A Self-Study

Lecture Notes and Calculations

Building Quantum Field Theory

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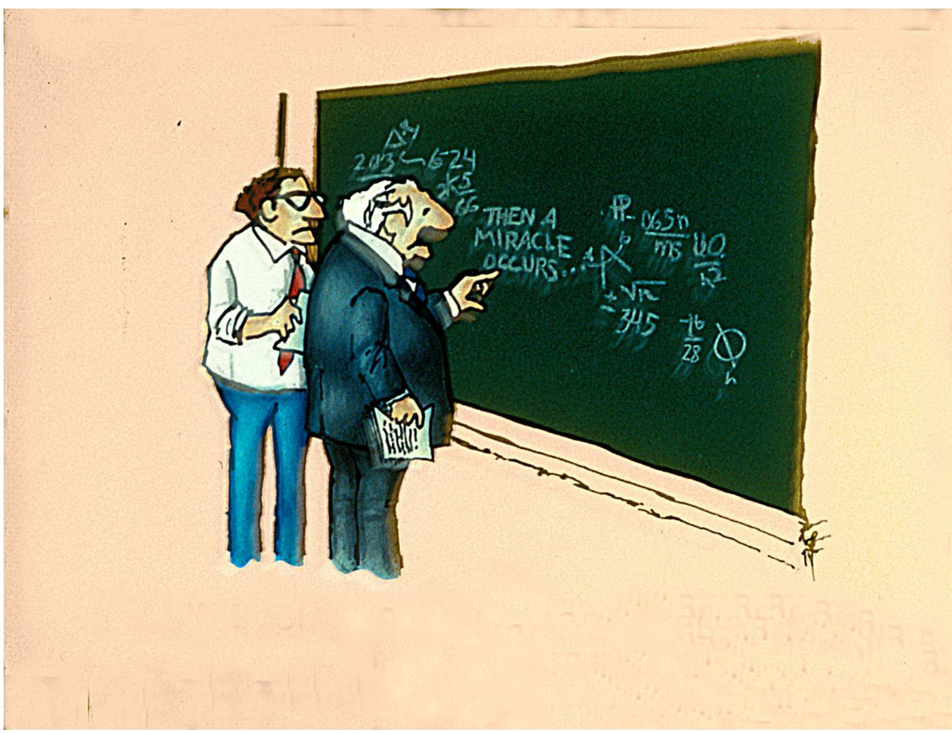

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A note to potential readers...

These pages are the author's personal notes and calculations. They have not been, nor has there ever been, any intention to distribute the works. The lecture style writing is for the author's personal benefit. The Feynman Diagrams were copied from Dr. A. Zee's book *Quantum Field Theory in a Nutshell*. Much of the work grew out of the extraordinary lecture notes of Dr. Michael Ramsey-Musolf from his courses in Advanced Particle Physics and Collider Physics at U.W Madison from 2012-2013, Dr. A. Zee's wonderful book *Quantum Field Theory in a Nutshell*, and J.J Sakurai's classic text, *Modern Quantum Mechanics*. Of course the motivation and inspiration was drawn from countless others, though it is the aforementioned Physicists which provided a clarity, rigor, and insight that has unmistakably influenced me for the better and encouraged me greatly in my endeavors. My endless gratitude and genuine best wishes goes to them and theirs.

1 Introduction and Motivation

I don't want to be misunderstood here - by an open mind I do not mean an empty mind - I mean that perhaps if we consider alternative theories which do not seem *a priori* justified, and we calculate what things would be like if such a theory were true, we might all of a sudden discover that's the way it really is!

-R.P Feynman

To date, the theory of quantum electrodynamics is the most accurate model of the universe ever constructed. The elegant mathematical machinery, as well as the physical insight and clarity necessitated in the derivation of the theory are enormously beneficial. Furthermore those same core concepts may be carried over into the realm of classical physics and therefore the laws of non-relativistic quantum mechanics and Newton may be rediscovered. It is for these reasons that the following discussion exists. The aim of the next pages will be to begin with only Newton's laws and some assumptions (which we will add to as we go, primarily based on physical insight more than mathematical rigor), and to derive the Feynman rules for the quantum field theory of interactions for scalars, bosons and fermions. Some of the mathematical definitions, derivations and subtleties will be addressed, though the primary effort will be on providing a clear physical account as to the nature of these fundamental interactions.

There are many ways by which the quantum theory of elementary particle interactions may be developed, generally by either the canonical formulation or the path integral formulation. Although the theory was first understood by means of the canonical route, the method of calculating amplitudes by summing over paths is much more elegant and oftern times more clear, albeit more technically involved, and so it is the direction of action which will be taken.

1.1 Newton's Laws and Classical Paths

Without a doubt, one of the most famous insights of all time was given by Sir Issac Newton in his book *Principia* and one of those strokes of genius is now referred to as his Second Law which states that the force applied to a body is equivalent to the mass of the body multiplied by its acceleration. Further, both the force applied and the acceleration are vectors in 3-D space which change in time. Formally,

$$\mathbf{F} = m\ddot{\mathbf{x}} \tag{1.1}$$

The units of force are therefore $[kg \cdot m \cdot s^{-2}]$. Upon the development of calculus, it was able to be determined that the amount of energy expended in moving a body from some point, a, to some point, b, would be equal to the force multiplied by the distance which the body traveled. The indefinite mathematical statement is

$$\int \mathbf{F}\; d^3\mathbf{x} = m \int \ddot{\mathbf{x}}\; d^3\mathbf{x} \tag{1.2}$$

Using the fact that

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = \frac{1}{\dot{x}}\frac{d}{dt} \tag{1.3}$$

then we can see that

$$\ddot{\mathbf{x}} = \frac{1}{2}\nabla\dot{\mathbf{x}}^2 \tag{1.4}$$

Therefore, we have

$$\int \mathbf{F}\; d^3\mathbf{x} = m \int \ddot{\mathbf{x}}\; d^3\mathbf{x} = m \int \frac{1}{2}\nabla\dot{\mathbf{x}}^2\; d^3\mathbf{x} = \frac{1}{2}m\dot{\mathbf{x}}^2 = E \tag{1.5}$$

Notice that $m\dot{\mathbf{x}} = \mathbf{p}$ which is a momentum. Let us now define two quantities (functionals), the Hamiltonian and the Lagrangian. The Hamiltonian, H, will be defined as the *sum* of the kinetic and potential energies in the system, whereas the Lagrangian, L, will be defined as the *difference* between the kinetic and potential energies, i.e,

$$H[\mathbf{p},\mathbf{q},t] = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{q},t), \quad L[\mathbf{q},\dot{\mathbf{q}},t] = \frac{1}{2}m\dot{\mathbf{q}}^2 - V(\mathbf{q},t) \tag{1.6}$$

Where $\mathbf{q}$ is simply the generalized position coordinate. It is important to notice that both of these functionals carry units of energy.

Let us now suppose that we have some particle which is going to travel from some point $q_a(t)$ to another point $q_b(t)$. The path which the particle takes will be, in part, constrained by the the end points, which we will take to be zero for simplicity since the surface terms will then vanish. We can also assert that the path will be a function of the difference in kinetic and potential energy. The reason for this is that we wish for the path between points to be a straight line when the kinetic and potential energies are equivalent. We can recognize this energy functional as the Lagrangian. Clearly, for the particle to travel to another spacetime point, some amount of time must have passed and therefore we ought in integrate the Lagrangian over time. The resulting quantity, let us call it S by convention, which has units of $[Energy \cdot time]$ is called the *action*,

$$S = \int_{q_a(t)}^{q_b(t)} L[\mathbf{q}, \dot{\mathbf{q}}, t] \; dt \tag{1.7}$$

Additionaly, it is convenient to define a Lagrangian density, $\mathcal{L}$,

$$L[\mathbf{q}, \dot{\mathbf{q}}, t] = \int \mathcal{L}[\mathbf{q}, \dot{\mathbf{q}}, t] \; d^3\mathbf{x} \tag{1.8}$$

So therefore, we may write

$$S = \int \mathcal{L}[q, \dot{q}] \; d^4x \tag{1.9}$$

Where the $q's$ are generalized to be spacetime coordinates. Since we have defined the endpoints to be zero (fixed), then the total variation of the action, which depends on these endpoints, had also better be zero.

$$\delta S = 0 = \int \delta\mathcal{L} \; d^4x = \int \left[\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} \right] \; d^4x \tag{1.10}$$

Integrating the second term by parts once, and letting the surface term vanish (since the end points vanish) means that the remaining integrand must also vanish, giving the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{1.11}$$

This this must represent the equations of motion chosen by the particle. Indeed, plugging in the most general Lagrangian $m\dot{\mathbf{x}}^2/2 - V(\mathbf{x}, t)$ gives

$$m\ddot{\mathbf{x}} = -\nabla V = \mathbf{F} \tag{1.12}$$

just as it should.

It is worth taking a moment to reflect on these ideas. A system of particles has some total energy given by a functional called the Hamiltonian. Knowing this, we may determine the Lagrangian density, which represents the difference in kinetic and potential energy. Nature has been clever enough to choose a path in which the extremum is minimized, that is, the "path of least resistance." When the variation of the path is null, you have discovered the particle's trajectory. If you know your Hamiltonian, you know your particle's path.

2 Quantum Mechanics

Let us fast-forward to the 1920's, when the laws of quantum mechanics were being formalized. Suppose that there is some state that a particle may be in which can be represented as $|\mathbf{x}\rangle$. The complex conjugate transpose (Hermitian conjugate) may be represented as $\langle\mathbf{x}|$. Further, suppose that some operator, $\hat{A}$ can measure the state without changing it. That is,

$$\hat{A}\,|\mathbf{x}\rangle = x\,|\mathbf{x}\rangle \tag{2.1}$$

Thus, measuring the system with $\hat{A}$ gives the *eigenvalue* x and returns the same state. It is not necessary that the operation return the same state, though for simplicity, it is all that we need consider at the moment. We make the requirement that all states are orthogonal and normalized to unity, i.e,

$$\langle\mathbf{x}'|\,\mathbf{x}\rangle = \delta^3_{\mathbf{x},\mathbf{x}'} \tag{2.2}$$

Additionally, there must be an identity operator, $\hat{I}$ in which

$$\hat{I}\,|\mathbf{x}\rangle = |\mathbf{x}\rangle \tag{2.3}$$

There are operations which are similar to taking the divergence which project out the part of the state that is within some new base state. Suppose that $\hat{\Lambda}$ is one such projection operator. Then,

$$\hat{\Lambda}\,|\mathbf{x}\rangle = c_a(\mathbf{x})\,|\mathbf{a}\rangle \tag{2.4}$$

Evidentally, if $c_a(\mathbf{x}) \neq 0$, then there is an amount of the state vector $|\mathbf{x}\rangle$, which lies in the same direction as the state vector $|\mathbf{a}\rangle$, precisely equal to $c_a(\mathbf{x})$.

If a vector is in some n-dimensional space and we project out each component of that space and sum them together, we must recover the original vector. Mathematically speaking,

$$\left(\sum_{n=0}^{\infty} |n\rangle \langle n|\right) |\psi\rangle = |\psi\rangle = \hat{I}\,|\psi\rangle \tag{2.5}$$

We therefore have our identity operator in a discrete basis. Passing to a continuous basis,

$$|\psi\rangle = \int dx' \;\; |\mathbf{x}'\rangle \langle \mathbf{x}'\, |\psi\rangle = \int dx' \;\psi(\mathbf{x}')\, |\mathbf{x}'\rangle \tag{2.6}$$

This also changes the delta function in (2.2) to

$$\langle \mathbf{x}' |\, \mathbf{x}\rangle = \delta^3(\mathbf{x} - \mathbf{x}') \tag{2.7}$$

2.1 Spatial Translations

Let us now look for a spatial translation operator, $\hat{\mathcal{T}}$ which acts to first order as follows[1],

$$\hat{\mathcal{T}}\,|\mathbf{x}\rangle = |\mathbf{x} + \delta\mathbf{x}\rangle \tag{2.8}$$

Requiring that $\hat{\mathcal{T}}^\dagger \hat{\mathcal{T}} = 1$, we have the requirement that $\delta^\dagger = -\delta$. We are also welcome to guess at the form of the translation operator to be

$$\mathcal{T} = 1 - i\mathbf{K} \cdot d\mathbf{x}' \tag{2.9}$$

Therefore,

$$\begin{aligned} \mathcal{T}^\dagger \mathcal{T} &= (1 + i\mathbf{K}^\dagger \cdot d\mathbf{x}')(1 - i\mathbf{K} \cdot d\mathbf{x}') && (2.10)\\ &= 1 - i(\mathbf{K} - \mathbf{K}^\dagger) \cdot d\mathbf{x}' + O(d\mathbf{x}'^2) && (2.11)\\ &\approx 1 && (2.12) \end{aligned}$$

In order to determine now the relationship of this vector quantity $\mathbf{K}$ and the $x's$, let's try

$$\mathbf{x}\hat{\mathcal{T}}\,|\mathbf{x}'\rangle = \mathbf{x}\,|\mathbf{x}' + d\mathbf{x}'\rangle = (\mathbf{x}' + d\mathbf{x}')\,|\mathbf{x}' + d\mathbf{x}'\rangle \tag{2.13}$$

and

[1] Ref. J.J. Sakurai *Modern Quantum Mechanics*

$$\hat{\mathcal{T}}\mathbf{x}\,|\mathbf{x}'\rangle = \mathbf{x}'\hat{\mathcal{T}}\,|\mathbf{x}'\rangle = \mathbf{x}'\,|\mathbf{x}' + d\mathbf{x}'\rangle \tag{2.14}$$

hence,

$$[\mathbf{x}, \hat{\mathcal{T}}] = d\mathbf{x}'\,|\mathbf{x}' + d\mathbf{x}'\rangle \approx d\mathbf{x}'\,|\mathbf{x}'\rangle \tag{2.15}$$

Now, since

$$[\mathbf{x}, \hat{\mathcal{T}}] = d\mathbf{x}' = -i\mathbf{x}\mathbf{K}\cdot d\mathbf{x}' + i\mathbf{K}\cdot d\mathbf{x}'\mathbf{x} \tag{2.16}$$

then we may obtain

$$[x_i, K_j] = i\delta_{ij} \tag{2.17}$$

It is clear here that the $K's$ must have units of inverse length. Using the relationship

$$\frac{2\pi}{\lambda} = \frac{p}{\hbar} \tag{2.18}$$

we suppose then that $K_j \to p_j/\hbar$ and so

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{2.19}$$

2.2 Time Evolution

Suppose now that we wish to introduce the notion of time into quantum mechanics. We would like to see (in the Schrödinger picture) some state ket evolve in time as follows

$$|\alpha, t_0\rangle \xrightarrow{time} |\alpha, t_0; t\rangle \tag{2.20}$$

Call the tine-evolution operator $\hat{\mathcal{U}}$. Then,

$$\hat{\mathcal{U}}\,|\alpha, t_0\rangle = |\alpha, t_0; t\rangle \tag{2.21}$$

Now surely, the probability that some state, which is entirely isolated in the universe, remains in that same state must be unity. Equivalently, the probability that some state, which has been allowed to evolve in time, must also be unitary when multiplied by its complex conjugate. In a mathematical sense

$$\langle \alpha, t_0 \mid \alpha, t_0 \rangle = 1 = \langle \alpha, t_0; t \mid \alpha, t_0; t \rangle \tag{2.22}$$

So then,

$$\langle \alpha, t_0 | \hat{\mathcal{U}}^\dagger \hat{\mathcal{U}} \, | \alpha, t_0 \rangle = 1 \quad \rightarrow \quad \hat{\mathcal{U}}^\dagger \hat{\mathcal{U}} = 1 \tag{2.23}$$

Therefore the time-evolution operator must be unitary. Some consider the conservation of probability to depend on it. There is another property which we assign to this operator which we call the *compositional* property. If some state is at a time t_0 and we move it first to time t_1 then to time t_2, this ought to be equivalent to moving that state from time t_0 directly to time t_2. To see this more clearly, look at the operator $\hat{\mathcal{U}}$ as a specific function of time,

$$\hat{\mathcal{U}}(t_2, t_0) \, |\alpha, t_0\rangle = |\alpha, t_0; t_2\rangle = \hat{\mathcal{U}}(t_2, t_1) \hat{\mathcal{U}}(t_1, t_0) \, |\alpha, t_0\rangle \tag{2.24}$$

Continuity requires

$$\lim_{dt \to 0} \hat{\mathcal{U}}(t_0 + dt, t_0) = 1 \tag{2.25}$$

All of these properties are satisfied if we assert that

$$\hat{\mathcal{U}}(t_0 + dt, t_0) = 1 - i\Omega dt \tag{2.26}$$

Then we must have from the unitary condition

$$\hat{\mathcal{U}}^\dagger(t_0 + dt, t_0) \hat{\mathcal{U}}(t_0 + dt, t_0) = \left(1 + i\Omega^\dagger dt\right)(1 - i\Omega dt) \simeq 1 \tag{2.27}$$

Let us veer from the mathematics and into physics to guess as the true character of Ω. It has the dimensions of frequency. We know that $E = \hbar\omega$, so that we may guess that the operator we seek is

$$\Omega = \frac{\hat{H}}{\hbar} \tag{2.28}$$

and so we have

$$\hat{\mathcal{U}}(t_0 + dt, t_0) = 1 - \frac{i\hat{H}dt}{\hbar} \tag{2.29}$$

We may of course do an infinite amount of infinitesimal translations and determine the operator to be the following

$$\lim_{N\to\infty}\left[1-\frac{(i\hat{H}/\hbar)(t-t_0)}{N}\right]^N = e^{-i\hat{H}(t-t_0)/\hbar} = \hat{\mathcal{U}}(t,t_0) \tag{2.30}$$

There is another way to derive this result. Using the fact that

$$\hat{\mathcal{U}}(t+dt,t_0) = \hat{\mathcal{U}}(t+dt,t)\hat{\mathcal{U}}(t,t_0) = \left(1-\frac{i\hat{H}dt}{\hbar}\right)\hat{\mathcal{U}}(t,t_0) \tag{2.31}$$

therefore,

$$\hat{\mathcal{U}}(t+dt,t_0) - \hat{\mathcal{U}}(t,t_0) = -i\left(\frac{\hat{H}}{\hbar}\right)dt\,\hat{\mathcal{U}}(t,t_0) \tag{2.32}$$

which is equivalent to

$$i\hbar\frac{\partial}{\partial t}\hat{\mathcal{U}}(t,t_0) = \hat{H}\hat{\mathcal{U}}(t,t_0) \tag{2.33}$$

This produces the same solution as above for the time evolution operator. We may multiply both sides by the state ket $|\alpha,t_0\rangle$ to get

$$i\hbar\frac{\partial}{\partial t}\left|\alpha,t_0;t\right\rangle = \hat{H}\left|\alpha,t_0;t\right\rangle \tag{2.34}$$

And therefore we have derived the much celebrated Schrödinger equation.

2.3 Path Integrals

Let us follow the path of A. Zee[2] in deriving the path integral formalism as it is extremely clear. If we split up the time interval into N components then we can write $T = N\delta t$. Ignoring the hats to signify operators and setting $\hbar = c = 1$, then the amplitude between an initial and final state is

$$\langle q_F|\, e^{-iHT}\, |q_I\rangle = \langle q_F|\, e^{-iH\delta t}e^{-iH\delta t}\cdots e^{-iH\delta t}\, |q_I\rangle \tag{2.35}$$

Inserting a complete set of base states between each infinitesimal time evolution operator gives

$$\langle q_F|\, e^{-iHT}\, |q_I\rangle = \left(\prod_{j=1}^{N-1}\int dq_j\right)\langle q_F|\, e^{-iH\delta t}\, |q_{N-1}\rangle\,\langle q_{N-1}|\, e^{-iH\delta t}\, |q_{N-2}\rangle\cdots\langle q_2|\, e^{-iH\delta t}\, |q_1\rangle\,\langle q_1|\, e^{-iH\delta t}\, |q_I\rangle$$

[2] Ref. A. Zee *Quantum Field Theory in a Nutshell*

Now we can shift into momentum space and begin to solve each of these integrals

$$\begin{aligned}
\langle q_{j+1}| e^{-i\delta tp^2/2m} |q_j\rangle &= \int \frac{dp}{2\pi} \langle q_{j+1}| e^{-i\delta tp^2/2m} |p\rangle \langle p \mid q_j\rangle \\
&= \int \frac{dp}{2\pi} e^{-i\delta tp^2/2m} \langle q_{j+1} \mid p\rangle \langle p \mid q_j\rangle \\
&= \int \frac{dp}{2\pi} e^{-i\delta tp^2/2m} e^{ip(q_{j+1}-q_j)} \\
&= \left(\frac{m}{2\pi i\delta t}\right)^{1/2} \exp\left[\frac{im\,\delta t}{2}\left(\frac{(q_{j+1}-q_j)}{\delta t}\right)^2\right] \quad (2.36)
\end{aligned}$$

And therefore

$$\langle q_F| e^{-iHT} |q_I\rangle = \left(\frac{m}{2\pi i\delta t}\right)^{N/2} \left(\prod_{k=1}^{N-1} \int dq_k\right) \exp\left[\frac{im\,\delta t}{2} \sum_{j=0}^{N-1} \left(\frac{(q_{j+1}-q_j)}{\delta t}\right)^2\right]$$

Now make the substitution in the continuum limit that $\left(\frac{(q_{j+1}-q_j)}{\delta t}\right)^2 \to \dot{q}^2$ and define the integral over all paths to be

$$\int \mathcal{D}q(t) \equiv \lim_{N\to\infty} \left(\frac{m}{2\pi i\delta t}\right)^{N/2} \left(\prod_{k=1}^{N-1} \int dq_k\right) \quad (2.37)$$

And so we now have the commonly used representation for path integrals,

$$\langle q_F| e^{-iHT} |q_I\rangle = \int \mathcal{D}q(t)\; e^{i\int_0^T dt \frac{1}{2}m\dot{q}^2} \quad (2.38)$$

which may be generalized to

$$\langle q_F| e^{-iHT} |q_I\rangle = \int \mathcal{D}q(t)\; e^{i\int_0^T dt \frac{1}{2}m\dot{q}^2 - V(q)} \quad (2.39)$$

We recognize the exponent as the classical Lagrangian. Following the methods derived earlier in the Classical Mechanics section, we may use instead the Lagrangian density. Let us also promote the spatial operator q to a field operator φ (again, the hat has been suppressed). Then, including the $\hbar$'s again,

$$Z \equiv \langle q_F| e^{-iHT/\hbar} |q_I\rangle = \int \mathcal{D}\varphi\; e^{\frac{i}{\hbar}\int d^4x\, \mathcal{L}(\varphi,\dot{\varphi})} \quad (2.40)$$

The dependence of φ on x is to be understood.

3 Quantum Field Theory for Spin 0 Particles

I would now like to dive into the somewhat messy calculation of the scattering amplitude for 4 scalar particles (two incoming and two outgoing) to first order perturbation in the interaction. Following that, an example will be calculated, again with 4 particles, but to second order perturbation in the interaction. The integrals remaining are done over the, potentially massive, spin 0 propagators which are known as "virtual" particles. These simple examples will demonstrate the need for renormalization, mass regularization or an upper bound cutoff, and the U.V catastrophe.

We begin with the quantum mechanical path integral integrated over the complete measure $D\varphi$ and insert the Lagrangian density for scalar fields interacting in a φ^4 theory. This Lagrangian only involves squared terms and therefore guarantees Lorentz invariance, and as we know, if the action is Lorentz invariant, then all of the physics derived from here, such as the equations of motion, will also be Lorentz invariant.

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \; e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] + J\varphi - \frac{\lambda}{4!}\varphi^4\right\}} \\ &= \int \mathcal{D}\varphi \; e^{i\int d^4x\left\{-\frac{1}{2}\varphi\left[\partial^2 + m^2\right]\varphi + J\varphi - \frac{\lambda}{4!}\varphi^4\right\}} \end{aligned} \tag{3.1}$$

This integral, at present, has proven to be insurmountably difficult to calculate in closed form. Fortunately, physicists are not determined to obtain infinite precision. Therefore we can expand the exponent and approximate the solution which turns out to be rather simple. Rewrite the integral as follows:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \; e^{i\int d^4x\left\{-\frac{1}{2}\varphi\left[\partial^2+m^2\right]\varphi + J\varphi\right\}} \left[1 - \frac{i\lambda}{4!}\int d^4\omega \; \varphi^4 + \cdots\right] \\ &= \int \mathcal{D}\varphi \; e^{i\int d^4x\left\{-\frac{1}{2}\varphi\left[\partial^2+m^2\right]\varphi + J\varphi\right\}} \left[1 - \frac{i\lambda}{4!}\int d^4\omega \; \left(\frac{\delta}{i\delta J(\omega)}\right)^4 + \cdots\right] \\ &= \int \mathcal{D}\varphi \; e^{\frac{-i\lambda}{4!}\int d^4\omega\left(\frac{\delta}{i\delta J(\omega)}\right)^4} e^{i\int d^4x\left\{-\frac{1}{2}\varphi\left[\partial^2+m^2\right]\varphi + J\varphi\right\}} \\ &= e^{\frac{-i\lambda}{4!}\int d^4\omega\left(\frac{\delta}{i\delta J(\omega)}\right)^4} \left(\frac{(2\pi i)^N}{\det[\partial^2 + m^2]}\right)^{1/2} e^{-\frac{i}{2}\int J(x)\Delta^{(0)}(x-y)J(y)d^4x d^4y} \end{aligned}$$

Defining

$$Z[0,0] \equiv \left(\frac{(2\pi i)^N}{\det[\partial^2+m^2]}\right)^{1/2}; \quad A \equiv (\partial^2 + m^2); \quad A^{-1} \equiv \Delta^{(0)}$$

and with the understanding that dx really means the four dimensional spacetime measure, then we can write

$$Z[J,\lambda] = Z[0,0]e^{-\frac{i\lambda}{4!}\int d^4\omega\left(\frac{\delta}{i\delta J(\omega)}\right)^4} e^{-\frac{i}{2}\int J(x)\Delta^{(0)}(x-y)J(y)dxdy} \tag{3.2}$$

Now when we want to solve some scattering problem, the approach is clear; just expand to the correct power of λ, plug in the propagator $\Delta^{(0)}$ and away we go. A few points to notice first: whenever we expand to the next power of λ, we add four derivatives with respect to J, which means that four powers of the field operator, φ, come down. Therefore, we introduce four particles. Each term also carries two J's with it, so we'd bring down eight of those. These correspond to the sources and sinks. Thus, each particle has its own source and sink. If we demand that the sinks of two particles and the sources of the other two particles (in a four particle configuration again) be in the same place, then the particles collide here, i.e. a scattering process. Also note that an integral which is odd in its powers of φ vanishes, so that 3-particle states cannot exist under these conditions.

3.1 First Order Expansion

Suppose that we start with a vacuum. We wish now to disturb the vacuum in such a way as to create two particles, let them interact, and then be returned to the same vacuum state as before. This means that we cannot have any source terms in our equations of motion. The sensible thing to do would be to simply set $J = 0$ in the action and go from there. However, in order to evaluate the integral, we had to expand in derivatives of J and therefore we cannot proceed in that way. What we must do is calculate out any interaction we wish and only at the end can we set $J = 0$. Taking the derivatives that are required obviously brings down many factors of J, therefore we need not worry about every term. Furthermore, it is possible to have *connected* and *disconnected* constructions. For the purposes of scattering processes, it is only relevant to examine connected structures. It may be worked out that the interactions which are connected may be constructed by expanding the exponent to the appropriate power of J so that none will remain upon setting J to zero. This is easier seen by example. Thus, let us finally consider the four point function of a scalar interaction. The graphical representation of the connected diagram is shown in Figure 1.

The (normalized) problem we need to solve is

$$Z[J,\lambda=1] = -\frac{i\lambda}{4!}\int d^4\omega\left(\frac{\delta}{i\delta J(\omega)}\right)^4 \frac{1}{4!}\left[\left(-\frac{i}{2}\right)\int J(x)\Delta^{(0)}(x-y)J(y)\,dx\,dy\right]^4$$

It needs to be understood here that the integrated measures are all four-dimensional. There are thirty-six integrals which need to be done here, fortunately, it is not as daunting as it may seem. Remember too that the functional

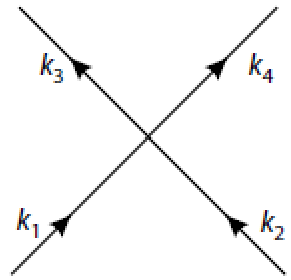


Figure 1: First order four-point scalar interaction

derivative will bring out a factor of two for each power since there are two ways in which the derivative can hit the source terms. Knowing that, we can get rid of those 2's in the denominator within the square brackets straight away. Also, $(-i)^4 = 1$. Therefore we have

$$Z[J,1] = -\frac{i\lambda}{(4!)^2}\int d^4\omega \left(\frac{\delta}{\delta J(\omega)}\right)^4 \left[\int J(x)\Delta^{(0)}(x-y)J(y)\ dx\, dy\right]^4$$

Since things will become somewhat cumbersome, I am going to introduce the following notation;

$$\int J(x)\Delta^{(0)}(x-y)J(y)\ d^4x\, d^4y \equiv \Gamma^0_{xy} \qquad \int J(x)\Delta^{(0)}(x-\omega)d^4x \equiv \Gamma^0_{x\omega} \tag{3.3}$$

And now we just turn the crank...

$$\begin{aligned} Z[J,1] &= -\frac{i\lambda}{(4!)^2}\int d^4\omega\left(\frac{\delta}{\delta J(\omega)}\right)^3 4\left(\Gamma^0_{xy}\right)^3\Gamma^0_{x\omega} \\ &= -\frac{i\lambda}{(4!)^2}\int d^4\omega\left(\frac{\delta}{\delta J(\omega)}\right)^2 4\cdot 3\left(\Gamma^0_{xy}\right)^2\left(\Gamma^0_{x\omega}\right)^2 \\ &= -\frac{i\lambda}{(4!)^2}\int d^4\omega\left(\frac{\delta}{\delta J(\omega)}\right) 4\cdot 3\cdot 2\Gamma^0_{xy}\left(\Gamma^0_{x\omega}\right)^3 \\ &= -\frac{i\lambda}{(4!)^2}\int d^4\omega\ 4\cdot 3\cdot 2\cdot 1\left(\Gamma^0_{x\omega}\right)^4 \\ &= -\frac{i\lambda}{4!}\int d^4\omega\ \left(\Gamma^0_{x\omega}\right)^4 \\ &= -\frac{i\lambda}{4!}\int d^4\omega\ \left[\int J(x)\Delta^{(0)}(x-\omega)d^4x\right]^4 \end{aligned} \tag{3.4}$$

As you surely noticed, a lot just happened. Most obviously, a lot of terms were not dealt with when taking the derivatives, like terms which contain $\left[\Delta^{(0)}(0)\right]^n$ where $n = 1, 2....$ Remember that this functional is the propagator for scalar particles, so if the argument is null, then a particle's source and sink are at the same spacetime point, i.e. a loop diagram. But here we are not considering loops since then, to this order, there is no interaction *between* particles, just one freely propagating particle and one vacuum contribution term.

Also, note that there still are plenty of J's wandering about, which is fine since we still have the integral over all ω and since we really want the scattering amplitude which will result in taking four more derivatives. It is easy to derive this connection from the canonical quantization standpoint and looking at the time-ordered products structure since the bra-ket formalism makes the appearance of an amplitude easy to spot. Obviously then, a term with less than four source terms would have been zero anyway.

The amplitude for the process we are considering has four particular spacetime trajectories, that of the incoming particles and outgoing particles, and a single interaction point which can occur anywhere in spacetime, and so this dummy variable (here it is ω) is integrated over all spacetime. Let's call the four momenta k_1, k_2, k_3 and k_4. Then the scattering amplitude, call it $\mathcal{M}$, can be written as:

$$\mathcal{M} = \lim_{J\to 0} \frac{\delta^4 Z[J,\lambda]}{i^4 \delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \tag{3.5}$$

For clarity, let $J(x_a) \equiv J_a$. Then,

$$\mathcal{M} = \lim_{J\to 0} \frac{\delta^4 Z[J,\lambda]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \tag{3.6}$$

One last piece of shorthand here: let $\Delta^{(0)}(x_a - \omega) \equiv \Delta^{(0)}_{a,\omega}$. Now we just turn the crank again.

$$
\begin{aligned}
\mathcal{M} &= \lim_{J\to 0}\left(-\frac{i\lambda}{4!}\right)\frac{\delta^4}{i^4\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\int d^4\omega\ \left(\Gamma^0_{x\omega}\right)^4 \\
&= \lim_{J\to 0}\left(-\frac{i\lambda}{4!}\right)\frac{\delta^3}{i^4\delta J(x_1)\delta J(x_2)\delta J(x_3)}\int d^4\omega\ 4\left(\Gamma^0_{x\omega}\right)^3\Delta^{(0)}_{4,\omega} \\
&= \lim_{J\to 0}\left(-\frac{i\lambda}{4!}\right)\frac{\delta^2}{i^4\delta J(x_1)\delta J(x_2)}\int d^4\omega\ 4\cdot 3\left(\Gamma^0_{x\omega}\right)^2\Delta^{(0)}_{4,\omega}\Delta^{(0)}_{3,\omega} \\
&= \lim_{J\to 0}\left(-\frac{i\lambda}{4!}\right)\frac{\delta}{i^4\delta J(x_1)}\int d^4\omega\ 4\cdot 3\cdot 2\Gamma^0_{x\omega}\Delta^{(0)}_{4,\omega}\Delta^{(0)}_{3,\omega}\Delta^{(0)}_{2,\omega} \\
&= \lim_{J\to 0}\left(-\frac{i\lambda}{4!}\right)\int d^4\omega\ 4\cdot 3\cdot 2\Delta^{(0)}_{4,\omega}\Delta^{(0)}_{3,\omega}\Delta^{(0)}_{2,\omega}\Delta^{(0)}_{1,\omega} \\
&= (-i\lambda)\int d^4\omega\ \Delta^{(0)}_{4,\omega}\Delta^{(0)}_{3,\omega}\Delta^{(0)}_{2,\omega}\Delta^{(0)}_{1,\omega} \qquad (3.7)
\end{aligned}
$$

Or in its full glory,

$$
\mathcal{M} = -i\lambda\int d^4\omega\ \Delta^{(0)}(x_1-\omega)\Delta^{(0)}(x_2-\omega)\Delta^{(0)}(x_3-\omega)\Delta^{(0)}(x_4-\omega) \qquad (3.8)
$$

The order of the propagators does not matter, and they enjoy an even symmetry, therefore it would not matter if we switched around the particular spacetime point x_a and the dummy variable ω. Solving the Klein-Gordon equation to obtain the propagators is almost trivially easy. The result is

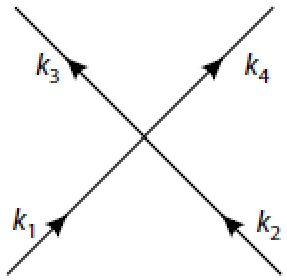
$$
\Delta^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon} \qquad (3.9)
$$

Let us associate some point in spacetime x_a with a momentum k_a. Furthermore, let us look at the 4-point vertex diagram and assign k_1 and k_2 a positive momentum and k_3 and k_4 a negative momentum. This is intuitively satisfying and will guarantee momentum conservation. We may then write the scattering amplitude as

$$
\mathcal{M} = -i\lambda\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}d^4\omega\ \frac{e^{ik_1(x_1-\omega)}}{k_1^2-m^2+i\epsilon}\frac{e^{ik_2(x_2-\omega)}}{k_2^2-m^2+i\epsilon}\frac{e^{-ik_3(x_3-\omega)}}{k_3^2-m^2+i\epsilon}\frac{e^{-ik_4(x_4-\omega)}}{k_4^2-m^2+i\epsilon}
$$

Doing the integral over $d^4\omega$ gives

$$
\mathcal{M} = -i\lambda(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)\times
$$



$$= \quad -i\lambda$$

Figure 2: Scalar interaction amplitude

$$\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}\ \frac{e^{ik_1x_1}}{k_1^2-m^2+i\epsilon}\frac{e^{ik_2x_2}}{k_2^2-m^2+i\epsilon}\frac{e^{-ik_3x_3}}{k_3^2-m^2+i\epsilon}\frac{e^{-ik_4x_4}}{k_4^2-m^2+i\epsilon} \tag{3.10}$$

The remaining integral will always be there in this way for any kind of spin 0 scattering problem. Therefore, it is not really necessary to carry it around all the time, so let's ignore it. (This process is sometimes called "amputating the external legs"). Now we have

$$\mathcal{M} = -i\lambda(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)$$

Furthermore, the momentum conserving delta funtion and it's factor of $(2\pi)^4$ will always be there, since momentum must be conserved. So we may take it as a given and not bother ourselves to carry it around either. So finally, we are left a delightfully simple answer shown diagrammatically in Figure 2

$$\mathcal{M} = -i\lambda \tag{3.11}$$

As a quick aside, the scattering matrix (*s-matrix*) is $\mathcal{S}_{fi} = 1 - i(2\pi)^4\delta^4(P_f - P_i)\mathcal{M} + \cdots$, so we see that we are really just calculating the first order term in the full scattering matrix expansion and where that momentum conserving delta function has gone to.

3.2 Second Order Expansion

Next, let us up the ante a little bit and calculate the scattering amplitude for four spin-0 particles to second order in the interaction which is shown pictorially in Figure 3.

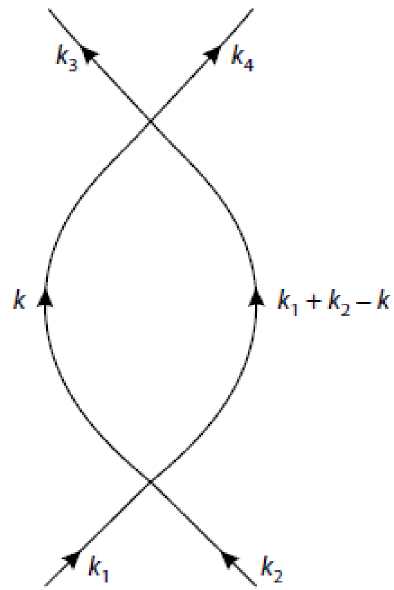


Figure 3: Scalar second order diagram

This problem is much more involved, and it will pay dividends in the end to think carefully about the setup of the problem. Work smart, not hard. Clearly this is second order in λ, so we understand which term in the series expansion around the interaction terms that we need to keep. To find Z, we will now have to take eight derivatives, and therefore, to find the scattering matrix we will have to take twelve. Therefore, we must have exactly twelve powers of J present at the start. There are no loops which close upon themselves, therefore any term which attains a $\Delta_0^{(0)}$ may be thrown out. However, we must have exactly *two* propagators present of the form $\Delta^{(0)}(x_a - x_b)$ wherein both points are to be integrated over. Remember from the last problem that the derivatives bring out factors of 2 since they can hit either of the two J's. Okay, let's see what we can do.

$$Z[J, \lambda = 2] = \frac{1}{2}\left(-\frac{i\lambda}{4!}\right)^2 \int d^4z\, d^4\omega \, \left(\frac{\delta}{i\delta J(z)}\right)^4 \left(\frac{\delta}{i\delta J(\omega)}\right)^4 \left(\frac{1}{6!}\right) \left[-i\int J(x)\Delta^{(0)}(x-y)J(y)\; d^4x\, d^4y\right]^6$$

Thanks to the shorthand notation introduced before we may write this as

$$Z[J, 2] = \frac{(-i)^2(-i\lambda)^2}{2\cdot 6!\cdot (4!)^2}\int d^4z\, d^4\omega \, \left(\frac{\delta}{i\delta J(z)}\right)^4 \left(\frac{\delta}{i\delta J(\omega)}\right)^4 \left(\Gamma^0_{xy}\right)^6 \qquad (3.12)$$

To make life easier yet again, a further notation will be used:

$$\frac{\delta}{\delta J(a)}\left(\Gamma^0_{xy}\right)^n = n\left(\Gamma^0_{xy}\right)^{n-1}\mathcal{A}_{ax}$$

$$\frac{\delta}{\delta J(b)}\mathcal{A}_{ax} = \Delta^{(0)}_{ab}$$

Notice that if a is used, so is $\mathcal{A}$. If we had chosen d then we'd also have to use $\mathcal{D}$ etc.

The thing to do now, is to divide the problem into chunks. First consider the derivatives w.r.t z. Call this chunk $\mathcal{I}$.

$$\begin{aligned}
\mathcal{I} &= \left(\frac{\delta}{i\delta J(z)}\right)^4 \left(\Gamma^0_{xy}\right)^6 \\
&= \left(\frac{\delta}{\delta J(z)}\right)^3 6\left(\Gamma^0_{xy}\right)^5 \mathcal{Z}_{zx} \\
&= \left(\frac{\delta}{\delta J(z)}\right)^2 6\cdot 5\left(\Gamma^0_{xy}\right)^4 \mathcal{Z}^2_{zx} \\
&= \left(\frac{\delta}{\delta J(z)}\right) 6\cdot 5\cdot 4\left(\Gamma^0_{xy}\right)^3 \mathcal{Z}^3_{zx} \\
&= \frac{6!}{2}\left(\Gamma^0_{xy}\right)^2 \mathcal{Z}^4_{zx}
\end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned}
Z[J,2] &= \frac{(-i)^2(-i\lambda)^2}{2\cdot 6!\cdot(4!)^2}\int d^4z\, d^4\omega\, \left(\frac{\delta}{i\delta J(\omega)}\right)^4 \mathcal{I} \\
&= \frac{(-i)^2(-i\lambda)^2}{4\cdot(4!)^2}\int d^4z\, d^4\omega\, \left(\frac{\delta}{\delta J(\omega)}\right)^4 \left(\Gamma^0_{xy}\right)^2 \mathcal{Z}^4_{zx}
\end{aligned} \tag{3.14}$$

Now do the derivatives w.r.t ω remembering that only terms which contain $\Delta^{(0)}_{\omega z}$ will be considered.

$$\left(\frac{\delta}{\delta J(\omega)}\right)^4 \left(\Gamma^0_{xy}\right)^2 \mathcal{Z}^4_{zx} = \left(\frac{\delta}{\delta J(\omega)}\right)^3 \left[4\Delta^{(0)}_{\omega z}\left(\Gamma^0_{xy}\right)^2 \mathcal{Z}^3_{zx} + 2\Gamma^0_{xy}\mathcal{Z}^4_{xz}\mathcal{W}_{\omega x}\right]$$

$$= \left(\frac{\delta}{\delta J(\omega)}\right)^2 \left[12\left(\Delta^{(0)}_{\omega z}\right)^2\left(\Gamma^0_{xy}\right)^2 \mathcal{Z}^2_{xz} + 8\Delta^{(0)}_{\omega z}\Gamma^0_{xy}\mathcal{W}_{\omega x}\mathcal{Z}^3_{xz} + 2\mathcal{W}^2_{\omega x}\mathcal{Z}^4_{xz} + 2\Gamma^0_{xy}\Delta^{(0)}_{\omega z}\mathcal{W}_{\omega x}\mathcal{Z}^3_{xz}\right]$$

$$= \frac{\delta}{\delta J(\omega)} \left[24 \left(\Delta^{(0)}_{\omega z}\right)^2 \mathcal{Z}^2_{xz} \mathcal{W}_{\omega x} \Gamma^0_{xy} + 8\Delta^{(0)}_{\omega z} \mathcal{Z}^2_{xz} \mathcal{W}^2_{x\omega} + 24 \mathcal{Z}^2_{xz} \mathcal{W}_{xz} \Gamma^0_{xy} \left(\Delta^{(0)}_{\omega z}\right)^2 + 8\mathcal{W}^2_{x\omega} \mathcal{Z}^3_{xz} \Delta^{(0)}_{\omega z} \right.$$

$$\left. + 24 \mathcal{W}_{x\omega} \mathcal{Z}^2_{x\omega} \left(\Delta^{(0)}_{\omega z}\right)^2 \Gamma^0_{xy} + 8\mathcal{W}^2_{x\omega} \mathcal{Z}^3_{x\omega} \Delta^{(0)}_{\omega z} \right] \tag{3.15}$$

$$= 6 \cdot 24 \mathcal{Z}^2_{xz} \mathcal{W}^2_{\omega x} \left(\Delta^{(0)}_{\omega z}\right)^2 \tag{3.16}$$

Therefore,

$$\begin{aligned} Z[J,2] &= \frac{(-i)^2(-i\lambda)^2}{4 \cdot (4!)^2} \int d^4z\, d^4\omega\ 6 \cdot 24 \mathcal{Z}^2_{xz} \mathcal{W}^2_{\omega x} \left(\Delta^{(0)}_{\omega z}\right)^2 \\ &= \frac{i^2(-i\lambda)^2}{16} \int d^4z\, d^4\omega\ \mathcal{Z}^2_{xz} \mathcal{W}^2_{\omega x} \left(\Delta^{(0)}_{\omega z}\right)^2 \end{aligned} \tag{3.17}$$

We can easily do the derivatives from (3.5) and (3.6) in order to end up with the scattering amplitude (a.k.a the Green's function) and we end up with

$$\mathcal{M} = \frac{i^2(-i\lambda)^2}{16} \int d^4z\, d^4\omega\ 4 \left(\Delta^{(0)}_{\omega z}\right)^2 \left[\Delta^{(0)}_{x_4,\omega} \Delta^{(0)}_{x_3,z} \Delta^{(0)}_{x_2,\omega} \Delta^{(0)}_{x_1,z} + \Delta^{(0)}_{x_4,\omega} \Delta^{(0)}_{x_3,z} \Delta^{(0)}_{x_2,z} \Delta^{(0)}_{x_1,\omega} \right.$$

$$\left. + \Delta^{(0)}_{x_4,\omega} \Delta^{(0)}_{x_3,\omega} \Delta^{(0)}_{x_2,z} \Delta^{(0)}_{x_1,z} + \Delta^{(0)}_{x_4,z} \Delta^{(0)}_{x_3,z} \Delta^{(0)}_{x_2,\omega} \Delta^{(0)}_{x_1,\omega} + \Delta^{(0)}_{x_4,z} \Delta^{(0)}_{x_3,\omega} \Delta^{(0)}_{x_2,\omega} \Delta^{(0)}_{x_1,z} + \Delta^{(0)}_{x_4,z} \Delta^{(0)}_{x_3,\omega} \Delta^{(0)}_{x_2,z} \Delta^{(0)}_{x_1,\omega} \right]$$

Now apply all of the insight that you have. The propagators have even symmetry. We may assign the sign of the momenta; let the ω's be positive and the z's be negative. Now, owing to the reflective symmetry, the third and fourth terms are the same. The rest of the terms are then to be omitted. This brings out a factor of two and so we have the following remaining:

$$\begin{aligned} \mathcal{M} &= \frac{4 \cdot 2 \cdot i^2(-i\lambda)^2}{16} \int d^4z\, d^4\omega\ \left(\Delta^{(0)}_{\omega z}\right)^2 \Delta^{(0)}_{x_4,\omega} \Delta^{(0)}_{x_3,\omega} \Delta^{(0)}_{x_2,z} \Delta^{(0)}_{x_1,z} \\ &= \frac{i^2(-i\lambda)^2}{2} \int d^4z\, d^4\omega\ \Delta^{(0)}(\omega - x_1) \Delta^{(0)}(\omega - x_2) \Delta^{(0)}_{q_1}(\omega - z) \Delta^{(0)}_{q}(\omega - z) \Delta^{(0)}(z - x_3) \Delta^{(0)}(z - x_4) \end{aligned} \tag{3.18}$$

The subscripts q and q_1 are the momenta to be integrated over for the virtual quanta once we transform into momentum space. Let's let $f(k_a) \equiv k_a^2 - m_a^2 + i\epsilon$

and let the integral measure d^4x be abbreviated as dx etc., then the scattering amplitude in momentum space reads

$$\mathcal{M} = \frac{i^2(-i\lambda)^2}{2} \int d^4\omega\, d^4z\, \frac{d^4k}{(2\pi)^4} \frac{d^4q_1}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{e^{-ik_1x_1}}{f(k_1)} \frac{e^{-ik_2x_2}}{f(k_2)} \frac{e^{ik_3x_3}}{f(k_3)} \frac{e^{ik_3x_3}}{f(k_3)} \frac{e^{i\omega(k_1+k_2-(k+q_1))}}{k^2-m^2+i\epsilon} \frac{e^{-iz(k_3+k_4-(k+q_1))}}{q_1^2-m^2+i\epsilon}$$

First do the integral over ω and let the delta function do the q_1 integral (don't forget that k_3 and k_4 are negative)

$$\mathcal{M} = \frac{i^2(-i\lambda)^2}{2}(2\pi)^4\delta^4(k_1+k_2-k-q_1) \int \frac{d^4k}{(2\pi)^4} \frac{d^4z}{(2\pi)^4}$$

$$\times \frac{1}{(k_1+k_2-k)^2-m^2+i\epsilon} \frac{e^{-iz(k_3+k_4-(k_1+k_2))}}{k^2-m^2+i\epsilon} \left(\int \prod_{j=1}^{4} \frac{d|k_j|}{(2\pi)^4} \frac{e^{ik_jx_j}}{f(k_j)} \right) \tag{3.19}$$

Now do the integral over z,

$$\mathcal{M} = \frac{i^2(-i\lambda)^2}{2}(2\pi)^4(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(2\pi)^4}$$

$$\times \frac{1}{(k_1+k_2-k)^2-m^2+i\epsilon} \frac{1}{k^2-m^2+i\epsilon} \left(\int \prod_{j=1}^{4} \frac{d|k_j|}{(2\pi)^4} \frac{e^{ik_jx_j}}{f(k_j)} \right) \tag{3.20}$$

We may rewrite this as

$$\mathcal{M} = \frac{i^2(-i\lambda)^2}{2}(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k_1+k_2-k)^2-m^2+i\epsilon} \frac{1}{k^2-m^2+i\epsilon} \prod_{j=1}^{4} F(k_j)$$

where the function $F(k_j)$ represents the external legs. We may ignore both the external legs and the momentum conserving delta function along with its factors of 2π and write the scattering amplitude as

$$\mathcal{M} = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k_1+k_2-k)^2 - m^2 + i\epsilon} \frac{i}{k^2 - m^2 + i\epsilon} \tag{3.21}$$

And that's all there is to it! A few points of interest: a factor of one-half has appeared which is known as a symmetry factor. It is due to the fact that there are multiple diagram possibilities for this order in the interaction. There are values which make the denominator extremely large. These are the values where $E^2 = \mathbf{p}^2 + m^2$, in other words, a real "on-shell" particle. When the particle is not on-shell, the denominator is larger meaning that the quantity is small, so it doesn't contribute much to the total integral. This off-shell particle is called a virtual particle. It really only contributes as it begins to come real. Looking at the work we have done so far, we can begin to write down the Feynman Rules:

- A factor of $-i\lambda$ gets associated with each vertex
- All internal lines get their momenta integrated over
- Amputate the external legs
- Ignore the momentum conserving delta function and its factors of 2π

So for a diagram as shown in Figure 4, we can immediately write down the scattering amplitude, barring those symmetry factors that we worked so hard to keep track of before:

$$\mathcal{M} \sim (-i\lambda)^4 \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{(k_1+k_2-p)^2 - m^2 + i\epsilon}$$
$$\times \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(p-q-r)^2 - m^2 + i\epsilon} \frac{i}{r^2 - m^2 + i\epsilon} \frac{i}{(k_1+k_2-r)^2 - m^2 + i\epsilon} \tag{3.22}$$

3.3 Mass Regularization of the 2-Point Function

Recall when deriving the Feynman Rules that we were only concerned about a few terms arising from the expansions and derivatives and the rest were cast out. But what happens when we include them? It turns out that the addition of these terms results in an extra term in the denominator of the propagator which may be absorbed into the mass term, giving a new, "regularized" mass. It is easier, at least in my humble opinion, to leave the sources in the exponent and only expand around the interaction term so that we see which terms remain as $J \to 0$. So we begin with

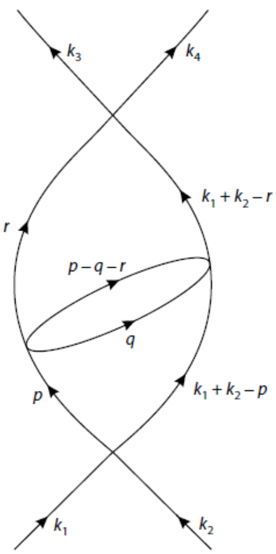


Figure 4: Fourth order scalar process

$$Z[J,1] = \frac{-i\lambda}{4!} \int d^4\omega \, \left(\frac{\delta^4}{i\delta J(\omega)} \right)^4 e^{-\frac{i}{2} \int J(x)\Delta^{(0)}(x-y)J(y)\, d^4x\, d^4y} \tag{3.23}$$

Upon taking the derivatives one ends up with

$$Z[J,1] = \frac{-i\lambda}{4!} \int d^4\omega \, \left[-3 \left(\Delta_0^{(0)} \right)^2 + 6i\Delta_0^{(0)} \left(\Gamma_{x\omega}^0 \right)^2 + \left(\Gamma_{x\omega}^0 \right)^4 \right] \tag{3.24}$$

The first term can be graphically represented as a figure which looks like a figure-8; there are crossed lines with no beginning or end. This term is purely virtual and therefore a vacuum contribution. The second term can be represented by a circle sitting on a line; there are two ends and only a single looped interaction. The third and final term may be represented graphically just as we had seen before, like an X; there are four end points and one interaction.

As you may have noticed, the labeling of the extra term which normalized Z has largely been absent. If we had carried it about, we would have observed that the vacuum contribution would be canceled out. We can argue anyway that since there are no sources or sinks, there can be no propagation and so no scattering amplitude. Additionally, we are free to set the zero-point energy of the system since all energy measurements are relative to some set zero, therefore by a rescaling of the zero-point energy, we may remove that term.

The second term has only two J's, so it can only be part of a two point function as mentioned above. We may calculate it out now. First we have

$$Z[J,1] = \frac{-i\lambda}{4!}(6i)\Delta_0^{(0)} \int d^4\omega \, \left(\Gamma^0_{x\omega}\right)^2 \tag{3.25}$$

And since the two-point scattering amplitude it just

$$\mathcal{M}^{(2)} = \frac{\delta^2 Z[J,\lambda=1]}{i^2 \delta J_1 \delta J_2} = -\frac{\delta^2 Z[J,\lambda=1]}{\delta J_1 \delta J_2} \tag{3.26}$$

where, as usual, $J_a = J(x_a)$. Plugging in we have

$$\mathcal{M}^{(2)} = (-1)\frac{-i\lambda}{4!}(6i)\Delta_0^{(0)} \int d^4\omega \, \frac{\delta^2}{\delta J_1 \delta J_2} \left(\Gamma^0_{x\omega}\right)^2 \tag{3.27}$$

The derivatives give

$$\frac{\delta^2}{\delta J_1 \delta J_2} \left(\Gamma^0_{x\omega}\right)^2 = 2\Delta^{(0)}(x_2-\omega)\Delta^{(0)}(x_1-\omega) \tag{3.28}$$

and so putting it together,

$$\begin{aligned}
\mathcal{M}^{(2)} &= (-1)\frac{-i\lambda}{4!}(6i)\Delta_0^{(0)} \int d^4\omega \; 2\Delta^{(0)}(x_2-\omega)\Delta^{(0)}(x_1-\omega) \\
&= \left(-\frac{i}{2}\right)(-i\lambda)\Delta_0^{(0)} \int d^4\omega \; \Delta^{(0)}(x_2-\omega)\Delta^{(0)}(x_1-\omega) \\
&= -\frac{\lambda}{2}\Delta_0^{(0)} \int d^4\omega \; \Delta^{(0)}(x_2-\omega)\Delta^{(0)}(x_1-\omega)
\end{aligned} \tag{3.29}$$

In momentum space, this can be written as

$$\begin{aligned}
\mathcal{M}^{(2)} &= -\frac{\lambda}{2}\Delta_0^{(0)} \int d^4\omega \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{e^{-ik_1(x_1-\omega)}}{k_1^2-m^2+i\epsilon}\frac{e^{-ik_2(x_2-\omega)}}{k_2^2-m^2+i\epsilon} \\
&= -\frac{\lambda}{2}\Delta_0^{(0)}(2\pi)^4\delta^4(k_1+k_2)\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{e^{-ik_1(x_1-x_2)}}{\left(k_1^2-m^2+i\epsilon\right)^2} \\
&= -\frac{\lambda}{2}\Delta_0^{(0)}\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{-ik_1(x_1-x_2)}}{\left(k_1^2-m^2+i\epsilon\right)^2}
\end{aligned} \tag{3.30}$$

But we can easily see that the most simple two-point function is just the propagator (multiplied by a factor of i), i.e.,

$$\mathcal{M}^{(2)} = i\Delta^{(0)}(x_2 - x_1) = i\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{-ik_1(x_1-x_2)}}{k_1^2 - m^2 + i\epsilon} \tag{3.31}$$

Therefore, the two-point scattering amplitude in a scalar theory, with the inclusion of a loop correction (the second term in (77)), can now be written. Switching the dummy variable from k_1 to just k, we get

$$\begin{aligned}\mathcal{M}^{(2)} &= i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2 - m^2 + i\epsilon}\left[1 + \frac{i\frac{1}{2}\lambda\Delta_0^{(0)}}{k^2 - m^2 + i\epsilon}\right] \\ &= i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2 - m^2 - i\frac{1}{2}\lambda\Delta_0^{(0)} + i\epsilon}\end{aligned} \tag{3.32}$$

where the term in brackets was taken to be the first term in the expansion of

$$\left[1 - \frac{i\frac{1}{2}\lambda\Delta_0^{(0)}}{k^2 - m^2 + i\epsilon}\right]^{-1}$$

Therefore, if we say that $m_R^2 = m^2 + \delta m^2$ and $\delta m^2 = i\frac{1}{2}\lambda\Delta_0^{(0)}$, then we may write (3.32) as

$$\mathcal{M}^{(2)} = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x_1-x_2)}}{k^2 - m_R^2 + i\epsilon} \tag{3.33}$$

This mass term is identified as the *regularized* or *physical* mass. We are still far from saved though. So far, every amplitude which we have found is at least logarithmically divergent at high momentum (the ultra-violet catastrophe) and it will take a full renormalization to solve this issue. We can see how corrections are added though; each new term in the series contributes some new amount which, if it gets small sufficiently quickly, will result in a summable series. The first scattering amplitude went like d^4k/k^2 which obviously diverges at high energy. The first correction to the scattering amplitude went like d^4k/k^4 which is logarithmically divergent. All of the other terms in the series converge. It is therefore the task of renormalization to wrestle these first few terms into place.

4 Fermions and Spin One-half

The next logical step from a theory of massive scalar particles, is to handle a theory of massive fermions. There are a few new ideas which need to be presented before we can move forward. There are two primary reasons for these extra steps. The first, and most obvious reason, is that the Dirac equation is a 4×4 matrix and so the field operators are four-component spinors and not scalars. The second reason is that, with the inclusion of spin, we have two degrees of freedom not allowed in the scalar theory (spin up & spin down) and so the field operator ψ must take these additional states into consideration. This would only require a two-component vector, but since we also must have positive and negative energy states, we need to have at least a four-component field operator $(+E\uparrow, +E\downarrow, -E\uparrow, -E\downarrow)$. This is good since the gamma matrices are also required to be at least 4×4. Let us begin with the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{4.1}$$

Now $\bar{\psi}$ and ψ need to each be integrated over. So inserting this into the kernel we have

$$Z[J] = \int D\bar{\psi}\, D\psi\; e^{i\int d^4x\; \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi} \tag{4.2}$$

This is fine, and so far, correct. But, the fermions, unlike the scalar fields, contribute negatively to the vacuum (the fundamental jumping-off point for supersymmetry) and so the kernal here had better turn out to be negative. If we forge ahead as we did with the Klein-Gordon equation, we'll get a positive energy contribution like the scalar particles had. Has field theory fallen apart? Not by a long shot. The problem evidently resides in the fact that we cannot treat the Dirac field as commuting complex numbers. We therefore try the next logical step, see what happens if we treat them as anti-commuting numbers. There's only one problem, we don't know how to evaluate those, but Grassman did.

Suppose we have some anti-commuting numbers, η and $\bar{\eta}$. Therefore

$$\eta\bar{\eta} + \bar{\eta}\eta = 0 \tag{4.3}$$

This clearly implies that the square of either of these must be zero. So, by Taylor expansion we must have that

$$\int d\eta\, d\bar{\eta}\; e^{\bar{\eta}a\eta} = \int d\eta\, d\bar{\eta}\;\left[1 + \bar{\eta}a\eta\right] \tag{4.4}$$

Let us set that the integration measure alone is zero and $\int d\eta\eta = 1$. Then

$$\int d\eta\, d\bar{\eta}\; e^{\bar{\eta}a\eta} = a \tag{4.5}$$

And since $a = e^{\log a}$, we may generalize to get

$$\int d\eta\, d\bar{\eta}\; e^{\bar{\eta}a\eta} = a = e^{Tr\,\log(a)} \tag{4.6}$$

The trace shows up since $\det[a] = e^{Tr\log(a)}$ and in general, if a is some matrix, then this is the form that we need. So we may now do (4.2) and find

$$\int D\bar{\psi}\, D\psi\; e^{i\int d^4x\; \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi} = C' e^{+Tr\log(i\partial\!\!\!/ - m)} \tag{4.7}$$

Whereas if we had done it the Gaussian way, we would have calculated

$$\int D\bar{\psi}\, D\psi\; e^{i\int d^4x\; \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi} = C'' \left(\frac{1}{\det[i\ \partial\!\!\!/ - m]}\right)^{1/2} = C'' e^{-Tr\log(i\partial\!\!\!/ - m)} \tag{4.8}$$

We see then that treating the fields as anticommuting indeed produces the correct answer, so there is a truly fundamental difference between particle states of scalars and fermions. Knowing this, then when we create particles, there must be two kinds of sources, one which creates regular particles and one which creates anti-particles.

Let us create an interaction then. We may let $\partial_\mu \to \partial_\mu - ieA_\mu$ and add the E.M fields and get the following QED Lagrangian density

$$\mathcal{L}_{QED} = \bar{\psi}\Big(i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\Big)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu^2 A^\mu A_\mu \tag{4.9}$$

This ignores the gauge fixing term necessitated by the massless photon which ends up creating the ghost fields of Faddev and Popov. For now, this is sufficient. Also, a photon must be massless in order to preserve electromagnetic gauge invariance, though we cannot set its mass (μ) to zero until the end of the problem. Examining the boson propagator will explain why. Since there are three fields being integrated over now, we cannot do the necessary integral and so we must expand in a series around the interaction again. The full integral is

$$Z[\bar{\eta}, \eta, J^\mu] = \int D\bar{\psi}\, D\psi\, DA_\mu\; e^{i\int d^4x\; [\mathcal{L}_{QED} + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu]} \tag{4.10}$$

The non-commuting nature of the fermionic field operators and the presence of that gamma matrix in the Dirac Lagrangian make this integral much more involved since the order of the terms is crucial. Words for the wise, label everything with indices. It is worth it if you don't have to start the problem over again. The interaction is clearly $-ij^\mu_{\text{em}} = e\bar{\psi}\gamma^\mu\psi$. The order here is of paramount importance. We can move around the A_μ if we wish, but the order $\bar{\psi}\gamma^\mu\psi$ (the bilinear covariant which transforms as a vector) must remain in this way.

Before we storm ahead and calculate an interaction, let's first dip out toes in the water with a simpler example; a spin 1/2 field interacting with a scalar field. The Lagrangian is

$$\mathcal{L}_{S,F} = \bar{\psi}\Big(i\gamma^\mu(\partial^\mu - ieA_\mu) - m\Big)\psi - \frac{1}{2}\varphi\left(\partial^2 + m^2\right)\varphi - \lambda\varphi^4 + f\varphi\bar{\psi}\psi \quad (4.11)$$

Again, we'll just expand around the interaction. This time, there are no indices which we must keep straight, so our task is made much simpler. Well, we need to remember to keep track of whether the sources producing the fermions were correlated to particles or anti-particles, but that's not too terribly tedious to do; just use the spinors. Let us define a few matrices:

$$(i\not{\partial} - m) \equiv K, \quad \partial^2 + m^2 \equiv A \quad (4.12)$$

Then the expanded kernal for the scalar-fermion interaction is

$$Z[\bar{\eta},\eta,J] = \int D\bar{\psi}\,D\psi D\varphi\; e^{if\int d^4\omega\,\left(\frac{\delta}{i\delta J}\frac{\delta}{i\delta\eta}\frac{\delta}{i\delta\bar{\eta}}\right)} e^{-i\lambda\int d^4z\,\left(\frac{\delta}{i\delta J}\right)^4}$$
$$\times e^{i\int d^4x\,\left[\bar{\psi}K\psi - \frac{1}{2}\varphi A\varphi + J\varphi + \bar{\eta}\psi + \bar{\psi}\eta\right]} \quad (4.13)$$

Now, since

$$(\bar{\psi} + \bar{\eta}K^{-1})K(\psi + K^{-1}\eta) - \bar{\eta}K^{-1}\eta \equiv \bar{Q}KQ - \bar{\eta}K^{-1}\eta \quad (4.14)$$

then

$$Z[\bar{\eta},\eta,J] = \int D\bar{Q}\,DQD\varphi\; e^{if\int d^4\omega\,\left(\frac{\delta}{i\delta J}\frac{\delta}{i\delta\eta}\frac{\delta}{i\delta\bar{\eta}}\right)} e^{-i\lambda\int d^4z\,\left(\frac{\delta}{i\delta J}\right)^4}$$
$$\times e^{i\int d^4x\,\left[\bar{Q}KQ - \bar{\eta}K^{-1}\eta - \frac{1}{2}\varphi A\varphi + J\varphi\right]} \quad (4.15)$$

There is one integral over the η's which we don't have to worry about, so that's nice. The integral over the Q's just gives a constant as does the quadratic integral over the φ's. Therefore we're left with

$$Z[\bar{\eta},\eta,J] = C' e^{if\int d^4\omega\,\left(\frac{\delta}{i\delta J}\frac{\delta}{i\delta\eta}\frac{\delta}{i\delta\bar{\eta}}\right)} e^{-i\lambda\int d^4z\,\left(\frac{\delta}{i\delta J}\right)^4}$$
$$\times e^{-i\int d^4x\,d^4y\;\bar{\eta}(i\not{\partial}-m)^{-1}\eta}\; e^{\frac{i}{2}\int d^4x\,d^4y\;J(x)(\partial^2+m^2)^{-1}J(y)} \quad (4.16)$$

We'll need the inverse of the Dirac equation. This is easy enough to do. The answer is

$$\Delta^{(1/2)}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{\not{p} - m + i\epsilon} \quad (4.17)$$

The inverse of the K.G equation, to remind the reader, is

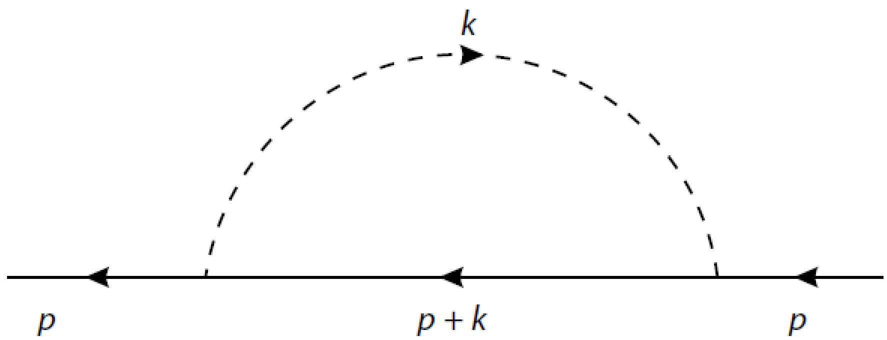


Figure 5: Scalar-Fermion Interaction

$$\Delta^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} \tag{4.18}$$

We have everything we need now to solve for the scattering amplitude for any process that we wish to calculate.

Suppose we have the diagram in Figure 5, where a dashed line is a scalar particle and the other solid lines are fermions.

There are no interactions in λ, so we can just as well set that to zero. There are two interactions in f. We may then write down the problem which we must solve

$$\begin{aligned} Z[\bar{\eta}, \eta, J] = C' \frac{1}{2} \left[if \int d^4\omega \, \left(\frac{\delta}{i\delta J} \frac{\delta}{i\delta \eta} \frac{\delta}{i\delta \bar{\eta}} \right) \right]^2 \\ \times e^{-i \int dx\, dy\ \bar{\eta} \Delta^{(1/2)}(x-y) \eta} e^{\frac{i}{2} \int d^4x\, d^4y\ J(x) \Delta^{(0)}(x-y) J(y)} \end{aligned} \tag{4.19}$$

Rather than assuming that momentum is conserved, let us determine this as we do the problem. So let the incoming momentum be p_i, the virtual fermion is $p_i + k = r$, the outgoing momentum is p_f and the virtual scalar particle is k. Additionally, let's ignore the overall factor for now. Consider the factors we are looking to end up with, i.e., the fermion and scalar propagators which must be in the scattering amplitude. We can therefore make our workload a little bit lighter. We write

$$Z[\bar{\eta}, \eta, J] = \frac{1}{2} \left[if \int d^4\omega \, \left(\frac{\delta}{i\delta J} \frac{\delta}{i\delta \eta} \frac{\delta}{i\delta \bar{\eta}} \right) \right]^2 \, \left(\frac{(-i)^3}{3!} \right) \mathcal{N}_{xy}^3 \left(\frac{i}{2} \right) \Gamma_{xy}^{(0)}$$

$$= \frac{(if)^2}{4\cdot 3!}\int d^4\omega\, d^4z\, \left(\frac{\delta}{i\delta J_z}\frac{\delta}{i\delta\eta_z}\frac{\delta}{i\delta\bar{\eta}_z}\right)\left(\frac{\delta}{i\delta J_\omega}\frac{\delta}{i\delta\eta_\omega}\frac{\delta}{i\delta\bar{\eta}_\omega}\right)\mathcal{N}_{xy}^3\,\Gamma_{xy}^{(0)} \tag{4.20}$$

Where the third term in the Taylor expansion of the Grassman variables was abbreviated as $\mathcal{N}_{xy}$ and each of the variables x and y are integrated over. This problem amounts to nothing more than just taking derivatives at this point. Doing the derivatives with respect to ω gives

$$Z[\bar{\eta},\eta,J] = \frac{(if)^2}{2}\int d^4\omega\, d^4z\, \left(\frac{\delta}{\delta J_z}\frac{\delta}{\delta\eta_z}\frac{\delta}{\delta\bar{\eta}_z}\right)\left[\mathcal{N}_{xy}\bar{\mathcal{N}}_{x\omega}\mathcal{N}_{x\omega}\right]\Gamma_{x\omega}^{(0)} \tag{4.21}$$

The bar dictates the function which has the antiparticle source and which has the regular particle source. Taking now the next set of derivatives with respect to z equals

$$Z[\bar{\eta},\eta,J] = \frac{(if)^2}{2}\int d^4\omega\, d^4z\, \left[\bar{\mathcal{N}}_{xz}\Delta^{(1/2)}(\omega-z)\mathcal{N}_{x\omega}\right]\Delta^{(0)}(\omega-z) \tag{4.22}$$

Two factors were combined to get this result. One may argue by symmetry (essentially swapping the names of the vertices) that there are two factors which are equivalent and therefore we may bring out a factor of two and only consider one of them. Therefore,

$$\mathcal{M} = i^2(if)^2\int d^4z\, d^4\omega\, \Delta^{(1/2)}(x_2-z)\Delta^{(1/2)}(\omega-z)\Delta^{(1/2)}(x_1-\omega)\Delta^{(0)}(\omega-z)$$

Now we just transform into momentum space and start doing the integrals.

$$\begin{aligned}\mathcal{M} = i^2(if)^2\int d^4z\, d^4\omega &\frac{d^4p_i}{(2\pi)^4}\frac{d^4p_f}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\\ \times&\frac{e^{-ip_f(z-x_1)}}{\not{p}_f-m}\frac{e^{-ir(\omega-z)}}{\not{r}-m}\frac{e^{ik(\omega-z)}}{k^2-m^2+i\epsilon}\frac{e^{ip_i(\omega-x_2)}}{\not{p}_i-m}\end{aligned}$$

$$= i^2(if)^2(2\pi)^4\delta^4(p_f-p_i)\int\frac{d^4k}{(2\pi)^4}\frac{1}{\not{k}+\not{p}_i-m}\frac{1}{k^2-m^2+i\epsilon}\prod_{j=1}^{2}\frac{d^4|p_j|}{(2\pi)^4}\frac{e^{-ip_jx^j}}{\not{p}_j-m}$$

$$\therefore \mathcal{M} = (if)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{\not{k}+\not{p}_i-m}\frac{i}{k^2-m^2+i\epsilon} \tag{4.23}$$

And that's that.

4.1 $\mathbf{e^- + e^- \rightarrow e^- + e^-}$

So far everything that has been done has involved scalar fields. The only fundamental scalar known is the Higgs, though it involves an electroweak interaction which has not yet been covered. We can treat mesons, such as the π^+, as a scalar particle, but since they are composites it will turn out that we'll have to add form factors which reveal information about the inner structure, i.e., the quarks. We can however progress in a very powerful way; consider purely QED interactions. From the lagrangian given in (4.9) we can expand around the interaction term and perform the integrations to get the transition amplitude in the same form as before. We get

$$Z[\bar{\eta}, \eta, J] = e^{ie\int d\omega\, \left(\frac{\delta}{i\delta J^\mu}\frac{\delta}{i\delta\eta^\lambda}(\gamma^\mu)_{\lambda\sigma}\frac{\delta}{i\delta\bar{\eta}^\sigma}\right)} e^{-i\int dx\; \bar{\eta}^\xi \Delta_F^{(1/2)}\eta^\rho} e^{-\frac{i}{2}\int dx\, dy\; J_\alpha(x) D^{\alpha\beta}(x-y) J_\beta(y)} \tag{4.24}$$

As always, all of the integrals are four dimensional. The index labeling is important so that we get things in the right order as the don't commute. The sources must keep track of the fermions and whether or not they are anti-particles. This is accomplished by the use of spinors. As before, a shorthand notation will be used and with judicious labeling. Let us make use of the following definitions:

$$\int dy\, dx\; \bar{\eta}^\xi(x)\Delta_F^{(1/2)}(x-y)\eta^\rho(y) \equiv \mathcal{N}_{xy}^{\xi\rho}$$

$$\left(\frac{\delta}{\delta\bar{\eta}^\lambda}\right)_\omega e^{-i\mathcal{N}_{xy}^{\xi\rho}} = -i\left(\int dx\; \bar{u}_\omega^\lambda \Delta_F^{(1/2)}(\omega - x)\eta^\rho(x)\right) e^{-i\mathcal{N}_{xy}^{\xi\rho}} \equiv -i\mathcal{N}_{\omega x}^{\lambda\rho}\, e^{-i\mathcal{N}_{xy}^{\xi\rho}}$$

$$\left(\frac{\delta}{\delta\eta^\lambda}\right)_\omega \mathcal{N}_{xy}^{\xi\rho} = -i\left(\int dx\; \bar{\eta}^\xi(x)\Delta_F^{(1/2)}(x-\omega)u_\omega^\lambda\right)\mathcal{N}_{xy}^{\xi\rho} \equiv -i\bar{\mathcal{N}}_{x\omega}^{\xi\lambda}\, e^{-i\mathcal{N}_{xy}^{\xi\rho}}$$

$$\left(\frac{\delta}{\delta\bar{\eta}^\sigma}\right)_z \bar{\mathcal{N}}_{x\omega}^{\xi\lambda} = \mathcal{N}_{\omega z}^{\sigma\lambda} = \bar{u}_\omega^\sigma\left(\Delta_F^{(1/2)}(\omega - z)\right)^{\sigma\lambda} u_z^\lambda$$

$$\int dx\, dy\; J_\alpha(x) D^{\alpha\beta}(x-y) J_\beta(y) \equiv \Gamma_{\alpha\beta}^{xy}$$

$$\left(\frac{\delta}{\delta J^\mu}\right)_\omega \Gamma_{\alpha\beta}^{xy} = 2\left(\int dx\; D^{\mu\beta}(\omega - x)J_\beta(x)\right)\Gamma_{\alpha\beta}^{xy} \equiv 2\Gamma_{\mu\beta}^{\omega x}\Gamma_{\alpha\beta}^{xy} = 2\Gamma_{\alpha\mu}^{\omega y}\Gamma_{\alpha\beta}^{xy}$$

$$\left(\frac{\delta}{\delta J^\nu}\right)_z \Gamma_{\mu\beta}^{\omega x} = D^{\mu\nu}(\omega - z) \tag{4.25}$$

The Γ function is symmetric in its upper indices and in its lower indices. The $\mathcal{N}$ function is symmetric in its lower indices but not in the upper pair. It's not even antisymmetric. Remember, these spinors are 4-component vectors and so multiplying them one way gives a scalar and reversing that gives a matrix. The ordering and careful labeling cannot be stressed enough! We can now rewrite Z,

$$Z[\bar{\eta},\eta,J] = e^{ie\int d^4\omega\,\left(\frac{\delta}{i\delta J^{\mu}}\frac{\delta}{i\delta\eta^{\lambda}}(\gamma^{\mu})_{\lambda\sigma}\frac{\delta}{i\delta\bar{\eta}^{\sigma}}\right)} e^{-i\mathcal{N}^{\xi\rho}_{xy}}\, e^{-\frac{i}{2}\Gamma^{xy}_{\alpha\beta}} \tag{4.26}$$

We can put this theory to work. Suppose that we'd like to see two electrons scatter via exchange of a photon. Since each electron will interact with the photon, the amplitude must be of the order e^2. There is an extra complication since we are scattering identical particles; we must account for more than one transition amplitude. Without knowledge of this second fact, let us use the path integral formalism and calculate the connected second-order amplitudes. We must have four powers of $\mathcal{N}^{\xi\rho}_{xy}$ and only one of Γ^{xy}. Additionally, let's assume that there are no loops and no fermion propagators, only the photon is virtual. Therefore,

$$Z[\bar{\eta},\eta,J] = \frac{1}{2}(ie)^2\int d^4z\, d^4\omega\,\left(\frac{\delta}{i\delta J^{\mu}}\frac{\delta}{i\delta\eta^{\lambda}}(\gamma^{\mu})_{\lambda\sigma}\frac{\delta}{i\delta\bar{\eta}^{\sigma}}\right)_z\left(\frac{\delta}{i\delta J^{\nu}}\frac{\delta}{i\delta\eta^{\Lambda}}(\gamma^{\nu})_{\Lambda\Sigma}\frac{\delta}{i\delta\bar{\eta}^{\Sigma}}\right)_{\omega}$$

$$\times\frac{(-i)^4}{4!}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^4\left(-\frac{i}{2}\right)\Gamma^{xy}_{\alpha\beta}$$

$$= \frac{i(ie)^2}{4\cdot 4!}\int d^4z\, d^4\omega\,\left(\frac{\delta}{\delta J^{\mu}}\frac{\delta}{\delta\eta^{\lambda}}(\gamma^{\mu})_{\lambda\sigma}\frac{\delta}{\delta\bar{\eta}^{\sigma}}\right)_z\left(\frac{\delta}{\delta J^{\nu}}\frac{\delta}{\delta\eta^{\Lambda}}(\gamma^{\nu})_{\Lambda\Sigma}\frac{\delta}{\delta\bar{\eta}^{\Sigma}}\right)_{\omega}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^4\Gamma^{xy}_{\alpha\beta} \tag{4.27}$$

Let's take it one chunk at a time. Notice the placement of the indices. We must keep them *in that order*. Not necessarily the whole time, of course, that is what an index is for after all.

$$\begin{aligned}&\left(\frac{\delta}{\delta J^{\nu}}\frac{\delta}{\delta\eta^{\Lambda}}(\gamma^{\nu})_{\Lambda\Sigma}\frac{\delta}{\delta\bar{\eta}^{\Sigma}}\right)_{\omega}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^4\Gamma^{xy}_{\alpha\beta}\\ &= 2\cdot 4!\,\Gamma^{\omega x}_{\nu\beta}\,\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,(\gamma^{\nu})_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^2\end{aligned} \tag{4.28}$$

Now we'll do the next batch of derivatives, this time with respect to z.

$$\begin{aligned}&\left(\frac{\delta}{\delta J^{\mu}}\frac{\delta}{\delta\eta^{\lambda}}(\gamma^{\mu})_{\lambda\sigma}\frac{\delta}{\delta\bar{\eta}^{\sigma}}\right)_z 2\cdot 4!\,\Gamma^{\omega x}_{\nu\beta}\,\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,(\gamma^{\nu})_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^2\\ &= 2\cdot 4!\left(\bar{\mathcal{N}}^{\xi\lambda}_{zx}(\gamma^{\mu})_{\lambda\sigma}\mathcal{N}^{\sigma\rho}_{zx}\right)D_{\mu\nu}(\omega-z)\left(\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}(\gamma^{\nu})_{\Lambda\Sigma}\mathcal{N}^{\Sigma\rho}_{\omega x}\right)\end{aligned} \tag{4.29}$$

Putting the pieces together, we find (so far) for the transition amplitude

$$Z = \frac{i(ie)^2}{2} \int d^4z\, d^4\omega \ \left(\bar{\mathcal{N}}^{\xi\lambda}_{zx} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma\rho}_{zx}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma\rho}_{\omega x}\right) \quad (4.30)$$

The arguments were left out of Z since we are going to be setting them to zero once we take the derivatives for the scattering matrix amplitude. The J's are already done, but we have four η's left. The two photon-electron exchange points are at z and ω. Based on the equation above, we see that the "z-side" electron is separate from the "ω-side" electron. This is good news, though we could always just turn our soon-to-be Feynman diagram and everything would have been alright anyway. OK, moving onward

$$\mathcal{M} = \frac{i(ie)^2}{2} \left(\frac{\delta^4}{\delta\eta^1 \delta\bar{\eta}^2 \delta\eta^3 \delta\bar{\eta}^4}\right) \int d^4z\, d^4\omega \ \left(\bar{\mathcal{N}}^{\xi\lambda}_{zx} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma\rho}_{zx}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma\rho}_{\omega x}\right) \quad (4.31)$$

Nothing to do but plow ahead. Fear not, it's not particularly difficult, just tedious.

$$\begin{aligned}
\mathcal{M} = \frac{i(ie)^2}{2} \int d^4z\, d^4\omega \Big\{ & \left(\bar{\mathcal{N}}^{0\lambda}_{zx_4} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma 0}_{zx_1}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_2} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma 0}_{\omega x_3}\right) \\
& + \left(\bar{\mathcal{N}}^{0\lambda}_{zx_4} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma 0}_{zx_3}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_2} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma 0}_{\omega x_1}\right) \\
& + \left(\bar{\mathcal{N}}^{0\lambda}_{zx_2} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma 0}_{zx_3}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_4} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma 0}_{\omega x_1}\right) \\
& + \left(\bar{\mathcal{N}}^{0\lambda}_{zx_2} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma 0}_{zx_1}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_4} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma 0}_{\omega x_3}\right) \Big\}
\end{aligned} \quad (4.32)$$

Let's be physicists again. We suppose that the electrons are identical, therefore if electron A zips from $x_1 \to x_2$ and B zips from $x_3 \to x_4$, that is indistinguishable from having electron A go from $x_3 \to x_4$ and electron B go from $x_1 \to x_2$. Therefore the first and third terms in the lines above are the same, as are the second and fourth lines. Similarly if electron A goes from $x_1 \to x_2$ and B goes from $x_3 \to x_4$, we cannot tell that from A going from $x_1 \to x_4$ and B going from $x_3 \to x_2$. Additionally, the superscript zeros let us know that there are no more η's lying around and to which side the spinors are associated. The momenta of the spinors correspond to it's spacetime point. For example, $x_1 \to p_1$ and so on. Thus we can write

$$\mathcal{M} = i(ie)^2 \int d^4z\, d^4\omega \Big\{ \left(\bar{\mathcal{N}}^{0\lambda}_{zx_4} \, (\gamma^\mu)_{\lambda\sigma} \, \mathcal{N}^{\sigma 0}_{zx_1}\right) D_{\mu\nu}(\omega - z) \left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_2} \, (\gamma^\nu)_{\Lambda\Sigma} \, \mathcal{N}^{\Sigma 0}_{\omega x_3}\right)$$

$$+ \left(\bar{\mathcal{N}}^{0\lambda}_{zx_4}\,(\gamma^\mu)_{\lambda\sigma}\,\mathcal{N}^{\sigma 0}_{zx_3}\right) D_{\mu\nu}(\omega - z)\left(\bar{\mathcal{N}}^{0\Lambda}_{\omega x_2}\,(\gamma^\nu)_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma 0}_{\omega x_1}\right)\Bigg\} \tag{4.33}$$

Splendid! The path integral formalism has managed to include the cross terms. We will be able to include crossing symmetry at any time, so we don't really need to strictly evaluate both terms. And really, it would only be necessary if we were calculating the cross section, but that is not the purpose here. So let's just worry about the first term. Put the whole mess into momentum space and then turn the crank.

$$\mathcal{M} = i(ie)^2 \int d^4z\, d^4\omega\, \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}$$

$$\times\Bigg\{\bar{u}(p_4)\left(\frac{e^{-ip_4(z-x_4)}}{\not{p}_4 - m}\right)^\lambda (\gamma^\mu)_{\lambda\sigma}\left(\frac{e^{ip_1(z-x_1)}}{\not{p}_1 - m}\right)^\sigma u(p_1)\,\frac{e^{ik(z-\omega)}}{k^2 - m^2 + i\epsilon}\left(k_\mu k_\nu/\mu^2 - g_{\mu\nu}\right)$$

$$\times\bar{u}(p_2)\left(\frac{e^{-ip_2(\omega-x_2)}}{\not{p}_2 - m}\right)^\Lambda (\gamma^\nu)_{\Lambda\Sigma}\left(\frac{e^{ip_3(\omega-x_3)}}{\not{p}_3 - m}\right)^\Sigma u(p_3)\Bigg\} \tag{4.34}$$

Do the integral over ω and let the remaining delta function do the k integral which sets $k \to p_3 - p_2$. Then do the integral over z to get the overall momentum conserving delta function

$$\mathcal{M} = i(ie)^2(2\pi)^4\delta^4(p_4 - p_1 - p_3 + p_2)\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}$$

$$\times\Bigg\{\bar{u}(p_4)\frac{e^{ip_4x_4}}{\not{p}_4 - m}\gamma^\mu\frac{e^{-ip_1x_1}}{\not{p}_1 - m}u(p_1)\frac{(k_\mu k_\nu/\mu^2 - g_{\mu\nu})}{k^2 - \mu^2 + i\epsilon}\bar{u}(p_2)\frac{e^{ip_2x_2}}{\not{p}_2 - m}\gamma^\nu\frac{e^{-ip_3x_3}}{\not{p}_3 - m}u(p_3)\Bigg\}$$

You should have noticed by now that although the spacetime points were arbitrary, the assignment to the momenta was somewhat particular. The momenta were defined, as you can see, so that the incoming particles have a positive momentum and the outgoing particles have a negative momentum. Additionally, there are no spinors which sit at the photon-electron exchange points. The spinors sole purpose is to differentiate between particles and anti-particles at their source, which we have taken to be infinitely far away. Therefore we only need the four spinors located at the four spacetime points x_1, x_2, x_3 and x_4. We may write the amplitude now as

$$\mathcal{M} = i(ie)^2(2\pi)^4\delta^4(p_4 - p_1 - p_3 + p_2)\left(\prod_{j=1}^{4}\int\frac{d^4|p_j|}{(2\pi)^4}\frac{e^{ip_jx^j}}{\not{p}_j - m}\right)$$

$$\times\bar{u}(p_4)\gamma^\mu u(p_1)\frac{(k_\mu k_\nu/\mu^2 - g_{\mu\nu})}{k^2-\mu^2+i\epsilon}\bar{u}(p_2)\gamma^\nu u(p_3) \tag{4.35}$$

Notice now that when the k_ν hits the spinors on the right, i.e., $\bar{u}(p_2)$ and $u(p_3)$, it gives zero. Thus the whole term $k_\mu k_\nu/\mu^2 = 0$ and so we may safely set the mass of the photon now to zero, and electromagnetic gauge invariance is conserved. The term in parenthesis is the external legs, so we may ignore that term. Finally, we can ignore the momentum conserving delta function and its factor of $(2\pi)^4$ so that we may finally write

$$\mathcal{M} = \bar{u}(p_4)(ie\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\bar{u}(p_2)(ie\gamma^\nu)u(p_3) \tag{4.36}$$

And that's all there is to it! Here's a quick recap of the Feynman rules for QED:

- Each vertex gets a factor of $ie\gamma^\mu$
- Incoming particles (anti-particles) get a u ($\bar{v}$) spinor and outgoing particles (anti-particles) get a $\bar{u}$ (v).
- Do not associate a spinor at the vertices
- Amputate the external legs
- There is an overall momentum conserving delta function of the form $\delta^4\left(\sum p_{in} - \sum p_{out}\right)$ which we may neglect to write, though use it to set the momentum of the virtual particle
- Photon propagators get a factor of $-ig_{\mu\nu}/k^2$, fermion propagators get $i/(\not{k}-m)$ and a massive boson gets $i(k_\mu k_\nu - g_{\mu\nu})/(k^2-m^2)$

Without even thinking, we can now write the amplitude for the crossed diagram which corresponds to (122) that we previously ignored ($p_4 \leftrightarrow p_2$)

$$\mathcal{M} = \bar{u}(p_2)(ie\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\bar{u}(p_4)(ie\gamma^\nu)u(p_3) \tag{4.37}$$

4.2 Compton Scattering: $\mathbf{e}^- + \gamma \to \mathbf{e}^- + \gamma$

Deriving theses amplitudes (don't forget the crossed one) is fundamentally no different that what we have already done in the electron-electron scattering. It's only a few turns different, so we'll breeze through it. The hardest part of this problem comes after we derive the amplitudes; the trace of the mixed amplitude part is a pain. And we'll do it in its full glory soon enough.

To start this problem, note that in this case we need three powers of $\mathcal{N}^{\xi\rho}_{xy}$ and two powers of $\Gamma^{xy}_{\alpha\beta}$. Then we just take a deep breath and dive in as before.

None of the indices in our shorthand notation are symmetric now, so take great care.

$$Z[\bar{\eta},\eta,J] = \frac{1}{2}(ie)^2 \int d^4z\, d^4\omega \, \left(\frac{\delta}{i\delta J^\mu}\frac{\delta}{i\delta\eta^\lambda}(\gamma^\mu)_{\lambda\sigma}\frac{\delta}{i\delta\bar{\eta}^\sigma}\right)_z \left(\frac{\delta}{i\delta J^\nu}\frac{\delta}{i\delta\eta^\Lambda}(\gamma^\nu)_{\Lambda\Sigma}\frac{\delta}{i\delta\bar{\eta}^\Sigma}\right)_\omega$$

$$\times \frac{(-i)^3}{3!}\left(\mathcal{N}^{\xi\rho}_{xy}\right)^3 \left(-\frac{i}{2}\right)^2 \frac{1}{2!}\left(\Gamma^{xy}_{\alpha\beta}\right)^2$$

$$= \frac{i(ie)^2}{3!\cdot 16}\int d^4z\, d^4\omega \, \left(\frac{\delta}{\delta J^\mu}\frac{\delta}{\delta\eta^\lambda}(\gamma^\mu)_{\lambda\sigma}\frac{\delta}{\delta\bar{\eta}^\sigma}\right)_z \left(\frac{\delta}{\delta J^\nu}\frac{\delta}{\delta\eta^\Lambda}(\gamma^\nu)_{\Lambda\Sigma}\frac{\delta}{\delta\bar{\eta}^\Sigma}\right)_\omega \left(\mathcal{N}^{\xi\rho}_{xy}\right)^3 \left(\Gamma^{xy}_{\alpha\beta}\right)^2 \tag{4.38}$$

Now let's start the derivatives...

$$\left(\frac{\delta}{\delta J^\nu}\frac{\delta}{\delta\eta^\Lambda}(\gamma^\nu)_{\Lambda\Sigma}\frac{\delta}{\delta\bar{\eta}^\Sigma}\right)_\omega \left(\mathcal{N}^{\xi\rho}_{xy}\right)^3 \left(\Gamma^{xy}_{\alpha\beta}\right)^2$$
$$= 3!\cdot 2\cdot 2\,\Gamma^{\omega x}_{\nu\beta}\,\Gamma^{xy}_{\alpha\beta}\,(\gamma^\nu)_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\,\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,\mathcal{N}^{\xi\rho}_{xy} \tag{4.39}$$

Next do the ones w.r.t z.

$$\left(\frac{\delta}{\delta J^\mu}\frac{\delta}{\delta\eta^\lambda}(\gamma^\mu)_{\lambda\sigma}\frac{\delta}{\delta\bar{\eta}^\sigma}\right)_z 3!\cdot 2\cdot 2\,\Gamma^{\omega x}_{\nu\beta}\,\Gamma^{xy}_{\alpha\beta}\,(\gamma^\nu)_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\,\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,\mathcal{N}^{\xi\rho}_{xy}$$

$$= 4\cdot 3!\,(\gamma^\mu)_{\lambda\sigma}\left(2\,\Gamma^{\omega x}_{\nu\beta}\,\Gamma^{zx}_{\beta\mu}\right)(\gamma^\nu)_{\Lambda\Sigma}\left(\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,\mathcal{N}^{\sigma\rho}_{zx}\,\mathcal{N}^{\Sigma\lambda}_{\omega z} + \mathcal{N}^{\sigma\Lambda}_{\omega z}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\,\bar{\mathcal{N}}^{\xi\lambda}_{zx}\right) \tag{4.40}$$

It is clearer to write this in a slightly different way

$$Z[\bar{\eta},\eta,J] = \frac{i(ie)^2}{2}\int d^4z\, d^4\omega \left\{ \left(\Gamma^{zx}_{\beta\mu}\,(\gamma^\mu)_{\lambda\sigma}\,\mathcal{N}^{\sigma\rho}_{zx}\right)\mathcal{N}^{\Sigma\lambda}_{\omega z}\left(\bar{\mathcal{N}}^{\xi\Lambda}_{\omega x}\,(\gamma^\nu)_{\Lambda\Sigma}\,\Gamma^{\omega x}_{\nu\beta}\right)\right.$$
$$\left. + \left(\bar{\mathcal{N}}^{\xi\lambda}_{zx}\,(\gamma^\mu)_{\lambda\sigma}\,\Gamma^{zx}_{\beta\mu}\right)\mathcal{N}^{\sigma\Lambda}_{\omega z}\left(\Gamma^{\omega x}_{\nu\beta}\,(\gamma^\nu)_{\Lambda\Sigma}\,\mathcal{N}^{\Sigma\rho}_{\omega x}\right)\right\} \tag{4.41}$$

Now we'll use the following to get the matrix amplitude

$$\mathcal{M} = \frac{\delta^4 Z[\bar{\eta},\eta,J]}{\delta\eta^1\delta J_2\delta\bar{\eta}^3\delta J_4} \tag{4.42}$$

As in the electron-electron scattering case, we will have four terms and we can combine them in pairs via a symmetry argument which will bring out a factor of two. One then obtains

$$\mathcal{M} = i(ie)^2 \int d^4z\, d^4\omega \left\{ \left(\Gamma^{zx_4}_{4\mu}\,(\gamma^\mu)_{\lambda\sigma}\,\mathcal{N}^{\sigma 1}_{zx_1}\right)\mathcal{N}^{\Sigma\lambda}_{\omega z}\left(\bar{\mathcal{N}}^{3\Lambda}_{\omega x_3}\,(\gamma^\nu)_{\Lambda\Sigma}\,\Gamma^{\omega x^2}_{\nu 2}\right)\right.$$

$$\left. + \left(\Gamma^{zx_2}_{2\mu}\,(\gamma^\mu)_{\lambda\sigma}\,\mathcal{N}^{\sigma 1}_{zx_1}\right)\mathcal{N}^{\Sigma\lambda}_{\omega z}\left(\bar{\mathcal{N}}^{3\Lambda}_{\omega x_3}\,(\gamma^\nu)_{\Lambda\Sigma}\,\Gamma^{\omega x^4}_{\nu 4}\right)\right\} \tag{4.43}$$

Remember now that $\delta Z/\delta J_\mu$ really just brings down a factor of A^μ. The index which remains is the polarization vector. The side which it appears on Γ determines whether or not it is the complex conjugate or not. This is why it is imperative to be so careful about the indices. Transforming into momentum space, and considering only the first term we get

$$\mathcal{M} = i(ie)^2 \int d^4z\, d^4\omega \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}$$

$$\left\{ \left(\epsilon^*_\mu(k)\frac{e^{ik(z-x_2)}}{k^2}\right)(\gamma^\mu)_{\lambda\sigma}\left(\frac{e^{ip(z-x_1)}}{\not p - m}u(p_1)\right)^\sigma \left(\frac{e^{-iq(z-\omega)}}{\not q - m + i\epsilon}\right)^{\lambda\Sigma}\right.$$

$$\left. \times \left(\bar{u}(p')\frac{e^{-ip'(\omega-x_3)}}{\not p' - m}\right)^\Lambda (\gamma^\nu)_{\Lambda\Sigma}\left(\frac{e^{-ik'(\omega-x_4)}}{k'^2}\epsilon_\nu(k')\right)\right\} \tag{4.44}$$

The following has been set here: $p_1 \to p,\ p_2 \to k,\ p_3 \to p',\ p_4 \to k'$. Do in the integral over ω and use it to set $q \to k' + p'$, then do the integral over z to get the momentum conserving delta function. This gives

$$\mathcal{M} = i(ie)^2(2\pi)^4\delta^4(k + p - k' - p')\left(\prod_{i=1}^{2}\prod_{j=1}^{2}\int \frac{d^4|k_i|}{(2\pi)^4}\frac{d^4|p_j|}{(2\pi)^4}\frac{e^{ik_ix^i}}{k_i^2}\frac{e^{ip_jx^j}}{\not p_j - m}\right)$$

$$\times \epsilon^*_\mu(k)\gamma^\mu u(p)\frac{1}{\not p + \not k - m}\bar{u}(p')\gamma^\nu\epsilon_\nu(k') \tag{4.45}$$

As always, amputate the external legs and ignore the overall delta function. Thus we finally have

$$\mathcal{M} = \epsilon^*_\mu(k)(ie\gamma^\mu)u(p)\frac{i}{\not p + \not k - m}\bar{u}(p')(ie\gamma^\nu)\epsilon_\nu(k') \tag{4.46}$$

And the crossed diagram is the same but with $\epsilon(k') \to \epsilon(k)$ and $k \to -k'$. See? Not so bad. The electron-positron annihilation into two photons is just a few turns different from this. I leave it to you to work out the details.

5 The Magnetic Moment of the Electron

It is worthwhile at this point to go ahead and tackle this problem. We start with the Dirac equation which interacts with an external electric potential A_μ by extending the derivative $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$. Then since

$$(i\gamma^\mu D_\mu - m)\psi = 0 \tag{5.1}$$

then multiplying on the left by $(i\gamma^\mu D_\mu + m)$ will also be zero. Rearranging this we can eventually arrive at

$$\left(D^\mu D_\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\psi = 0 \tag{5.2}$$

Within the $\sigma^{\mu\nu}F_{\mu\nu}$ there is a B-field and some angular momentum operators, namely $\hat{L}$ and $\hat{S}$. In the nonrelativistic limit the equation becomes

$$\left(-2im\partial_0 - \nabla^2 - e\vec{B}\cdot(\hat{L} + 2\hat{S})\right)\psi = 0 \tag{5.3}$$

And so we plainly see the interaction wth a magnetic field. The most interesting part of this is to notice that the spin interacts twice as strongly as the orbital angular momentum. Actually, the whole process of getting here is interesting, and a tremendous triumph of 20th century physics. There is therefore a huge amount of information contained in that $F_{\mu\nu}$ term. We can make use of this information. Consider the Gordon decomposition,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right)u(p) \tag{5.4}$$

The second term on the r.h.s of the equal sign contains the magnetic moment. We would like to see if we can extract some information about it and possibly reveal a correction to the anomolous magnetic moment using quantum field theory. We can write our task as the following

$$\langle p', s'|J^\mu(0)|p, s\rangle = \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{(p'+p)^\mu}{2m}F_1(q^2) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}F_2(q^2)\right)u(p) \tag{5.5}$$

To zeroth order, and using the Gordon decomposition, we can rewrite this as

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{(p'+p)^\mu}{2m}F_1(0) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\left[F_1(0) + F_2(0)\right]\right)u(p) \tag{5.6}$$

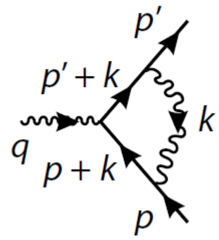


Figure 6: Diagram for the magnetic moment of the electron

or

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\gamma^\mu\left[F_1(0)+F_2(0)\right] - \frac{(p'+p)^\mu}{2m}F_2(0)\right)u(p) \tag{5.7}$$

By definition, $F_1(0) = 1$. Our task is greatly reduced now. In this second form, $F_2(0)$ shows up twice. We can ignore one of them. There are a lot of diagrams of interaction which are proportional to γ^μ, so it would be wise to throw that term away and only consider the term on the right. What's better, there is only one diagram which contributes and it is shown in Figure 6.

We can assume that the vertex function goes something like $ie\Gamma^\mu(p,q,p') \simeq ie(\gamma^\mu + \Lambda^\mu(p,q,p') + \cdots)$ All terms will have that factor of ie in them, so we can leave it out of the amplitude which we calculate. Moving forward, let us now calculate the amplitude of the vertex function Λ^μ,

$$\Lambda^\mu = -ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\nu\,(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\,\gamma_\nu}{[(p'+k)^2-m^2][(p+k)^2-m^2]k^2} \equiv -ie^2\int\frac{d^4k}{(2\pi)^4)}\frac{\mathcal{N}^\mu}{\mathcal{D}} \tag{5.8}$$

For as innocuous as this looks, (or doesn't look for that matter), it requires quite a bit of care. We can use Feynman's triangle integrals to rewrite the denominator. The structure of the integral is as follows

$$\frac{1}{xyz} = 2\int_0^1 d\beta\int_0^{1-\beta} d\alpha\frac{1}{[(1-\alpha-\beta)z+\alpha x+\beta y]^3} \tag{5.9}$$

This region is bounded by $\alpha = 0$, $\beta = 0$, and $\alpha+\beta = 1$, so we can see why it is called the triangle integral. We can therefore write

$$\begin{aligned}\mathcal{D} &= [k^2+2k(\alpha p'+\beta p)-m^2(\alpha+\beta)]^3\\ &= [\ell^2-(\alpha p'+\beta p)^2]^3\\ &= [\ell^2-m^2(\alpha+\beta)^2]^3\end{aligned} \tag{5.10}$$

where in the second line, the fact that this is all smushed between spinors was used and so $\not{p}u(p) = mu(p)$. Also the variable of integration was shifted to $k = \ell - (\alpha p' + \beta p)$. We now have

$$\Lambda^\mu = -2ie^2 \int_0^1 \int_0^{1-\beta} d\alpha d\beta \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}^\mu}{[\ell^2 - m^2(\alpha+\beta)^2]^3} \tag{5.11}$$

So far so good. Now we come to a slightly larger pain in the neck, that $\mathcal{N}^\mu$ term. Remember, any term proportional to γ^μ is of no use and may be ignored. We have

$$\begin{aligned} \mathcal{N}^\mu &= \gamma^\nu(\not{p}' + \not{\ell} - \alpha \not{p}' - \beta \not{p} + m)\gamma^\mu(\not{p} + \not{\ell} - \alpha \not{p}' - \beta \not{p} + m)\gamma_\nu \\ &\equiv \gamma^\nu(\not{\ell} + \not{P}_1 + m)\gamma^\mu(\not{\ell} + \not{P}_2 + m)\gamma_\nu \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} P_1 &= (1-\alpha)p' - \beta p \\ P_2 &= (1-\beta)p - \alpha p' \end{aligned}$$

By symmetry, any term linear in ℓ will integrate to zero. Following some sage advice, let's consider terms by powers of m. First, the m^2 term,

$$m^2\gamma^\nu\gamma^\mu\gamma_\nu = 4m^2\gamma^\mu \to 0 \tag{5.13}$$

Next, the terms proportional to m,

$$m\gamma^\nu \left[\gamma^\mu \not{P}_2 + \not{P}_1\gamma^\mu\right]\gamma_\nu \tag{5.14}$$

Consider just the first term,

$$m\gamma^\nu\gamma^\mu \not{P}_2\gamma_\nu$$

$$= 4mP_2^\mu = 4m[(1-\beta)p^\mu - \alpha p'^\mu] \tag{5.15}$$

The same procedure goes for the second term. Adding them we get

$$= 4m[p'^\mu(1-\alpha-\alpha) + p^\mu(1-\beta-\beta)] = 4m(1-\alpha-\beta)(p'+p)^\mu \tag{5.16}$$

where the fact that the integral is symmetric under interchange of α and β was used. Now we do the terms to zeroth order in m. We start with

$$\gamma^\nu(\not{\ell} + \not{P}_1)\gamma^\mu(\not{\ell} + \not{P}_2)\gamma_\nu \to \gamma^\nu \not{P}_1\gamma^\mu \not{P}_2\gamma_\nu \tag{5.17}$$

since the terms linear in ℓ integrate to zero and the term quadratic in ℓ is proportional to γ^μ. We are left with

$$\begin{aligned} &-2[(1-\beta)\not{p}-\alpha m]\gamma^\mu[(1-\alpha)\not{p}'-\beta m] \\ &= -2(1-\alpha)(1-\beta)\not{p}\gamma^\mu\not{p}' + 2m\beta(1-\beta)\not{p}\gamma^\mu + 2m\alpha(1-\alpha)\gamma^\mu\not{p}' \end{aligned} \tag{5.18}$$

It's a straightforward procedure to work out. You should end up with three terms out of this,

$$\begin{aligned} &4m[\beta(1-\beta)p^\mu + \alpha(1-\alpha)p'^\mu] \\ &-4m(1-\beta)(1-\alpha)(p'+p)^\mu \\ &4m(1-\alpha-\beta)(p'+p)^\mu \end{aligned} \tag{5.19}$$

Adding them all together we get

$$\mathcal{N}^\mu = 2m(1-\alpha-\beta)(\alpha+\beta)(p'+p)^\mu \tag{5.20}$$

Therefore,

$$\Lambda^\mu = -2ie^2\int_0^1\int_0^{1-\beta} d\alpha d\beta \int \frac{d^4\ell}{(2\pi)^4}\frac{2m(1-\alpha-\beta)(\alpha+\beta)(p'+p)^\mu}{[\ell^2-m^2(\alpha+\beta)^2]^3} \tag{5.21}$$

The integral over ℓ is a snap now; it gives $-i(32\pi^2 m^2(\alpha+\beta)^2)^{-1}$. Thus

$$\Lambda^\mu = -\frac{e^2}{8\pi^2}\frac{(p'+p)^\mu}{m}\int_0^1\int_0^{1-\beta} d\alpha d\beta \frac{1-\alpha-\beta}{\alpha+\beta} \tag{5.22}$$

The integral gives a value of 1/2. And so we finally arrive at our answer,

$$\Lambda^\mu = -\frac{e^2}{8\pi^2}\frac{(p'+p)^\mu}{2m} = -\frac{\alpha}{2\pi}\frac{(p'+p)^\mu}{2m} \tag{5.23}$$

This implies (from Eq. 5.6 & Eq. 5.7) that, drum roll please...

$$F_2(0) = \frac{\alpha}{2\pi} \simeq 0.00116 \tag{5.24}$$

What a wonderfully simple answer! It shows that the contribution to the magnetic moment, which was found to have a coefficient of $F_1(0) + F_2(0) = 1 + F_2(0)$ indeed has an offset from the prediction of the Dirac theory. What's more, it turned out to be right. This was a huge feather in the cap of quantum field theorists as it suggested the viability of, at least, quantum electrodynamics.

This was first worked out by J. Schwinger in 1948 and has since become part of a measurement of the magnetic moment of the electron which is accurate to nine decimal places at current. The theory may be more accurate than that, but we have not been able to engineer an experiment which can measure with that kind of sensitivity. It is all good and well to charge ahead at this point and calculate all diagrams to all orders, complete with loops and every strange kind of diagram or interaction that you can cook up. But take a page out of Schwinger's book and figure out which terms contain the interesting new physics, and just do the one integral.