Electrostatics Kung Fu

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1 Useful Things Before Starting

I'll keep referring back to these over and over by equation number. They're just here for reference so if you already know these things, you can skip ahead.

1.1 Integrals

$$\int \frac{dx'}{\sqrt{x'^2 + z^2}} = \ln\left(x' + \sqrt{x'^2 + z^2}\right) \tag{1}$$

$$\int \frac{x'dx'}{\sqrt{x'^2 + z^2}} = \sqrt{x'^2 + z^2} \tag{2}$$

$$\int \frac{dx'}{(x'^2 + z^2)^{3/2}} = \frac{1}{z^2}\frac{x'}{\sqrt{x'^2 + z^2}} \tag{3}$$

$$\int \frac{x'dx'}{(x'^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{x'^2 + z^2}} \tag{4}$$

1.2 Fields

$$\mathbf{E} = k\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2}\, d^3x' = k\int \rho(\mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x' \tag{5}$$

$$\Phi = k\int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x' \tag{6}$$

$$\mathbf{E} = -\nabla\Phi = -\left(\hat{x}\frac{d\Phi}{dx} + \hat{y}\frac{d\Phi}{dy} + \hat{z}\frac{d\Phi}{dz}\right) \tag{7}$$

Note - The *unprimed* coordinates (like $\mathbf{x}$)are where *we* are sitting. The *primed* coordinates (like $\mathbf{x}'$) are where the *source* is and where we are going to integrate over.

1.3 Delta Function

This little guy? I wouldn't worry about the details of this little guy (Super Troopers anyone?). Just know that it does the integrals for you automatically and that it is zero everywhere *except* where the argument is zero AND that is has dimensions of length^{-1} for us. For example,

$$\int_0^{10} f(x')\delta(x'-4)\,dx' = f(4) \tag{8}$$

The delta function $\delta(x'-4)$ is zero everywhere except where $x'=4$ in which case it is one. The shorthand for this is

$$\delta(x') = \begin{cases} 1 & \text{if}\, x'=0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Whenever you see one of these delta functions, figure out what it takes to make the argument (the stuff in parenthesis) zero then set every occurance of that variable to whatever that number is. Here are two more examples

$$\int_0^{50} x'^3\delta(x'-2)\,dx' = 2^3 = 8 \tag{10}$$

$$\iint x'y'^2\delta(x'-5)\delta(y'-4)\,dx'\,dy' = \int x'4^2\delta(x'-5)\,dx' = 5\cdot 4^2 = 80 \tag{11}$$

See? You're already as good at this as I am. Easy peasy.

2 Point Charges The Real Way

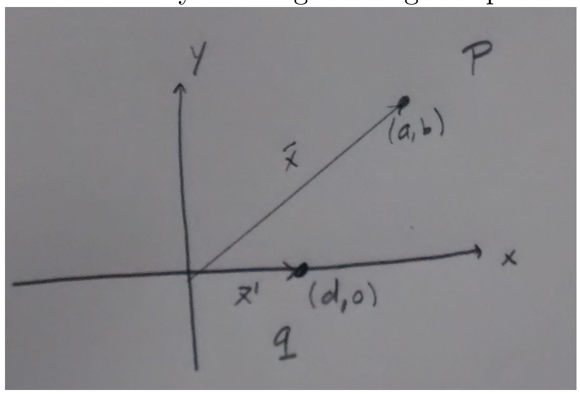
Be aware!! Throughout this I'm going to be writing $\mathbf{x}'$ and x' but please remember that these are not the same thing. In fact, $\mathbf{x}'$ is some generic vector that can point in any and all directions. But x' is just a number. It may seem awkward that the name of the vector is the same as one of its components, unfortunately every book does this so it seems like we're stuck with it. I'll always bold the vectors so just keep in mind that they can point anywhere.

Suppose that there is a point charge $+q$ sitting on the x-axis at a distance d away (to the right) from the origin, so that $(x',y')=(d,0)$. Also, suppose that you're sitting at the point $(x,y)=(a,b)$. It will look something like Fig. 1.

Let's use the r.h.s of Eq. 5. If we're going to use that, then we need to know what $\mathbf{x}', \mathbf{x}$, and $\rho(\mathbf{x}')$ are. So let's start there.

$\mathbf{x}$ is the distance from the origin to us. Always. Since we are at $(x,y)=(a,b)$, then we're "over" a and "up" b, right? To be more fancy-pantsy about it

Figure 1: Behold my amazing drawing of a point charge!



$$\mathbf{x} = a\hat{x} + b\hat{y} \tag{12}$$

Cool. Now let's figure out $\mathbf{x}'$. We don't have to assume anything about where the charge is, that's what the delta function is for. Let's keep it generic,

$$\mathbf{x}' = x'\hat{x} \tag{13}$$

for some x'. Two down and one to go! What is the charge density, $\rho(\mathbf{x}')$? Well, the charge is sitting at on the x-axis at a point d, so $(x', y', z') = (d, 0, 0)$. Whenever we see an integral with x's in it, we just set them to d. We set y' and z' to zero. We can express this mathematically as

$$\rho(\mathbf{x}') = q\delta(x' - d)\delta(y')\delta(z') \tag{14}$$

This says that the charge density is zero everywhere except where $x' = d$, $y' = 0$ and $z' = 0$ (vis-à-vis Eq. 9). Also check out the dimensions. If the delta functions have units of length^{-1} then we have

$$\rho \sim \text{charge } \cdot \text{length}^{-1} \cdot \text{length}^{-1} \cdot \text{length}^{-1} = \text{charge}/\text{length}^3 = \text{charge}/\text{volume} \tag{15}$$

which is what we want, so everything seems to be ok. To recap,

$$\mathbf{x} = a\hat{x} + b\hat{y} \tag{16}$$
$$\mathbf{x}' = x'\hat{x} \tag{17}$$
$$\rho(\mathbf{x}') = q\delta(x' - d)\delta(y')\delta(z') \tag{18}$$

In some sense, we're done. We have all of the information that we need. Let's shove all of this stuff into the r.h.s of Eq. 5 and turn the crank.

$$\mathbf{E} = k\int \rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x \tag{19}$$

$$= k\int \frac{q\delta(x'-d)\delta(y')\delta(z')\left((a-x')\hat{x}+b\hat{y}\right)}{((a-x')^2+b^2)^{3/2}}\,dx'dy'dz' \tag{20}$$

$$= kq\int \frac{\delta(x'-d)\left((a-x')\hat{x}+b\hat{y}\right)}{((a-x')^2+b^2)^{3/2}}\,dx' \tag{21}$$

$$= kq\frac{(a-d)\hat{x}+b\hat{y}}{((a-d)^2+b^2)^{3/2}} \tag{22}$$

Ok what happened here? To get from line 19 to line 20 I just substituted in everything that we found in Eq. 16-18. Then in line 20 I used the y' and z' delta functions to do the integrals for me - all y-primes and z-primes were set to zero (not like there were any around this time... but they would have been set to zero, by golly). In line 21 I let the x' delta function work its magic - it does the remaining integral for us and sets all of the x-primes to d. After that, we have the answer!

All this? For a point charge? Yeah, you can call me Captain Overkill. That's fair. That was a long walk for a problem you already know how to do ten times faster. The first is the worst, I promise. Fortunately, we can use *exactly* the same procedure every time for every electrostatics problem and it always works. Before moving on, let's make use of Eq. 7 and calculate the potential first and then use it to find the electric field.

We already know all of the things that we need, namely, Eq. 16-18. So let's just dive in!

$$\Phi = k\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \tag{23}$$

$$= k\int \frac{q\delta(x'-d)\delta(y')\delta(z')}{\sqrt{(a-x')^2+b^2}}\,dx'dy'dz' \tag{24}$$

$$= kq\int \frac{\delta(x'-d)}{\sqrt{(a-x')^2+b^2}}\,dx' \tag{25}$$

$$= kq\frac{1}{\sqrt{(a-d)^2+b^2}} \tag{26}$$

The same steps happened here as in the calculation of the electric field. To get from Eq. 23 to Eq. 24 I just substituted in everything. Then to get to line 25 I used the y' and

z' delta functions to set all $y's$ and $z's$ to zero and do those integrals for us. Lastly, I used the x' delta function to set all $x's$ to d and do the last integral for us. See? You're already getting the hang of it.

So what good did this do? Well, the potential is often much, much easier to calculate. Once we have it, we just take some derivatives and then we know the electric field. Derivatives tend to be easier than integrals, not to mention scalers are easier to handle than vectors, so it definitely has a place in our hearts. Let's get the electric field from the potential now. Be aware that $a-d$ is the part in the $\hat{x}$ direction and b is in the y direction. So our derivative w.r.t x will really be w.r.t $(a-d)$ and the derivative w.r.t y will really be a derivative w.r.t b. Got it? Ok, so we get

$$\mathbf{E} = -\left(\hat{x}\frac{d}{d(a-d)} + \hat{y}\frac{d}{db}\right)\Phi \tag{27}$$

$$= -kq\left(\hat{x}\frac{-2(a-d)}{2((a-d)^2+b^2)^{3/2}} + \hat{y}\frac{-2b}{2((a-d)^2+b^2)^{3/2}}\right) \tag{28}$$

$$= kq\frac{(a-d)\hat{x}+b\hat{y}}{((a-d)^2+b^2)^{3/2}} \tag{29}$$

This is exactly the same as Eq. 22 as promised. So now you have two ways to calculate the electric field. You can either

- Calculate the electric field directly (may involve trickier integrals and you have to worry about vecors)

- Calculate the potential first and then take derivative

There is no "better" way. It's a matter of preference. If you can do some sneaky sleight of hand and still get the right answer, by all means, do that instead! This isn't so much about clever tricks as it is sure-fire methods of getting to the answer.

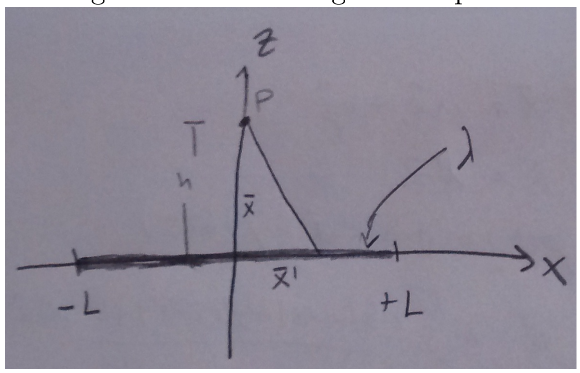
Right, let's turn up the heat a little...

3 Line Charge with Uniform Charge Density

> Boy, that escalated quickly... I mean, that really got out of hand fast.
>
> Ron Burgundy

Let's work out the electric field over a wire of length $2L$ that goes from $x' = -L$ to $x' = L$ and has a linear charge density λ as in Fig. 2. Let the coordinate system have the

Figure 2: A line charge masterpiece



x-direction pointing to the right and the z-direction pointing upwards. Imagine that we are sitting at $(x, z) = (0, h)$. We again need to find three things:

1. $\mathbf{x}$ - The vector from the origin to where we are sitting

2. $\mathbf{x}'$ - The vector from the origin to the source

3. $\rho(\mathbf{x}')$ - the charge density

Well, where are we sitting? Yup, we're at a height h in the z-direction over the line charge,

$$\mathbf{x} = h\hat{z} \tag{30}$$

And where is the source? Again, let's be generic and let the delta functions worry about where exactly the line charge is. That being said, we can write

$$\mathbf{x}' = x'\hat{x} \tag{31}$$

So far so good. Now the charge density. Is the charge density in the z-direction at all? Nope, it's at $z' = 0$. What about in the y-direction? Nope. There aren't even any y's around to worry about (the y-direction is into the page). The line charge *is* in the x-direction, so we won't have delta function for that direction. In other words, we should expect that we'll have to do an integral over dx'. The charge density is

$$\rho(\mathbf{x}') = \lambda\delta(z')\delta(y') \tag{32}$$

Do the dimensions make sense? We have

$$\rho \sim (\text{charge/length}) \cdot \text{length}^{-1} \cdot \text{length}^{-1} = \text{charge/length}^3 = \text{charge/volume} \tag{33}$$

Everything checks out, dimensionally anyway. For the sake of clarity, here is what we have

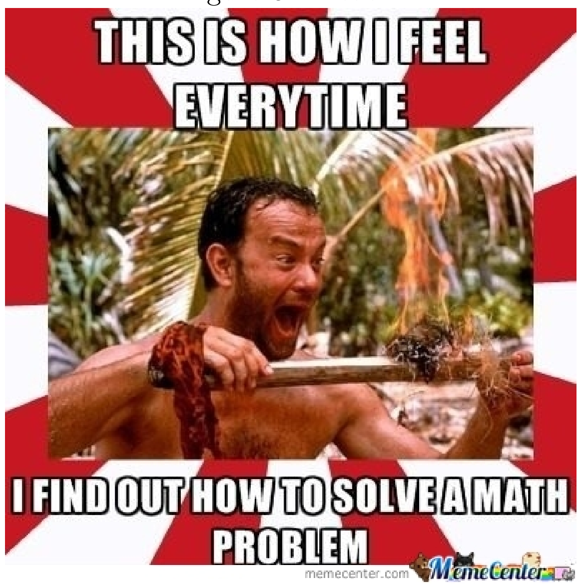
$$\mathbf{x} = h\hat{z} \tag{34}$$
$$\mathbf{x}' = x'\hat{x} \tag{35}$$
$$\rho(\mathbf{x}') = \lambda\delta(z')\delta(y') \tag{36}$$

Before we get too far into this, what should the answer look like? We know that the electric field will only point in $\hat{z}$ so for any other direction that shows up in the integral, we can simply throw out since it can't contribute to the final answer. Also, at $h = 0$, the field had better be zero and far, far away the line charge should look like a point charge (with $q = 2L\lambda$). If we don't get that, then something went terribly wrong.

Now just turn the crank again! Stuff everything into Eq. 5 and start simplifying things

$$\mathbf{E} = k\int \rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x' \tag{37}$$
$$= k\int \frac{\lambda\delta(z')\delta(y')\,(h\hat{z}-x'\hat{x})}{(x'^2+h^2)^{3/2}}\,dx'dy'dz' \tag{38}$$
$$= k\lambda\int_{-L}^{L}\frac{(h\hat{z}-x'\hat{x})}{(x'^2+h^2)^{3/2}}\,dx' \quad \text{(Only keep the } \hat{z} \text{ stuff!)} \tag{39}$$
$$= k\lambda h\hat{z}\int_{-L}^{L}\frac{1}{(x'^2+h^2)^{3/2}}\,dx' \quad \text{(Use Eq. 3)} \tag{40}$$
$$= k\lambda h\hat{z}\left(\frac{x'}{h^2\sqrt{x'^2+h^2}}\bigg|_{-L}^{L}\right) \tag{41}$$
$$= k\lambda h\hat{z}\left(\frac{L}{h^2\sqrt{L^2+h^2}} - \frac{-L}{h^2\sqrt{L^2+h^2}}\right) \tag{42}$$
$$= k\lambda h\hat{z}\left(\frac{2L}{h^2\sqrt{L^2+h^2}}\right) \tag{43}$$
$$= k(2L\lambda)\frac{1}{h\sqrt{L^2+h^2}}\hat{z} \tag{44}$$
$$= kQ_{Total}\frac{1}{h\sqrt{L^2+h^2}}\hat{z} \tag{45}$$

Figure 3: Preach!



We did it! Tom Hanks (Fig. 3) would be proud. By the way, if we *hadn't* thrown out the stuff in the $\hat{x}$ direction and bothered to do the integral, it would have come out to zero (as it had to). So we, being sophisticated physicists, took the clever way around.

Let's start all over and do it with the potential. Use Eq. 6 and then just plug 'n chug

$$\Phi = k \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x' \tag{46}$$

$$= k \int \frac{\lambda \delta(z')\delta(y')}{\sqrt{x'^2 + h^2}} \, dx' dy' dz' \tag{47}$$

$$= k\lambda \int_{-L}^{L} \frac{1}{\sqrt{x'^2 + h^2}} \, dx' \quad \text{(Use Eq. 1)} \tag{48}$$

$$= k\lambda \ln\left(x' + \sqrt{x'^2 + h^2}\right)\Big|_{-L}^{L} \tag{49}$$

$$= k\lambda \left[\ln\left(L + \sqrt{L^2 + h^2}\right) - \ln\left(-L + \sqrt{L^2 + h^2}\right)\right] \tag{50}$$

Now, notice that h is in the $\hat{z}$ direction and that's the direction of the electric field. So we need to find

$$
\begin{aligned}
\mathbf{E} &= -\hat{z}\frac{d\Phi}{dh} && (51)\\
&= -\hat{z}k\lambda\left[\left(\frac{1}{L+\sqrt{L^2+h^2}}\frac{2h}{2\sqrt{h^2+L^2}}\right)-\left(\frac{1}{-L+\sqrt{L^2+h^2}}\frac{2h}{2\sqrt{h^2+L^2}}\right)\right] && (52)\\
&= -\hat{z}k\lambda\frac{h}{\sqrt{h^2+L^2}}\left[\frac{1}{L+\sqrt{L^2+h^2}}-\frac{1}{-L+\sqrt{L^2+h^2}}\right] && (53)\\
&= -\hat{z}k\lambda\frac{h}{\sqrt{h^2+L^2}}\left[\frac{(-L+\sqrt{L^2+h^2})-(L+\sqrt{L^2+h^2})}{(-L+\sqrt{L^2+h^2})(L+\sqrt{L^2+h^2})}\right] && (54)\\
&= -\hat{z}k\lambda\frac{h}{\sqrt{h^2+L^2}}\left[\frac{-2L}{-L^2+L^2+h^2}\right] && (55)\\
&= \hat{z}k(2L\lambda)\frac{1}{h\sqrt{h^2+L^2}} && (56)\\
&= kQ_{Total}\frac{1}{h\sqrt{h^2+L^2}}\hat{z} && (57)
\end{aligned}
$$

Miracle of miracles, it's the same as Eq. 45! Huzzah! The derivative was, well, pretty gross actually. But the integral wasn't *too* bad. Intrepid as we are, we move on to a ring of charge.

4 Lord of the Charged Rings

"Take physics" they said. "It will be easy" they said.

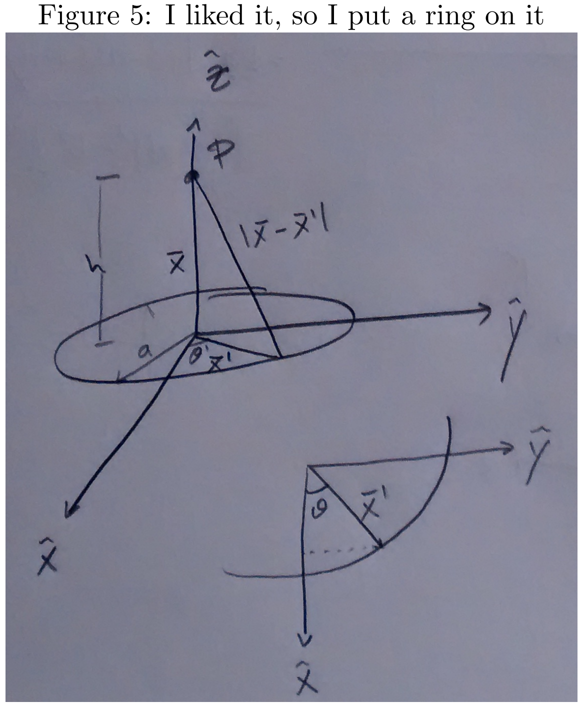
Maniac in Fig. 4

We're now ready to move on to a ring of charge. This isn't too different conceptually from the line charge. It is still just a line charge after all. The biggest mathematical difference come from the inevitable change to cylindrical coordinates($dx'dy'dz' \to r'dr'd\theta'dz'$). Short of that, it's business as usual. First things first, let's gather the necessary information that we need for Eq. 5. Use the specs in Fig. 5 which also shows a "top" view to make it easier to see how much of $\mathbf{x}'$ is on the x-axis and how much is on the y-axis (the θ in the "top" view should be a θ')

Before we get carried away, take a moment and think about what the answer needs to look like. Of course it should be like q/r^2, but what else? There is a lot of symmetry here and I trust that you can convince yourself that the final result must be only in the

Figure 4: And yet we soldier on

Figure 5: I liked it, so I put a ring on it



$\hat{z}$ direction so we can throw the other stuff away. Also, when $h = 0$, that is, when we are sitting right in the middle of the loop, the field had better drop to zero. Ok, let's hop to it.

$$\mathbf{x} = h\hat{z} \tag{58}$$

$$\mathbf{x}' = r'\hat{r} = r'cos\theta'\hat{x} + r'sin\theta'\hat{y} \tag{59}$$

$$\rho(\mathbf{x}') = \lambda\delta(z')\delta(r' - a) \tag{60}$$

There's some explaining worth doing here I think, primarily about $\mathbf{x}'$. Picking r' as a variable was totally arbitrary. You could have picked x' or anything else for that matter. I'm looking into the future and am keeping the notation consistent with cylindrical coordinates, but that's not particularly important. Secondly, what is $\hat{r}$? The charged ring is in some plane and the radial vector always points outward from the origin which is what $\hat{r}$ does. The trouble is, we're integrating over θ' and we know that the vector $\mathbf{x}'$ is spinning around the hoop and must have something to do with angles - so we decompose $\hat{r}$ into it's components in the same way that we break down a hypotenuse into a horizontal and a vertical component. Lastly, about the delta function: the ring is only at $z' = 0$ and at $r' = a$. At any other height or radius, you'll miss the ring. We don't have a delta function to do the integral over θ', so we'll have to do that one by hand. Fortunately, this one is as easy as it gets. Let's dive in!

$$\mathbf{E} = k\int \rho(\mathbf{x}')\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x' \tag{61}$$

$$= k\int \frac{\lambda\delta(z')\delta(r' - a)\,(h\hat{z} - r'cos\theta'\hat{x} - r'sin\theta'\hat{y})}{(h^2 + r'^2)^{3/2}}\, r'dr'd\theta'dz' \tag{62}$$

$$= k\lambda\int_0^{2\pi} \frac{(h\hat{z} - a\cos\theta'\hat{x} - a\sin\theta'\hat{y})}{(h^2 + a^2)^{3/2}} a d\theta' \quad \text{(Throw away everything that isn't in } \hat{z}\text{)} \tag{63}$$

$$= k\lambda\int_0^{2\pi} \frac{h\hat{z}}{(h^2 + a^2)^{3/2}} a d\theta' \tag{64}$$

$$= k\lambda\frac{2\pi a h\hat{z}}{(h^2 + a^2)^{3/2}} \tag{65}$$

$$= k(2\pi a\lambda)\frac{h}{(h^2 + a^2)^{3/2}}\hat{z} \tag{66}$$

$$= kQ_{Total}\frac{h}{(h^2 + a^2)^{3/2}}\hat{z} \tag{67}$$

And that's all there is to it. We should verify that our intuition is intact. What happens when we sit right in the middle of the loop ($h = 0$)? As hoped, the electric field vanishes. What about when we're really far away? That is, consider when $h \gg a$. In that case we get $\mathbf{E} = (kQ_{Total}/h^2)\hat{z}$ which is Coulomb's law. Physically, this means that when we're far enough away from the ring, it doesn't "look" like a ring anymore, it just looks like some lump of charge. This seems completely reasonable and indeed it is.

As always, let's get this answer using the potential instead. We already have the info we need, so plug it in and get going

$$\Phi = k \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x' \tag{68}$$

$$= k \int \frac{\lambda\delta(z')\delta(r'-a)}{\sqrt{h^2 + r'^2}} r'dr'd\theta'dz' \tag{69}$$

$$= k\lambda \int_0^{2\pi} \frac{ad\theta'}{\sqrt{h^2 + a^2}} \tag{70}$$

$$= k(2\pi a\lambda)\frac{1}{\sqrt{h^2 + a^2}} \tag{71}$$

$$= kQ_{Total}\frac{1}{\sqrt{h^2 + a^2}} \tag{72}$$

That wasn't so bad. Now we need to take a derivative of the "z part" (which is w.r.t h of course) so we have

$$\mathbf{E} = -\hat{z}\frac{d\Phi}{dh} \tag{73}$$

$$= -\hat{z}kQ_{Total}\left(\frac{-2h}{2(h^2 + a^2)^{3/2}}\right) \tag{74}$$

$$= kQ_{Total}\frac{h}{(h^2 + a^2)^{3/2}}\hat{z} \tag{75}$$

We hardly need to be surprised at this point; we've reproduced the electric field. :) Go us.

5 DISCpicable Me

> A mystic is someone who wants to understand the universe, but is too lazy to study physics.
>
> Unknown

Last but not least, let us calculate the electric field over the center of a uniformly charged disc with a surface density σ and an outer radius of a. We can continue to use Fig. 5, simply pretend that the ring is a flat disc. In that case, We still want Eq. 58-59. The charge density is

$$\rho(\mathbf{x}') = \sigma\delta(z') \tag{76}$$

Check the units to convince yourself that everything is ok! (Hint: $\sigma \sim$ charge/length^2). We only get one delta function this time. Why? Well, a disc is a two dimensional thing, so two of the three integrals need to stick around. The disc is definitely confined to $z' = 0$, but it has a variable radius and angular component, so we need to do two integrals and that's just the way it is this time. We *could* cheat a little and jump back to Eq. 63 and say "I know what we don't just have radius 'a', but rather we need to integrate up to 'a', so rename $a \to r'$ and integrate from 0 to a". Pretty smart! Yes, you could do that. But for the sake of completeness, I"m going to work it out form the ground up. So here is what we know:

$$\mathbf{x} = h\hat{z} \tag{77}$$

$$\mathbf{x}' = r'\hat{r} = r'cos\theta'\hat{x} + r'sin\theta'\hat{y} \tag{78}$$

$$\rho(\mathbf{x}') = \sigma\delta(z') \tag{79}$$

What about the physics? Again, the field can only point in $\hat{z}$ and must vanish when $h = 0$. Additionally, far, far away the disc should look like a point charge. And with that, let's do it!

$$\mathbf{E} = k\int \rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x' \tag{80}$$

$$= k\int \frac{\sigma\delta(z')\,(h\hat{z} - r'cos\theta'\hat{x} - r'sin\theta'\hat{y})}{(h^2+r'^2)^{3/2}}\, r'dr'd\theta'dz' \tag{81}$$

$$= k\sigma\int \frac{(h\hat{z} - r'cos\theta'\hat{x} - r'sin\theta'\hat{y})}{(h^2+r'^2)^{3/2}}\, r'dr'd\theta' \quad \text{(Throw out stuff not in } \hat{z}\text{)} \tag{82}$$

$$= k\sigma\int_0^{2\pi}\int_0^{a} \frac{h\hat{z}}{(h^2+r'^2)^{3/2}} r'dr'd\theta' \tag{83}$$

$$= k\sigma(2\pi)h\hat{z}\int_0^{a} \frac{r'dr'}{(h^2+r'^2)^{3/2}} \quad \text{(Use Eq. 4)} \tag{84}$$

$$= k\sigma(2\pi)h\hat{z}\left(\frac{-1}{\sqrt{h^2+r'^2}}\bigg|_0^a\right) \tag{85}$$

$$= k\sigma(2\pi)h\hat{z}\left(\frac{-1}{\sqrt{h^2+a^2}} + \frac{1}{|h|}\right) \tag{86}$$

$$= k\sigma(2\pi)\hat{z}\left(1 - \frac{h}{\sqrt{h^2+a^2}}\right) \tag{87}$$

$$= \frac{\sigma}{2\epsilon_0}\left(1 - \frac{h}{\sqrt{h^2+a^2}}\right)\hat{z} \tag{88}$$

In the last step I used the fact that $k = 1/(4\pi\epsilon_0)$ to write the field in a more "traditional" way. Be very careful about line 86. Remember that $1/\sqrt{h^2}$ is guaranteed to be a positive number here. So I *cannot* say that $1/\sqrt{h^2} = 1/h$ because h could be negative. The absolute value bars are... well, absolutely necessary (ba-dum tsch!).

The last thing to do would be to calculate $\mathbf{E}$ from the potential. But I will leave this up to you, dear reader. You are well equipped at this point to tackle problems like this on your own. Congratulations for making it this far! I hope this was at least somewhat helpful. If it is a confusing nightmare, let me know and I'll try my best to make it more clear. Also, there *may* be typos in here. If you think I messed something up, you're probably right. Send me an email and let me know so I can send around the correction and an updated pdf to everyone if necessary.

As a final note, don't be discouraged. Rome wasn't built in a day. This stuff takes practice. There are no special people and nobody was born knowing how to do these types of problems. Hang it there. It *will* come to you more and more every day.