $x \subseteq W(h)$.

Second, suppose that there is no $w \in W$ such that $w \in R_s^h(w)$. Since $U^h(s) \neq \emptyset$, there exists a maximal chain u in $U^h(s)$ by the Hausdorff Maximality Principle. Let $w \in W$ be such that $u \subseteq \uparrow\{w\}$. If $u = \uparrow\{w\} \setminus h$, then $\uparrow\{w\} = u \cup h$ and, by construction, $w \in R_s^h(w)$, a contradiction. Thus, $u \subset \uparrow\{w\} \setminus h$ and $u \cup h$ is a history.

Third, u has no minimum. If it had, say, $x = \min u$, then $x = \min u \cup h$. By weak up-discreteness and Lemma 4, every continuation of $u \cup h$ would then have a maximum. Let $w \in s(x)$, and let z be the maximum of the continuation $\uparrow\{w\} \setminus (u \cup h)$. Hence, $P(z) = \{x\}$. Since $z \subseteq s(x)$ by (EDP.i) and (EDP.iv), and $x \in U^h(s)$, it follows from the fact that $U^h(s)$ is a filter that $z \in U^h(s)$, a contradiction to maximality of u.

Since u has no minimum, coherence implies that there exists a continuation g of $u \cup h$ which has a maximum, $z' = \max g$. Let $w \in W$ be such that $\uparrow\{w\} = u \cup h \cup g$. Since $w \in x$ for all $x \in u$, it follows that $w \in s(x)$ for all $x \in u$. For, since u has no minimum, for any $x \in u$ there is $x' \in u$ such that $x' \subseteq x$. Since $x' \in u \subseteq U^h(s)$, it follows that $w \in x' \subset s(x)$, using (EDP.i) and (EDP.iv) again. But then that $w \in z'$ implies $z' \subseteq s(x)$ for all $x \in u$. Since $\uparrow z' \setminus \{z'\} \subseteq u \cup h$, it follows that $z' \in U^h(s)$, a contradiction.

"Only if": It has to be shown that if either weak up-discreteness or coherence fail, then some strategy induces no outcome after some history. The EDP used is again $\Pi(T)$.

Suppose, first, that weak up-discreteness fails. Then, by Lemma 4, there exists a history h which has a minimum, $z = \min h = W(h)$ (by Lemma 3), and a continuation g of h which has no maximum. Let $w^* \in W$ be such that $\uparrow\{w^*\} = h \cup g$. Define a strategy s as follows. For every

$x \in h$ (which includes $z = \min h$) set $s(x) = \gamma(x, w^*)$. For every $x \in g$, choose $s(x) \neq \gamma(x, w^*)$. Choose arbitrary choices at all other nodes. Obviously, $w^* \notin R_s^h(w^*)$. If $w \in z \setminus \gamma(z, w^*)$, also $w \notin R_s^h(w)$ because $s(z) = \gamma(z, w^*)$.

Let $w \in \gamma(z, w^*)$ be such that $w \neq w^*$, and consider the choice $\gamma(z, w)$. Since $w \in \gamma(z, w^*) = \bigcup\{x \in \downarrow z \setminus \{z\} \mid w^* \in x\}$, there exists $x \in N$ such that $x \subset z = W(h)$ and $w, w^* \in x$. Since $x \subset z$ and $w^* \in x$, hence, $x \in g$ (which has no maximum), there exists $y \in g$ such that $x \subset y \subset z$. Hence, $\gamma(y, w) \cap \gamma(y, w^*) \neq \emptyset$ and it follows from Proposition 1(a) that $\gamma(y, w) = \gamma(y, w^*)$. Thus, $s(y) \neq \gamma(y, w)$, implying that x is discarded at y and hence $w \notin R_s^h(w)$. Therefore, s does not induce an outcome after the history h.[16]

Suppose, second, that coherence fails. Then, there exists a history h without minimum, such that no continuation g of h has a maximum. Define a relation on $W(h)$ as follows. Given $w, w' \in W(h)$, say that wRw' if there exists $x \in N$ such that $x \subseteq W(h)$ and $w, w' \in x$. This is an equivalence (it is clearly reflexive and symmetric; transitivity follows from TI) and, hence, its quotient set $W(h)/R$ induces a partition of the set $W(h)$. Thus, given any $x \in N$ such that $x \subseteq W(h)$, there exists a unique $z \in W(h)/R$ such that $x \subseteq z$.

For each $z \in W(h)/R$, choose an element $w(z) \in z$ and let $g(z) = (\uparrow\{w(z)\}) \setminus h$. Define a strategy s as follows. For every $x \in h$, set $s(x) = \gamma(x, w(z))$ for any $w(z)$. For every $z \in W(h)/R$ and every $x \in g(z)$, choose $s(x) \neq \gamma(x, w^*)$.

Consider an arbitrary $z \in W(h)/R$. Clearly, $w(z) \notin R_s^h(w(z))$. Now consider any $w \in z$ such that $w \neq w(z)$. By definition of R, there exists a node $x \in N$ such that $x \subseteq W(h)$ and $w, w' \in x$. Since $x \in g(z)$ (which has no maximum), there exists $y \in g(z)$ such that $x \subset y \subset z$. Hence, $\gamma(y, w) \cap \gamma(y, w(z)) \neq \emptyset$ and it follows from Proposition 1(a) that $\gamma(y, w) = \gamma(y, w(z))$. Thus, $s(y) \neq \gamma(y, w)$, implying that x is discarded at y and, hence, $w \notin R_s^h(v)$. Therefore, s does not induce an outcome after the history h.[17] $\square$

The theorem fails, if coherence is replaced by regularity. In Example 4, with $\{1/2\} \notin N$ and $1/2 \notin W$, the strategy that "continues" at all x_t and "stops" otherwise fails to induce a play, but the tree is regular and weakly up-discrete. On the other hand, the theorem applies to some non-regular trees. Add in Example 5, where $\{1/2\}$ is a strange node and $1/2 \in W$, the (also strange) nodes $y = [1/4, 1/2)$ and $y' = (1/2, 3/4]$. An EDP can be defined, but the three strange nodes are unavailable choices. Yet, the tree is weakly up-discrete and coherent. Thus, any EDP defined on it is everywhere playable, despite the presence of strange nodes.

The distinction between "everywhere playable" and "playable" may first appear puzzling. It may seem that if every strategy induces outcomes, then every strategy should induce outcomes after every history. An intuitive (but false) supportive argument could be as follows. Suppose every strategy induces an outcome, but there is a strategy that induces no outcome after history h. Then, construct a new strategy that coincides with the first after history h, but "selects" h before. Since the new strategy must induce an outcome, and this outcome must necessarily "come after" h, the original strategy combination must induce an outcome after history h.

This argument is an instance of an intuition that is guided by the finite case, but fails in the general case. The following example presents a *playable* EDP, where not every strategy induces an outcome after every history.

[16] This does not mean that it does not induce an outcome in the whole game. The strategy could have selected a play which bifurcates from h before z, e.g. if there is a proper subhistory of h which has a continuation that is simply a terminal node and no other continuation has a maximum.

[17] A similar comment as in the previous footnote applies.

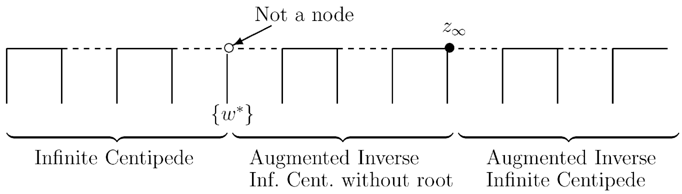

Fig. 1. Lexicographic centipede.

Example 11 *(Lexicographic centipede).* The idea is to start with an infinite centipede, replace its infinite terminal node with an augmented inverse infinite centipede (Example 9) without its root, and add another augmented inverse infinite centipede, now with root (see Fig. 1). Then every strategy induces an outcome, but not after every history. For, roughly, if it does not end during the first centipede, it is forced to select the first augmented inverse infinite centipede. That the latter "has no beginning" generates an "unavoidable" outcome, but it does not impose anything on the rest of the tree. In particular, in the rest things can be arranged so as to violate playability. Formally, let $W = \{((-1)^\tau t, \tau) \mid t = 1, 2, \dots,\ \tau = 0, 1, 3\} \cup \{(\infty, 0)\}$, endow W with the natural lexicographic order, consider the associated W-centipede, and remove the node $\{(-t, \tau) \mid \tau = 1, 3,\ t = 1, 2, \dots\} \cup \{(\infty, 0)\}$ viz. the root of the first augmented inverse infinite centipede.[18] Denote $w^* = (\infty, 0)$. Note that the ordering of the nodes for $\tau = 1, 3$ is the reverse than for $\tau = 0$, and that w^* does not belong to any one of the moves $y_t = \{(-k, \tau) \in W \mid \tau = 1, 3,\ \tau = 1 \Rightarrow k = 1, \dots, t\}$ or $z_t = \{(-k, 3) \in W \mid k = 1, \dots, t\}$, but to all moves $x_t = \{(k, \tau) \in W \mid \tau = 0 \Rightarrow k \geqslant t\}$.

At each move two choices (for $\Pi(T)$) are available, a singleton and the remainder of the plays in the move. That is, choices at x_t are $\{(t, 0)\}$ and x_{t+1} for $t = 1, 2, \dots$, at y_t they are $\{(-t, 1)\}$ and y_{t-1} for $t = 2, 3, \dots$, at y_1 they are $\{(-1, 1)\}$ and $\{(-t, 3) \in W \mid t = 1, 2, \dots\} = \gamma(y_1, (-1, 3))$, at z_t they are $\{(-t, 3)\}$ and z_{t-1} for $t = 2, 3, \dots$, and at z_1 they are $\{(-2, 3)\}$ and $\{(-1, 3)\}$.

The strategy s^* given by $s^*(x_t) = x_{t+1}$ for $t = 1, 2, \dots$, $s^*(y_t) = y_{t-1}$ for $t = 2, 3, \dots$, $s^*(y_1) = \gamma(y_1, (-1, 3))$, and $s^*(z_t) = \{(-t, 3)\}$ does not induce an outcome after the history $\{x_1, x_2, \dots, y_3, y_2\}$. This is so, because the remainder of the tree is an augmented inverse infinite centipede with a strategy like in Example 9: "continue" at the root, but "stop" everywhere else. So, this EDP is *not* everywhere playable.

Let, on the other hand, s be any strategy. If there is $t = 1, 2, \dots$ such that $s(x_t) = \{(t, 0)\}$, let t^* be the smallest such t. Then, s induces the outcome $(t^*, 0)$. If $s(x_t) = x_{t+1}$ for all $t = 1, 2, \dots$, the strategy s induces the outcome w^*, because $\uparrow\{w^*\} \setminus \{w^*\} = \{x_t \mid t = 1, 2, \dots\}$ and $w^* \in x_t$ for all $t = 1, 2, \dots$.

The crux of the matter is that the node $\{w^*\}$ is strange. For a finite node x, a strategy that continues towards x along $\uparrow x \setminus \{x\}$ will reach the immediate predecessor of x and may or may not choose x. For an infinite node x, a strategy that continues towards x along $\uparrow x \setminus \{x\}$ must select x, but cannot select anything else. For a strange node, a strategy that continues towards x along $\uparrow x \setminus \{x\}$ must select x, but also selects anything else "after" (contained in the union over) $\uparrow x \setminus \{x\}$. If x were infinite and terminal, the strategy would select it and the game would end. If x is strange and terminal, as in the example, the strategy selects x, but the game goes on.

[18] The resulting tree is technically not a centipede, because it not regular anymore.

5.5. Up-discrete trees

If there were no strange nodes, the two conditions in Theorem 2 would simplify. In order theory [20] and theoretical computer science it is often assumed that for trees the sets $\uparrow x$ are (dually) well-ordered: all their subsets have a first element (a maximum; see [19, chp. 6]). This is equivalent to the following:

Definition 5.1. A game tree is *up-discrete* if all (nonempty) chains have a maximum.

For, if $(N, \supseteq)$ is an up-discrete tree, all subsets $g \subseteq \uparrow x$ for $x \in N$ have maxima. Conversely, let $(N, \supseteq)$ be a tree, where all subsets of $\uparrow x$ have a maximum, for all x. Let g be a chain in N and $x \in g$. The chain $g \cap \uparrow x$ must have a maximum, z. But, if $y \in g$ and $y \notin \uparrow x$, then (as g is a chain), $z \supseteq x \supseteq y$, which shows that $z = \max g$. This proves the equivalence between up-discreteness and (dually) well-ordered sets $\uparrow x$. A further characterization strengthens Lemma 4(b):

Lemma 5. *A game tree $T = (N, \supseteq)$ is up-discrete if and only if for every history, every continuation has a maximum.*

Theorem 2 works even in the presence of strange nodes. If regularity, $S(N) = \emptyset$, were assumed, the previous lemmata imply that weak up-discreteness and coherence would collapse to up-discreteness.

Corollary 3. *A regular game tree $T = (N, \supseteq)$ is up-discrete if and only if it is weakly up-discrete and coherent.*

The "only if"-part of Corollary 3 holds even in the presence of strange nodes. But its "if"-part is false without regularity: In Example 5 the chains $\{y_t\}_{t=1}^{\infty}$ and $\{y'_t\}_{t=1}^{\infty}$ have no maxima, even though the tree is weakly up-discrete and coherent.

Recall that in Example 11 the strange node $\{w^*\}$ generated an outcome even though the EDP was not everywhere playable. This is suggestive: If there are no strange nodes and a strategy induces no outcome after history h, then there exists another strategy which prescribes to go along the history h and *nothing else*. Hence, this new strategy does not induce an outcome. In other words, playability may indeed be equivalent to playability everywhere—but only for regular trees. Regularity is necessary for this, as Example 11 demonstrates.

That playability and everywhere playability are equivalent for regular game trees will now be formally demonstrated. The key technical step before Theorem 3 shows how to construct a strategy that discards all the nodes of a continuation without maximum. In the augmented inverse infinite centipede (Example 9) this is accomplished by "continuing" towards a continuation without maximum (at the root) and "stopping" everywhere else. In the differential game (Example 10) we construct a strategy such that for every node (after the root) there is a previous node, where the former gets discarded. The Supremum Axiom allows us to do this consistently for all nodes in the tree (except the root). Analogous constructions yield non-existence in Examples 3 and 4. Generalizing this to arbitrary game trees requires the Axiom of Choice (an application of Zorn's Lemma in the proof of Proposition 4; see e.g. [17]).

Formally, for a game tree with available choices $T = (N, \supseteq)$ and its associated problem $\Pi(T)$, define a *partial strategy* on a set of nodes $Y \subseteq N$ to be the restriction of a strategy to $Y \cap X$, i.e.

a mapping that assigns to every $y \in Y \cap X$ a choice available at y. The set of partial strategies on Y is denoted S_Y.

Proposition 4. *Let $T = (N, \supseteq)$ be a game tree with available choices and consider the associated problem $\Pi(T)$. Let h be a history and g a continuation that has no maximum. Then, there exists a partial strategy s on $N(h, g) = \{x \in N \mid x \subset \bigcup_{y \in g} y\}$ such that all nodes in $N(h, g)$*[19] *are discarded under s, i.e. for all $x \in N(h, g)$ there is $y \in N(h, g)$ such that $x \subset y$ and $x \cap s(y) = \emptyset$.*

Theorem 3. *For a regular* (*rooted*) *game tree $T = (N, \supseteq)$ with available choices the following statements are equivalent*:

(a) *every EDP (T, C) is playable everywhere*;
(b) *every EDP (T, C) is playable*;
(c) *T is up-discrete.*

Proof. (a) implies (b) is trivial. Furthermore, (c) implies (a) by Theorem 2 and Corollary 3. It remains to show that (b) implies (c). By Corollary 3, it suffices to establish weak up-discreteness and coherence. Three observations are useful for that:

Claim A. *The sets $W(h, g) = \bigcup\{x \in N \mid x \in g\}$, where g is a continuation of h, form a partition of $W(h)$.*

To see this, let g be a continuation of h. Since $x \subseteq y$ for any $x \in g$ and $y \in h$, it follows that $W(h, g) \subseteq W(h)$. That $\bigcup\{W(h, g) \mid g$ is a continuation of $h\} = W(h)$ follows since, for any $w \in W(h)$, $\uparrow \{w\} \setminus h$ is a continuation of h. To see that the union is disjoint, let g, g' be continuations of h such that $W(h, g) \cap W(h, g') \neq \emptyset$ and $w \in W(h, g) \cap W(h, g')$. Then, there are $x \in g$ and $x' \in g'$ such that $w \in x \cap x'$, hence $x \cap x' \neq \emptyset$. By TI either $x \subseteq x'$ or $x' \subseteq x$. In the first case (the second is analogous), since $h \cup g$ is a play, it follows that $x' \in g$. This implies that $g \cap \uparrow x' = g' \cap \uparrow x'$ and thus $W(h, g) = W(h, g')$.

Claim B. *If h has no minimum and no continuation of h has a maximum, then h has no infimum and the set $W(h) = \bigcap\{x \in N \mid x \in h\}$ is not a node in N.*

Let g be a continuation of h. If h had an infimum z, then $z \in g$, because $h \cup g$ is a play and h has no minimum. Thus, h has no infimum and, by Lemma 3, $W(h)$ is not a node.

Claim C. *Let $(N, \supseteq)$ be a regular game tree and h a history without minimum. Then, for any subhistory $h' \subset h$ which also has no minimum, no alternative continuation of h'* (*i.e. g' with $g' \cap (h \setminus h') = \emptyset$*) *has a maximum.*

To see this, suppose h' is a subhistory of h such that one continuation g' with $g' \cap (h \setminus h') = \emptyset$ has a maximum, $x = \max g'$. Then, by construction $\uparrow x \setminus \{x\} = h'$ and, since $g' \cap (h \setminus h')$ is empty, h' has no infimum. Thus, x is strange, in contradiction to regularity.

[19] Note that the defining inclusion in $N(h, g)$ is strict. If h had a minimum, then the minimum would not be in this set. Also, if h had an infimum but not a minimum, then there could not exist a continuation without maximum.

Coherence. Start now with verifying coherence. Suppose, by contradiction, that coherence is violated. We construct a strategy that induces no outcome.

Let h be a history without minimum such that no continuation has a maximum. Define a strategy s as follows. Fix $w \in W(h)$ and, for any $x \in h$, let $s(x) = \gamma(x, w)$. Thus, $h \subseteq U(s)$. For the nodes in $N(h) = \{x \in N \mid x \subset W(h)\}$, proceed as follows. Partition $W(h)$ into the sets $W(h, g)$ (apply Claim A) where g are the possible continuations of h. Clearly, every node in $N(h)$ belongs to exactly one $N(h, g)$. For every $W(h, g)$, define s on $N(h, g)$ as the partial strategy identified by Proposition 4. Thus, $N(h) \subseteq D(s)$.

Consider any subhistory h' of h having no minimum. By Claim C no continuation has a maximum. For all $W(h', g)$ such that g is a continuation of h' and such that $W(h', g) \neq W(h', \uparrow\{w\} \setminus h')$, define s on $N(h', g)$ as the partial strategy identified by Proposition 4. Thus $N(h', g) \subseteq D(h)$ for all such g. At all other nodes s is arbitrary.

Let $x \notin h$ be such that $x \notin N(h)$. Choose any $w' \in x$. Define $g = \uparrow\{w'\} \setminus h$ and $h' = \uparrow\{w'\} \cap h$ (which is nonempty, because h is a history). Then, h' is a subhistory of h and g is a continuation of h'. If h' has no minimum, it follows by construction that $x \in D(s)$. Suppose, then, that there exists $z = \min h'$. Since $z \in h$, we have $s(z) = \gamma(z, w)$. Then $\gamma(z, w') \cap \gamma(z, w) = \emptyset$. For, if not, $\gamma(z, w') = \gamma(z, w)$ by Proposition 1(a) and there would exist $y \in \uparrow\{w'\}$ with $y \subset z$ such that $w \in y$. It follows that $w \in y \in g = \uparrow\{w'\} \setminus h$. If $y \in h$, then $y \in \uparrow\{w'\} \cap h = h'$, in contradiction to $y \subset z = \min h'$. Thus $y \notin h$. Then $w \in y$ implies $y \in \uparrow\{w\} \setminus h$, thus $w' \in W(h)$, in contradiction to $x \notin N(h)$.

In conclusion, $s(z) = \gamma(z, w)$ and $x \subseteq \gamma(z, w')$ with $\gamma(z, w') \cap \gamma(z, w) = \emptyset$, i.e. $x \in D(s)$. It has been shown that $U(s) = h$ and, hence, $U(s)$ contains no play, i.e. s induces no outcome by Proposition 2.

Weak Up-Discreteness. Next, turn to weak up-discreteness. The proof again proceeds by contradiction. Let h be a history with a minimum such that one continuation g of h has no maximum (recall Lemma 4). By Claim B, $W(h, g)$ is not a node. The construction is similar to the one in the coherence part above. Let $z^* = \min h$, and define a strategy s as follows. Fix $w \in W(h, g)$. For any $x \in h$, including z^*, let $s(x) = \gamma(x, w)$. Thus all nodes in h are undiscarded, $h \subseteq U(s)$.

For the nodes in $N(h) = \{x \in N \mid x \subset W(h)\}$, proceed as follows. Define s on $N(h, g)$ as the partial strategy given by Proposition 4. For every $x \in N(h)$ not in $W(h, g)$, specify s arbitrarily. Since $s(z^*) = \gamma(z^*, w)$, it follows that $N(h) \subseteq D(s)$.

For subhistories h' of h having no minimum, proceed as in the proof of coherence to obtain $N(h', g') \subseteq D(h)$ for all alternative continuations g' of h'. At all other nodes s is specified arbitrarily. As in the coherence part above, for any $x \notin h$ such that $x \notin N(h)$, we obtain $x \in D(s)$. This shows that $U(s) = h$ and, hence, $U(s)$ contains no play, i.e. s induces no outcome by Proposition 2. $\square$

That not every playable EDP is necessarily everywhere playable may be regarded as an argument in favor of focusing on the class of regular games trees, banning strange nodes. Thus, (A1) would lead to up-discrete game trees.

5.6. *A converse to (A1)*

A remaining issue is whether every play is reachable by some pure strategy combination (a condition imposed e.g. by Stinchcombe [26, p. 236]). This is a natural converse to (A1). It would fail, for instance, if the game had absent-mindedness [22]. Since the current definition of an EDP rules out absent-mindedness [3, Proposition 13],[20] every play is reachable by some strategy combination, without further assumptions on the EDP. When (A1) and (A2) are fulfilled, this implies that the function $\phi : S \to W$ defined by the desiderata is onto.

Theorem 4. *If (T, C) is an EDP, then for every play $w \in W$ there is a pure strategy combination $s \in S$ such that $w \in R_s(w)$.*

Proof. Given $w \in W$, we construct $s \in S$ in three steps. First, consider the moves $x \in X$ with $w \in x$ and let $i \in J(x)$. Let $y \in \downarrow x \setminus \{x\}$ with $w \in y$. By (EDP.iv), there exists a choice $c_i \in A_i(x)$ such that $y \subseteq c_i$, so $w \in c_i$. Define $s_i(x) = c_i$.

Second, consider the moves $x \in X$ with $w \notin x$. Suppose there exists $x' \in X$ with $w \in x'$ and $A_i(x) \cap A_i(x') \neq \emptyset$. Then, necessarily $A_i(x) = A_i(x')$. (Let $c^* \in A_i(x) \cap A_i(x')$, so that $x, x' \in P(c^*)$. Now let $c \in A_i(x)$. Then $x \in P(c^*) \cap P(c)$ and (EDP.i) imply that $x' \in P(c) = P(c^*)$ and $c \in A_i(x')$.) Hence, by (EDP.iv) and (EDP.i) there exists a unique $c \in A_i(x) = A_i(x')$ with $w \in c$. Set $s_i(x) = c$. To see that this definition is consistent, suppose there exist two different nodes x', x'' with $w \in x', x''$, $A_i(x) \cap A_i(x') \neq \emptyset$ and $A_i(x) \cap A_i(x'') \neq \emptyset$. The latter imply $A_i(x') = A_i(x'')$. By TI x' and x'' are ordered, and then Proposition 13 of [3] implies that $x' = x''$, a contradiction.

Third, consider the moves $x \in X$ with $w \notin x$ such that for all $x' \in X$ with $w \in x'$ it follows that $A_i(x) \cap A_i(x') = \emptyset$. Then, for every $i \in J(x)$ choose $s_i(x)$ arbitrarily. Then, $w \in R_s(w)$ holds by construction. □

6. Uniqueness

This section is devoted to a characterization of the class of EDPs for which the uniqueness criterion (A2) holds: "extensive forms." Those satisfy both an extra condition on the choice (information) structure and one on the tree. The condition on choices becomes redundant, though, in the class of EDPs that satisfy (A1).

6.1. *Examples with multiple outcomes*

(A2) imposes further restrictions on the tree. This is illustrated by the third "hole in the middle" (Example 5), where a strategy induces multiple outcomes. This holds even in Example 4, where the tree is turned regular by removing $1/2$ from W and $\{1/2\}$ from N. The absence of a node $[1/4, 3/4]$ still dictates that the strategy described in Example 5 induces multiple plays, $\{1/4\} = R_s(1/4)$ and $\{3/4\} = R_s(3/4)$.

The same argument applies in a variant of Example 4, when the (strange) nodes $[1/4, 1/2)$ and $(1/2, 3/4]$ are added. This tree is then weakly up-discrete and coherent, therefore, (everywhere)

[20] A decision theory for games with absent-mindedness would require allowing the state that obtains to depend on the decision maker's choice. Since we do not know how to handle that, we find it worthwhile to follow Kuhn [21] by excluding absent-mindedness in the definition of an EDP.

playable by Theorem 2. But a strategy that always "continues" still induces multiple plays. Thus, (A1) holds but (A2) fails.

The defect in these examples may appear to be the absence of a minimum for the chain $\uparrow y_t \cap \uparrow y'_t$, like in the "Twins" example [3, Example 13].[21] The problem is deeper, though. In the following example *every* chain of the form $\uparrow x \cap \uparrow y$ with $x, y \in N$ has a minimum,[22] and still (A2) fails. (Anderson [6], Simon and Stinchcombe [25, p. 1172], and Stinchcombe [26, p. 239] provide similar examples.)

Example 12. Reconsider the differential game (Example 2). Perfect information choices are $c_t(f, a) = \{g \in x_t(f) \mid g(t) = a\} = \gamma(x_t(f), g)$ for any $g \in x_t(f)$ with $g(t) = a \in A$ (see Example 7). Every chain of the form $\uparrow x \cap \uparrow y$ with $x, y \in N$ has a minimum [4, Example 15]. Yet, some strategies still induce multiple outcomes.

Let $A = \{0, 1\}$ and consider $\Pi(T)$. Let $\mathbf{1} \in W$ be constant 1. Note that $g \in x_t(f) \Leftrightarrow f \in x_t(g)$, for all t and all $f, g \in W$. Define $s \in S$ by $s(x_t(f)) = c_t(f, 1)$ if $f \in x_t(\mathbf{1})$ and $s(x_t(f)) = c_t(f, 0)$ otherwise. Consider the function $f_r \in W$ defined by $f_r(t) = 1$ for all $t \in [0, r]$ and $f_r(t) = 0$ for all $t > r$, for any $r > 0$. Then,

$$R_s(f_r) = \bigcap_{g \in x_t(f_r)} s\big(x_t(g)\big) = \left[\bigcap_{t \leqslant r} c_t(\mathbf{1}, 1)\right] \cap \left[\bigcap_{r<t} \big\{h \in x_t(f_r) \,\big|\, h(t) = 0\big\}\right]$$

$$= \left[\bigcap_{t \leqslant r} c_t(\mathbf{1}, 1)\right] \cap \left[\bigcap_{r<t} \big\{h \in x_t(f_r) \,\big|\, h(\tau) = 0,\ \forall \tau \in (r, t]\big\}\right] = \{f_r\}.$$

In other words, for every $r > 0$ the play $f_r \in W$ is a fixed point of R_s, so $s \in S$ induces a continuum of outcomes![23]

6.2. *Extensive forms*

In order to understand the origin of multiplicity of outcomes, the following is introduced.

Definition 6.1. An **extensive form** (EF) with player set I is an EDP which, instead of (EDP.iii), satisfies

(EDP.iii′) if $y \cap y' = \emptyset$, then there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and $P(c) \cap P(c') \neq \emptyset$, for all $y, y' \in N$.

(EDP.iii′) is almost identical to (EDP.iii) except that the two choices $c, c' \in C_i$ have to be available simultaneously at some move, i.e. $P(c) \cap P(c') \neq \emptyset$. Without this stronger property *outcome uniqueness* fails, that is, there are $s \in S$ and $w, w' \in W$ with $w \neq w'$ such that $w \in R_s(w)$ and $w' \in R_s(w')$.

[21] There, $\uparrow \{0\} \cap \uparrow \{1\}$ has no minimum. A strategy (for $\Pi(T)$) that assigns to each move its immediate successor among the moves (and not an available singleton) induces two distinct plays.

[22] If the chain $\uparrow x \cap \uparrow y$ has a minimum for all $x, y \in N$, the tree is "well-joined," the dual of the concept of a "well-met" (pseudo)tree in [11]. If the tree is well-joined, then $N \cup \{\emptyset\}$ is a *lattice* [2].

[23] We are grateful to Nicolas Vieille for suggesting this example.

Proposition 5. *If for an EDP (T, C) condition* (EDP.iii′) *is violated, then outcome uniqueness fails.*

As with (A1), condition (EDP.iii′) combines a restriction on the tree with one on choices. To make this precise the following is introduced:

Definition 6.2. A game tree $T = (N, \supseteq)$ is *selective* if, for all $w, w' \in W$, that $w \neq w'$ implies $\exists x \in X$ such that $w, w' \in x$ and $\gamma(x, w) \neq \gamma(x, w')$.

Proposition 6.

(a) *If a game tree $T = (N, \supseteq)$ is selective, then it is regular.*
(b) *If a game tree $T = (N, \supseteq)$ is regular and every history h has a continuation g with a maximum $z \in g$, then T is selective.*

Selectiveness is purely a condition on the tree. Proposition 6(a) shows that it is a strong one, though. Part (b) gives a sufficient condition for it that is reminiscent of the conditions for playability. In fact, it implies coherence, but not weak up-discreteness.

Selectiveness and coherence are independent properties, however: The third variant of the "hole in the middle" (Example 5) is coherent, but not selective. On the other hand, replace the terminal node $\{\infty\}$ of an infinite centipede (Example 6) by an inverse infinite centipede (Example 8) without root. This yields a selective tree that is not coherent, because all plays that "continue" through the (first) infinite centipede "continue" into the (second) inverse infinite centipede that has no maximum.

Example 13. Reconsider $\Pi(T)$ for the differential game (Examples 2 and 7). Let $f, g \in W$ be any two functions that agree on $[0, r]$ and differ thereafter, for some $r > 0$. If $g \in x_t(f)$, then $t \leqslant r$ and $g \in \{h \in x_t(f) \mid h(t) = f(t)\} = \gamma(x_t(f), f)$, so that $\gamma(x_t(f), g) = \gamma(x_t(f), f)$ by Proposition 1(a). Therefore, the tree of the differential game is not selective, as there is no move where f and g get "sorted out." But it is regular, showing that the converse of Proposition 6(a) is false. (To see that it is regular observe that all nodes except the root are infinite, i.e. $x_t(f) = \inf \uparrow x_t(f) \setminus \{x_t(f)\}$ for all $f \in W$ and all $t > 0$.)

The following is a characterization for when an EDP is an EF in terms of one condition on the tree and another on choices.

Proposition 7. *The EDP (T, C) is an EF if and only if T is selective and*

(EDP.ii′) $x \cap [\bigcap_{i \in J(x)} c_i] = \gamma(x, w)$ *for some* $w \in x \cap [\bigcap_{i \in J(x)} c_i]$, *for all* $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ *and for all* $x \in X$.[24]

This disentangles what (A2) requires on the tree and on the choices: respectively selectiveness and (EDP.ii′). Since (EDP.ii′) holds trivially for $\Pi(T)$ on the tree of the differential game,

[24] In (EDP.ii′) the equality may be replaced by weak inclusion $\subseteq$, as Lemma 7 and Proposition 1(b) imply the reverse inclusion.

Proposition 7 provides an indirect verification that the tree of the differential game is not selective. Thus, the failure of (A2) in the differential game is due to a property of the tree, and not to "misspecified" choices.

(EDP.ii′) may fail even when the tree is selective. To see this, reconsider the "hole in the middle" (Example 3), with $[1/4, 3/4] \in N$. Make this into an EDP by assigning all nodes except the root plus $[1/4, 1/2) \cup (1/2, 3/4]$ as choices to a single player. The singleton $\{1/2\}$ is a choice available at $\bar{x} = [1/4, 3/4] \in X$, so the decision at $\bar{x}$ is nontrivial. Since $\gamma(\bar{x}, 1/4) = [1/4, 1/2)$ and $\gamma(\bar{x}, 3/4) = (1/2, 3/4]$, the tree is selective, but the construction of choices violates (EDP.ii′).

The next result (whose proof is omitted) states that selectiveness and (EDP.ii′) are separately necessary for (A2).

Proposition 8.

(a) *If a game tree T with available choices is not selective, then there is an EDP on T where outcome uniqueness fails.*
(b) *If for an EDP condition* (EDP.ii′) *is violated, then outcome uniqueness fails.*

Corollary 4. *The tree of an EF is selective and, hence, regular.*

The corollary follows by combining the "only if"-part of Proposition 7 with Proposition 6(a). Going from an EDP to an EF has further implications. First, it can be shown that for an EF the set of undiscarded nodes for any strategy combination, $U(s)$, becomes a chain. Second, if every strategy combination for an arbitrary EF with tree T induces outcomes after a history h, then so does every strategy for the problem $\Pi(T)$ [4, Proposition 9]. This says that if an EF is playable resp. everywhere playable, then *every* EF with the same tree is playable resp. everywhere playable.

6.3. A uniqueness result

Up to this point it has been argued that an EF—replacing (EDP.iii) by (EDP.iii′)—is *necessary* for (A2). It will now be shown that an EF is also sufficient for (A2).

Theorem 5. *Consider an EF as in Definition* 6.1 *and fix a pure strategy combination $s \in S$. If $w \in R_s(w)$, then (a) $R_s(w) = \{w\}$, and (b) if $w' \in R_s(w')$ then $w' = w$.*

Proof. First, we claim that if $w \in R_s(w)$ and $w' \in W \setminus \{w\}$, then there are $x \in X$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$. For, by IR there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$. By (EDP.iii′) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and there is $x \in P(c) \cap P(c')$. Because $w \in R_s(w)$, we have $s_i(x) = c$ and thus $w' \in y' \subseteq c' \neq s_i(x)$ implies that $w' \notin s_i(x)$.

(a) Let $s \in S$, assume that $w \in R_s(w)$, and consider any $w' \in R_s(w)$. If $w' \in W \setminus \{w\}$, then by the claim above there are $x \in X$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$. But this contradicts $w' \in R_s(w)$. Hence, $w' = w$.

(b) Let again $s \in S$ and assume $w \in R_s(w)$. Consider any $w' \in W$ such that $w' \in R_s(w')$. If w' were not equal to w, then, again by the claim, there would be $x \in N$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$. Since the latter would contradict $w' \in R_s(w')$, it follows that $w' = w$. □

Since the strengthening of (EDP.iii) to (EDP.iii′) is necessary for (A2), this result means that (EDP.iii′) is *precisely* what is needed for strategies to induce unique outcomes. Recall, however, that (EDP.iii′) is a combination of a condition on the tree, namely selectiveness, and one on choices, namely (EDP.ii′). The condition on choices, (EDP.ii′), becomes redundant, though, when the tree is weakly up-discrete.

Proposition 9. *An EDP (T, C) with a weakly up-discrete tree $T = (N, \supseteq)$ is an EF if and only if T is selective.*

In the class of weakly up-discrete trees (A2) is, therefore, characterized by selective trees. Note that weak up-discreteness and selectiveness are independent properties. The "hole in the middle," Example 5, has a weakly up-discrete tree that is not selective: There is no $x \in X$ such that $1/4, 3/4 \in x$ and $\gamma(x, 1/4) \cap \gamma(x, 3/4) = \emptyset$, because a node $[1/4, 3/4]$ is absent from this tree. For, $1/4, 3/4 \in x$ implies $x \in \{x_t\}_{t=1}^{\infty}$, yet $\gamma(x_t, 1/4) = \gamma(x_t, 3/4) = x_{t+1}$ for all $t = 1, 2, \ldots$. But the first "hole in the middle," Example 3, has a selective tree, because $[1/4, 3/4]$ is a node. It is not weakly up-discrete, though, because nodes of the form $[1/4, 1/2)$ and $(1/2, 3/4]$ are still missing, i.e., the chains $\{y_t\}_{t=1}^{\infty}$ and $\{y'_t\}_{t=1}^{\infty}$ have no maxima.

7. A joint characterization

Combining Theorems 2, 3, and 5 yields a characterization of EDPs that satisfy the desiderata (A1) and (A2) purely in terms of easily verifiable properties of the tree.

Theorem 6. *An EDP (T, C) satisfies* (A1) *and* (A2) *if and only if the* (*rooted*) *game tree $T = (N, \supseteq)$ is regular, weakly up-discrete, and coherent.*[25]

Proof. "If": By weak up-discreteness and Corollary 1 the tree has available choices. As the tree is weakly up-discrete and coherent by hypothesis, Theorem 2 implies that every strategy induces an outcome after every history. In particular, every strategy induces an outcome after the null history that consists only of the root. Hence, (A1) holds true.

To see (A2), observe first that every history without minimum has a continuation with a maximum by coherence. Second, for any history with a minimum all continuations have maxima by Lemma 4(b) and weak up-discreteness. Then regularity and Proposition 6(b) imply that the tree is selective. Consequently, the EDP (T, C) is an EF by (the "if" part of) Proposition 9. It follows from Theorem 5 that (A2) holds.

"Only if": The uniqueness criterion (A2) implies by Proposition 5 that the EDP (T, C) must be an EF. Thus the tree is regular by Corollary 4. By Corollary 2 and Proposition 10 in Appendix A, the playability criterion (A1) for (T, C) implies playability for all EDPs on T. Since T is regular, Theorem 3 implies weak up-discreteness and coherence. $\square$

As shown by the examples in Section 2.2, the three characterizing properties are independent. By Corollary 3 a regular tree is up-discrete if and only if it is weakly up-discrete and coherent. Combining this with Theorems 3 and 6 yields:

[25] Note that Theorem 6 implies that the "Lexicographic Centipede" (Example 11) fails (A2).

Corollary 5.

(a) *If an EDP satisfies* (A1) *and* (A2)*, then so does every EDP with the same tree.*
(b) *An EDP satisfies* (A1) *and* (A2) *if and only if its tree is regular and up-discrete. Furthermore, the EDP is then everywhere playable.*

8. Conclusions

The concept of a non-cooperative game—that is, a game with complete rules—extends well beyond the confines of finite games. Infinitely repeated games, stochastic games [24], and even differential games can be rigorously defined in extensive form. This step verifies that the rules of these games are complete.

Whether the rules of such games are also consistent is a different matter, though. It may well be that at each "when" it is fully specified "who can do what," but at the same time a global specification of such instructions—that is, a strategy combination—may not yield an outcome at all or multiple outcomes. In this paper we characterize extensive form games that satisfy two global criteria: (A1) every strategy combination does induce some outcome/play; (A2) the outcome/play induced by a strategy combination is unique.

It is shown that EDPs satisfying (A1) and (A2) are fully characterized by three properties of the underlying game tree: regularity, coherence, and weak up-discreteness. The latter two together characterize the class of "everywhere playable" EDPs. Those are the ones which guarantee that strategies induce outcomes after every possible history—a principle underlying any backward induction procedure. Regularity of the tree is added via the uniqueness criterion (A2).

The characterization result allows us to draw a dividing line between those games that can be defined on game trees with the above three properties, and those that cannot. As a rule, almost all games in the literature turn up on the safe side of this line, inclusive of Aumann and Hart's [10] transfinite cheap-talk game. The only exception concerns differential games. Though the tree of a differential game is regular, it fails to be up-discrete. As a consequence, differential games allow for strategy combinations that do not induce any outcome/play at all—differential games are not "playable." Furthermore, they allow for strategy combinations that induce continua of outcomes/plays—differential games are not "extensive forms."

This, of course, raises the issue of how to interpret existing applications of differential games. One possible reaction is to give up their dynamic interpretation and view them as "one-shot" normal form games. In contrast to extensive form games, in a normal form game strategies are primitives. Therefore, one is free to restrict the strategy sets to those strategies that induce unique outcomes.

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Appendix A

Proof of Lemma 1. The "if"-part is trivial. For the "only if" one, let x be terminal and let $v, w \in x$. If $v \neq w$, then by IR there are $x', y' \in N$ with $v \in x' \setminus y'$ and $w \in y' \setminus x'$. Hence,

$v \in x \cap x'$ and $w \in x \cap y'$. Since $v \notin y'$ and $w \notin x'$, neither $x \subseteq x'$ nor $x \subseteq y'$. Hence, by TI $x' \subset x$ and $y' \subset x$, in contradiction to x being terminal. Thus, $v = w$. □

Proof of Lemma 2. (a) The "if"-part is trivial. For the "only if" part, let $x = \inf h$. Obviously, $x \subseteq \bigcap_{y \in h} y$. Fix $w \in x$, and suppose there exists $w' \in \bigcap_{y \in h} y$ such that $w' \notin x$. By IR there are $z, z' \in N$ such that $w \in z \setminus z'$ and $w' \in z' \setminus z$.

Consider any $y \in h$. Since $w' \in z' \setminus z$ and $w' \in y$, it follows from TI that either $y \subseteq z'$ or $z' \subseteq y$. In the first case, it would follow that $x \subseteq z'$, in contradiction to $w \in z \setminus z'$. Hence, $z' \subseteq y$ for all $y \in h$, i.e., z' is a lower bound for the chain h. Since x is its infimum, $z' \subseteq x$, a contradiction to $w' \notin x$. This shows that $x = \bigcap_{y \in h} y$.

(b) follows from (a).

(c) "If": If $z = \inf \uparrow x \setminus \{x\}$, it follows from (a) that $z = \bigcap_{y \in \uparrow x \setminus \{x\}} y$. By hypothesis, $x \subset z$ and thus $z = \min \uparrow x \setminus \{x\}$, a contradiction.

"Only if": If $x \in N$ is strange, then $\uparrow x \setminus \{x\}$ has no infimum, in particular, no minimum. If $x = \bigcap_{y \in \uparrow x \setminus \{x\}} y$ would hold, then by (a) $x = \inf \uparrow x \setminus \{x\}$ and x could not be strange. □

The following lemma, the proof of which is omitted, is needed for Proposition 1.

Lemma 6. *Let $(N, \supseteq)$ be a game tree with set of plays W. If $w, w' \in x \in X$, then there are $z, z' \in \downarrow x \setminus \{x\}$ such that $w \in z$ and $w' \in z'$, where $w \neq w'$ implies $z \cap z' = \emptyset$.*

Proof of Proposition 1. (a) Let $w'' \in \gamma(x, w) \cap \gamma(x, w')$. Then there are nodes $z, z' \in \downarrow x \setminus \{x\}$ such that $w, w'' \in z$ and $w', w'' \in z'$. Since $w'' \in z \cap z'$, TI implies either $z \subset z'$ or $z' \subseteq z$. If $z \subset z'$, then $w \in z'$ implies $\gamma(x, w') = \bigcup_{y \in \uparrow z' \setminus \uparrow x} y = \gamma(x, w)$. If $z' \subseteq z$, then $w' \in z$ implies $\gamma(x, w) = \bigcup_{y \in \uparrow z \setminus \uparrow x} y = \gamma(x, w')$.

(b) Denote $c = \gamma(x, w) \subset x$. Since $x \in X$, there is some $y \in \downarrow x \setminus \{x\}$ such that $w \in y$ by Lemma 6 and, therefore, $y \in \downarrow c$. If $z \in \uparrow x$, then $y \subseteq c \subset x \subseteq z$ implies $z \in \uparrow y$ and $z \notin \downarrow c$, so $z \in \uparrow y \setminus \downarrow c$. Since $z \in \uparrow x$ was arbitrary, $\uparrow x \subseteq \uparrow y \setminus \downarrow c$. If $z \in \uparrow y \setminus \downarrow c$, i.e. $y \subseteq z$ and $z \setminus c \neq \emptyset$, then $x \subseteq z$. For, $y \subseteq z \cap x$ implies $z \subset x$ or $x \subseteq z$ by TI; but $z \subset x$ would imply $z \subseteq c$, because $w \in y \subseteq z \in \downarrow x \setminus \{x\}$, in contradiction to $z \setminus c \neq \emptyset$, so that $x \subseteq z$ must obtain. Since $z \in \uparrow y \setminus \downarrow c$ was arbitrary, $\uparrow x \supseteq \uparrow y \setminus \downarrow c$. Hence, $x \in P(c)$.

Let $x' \in P(c)$. Then there is $y' \in \downarrow c$ such that $\uparrow x' = \uparrow y' \setminus \downarrow c$. Since $y' \in \downarrow c$, there exists some $z \in \downarrow x \setminus \{x\}$ such that $w \in z$ and $y' \subseteq z$. Thus $z \in \downarrow c$. Since $y' \subseteq x'$ but $x' \notin \downarrow c$, TI implies $z \subseteq x'$, in particular $w \in x'$. Then, $w \in x' \cap x$ implies, again by TI, that $x \subset x'$ or $x' \subseteq x$. If $x \subset x'$, then $x \in \uparrow y' \setminus \downarrow c$ but $x \notin \uparrow x'$, a contradiction. If $x' \subset x$, that $w \in x'$ implies $x' \in \downarrow c$, a contradiction. Thus $x = x'$. It follows that $P(c) = \{x\}$.

To prove the second statement, choose $w' \in x \setminus c$ and let $c' = \gamma(x, w')$. If $c \cap c' \neq \emptyset$ would hold, part (a) would imply that $c = c'$, yielding the contradiction $w' \in c$. Therefore, $c \cap c' = \emptyset$. Since $c' \subseteq x \setminus c$, it also follows that $x \in P(c')$.

(c) Suppose that $\gamma(x, w) = x$. If the chain $h = \{z \in \downarrow x \setminus \{x\} \mid w \in z\}$ had a maximum $y \in \downarrow x \setminus \{x\}$, then $\bigcup_{w \in z \in \downarrow x \setminus \{x\}} z = \gamma(x, w) = y \subset x$, in contradiction to the hypothesis. To see the second part of the claim, suppose that an EDP is defined on this tree. Because $x \in X$ and $w \in x$, there is $z \in N$ such that $w \in z \subset x$ by Lemma 6, hence, $\gamma(x, w) = \bigcup_{y \in \uparrow z \setminus \uparrow x} y$. Since $z \subset x$, there is $c_i \in A_i(x)$ such that $z \subseteq c_i$ for all $i \in J(x) \neq \emptyset$ by (EDP.iv). By $x \in P(c_i)$ there is $y_i \in \downarrow c_i$ such that $\uparrow x = \uparrow y_i \setminus \downarrow c_i$ for all $i \in J(x)$. Since $x \notin \downarrow c_i$, there is $w_i \in x \setminus c_i$ for all $i \in J(x)$. Yet, by hypothesis there is $x_i \in \uparrow z \setminus \uparrow x$ such that $\{w_i\} \cup y_i \subseteq x_i$ for all $i \in J(x)$. Because $x_i \in \uparrow y_i$, it follows from $\uparrow x = \uparrow y_i \setminus \downarrow c_i$ that $x_i \in \downarrow c_i$ (i.e. $x_i \subseteq c_i$), in contradiction to $w_i \notin c_i$, for all $i \in J(x)$. □

Proof of Lemma 3. (a) Let $w \in W(h)$. Then, $w \in x$ and hence $x \in \uparrow\{w\}$ for all $x \in h$. Hence, $\uparrow\{w\}$ is a play containing h and $\uparrow\{w\} = h \cup g$ for the continuation $g = \uparrow\{w\} \setminus h$. Conversely, let $w \in W$ be such that $\uparrow\{w\} = h \cup g$ for some continuation g of h. Then, $x \in \uparrow\{w\}$ and hence $w \in x$ for all $x \in h$. Hence, $w \in W(h)$. This proves the equality. $W(h)$ is nonempty by definition of a game tree.

(b) is an immediate consequence of Lemma 2(a). □

Proof of Proposition 2. "If": If $u \subseteq U^h(s)$ is a maximal chain in $\downarrow W(h)$, then $u \cup h$ is a maximal chain in N. By BD, there is $w \in W$ such that $w \in \bigcap_{x \in u \cup h} x$. By Theorem 3(c) of [3], $\uparrow\{w\} = u \cup h$. Hence, $w \in R_s^h(w) = \bigcap_{x \in u} \bigcap_{i \in J(x)} s_i(x)$.

"Only if": Let $w \in W$ be such that $w \in R_s^h(w)$. By construction, $u = \uparrow\{w\} \setminus h \subseteq U^h(s)$. Since $\uparrow\{w\}$ is a play and h is a history, it follows that u is maximal in $\downarrow W(h)$. □

The next lemma, whose proof is omitted, is needed for Propositions 3, 7, and 9.

Lemma 7. *For (T, C): If $x \in P(c)$, with $c \in C_i$ for some $i \in I$, and $w \in x \cap c$, then $\gamma(x, w) \subseteq x \cap c$.*

Proof of Proposition 3. Pick any $s \in S$ and construct $s' \in S'$ as follows: By (EDP.ii) and the Axiom of Choice there is $w_x \in x \cap [\bigcap_{i \in J(x)} s_i(x)]$ for every $x \in X$; set $s'(x) = \gamma(x, w_x)$ for all $x \in X$. Then $s'(x) \subseteq x \cap [\bigcap_{i \in J(x)} s_i(x)]$ for all $x \in X$ by Lemma 7.

Next, we claim that if $s \in S$ and $s' \in S'$ are such that $s'(x) \subseteq s_i(x) \cap x$ for all $i \in J(x)$ for all $x \in X$, then $U^h(s') \subseteq U^h(s)$, for any history h. To see this, let $y \in D^h(s)$, i.e., there are $x \in \uparrow y \setminus \{y\}$ with $x \subseteq W(h)$, $i \in J(x)$, and $c \in A_i(x)$ such that $y \subseteq c \neq s_i(x)$. Then $y \cap s_i(x) = \emptyset$. Because $s'(x) \subseteq x \cap [\bigcap_{i \in J(x)} s_i(x)]$ by hypothesis, $s'(x) \cap y = \emptyset$. Hence, $y \in D^h(s')$. Since $y \in D^h(s)$ was arbitrary, $D^h(s) \subseteq D^h(s')$ or, equivalently, $U^h(s') \subseteq U^h(s)$, verifying the claim.

This implies that $U^h(s') \subseteq U^h(s)$. By hypothesis and the "only if"-part of Proposition 2, $U^h(s')$ contains a maximal chain u that is also maximal in $\{x \in N \mid x \subseteq W(h)\}$. By $U^h(s') \subseteq U^h(s)$ this chain u is also contained in $U^h(s)$. Hence, the "if"-part of Proposition 2 implies the statement. □

Proof of Lemma 4. "(a) implies (b)": If h is a history with a minimum x, then by Lemma 3, $W(h) = x$ and the continuations of h are maximal chains in $\downarrow x \setminus \{x\}$.

"(b) implies (c)": Let $w \in x \in X$. As $\uparrow\{w\}$ is a play, $u = \{z \in N \mid w \in z \in \downarrow x \setminus \{x\}\}$ is a maximal chain in $\downarrow x \setminus \{x\}$. Obviously, $h = \uparrow x$ is a history with minimum x and u is a continuation of h. Hence, u has a maximum by (b), $y = \max u$, and $y \subset x$. It follows that $\gamma(x, w) = \bigcup\{z \mid z \in u\} = y \in N$ and $\gamma(x, w) \subset x$.

"(c) implies (a)": Suppose that $\gamma(x, w) \in N$ for all $w \in x$ and all $x \in X$. Consider $x \in X$ and let u be a maximal chain in $\downarrow x \setminus \{x\}$. Since T is a game tree, u is contained in a maximal chain in N (play), i.e. there exists $w \in W$ such that $w \in y$ for all $y \in u$. By hypothesis $w \in \gamma(x, w) \in \downarrow x \setminus \{x\}$ and thus $\gamma(x, w) \in u$. By the construction of γ also $y \subseteq \gamma(x, w)$ for any $y \in u$. Therefore, $\gamma(x, w) = \max u$. □

Proof of Lemma 5. The "only if"-part is obvious, because every continuation is a chain. To see the "if"-part, suppose every continuation of every history has a maximum. Let g be an arbitrary (nonempty) chain in T. If the root W is in g, then $W = \max g$. Thus, suppose $W \notin g$. Let w be a play such that $g \subseteq \uparrow\{w\}$. Let $g^* = \{x \in \uparrow\{w\} \mid x \subseteq x' \text{ for some } x' \in g\}$. Then, $h = \uparrow\{w\} \setminus g^*$ is a history and g^* is a continuation of h. (To see that h is a history, let $x \in h$ and $y \in \uparrow x \subseteq \uparrow\{w\}$. If $y \notin h$, then $y \in g^*$, hence there exists $y' \in g$ with $x \subseteq y \subseteq y'$ and thus $x \in g$, a contradiction.) By hypothesis, g^* has a maximum, $z = \max g^*$. Then, $z \in g^*$ and there exists $z' \in g$ such that $z \subseteq z'$. But, since $g \subseteq g^*$, it follows that $z = z'$, i.e. z is also the maximum of g. □

Proof of Corollary 3. The result follows from Lemmata 4 and 5 if it can be shown that under regularity coherence is equivalent to the property that for every history without minimum *every* continuation has a maximum. The latter clearly implies coherence. To see the converse, let h be a history without minimum and suppose g, g' are two continuations such that g has a maximum, $x = \max g$, but g' has no maximum.

Obviously, $\uparrow x \setminus \{x\} = h$. By hypothesis, h has no minimum, but, by regularity, there exists $z = \inf h$. Since x is a lower bound for h, it follows that $x \subseteq z$. If $x \neq z$, then $z \in \uparrow x \setminus \{x\} = h$ and $z = \min h$, a contradiction. Thus, $x = z = \inf h$.

Let $y \in g'$. Again, since y is a lower bound for h and $x = \inf h$, it follows that $y \subseteq x$. This shows that $x \in g'$ and $x = \max g'$, a contradiction. □

Proof of Proposition 4. The proof proceeds in several steps. First, observe that for every $x \in N(h,g)$, there exists $y \in N(h,g)$ such that $x \subset y$. That is, every node in $N(h,g)$ has a proper predecessor in $N(h,g)$. To see this, suppose $x \in N(h,g)$ has no immediate predecessor in $N(h,g)$. Let $w \in x$. Then, $g' = \uparrow\{w\} \cap N(h,g)$ is a continuation of h with $\max g' = x$. Clearly, $W(h,g) = W(h,g') = x$. It follows that $x \in g$ (because $x \notin h$) and $\max g = x$, a contradiction.

Second, define the set

$$A = \left\{ (Y,s) \,\middle|\, \begin{array}{l} Y \subseteq N(h,g),\, Y \cap X \neq \emptyset, \text{ and } s \in S_Y \text{ such that} \\ \forall x \in Y\ \exists y \in Y \text{ with } x \subset y \text{ and } x \cap s(y) = \emptyset \end{array} \right\}$$

and the partial order on A given by $(Y,s) \geqslant (Y',s') \Leftrightarrow Y \supseteq Y'$ and $s|_{Y'} = s'$ (reflexivity and transitivity are obvious; antisymmetry follows by construction).

Third, for every $x \in N(h,g)$, there exists $(Y,s) \in A$ such that $x \in Y \subseteq \uparrow x$. To see this, let $x \in N(h,g)$. By the first step, there exists $x_1 \in N(h,g)$ such that $x \subset x_1$. By (EDP.iv), there exists a choice c, available at x_1, such that $x \subseteq c$. By available choices, there exists an available choice at x_1 which is disjoint from c. Thus one can define $s(x_1) \neq c$. Analogously, there is $x_2 \in N(h,g)$ such that $x_1 \subset x_2$ and an available choice at x_2 that discards x_1. The conclusion follows from an induction argument.[26]

Fourth, apply Zorn's Lemma to $(A, \geqslant)$. Observe that A is nonempty by the third step. We have to show that every chain in $(A, \geqslant)$ has an upper bound in A. Let C be a chain in A. That is, for every $(Y,s), (Y',s') \in C$, either $(Y,s) \geqslant (Y',s')$ or $(Y',s') \geqslant (Y,s)$. Define $Z = \bigcup\{Y \mid (Y,s) \in C\}$ and construct $\bar{s}$ as follows. Given $z \in Z$, define $\bar{s}(z) = s(z)$ for any $(Y,s) \in C$ such that $z \in Y$. Such a (Y,s) exists by construction of Z, and $\bar{s}$ is well-defined because C is a chain. Further, given $z \in Z$, taking $(Y,s) \in C$ such that $z \in Y$ shows that there exists $y \in Y$ with $x \subset y$ and $x \cap s(y) = \emptyset$, and hence $x \cap \bar{s}(y) = \emptyset$. Thus $(Z, \bar{s})$ is an upper bound for C in A.

[26] More precisely, this step takes the Axiom of Dependent Choices, a consequence of the Axiom of Choice.

Zorn's Lemma implies that $(A, \geqslant)$ has a maximal element (Z^*, s^*). Then, for any $x \in N(h)$, there exists $z \in Z^*$ such that $x \cap s^*(z) = \emptyset$. That is, any node in $N(h, g)$ (and not only in Z^*) is discarded under the partial strategy given by s^*.

If $\uparrow x \cap Z^* \neq \emptyset$, then let $z \in \ \uparrow x \cap Z^*$. Since $(Z^*, s^*) \in A$, there exists $z' \in Z^*$ such that $z \cap s^*(z') = \emptyset$, and thus $x \cap s^*(z') = \emptyset$ (because $x \subseteq z$). Suppose, then, $\uparrow x \cap Z^* = \emptyset$. By the third step above, there exists $(Y, s) \in A$ such that $x \in Y \subseteq \ \uparrow x$. Define now $Z_1 = Z^* \cup Y$. Clearly, $Z^* \subset Z_1$. Define also $s_1 \in S_{Z_1}$ as follows. For every $y \in Y$, let $s_1(y) = s(y)$. For every $z \in Z^*$, define $s_1(z) = s^*(z)$. Since $Y \cap Z^* = \emptyset$, s_1 is well defined. It follows that $(Z_1, s_1) \in A$ and $(Z_1, s_1) \geqslant (Z^*, s^*)$ but $Z \subset Z_1$, a contradiction with the maximality of (Z^*, s^*).

The conclusion now follows by specifying $s^*(x)$ arbitrarily for any $x \in N(h, g) \setminus Z^*$. □

Proof of Proposition 5. If (EDP.iii′) fails, then there are $y, y' \in N$ with $y \cap y' = \emptyset$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$ imply $P(c) \cap P(c') = \emptyset$ for all $c, c' \in C_i$ and all $i \in I$. We claim that then for every pair $(w, w') \in y \times y'$ there is $s \in S$ such that $w \in R_s(w)$ and $w' \in R_s(w')$.

First, for every $x \in \ \uparrow y \cap \ \uparrow y'$ and every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \cup y' \subseteq c$. To see this, note that $y \cap y' = \emptyset$ and $x \in \ \uparrow y \cap \ \uparrow y'$ imply $y \subset x$ and $y' \subset x$. Therefore, by (EDP.iv), for every $i \in J(x)$ there are $c, c' \in A_i(x)$ such that $y \subseteq c$ and $y' \subseteq c'$. Since $c \cap c' = \emptyset$ would imply $P(c) \cap P(c') = \emptyset$ in contradiction to $x \in P(c) \cap P(c')$, it follows that $c \cap c' \neq \emptyset$. But then (EDP.i) and $x \in P(c) \cap P(c')$ imply $c = c'$, as desired.

Fix any pair $(w, w') \in y \times y'$. Construct a strategy profile $s \in S$, by specifying $s_i(x)$ for each $x \in X$ and $i \in J(x)$, as follows. If $w, w' \notin x$, specify $s_i(x)$ arbitrarily. If $w \in x$ but $w' \notin x$, define $s_i(x)$ such that $w \in s_i(x)$ (which is possible by (EDP.iv)). If $w' \in x$ but $w \notin x$, define $s_i(x)$ such that $w' \in s_i(x)$. Last, suppose $w, w' \in x$. Then, $x \cap y \neq \emptyset \neq x \cap y'$. Since $y \cap y' = \emptyset$, TI implies that $y \cup y' \subset x$. By the above, for every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \cup y' \subseteq c$. Set $s_i(x) = c$. Clearly, $w \in s_i(x)$ for all $i \in J(x)$ whenever $w \in x$, and analogously for w'. Thus $w \in R_s(w)$ and $w' \in R_s(w')$. □

The following lemma, the proof of which is omitted, is used for Proposition 6.

Lemma 8. *For a game tree $T = (N, \supseteq)$, a history h has a continuation with a maximum if and only if there is $z \in N$ such that $h = \ \uparrow z \setminus \{z\}$.*

Proof of Proposition 6. (a) We first claim that, if a game tree $(N, \supseteq)$ is selective, every chain of the form $\uparrow x \cap \ \uparrow y$ for $x, y \in N$ has a minimum.[27] Let $x, y \in N$ be such that $x \cap y = \emptyset$ (else the result is obvious) and choose $w \in x$ and $w' \in y$. Because the tree is selective, there is $z \in X$ with $w, w' \in z$ such that $\gamma(z, w) \cap \gamma(z, w') = \emptyset$ by Proposition 1(a). Let $z' \in \ \uparrow x \cap \ \uparrow y$. Since $\{w, w'\} \subseteq z \cap z'$, either $z' \subset z$ or $z \subseteq z'$ by TI. If $z' \subset z$, that $w, w' \in z' \in \ \downarrow z \setminus \{z\}$ would imply $z' \subseteq \gamma(z, w) \cap \gamma(z, w')$ in contradiction to $\gamma(z, w) \cap \gamma(z, w') = \emptyset$. Therefore, $z \subseteq z'$ for all $z' \in \ \uparrow x \cap \ \uparrow y$ implies together with $z \in \ \uparrow x \cap \ \uparrow y$ that $z = \min \uparrow x \cap \ \uparrow y$.

Suppose $x \in S(N)$. Since $x \neq \inf \uparrow x \setminus \{x\}$, there is a node x' which is a lower bound of $\uparrow x \setminus \{x\}$ but such that $x' \nsubseteq x$. If $x \subset x'$, then $x' = \min \uparrow x \setminus \{x\}$, a contradiction with $x \in S(N)$. By TI this implies $x \cap x' = \emptyset$. Since x' is a lower bound of $\uparrow x \setminus \{x\}$, it follows that $\uparrow x \setminus \{x\} \subseteq \ \uparrow x \cap \uparrow x'$. By the claim above there exists $z = \min \uparrow x \cap \uparrow x'$. In particular, $z \supseteq x'$ and, therefore, $z \in \ \uparrow x \setminus \{x\}$. It follows that $\uparrow x \cap \ \uparrow x' \subseteq \ \uparrow x \setminus \{x\}$ and, hence, we have equality. But then $z = \min \uparrow x \setminus \{x\}$, in contradiction to $x \in S(N)$.

[27] In other words, every selective game tree is well-joined (see footnote 22).

(b) Assume that T is regular and every history has a continuation with a maximum. Let $w, w' \in W$ be such that $w \neq w'$ and $h = \uparrow\{w\} \cap \uparrow\{w'\}$, so that h is a history. Suppose that h has no minimum. By Lemma 8 there is $z \in N$ such that $h = \uparrow z \setminus \{z\}$. If $z = \bigcap_{x \in h} x$, then $w, w' \in z$ imply $z \in h$, a contradiction. Therefore $z \subset \bigcap_{x \in h} x$. But then, that h has no minimum and Lemma 2(c) imply that $z \in S(N)$, a contradiction.

Therefore, regularity of the tree implies that $h = \uparrow\{w\} \cap \uparrow\{w'\}$ has a minimum $x \in h$. If $\gamma(x, w) \cap \gamma(x, w')$ were nonempty, $\gamma(x, w) = \gamma(x, w')$ would hold by Proposition 1(a). But that would imply that there is a node $y \in \downarrow x \setminus \{x\}$ with $w, w' \in y$, so that $y \in h$ by the construction of h. As this contradicts $x = \min h$, $\gamma(x, w) \cap \gamma(x, w') = \emptyset$ follows. □

Proof of Proposition 7. "If": Assume selectiveness and (EDP.ii′). Let $y, y' \in N$ be such that $y \cap y' = \emptyset$ and $w \in y$ and $w' \in y'$. By selectiveness, there is $x \in X$ with $w, w' \in x$ such that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$. For each $i \in J(x)$ choose $c_i, c_i' \in A_i(x)$ such that $y \subseteq c_i$ and $y' \subseteq c_i'$ which is possible by (EDP.iv), since $w, w' \in x$, $w \in y$, $w' \notin y$, $w' \in y'$, and $w \notin y'$ imply $y, y' \in \downarrow x \setminus \{x\}$ by TI. By the hypothesis and Proposition 1(b) $\gamma(x, w) = x \cap [\bigcap_{i \in J(x)} c_i]$ and $\gamma(x, w') = x \cap [\bigcap_{i \in J(x)} c_i']$, because $\gamma(x, w) \subseteq x \cap [\bigcap_{i \in J(x)} c_i]$ and $\gamma(x, w') \subseteq x \cap [\bigcap_{i \in J(x)} c_i']$ by Lemma 7. In particular, for some $i \in J(x)$ it must be the case that $y \subseteq \gamma(x, w) \subseteq c_i$, $y' \subseteq \gamma(x, w') \subseteq c_i'$, and $c_i \cap c_i' = \emptyset$. For, if $c_i \cap c_i' \neq \emptyset$ for all $i \in J(x)$, then (EDP.i) implies that $c_i = c_i'$ for all $i \in J(x)$ in contradiction to what has been shown, as $x \in P(c_i) \cap P(c_i')$ by construction.

"Only if": Suppose that (EDP.iii′) holds. Let $w, w' \in W$ be such that $w \neq w'$. By IR there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$. By (EDP.iii′) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and $x \in P(c) \cap P(c') \neq \emptyset$, say. By Lemma 7 $\gamma(x, w) \subseteq x \cap c$ and $\gamma(x, w') \subseteq x \cap c'$. Since $c \cap c' = \emptyset$, it follows that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$. Because $x \in P(c) \cap P(c')$, it follows that $\uparrow x = \uparrow y \setminus \downarrow c = \uparrow y' \setminus \downarrow c'$ by (2), in particular, $w, w' \in x$. Thus, the tree is selective.

Let $x \in X$ and $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$. By (EDP.ii) there is $w \in x \cap [\bigcap_{i \in J(x)} c_i]$ and by Lemma 7, $\gamma(x, w) \subseteq x \cap c_i$ for all $i \in J(x)$. Choose $w' \in x \cap [\bigcap_{i \in J(x)} c_i] \setminus \gamma(x, w)$. By IR there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$, so that $y, y' \in \downarrow x \setminus \{x\}$ by TI. By (EDP.iii′) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and $x' \in P(c) \cap P(c')$, say. Therefore, $w, w' \in x' \cap x$ implies either $x \subseteq x'$ or $x' \subset x$ by TI. In the latter case $w \in x'$ implies $x' \subseteq \gamma(x, w)$ in contradiction to $w' \in x'$. Therefore, $x \subseteq x'$. If $x \subset x'$ would holds, then $y \cup y' \subseteq x \subseteq \gamma(x', w) \subseteq x' \cap c$ (by Lemma 7) would contradict $c \cap c' = \emptyset$. Thus, $x' = x$. But then $c' \cap c_i = \emptyset$ implies $w' \notin c_i$, a contradiction. Hence, $x \cap [\bigcap_{i \in J(x)} c_i] \setminus \gamma(x, w) = \emptyset$ yields (EDP.ii′). □

Proof of Proposition 9. Let $x \in X$ and $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$. By (EDP.ii) there is $w \in x \cap [\bigcap_{i \in J(x)} c_i]$. By Lemma 7, $\gamma(x, w) \subseteq x \cap [\bigcap_{i \in J(x)} c_i]$ and, as T is weakly up-discrete, $y \equiv \gamma(x, w) \in N$ by Lemma 4(c). Suppose there is $w' \in x \cap [\bigcap_{i \in J(x)} c_i] \setminus \gamma(x, w)$. Then, $y' \equiv \gamma(x, w') \in N$ by Lemma 4(c), $y' \subseteq x \cap [\bigcap_{i \in J(x)} c_i]$ by Lemma 7, and $y \cap y' = \emptyset$ by Proposition 1(a). By (EDP.iii) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$. Thus, $y' \subseteq c' \cap x$ implies that $x \setminus c \neq \emptyset$ and from $y \subseteq c \cap x$ it follows that $x \setminus c' \neq \emptyset$, i.e. $x \notin \downarrow c$ and $x \notin \downarrow c'$. Thus, $\uparrow x \subseteq \uparrow y \setminus \downarrow c$ and $\uparrow x \subseteq \uparrow y' \setminus \downarrow c'$. By construction of y and y' these inclusions are equalities, though, so that $\uparrow x = \uparrow y \setminus \downarrow c$ and $\uparrow x = \uparrow y' \setminus \downarrow c'$, i.e. $x \in P(c) \cap P(c')$. But then $i \in J(x)$ and $c = c_i \supseteq y' = \gamma(x, w')$ contradicts $c \cap c' = \emptyset$. Thus, $x \cap [\bigcap_{i \in J(x)} c_i] \setminus \gamma(x, w) = \emptyset$ yields, together with Lemma 7, $\gamma(x, w) = x \cap [\bigcap_{i \in J(x)} c_i]$. As x and $(c_i)_{i \in J(x)}$ were arbitrary, (EDP.ii′) holds and the statement follows by Proposition 7. □

The following is a converse to Proposition 3. That is, if an arbitrary EF is playable resp. everywhere playable, then *every* EDP with the same tree is playable resp. everywhere playable. This is needed for the "only if"-part of Theorem 6.

Proposition 10. *Fix a history h for a game tree $T = (N, \supseteq)$. If for an arbitrary EF (T, C) every strategy combination induces outcomes after h, then for the problem $\Pi(T)$ every strategy induces outcomes after h.*

Proof. Pick any strategy $s' \in S'$ for $\Pi(T)$ and choose $s \in S$ for (T, C) as follows: If $s'(x) = \gamma(x, w)$ for $w \in x$, then for (T, C) pick the choice combination $(c_i^w)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ such that $x \cap [\bigcap_{i \in J(x)} c_i^w] \subseteq \gamma(x, w) = s'(x)$ (which exists by (EDP.iv) in the definition of an EDP and by the "only if"-part of Proposition 7) and declare $s_i(x) = c_i^w$ for all $i \in J(x)$; doing this for all $x \in X$ with $x \subseteq W(h) \equiv \bigcap_{y \in h} y$ determines s for the relevant part of the tree. It is then straightforward to see that $U^h(s) \subseteq U^h(s')$. By the "only if"-part of Proposition 2 the hypothesis implies that $U^h(s)$ contains a chain u that is maximal in the set of nodes $x \in N$ that are contained in $W(h)$; by the previous finding $u \subseteq U^h(s')$. Therefore, the "if"-part of Proposition 2 yields that s' induces outcomes after h. Since the strategy s' for $\Pi(T)$ was arbitrary, the desired conclusion follows. □

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