Lecture 1: Prelude

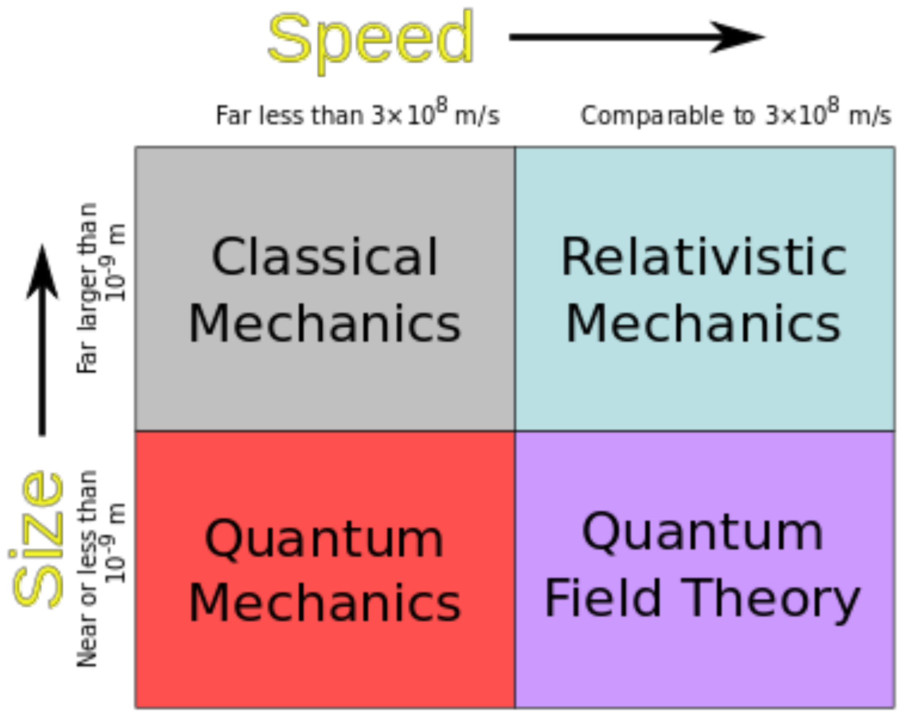
November 14, 2015

1 General Survey

Physics is often called the fundamental science. What is meant is that all natural phenomena can ultimately be understood in terms of physics. Although that statement is technically true so far as we can tell, it often has very little utility. Suppose a biologist is trying to study the migratory patterns of some new species of bird. If the bird's actions are controlled by the combination of voluntary actions (such as snatching up a worm) and involuntary actions (such as a heartbeat) and those actions are caused by the motion of the atoms within the bird, shouldn't the biologist study the motion of the atoms in order to fully understand the inner workings of the bird? Well, let's see how reasonable that is: There are roughly 10^{23} atoms in a bird, so the biologist would have to set up 10^{23} equations and solve them all at the same time in order to understand the next split second decision of the bird, and even then, it will still be a probabilistic outcome. This is clearly not a possibility and not worth such massive effort! From a practical point of view, it makes more sense to develop a new, macroscopic model for the bird than to try and use atomic physics, and thus we have biology and biologists.

When we look at some physical process, we have to decide on what "scale" we are using. Are we dealing with only one or two atoms, several dozen atoms, or trillions of them? We can imagine watching some atoms bang into each other (which we use physics to understand) and as we slowly zoom out we start to see entire chemicals (which we use chemistry to study) and as we zoom out even further we start to see proteins and cells and eventually a bird or a human or a skunk etc. and we rely on biologists to tell us how they work.

For systems which don't have too many moving parts, we can use the laws of physics that have been discovered to understand an enormous range of phenomena. The figure illustrates the fantastic range of speeds and sizes which may be understood by Newton's Laws alone (a.k.a Classical Mechanics).



Therefore, although we are limited by the complexity of the system, we have relatively few limitations in terms of the size or speed of the system. In fact, knowing only Newton's Second Law, $\vec{F} = m\vec{a}$, you can build a rocket and put a man on the moon. Don't be fooled by the simplicity of the expression! Most major laws have a delightfully simple form. In fact, strictly for your amusement, here are the fundamental equations in electrodynamics (EM), quantum mechanics (QM), and quantum electrodynamics (QED) respectively

$$\partial_\mu F^{\mu\nu} = J^\nu \quad \text{(EM)}$$
$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle \quad \text{(QM)}$$
$$(\not{p} - m)\phi(x) = 0 \quad \text{(QED)}$$

Each equation is no more complicated than Newton's Law, yet those equations possess almost all that is known in physics. An enormous amount of informations is just a slide up or a slide down from these simple expressions. Anything you have ever wondered about magnets, photons, electrons, scanning electron microscopes, MRI's, X-ray's, microwaves, lightning, mirrors etc., it's all in there.

It should also be noted that although it is necessary to have a solid grasp of mathematics to fully understand physical phenomena, physics is not math. There are many mathematically sound equations which may be written down, however when checked against experiment they will be dead wrong. Why? Because the natural world is presumably not interested in *how* you evaluate something, only that you *do*. As a simple example, suppose I tell you that the square of the length of a room is $\ell^2 = 144$ feet squared. What, then, is the length of the room? The mathematical answer is of course

$$\ell = \pm 12\,\text{feet}$$

So the mathematician insists that the room can be negative 12 feet long! Have we made a mistake? Well, no, not exactly. If we ask a physicist the same question they will return with the answer "12 feet" having already understood the fact that lengths are either greater than or equal to zero. It appears to be so that the physicist has extra information on hand. Indeed, this is the case; the scientist uses *experience*. The word "experiment" actually derives from the Old French word for experience. So in a very literal sense, one must account for the world according to known experiments, and therefore must guide their calculations with these facts. If a statement disagrees with experiment, it's wrong. It is that statement which underlies all of science. One may employ all of the imagination possible on a problem, but it must be imagination in a straight jacket so to speak because there are certain principles in nature which must be true, no matter how fancy your theory may be. We can now start to understand why we can throw out the negative length found by the mathematician: experimenters only ever measure positive lengths. It may sound silly, but it's really how we do it.

As a short aside, the fact that taking a square root of something spits out a $\pm$ sign is highly non-trivial. Consider three quantities, energy, momentum and mass (E, p, m) respectively. There is an equation with looks as follows

$$E^2 = m^2 + p^2$$

This seems harmless enough. It looks a bit like the Pythagorus Theorem for a right triangle. Since the mass and momentum are real (as opposed to complex), then the square of them is guaranteed to be positive. Both sides of the equation are therefore said to be positive definite. But now, let's take a square root

$$E = \pm\sqrt{p^2 + m^2}$$

The energies can be positive or negative! A negative energy will allow us to get work out of it, which would make the energy more negative and allow us to get more work out which makes it even more negative and so on. We'd therefore have perpetual motion! This troubling result came about in the 1920's and was first encountered by Paul Dirac while formulating a relativistic theory of quantum electrodynamics. The negative solutions couldn't be thrown out if the equations were to remain internally consistent, and it was therefore supposed that there are "anti-particles" which occupy these negative energy states. And then, almost miraculously, they were discovered soon after. (The energies aren't actually negative, they're only interpreted that way. Anti-particles are completely described mathematically by a regular particle with opposite charge going backwards in time).

Lastly, it's important to know that a physical theory or even a law, can never be shown to be exactly correct. Rather it can only be shown to be right *so far*. Axioms are proposed which appear to always work, then calculations are made to determine what the world would be like if such things were true. If these predictions match all experimental data, then the axiom is kept and further tested. If a contradiction arises, the axiom is thrown away. We have to suppose *something* to be true in order to get anywhere. For example, we assume that the universe exists, we assume that today will be more or less like yesterday and that tomorrow will be essentially the same as well, we assume that no gods or angels or devils or fairies will interfere with our experiments etc. Now, it COULD BE that tomorrow is vastly different somehow or that a fairy interferes with our data collection. However, such a thing is useless to assume since we'd never be able to trust any results we get. Instead, we suppose that no such things exist and calculate away under that assumption. If all experimental data agrees with our calculations, we begin to feel comfortable with the axioms and rejecting our former worries, however we can never know for sure if they are absolutely correct. This process of making fundamental assumptions (and rejecting superfluous suggestions) in order to continue on finding results is often called "methodological naturalism." The process of science is then as follows: Make a guess, Calculate out the way things would be assuming that the guess is true, Compare the prediction to experiment.

A nice example of this mindset comes from Carl Sagan and is included in full below.

2 The Dragon In My Garage - C. Sagan

"A fire-breathing dragon lives in my garage" Suppose (I'm following a group therapy approach by the psychologist Richard Franklin) I seriously make such an assertion to you. Surely you'd want to check it out, see for yourself. There have been innumerable stories of dragons over the centuries, but no real evidence. What an opportunity!

"Show me," you say. I lead you to my garage. You look inside and see a ladder, empty paint cans, an old tricycle – but no dragon.

"Where's the dragon?" you ask.

"Oh, she's right here," I reply, waving vaguely. "I neglected to mention that she's an invisible dragon."

You propose spreading flour on the floor of the garage to capture the dragon's footprints.

"Good idea," I say, "but this dragon floats in the air."

Then you'll use an infrared sensor to detect the invisible fire.

"Good idea, but the invisible fire is also heatless."

You'll spray-paint the dragon and make her visible.

"Good idea, but she's an incorporeal dragon and the paint won't stick." And so on. I counter every physical test you propose with a special explanation of why it won't work.

Now, what's the difference between an invisible, incorporeal, floating dragon who spits heatless fire and no dragon at all? If there's no way to disprove my contention, no conceivable experiment that would count against it, what does it mean to say that my dragon exists? Your inability to invalidate my hypothesis is not at all the same thing as proving it true. Claims that cannot be tested, assertions immune to disproof are veridically worthless, whatever value they may have in inspiring us or in exciting our sense of wonder. What I'm asking you to do comes down to believing, in the absence of evidence, on my say-so. The only thing you've really learned from my insistence that there's a dragon in my garage is that something funny is going on inside my head. You'd wonder, if no physical tests apply, what convinced me. The possibility that it was a dream or a hallucination would certainly enter your mind. But then, why am I taking it so seriously? Maybe I need help. At the least, maybe I've seriously underestimated human fallibility. Imagine that, despite none of the tests being successful, you wish to be scrupulously open-minded. So you don't outright reject the notion that there's a fire-breathing dragon in my garage. You merely put it on hold. Present evidence is strongly against it, but if a new body of data emerge you're prepared to examine it and see if it convinces you. Surely it's unfair of me to be offended at not being believed; or to criticize you for being stodgy and unimaginative – merely because you rendered the Scottish verdict of "not proved."

Imagine that things had gone otherwise. The dragon is invisible, all right, but footprints are being made in the flour as you watch. Your infrared detector reads off-scale. The spray paint reveals a jagged crest bobbing in the air before you. No matter how skeptical you might have been about the existence of dragons – to say nothing about invisible ones – you must now acknowledge that there's something here, and that in a preliminary way it's consistent with an invisible, fire-breathing dragon.

Now another scenario: Suppose it's not just me. Suppose that several people of your acquaintance, including people who you're pretty sure don't know each other, all tell you that they have dragons in their garages – but in every case the evidence is maddeningly elusive. All of us admit we're disturbed at being gripped by so odd a conviction so ill-supported by the physical evidence. None of us is a lunatic. We speculate about what it would mean if invisible dragons were really hiding out in garages all over the world, with us humans just catching on. I'd rather it not be true, I tell you. But maybe all those ancient European and Chinese myths about dragons weren't myths at all.

Gratifyingly, some dragon-size footprints in the flour are now reported. But they're never made when a skeptic is looking. An alternative explanation presents itself. On close examination it seems clear that the footprints could have been faked. Another dragon enthusiast shows up with a burnt finger and attributes it to a rare physical manifestation of the dragon's fiery breath. But again, other possibilities exist. We understand that there are other ways to burn fingers besides the breath of invisible dragons. Such "evidence" – no matter how important the dragon advocates consider it – is far from compelling. Once again, the only sensible approach is tentatively to reject the dragon hypothesis, to be open to future physical data, and to wonder what the cause might be that so many apparently sane and sober people share the same strange delusion.

3 Mathematical Assessment

1. Solve $x^2 - x - 6 = 0$ for x
2. $\frac{d}{dx}x^2 =$
3. $\frac{d}{dt}\sin(\omega t) =$
4. $\int x\,dx =$
5. $\int \cos(\omega t)\,dt =$
6. If $\vec{a} = (3xy, y^3) \equiv 3xy\hat{x} + y^3\hat{y}$ and $\vec{b} = (-3xy, \cos(x))$, then what is $\vec{a} + \vec{b}$?
7. Using the vectors above, what is $\vec{a} \cdot \vec{b}$?
8. $\sum\limits_{n=1}^{3} n^2 =$
9. If $\vec{v} = (1, 2, 3)$, what is the magnitude of $\vec{v}$, i.e, what is $||\vec{v}||$?
10. $\cos^2(x) + \sin^2(x) =$
11. Sketch $y(x) = x^2$

4 Physical Assessment

1. What are the dimensions of the following: position, velocity, acceleration, energy, force, momentum?
2. Given that Newton's second Law can be expressed in 1D mathematically as $F = ma$, what does this mean physically?
3. You find yourself in the middle of a frictionless ice rink and you are perfectly still, that is, you have 0 velocity. Armed only with the apple you were about to eat, how do you get off of the ice?
4. A spaceship is deep in outer space and has zero velocity when suddenly the rockets turn on. Will the spaceship start moving?
5. Is it possible to heat up a cup of coffee by yelling at it? Why or why not?
6. You are spinning a yo-yo around your head by the string when suddenly the string breaks. Describe the subsequent motion of the yo-yo.

7. If I tell you that the only quantities in a system consisting of a pendulum and string are the acceleration due to gravity, g in meters per second squared and the length of the string, l, in meters, then the frequency (which has units of 1/s) must be proportional to what?
8. Roughly how many grains of sand are on the earth?
9. Given that there are roughly 3×10^7s in a year, how many seconds old is the universe?
10. You drive exactly once around a circular 1 mile race track at precisely 60mph. What is your average speed? What is your average velocity?

Newton's Laws in 1D

1 Introduction

Here we are at the start of our discussions on classical mechanics. It is my hope that we not only cover the topic as thoroughly as possible within the time allotted, but also to spark some interest, imagination, and to provide the tools necessary to understand and enjoy an enormous amount of phenomena. If we will have seen further, it is by standing on the shoulders of giants.

The first few sections are going to be a terse look at the quantities that we care about, their mathematical relationships and some insight as to what they represent. If not everything is clear in those sections, don't worry; it shouldn't be yet. Following the outline on formalism, we'll dive right in and start doing calculations.

2 Mass, Acceleration, and Force

To some degree, merely writing out Newton's Second Law, $\mathbf{F} = m\mathbf{a}$, that is, that an object accelerates in the direction of an applied force with a slope inversely proportional to the mass, and additionally supplying the information that the gravitational force acts radially and varies inversely as the square of the distance would be enough to calculate the motions of the planets, bouncing balls, gyroscopes etc. However, we don't yet have an understanding of what it means to call something a force. You may object; you know that a force is equal to the product of a mass and an acceleration. Yes, but this is a merely a definition. What does it actually *mean* to impart a force on something? There are a few hurdles we must first clear here.

The first point is somewhat pedantic, though crucial: Newton's Laws are an idealization. They are not exact. Compiled together, they form a model which describes roughly how objects of a certain size and velocity behave under various circumstances. Let's consider the mass term first. To start, the mass, or *inertial mass*, is not, in and of itself, understood. The origin of mass is not known and therefore the properties that derive from it (such as inertia) are not actually understood. This is reminiscent of the fact that the Babylonians for example could very accurately predict the motion of Venus, however they had absolutely no idea of why it moved the way it did, they simply knew *that* it moved in such a way. Newton himself stated the inverse square law of gravitation without an understanding of its origin. History certainly seems to suggest however that we can proceed to get accurate and useful results without a complete understanding of the process, so we accept that mass exists for now and simply move on. Sort of.

Suppose you have a billiard ball in front of you. What is the mass of the ball? You may get a scale and determine that the ball is 0.1kg. On a more accurate scale you determine the ball to be 0.112kg. We could continue this process, though we will eventually come to realize that atoms are coming and going, dust is settling, breezes are blowing, different altitudes feel different gravitational effects... there is a finite accuracy with which we may measure the mass of the ball, even with hypothetical devices capable of infinite precision. It usually contents us to know the mass to one part in 1000 or so for most macroscopic objects, which is well within reach. Our total accuracy of any calculation is now brought down to one part in 1000, unless of course the mass is ultimately irrelevant. In fact, no physical process is entirely isolated such that if we look closely enough at, say a ball bouncing off of the floor, we will notice that when the ball bangs into the atoms in the air, energy is transferred to them and they speed up (causing the ball to slow down of course), the net result of which is a temperature increase in the air. Just as the air warms, so too does the floor when the ball strikes it. The atoms in the floor, initially still, become a complicated wiggling mess which radiates photons around the room. The ball itself is not perfectly rigid and it oscillates about, particularly so after hitting the floor. The system is becoming fantastically complicated and for all practical purposes, impossible to model. The microscopic phenomena are often uninteresting to us when we're considering the motion of macroscopic bodies, and so we content ourselves to ignore these effects. They do however exist! We are making approximations and idealizations when we use Newton's Laws.

The second point is more mathematical and has to do with the acceleration term. In fact the problem is not specific to acceleration, but derivatives in general. The acceleration is the change in velocity per unit time, or (ignoring the vector nature for the moment)

$$a(t) = \lim_{t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \tag{1}$$

While this definition is surely rather familiar to you, it may seem questionable that we are taking the limit as the denominator of a fraction goes to zero. The assumption is that the numerator is then also going to zero in such a way that the ratio of the two converges to a number which is hopefully finite. We can open up this definition a little more rigorously to better understand what is going on and why we believe in such definitions.

Consider the following "well behaved" function, $f(x) = x^n$. By well behaved we mean that it is continuous everywhere (smooth) and infinitely differentiable. You already know *that* the derivative of this function is nx^{n-1}, but *how* do you know? Let's use the formal definition of a derivative:

$$\frac{df(x)}{dx} = \frac{dx^n}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \tag{2}$$

It is already clear that as Δx tends towards zero, so does the numerator. The convergence is not yet obvious. We need to simplify the numerator, and to this end, let's take some simpler powers and try to notice a pattern.

$$\begin{aligned} (x+\Delta x)^1 &= x + \Delta x \\ (x+\Delta x)^2 &= x^2 + 2x\Delta x + (\Delta x)^2 \\ (x+\Delta x)^3 &= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \end{aligned} \tag{3}$$

Remember now that Δx is becoming infinitesimally small such that $\Delta x \gg (\Delta x)^2 \gg (\Delta x)^3$... In other words, in the limit that we take this quantity to zero, the squared terms, cubed terms and so on can be ignored as they approach zero at an exponential rate. Notice that the coefficient in front of the term linear in Δx (a.k.a the "leading order" term in the expansion), is just the power that the function is raised to. And so we're led to the following approximation to leading order

$$(x+\Delta x)^n \approx x^n + nx^{n-1}\Delta x \tag{4}$$

Substituting this into Eqn. (2) we find

$$\frac{df(x)}{dx} = \frac{dx^n}{dx} \approx \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x) - x^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x}{\Delta x} = nx^{n-1} \tag{5}$$

Hardly a surprising result. But notice, there *are* other terms, we just chose to ignore them considering the limit that we're taking. The denominator will always cancel, so indeed it is not unreasonable to ask that a derivative of a smooth function exists. The drawn out definition of a force is now a mass, whatever that means, measured to some finite degree of accuracy, is multiplied by the ratio of the first order correction of the velocity over an infinitesimal time interval, and in the direction of the change in velocity. That is,

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{x}}{dt^2} \tag{6}$$

Any change in velocity implies that a force was present in the direction of the velocity change. Since velocity is a vector, then changing direction corresponds to a change in velocity which requires a force. Imagine spinning a rock on a string around your head at a constant frequency in a counterclockwise direction as viewed from above looking down. The rock is getting no further away from you, however if no force were present, the rock would go flying off in a straight line tangential to the curve of which it was moving. Instead, the rock has stayed closer to you, it has “fallen in” towards you. Therefore the rock is accelerating radially inwards towards you. This is true too of planets. The gravitational force acts radially, a point which was missed for quite some time as planets never seem to get closer but they do seem to move about tangentially. However now you know better- they are actually falling towards an object. The reason we don’t expect the moon to immediately spiral in towards the earth then is due to the tangential velocity of the moon about the earth which causes an apparent force, called centripetal force, attempting to toss the moon away from the earth. Orbiting bodies are therefore locked in a dance of balancing the gravitational forces and centripetal forces. Left to themselves, friction will slow the tangential velocity and the bodies will collide.

3 Force, Momentum and Conservation

Before diving into the physics, let us simplify our notations. This not only saves writing but introduces clarity and elegance. Consider and arbitrary function $f(t)$. Let

$$\frac{df(t)}{dt} \equiv \dot{f}(t) \tag{7}$$

In words, “The derivative of f with respect to t is defined to be ‘f dot.’” This is true also for vectors. Let $\mathbf{q}$ be some vector, then

$$\frac{d\mathbf{q}}{dt} \equiv \dot{\mathbf{q}} \tag{8}$$

We can continue this process; each dot means that we take another time derivative. Therefore, since acceleration is the second derivative of position with respect to time, we may write,

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = \ddot{\mathbf{x}} \tag{9}$$

We read $\ddot{\mathbf{x}}$ as “x double dot.” You can thank physicists for their linguistic perspicacity.

3.1 Linear Momentum

Let us definite a vector quantity called the *linear momentum*. It has units of $kg \cdot m \cdot s^{-1}$ and is defined as follows,

$$\mathbf{p} = m\mathbf{v} \tag{10}$$

where m is the mass of the object and $\mathbf{v}$ is its velocity. The adjective "linear" is to distinguish it from angular momentum, analogous to the way that $\mathbf{v}$ is a linear velocity. However, by convention, when we speak of momentum, we almost always mean linear momentum unless we specifically say that we mean angular momentum. Recall now that a force can be written as $\mathbf{F} = m\dot{\mathbf{v}}$ and therefore, making use of our new quantity, we have the succinct statement,

$$\mathbf{F} = \dot{\mathbf{p}} \tag{11}$$

This statement is undoubtably superior to that in Eq. (6) not only because of its brevity, but because Eq. (6) does not allow the mass to change. However, since $\mathbf{p} = m\mathbf{v}$, then by a direct application of the chain rule,

$$\dot{\mathbf{p}} = \frac{d}{dt}(m\mathbf{v}) = \dot{m}\mathbf{v} + m\dot{\mathbf{v}} \tag{12}$$

Alright, well, when does the mass change? Consider the motion of a rocket at takeoff. The rocket is burning a tremendous amount of fuel and therefore losing mass. This ejected mass is being accelerated towards the ground, and according to Newton, there must be an equal and opposite reaction, so there must be an acceleration on the rocket in the opposite direction of the ejected mass, to wit, straight up. Strictly speaking, Eq. (6) doesn't allow the rocket to ever lift off as it says that masses are constant and so its derivative, $\dot{m}$, is zero. Notice too that we never made the claim that Newton's Laws are terrestrial. We assume that they are equally as valid here as they are on the moon or in a different galaxy. Therefore if a rocket at rest in space turns on its boosters, it will accelerate in a direction opposite to that of the fuel mass ejection.

3.2 Angular Momentum

The angular momentum is not a relationship between mass and linear velocity as the linear momentum is, but rather a moment of inertia and an angular velocity, i.e, spinning things.

If something is rotating, it has some angular momentum. It can also be represented as a cross product of the radius vector of the object and its momentum. Let $\mathbf{r}$ be the radius vector or some object and $\mathbf{p}$ be the linear momentum of the same object, then the angular momentum, $\mathbf{L}$, is defined as follows,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{13}$$

The units are therefore $kg \cdot m^2 \cdot s^{-1}$. Since this quantity is a cross product, it necessarily exists in three at least three dimensions and is often more cumbersome. There is another useful definition of this quantity using the moment of inertia, I and angular velocity, ω

$$\mathbf{L} = I\boldsymbol{\omega} \tag{14}$$

The angular velocity is in units of s^{-1} and so the strange quantity we're calling the moment of inertia must have units of $kg \cdot m^2$. Therefore, loosely speaking, the angular momentum must have the following proportionality by dimensional analysis,

$$L \sim mr^2\omega \tag{15}$$

We will make use of this shortly. We can also connect $\mathbf{L}$ to other useful quantities by taking a time derivative.

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} \tag{16}$$

However, notice that

$$\dot{\mathbf{r}} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = m(\mathbf{v} \times \mathbf{v}) = 0 \tag{17}$$

Also, since $\dot{\mathbf{p}} = \mathbf{F}$, then we finally have

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} \tag{18}$$

There are a few key features to note: if the radius of an object is zero, then the angular momentum (and therefore the rate of change of the angular momentum) is identically zero. In other words, a "point particle" cannot have any angular momentum. As we will soon see, an enormous amount of calculations in classical mechanics treat objects as point particles even if they are not, and this dramatically simplifies the work. Second, if the net force of the system is zero, then the angular momentum cannot change in time. We therefore say that it is *conserved*.

4 Conservation Laws

The most useful quantities to keep track of in a physical system are almost always the conserved quantities. By conserved we simply mean that a given quantity in some system at some time has a particular value and if we measure that same quantity later on, we will get the same value. In other words, a conserved quantity doesn't change in time and so its time derivative is zero.

To elucidate this, consider a system in which the sum of all of the forces is exactly zero. So then,

$$\mathbf{F} = 0 = \dot{\mathbf{p}} \tag{19}$$

This means that the momentum must be a constant in time. Therefore a system with no net force is equivalent to the statement of conservation of momentum. If the momentum is constant, and since $\mathbf{p} = m\mathbf{v}$, then for an object with constant mass, the velocity must also be constant in time. Remember the twirling rock on the end of the string? We had determined that without a force present it would fly away in a straight line. Indeed, now we see that in the absence of a force, the velocity is necessarily constant which means that no change in direction can occur and our rock (or planet for that matter) would happily sail away in a straight line at a constant velocity forever and ever. In a similar fashion,

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times 0 = 0 \tag{20}$$

Thus the angular momentum must also be a constant and is therefore a conserved quantity. To see this in action consider the canonical figure skater spinning in circles with her arms out at some radius, r_s and with some angular velocity, ω_s. Her angular momentum is therefore what we determined in Eq. (15),

$$L_s \sim m r_s^2 \omega_s \tag{21}$$

But now, if I insist that there are no forces around (nobody is spinning her around and she's not pushing off of the ground), then angular momentum is conserved and L_s is a constant. So what happens when she pulls in her arms so that her radius is half of what it used to be? Well, we have

$$L_s \sim mr_s^2\omega_s = m(\frac{1}{2}r_s)^2\omega_{new} = \frac{1}{4}mr_s^2\omega_{new} \implies \omega_{new} = 4\omega_s \tag{22}$$

So her angular velocity must have increased by a factor of four. Our figure skater has sped up! A result we no doubt could have predicted ahead of time, but now we're building a framework to determine exactly how that works.

Before moving on, here's another interesting consequence of angular momentum conservation that I can't resist. The cross product of two vectors gives a third vectors which points perpendicularly to both. Imagine a vector $\mathbf{x}$ pointing in the x-direction and a vector $\mathbf{y}$ pointing in the y-direction. Then $\mathbf{x} \times \mathbf{y}$ is in the z-direction. We can imagine then that $\mathbf{x}$ and $\mathbf{y}$ form a plane, the x-y plane. Still with me? Now if angular momentum is conserved, then we can think of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, or if you prefer, $\mathbf{L} = I\omega$, as forming a plane which doesn't change in time. Thus a system with some angular momentum ($\omega \neq 0$) that is conserved will tend to form a plane. This is why your spinning rock on a string will fight gravity to try and form a flat plane and also why rotating galaxies tend to form discs.

5 Free Falling Objects

Now that we've had a chance to look at some of the mathematical machinery, let's put it to use. We'll start by considering free falling masses, that is, masses which are subject only to a gravitational force. Until we study moments of inertia, we will idealize all objects to be "point particles" where all of the mass is presumed to be at the center of the object. In this way, we may treat the object as having zero radius, and it therefore has no angular momentum.

Let us suppose that we have a falling object of mass m and the acceleration due to gravity is g (the numerical value is roughly $9.8\,m \cdot s^{-2}$ at sea level). Since force and acceleration are both vectors, we need to specify some coordinate system. Any system will do, however there are choices which certainly make our lives easier. In that spirit, let's call the vertical axis the $\hat{z}-$axis and let gravity point *down*, or in negative $\hat{z}$. Therefore we may write Newton's Second Law as

$$\mathbf{F} = mg(-\hat{z}) = -mg\hat{z} = m\ddot{\mathbf{r}} \tag{23}$$

where $\ddot{\mathbf{r}}$ is some arbitrary acceleration in our Cartesian coordinate system. It will be our goal to find $\mathbf{r}(t)$, for if we know the position of the object with respect to time, then we know everything. The most obvious simplification we can make at this point is canceling the masses.

$$-g\hat{z} = \ddot{\mathbf{r}} \tag{24}$$

Wait, what?! All this trouble about realizing that we don't know what mass is and it ends up canceling out anyway? This is an extremely strong statement. It says that *all* objects fall at the same rate (in a vacuum). A wrecking ball and a pebble dropped from the same height will hit the ground at the same time. I'm sure you can think of at least one notable figure that thought this was too bizarre to be true.

The next step is to do something with $\ddot{\mathbf{r}}$. Let's write out its components,

$$\ddot{\mathbf{r}} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z} = \ddot{\mathbf{x}} + \ddot{\mathbf{y}} + \ddot{\mathbf{z}} = (\ddot{x}, \ddot{y}, \ddot{z}) \tag{25}$$

The form you pick to write the vector is entirely up to. I will switch between them from time to time depending on the application so I've written them all out this time. Each vector component is orthogonal (perpendicular) to the other two. The $\hat{x}-$axis is surely perpendicular to the y and z axes, right? That means that whatever is happening in the $x-$direction has no influence whatever on whats happening in the y and z directions. Let's then write our equation again using this idea,

$$-g\hat{z} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z} \tag{26}$$

Since the components in x can only equal other components in x (same goes for y and z), we really have three equations,

$$\ddot{x}\,\hat{x} = 0 \tag{27}$$
$$\ddot{y}\,\hat{y} = 0 \tag{28}$$
$$\ddot{z}\,\hat{z} = -g\hat{z} \tag{29}$$

The object is not accelerating in x or y and it feels a downward acceleration of $-g$ in z. The next step is to solve the equation so that we end up with the vector $\mathbf{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$. To do this we simply integrate twice with respect to time. I'm going to suppress the time dependence and vector direction as not to clutter things up, so remember that x really means $x(t)\,\hat{x}$. Integrating the left side of (27) with respect to time twice just gives us $x(t)$,

$$\int dt \int dt\,\ddot{x} = x(t) \tag{30}$$

The first integral on the right hand side of (27) over 0 will give a constant, call it c_1 (The derivative of a constant is zero, right?). The second integration gives

$$\int dt\,c_1 = c_1 t + c_2 \tag{31}$$

And so we have the general equation for $x(t)$,

$$x(t) = c_1 t + c_2 \tag{32}$$

The second derivative is indeed zero, so things seem to be working out. The procedure for finding $y(t)$ is identical and so we can write down the answer right away,

$$y(t) = c_3 t + c_4 \tag{33}$$

Lastly, let's find $z(t)$. Integrating the left side or (29) is the same as the others,

$$\int dt \int dt\, \ddot{z} = z(t) \tag{34}$$

The right hand side is just as easy, so I'll simply write the answer for this one as well

$$z(t) = -\frac{1}{2} g t^2 + c_5 t + c_6 \tag{35}$$

In a sense, we're done. But there are quite a lot of arbitrary constants around. When this happens, it means that we have underspecified our system. In general, for each time derivative present, you need to specify a condition if you don't want have to have constants everywhere. The conditions are determined by how your system is to behave. Let us then make two requirements.

1) Let the object be dropped from a height h, $\implies \mathbf{r}(t=0) = h\,\hat{z}$

2) Let the object be dropped from rest, $\implies \dot{\mathbf{r}}(t=0) = 0$

Consider first Eq. (32) for $x(t)$. The first condition gives

$$x(0) = 0 = c_2 \tag{36}$$

Thus $c_2 = 0$. The second condition implies

$$\dot{x}(0) = 0 = c_1 \tag{37}$$

and so c_1 is also zero. Again, the conditions for determining $y(t)$ are identical. The results are

$$\begin{aligned} x(t) &= 0 \\ y(t) &= 0 \end{aligned} \tag{38}$$

Lastly we can do the z direction. Condition 1 sets $c_6 = h$. Condition 2 makes c_5 zero. At long last, we can collect our answer. Given a free falling object that starts from rest at a height h above the ground, the position at all times will be determined by

$$
\begin{aligned}
x(t) &= 0 \\
y(t) &= 0 \\
z(t) &= -\frac{1}{2} g t^2 + h
\end{aligned}
\tag{39}
$$

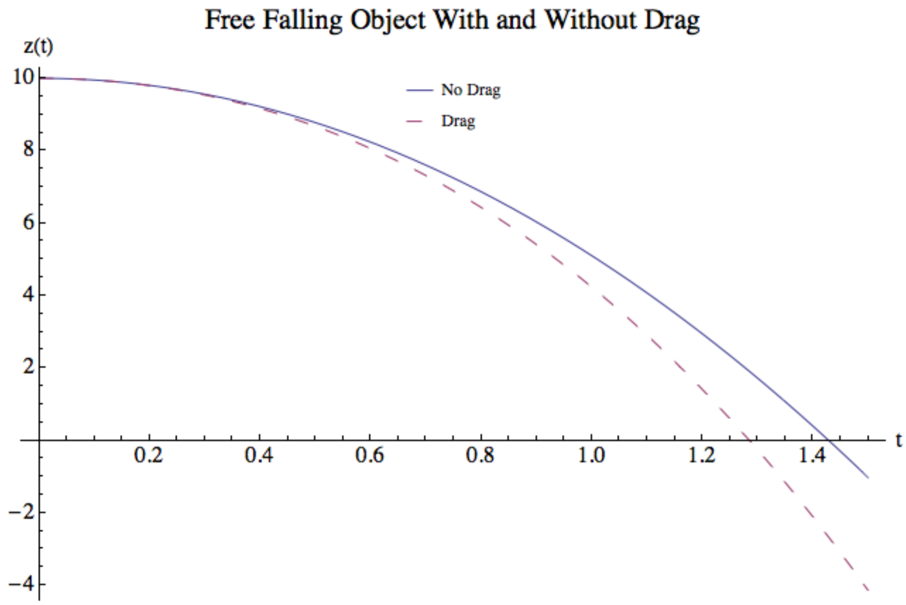
Or as a complete vector,

$$
\mathbf{r}(t) = (-\frac{1}{2} g t^2 + h)\,\hat{z} \tag{40}
$$

Now we're really done. The object is falling straight down, and with no other accelerations except g, the object never moves in the x and y directions. We are now in a position to ask the question, "How long will it take a ball that has been dropped from rest from a height h_0 to hit the ground?" We need to determine the time elapsed before the ball hits the ground and for that, we need to know when the ball hits the ground. Well, if $z(t)$ determines how high up we are, then when this equals zero, the ball must have zero height; it's on the ground. So,

$$
z(t_0) = 0 = -\frac{1}{2} g t_0^2 + h_0 \implies t_0 = \sqrt{\frac{2h_0}{g}} \tag{41}
$$

And there we have it, our first result! In a vacuum, this simple result is fantastically accurate. In the real world, it does quite well for short drops, under 10 meters or so. After that, the effects of drag become enormous and have to be taken into account. For a numerical result, consider having dropped the ball from 10 meters, then $t_0 = \sqrt{2 \cdot 10/9.8} \approx$ 1.43 seconds. Below is a plot of the ball's trajectory for the 10 meter drop with and without drag present.



6 Problems in 1D

In the last section we considered a free falling body and took pains to keep track of all of the vector components. Our task could have been made easier by realizing ahead of time that we only care about what is happening in one dimension. Also, based on the work above, there is nothing keeping us from using those same equations to describe a car on a race track or the flight of a bee or a frisbee etc. They are just models of a constantly accelerating object; it is up to us to interpret what the equations represent physically. In consideration of that, let's take a look at some other one dimensional problems.

6.1 Example: Car on a Track

Suppose that the acceleration of a car on a track is a, it's initial position is at x_0 and it's initial velocity is v_0. What is the position of the car as a function of time? As always, we begin with Newton's Law,

$$\ddot{x}(t) = a \tag{42}$$

Integrating twice is a simple matter. We get

$$x(t) = \frac{1}{2}at^2 + c_1 t + c_2 \tag{43}$$

Now just apply the initial conditions. If the initial position is zero, then $z(0) = x_0 = c_2$. This determines c_2. The second condition gives $\dot{z}(0) = v_0 = c_1$, and so the final equation is

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0 \tag{44}$$

And that's all there is to that.

6.2 Example: Tossing a Water Balloon

If you were on top of a 20 meter building and threw a water balloon vertically downward at 5m/s, how long would it take for the water balloon to hit the top of the head of your 2m tall friend down below?

Since the balloon is being thrown vertically downward, and the only force present is gravity which also points downward, then we may as well only worry about finding $z(t)$. Newton's Law tells us that $\ddot{z}(t) = -g$ and straightforward integration gives

$$z(t) = -\frac{1}{2}gt^2 + c_1 t + c_2 \tag{45}$$

Now we use our initial conditions. The ball was 20 meters high initially, so $z(0) = 20 = c_2$. We also know that initial velocity is 5m/s, therefore $\dot{z}(0) = 5m/s = c_1$. Our final solution is therefore

$$z(t) = -\frac{1}{2}gt^2 + 5t + 20 \tag{46}$$

We'd like to find the time, t_0 when the balloon hits the top of your friend's head 2 meters off of the ground. So we need to solve the following for t_0,

$$z(t_0) = 2 = -\frac{1}{2}gt_0^2 + 5t_0 + 20 \tag{47}$$

This is just a quadratic equation which can be easily solved. The answers are

$$t_0 = \frac{5 \pm \sqrt{25 + 36g}}{g} \tag{48}$$

Which sign do we pick? If we plug in $g = 9.8$ we find

$$t_0 = 2.49s \text{ or } -1.47s \tag{49}$$

In this context, picking a negative time (something that happened before you threw the ballon) doesn't make any sense. Therefore we go with the other choice and conclude that your friend gets a mini shower about 2.5 seconds after you threw the balloon.

Differential Equations

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Due to your level of understanding, we can advance our discussion to a more sophisticated level and begin to think about Newton's Laws and classical kinematics in their full glory. Physics is all about how quantities change with respect to other quantities. Therefore, differential equations are essentially what we consider when we think about a physical processes. Entire courses are dedicated to understanding problems of this sort, however that kind of detailed understanding isn't necessary in order to begin to use these new tools. If any step seems suspicious, ask me about it and we'll work through the proofs together.

Differential equations are just expressions that involve a function and its derivatives. The term which has the most derivatives determines the *order* of the equation. For what follows, consider an arbitrary function of one variable, $f(x)$, and constants $a_0, a_1, a_2 ... a_n$. The case of several variables is a little different and will be discussed later.

1 Zeroth Order Equations

A zeroth order differential equation in $f(x)$ has, at most, zero derivatives. So it will look something like

$$f(x) = a_0 \tag{1}$$

In this case, we don't have to do any work to solve for $f(x)$. We're already done: $f(x) = a_0$. Since we are using general expressions here, we conclude that the zeroth order term is always a trivial case; the function equals a constant (zero included). Onward to first order!

2 First Order Equations

A general first order equation will look like

$$\frac{df(x)}{dx} + a_1 f(x) = a_0 \tag{2}$$

This one is substantially more difficult to solve for $f(x)$. Let's first consider three special scenarios:

1. $a_0 = 0, a_1 = 0$
2. $a_0 \neq 0, a_1 = 0$
3. $a_0 = 0, a_1 \neq 0$

For **Case 1**, the differential equation reduces to

$$\frac{df(x)}{dx} = 0 \tag{3}$$

This says that the derivative of some function is zero, and so $f(x)$ is equal to a constant. To be explicit, we could integrate both sides of (3) with respect to x to get

$$\int \frac{df(x)}{dx}\,dx = \int df(x) = f(x) = \int 0\,dx = \text{constant} \tag{4}$$

And so we get the expected result that

$$f(x) = \text{constant} \tag{5}$$

For **Case 2** we have the more interesting relationship that the change in a function with respect to some variable is proportional to itself. Mathematically,

$$\frac{df(x)}{dx} + a_1 f(x) = 0 \tag{6}$$

or equivalently,

$$\frac{df(x)}{dx} = -a_1 f(x) \tag{7}$$

The trick now is to get an expression that looks like

$$df(x)\,(f(x) \text{ times stuff}) = dx\,(x\text{'s times other stuff}) \tag{8}$$

The reason for this is because now when we integrate both sides, the left side will give only answers in terms of $f(x)$ and the right will only give some combination of x's and thus we have an expression for $f(x)$. For this case, multiply both sides of (7) by dx and divide by $f(x)$ to get

$$\frac{df(x)}{f(x)} = -a_1 dx \tag{9}$$

Now we just integrate both sides

$$\int \frac{df(x)}{f(x)} = -a_1 \int dx \tag{10}$$

If the integral on the left hand side bothers you because it is a function and not just a variable, then we can make a substitution. Let $g = f(x)$ so that $dg = df(x)$. Therefore

$$\int \frac{df(x)}{f(x)} = \int \frac{dg}{g} \tag{11}$$

This is a standard integral which you already know, it gives a logarithm.

$$\int \frac{dg}{g} = \log[g] = \log[f(x)] \tag{12}$$

So we have

$$\int \frac{df(x)}{f(x)} = \log[f(x)] = -a_1 \int dx = -a_1 x + C \tag{13}$$

Where C is a constant of integration. Therefore we see that

$$\log[f(x)] = -a_1 x + C \tag{14}$$

We'd like to solve for $f(x)$, so we should exponentiate both sides to get

$$f(x) = e^C e^{-a_1 x} = f_0 e^{-a_1 x} \tag{15}$$

where $f_0 = e^C$. Since C is just a constant, then e^C is also a constant and there's no reason to carry around extra stuff, so let's just call it f_0.

So we've done it! We considered the first order differential equation

$$\frac{df(x)}{dx} + a_1 f(x) = 0 \tag{16}$$

and found that

$$f(x) = f_0 e^{-a_1 x} \tag{17}$$

Is this correct? We can check by simply taking a derivative and comparing to (7),

$$\frac{d}{dx} f(x) = \frac{d}{dx} f_0 e^{-a_1 x} = -a_1 f_0 e^{-a_1 x} = -a_1 f(x) \quad \longrightarrow \quad \frac{df(x)}{dx} = -a_1 f(x) \tag{18}$$

This is exactly what we wanted and therefore the solution we proposed is indeed the correct one. The constant f_0 would be set by the conditions of the problem. Although we don't know anything about this system, there is something to notice: as x gets huge, $f(x)$ goes to zero. In physics this is a good thing since 0 is a a nice, finite number. If the exponent had a positive sign, then as x gets really, really big, $f(x)$ would blow up too. As a general rule, if your answer to a physics problem is infinite, then something has gone wrong. More often than not, you'll see negative exponents in physics.

For **Case 3**, we have the differential equation

$$\frac{df(x)}{dx} = a_0 \tag{19}$$

Following formula (8) we are led to rewrite this as

$$df(x) = a_0 \, dx \tag{20}$$

Now we integrate both sides and get the simple result

$$f(x) = a_0 x + C \tag{21}$$

As always, the arbitrary constants will be set by the boundary conditions of the problem. Having warmed up with these, let's attack the full problem, Eq. (2),

$$\frac{df(x)}{dx} + a_1 f(x) = a_0 \tag{22}$$

There are two ways of approaching this: as a mathematician and as a hand-waving physicist. For the sake of completeness, we'll do both and you can decide on which method you prefer. Let us begin as mathematicians and put the equation in the form of Eq. (8).

$$\begin{aligned} &\frac{df(x)}{dx} + a_1 f(x) = a_0 \\ &\longrightarrow df(x) = -a_1 (f(x) - a_0/a_1) \, dx \end{aligned} \tag{23}$$

It's not necessary to get $f(x)$ alone by factoring out a_1; I like doing it so that I don't have to worry about losing a coefficient when I do the integration, but you're welcome to do it any way you wish, of course. We may write this as

$$\frac{df(x)}{f(x) - a_0/a_1} = -a_1\, dx \tag{24}$$

It looks just like our master recipe, Eq. (8), which means that we can integrate both sides and then reign victorious over yet another problem.

$$\int \frac{df(x)}{f(x) - a_0/a_1} = \log[f(x) - a_0/a_1] = -a_1 \int dx = -a_1 x + C \tag{25}$$

Exponentiating both sides and solving for $f(x)$ we get

$$\begin{aligned} f(x) &= e^C e^{-a_1 x} + a_0/a_1 \\ &= f_0 e^{-a_1 x} + a_0/a_1 \end{aligned} \tag{26}$$

This is the complete solution. As always, we should check to make sure that everything is indeed correct. Since the derivative is the same as in Eq. (18), we can just plug everything in,

$$\frac{df(x)}{dx} + a_1 f(x) = -a_1 f_0 e^{-a_1 x} + a_1 \left(f_0 e^{-a_1 x} + a_0/a_1 \right) = a_0 \tag{27}$$

where the first two terms have canceled. Everything checks out, so our solution is correct. You may have noticed that the first term of the complete solution, Eq. (26), looks just like Eq. (17). This is no coincidence. In fact, equations (17) and (26) have names: the *homogeneous* solution and the *particular* solution, respectively.

Let's let $f_p(x)$ be the particular solution and $f_h(x)$ be the homogeneous solution to Eq. (2). We can then write two differential equations

$$\frac{df_h(x)}{dx} + a_1 f_h(x) = 0 \tag{28}$$
$$\frac{df_p(x)}{dx} + a_1 f_p(x) = a_0 \tag{29}$$

To get the homogeneous differential equation, set every term in the full differential equation to zero if it doesn't have the function or its derivatives multiplying it. That's why

$a_0 = 0$ here; it doesn't have any f's multiplying it. All in all, this doesn't seem better, it seems worse. Believe it or not, it's a very useful step. The reason is that the homogeneous solution $f_h(x)$ is usually much easier to find than the particular solution. Now, suppose that $f_p(x) = f_h(x) + g$ where g is some unknown constant. Let's just plug this into (28) and see what happens.

$$\frac{d}{dx}(f_h(x) + g) + a_1(f_h(x) + g) = \left(\frac{df_h(x)}{dx} + a_1 f_h(x)\right) + \left(\frac{dg}{dx} + a_1 g\right) = a_0 \tag{30}$$

But by Eq. (27), the first term in the middle with $f_h(x)$ only is identically zero. Using the fact that the derivative of g is also zero, all that we have left is

$$a_1 g = a_0 \tag{31}$$

or

$$g = a_0/a_1 \tag{32}$$

This determines g, which means that the particular solution is

$$f_p(x) = f_h(x) + g = f_h(x) + a_0/a_1 \tag{33}$$

In doing this, we managed to show that once we have the homogeneous solution, the particular solution is essentially known. We can go one step further and really throw some tricks at this problem - we can *guess*. You may have gotten the sense by now that with first order differential equations, the answers is usually an exponent raised to some power as in (17) and/or a linear term with a constant added on as in (21). That is correct. Let's then guess the following form as a solution to Eq. (2)

$$f_{guess}(x) = Ae^{\omega x} + Bx + C \tag{34}$$

Now plug this into Eq. (2)

$$\begin{aligned}
\frac{d}{dx}f_{guess}(x) + a_1 f_{guess}(x) &= (\omega A e^{\omega x} + B) + a_1\left(Ae^{\omega x} + Bx + C\right) &(35)\\
&= Ae^{\omega x}(\omega + a_1) + Bx + (B + a_1 C) &(36)\\
&= a_0 &(37)
\end{aligned}$$

Line (36) is just a rearrangement of line (35). What does it take to make Eq. (37) = Eq. (36)? Well, Eq. (37) is just a constant with no x dependence, so anything with an x dependence has to go away somehow. This evidently means that $B = 0$. To get rid of the exponent, we require that $\omega + a_1 = 0$. We're then left with $a_1 C = a_0$. This determines all of the unknowns! To wit,

$$\begin{aligned} B &= 0 \\ \omega &= -a_1 \\ C &= a_0/a_1 \end{aligned} \tag{38}$$

Thus,

$$f_{guess}(x) = Ae^{-a_1 x} + a_0/a_1 \tag{39}$$

Just as we had expected from Eq. (26), only this time with considerably less effort. (Instead of f_0 we have A, but they're both 'unknown' until we have some initial conditions. If it bothers you, set $f_0 = A$). As hokey as this method my appear, it's one of the most common ways of doing it, particularly when it comes to second order equations.

3 Second Order Differential Equations

As we learned in the section above, the "order" of a differential equation tells us the term with the largest amount of derivatives. A second order differential equation therefore has a function with two derivatives at the most. The general *homogeneous* second order differential equation is therefore

$$\frac{d^2\phi(x)}{dx^2} + b_1 \frac{d\phi(x)}{dx} + b_0\,\phi(x) = 0 \tag{40}$$

I'm going to make a change of variables here for strictly aesthetic reasons. Let $x \to t$, $b_0 \to \omega_0^2$, and $\phi \to r$. Then, by making use of our dot-notation, Eq. (40) becomes

$$\boxed{\ddot{r}(t) + b_1 \dot{r}(t) + \omega_0^2\, r(t) = 0} \tag{41}$$

This is a *major* milestone! Equation (41) is one of the most important equations in all of physics. It represents a harmonic system that has damping (the $b_1 \dot{r}(t)$ term does the damping). The word "harmonic" just means that something is oscillating back and forth in a periodic way. Without damping, the thing will oscillate forever.

Imagine a perfect bouncing ball that can bounce back to the height at which it was dropped. This system can be represented by a harmonic oscillator. (In fact, *any* system which has a repetitive, regular motion can be modeled using harmonic oscillators). A real bouncing ball, however, loses some height after each bounce; this is the effect of the damping term. We can also imagine a pendulum that swings back and forth and is gradually slowing down. Again, this is nothing more than a dampened harmonic oscillator.

Not everything is dampened. The electromagnetic field is a product of oscillating or accelerating charges wiggling about. Weights hanging from springs are oscillators. Crystal lattices are imagined to be a bunch of weights connected by pretend springs, a bit like a mattress. The most accurate predictions in physics comes from imagining that each point in space is a tiny oscillator (that is, we have a *field* of oscillators) that can either be made to oscillate at particular frequencies, or just sit there and mind their own business. We identify the oscillators that are wiggling as having created a particle. You read correctly, particles are just resonances of a field. (The wiggling field has some energy, and Einstein said that energy and mass are the same thing. We can therefore assign a mass to the resonance). Have you ever heard of the "wave-particle duality?" This is what people are referring to. The study of how this works is called quantum field theory and it is the subject of much ongoing research.

Let us now begin to unravel Eq. (41). We'll warm up by treating special cases first. In that spirit, let $b_1 = 0$ so that we have

$$\boxed{\ddot{r}(t) = -\omega_0^2\, r(t)} \tag{42}$$

This is the "free" harmonic oscillator. It is staggeringly important. Commit this to memory. How can we solve this? Using a notation where a superscript in parenthesis denotes a time derivative, i.e,

$$r^{(2)}(t) = \ddot{r}(t) = \frac{d^2 r(t)}{dt^2} \tag{43}$$

then (42) can be written as

$$r^{(2)}(t) = -\omega_0^2\, r(t) \tag{44}$$

Let's take two more derivatives of both sides of this. We get

$$r^{(4)}(t) = -\omega_0^2\, r^{(2)}(t) = -\omega_0^2\, \left(-\omega_0^2\, r(t)\right) = \omega_0^4\, r(t) \tag{45}$$

where (44) was used to get rid of the $r^{(2)}(t)$ term. This means then that we can take four derivatives and the function still isn't zero. If we take two more derivatives we find

$$r^{(6)}(t) = -\omega_0^6\, r(t) \tag{46}$$

So the function doesn't vanish after taking six derivatives. In fact, we can continue this pattern to show that for $n = 0, 1, 2...$

$$r^{(2n)}(t) = (-\omega_0^2)^n\, r(t) \tag{47}$$

It doesn't matter how many derivatives we take; the function still exists. That is, we need to find a function that is infinitely differentiable. Fortunately, there aren't too many of them. The first function type that may come to mind is an exponent. In the spirit of the last section, let's venture a guess of the solution

$$r_{guess}(t) = \alpha e^{\omega t} \tag{48}$$

Plugging this into Eq. (42) we discover

$$\ddot{r}_{guess}(t) = \omega^2 \alpha e^{\omega t} = -\omega_0^2 \alpha e^{\omega t} \tag{49}$$

For this to be true, we require

$$\omega^2 = -\omega_0^2 \;\longrightarrow\; \omega = \sqrt{-\omega_0^2} = \pm\omega_0\sqrt{-1} = \pm i\omega_0 \tag{50}$$

To get to the last step, the identity $\sqrt{-1} = i$ was used. This value, i, is known as a *complex* or *imaginary* number. The specific properties aren't important right now. All you need to know is that $i^2 = -1$. Inserting Eq. (50) into Eq. (48) gives

$$r_{guess}(t) = \alpha e^{\pm i\omega_0 t} \tag{51}$$

Believe it or not, this result changed the world. The coefficient α has a special name, we call it the *amplitude*. Since $0 \le e^{i\omega_0 t} \le 1$, then α will determine the "size" of $r(t)$. In terms of a wave, the height of the wave is called the amplitude. This quantity is very useful. The square of the amplitude is proportional to the energy in the wave. And in terms of quantum mechanics, the amplitude is a "probability amplitude" wherein the square of it tells us the probability that a particle will be found in a particular state. In some sense, all that quantum mechanics amounts to is finding these amplitudes.

Notice also that the argument of an exponent (and log, and sines and cosines etc.) must be dimensionless. That is, if we have e^{stuff} or cos(stuff)... the 'stuff' has to be dimensionless. Since the exponent in Eq. (51) is proportional to $\omega_0\, t$, then ω_0 must have units of s^{-1}. This quantity is called the *frequency*. It tells us how quickly something is oscillating back and forth. Remember the comment about wanting negative exponents? It doesn't matter here since the function is only as big as its amplitude. A large exponent just means that the object in question oscillates faster, i.e, ω_0 is large. There's one last point to be made: keep in mind the identities

$$e^{\pm 2n\pi i} = 1 \;\; (n = 0, 1, 2...) \quad \text{and} \quad e^a e^b = e^{a+b} \tag{52}$$

Multiplying Eq. (51) by the identity for 1 allows us to write

$$r_{guess}(t) = \alpha e^{\pm i\omega_0 t} e^{\pm 2n\pi i} = \alpha e^{\pm i\omega_0 (t + 2\pi n/\omega_0)} \tag{53}$$

It is as though we changed $t \to t + 2\pi n/\omega_0$. Realizing that $2\pi n/\omega_0$ is just some number, we can guess that if we started at time t_0 and let the time run long enough, eventually it would reach $t_0 + 2\pi n/\omega_0$ anyway, so adding it on doesn't change anything. It does however let us know that just because the exponent is growing, that doesn't mean that anything is oscillating faster since every time t advances by $2\pi/\omega_0$, it just puts us back where we started. To get a physical representation of this, picture the amplitude as the second hand on a watch face. Once the second hand has rotated in a full circle (2π radians), it is back where it started. Once the second hand has gone around three times, it has rotated $2\pi * 3$ radians, but is still back where it began. After a very long time, the second hand has spun through an enormous angle, however it's still only moving in a circle with the same frequency it had when it started.

Ok, but what about the $\pm$ sign in Eq. (51)? In order to take advantage of both possible solutions (positive and negative exponents), we may write this as

$$r_{guess}(t) = \alpha_+ e^{i\omega_0 t} + \alpha_- e^{-i\omega_0 t} \tag{54}$$

If we only want solutions with a positive (negative) exponent then we simply set α_- (α_+) to zero. Suppose we have the interesting case where $\alpha_+ = \alpha_- = A/2$. Then,

$$r_{guess}(t) = A\,\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} = A\cos(\omega_0 t) \tag{55}$$

Similarly, if $\alpha_+ = -\alpha_- = B/2i$ then we get

$$r_{guess}(t) = B\,\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} = B\sin(\omega_0 t) \tag{56}$$

In general then, we have two ways of writing the solution: like Eq. (54) (and dropping the "guess" subscript for brevity)

$$r(t) = \alpha_+ e^{i\omega_0 t} + \alpha_- e^{-i\omega_0 t} \tag{57}$$

Or by making use of the Euler identities, we are free to write the following instead

$$r(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{58}$$

The forms are equivalent.

Exercise 1

Prove that Eq. (57) and Eq. (58) are the same by expressing A and B in terms of α_+ and α_- and vice versa.

Though they are the same, often times we prefer to write (58) instead. The reason is due to the initial conditions. Suppose that we are considering a particle oscillating back and forth in an electromagnetic field, and at time $t = 0$, the particle has an amplitude r_0 and a velocity of 0. The first condition gives

$$r(t=0) = r_0 = A\cos(0) + B\sin(0) = A \;\longrightarrow\; A = r_0 \tag{59}$$

The second condition tells us that the velocity (or derivative) of $r(t)$ at $t = 0$ is 0. This means that the system is initially not moving. This condition implies

$$\dot{r}(t=0) = 0 = -\omega_0 A\sin(0) + \omega_0 B\cos(0) = \omega_0 B \;\longrightarrow\; B = 0 \tag{60}$$

Putting it together we get the nice result

$$r(t) = r_0\cos(\omega_0 t) \tag{61}$$

It is therefore rather painless to use the trig functions because we are familiar with them and often know some special values off of the top of our head, like where they are zero or one.

Exercise 2

Prove that by starting with Eq. (54) and applying the same conditions as above, you get the same answer, Eq. (61).

As a quick recap before moving on, we have found the following solutions to this (highly important) differential equation,

$$\boxed{\ddot{r}(t) = -\omega_0^2\, r(t) \;\longrightarrow\; r(t) = \alpha_+ e^{i\omega_0 t} + \alpha_- e^{-i\omega_0 t} = A\cos(\omega_0 t) + B\sin(\omega_0 t)} \tag{62}$$

4 Projectiles with Drag

Drag, or more colloquially, air resistance, typically manifests itself in two ways: either the drag force is proportional to the velocity, or proportional to the square of the velocity. We consider first a drag force proportional to the velocity and apply these conditions to an object of mass m shot vertically upwards with a velocity v_0 and from a height h. The force due to drag is

$$F_d = \lambda v(t) \tag{63}$$

where λ is the unknown proportionality factor. Newton's Law for a particle moving upwards with drag is

$$m\ddot{y}(t) = -mg - \lambda v(t) \tag{64}$$

Gravity points down, as always. The drag, like friction, will always oppose the direction of motion. If the particle is falling, drag will try to push it back up. If we shoot the particle upwards, drag will be trying to push it back down. Since the particle is moving upwards first, we let λ be negative. We can clean things up by dividing by m first,

$$\ddot{y}(t) = -g - \frac{\lambda}{m} v(t) \tag{65}$$

Now, $\dot{y}(t) = v(t)$ so we *could* write

$$\ddot{y}(t) = -g - \frac{\lambda}{m} \dot{y}(t) \tag{66}$$

We haven't solved anything like this yet, and it looks pretty awful. We can choose a better substitution though if we recall that $\ddot{y}(t) = a(t) = \dot{v}(t)$ and use this in Eq. (65). Doing so gives

$$\dot{v}(t) = -g - \frac{\lambda}{m} v(t) \tag{67}$$

or equivalently,

$$\dot{v}(t) + \frac{\lambda}{m} v(t) = -g \tag{68}$$

Behold! This is exactly the form of Eq. (22) if we just make the following simple changes

$$\begin{aligned} f &\to v \\ x &\to t \\ a_1 &\to \lambda/m \\ a_0 &\to -g \end{aligned} \tag{69}$$

The general solution is given by Eq. (26). Making our necessary substitutions we immediately determine

$$v(t) = f_0 e^{-\lambda t/m} - \frac{mg}{\lambda} \tag{70}$$

We don't know what f_0 actually is yet, but it may be determined from the initial conditions. The object was launched with velocity v_0. This implies

$$v(0) = v_0 = f_0 - \frac{mg}{\lambda} \longrightarrow f_0 = v_0 + \frac{mg}{\lambda} \tag{71}$$

And so we have

$$\boxed{v(t) = \left(v_0 + \frac{mg}{\lambda}\right) e^{-\lambda t/m} - \frac{mg}{\lambda}} \tag{72}$$

Looking at the exponent, we see that λ must have units of kg$\cdot$ s^{-1}. This means that the product mg/λ has units of velocity, as indeed it must. What is this velocity? Well at $t = 0$ we have $v(0) = v_0$ (since $e^0 = 1$), but as $t \to \infty$ the exponent goes to zero and so we get

$$v(t \to \infty) = -\frac{mg}{\lambda} \tag{73}$$

Our falling object keeps picking up speed but eventually plateaus at this constant velocity. This velocity is known as the *terminal velocity*, often abbreviated v_T or v_∞. Unlike the free fall problems that we're used to, now the mass plays a role. Galileo was right to be skeptical - if a drag force proportional to the velocity is present, then doubling the mass will double the terminal velocity of a freely falling object.

Exercise 3

Assuming that the height of the object is h at $t = 0$, obtain the position as a function of time.

Exercise 4 (hard)

Suppose that the drag force were instead proportional to the square of the velocity. What would be the terminal velocity in this case? You may find the following relations useful:

$$\cosh(x)^2 - \sinh(x)^2 = 1, \qquad \frac{1}{x^2 - a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

Hint: You won't need both relations, just one or the other

5 Closing Words

A lot has been said here. Entire books are written on the topic, and so the material here is far from complete. It does however give you a good idea of how these equations work from a mathematical perspective. Now when you see a statement such as

$$\mathbf{E}(t) = E_0 \cos(\omega t)\hat{z} \tag{74}$$

where $\mathbf{E}$ is the electric field, you know that the amplitude of the field is E_0 and that the field oscillates at a frequency ω and moves in the direction $\hat{z}$. Similarly if we replace $\omega t \to kx$ so that

$$\mathbf{E}(t) = E_0 \cos(kx)\hat{z} \tag{75}$$

then, since x is a length, k must be an inverse length, i.e, a wavelength. Therefore this oscillating electric field has an amplitude E_0 and a wavelength k. These could even be written together. In terms of exponents, we could expect to find

$$\mathbf{E}(t) = E_0 e^{i(kx+\omega t)}\hat{z} \tag{76}$$

We know this field's frequency, wavelength, amplitude and direction! That's quite a lot packed into such a concise statement. But there's nothing too complicated about it. If $x = 0$ and $t \neq 0$, the field just oscillates in time. If $t = 0$ and $x \neq 0$, then the field oscillates in space. If neither are zero, then the field oscillates in space and time. How do we know it oscillates? Because it satisfies Eq. (42), and any function that satisfies a differential equation of this form is harmonic, i.e, it oscillates.

Exercise 5

Is it possible to pick coefficients for the following function which would make it harmonic? Why or why not?

$$f(x) = \sum_{j=0}^{n} a_j x^j = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots a_n x^n$$

Ordinary Differential Equations Homework Supplement

February 3, 2016

Solve the following problems. If an initial condition is given, be sure to use it in order to determine any unknowns. Assume for all problems that a prime denotes a derivative with respect to x. That is,

$$f'(x) = \frac{df(x)}{dx}$$
$$f''(x) = \frac{d^2 f(x)}{dx}$$
$$\vdots$$
$$f^n(x) = \frac{d^n f(x)}{dx}$$

Problem 1

Solve the following for $y(x)$:

$$xy'(x) + y(x) = 0 \tag{1}$$

with the initial condition that $y(1) = 3$.

You may use the identity

$$a \log(x) = \log(x^a)$$

In particular, for $a = -1$,

$$(-1)\log(x) = \log\left(x^{-1}\right) = \log\left(\frac{1}{x}\right)$$

Problem 2

Solve the following for $y(x)$:

$$y'(x) + \frac{1}{2}y(x) = \frac{3}{2} \tag{2}$$

with the initial condition

$$y(x=0) = 0$$

Problem 3

Electrical circuits are also modeled by differential equations. There are, of course, many different types of components that we may hook into our circuits, but there are some circuit types that are extremely common and are worth looking at. First, some basics:

An **electric current**, I (measured in amps after Ampère, abbreviated as A), is produced by flowing electric charges. In fact, if the charge is called q, then $I = \dot{q}$. Just having one charge does *not* constitute a current (Can you guess why that may be?).

A **voltage**, V or $\mathcal{E}$ (measured in volts, abbreviated as V), makes an uneven distribution of charge. A battery, for example, has positive charges on one side and negative charges on the other. This uneven distribution of charge is called a **potential difference** since it is a difference in electric potential energy. If we connect a wire across the terminals of the battery, the charges will flow in an attempt to equalize each other. Once this equalization is met, the potential difference is zero and your battery is dead.

The **resistance**, R (measured in Ohms, abbreviated as Ω), is essentially a measure of how hard it is for the charges to flow around the circuit. A high resistance means that the charges have a hard time flowing around. Instead of a nice flow around the wire, they chaotically bump into each other and lose their kinetic energy (which they got from the potential difference) in these collisions. The energy is lost as heat. So if you want to heat something up, put a large resistor your circuit. A toaster is basically one big resistor. Neat, huh? :) If you don't want to make any heat, make the resistance as low as possible. As the resistance approaches zero, the material becomes a superconductor. CERN uses superconductors so that they can have tremendously high currents and very little losses due to heat. Unfortunately, back in 2007, one superconductor suddenly became resistive and due to the massive rate at which charges were flowing, got extremely hot extremely fast causing the liquid helium to vaporize and expand rapidly. This is essentially how bombs work. Needless to say, it was a major setback!

A **capacitor** stores a potential difference, kind of like a voltage in a small battery, except that the stored potential can be released almost instantaneously. For this reason, giant banks of capacitors are used for certain kinds of weapons, like pulsed lasers and railguns (It's also why you have to be careful with them!). Once a capacitor in a circuit reaches an equilibrium with the voltage source, no more current can pass through the capacitor or anything connected in the circuit with it, and so the circuit is effectively "off" (analogous to the dead battery). This can only happen with a DC (direct current) circuit since there is an equilibrium voltage to be reached. An AC (alternating current) source constantly reverses its direction so the capacitor is never in equilibrium. Thus capacitors can be used to block DC current altogether. The shuffling about of charges in the capacitor is much slower than in the wire itself, so large variances from the voltage source are evened out if they have to go through a capacitor. This is why you will often find capacitors in power sources - you get a nice smooth output voltage.

Lastly, an **inductor**, L (measured in henries, abbreviated as H) is the electrical analog of inertia. Imagine that the current in a circuit is initially off. When we turn the current on, the inductor will do everything it can to keep it off. It will lose, of course, but it makes things turn on nice and slowly. In the reverse case, if a current is flowing and we suddenly cut the power, the inductor will try desperately to keep the current going. This is the reason that you sometimes see sparks when you unplug something from the wall - the inductance wants the current to continue so badly that it tries to connect the circuit through the air! Due to other side effects however, inductors are becoming less and less common.

One final comment on circuits (for now): Voltages aren't really that dangerous. However, currents most definitely are. An important equation relating current, voltage and resistance is

$$V = IR$$

When you get zapped by a door knob, you experience roughly 25,000 volts. But the voltage from your outlet is only about 120 volts, so what's the difference? Since $V/R = I$, then if R is massive (about $10^{16}\,\Omega$ for air), the current is pretty much zero and nothing much happens. In your wall, the current flows down copper wires which have a resistance of about $10^{-8}\,\Omega$. That's 24 orders of magnitude less than in air! There are resistors along the way to prevent shorting out, and so most houses in the U.S have 120 volts and 15 amps running to them. Putting a fork in the outlet means that you feel almost all of that 15 amps. About 0.2 amps is considered lethal. So next time you see a "Danger: High Voltage" sign, you can laugh at the idiocy of that statement. The sign really ought to read, "Danger: High Current."

Imagine a circuit with a voltage source held at $\mathcal{E}_0$, a resistor R, a current $I(t)$, and an inductor L. This is called an *LR circuit.* It may be described by

$$L\dot{I}(t) + RI(t) = \mathcal{E}_0 \tag{3}$$

If the current is zero at $t = 0$, find $I(t)$. What happens as $t \to \infty$? What quantity (other than t) has units of time?

Problem 4

Suppose that we have a circuit with a voltage source held at $\mathcal{E}_0$, a resistor R, a capacitor C and a charge $q(t)$. This is called an *RC circuit*. It may be described by

$$R\dot{q}(t) + \frac{1}{C}q(t) = \mathcal{E}_0 \tag{4}$$

If the charge is zero at $t = 0$, what is the charge as a function of time? What quantity (other than t) has units of time?

Problem 5

Suppose that we have a circuit with no voltage, a capacitor C, inductor L, and a charge $q(t)$. This is called an *LC circuit*. It may be described by

$$L\ddot{q}(t) + \frac{1}{C}q(t) = 0 \tag{5}$$

If the charge is q_0 at $t = 0$ and the current ($\dot{q}$) is zero at $t = 0$, what is the charge as a function of time? What quantity (other than t) has units of time?

Physics Homework 1

January 23, 2016

1D Motion

For all problems which require taking the acceleration due to gravity into account, use $g = 9.8$ m/s^2.

Problem 1. Suppose a rock is dropped from rest from a height of 20 meters. How long will it take the rock to hit the ground?

Problem 2. Now suppose that instead of being simply dropped, the rock is thrown vertically downward at a velocity of 25 m/s. How long will it take the rock to hit the ground this time?

Problem 3. A car is driving in a straight line at an initial velocity of 45 m/s. Suddenly (at $t = 0$ and $x = 0$) the driver starts accelerating at a rate of 10 m/s^2. When the car is finally traveling at 55 m/s, how far has the car traveled since it began accelerating?

Problem 4. Another driver is heading down a straight track at only 15 m/s. If this driver also wants to reach 55 m/s and accelerates at the constant rate of 10 m/s^2, what is the minimum length of track necessary such that the car never drives off of the track before reaching this velocity?

Problem 5. Explain briefly whether or not the mass of an object alters the rate at which it falls according to Newton's Laws. In the real world (that is, considering drag, friction, air density etc.) do you think that the rate at which objects fall is independent or dependent on the mass. Can the shape of the object change its acceleration? Qualitatively defend your answer.

Problem 6. Suppose you were to fire a bullet one meter off of the ground. There is a speed at which the earth curves away from the falling bullet such that the bullet never touches the ground and is therefore essentially in orbit. We will do this problem explicitly next time, but try to list the quantities that will be relevant for this calculation (i.e, velocities, masses etc.)

2D Motion

A rocket is launched at an angle of 30 degrees relative to the horizon and at an initial velocity of 100 m/s. The only force acting on the rocket is gravity. Use $g = 9.8$ m/s^2.

Problem 7. What is the maximum height attained by the rocket?

Problem 8. How long does it take the rocket to reach this maximum height?

Problem 9. How far from the launch point will the rocket land?

Problem 10. Suppose the rocket were launched vertically, what is the maximum height now?

Fun Fact: The *gradient* of a function, say $f(x, y, z)$, is defined as follows

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,\hat{x} + \frac{\partial f(x,y,z)}{\partial y}\,\hat{y} + \frac{\partial f(x,y,z)}{\partial z}\,\hat{z}$$

This operation gives us a vector which tells us the direction in which the function is changing the fastest. (The function $f(x, y, z)$ is *not* a vector itself. However, the gradient returns a value for each direction, and therefore the gradient of $f(x, y, z)$ *is* a vector). Suppose for example there was a fire in the the center of a cold room. The gradient of the temperature in the room would always point towards the fire since the temperature changes quickest as we approach the fire directly. This is how heat-seeking missiles work. A program in the missile constantly calculates the temperature gradient and changes course so that it is always heading towards the hottest object nearby. This is also why you may have heard of countermeasure flares on planes. The flares put out bigger infrared signals than the planes do, and so any surface-to-air heat-seeking missiles would be steered off course.

Multivariable Calculus, Energy Conservation and Lagrangians

February 26, 2016

1 Multivariable Calculus

This section will introduce the idea of partial derivatives. The ideas behind them introduce no "new" mathematics, they only expand the rules of ordinary derivatives by a little bit.

1.1 Partial Derivatives

Suppose that we have a function of just one variable called $f(x)$. Finding the derivative of this function is a simple matter. Formally,

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

In general, we rarely write the part on the right hand side even though it is the part of the definition that actually tells us how to compute the derivative. Nevertheless, our job of finding the rate of change of the function is essentially done. But suppose now that instead of $f(x)$, we have $g(x, y, z)$. What is the rate of change of this function?

As it turns out, there are a few different answers depending on what exactly we mean by how $g(x, y, z)$ changes. It's easiest to understand by example. Let us therefore consider the following definition for g,

$$g(x, y, z) = 2x^2 y + xy\cos(z) + x^3 z^2 \tag{2}$$

There are no vector components here, so g is just some scalar function, albeit a horrible one. It is now asked how this function changes with respect to x. You may be tempted to write

$$\frac{d}{dx} g(x, y, z) \tag{3}$$

and begin computing derivatives while treating y and z as ignorable constants. You would be correct. However, when we take a derivative of a function with several variable, we use a Greek d instead of a regular Latin one as to explicitly imply the multivariate nature of the function. Therefore, we should really write

$$\frac{\partial}{\partial x} g(x, y, z) \tag{4}$$

instead. The actual mechanics of how the derivative works is completely unchanged. When we take the derivative of a function with several variables, we are taking *partial derivatives*. The partial derivative of Eq. (2) with respect to x is

$$\frac{\partial}{\partial x} g(x, y, z) = \frac{\partial}{\partial x}\left(2x^2 y + xy\cos(z) + x^3 z^2\right) = 4xy + y\cos(z) + 3x^2 z^2 \tag{5}$$

The derivative of g with respect to y would be

$$\frac{\partial}{\partial y} g(x, y, z) = \frac{\partial}{\partial y}\left(2x^2 y + xy\cos(z) + x^3 z^2\right) = 2x^2 + x\cos(z) \tag{6}$$

And finally, the derivative of g with respect to z is

$$\frac{\partial}{\partial z} g(x, y, z) = \frac{\partial}{\partial z}\left(2x^2 y + xy\cos(z) + x^3 z^2\right) = -xy\sin(z) + 2x^3 z \tag{7}$$

There are no surprises here so far. There are some special kinds of partial derivatives, however, and they have special names: the divergence (abbreviated "div"), the gradient (abbreviated "grad"), and the curl. We need not overly concern ourselves with these, but you should get familiar with them. Just to show you how they work and that they're not much different from what you already know, I'll show you a few quick examples.

1.1.1 The Gradient

Consider first the gradient. It is a derivative which acts on a scalar but returns a vector. Basically, the gradient finds the rate of change of the function in each direction. For $g(x, y, z)$, the gradient finds the rate of change of x in the x direction plus the rate of change of y in the y direction plus the rate of change of z in the z direction. The gradient has the following structure

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) g(x, y, z) \equiv \vec{\nabla} g(x, y, z) \tag{8}$$

The ∇ symbol is called "del" and the arrow above it is to remind you that it returns a vector. However, almost always, the arrow is absent and it is simply assumed that you understand that the gradient returns a vector. I too will drop the arrow. The hats just come along for the ride; they don't actually *do* anything. Let's again take $g(x, y, z)$ as defined in Eq. (2) and calculate the gradient. We already know what the partial derivatives are from Eq. (5) - (7). Therefore we can use these and just add in the direction for each component. Thus,

$$
\begin{aligned}
\nabla g(x,y,z) &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(2x^2y + xy\cos(z) + x^3z^2\right) \\
&= \hat{x}\left(4xy + y\cos(z) + 3x^2z^2\right) \\
&+ \hat{y}\left(2x^2 + x\cos(z)\right) \\
&+ \hat{z}\left(-xy\sin(z) + 2x^3z\right) \qquad (9)
\end{aligned}
$$

As a somewhat simpler example, take the following function of two variables

$$\psi(x,y) = \frac{1}{2}y^2\sin(kx) \qquad (10)$$

Then, the gradient is just

$$
\begin{aligned}
\nabla\psi(x,y) &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\left(\frac{1}{2}y^2\sin(kx)\right) \\
&= \hat{x}\left(\frac{k}{2}y^2\cos(kx)\right) + \hat{y}\Big(y\sin(kx)\Big) \qquad (11)
\end{aligned}
$$

That's it for the gradient.

1.1.2 The Divergence

The divergence (a.k.a "dot product") acts on a vector and returns a scalar. It tells us how much of the function is exiting some area or coming into some area. People often use the terms "sources" and "sinks" when they talk about the divergence. Imagine a bucket full of water. If we poke a hole in it so that more water is flowing out than is flowing in, we call this a "sink." If our bucket is filling up, we then say that we have a "source." Mathematically, if the divergence is negative (positive) we have a sink (source). If the divergence is zero, that means that just as much water is flowing in as is flowing out. Let's use the following multivariable vector function

$$\mathbf{A}(x,y,z) = x^2y^3\,\hat{x} - \frac{1}{2}xy^4\,\hat{y} + ze^{-\gamma x}\,\hat{z} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \qquad (12)$$

where I defined: $x^2y^3 \equiv A_x$, $-\frac{1}{2}xy^4 \equiv A_y$, and $ze^{-\gamma x} \equiv A_z$ for brevity (an equals sign with three lines means "defined as"). The action of the divergence is as follows

$$\nabla\cdot\mathbf{A}(x,y,z) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\mathbf{A}(x,y,z) \qquad (13)$$

But now, recall something very important about the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$: They are all perpendicular to each other.[1] Therefore,

$$\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0 \tag{15}$$

and

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \tag{16}$$

This means that all of the terms in Eq. (13) that are of the form

$$\hat{i}\frac{\partial}{\partial i} \cdot A_j \hat{j} = \left(\hat{i} \cdot \hat{j}\right) \frac{\partial}{\partial i} A_j = 0 \tag{17}$$

We are then left with the simple result

$$\nabla \cdot \mathbf{A}(x, y, z) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \mathbf{A}(x, y, z) = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \tag{18}$$

Let us now compute the divergence using the definition of $\mathbf{A}$ in Eq. (12). Taking one derivative at a time we find

$$\begin{aligned}
\frac{\partial}{\partial x} A_x &= \frac{\partial}{\partial x}\left(x^2 y^3\right) = 2xy^3 \\
\frac{\partial}{\partial y} A_y &= \frac{\partial}{\partial y}\left(-\frac{1}{2} x y^4\right) = -2xy^3 \\
\frac{\partial}{\partial z} A_y &= \frac{\partial}{\partial z}\left(z e^{-\gamma x}\right) = e^{-\gamma x}
\end{aligned} \tag{19}$$

Putting it all together,

$$\nabla \cdot \mathbf{A}(x, y, z) = 2xy^3 + (-2xy^3) + e^{-\gamma x} = e^{-\gamma x} \tag{20}$$

For a final example of the divergence, consider

$$\mathbf{E}(x, y, t) = E_0 e^{-i\omega t}\left(\sin(k_1 x)\hat{x} - \cos(k_2 y)\hat{y}\right) \tag{21}$$

Then, simply doing the defined operation we find

[1]This is true in general. For two arbitrary unit vectors $\hat{i}$ and $\hat{j}$ we have

$$\hat{i} \cdot \hat{j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{14}$$

$$
\begin{aligned}
\nabla \cdot \mathbf{E}(x,y,t) &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \cdot E_0 e^{-i\omega t}\bigg(\sin(k_1 x)\hat{x} - \cos(k_2 y)\hat{y}\bigg) \\
&= E_0 e^{-i\omega t}\left(\frac{\partial}{\partial x}\sin(k_1 x) - \frac{\partial}{\partial y}\cos(k_2 y)\right) \\
&= E_0 e^{-i\omega t}\left(k_1\cos(k_1 x) + k_2\sin(k_2 y)\right)
\end{aligned} \tag{22}
$$

1.1.3 The Curl

Lastly, we come to the curl. This derivative operation acts on a vector and returns another vector. It tells us how much something is being turned. Imagine a stick floating in a river. If the water is pushing on the stick harder on one side than the other, the stick will rotate. We would say then that the curl of the water is nonzero. Sometimes people refer to a field which has zero curl as "irrotational."

The operation is a pretty horrible one, especially in other coordinate systems. I'll write out the general form, but we're going to skip it for now. We can come back to it later if there's time. Given some multivariable vector function $\mathbf{B}(x,y,z)$ defined only as

$$
\mathbf{B}(x,y,z) = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \tag{23}
$$

then the curl is

$$
\nabla \times \mathbf{B}(x,y,z) = \left(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y\right)\hat{x} - \left(\frac{\partial}{\partial x}B_z - \frac{\partial}{\partial z}B_x\right)\hat{y} + \left(\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x\right)\hat{z} \tag{24}
$$

1.2 Final Notes on Partial Derivatives

Partial derivatives are often abbreviated as to make for less writing. I will usually write ∂_t instead of $\frac{\partial}{\partial t}$. Then, for example, instead of writing out Eq. (13) every time, we could write

$$
\nabla \cdot \mathbf{A}(x,y,z) = (\hat{x}\,\partial_x + \hat{y}\,\partial_y + \hat{z}\,\partial_z) \cdot \mathbf{A}(x,y,z) \tag{25}
$$

If we had a differential equation that depended on more than one variable, we'd have to use partial derivatives. Such an equation is then called a *partial differential equation* and they are what physics is all about. Except for a few axiomatic statements and some derivations in statistical mechanics, just about everything else in the physical universe is expressed as a differential equation. You can even consider ordinary differential equations (using regular derivatives and functions of only one variable) as just a special case of a partial differential equation.

Remember when you first looked at an ordinary derivative; you concerned yourself a great deal with the infinitesimal increment of change. That is, if your function was $f(x)$, you were interested in knowing $df(x)$. Take $f(x) = x^2$ for example. Then

$$df(x) = d(x^2) = 2x\,dx \tag{26}$$

Then dividing by dx on each side we get the expected result

$$\frac{df(x)}{dx} = 2x \tag{27}$$

But what about when we have a function of several variables, say, $g(x, y, z)$? In this case, we write $\delta g(x, y, z)$ instead of $\partial g(x, y, z)$ like you may have expected, but $\delta g/\delta x$ means the same thing as $\partial g/\partial x$. We're sort of just stuck with an odd historical convention. What then is δg? The answer is

$$\delta g(x, y, z) = \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z}\right) g(x, y, z) \tag{28}$$

This will be very important later on. Consider if δy and δz were zero. Then we'd just have

$$\delta g(x, y, z) = \delta x \frac{\partial}{\partial x} g(x, y, z) \longrightarrow \frac{\delta g}{\delta x} = \frac{\partial g}{\partial x} \tag{29}$$

as mentioned above. If we reduce $g(x, y, z)$ to only one variable, i.e, $g(x, y, z) \to g(x)$, then

$$\delta g(x) = \delta x \frac{\partial}{\partial x} g(x) = dg(x) = dx\,\frac{dg(x)}{dx} \longrightarrow \frac{dg(x)}{dx} = \frac{dg(x)}{dx} \tag{30}$$

which is obviously true. So there isn't anything too bizarre going on here. I'm not going to derive Eq. (30). We will take it as a given and simply use it to find out more interesting things.

2 Energy and Momentum

As an external reference, I suggest Chapters 13 and 14 from Feynman and Chapter 6 from Shankar. They also cover the topic of "work" a bit, but you can skip that part for now if you feel so inclined.

When we considered the conservation of energy and the conservation of momentum, we saw that we could find velocities and momenta of objects much more simply than if we trued to apply Newton's Laws directly. For a single object with kinetic energy T, potential energy U, and total energy $E = T + U$, the statement of conservation of energy is

$$E_i = E_f \tag{31}$$

or

$$T_i + U_i = T_f + U_f \tag{32}$$

where the subscripts i and f stand for *initial* and *final* respectively. The sum off all of the energies at the start of some process MUST equal the sum of the energies at the end of the process. Suppose that the whole process takes a time Δt and we started the clock at some time t. Then we may write Eq. (33) as

$$E(t) = E(t + \Delta t) \tag{33}$$

Where the energy at time t is $E(t) = E_i$ and the energy at time $t+\Delta t$ is $E(t+\Delta t) = E_f$. Moving everything to one side on the equals sign we get

$$E(t) - E(t + \Delta t) = 0 \tag{34}$$

Now we can divide and multiply both sides by Δt.

$$\Delta t \frac{E(t) - E(t + \Delta t)}{\Delta t} = 0 \tag{35}$$

But this implies that

$$\frac{E(t) - E(t + \Delta t)}{\Delta t} = 0 \tag{36}$$

This is the definition of a derivative! So we have the lovely result

$$\frac{dE}{dt} = 0 \tag{37}$$

The energy does not change in time and is therefore conserved. If we measure it at any point in time, we will get the same exact answer. If we have two objects, the story is exactly the same.

$$E_{1i} + E_{2i} = E_{1f} + E_{2f} \tag{38}$$

or

$$T_{1i} + U_{1i} + T_{2i} + U_{2i} = T_{1f} + U_{1f} + T_{2f} + U_{2f} \tag{39}$$

where T_{1i} is the initial kinetic energy of object 1 etc. In fact, we can easily generalize this for N objects

$$\sum_{n=1}^{N} (T_{ni} + U_{ni}) = \sum_{n=1}^{N} (T_{nf} + U_{nf}) \tag{40}$$

The kinetic energy term is always of the form

$$T = \frac{1}{2}mv^2 \tag{41}$$

and the potential energy always has the form

$$U = mgh \tag{42}$$

We could therefore rewrite Eq. (42) as

$$\sum_{n=1}^{N}\left(\frac{1}{2}m_n v_{ni}^2 + m_n g h_{ni}\right) = \sum_{n=1}^{N}\left(\frac{1}{2}m_n v_{nf}^2 + m_n g h_{nf}\right) \tag{43}$$

Here we have that m_n is the mass of the n-th object, v_{ni}^2 is the square of the initial velocity of the n-th object etc. Let us then consider the scenario of a ball of mass m which starts at rest and falls from a height D. What is the velocity of the ball when it hits the ground? Consider first the left hand side of Eq. (45) for $N = 1$,

$$T_i = \frac{1}{2}mv_i^2 = 0, \quad U_i = mgD \tag{44}$$

What about the final energies? The moment before the ball hits the ground, it has essentially zero height but some nonzero velocity. So,

$$T_f = \frac{1}{2}mv_f^2, \quad U_f = 0 \tag{45}$$

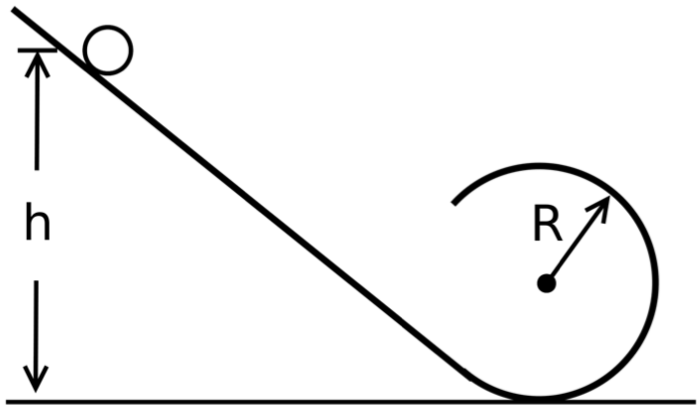
Putting it all together we get

$$0 + mgD = \frac{1}{2}mv_f^2, +0 \tag{46}$$

And therefore we now know the velocity,

$$v_f = \sqrt{2gD} \tag{47}$$

That was embarrassingly simple. Consider the following scenario where a ball of mass m, initially at rest and at a height h, is rolling down a track with a loop of radius R at the bottom. We'd now like to know, what is the minimum height that the ball must start at so that it *just* completes the loop? The figure below may be helpful



First, what does it mean to say that the ball *just* completes the loop? Consider the forces on the ball when it is zipping around the loop. The centripetal force is

$$F_c = m\frac{v^2}{R} \tag{48}$$

and is throwing the ball outward to that it sticks to the track. The faster the ball goes, the bigger F_c gets and the more it sticks to the track. There is another force present, $F_g = mg$. Right at the top of the loop, the gravitational force is trying to pull the ball straight down away from the track and the centripetal force is pointing straight up trying to keep the ball mushed against the track. If the ball *just* makes it around the loop, then it essentially feels weightless right at the top of the loop. In other words, the amount of force pulling downwards is precisely equal to the amount of force pushing upwards,

$$mg = m\frac{v^2}{R} \tag{49}$$

Solving for v we obtain the minimum velocity that we have to have when we start going around the loop. Therefore $v = \sqrt{Rg}$. Now, what are our initial energies? As before we have

$$T_i = \frac{1}{2}mv_i^2 = 0, \quad U_i = mgh \tag{50}$$

The final energy at the top of the loop (height 2R) is

$$T_f = \frac{1}{2}mv_f^2 = \frac{1}{2}mRg, \quad U_f = mg2R \tag{51}$$

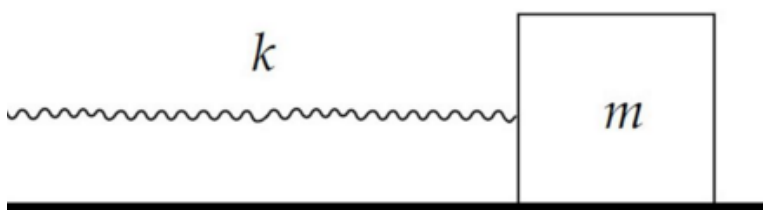
Assembling the pieces we get

$$mgh = \frac{1}{2}mRg + mg2R \tag{52}$$

This can be easily solved for h and we find

$$h = \frac{5}{2}R \tag{53}$$

By any other means, this would be a deceptively tricky problem. Imagine now a spring sticking out of the wall and a block of mass m attached to the spring of spring constant k (k tells us how stiff the spring is). Assume that there isn't any friction between the block and the table.



Hooke's law describes this situation by the following

$$F = -kx = m\ddot{x} \tag{54}$$

If you set $k/m = \omega_0^2$, then you see that the differential equation for this problem is just our old friend

$$\ddot{x} = -\omega_0^2 x \tag{55}$$

You already understand what is going to happen: the block will oscillate back and forth with a frequency $\omega_0 = \sqrt{k/m}$. The kinetic energy is easy enough to find. As always, it will be

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \tag{56}$$

What about the potential energy? This isn't quite as clear. There is an extremely useful relationship between the force and the potential. In one dimension.

$$F = -\frac{dU}{dx} \tag{57}$$

The most general expression we can make is

$$\mathbf{F} = -\nabla U \tag{58}$$

We don't have to worry ourselves about this level of generality just yet. Let us use Eq. (59) to find the potential. If

$$F = -kx = -\frac{dU}{dx} \tag{59}$$

then, following our standard treatment of differential equations, we can write this as

$$-kx\,dx = -dU \tag{60}$$

And therefore

$$U = \frac{1}{2}kx^2 \tag{61}$$

I let the constant of integration be zero. Suppose I pull the spring out 2 meters and then let it go, what will the velocity of the block be after it travels two meters and gets back to its equilibrium length (where the spring is un-stretched)? Well,

$$T_i = 0, \quad U_i = \frac{1}{2}k2^2 = 2k \tag{62}$$

and

$$T_f = \frac{1}{2}mv^2, \quad U_i = \frac{1}{2}k0^2 = 0 \tag{63}$$

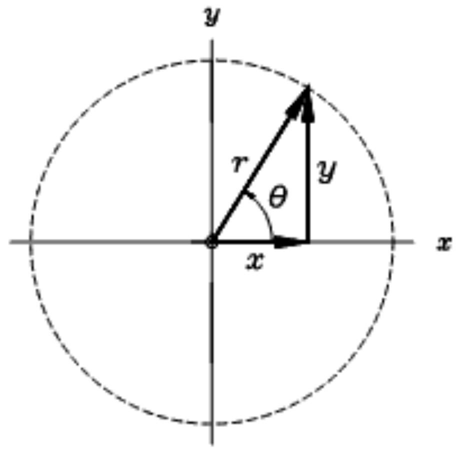
Therefore,

$$2k = \frac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{4k}{m}} \tag{64}$$

These examples are certainly pretty simple so let's take up something a bit more interesting and consequential: Suppose we had a particle of mass m moving in a circle in the $x - y$ plane. What is its kinetic energy? The particle is moving in both x and y, so we guess that the kinetic energy is just

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \tag{65}$$

and we'd be exactly right. There is however a better way to do it. If we are moving around in a circle, it makes more sense to use some coordinate system more natural to circles. Look at the following diagram



If the radius r has a constant length, but we let θ change in time, then r just traces out a circle over and over. This is exactly the kind of thing we are looking for. Can we express x and y in terms of r and θ? Of course! (Assume only x, y, and θ depend on time)

$$x = r\cos\theta \quad y = r\sin\theta \tag{66}$$

In order to get this in the form of Eq. (67), we need $\dot{x}$ and $\dot{y}$. If r is constant, then $\dot{r} = 0$. Also, by the chain rule for derivatives,

$$\frac{d}{dt}\cos\theta = -\sin\theta\frac{d\theta}{dt} = -\dot{\theta}\sin\theta \tag{67}$$

Therefore,

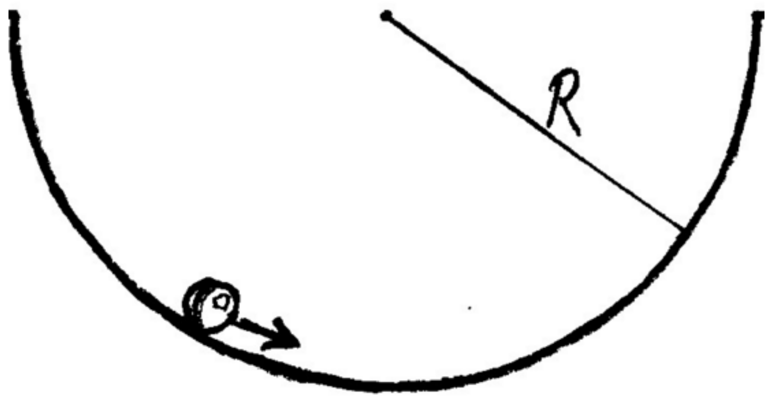
$$\dot{x} = -r\dot{\theta}\sin\theta \quad \dot{y} = r\dot{\theta}\cos\theta \tag{68}$$

Sticking this into Eq. (67) we find

$$\begin{aligned} T &= \frac{1}{2}m\left(-r\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}m\left(r\dot{\theta}\cos\theta\right)^2 \\ &= \frac{1}{2}mr^2\dot{\theta}^2\left(\sin^2\theta + \cos^2\theta\right) \\ &= \frac{1}{2}mr^2\dot{\theta}^2 \end{aligned} \tag{69}$$

The trig relation $\sin^2\theta + \cos^2\theta = 1$ helped us out here. This is a delightfully simple result. Comparing it to $mv^2/2$ we see that our velocity is now $r\dot{\theta}$ (you can check the units to be sure). But other than that, from here on out it is the same old song and dance: set the initial and final energies equal and solve for the quantity you're looking for.

Return to the problem before of a ball rolling down a track. Only this time, the ball is stuck in the loop and rocks back and forth. You could instead imagine a skateboarder in a half pipe or even a pendulum merrily swinging back and forth. When the radius vector is vertical, the angle is zero degrees. When it is parallel to the ground, it is at 90 degrees. In other words, x is the horizontal direction and y is the vertical direction.



The kinetic energy for circular motion can be simply written now as

$$T = \frac{1}{2} m R^2 \dot{\theta}^2 \tag{70}$$

The potential is still going to be $U = mgh$ essentially, but we need h in terms of r and θ. Taking the center of the circle to be at height 0, then when the angle is 0 degrees, the height is $-R$. When the angle is 90 degrees, the height is 0. Which function is -1 at zero degrees and 0 at 90 degrees? Well $-\cos\theta$ fits the bill.[2] The height is therefore $h = -R\cos\theta$, and so with this coordinate system,

$$U = -mgR\cos\theta \tag{71}$$

The total energy is then

$$E = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) + mgy = \frac{1}{2} m R^2 \dot{\theta}^2 - mgR\cos\theta \tag{72}$$

Remember Eq. (39) which says that $\dot{E} = 0$? What does that imply here?

[2]If we had chosen a different zero point, we would have gotten a different answer for the potential energy. There is no inconsistency, it is all just a matter of how you define things which is based on preference. Here, gravity is negative

$$
\begin{aligned}
\dot{E} &= \frac{d}{dt}\left(\frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta\right) = \frac{1}{2}mR^2\frac{d}{dt}\dot{\theta}\dot{\theta} - mgR\frac{d}{dt}\cos\theta \\
&= \frac{1}{2}mR^2(\dot{\theta}\ddot{\theta} + \ddot{\theta}\dot{\theta}) - mgR(-\sin\theta)\dot{\theta} \\
&= mR^2\dot{\theta}\ddot{\theta} + mgR\dot{\theta}\sin\theta \\
&= mR^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{R}\sin\theta\right) \\
&= 0
\end{aligned} \tag{73}
$$

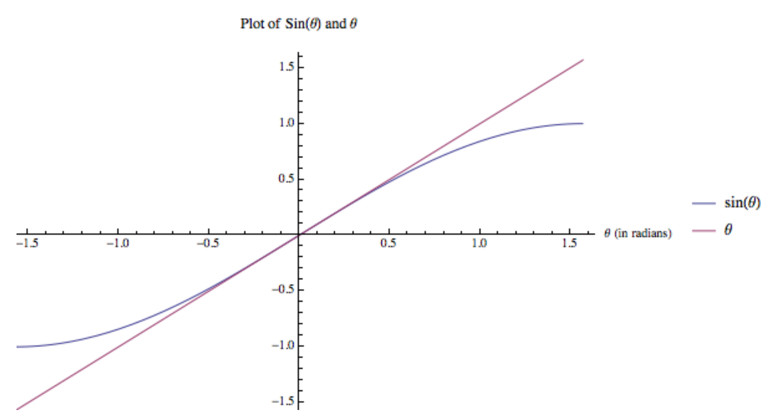
Now since neither m nor r nor $\dot{\theta}$ are zero, then it must be so that

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0 \tag{74}$$

or

$$\ddot{\theta} = -\frac{g}{R}\sin\theta \tag{75}$$

What is the point of all of this? Well, we've found the equations of motion, so in some sense, we now know everything there is to know. Consider the case where the ball is only barely rolling along the bottom of the half pipe. In this case, the angle that the ball travels up the sides is small and so we may make the approximation $\sin\theta \approx \theta$. The plot below may help to convince you that this is indeed the case.



Armed with this information, then Eq. (77) becomes (brace yourself!)

$$\ddot{\theta} = -\omega^2\theta \tag{76}$$

where I set $\omega^2 = g/R$.

Yes ma'am, indeed! This is the harmonic oscillator yet again (I told you, just about everything is a harmonic oscillator) and what you would have gotten if you applied Newton's

Laws directly. The ball oscillates about along the bottom with a frequency that depends on gravity and the radius of the half pipe. If you instead imagined a pendulum swinging back and forth, then the radius is the length of the pendulum. Curiously, the mass doesn't matter... again. When the radius of the bowl (or pendulum) shrinks, the frequency goes up. This seems perfectly reasonable, and it is. We will derive this result again and again in a number of different ways. We are also on the cusp of understanding what I consider to be the single most important function in physics, the *Lagrangian*.

Before moving on to the Lagrangian, let's work out more examples. We worked out the previous example under the assumption that $\dot{r} = 0$, but suppose that is isn't. What then? The physical interpretation of this would be that the pendulum is changing its length or that the half pipe is changing its radius. Look back at Eq. (68) when we changed coordinates from x's and y'x to r's and θ's. If we assume that r also depends on time, then $\dot{x}$ and $\dot{y}$ become

$$\begin{aligned} \dot{x} &= \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \frac{d}{dt}(r\sin\theta) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \tag{77}$$

Now stuff this whole mess into Eq. (67) just as before.

$$\begin{aligned} T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) &= \frac{1}{2}m\Bigg(\Big(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\Big)^2 + \Big(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\Big)^2\Bigg) \\ &= \frac{1}{2}m\Bigg(\dot{r}^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right)\Bigg) \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \end{aligned} \tag{78}$$

This is just what we got in Eq. (71) except now we see the extra term which accounts for a changing radius. The potential energy is unchanged, i.e, $U = -mgr\cos\theta$. Therefore, the total energy is now

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \tag{79}$$

If we again apply the conservation of energy and use $\dot{E} = 0$, we'd get the equations of motion that Newton's Law would have provided, but with hardly any pain! Taking the derivative with respect to time leads to a mess. You can try it if you like, but the equations are not simple to solve. I don't think I've ever seen an exact solution. The Lagrangian method that we'll soon learn about will give us some symmetries to work with that will make the solution much more tractable (though still quite difficult).

Let us now go all in. We have thought about the equations of motion for particles moving in one dimension and in two dimensions, but what if we have a particle moving in some 3D cylindrical region?

We just found the general expression for the kinetic energy for a particle moving in a circle, i.e, Eq. (80). A cylinder is just a circle that has some height to it. Call the vertical direction z. The kinetic energy in the z-direction is simply

$$T_{vertical} = \frac{1}{2}m\dot{z}^2 \tag{80}$$

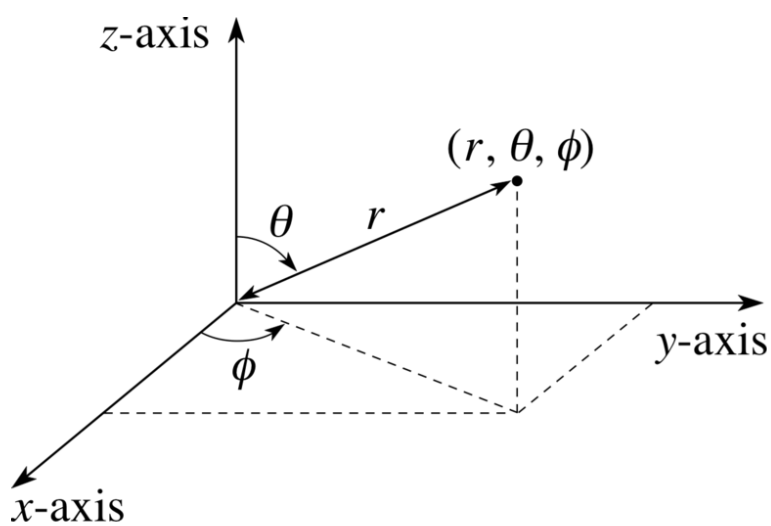
The total kinetic energy is just this term added on to Eq. (80),

$$T_{cylindrical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \tag{81}$$

So much for that. What about the potential? Well if z is the vertical direction, then $U = -mgh \rightarrow U = -mgz$. And so,

$$E_{cylindrical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \tag{82}$$

Again, gravity is negative. That wasn't too painful. The final case is if we consider motion in a spherical region. In this instance, we have to start over with the coordinate transformations as in Eq. (68), except for spherical coordinates. The typical way of defining the coordinates is from the graph below.



The transformations are

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
\tag{83}
$$

I'll leave it to you to fill in the steps, but to get you going,

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi \tag{84}$$

After the dust settles, you will determine the kinetic energy to be

$$T_{spherical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \tag{85}$$

Since the potential is $-mgz$ and $z = r\cos\theta$, then

$$U_{spherical} = -mgr\cos\theta \tag{86}$$

And at long last we arrive at the total energy for a particle moving in three dimensions in spherical coordinates,

$$E_{spherical} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - mgr\cos\theta \tag{87}$$

Fundamentally, this is identical to

$$E_{cartesian} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \tag{88}$$

just taken from a different coordinate system.

To summarize, we have Cartesian, or rectangular, coordinates (x, y, z); polar coordinates (r, θ); cylindrical coordinates (r, z, θ); and spherical coordinates (r, θ, ϕ). They all have different uses. For straight lines, use Cartesian coordinates. For problems with circular symmetry, use polar coordinates. When you have cylindrical symmetry, use cylindrical coordinates, and lastly, when the problem has spherical symmetry, use spherical coordinates.

You could of course argue that spherical coordinates trace out circles too, so why not use those instead of polar coordinates? You certainly can. In fact, polar coordinates are just a special case of spherical coordinates. Look at the figure that shows the spherical coordinates. If $\theta = 90°$, then we are confined to the $x - y$ plane and Eq. (89) reduces to Eq. (81) since $\sin(90°) = 0$ and $\dot{\theta} = \frac{d}{dt}90° = 0$.

For your reference, here are the energies we derived for each coordinate system (gravity is negative):

$$
\begin{aligned}
E_{cartesian} &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz \\
E_{polar} &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \\
E_{cylindrical} &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \\
E_{spherical} &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - mgr\cos\theta
\end{aligned} \tag{89}
$$

3 The Lagrangian

Well, Charlotte, we have finally arrived. We are about to revel the most important function in all of physics. First, recall that the kinetic energy is a function of $\dot{q}$ where q could stand for $x, y, z, r \ldots$. The potential on the other hand is just a function of this general coordinate q. Therefore, all along we could have been writing

$$T(\dot{q}) = \frac{1}{2}m\dot{q}^2, \quad U(q) = mgq \tag{90}$$

Technically, q depends on time, so really,

$$T[\dot{q}(t)] = \frac{1}{2}m\dot{q}(t)^2, \quad U[q(t)] = mgq(t) \tag{91}$$

I switched to square brackets only so that the parenthesis didn't become confusing. So the energies are functions of functions. We have a name for such things: *functionals*. The Lagrangian precisely that. And with that, here is the definition of the most important function in physics, the *Lagrangian*:

$$\boxed{\mathscr{L}[\dot{q}(t), q(t)] = T[\dot{q}(t)] - U[q(t)]} \tag{92}$$

I know... you're thoroughly underwhelmed. The Lagrangian is simply the difference of the kinetic energy and potential energy. Here's where the magic comes from: if we integrate the Lagrangian over time, we call it the *action*, S,

$$\boxed{S = \int \mathscr{L}[\dot{q}(t), q(t)]\,dt} \tag{93}$$

We then require that the infinitesimal variation is 0. That is, $\delta S = 0$. This implies

$$\delta S = 0 = \int \delta\mathscr{L}[\dot{q}(t), q(t)]\,dt = \int \left(\delta q\,\frac{\partial \mathscr{L}}{\partial q} + \delta\dot{q}\,\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) dt \tag{94}$$

We can rearrange the second term by doing something called "integrating by parts." I wont bother you with the details, but the outcome is the following

$$\delta S = 0 = \int \left(\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} \right) \delta q \, dt \tag{95}$$

And because this is equal to zero, then the integrand must be zero. Therefore, we have the unbelievably, fantastically, magnanimously important result known as the Euler-Lagrange equations:

$$\boxed{\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} = 0} \tag{96}$$

It may look a bit spooky, but I assure you that it's all smoke and mirrors. You know how to find the energies, and you know how to take a derivative. That's all you need to know. When you use the Euler-Lagrange equations, you get the equations of motion that Newton worked tirelessly for. But now we don't need to worry ourselves with vector functions, just scalars. Better yet, the kinetic energy and potential energy are always of the same form. We're also able to determine conserved quantities using the Lagrangian and it is manifestly invariant, which is why it appears throughout relativity and quantum field theory.

Let us learn how to use this new tool by example. Consider a free falling particle of mass m in one dimension, say the z-direction. The kinetic energy is

$$T = \frac{1}{2} m \dot{z}^2 \tag{97}$$

The potential is also easy to find. Letting gravity be negative,

$$U = -mgz \tag{98}$$

And therefore

$$\mathscr{L}[\dot{z}, z] = T - U = \frac{1}{2} m \dot{z}^2 + mgz \tag{99}$$

Now let's use Eq. (98). The first term is

$$\frac{\partial \mathscr{L}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{2} m \dot{z}^2 + mgz \right) = mg \tag{100}$$

The second term is

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{z}} = \frac{d}{dt}\frac{\partial}{\partial \dot{z}}\left(\frac{1}{2} m \dot{z}^2 + mgz \right) = \frac{d}{dt} m \dot{z} = m \ddot{z} \tag{101}$$

Therefore, we determine the equations of motion to be

$$mg = m\ddot{z} \tag{102}$$

Almost magically, we've recovered Newton's Law!

Homework 3

March 29, 2016

Problem 1

Suppose that the one-dimensional potential of a system is given by

$$U(r) = \frac{\alpha}{r} + \sin\left(\frac{n\pi r}{l}\right)$$

What is the force on this system?

Problem 2

Consider a particle of mass m falling under the influence of gravity.

a) What are the kinetic and potential energies? (Feel free to arrange your coordinate system in any way you prefer)

b) Calculate the force on the system.

c) What is the energy in the system?

d) Using the result in part (c) and the fact that the energy is a conserved quantity, find the equation of motion.

Problem 3

We saw that the kinetic energy is always of the form $T = \frac{1}{2}mv^2$. In the notes, we showed what this term looks like in spherical coordinates. Derive this term first by replacing $\mathbf{v} = \dot{x}\,\hat{\mathbf{x}} + \dot{y}\,\hat{\mathbf{y}} + \dot{z}\,\hat{\mathbf{z}}$ and then using the appropriate transformation from Cartesian to spherical coordinates.

Problem 4

Consider the Lagrangian given by

$$\mathscr{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\alpha}{r}$$

a) What are the units of α?

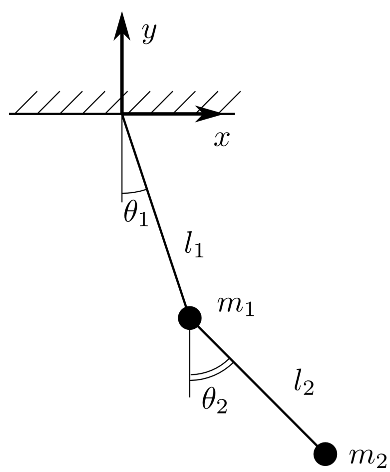
b) Show that the angular momentum given by $\ell = mr^2\dot{\theta}$ is a conserved quantity.

c) The incremental change in area being swept out by the radius is $dA = \frac{1}{2}r^2 d\theta$. What does this imply about the rate of change of the area with respect to time?

d) It may seem extraordinary (and it is), but this Lagrangian contains nearly everything that there is to know about planetary motion. Use the Euler-Lagrange equations to find the equations of motion for the radial coordinate. You should be able to do this about 9 years quicker than it took Kepler.

Problem 5

a) Use the image below to construct the Lagrangian for the double pendulum assuming that $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. Also, use the small-angle approximation. That is, assume that $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. (Drop terms with θ^3 and higher in them)



b) Write the equations of motion for both θ_1 and θ_2. You will notice that the equation for θ_1 still has θ_2's in it and vice versa. When this occurs, we say that the equations are *coupled*. We will discuss these much more in the lessons to come.

c) Suppose now that the masses are different. If $m_1 \to \infty$, describe the subsequent motion for m_2. Do the same thing supposing that $m_1 \to 0$. How does m_1 move when

$m_2 \to 0$? (We don't technically need m_1 to be infinite, we just mean that $m_1 \gg m_2$. Similarly, we don't need m_2 to be zero, we just want $m_2 \ll m_1$ etc.)

Problem 6

From the Lagrangian for a hanging pendulum, show that energy is conserved.

Problem 7

Consider the general kinetic energy $T = \frac{1}{2}m\dot{x}^2$ and the general potential energy $U(x)$. Can you show that energy is conserved in this general case?