An Introduction to Calculus

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1 Introduction to Functions

There's a certain amount of lingo that gets thrown around, so let's quick take care of that with a little 'pre-calc.' First, what is a *function* exactly? A function is really just a shorthand way of writing some equation. It doesn't actually tell you anything except what variables are inside that are independent, that is, you get put in those values at will. If it's not in the function name, then either it's not in the function at all, or it's just a number that we don't much care about. Let's look at a few examples then.

Let's say we have a parabola with equation $y = x^2$. We know just how this works; we plug in values of x, then it spits out a value for y. We say $x = 3$ and then it says "Then y is 9". I'm boring you already, I can tell. So then x is the independent variable (we pick it), and y is dependent on whatever we chose for x, so it's the dependent variable. But if y depends on x, we may say that it is a function of x. Algebra is just a sneaky trick; y was a function all along. We should have written: $y(x) = x^2$. Also, there's nothing special about using one letter over another. We could have written $\Lambda(\xi) = \xi^2$. Aside from looking a bit spiffier, it is exactly the same.

We can go a little further. Suppose the parabola now has the form $x^2 - 3$. What should our function name look like now? Well the function just tells us what the independent variables are. We know that x is one, but what about 3? Nope. That's just a number. So the function is still just $y(x)$, or in it's complete glory, $y(x) = x^2 - 3$. Alright wise guy, now try to find a function name for $2x^2 - 3$. Yup, it's still the same. No matter how many numbers we throw in, we couldn't care less. It can still be written as $y(x)$ where now $y(x) = 2x^2 - 3$. Let's turn up the heat a little...

Now we've got a parabola that *is* 3D and it has the equation $x^2 + y^2$. What now? If both x and y are independent variables, then the function must tell us about both. So let's try out $y(x, y) = x^2 + y^2$. That's no good at all! The y is now the name and the definition! Let's use a different letter: f for *function*. So we have $f(x, y) = x^2 + y^2$. Looks good to me! Getting the hang of it? Ok so naming $2x^2 + 3y^2 + xy$ is a cake walk right? Of course it is. $f(x, y)$ will do just fine giving us $f(x, y) = 2x^2 + 3y^2 + xy$. Now a curveball! Suppose I insist that this is NOT a 3D parabola, but just a regular 2D one and that only x is the independent variable. What now? Following the scheme, we'd suggest that $f(x) = 2x^2 + 3y^2 + xy$ is valid, and we'd be right. In this case, y must just be some number, so it is to be treated no different than if it were a 6 or a 1.5 or whatever. Suppose we were faced with $x^3 + \lambda^2$ and you know that λ was just a number. What can we call it? How about $f(x) = x^3 + \lambda^2$? Yes sir indeed!

So what does the function name tell us? It tells us what the independent variable is. There may be tons of other numbers in there, but we'd have no idea. "Then what good is this naming scheme?" I hear you say. Well, Consider some random curve. If we knew it was labeled as "$f(x)$" then we know that it only depends on x somehow. So in order to guess the correct form we know it must be something like $a + bx + cx^2 + dx^3 + ...$ with a, b, c and d just some arbitrary numbers. It get's us close to the right equation anyway. We'll revisit the benefits and add to them when we get to derivatives in a little bit.

2 With Respect to...

The most common phrase in all of calculus has to be "with respect to" so it's good to know what we mean. Rather than jib-jab too much, how about we just look at some examples? Dust off our favorite $f(x) = x^2 - 3$. Suppose now that we want to change f somehow. Could we alter Δ to change f? WHAT? There's not a single triangle-thingy in there, so how could it change f?! Right, it can't. How about altering y in order to change f? Not a single y in there either, so it cannot change anything.

This isn't as worthless as it may first seem. When we say "with respect to" we mean "on account of" or "due to". So, would it make sense to attempt to change a function $f(x)$ with respect to Δ? No, of course not. In fact, there is only 1 argument that makes sense, and you already know what it is because there is only one thing that f depends on, and that's it's argument, x. So we can only change $f(x)$ with respect to x. If we tried to change $f(x)$ with respect to ρ, we had better get zero because there is 0 change. How about a few more?

2.1 Pop Quiz!

Of the following operations, which will be zero and which will be non-zero?

1. Changing $f(x)$ with respect to y?
2. Changing $f(x)$ with respect to x?
3. Changing $f(x, y)$ with respect to z?
4. Changing $f(x, y)$ with respect to x?
5. Changing $f(x, y)$ with respect to y?
6. Changing $f(\lambda, \sigma)$ with respect to α?

Answers:

1. This will be 0 since f doesn't depend on y here.
2. Non-zero. We don't know what it will be exactly, but we know it's not 0
3. Zero for the same reason as #1
4. Non-zero for the same reason as #2
5. Non-zero for the same reason as #2
6. Zero for the same reason as #1

3 Geometry

Let us suppose that you threw a ball and it flew from point A to point B as shown in Fig. 1. Each point is given some value of time and some position. Suppose you also know that someone started a stopwatch at $t = 0$ seconds (the moment the ball was at point A) and the watch was stopped when the ball was at point B, which happened to be at $t = 2$ seconds. What is it *average speed* of the ball? Well, x started at 0 and went to $x = 4$ and is obviously a some unit of distance. We know a speed must have the units "distance/time" so it looks like we have what we need. Let's call the average speed in x, s_x. We can see then that

$$s_x = \frac{\text{x coordinate at B- x coordinate at A}}{t = 2 - t = 0} = \frac{4-0}{2-0} = 2 \tag{1}$$

This is fine, but its a little cluttered. Call the final position in the x-direction x_f and the initial point x_i. The same thing can be done with the time part which suggests that we use t_f and t_i for the final and initial times respectively. Now we can write the average speed as

$$s_x = \frac{x_f - x_i}{t_f - t_i} = \frac{4-0}{2-0} = 2 \tag{2}$$

That's way better. Notice that we didn't even have to specify which points we were considering at first, we just used general variables. Therefore, we've found the equation that determines the average speed in all cases. Yay for us. We can go a step further. It is clear that all we care about is the *change* in x and the *change* in t. If x doesn't change, then the ball hasn't moved and so $s = 0/(t_f - t_i) = 0$ as we'd expect. If the change in t were 0, then the ball has moved a certain distance in 0 time and must have gone infinitely fast, that's the 'divide by zero' creeping up. It all seems to make sense.

Since "change" is the name of the game, let us permanently define Δ (delta) to be read as "small change in." For example, Δr can be read as "the small change in r" (the "small"

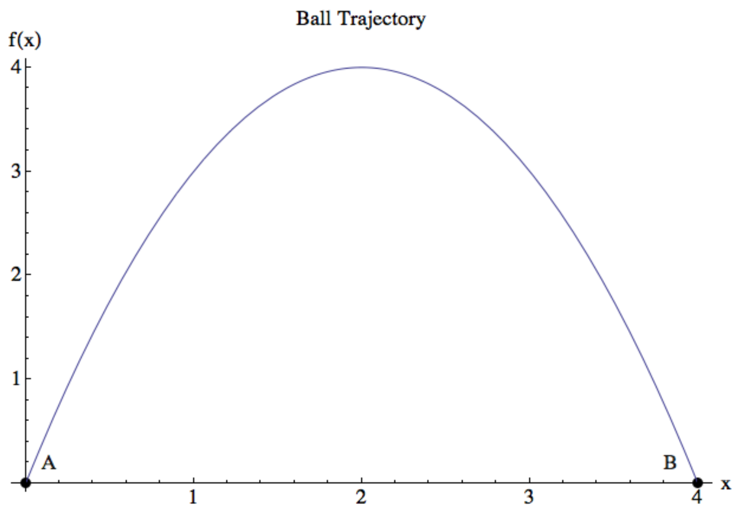

Figure 1: Trajectory of a thrown ball. Point A is at (0,0) and point B is at (4,0)

part is usually understood, so we can just say “change in”). Armed with this, we can again rewrite the average speed in x.

$$s_x = \frac{\Delta x}{\Delta t} = \frac{4}{2} = 2 \tag{3}$$

Much better! The average speed is the change in x divided by the change in t, or, the change in position divided by the change in time. Notice I keep say ”average.” Why? Well the ball didn’t zip from point A to point B in a straight line, so it’s position actually changed more than we gave it credit for. For example, consider the slope of the line between points A and B. It started on the ground and ended on the ground, and $0 - 0 = 0$ so the slope (rise over run) is 0 since the rise is zero! The ball didn’t just roll along the ground. Our approximation with two points it too rough evidently. Let’s try to weasel our way out of this by adding another point like in Fig. 2.

We could now use our equation (with $f(x)$ in place of x and x in place of t) to find the slope by figuring it out from A to C, then from C to B. If the x coordinate at point A is x_A and the x-coordinate at point B is x_B etc, we get

$$s_{f(x)} = \frac{1}{2}\left[\frac{\Delta f(x)}{\Delta x} = \frac{f(x_C) - f(x_A)}{x_C - x_A} + \frac{f(x_B) - f(x_C)}{x_B - x_C}\right] \tag{4}$$

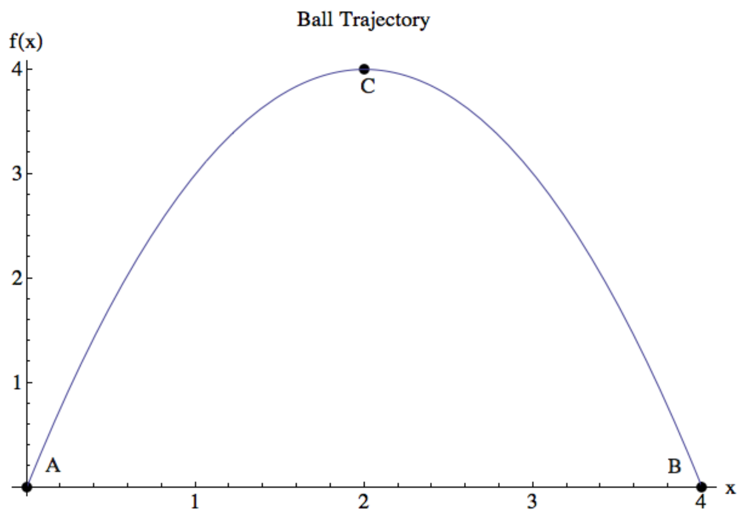

Figure 2: Trajectory of a thrown ball. Point A is at (0,0) and point B is at (4,0) and point C is at (2,4)

Ick. It's substantially uglier now. And it get's worse...that's still an approximation. We need to add more points. Back on this is a second.

Let's look at Fig. 3 for a moment. If this were a trajectory, and x were a position and t were a time, then the slope is the change in x divided by the change in time or

$$\text{slope} = \frac{\Delta x}{\Delta t} \tag{5}$$

We know that one! Here comes a very important statement "The slope of a curve gives the rate of change of that curve." We can elaborate on it more later but it's good for now. When we had a plot of position versus time, the slope gave us a speed in units of position/time. If we had a plot of speed versus time, the slope would be in units of speed/time or position/time/time which is an acceleration. We will hammer this point home, so if it's not clear, you needn't worry. Let's get back to the thrown ball. We want the slope of this curve. We found the slope from point A to B, but determined it to be zero so we added another point, but it's still not enough. More points!

Check out Fig. 4. Now we need to find the differences from A to D, D to C, C to E, and E to B. Better, but it's still an approximation! Ugh... You mean we have to add more points? Yes sir, we do. But you see that if we were to connect the dots we're drawing, we're resembling more and more the actual curve. Our error shrinks as we add points. Well then, how about we add an infinite amount of points? That's exactly the idea. Imagine

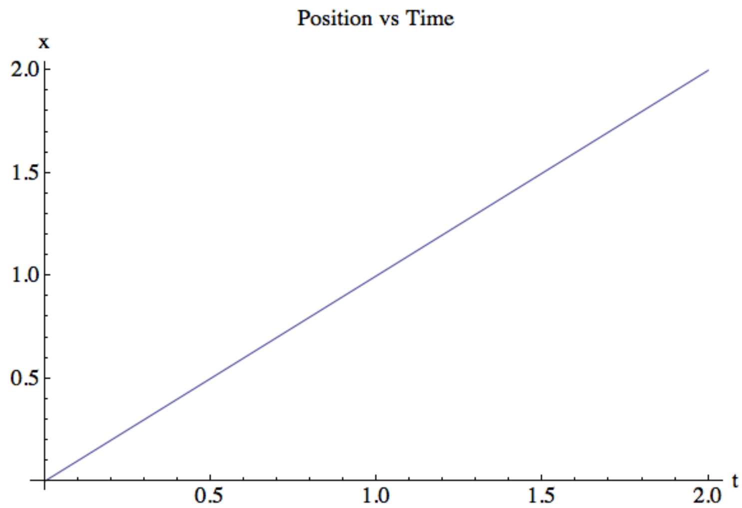

Figure 3: A straight line given by $x(t) = t$

looking at just a small chunk of the curve and suggesting that we find the slope at just one point. Fair enough. If there were a point between D and C, called P, then the slope from D and C should give us *roughly* the slope at P (Fig. 5). To get a better approximation, scoot D and C closer and closer to point P. The approximation of the slope at P becomes exact as the distance between D and C becomes infinitely small. If we had points infinitely close together everywhere on the line, then we should be able to know the slope at every point on the line. We could keep using our geometry, but its becoming infinitely tedious and horribly messy. There is a better way...

4 The Derivative

We are now ready to establish 50% of calculus, the differential, or derivative. Continuing on our last example, let us suppose that we know two coordinates and we want to know the slope between those points. The values we will use are

$$x \text{ and } x + \Delta x$$

and then the function would spit out

$$f(x) \text{ and } f(x + \Delta x)$$

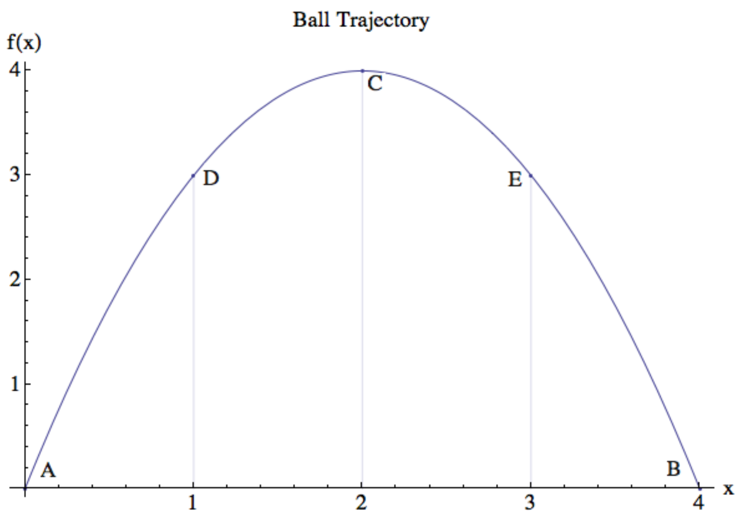

Figure 4: Trajectory of a thrown ball with several points marked

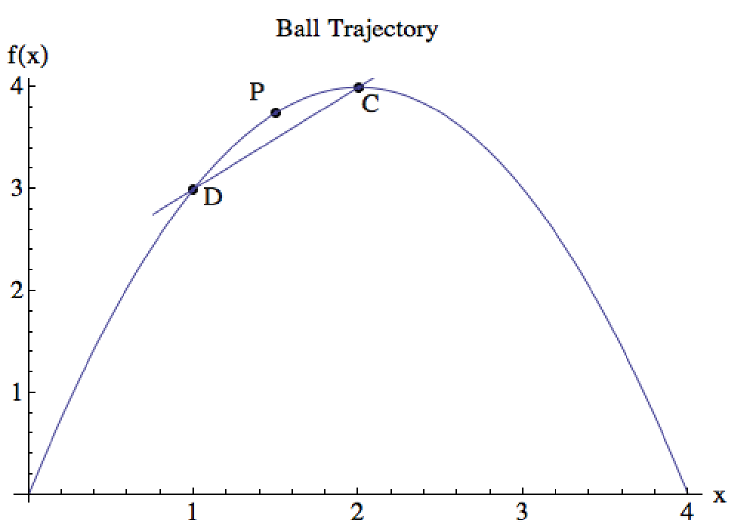

Figure 5: Estimation of the slope

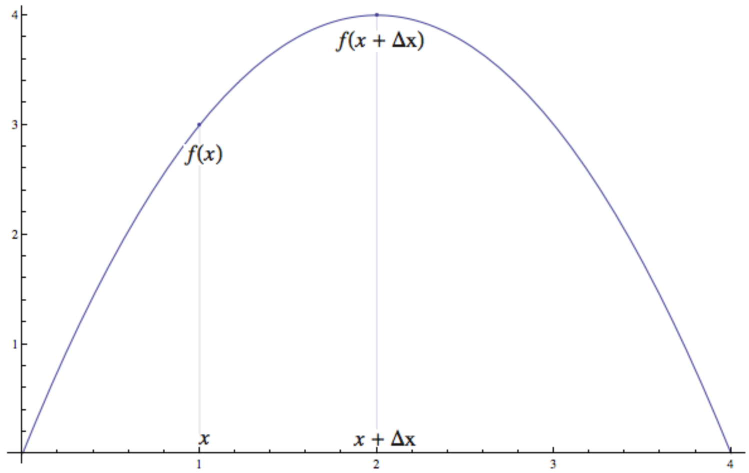

Figure 6: Determination of the exact slope

Remember that Δ means "small change in" so if one point is at x, then a small amount away from x is the point $x + \Delta x$. This is shown in Fig. 6. What would then be the slope? It's still rise over run like we've been doing. This time we get

$$\text{slope} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{6}$$

We are nearly there. Remember that we got more accurate as the separation between points became small, in other words, we want Δx to shrink towards zero and be infinitely small. In the limit that Δx approaches zero (since it can't actually be zero because its in the denominator) we have an exact slope.

$$\text{slope} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{7}$$

Notice that the numerator gets tiny along with the denominator, so we don't have to worry about blowing up when we approach zero. We have a short-hand notation for this expression. Rather than using Δ, we use a lowercase d to encompass not only the change in x, but in the function as well. We can permanently define d to mean "an infinitesimal change in." Again, then "dr" would be "an infinitesimal change in r." Putting it together,

$$\text{slope of f(x)} = \frac{d\,f(x)}{dx} = \frac{d}{dx}\,f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{8}$$

Well, we've done it. This is the formal definition of a derivative. A staggeringly important result. The derivative is really just an operation that we perform on a function which returns a new function. Pretend that there are 3 eggs in a line in this order: red, green, blue. Suppose further that you switch the green and the blue egg around so the order is: red, blue, green. You have performed an operation on the system. If we named your operation "*Op*" and called the system $f(R, G, B)$ where R=red, G=green and B=blue and they necessarily come in that order, then we could mathematically write what you did as

$$Op\ f(R, G, B) = f(R, B, G)$$

In words, "the operation of *Op* on the function $f(R, G, B)$ resulted in the new function $f(R, B, G)$." A derivative is exactly synonymous. Alright, let's review how we got here real quickly.

We wanted to find the slope at some point on *line*. This was a cinch since the slope was always the same. Next, we wanted to find the slope at some point on a *curve*. This wasn't so obvious. First, we tried the same approach as with the line but fell short. Realizing that more dots gives more accuracy, we added another point on the curve to consider. Continuing the pattern we eventually added so many points, that we essentially reformed our curve. But now, understanding that a curve is really just individual points infinitely close together, we took the slope between two of these points. Now, thanks to our handy new notation, given and *single* point on the curve, we can find it's slope, or rate of change (notice that the only argument that appears now is x). It still may sound strange, but the key to understanding comes by just doing examples and seeing how the thing works. From here on out, equations should outnumber words, but it's good to understand what is really happening and where all the dx's and whatnot come from.

4.1 Examples

One test of any theory is that it must produce the results that are already understood to be true. With that, let's consider the line given by the equation $y(x) = x$. We wish to find the slope at the point $(1, 1)$. Doing it the old fashioned way, we first collect some values in Table. 1.

Next we compute the slope, or rate of change, using any two of these points that we like:

Table 1: Values from $y(x) = x$

$y(x)$	x
0	0
1	1
2	2

$$\text{slope} = \frac{y(x_f) - y(x_i)}{x_f - x_i} = \frac{y(2) - y(1)}{2 - 1} = \frac{2 - 1}{2 - 1} = 1 \tag{9}$$

Therefore, the slope of the line, or the rate of change of the line, or the rate of change of the function $y(x)$ with respect to x, is 1. Nothing new here. Remember though that from (Eq. 8)

$$\text{slope of } y(x) = \frac{d}{dx}\, y(x) \tag{10}$$

BUT, and here comes the magic, $y(x) = x$, so then

$$\text{slope of } y(x) = \frac{d}{dx}\, y(x) = \frac{dx}{dx} = 1 \tag{11}$$

Success! Our new formulation worked. Let's try it with a parabola now, a task our geometry just could not handle. The equation is given simply by $y(x) = x^2$. This is a parabola centered at $(0, 0)$ that opens upward. What is the rate of change of the function $y(x)$ with respect to x at the point $(1, 1)$? Well, it turns out we need to find the pattern for finding slopes (derivatives) first. We just found that the derivative of x with respect to x is 1. Let's see if we can find a pattern, given only one piece of the puzzle. We need to think of all the ways we can do something to an x and turn it into a 1.

$$x^0 = 1 \quad x/x = 1$$

That's all I can think of. Fortunately we have the right answer. The general pattern for the operation of a derivative is

$$\frac{d}{dx}\, x^n = n\, x^{n-1} \tag{12}$$

where n is any arbitrary number or function not involving x that we like. Needless to say, this is another important result. Commit this to memory. Since $x = x^1$ let's verify that this works.

$$\frac{d}{dx}\, x^1 = 1x^{1-1} = x^0 = 1 \tag{13}$$

Awesome. The derivative of x is 1, as it should be. Now we can do $y(x) = x^2$

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\, x^2 = 2x^{2-1} = 2x \tag{14}$$

Bring the exponent down to the front, then subtract one from the power and you're done. How about finding the derivative of $y = 5$?

$$\frac{d}{dx}\, y = \frac{d}{dx}\, 5 = 0 \tag{15}$$

Remember, numbers don't depend on x, so 5 does not change with respect to x, it's just a constant. The derivative of a number is always zero. Ok then, find the derivative of $y(x) = 5x$.

$$\frac{d}{dx}\, y(x) = \frac{d}{dx} 5x = 5\frac{d}{dx}\, x = 5\frac{dx}{dx} = 5 \tag{16}$$

Since the derivative of a number is zero, only the derivative of x matters there and the 5 can be factored out. What is the derivative of $y(x) = 3x - 3$? Easy,

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\, (3x - 3) = \frac{d}{dx}\, 3x - \frac{d}{dx}\, 3 = 3\frac{dx}{dx} - 0 = 3 \tag{17}$$

How about the derivative of $y(x) = 4x^2 + 2$?

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\, (4x^2 + 2) = 4\frac{d}{dx}\, x^2 + \frac{d}{dx}\, 2 = 4 \times 2x^{2-1} + 0 = 8x \tag{18}$$

Let's do one more, $y(x) = 2x^2 + 3x + 4$

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\, (2x^2 + 3x + 4) = 2\frac{d}{dx}\, x^2 + 3\frac{dx}{dx} + \frac{d}{dx}\, 4 = 2 \times 2x^{2-1} + 3x^{1-1} + 0 = 4x + 3 \tag{19}$$

Ok cool. Here is the general rule: (n and q are arbitrary constants)

$$\frac{d}{dx}\left(n\, x^{q}\right) = n\frac{d}{dx}\, x^{q} + x^{q}\, \frac{d}{dx}\, n = n\frac{d}{dx}\, x^{q} = n\; q\; x^{q-1} \tag{20}$$

If part of that doesn't make sense, don't panic. It shouldn't yet. Notice a few things: first, the way the derivative operates. It acts on each part of the function separately, this is called the *Product Rule*. It's very important, so much on that later. In my head I always say "First times the derivative of the second, plus second times the derivative of the first" to remember the product rule. Not a bad jingle to remember and you'll be sick of it before long. Next, you can notice the rule we already found, where the power comes to the front and gets 1 subtracted from it. That's all there is to it. It's more complicated looking in its general form I admit, but it may be helpful to look back on. What you should take away from it is the first and last terms. The middle part we can worry about down the road a little.

4.2 Pop Quiz!

Try a few on your own. The answers are given too. Use the "Feynman Problem Solving Algorithm" if you get stuck:
Step 1: Write down the problem
Step 2: Think very hard
Step 3: Write down the answer

Find the derivatives with respect to x of the following:

1.) $y(x) = x^4 - 1$

2.) $y(x) = 6x$

3.) $y(x) = x^2 + x$

4.) $y(x) = x^4 + x^3$

5.) $y(g) = g^2$

6.) $y(x) = 5x^{10}$

Answers:

1.) $\dfrac{d}{dx}\,(x^4 - 1) = 4x^3 - 0 = 4x^3$

2.) $\dfrac{d}{dx}\,6x = 6\dfrac{dx}{dx} = 6$

3.) $\dfrac{d}{dx}\,(x^2 + x) = \dfrac{d}{dx}\,x^2 + \dfrac{dx}{dx} = 2x + 1$

4.) $\dfrac{d}{dx}\,(x^4 + x^3) = \dfrac{d}{dx}\,x^4 + \dfrac{d}{dx}\,x^3 = 4x^3 + 3x^2$

5.) $\dfrac{d}{dx}\,g^2 = 0$ (g is just a constant that doesn't depend on x!)

6.) $\dfrac{d}{dx}\,(5x^{10}) = 5\dfrac{d}{dx}\,x^{10} = 5 \times 10x^9 = 50x^9$

4.3 More Derivatives and Applications in Science

We're about to get into cooler stuff, so bare with me. First, let's conquer a few more examples. Remember, the only variables that matters are the ones named in the function, everything else is just treated as a number. What would be the answer to

$$\frac{d}{dx}\, f(p) = \ ?$$

Its just zero again since we'd be taking derivatives of p with respect to x, but p does not depend on x.. How about

$$\frac{d}{dx}\, (f(g) + \lambda(\kappa) + \xi(r,\theta,\phi) + 2^{98}) = \ ?$$

You got it, zero again. For all we care, all of those 'x-less' functions are just numbers. Alright, here's another,

$$\frac{d}{dt}\, (\frac{1}{2} g\, t^2) = \ ?$$

Hopefully you got the same thing as me, gt. "Why is this important?" you say. Suppose I tell you that t up there is time and g is gravity (which has units of m/s^2 or ft/s^2 or whatever unit of distance and time you like). So then gt^2 has units of $s^2\ \times$ position$/s^2 =$ position. In that spirit, let's give the function a name. How about,

$$y(t) = \frac{1}{2} g t^2$$

Now, the units on the function $y(t)$ is position, let's say it is in feet. This means that if you give me any time t, I can give you the new position. Awesome, awesome, awesome. For example, what is the position after 3 seconds ($t = 3$)?

$$y(t=3) = \frac{1}{2} g(3^2) = \frac{9}{2} g \text{ feet}$$

It gets better. Let's take a derivative of $y(t)$ again.

$$\frac{d}{dt}\, y(t) = \frac{d}{dt}\, (\frac{1}{2} g t^2) = \frac{1}{2} g \frac{d}{dt}\, t^2 = gt \tag{21}$$

We are taking time derivatives here, as you know. What then, becomes of the units for the derivative of $y(t)$? If $y(t)$ has dimensions of feet and t is time, then $dy(t)/dt$ is feet/second, a velocity! So, the derivative of position with respect to time is a velocity. Now, if I give you a time t, you can give me the position and velocity at that time by just plugging in that time. We have one more step to go. Let's take a derivative of the derivative. First, we found that the derivative of $\frac{1}{2}gt^2$ was gt. So now let's take the derivative of gt with respect to t. No problem!

$$\frac{d}{dt}\left(gt\right) = g\frac{dt}{dt} = g \tag{22}$$

And the units of gravity are.....feet/s^2, an acceleration of course. So the derivative with respect to time of a velocity is an acceleration. Recap,

$$\frac{d}{dt}\text{ position} = \text{velocity}$$

$$\frac{d}{dt}\text{ velocity} = \text{acceleration}$$

Let's put it to use. Suppose I tell you to give me the position, velocity, and acceleration of $y(t) = \frac{1}{2}gt^2$ at the time $t = 4$. What do you do? The position is trivial, just plug in the time:

$$y(t = 4) = \frac{1}{2}g(4^2) = 8g\text{ ft} \approx 258\text{ ft} \tag{23}$$

To find the velocity, take a derivative, then plug in $t = 4$.

$$\frac{d}{dt}\,y(t) = gt \to 4g\text{ ft/s} \approx 129\text{ ft/s} \tag{24}$$

We can write this in a little better way:

$$\frac{d}{dt}\,y(t)\Big|_{t=4} = gt\Big|_{t=4} = 4g\text{ ft/s} \tag{25}$$

The vertical bar means "evaluated at." So we can read the above as "The derivative of y with respect to t, evaluated at t=4 is ..." The acceleration is again, just g. Great, let's do another one. Suppose there is a car on a track, swerving like a maniac and his position in feet as a function of time can be modeled as

$$x(t) = 3 + 2t + t^2$$

What is the driver's position, velocity and acceleration at the time $t = 2$? First, the position calculation. Just plug-and-chug.

$$x(t)\big|_{t=2} = 3 + 2(2) + 2^2 = 11 \text{ ft} \tag{26}$$

Next, find the velocity by taking a derivative with respect to time,

$$\frac{d}{dt}\, x(t)\big|_{t=2} = \frac{d}{dt}\,(3 + 2t + t^2)\big|_{t=2} = (0 + 2\frac{dt}{dt} + \frac{d}{dt}\, t^2)\big|_{t=2} = (2 + 2t)\big|_{t=2} = 6 \text{ ft/s} \tag{27}$$

Almost done! Now we need to find the acceleration, which is the derivative of the velocity, or equivalently, the derivative of the derivative of the position (2^{nd} derivative). Let's take the derivative of the velocity then.

$$\frac{d}{dt}\,(2 + 2t)\big|_{t=2} = 0 + 2\frac{dt}{dt}\big|_{t=2} = 2\big|_{t=2} = 2 \text{ ft}/s^2 \tag{28}$$

One more thing before we close out this section. Look at the following equation

$$\frac{d}{dt}\, p(t) = F$$

There is nothing in there that you don't understand, at least mathematically. The derivative of some function $p(t)$ with respect to time gives the value F. So what? This is Newton's second law (essentially). Everything in classical mechanics, from waves to rockets, hydrodynamics to gyroscopes, and all the rest is all contained (and derived) from that equation. With only that equation and a little wit, you can figure out how to put a man on the moon. You ought to be impressed.

Try to do the following: (-hint- watch out for units!)

1.) Find the *expression* for the velocity and acceleration from the function $x(t) = t^3$ ft

2.) Evaluate the velocity and acceleration of Problem 1 at $t = 3$.

3.) Find the acceleration at $t = 1$ from the function $f(t) = at$ ft

4.) Find the velocity at $t = 2$ from $f(t) = bt^4 + t$ ft

5.) Find the *expression* for the acceleration from the function in Problem 4.

6.) From $x(t) = x_0 + st + gt^2$ ft calculate the velocity and the acceleration

7.) What is the position for the function in Problem 6 when $t = 5$s?

8.) What is the acceleration of the function $r(t) = t^{10}$ ft?

9.) What is the acceleration of the function $p(t) = t^2$ ft/s

Answers:

1.) "Expression" means that we don't evaluate it any time. The function $x(t)$ is a position, therefore we take one derivative to get the velocity:

$$\frac{d}{dt}\, x(t) = \frac{d}{dt}\, t^3 = 3t^2 \text{ ft/s.}$$

To get the acceleration, we take another derivative:

$$\frac{d}{dt}\,(3t^2) = 6t \text{ ft}/s^2$$

2.) Since the velocity is (let's give it a name while we're at it) $v(t) = 3t^2$ then

$$v(t)\big|_{t=3} = 3(3^2) = 27 \text{ ft}/s$$

The acceleration is (here I go naming things again) $a(t) = 6t$ ft$/s^2$ and so

$$a(t = 3) = 6(3) = 18 \text{ ft}/s^2$$

3.) We need to take two derivatives of $f(t) = at$ in order to get the acceleration. The first derivative gives $\frac{d}{dt}\,(at) = a$ ft/s. Taking the next derivative, we get $\frac{d}{dt}\, a\big|_{t=2} = 0$ ft$/s^2$ Evidently, this object is cruising at a constant velocity and not accelerating.

4.) $\frac{d}{dt}\,(bt^4 + t)\big|_{t=2} = (4bt^3 + 1)\big|_{t=2} = (32b + 1)$ ft$/s$

5.) All we do is take another derivative of the velocity:

$$\frac{d}{dt}\,(4bt^3 + 1) = 4b(3t^2) = 12b^2 \text{ ft}/s^2$$

6.) First, as always, the velocity:

$$v(t) = \frac{d}{dt}\,(x_0 + st + gt^2) = s + 2gt$$

Next, the acceleration:

$$a(t) = \frac{d}{dt}\, v(t) = \frac{d}{dt}\,(s + 2gt) = 2g \text{ ft}/s^2$$

7.) Just plug-and-chug again. $x(t=5) = x_0 + 5s + 25g$ ft

8.) Take two time-derivatives to get the acceleration. The first one gives us the velocity:

$$v(t) = \frac{d}{dt}\, t^{10} = 10t^9 \text{ ft/s}$$

and the second gives the acceleration, what we wanted:

$$a(t) = \frac{d}{dt}\, 10t^9 = 10(9t^8) = 90t^8 \text{ ft}/s^2$$

9.) Here is where the hint matters. The function is already a velocity, so we only have to take one more derivative to get the acceleration.

$$\frac{d}{dt}\, t^2 = 2t \text{ ft}/s^2$$

4.4 Second Derivatives

In the last section, we found the expression for the acceleration by taking two derivatives of the position, or if you like, by taking one derivative of the velocity. All that is introduced here is a notation, nothing more. Say we have the function $x(t) = 3t^2$ in feet. To find the acceleration, we take two consecutive derivatives of this. If $v(t)$ is the velocity, then we can write it as

$$a(t) = \frac{d}{dt}\,v(t) = \frac{d}{dt}\left(\frac{d}{dt}\,x(t)\right) = \frac{d^2}{dt^2}\,x(t) \rightarrow \frac{d^2}{dt^2}\,(3t^2) = 6\text{ ft}/s^2 \tag{29}$$

Well that's it. The '2' on the derivative doesn't mean that anything is squared or something like that. It is just an index to let us know that we have to take two derivatives. You still have to go through the same process as before; take one derivative, than take another one, but at least it is a little more concisely written now. The only thing that's particularly noteworthy is that in the numerator, the d gets the index, but in the denominator, the variable gets it. Strange and senseless I know, but we're stuck with it. If we had to take a third derivative, we could write

$$\frac{d^3}{dx^3}$$

if we like. In fact, to be as general as possible, here is the basic format

$$\frac{d^n}{dx^n} f(x) = \left(\frac{d}{dx}\right)^n f(x)$$

You can write it however you like. Putting it in parenthesis is a lot less common, but completely acceptable.

More Problems:

Try your hand at solving the following. The answers are on the next page.

1.) $\dfrac{d^2}{dt^2}\,t =$

2.) $\dfrac{d^2}{dt^2}\,(\pi t^4) =$

3.) $\dfrac{d^2}{dt^2}\,(\lambda t^2 + \lambda) =$

4.) $\dfrac{d^2}{dt^2}\left(\sigma t^3 + 4\right) =$

5.) $\dfrac{d^2}{dt^2}\left(bt^5 + ct^4 + gt^3\right) =$

Answers:

1.) 0. The first derivative gives an answer of 1, and the derivative of a number is 0.

2.) $12\pi t^2$

3.) 2λ

4.) $6\sigma t$

5.) $20bt^3 + 12c^2 + 6gt$

5 The Chain Rule and Derivatives of Trigonometric Functions

Up until now, we've figured out how to take derivatives when the independent variable was either alone (i.e x), raised to a power (i.e x^2), multiplied by a constant (i.e $3x$), or some combination thereof. But what about if we wanted to find a derivative like this:

$$\frac{d}{dx}\sin x = \ ? \tag{30}$$

They're not any harder, unfortunately, I suggest that you just memorize a few for now since proving it is a bit of work and not necessary yet. I remember the derivatives of sine and cosine in a little ladder-like loop like this

$$\begin{aligned} \frac{d}{dx} \quad \sin x &= \quad \cos x & (31) \\ \frac{d}{dx} \quad \cos x &= -\sin x & (32) \\ \frac{d}{dx} -\sin x &= -\cos x & (33) \\ \frac{d}{dx} -\cos x &= \quad \sin x & (34) \end{aligned}$$

Fortunately, its pretty simple to remember once you've worked it over it a few times, and really, you only need to know the first two since the rest can be determined by moving around some minus signs. There are of course other trig functions, like tangent, cotangent, cosecant, etc... but nobody ever remembers those off hand. Everyone looks them up in a table or does them on their computer. Even better, they don't come up that often. sine and cosine are everywhere though. The others you can look up if you like, but I don't see that as being particularly important (spoken as a true physicist!). Perhaps though, I can elaborate on that by using some trig identities. Recall

$$\tan x = \frac{\sin x}{\cos x} = (\sin x)(\cos x)^{-1} \tag{35}$$

$$\csc x = \frac{1}{\sin x} = (\sin x)^{-1} \tag{36}$$

$$\sec x = \frac{1}{\cos x} = (\cos x)^{-1} \tag{37}$$

$$\cot x = \frac{1}{\tan x} = (\cos x)(\sin x)^{-1} \tag{38}$$

So really, the "other" trig functions are just combinations of sine and cosine. Aside from knowing that tangent is just sine over cosine, there is one other trig identity that you must know,

$$(\sin x)^2 + (\cos x)^2 = \sin^2 x + \cos^2 x = 1$$

Two things to note: 1) we have some liberty with how we write squares, cubes, quartics... on our trig function. As with the derivatives in parenthesis, its a lot less common that way, probably since it's more writing and a little more clutterd. 2) sine squared plus cosine squared is one. Always. That being said, solve this problem

$$\frac{d}{d\theta}\left(\sin^2\theta + \cos^2\theta\right)$$

There's a reason why I rewrote (Eq. 35-38) as a power too, rather than just a fraction. I suppose now is as good a time as any to dive into that. Take for example

$$\frac{d}{dx}(x+2)^2$$

How do we find the derivative with respect to x? We could square it out and do it that way, but for the sake of argument, suppose we didn't think of that. What if I told you that $y = x + 2$ so that we can rewrite it as

$$\frac{d}{dx}\,y^2$$

That looks a bit more familiar! BUT! This looks like it would be zero, and that cannot be right. Think 'sneaky algebra teachers.' y is actually now a function of x, $y(x) = x + 2$. So we really have

$$\frac{d}{dx}\, y(x)^2$$

Ok... What now?

5.1 The Chain Rule

Here's what to do, we work from the outside in, taking derivatives as we go. There's a rule called "The Chain Rule." In a mathematical sense, it says

$$\frac{d}{dx}\, f(g(x)) = \frac{dg}{dg}\frac{d}{dx}\, f(g(x)) = \frac{dg}{dx}\frac{d}{dg}\, f(g(x)) \tag{39}$$

where I let $f(g(x)) = f$ and $g(x) = g$ for short. If you look at the middle, all we did was take the left side and multiply it by dg/dg which is of course just 1. Then, to get to the right side, just swapped places.

We need one more tiny detail that I left out way back in the beginning. Suppose we have $y(x) = x$. What is the *differential* of each side? The differential is the infinitesimal displacement that we derived earlier. For example, dr was the infinitesimal displacement of r. All it really means then is to take the derivative of each side, but don't divide out the differentials so that they're all on one side. Just keep them where they are at. It's easier to understand by example.

Consider $y(x) = x$. What is the derivative with respect to x? You already know that it is 1. How? Because you know

$$\frac{d}{dx}\, y(x) = \frac{d}{dx}\, x = \frac{dx}{dx} = 1$$

Yes, but, we're actually doing a few steps at once. Really we found the differential of each side first by sticking the d in front

$$d(y(x)) = d(x)$$

Then we divide both sides by the differential, dx, to get what we already know

$$\frac{dy(x)}{dx} = 1$$

To be sure, we can do the same with $f(x) = x^2$. In this case

$$d(f(x)) = d(x^2) = df(x) = 2x\ dx$$

Diving the differential on both sides gives

$$\frac{d}{dx}\ x^2 = 2x$$

Now we can move on.

Let's see it some more in practice. Consider $(x+2)^2$. Set $y(x) = x + 2$. Then we can rewrite this as

$$(x+2)^2 = y(x)^2$$

So far so good. Now we take the derivative. Using the chain rule, we can see that

$$\frac{d}{dx} = \frac{dy(x)}{dx}\frac{d}{dy(x)} \tag{40}$$

right? All I did was multiply by 1, or rather, $dy(x)/dy(x)$. Ok cool. So,

$$\frac{d}{dx}\ y(x)^2 = \frac{dy(x)}{dx}\frac{d}{dy(x)}\ y(x)^2 \tag{41}$$

You with me so far? Now we're taking a derivative of $y(x)$ with respect to $y(x)$. Perfect! Onward we go,

$$\frac{dy(x)}{dx}\frac{d}{dy(x)}\ y(x)^2 = \frac{dy(x)}{dx}\ 2y(x) \tag{42}$$

Still hanging in there? Good. We're pretty much done. Actually, we are done, but we got the problem handed to us in terms of x, so we should put it back in that form. We know that $y(x) = x + 2$, so that part is done. All we need now is $dy(x)/dx$. Easy!

$$\frac{dy(x)}{dx} = \frac{d}{dx}\ (x+2) = 1 \tag{43}$$

All that is left now is to put the pieces together.

$$\frac{dy(x)}{dx}\frac{d}{dy(x)}\,y(x)^2 = \frac{dy(x)}{dx}\,2y(x) = 1 \times 2(x+2) = 2(x+2) \tag{44}$$

We're done! Finally! The first one is the worst one, I promise. It looks like a lot more work, but once you see the pattern, you'll see that they're really no different than what we've been doing. Remember the pattern:

$$\frac{d}{dx}\,x^n = n\,x^{n-1}$$

Of course you do. What we did was bring down the power to the front and then subtract one. Well look at what we just did

$$\frac{d}{dx}\,(x+2)^2 = 2(x+2)$$

See it? We brought the exponent down to the front, subtracted one from it, THEN took the derivative of the inside, which was 1. So we took derivatives 'from the outside in.' First worry about the exponent and differentiate as if the inside we just a regular variable, then differentiate the inside for what it is. Let's look at another one. I'll do this one for you. Just notice the pattern. Forget the math. Forget the symbols. Try and catch the pattern

$$\frac{d}{dx}\,(x-1)^3 = 3(x-1)^2 \tag{45}$$

Exactly as you'd expect, right? The exponent came to the front and then one was subtracted from the power. How about another:

$$\frac{d}{dx}\,(x+\xi)^5 = 5(x+\xi)^4 \tag{46}$$

Now you've got it! Two more to be sure

$$\frac{d}{dx}\,(x+r)^{3.5} = 3.5(x+r)^{2.5} \tag{47}$$

Yup, same old, same old. One more

$$\frac{d}{dx}\ (2x+1)^2 = 4(2x+1) \tag{48}$$

Hmm, what happened this time? Remember, we take derivatives from the outside in. So first we treat $(2x+1)^2$ just as if it were x^2 and get back what we'd expect, $2(2x+1)$. And now we have to take the derivative of the inside! (the dy/dx is still lingering there, we're just not writing it. Strictly speaking, this should be written as $2(2x+1)\times dy/dx$) You know as well as I do now that the derivative of $(2x+1)$ with respect to x is 2. Therefore answer is

$$\frac{d}{dx}\ (2x+1)^2 = \frac{dy}{dx}\ 2(2x+1) = 2\times 2(2x+1) = 4(2x+1) \tag{49}$$

If you don't believe me, square it out and then take the derivative

$$\frac{d}{dx}\ (2x+1)^2 = \frac{d}{dx}\ (4x^2+4x+1) = 4\frac{d}{dx}\ x^2 + 4\frac{dx}{dx} + 0 = 8x+4 = 4(2x+1) \tag{50}$$

Alright let's do another one.

$$\frac{d}{dx}\ (4x-9)^2 \tag{51}$$

What do we do? 'Outside in,' right. First it becomes $2(4x-9)$. Then what? The derivative of the inside, exactly. Since the derivative of the inside, $(4x-9)$, with respect to x is just 4, then the answer must be

$$\frac{d}{dx}\ (4x-9)^2 = 4\times 2(4x-9) = 8(4x-9) \tag{52}$$

Right you are. We'll do two more before I turn you loose.

$$\frac{d}{dx}\ (nx+2)^2 \tag{53}$$

Don't be thrown by the n. It's no different than any other number that we've already used. Going from the outside in we get first $2(nx+2)$. Then take the derivative of the inside, which is simply n, and multiply them together Thus,

$$\frac{d}{dx}\,(nx+2)^2 = 2n(nx+2) \tag{54}$$

And lastly

$$\frac{d}{dx}\,(x^2+2)^2 = \tag{55}$$

Proceed as usual. First the exponent comes down and we subtract 1 from it, leaving us with $2(x^2+2)$ (Don't forget about the 'invisible' dy/dx). Next, take the derivative of the inside. Since

$$\frac{d}{dx}\,(x^2+2) = 2x$$

Then the final result is

$$\frac{d}{dx}\,(x^2+2)^2 = 2x \times 2(x^2+2) = 4x(x^2+2) \tag{56}$$

I did play a little fast and loose with the math here, so if anything isn't clear or you'd like a better description, let me know. Ok. Your turn...

Problems:

Solve:

1.) $\frac{d}{dx}\,(x+7)^3 =$

2.) $\frac{d}{dx}\,(2x+2)^3 =$

3.) $\frac{d}{dx}\,(-x+1)^2 =$

4.) $\frac{d}{dx}\,(-2x-2)^3 =$

5.) $\frac{d}{dx}\,(x^2+1)^2 =$

Answers:

1.) Doing the outside we get $3(x+7)^2$. Since the derivative of $(x+7)$ with respect to x is just 1, then the answer is

$$\frac{d}{dx}\,(x+7)^3 = 3(x+7)^2$$

2.) Following the same procedure we get first $3(2x+2)^2$ from doing the outside derivative. The inside derivative gives a 2 this time. So,

$$\frac{d}{dx}\,(2x+2)^3 = 6(2x+2)^2$$

3.) The outer derivative gives $2(-x+1)$. The inside derivative is just -1. Therefore we get

$$\frac{d}{dx}\,(-x+1)^2 = -2(-x+1)$$

4.) Doing the outside part we get $3(-2x-2)^2$. The inside part will work out to be -2. So the answer is

$$\frac{d}{dx}\,(-2x-2)^3 = -6(-2x-2)^2$$

5.) The exponential part is $2(x^2+1)$. The derivative of (x^2+1) with respect to x is $2x$. Putting it all together we find

$$\frac{d}{dx}\,(x^2+1)^2 = 4x(x^2+1)$$

5.2 Including Trig Functions

At long last we can do the trig functions. I already gave you the list of the ones you should memorize, but here it is again.

$$\frac{d}{dx} \sin x = \cos x \tag{57}$$
$$\frac{d}{dx} \cos x = -\sin x \tag{58}$$
$$\frac{d}{dx} -\sin x = -\cos x \tag{59}$$
$$\frac{d}{dx} -\cos x = \sin x \tag{60}$$

First, let's try

$$\frac{d}{dx}\frac{1}{\sin x} = \frac{d}{dx}(\sin x)^{-1} = \ ? \tag{61}$$

This is just the same as the problems done in the last section. I'll do this one in all of it's gory detail but then we'll catch the pattern and rip through them after that. We first write $y(x) = \sin x$. So now we have

$$\frac{d}{dx}(y(x))^{-1} = \ ? \tag{62}$$

Ringing some bells? Next, rewrite the derivative using the chain rule

$$\frac{d}{dx} y(x)^{-1} = \frac{dy(x)}{dx}\frac{d}{dy(x)} y(x)^{-1} = \tag{63}$$

We have two chunks to evaluate again. First,

$$\frac{d}{dy(x)} y(x)^{-1} = -1y(x)^{-2} \tag{64}$$

Same rules applied there as always. The exponent comes down, then we subtract one from it. The part that is left is

$$\frac{d}{dx}\ y(x) = \frac{d}{dx}\ \sin x = \cos x \tag{65}$$

thanks to the trig derivative ladder. Putting it all together

$$\frac{d}{dx}\ (\sin x)^{-1} = \frac{d}{dx}\ y(x)^{-1} = \frac{dy(x)}{dx}\frac{d}{dy(x)}\ y(x)^{-1} = -1y(x)^{-2}\cos x = -(\sin x)^{-2}(\cos x) \tag{66}$$

Not too bad. I prefer to work in powers rather than fractions as you can tell; they're easier. Real quick, how would you do

$$\frac{d}{dx}\ \sqrt{x}$$

If you're like me, write it as a power. Then it's a breeze.

$$\frac{d}{dx}\ \sqrt{x} = \frac{d}{dx}\ x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

Ok, enough of that. How about

$$\frac{d}{dx}\ (\cos x + 2)^2 \tag{67}$$

We did ones just like this, except now, $x \to \cos x$. The operation is identical to what you already know. Do the 'outside' part, then do the 'inside' part. So first we get

$$\frac{d}{dx}\ (\cos x + 2)^2 = 2(\cos x + 2)\frac{dy}{dx} \tag{68}$$

Now we do the inside part (dy/dx).

$$\frac{d}{dx}\ (\cos x + 2) = -\sin x \tag{69}$$

Assembling the pieces

$$\frac{d}{dx}\ (\cos x + 2)^2 = 2(\cos x + 2)\frac{dy}{dx} = 2(\cos x + 2)(-\sin x) = -2(\cos x + 2)\sin x \tag{70}$$

How about one more?

$$\frac{d}{dx}\ (-3\sin x + 1)^3 \tag{71}$$

Just turn the crank. First we do the 'outer' derivative and get $3(-3\sin x + 1)^2$. The derivative of the inside is

$$\frac{d}{dx}\ (-3\sin x + 1) = -3\frac{d}{dx}\ \sin x = -3\cos x$$

So,

$$\frac{d}{dx}\ (-3\sin x + 1)^3 = 3(-3\sin x + 1)^2(-3\cos x) = -9(-3\sin x + 1)^2\cos x \tag{72}$$

There really isn't much more to it. It's just a matter of memorizing that ladder and remembering 'outside to inside.' There are a few more things I want to cover real quick. The first is, what if the argument of a trig function isn't alone? Take for instance,

$$f(x) = \sin ax$$

How do we find the derivative? I'll show you the long way (the mathematically proper way), but then you'll see the pattern, and the proper stuff can fly out the window. So let's start by being be proper:

Start by using substitution to turn the problem into one we've seen before. We could say $ax \equiv \theta$ for example (three lines just means "defined to be". Its like the equals sign, but a little stronger). Then we could change $\sin ax \to \sin\theta$. That's good. We have to change the function too. It would change to $f(x) \to f(\theta/a)$, right? Right. There's one more crucial thing though. Let's write out what we have so far and see if you can spot the problem.

$$\frac{d}{dx}\ f(x) = \frac{d}{dx}\ \sin ax \to \frac{d}{dx}\ f(\theta/a) = \frac{d}{dx}\ \sin\theta \tag{73}$$

See it? It appears that now the answer should be zero since we're taking a derivative with respect to x but the only variable around is θ! Whoops! We forgot to change the differential dx to be in terms of θ when we were changing everything else. Since $\theta \equiv ax$ then

$$d(\theta) = d(ax) = d\theta = a\ dx \tag{74}$$

And therefore, $dx = d\theta/a$. Ah-ha! That means that

$$\frac{d}{dx} \rightarrow \frac{d}{(d\theta/a)} = a\frac{d}{d\theta} \tag{75}$$

This turns our problem into the following:

$$\frac{d}{dx}\ f(x) = \frac{d}{dx}\ \sin ax = a\frac{d}{d\theta}\ f(\theta/a) = a\frac{d}{d\theta}\ \sin\theta = a\cos\theta = a\cos ax \tag{76}$$

Comparing the two first terms and last term

$$\frac{d}{dx}\ f(x) = \frac{d}{dx}\ \sin ax = a\cos ax \tag{77}$$

And that's that. We can do it (still using the chain rule) in a perhaps more step-by-step oriented way. Remember that we can always multiply something by 1 without changing anything. That means that

$$\frac{d}{dx} = \frac{d\theta}{d\theta}\ \frac{d}{dx} = \frac{d\theta}{dx}\ \frac{d}{d\theta} \tag{78}$$

The $d\theta$'s still cancel, so we haven't done anything illegal. If we change the rest of the equation to be in terms of θ as well, that is, $f(x) \rightarrow f(\theta/a)$ and $\sin ax \rightarrow \sin\theta$, then we have the following

$$\frac{d\theta}{dx}\ \frac{d}{d\theta}\ f(\theta/a) = \frac{d\theta}{dx}\ \frac{d}{d\theta}\ \sin\theta = \frac{d\theta}{dx}\ \cos\theta \tag{79}$$

All we need now is $d\theta/dx = a$ which is just what we did in (Eq. 74). Therefore we have

$$\frac{d\theta}{dx}\frac{d}{d\theta}\, f(\theta/a) = \frac{d\theta}{dx}\frac{d}{d\theta}\, \sin\theta = \frac{d\theta}{dx}\, \cos\theta = a\cos\theta \tag{80}$$

Putting it back in terms of x, we indeed get the same result. It's just two different ways of looking at the same process. Let's do another one. Find the derivative of $\cos bx$. No sweat. Let $\theta \equiv bx$, then

$$\frac{d}{dx}\, \cos bx = \frac{d\theta}{dx}\frac{d}{d\theta}\, \cos\theta = \frac{d\theta}{dx}\, (-\sin\theta) = b(-\sin\theta) = -b\sin bx \tag{81}$$

Seeing the pattern? Quick, what is the derivative of $\sin 2x$? Use the pattern to your advantage. Look at what we have so far

$$\frac{d}{dx}\, \sin ax = a\cos ax \tag{82}$$
$$\frac{d}{dx}\, \cos bx = -b\sin bx \tag{83}$$

By virtue of the pattern, we'd guess that

$$\frac{d}{dx}\, \sin 2x = 2\cos 2x \tag{84}$$

and we'd be right. You could just set $a = 2$ in (Eq. 82). It's still the "outside to inside" business. First take the derivative of the function like nothing new is going on, then take the derivative of the inside. Multiply them together and wah-lah! So what is the derivative of $\cos\pi x$? The derivative of cosine is negative sine, and the derivative of the inside (πx) is just π. So the answer is

$$\frac{d}{dx}\, \cos\pi x = -\pi\sin\pi x \tag{85}$$

Two more of these. Find the derivative of $-\cos\alpha x$. The derivative of negative cosine is positive sine, and the derivative of the inside (αx) is just α. So

$$\frac{d}{dx}\ -\cos\alpha x = \alpha\sin\alpha x \tag{86}$$

Last but not least, find the derivative of $\sin\left(\frac{n\pi x}{l}\right)$. If it makes you more comfortable, call $\frac{n\pi}{l}$ a new variable, like a or something. The derivative of sine is cosine, the derivative of the inside is $\frac{n\pi}{l}$ so we have

$$\frac{d}{dx}\ \sin\left(\frac{n\pi x}{l}\right) = \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right) \tag{87}$$

It wouldn't matter what the coefficient of the argument is, the procedure doesn't change, even if the coefficient were a function of the independent variable. Try out

$$\frac{d}{dx}\ \sin(\cos x) \tag{88}$$

I'll do it both ways again, the proper way, and the "outside to inside" down 'n dirty way. Rename $\cos x = f(x)$ so then $\sin(\cos x) = \sin(f(x))$. Now use the chain rule to write the derivative in terms of $f(x)$ rather than just x.

$$\frac{d}{dx} = \frac{df(x)}{df(x)}\frac{d}{dx} = \frac{df(x)}{dx}\frac{d}{df(x)} \tag{89}$$

Now plug in the function

$$\frac{df(x)}{dx}\frac{d}{df(x)}\sin(f(x)) = \frac{df(x)}{dx}\cos(f(x)) \tag{90}$$

Now we need the $df(x)/dx$ part. Since $f(x) = \cos x$, then

$$\frac{d}{dx}\ f(x) = \frac{d}{dx}\ \cos x = -\sin x \tag{91}$$

Assembling the pieces,

$$\frac{df(x)}{dx}\frac{d}{df(x)}\sin(f(x)) = \frac{df(x)}{dx}\cos(f(x)) = -\sin x\cos(f(x)) = -\sin x\cos(\cos x) \qquad (92)$$

And we're done. Now let's do it the easy way. Look at the function as if it were 'sin(stuff)'. The outside derivative spits back 'cos(stuff)'. Now take the derivative of the inside, (stuff), and multiply them together. Since 'stuff=$\cos x$,' then the derivative of 'stuff' is $-\sin x$. Sticking it all together,

$$\frac{d}{dx}\sin(\cos x) = -\sin x\cos(\cos x) \qquad (93)$$

The best way to learn is to just do it. So here you go!

Problems:

Solve: (-hint- pay attention to the inside of #3 and #4)

1.) $\frac{d}{dx}(4\sin x + 2)^3$

2.) $\frac{d}{dx}(\cos x - 1)^6$

3.) $\frac{d}{dx}(\sin x + x)$

4.) $\frac{d}{dx}(\sin x + x)^2$

5.) $\frac{d}{dx}\sin 4\pi x$

6.) $\frac{d}{dx}\sin(\sin x)$

Answers:

1.) $\dfrac{d}{dx}\,(4\sin x+2)^3 = \dfrac{dy}{dx}(3(4\sin x+2)^2) = (4\cos x)(3(4\sin x+2)^2)$

$$= 12(4\sin x+2)^2\cos x$$

2.) $\dfrac{d}{dx}\,(\cos x-1)^6 = \dfrac{dy}{dx}(6(\cos x-1)^5) = (-\sin x)(6(\cos x-1)^5) = -6(\cos x-1)^5\sin x$

3.) $\dfrac{d}{dx}\,(\sin x+x) = \dfrac{d}{dx}\,\sin x + \dfrac{d}{dx}\,x = \cos x + 1$

4.) $\dfrac{d}{dx}\,(\sin x+x)^2 = \dfrac{dy}{dx}(2(\sin x+x)) = (\cos x+1)(2(\sin x+x)) = 2(\sin x+x)(\cos x+1)$

5.) $\dfrac{d}{dx}\,\sin 4\pi x = 4\pi\cos 4\pi x$

6.) Define $g(x)\equiv\sin x$. Then,

$\dfrac{d}{dx}\,\sin(\sin x) = \dfrac{d}{dx}\,\sin(g(x)) = \dfrac{dg(x)}{dx}\dfrac{d}{dg(x)}\,\sin(g(x)) = \dfrac{dg(x)}{dx}\cos(g(x))$

And since

$\dfrac{dg(x)}{dx} = \dfrac{d}{dx}\,\sin x = \cos x$, then the answer is

$\dfrac{d}{dx}\,\sin(\sin x) = \cos x\cos(\sin x)$

6 Exponents and Natural Logs

Now that you've mastered trig functions and polynomials, let's knock out the rest of the 'standard functions.' The ones we have left are exponents and natural logs.

First, the exponents. The most basic function would be like $y(x)=e^x$. The goal now is to find the derivative. It's the easiest one you'll ever do. And in physics and engineering, it's probably the most common too. Sweet. Brace yourself.

$$\frac{d}{dx}\,e^x = e^x \tag{94}$$

That's right. Nothing happens. Sometimes it is called "the indestructible function" since you could take derivatives of it all day but it would always be the same. We can spice it up a little bit though.

$$\frac{d}{dx}\, e^{ax} = ae^{ax} \tag{95}$$

Remember that a can be anything as long as it has no x dependence. We could set $a = \cos y$ or $a = e^y$ or some massive mess of whatever we can cook up. It could also be imaginary which is not a trivial aside. It turns out that

$$e^{ix} = \cos x + i \sin x \tag{96}$$

where i is imaginary. So if you know your exponential rules, you can bypass a hell of a lot of trig. We'll worry about that later, but it's going to be indispensable. There is one more configuration that shows up quite a bit in exponents, and that is if we had something like

$$f(x) = e^{x^2}$$

or really any function of our independent variable stuffed up there. To handle these, we turn back to the chain rule, as always. Let's rename $x^2 \equiv g(x)$. Then we have

$$f(x) = e^{g(x)}$$

Using the chain rule we can write

$$\frac{d}{dx}\, e^{x^2} = \frac{d}{dx}\, e^{g(x)} = \frac{dg(x)}{dx} \frac{d}{dg(x)}\, e^{g(x)} \tag{97}$$

Now we just proceed as usual

$$\frac{dg(x)}{dx} \frac{d}{dg(x)}\, e^{g(x)} = \frac{dg(x)}{dx} e^{g(x)} \tag{98}$$

And since

$$\frac{dg(x)}{dx} = \frac{d}{dx}\, x^2 = 2x \tag{99}$$

then the final result will be

$$\frac{d}{dx}\ e^{x^2} = \frac{dg(x)}{dx}\frac{d}{dg(x)}\ e^{g(x)} = \frac{dg(x)}{dx}e^{g(x)} = 2xe^{g(x)} = 2xe^{x^2} \tag{100}$$

or more succinctly

$$\frac{d}{dx}\ e^{x^2} = 2xe^{x^2} \tag{101}$$

We'll do one more of these types.

$$\frac{d}{dx}\ e^{\cos x} \tag{102}$$

First, rename $\cos x = \lambda(x)$ or whatever name you like. Then use the chain rule to rewrite it in a more obvious form

$$\frac{d}{dx}\ e^{\cos x} = \frac{d}{dx}\ e^{\lambda(x)} = \frac{d\lambda(x)}{dx}\frac{d}{d\lambda(x)}e^{\lambda(x)} = \frac{d\lambda(x)}{dx}e^{\lambda(x)} \tag{103}$$

And since

$$\frac{d\lambda(x)}{dx} = \frac{d}{dx}\ \cos x = -\sin x \tag{104}$$

Then the result is

$$\frac{d}{dx}\ e^{\cos x} = -\sin x\ e^{\lambda(x)} = -\sin x\ e^{\cos x} \tag{105}$$

Since exponents are "indestructible," then it should always reappear at the end *as it was* initially. We started with $e^{\cos x}$ and ended with it too, as we ought too. If you find that you have lost that original function, something has gone wrong.

The next function we need to deal with is natural logs, like $y(x) = \ln(x)$. It's derivative is another one to commit to memory. They come up a fair amount. The rule is

$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{106}$$

We could spice this one up a fuzz too. Before we do, a few identities

$$e^a e^b = e^{a+b} \tag{107}$$
$$e^{ax} e^{bx} = e^{x(a+b)} \tag{108}$$
$$(e^a)^b = e^{ab} \tag{109}$$
$$e^{\ln x} = x \tag{110}$$
$$\ln(ab) = \ln a + \ln b \tag{111}$$
$$\ln(a/b) = \ln a - \ln b \tag{112}$$
$$\ln(x^n) = n \ln x \tag{113}$$

Understanding these, let's try to determine

$$\frac{d}{dx} \ln(3x)$$

Using (Eq. 111) we can rewrite it as

$$\frac{d}{dx} \ln(3x) = \frac{d}{dx} (\ln x + \ln 3) = \frac{d}{dx} \ln x + 0 = \frac{1}{x}$$

Turns out this one is pretty easy too. What if we had

$$\frac{d}{dx} \ln(x^2)$$

Use the identity (Eq. 113) so that you can write

$$\frac{d}{dx} \ln(x^2) = \frac{d}{dx} 2 \ln x = \frac{2}{x}$$

You're practically bored again, aren't you? Let's press on.

7 The Product Rule and The Quotient Rule

When two or more functions of the independent variable exist as either being multipled by each other or divided by each other, we must find an alternative way to take derivatives since we can't do it in one fell swoop. To take a derivative when functions of the independent variable are multiplied by each other [i.e $f(x)g(x)$], we use something called the product rule. When these functions are diving each other [i.e $f(x)/g(x)$], we use the quotient rule.

7.1 The Product Rule

We've already touched on the product rule earlier. Remember the mantra, "first times the derivative of the second, plus second times the derivative of the first." It's extraordinarily handy. For, suppose we wanted to take a derivative of

$$y(x) = xe^x$$

What do we do first? Sing the mantra, then apply it. I'm calling x 'first' and e^x 'second' here.

$$\frac{d}{dx}\left(xe^x\right) = x\,\frac{d}{dx}\,e^x + e^x\,\frac{d}{dx}\,x = xe^x + e^x = e^x(x+1) \tag{114}$$

Easy enough. How about

$$y(x) = x\sin x$$

Are you singing yet? "First times the derivative of the second, plus second times the derivative of the first."

$$\frac{d}{dx}\left(x\sin x\right) = x\,\frac{d}{dx}\,\sin x + \sin x\,\frac{d}{dx}\,x = x\cos x + \sin x \tag{115}$$

Still not too bad, right? Let's do

$$y(x) = \cos x \sin x$$

Nothing new, so turn the crank...

$$\frac{d}{dx}\left(\cos x\sin x\right) = \cos x\,\frac{d}{dx}\,\sin x + \sin x\,\frac{d}{dx}\,\cos x = (\cos x)^2 - (\sin x)^2 = \cos^2 x - \sin^2 x$$

If there were more than two functions of x in there, we can keep using the product rule to break them apart in the chunks of 2. For example, let

$$y(x) = xe^x \sin x$$

So here's what we do: divide the three functions of x into a group of 1 and a group of 2. I'll pick $y(x) = (xe^x)(\sin x)$ arbitrarily. I now have a group of two functions of x and another one all by itself. Let's use the product rule on the two *groups*.

$$\frac{d}{dx}\ \{(xe^x)(\sin x)\} = (xe^x)\ \frac{d}{dx}\ \sin x + \sin x\ \frac{d}{dx}\ (xe^x) = T1 + T2 \tag{116}$$

where I labeled the two terms $T1$ and $T2$ respectively solely out of convenience. See what's going on here? It's just like the rest of the ones we did. We can already finish the $T1$ part,

$$T1 = (xe^x)\ \frac{d}{dx}\ \sin x = (xe^x) \cos x \tag{117}$$

As for the $T2$ part, we have to use the product rule again,

$$T2 = \sin x\ \frac{d}{dx}\ (xe^x) = \sin x \left(x\ \frac{d}{dx}\ e^x + e^x\ \frac{d}{dx}\ x \right) = \sin x\,(xe^x + e^x) = e^x \sin x(x+1) \tag{118}$$

And so at long last adding $T1 + T2$ gives us the final result

$$\frac{d}{dx}\ xe^x \sin x = e^x \sin x(x+1) + (xe^x) \cos x \tag{119}$$

If you like, you could simplify it as you see fit, but that's the answer. Say it proud! You've come a long way in just a few pages.

One final problem. Let

$$y(x) = (x+4)^2(2x+1)$$

Sing the mantra and go to town,

$$\frac{d}{dx}\,(x+4)^2(2x+1) = (x+4)^2\,\frac{d}{dx}\,(2x+1) + (2x+1)\,\frac{d}{dx}\,(x+4)^2 = T1 + T2 \tag{120}$$

Looking at the two parts individually we see

$$T1 = (x+4)^2\,\frac{d}{dx}\,(2x+1) = 2(x+4)^2 \tag{121}$$

$$T2 = (2x+1)\,\frac{d}{dx}\,(x+4)^2 = (2x+1)(2(x+4)) = 2(2x+1)(x+4) \tag{122}$$

And so finally

$$\frac{d}{dx}\,(x+4)^2(2x+1) = T1 + T2 = 2(x+4)^2 + 2(2x+1)(x+4) \tag{123}$$

Again, you can simplify it if you want, but the problem is completely solved, so we're as done as we need to be.

7.2 The Quotient Rule

Suppose that we have

$$y(x) = \frac{x+1}{x+4}$$

and we'd like to find the derivative. What do we do? "Well," you start, "I know that I can rewrite it as

$$y(x) = (x+1)(x+4)^{-1}$$

so I can just use the product rule and bypass learning a new rule altogether."

Indeed, sir! The need for a quotient rule can always be turned into a need for the product rule. And, the product rule is almost always a lot less work. So forget it, we don't have to bother with this new rule. It's a waste of time. Let's do the example the way you said we should do it, just for fun.

$$\frac{d}{dx}\ (x+1)(x+4)^{-1} = (x+1)\ \frac{d}{dx}\ (x+4)^{-1} + (x+4)^{-1}\ \frac{d}{dx}(x+1) = T1 + T2 \tag{124}$$

$$T1 = (x+1)\ \frac{d}{dx}(x+4)^{-1} = (x+1)(-1(x+4)^{-2}) = -(x+1)(x+4)^{-2} \tag{125}$$

$$T2 = (x+4)^{-1}\ \frac{d}{dx}(x+1) = (x+4)^{-1} \tag{126}$$

And thus the answer is

$$\frac{d}{dx}\ (x+1)(x+4)^{-1} = -(x+1)(x+4)^{-2} + (x+4)^{-1} \tag{127}$$

7.3 Recap

First of all, congratulations to you! At this point, you know everything that you need to know in order to figure out any kind of derivative you may come across, no matter how ugly. It's just a matter of breaking the problem into small, workable steps so that it looks like problems you've seen before. One little anecdote to make that point:

There's a blueprint of a kitchen and on the west wall, there is a sink, on the east wall there is a stove with an empty teapot on it, and directly in the center of the room is a table.

A mathematician is asked, "How would you go about heating up a teapot full of water?"

He replies, "I'd move the teapot to the table, then to the sink to fill it, back to the table and then put it on the stove."

"Ok then. What would you do if the teapot were on the table first instead of on the stove?"

The mathematician says, "Move the teapot to the stove, then to the table, then to the sink to fill it, back to the table and then to the stove."

"But why put the pot back on the stove if its already on the table?" asks the man.

"So I could turn it into a problem I've seen before," replies the cheeky mathematician.

Maybe not the best story in the world, but it sums up the idea. You have all the necessary tools, so no matter how something may look, with sufficient cleverness you can breeze through it. In spirit of that, I've thrown together a list of problems covering everything that we've done so far. There won't be anything new, no tricks, no special cases, just straight forward problems with a straight forward "turn the crank" procedure. There is an initial "hump" that one has to get over before the math becomes just a tool to solve cool physics and engineering problems. Initially, it's just math, which sucks, I know. But you're damn near there. We've done a few physics problems, and we'll do many more. Soon, very soon. With that, away we go...

8 Problems, Problems, and More Problems

"If I had six hours to cut down a tree, I'd spend the first four sharpening my axe." -Honest Abe

1. $\frac{d}{dx}\, 5$
2. $\frac{d}{dx}\, n$
3. $\frac{d}{dx}\, x$
4. $\frac{d}{dx}\, 3x$
5. $\frac{d}{dx}\, (x^2 + x)$
6. $\frac{d}{dx}\, (2x^2 + 4x + 7)$
7. $\frac{d}{dx}\, (x^3 + \frac{1}{2}x^2)$
8. $\frac{d}{dx}\, g(z)$
9. $\frac{d}{dx}\, \frac{1}{50}x^{50}$
10. $\frac{d}{dx}\, (\Lambda(\theta, \phi) + \epsilon^{\alpha\beta\sigma\delta})$
11. $\frac{d}{dz}\, (3z^3 + z)$
12. $\frac{d}{d\theta}\, \sin\theta$
13. $\frac{d}{d\xi}\, x^2\xi^2$
14. $\frac{d}{dr(s)}\, r(s)$
15. $\frac{d}{dr(s)}\, 2r(s)^2$
16. $\frac{d}{d(milk)}\, \big(3(milk)^2 + 2(milk)\big)$
17. $\frac{d}{dx}\, \cos x$
18. $\frac{d}{dx}\, \sin x$
19. $\frac{d}{dx}\, (x+1)^2$
20. $\frac{d}{dr}\, \big(x^x \sin x \ln(e^{\tanh x})\big)$
21. $\frac{d}{dx}\, (3x+5)^2$
22. $\frac{d}{dx}\, (x^2)^2$
23. $\frac{d}{dx}\, (x^2+1)^2$
24. $\frac{d}{dm}\, \sin m$
25. $\frac{d}{d\theta}\, \frac{1}{\sin\theta}$
26. $\frac{d}{d\theta}\, (\cos\theta)^{-1}$
27. $\frac{d}{dx}\, \ln(x)$
28. $\frac{d}{dt}\, \left(st + gt^2 + \frac{3}{4}\right)$
29. $\frac{d}{dt}\, (\sin t + \pi)^2$
30. $\frac{d}{dt}\, (\cos t + \pi)^2$

31. $\dfrac{d}{dx}\ x\sin x$

32. $\dfrac{d}{dx}\ x\ln(x)$

33. $\dfrac{d}{dx}\ \sin x\cos x$

34. $\dfrac{d}{dx}\ xe^x$

35. $\dfrac{d}{d\alpha}\ \alpha e^\alpha$

36. $\dfrac{d}{dx}\ e^x\sin x$

37. $\dfrac{d}{dx}\ e^x\cos x$

38. $\dfrac{d}{dx}\ (x^5+x^4+x^3+x^2+x)$

39. $\dfrac{d}{dx}\ (x+3)(x+2)$

40. $\dfrac{d}{dx}\ (x+3)^2(x+2)$

41. $\dfrac{d}{dx}\ (x+3)^3(x+2)^2$

42. $\dfrac{d}{dx}\ (\sin x+x)^2$

43. $\dfrac{d}{dx}\ (\ln(x)+x)^2$

44. $\dfrac{d}{dx}\ (\cos x+x)^2$

45. $\dfrac{d}{dx}\ (\sin x+\cos x)^2$

46. $\dfrac{d}{dx}\ (\sin^2 x+\cos^2 x)$

47. $\dfrac{d}{dx}\ \left(\dfrac{x+2}{x+3}\right)$

48. $\dfrac{d}{dx}\ \dfrac{1}{(x+1)^2}$

49. $\dfrac{d}{dx}\ \left(\dfrac{\sin x}{x+1}\right)$

50. $\dfrac{d}{dx}\ \left(\dfrac{x+1}{\sin x}\right)$

51. $\dfrac{d}{dx}\ \dfrac{\sin x}{\cos x}$

52. $\dfrac{d}{dx}\ \dfrac{1}{\ln(x)}$

53. $\dfrac{d}{dx}\ \ln(x^5)$

54. $\dfrac{d}{dx}\ e^{\ln(x)}$

55. $\dfrac{d}{dx}\ \dfrac{(x+1)^2}{(x+2)^2}$

56. $\dfrac{d}{dx}\ \left(\dfrac{x+1}{x+2}\right)^2$

57. $\dfrac{d}{dx}\ xe^x\sin x$

58. $\dfrac{d}{dx}\ xe^x\cos x$

59. $\dfrac{d}{dx}\ x^2e^x$

60. $\dfrac{d}{dx}\ x^2(x+6)$

61. $\dfrac{d}{dx}\ x^2(x+4)^2$

62. $\dfrac{d}{dx}\ x^2(x^2+\beta)$

63. $\dfrac{d}{dx}\ x^2e^x\sin x$

64. $\dfrac{d}{dx}\ \left(\dfrac{x^2e^x}{\sin x}\right)$

65. $\frac{d}{dx}\ (xe^x + 1)^2$

66. $\frac{d}{dx}\ (xe^x + x)^2$

67. $\frac{d}{dx}\ \sin 3x$

68. $\frac{d}{dx}\ \cos 4x$

69. $\frac{d}{dx}\ (\sin x)(\cos \pi x)$

70. $\frac{d}{dx}\ \sin\left(\frac{n\pi x}{l}\right)$

71. $\frac{d}{dx}\ (x \sin x + a)^2$

72. $\frac{d}{d\theta}\ \frac{1}{\sin\theta}$

73. $\frac{d}{dt}\ (x_0 + vt + \frac{1}{2}gt^2)$

74. $\frac{d}{dx}\ e^{3x}$

75. $\frac{d}{dx}\ e^{x^2}$

76. $\frac{d}{dx}\ (\sqrt{x})^3$

(hint: $(x^a)^b = x^{ab}$)

Answers:

These are going to be answers and not solutions since there are so many. If you get hung up on some, or if I screwed up something, let me know and we'll sort it out. Also, I tried not to simplify things so that if you do get stuck, you may see why,, but it's likely that our answers, although the same, won't look identical. Try to mess around with them and see if you can determine if we got the same answer. If not, you know where to find me. Sorry for the whirlwind of parenthesis.

1. 0
2. 0
3. 1
4. 3
5. $2x+1$
6. $4x+4$
7. $3x^2+x$
8. 0
9. x^{49}
10. 0
11. $9z^2+1$
12. $\cos\theta$
13. $2x^2\xi$
14. 1
15. $4r(s)$
16. $6(milk)+2$
17. $-\sin x$
18. $\cos x$
19. $2(x+1)$
20. 0
21. $6(3x+5)$
22. $4x^3$ (powers add, so $(x^2)^2 = x^4$)
23. $4x(x^2+1)$
24. $\cos m$
25. $-\cos\theta(\sin\theta)^{-2}$
26. $(\cos\theta)^{(}-2)\sin\theta$
27. $\dfrac{1}{x}$
28. $s+2gt$
29. $2\cos t(\sin t+\pi)$
30. $-2\sin t(\cos t+\pi)$
31. $x\cos x+\sin x$
32. $1+\ln(x)$
33. $-\sin^2 x+\cos^2 x$
34. xe^x+e^x
35. $\alpha e^\alpha+e^\alpha$
36. $e^x\cos x+\sin xe^x$
37. $-\sin xe^x+\cos xe^x$
38. $5x^4+4x^3+3x^2+2x+1$
39. $(x+3)+(x+2)=2x+5$
40. $(x+3)^2+(x+2)3(x+3)$

41. $(x+3)^3 2(x+2)+(x+2)3(x+3)^2$

42. $2(\sin x+x)(\cos x+1)$

43. $2(\ln(x)+x)(\frac{1}{x}+1)$

44. $2(\cos x+x)(-\sin x+1)$

45. $2(\sin x+\cos x)(\cos x-\sin x)$

46. 0

47. $(x+2)(-1(x+3)^{-2})+(x+3)^{-1}$

48. $-2(x+1)^{-3}$

49. $\sin x(-1(x+1)^{-1})+(x+1)^{-1}\cos x$

50. $(x+1)(-\cos x(\sin x)^{-2})+(\sin x)^{-1}$

51. $\sin x(\sin x(\cos x)^{-2})+(\cos x)^{-1}\cos x$

52. $-1\ln(x)^{-2}\frac{1}{x}=\frac{-1}{x\ln(x)^2}$

53. $\frac{5}{x}$

54. 1

55. $(x+1)^2(-2(x+2))+(x+2)^{-2}2(x+1)$

56. $(x+1)^2(-2(x+2))+(x+2)^{-2}2(x+1)$

57. $xe^x\cos x+e^x\sin x+xe^x\sin x$

58. $e^x\cos x+xe^x\cos x-e^x\sin x$

59. $2xe^x+x^2e^x$

60. $x^2+2x(x+6)$

61. $x^2 2(x+4)+2x(x+4)^2$

62. $x^2(2x)+2x(x^2+\beta)$

63. $x^2e^x\cos x+2xe^x\sin x+e^2e^x\sin x$

64. $x^2e^x(-\cos x(\sin x)^{-2})+(\sin x)^{-1}(x^2e^x+2xe^x)$

65. $2(xe^x+1)(xe^x+e^x)$

66. $2(xe^x+x)(xe^x+e^x+1)$

67. $3\cos x$

68. $-4\sin x$

69. $\sin x(-\pi\sin x)+\cos\pi x\cos x$

70. $\frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)$

71. $2(x\sin x+a)(x\cos x+\sin x)$

72. $-\cos\theta(\sin\theta)^{-2}$

73. $v+gt$

74. $3e^{3x}$

75. $2xe^{x^2}$

76. $\frac{3}{2}x^{1/2}$