Math From a Physicist's Point of View

R. Ormiston

November 4, 2014

1 Intro Remarks

One of the most important things you can do before starting a problem is to know what the answer is. It sounds facetious, but it isn't. Always know what the answer should look like before you start calculating things so that when you get your final answer you already know whether or not it is right. This little blurb here is a way to determine just that; what the answer looks like or what it is "roughly equal to." There will be some numerical factors that are left out, but so what, those don't contain any of the interesting parts.

Consider $f(x) = 2x^2 + 5$. The important part of this is to observe that $f(x) \sim x^2$. The numbers out in front and the extra constant aren't really that important. You can understand the answer more or less exactly because you know how the function changes with the argument and what the shape of the curve is. You can get a long way just finding out how things are approximately related. My General Relativity professor once said to me, "Never underestimate the power of a squiggly." By that he of course meant relating objects via the "$\sim$" symbol.

Ok so how do we do this? Dimensional analysis. Whatever units are on one side of the equation must be the same on the other side of the equation. So if the left hand side is in seconds, the right hand side had better be in seconds. The best way to see how it works is to just do a bunch of examples. You'll be amazed how far you can get and how fast you can come do it.

2 Derivatives

Let's count powers of x to determine the answer (to within a multiplicative constant that we don't care about) to some derivatives. Each x counts as one "unit" of x. So x^{10} has ten units of x. Don't forget that the dx in integrals and derivatives counts as a power. Let's see how that works:

$$\frac{d}{dx}x^2 \sim x \tag{1}$$

How did we know? There are two powers of x in the numerator and one power of x in the denominator (the dx). The grand total then is that the left side has one power of x present in the numerator, therefore the right hand side must have only one power of x in it, and also in the numerator. Here's another one

$$\frac{d}{dx}x^9 \sim x^8 \tag{2}$$

Same idea. There are nine powers of x on top and one power on the bottom. Canceling the powers out leaves eight powers on the top, hence the answer. This trick works for almost any power of x, the exception being that the derivative of x^0 is zero, not $1/x$.

$$\frac{d}{dx}\left(\frac{1}{x^4}\right) \sim \frac{1}{x^5} \tag{3}$$

In this case, there are four powers of x in the denominator due to the function and the extra power due to the dx so in total there are five powers in the denominator on the left and therefore the right hand side must go as $1/x^5$. Not too bad, right? Here are a few more just for fun:

$$\frac{d}{dx}x^{3.5} \sim x^{2.5} \tag{4}$$

$$\frac{d}{dx}x^n \sim x^{n-1} \tag{5}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) \sim \frac{1}{x^2} \tag{6}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) \sim \frac{1}{x^{n+1}} \tag{7}$$

$$\frac{d}{dx}\left(\frac{1}{x^8}\right) \sim \frac{1}{x^9} \tag{8}$$

Again, we don't care about exact factors here. Just add up the powers of x on the left and then that's approximately the answer. This even works for multiple derivatives. Consider

$$\frac{d^3}{dx^3}x^8 \sim x^5 \tag{9}$$

We have eight powers of x in the numerator and 3 powers of x in the denominator so the result is five powers in the numerator. Simple, eh?

Integrals work the same way.

3 Integrals

Nothing changes with integrals. Count all of the powers of x and remember that the dx counts too.

$$\int x\,dx \sim x^2 \tag{10}$$

See it? The left side has two powers of x and therefore the right hand side must have two powers of x. There's really nothing more to it. Let's see some more.

$$\int x^7\,dx \sim x^8 \tag{11}$$

$$\int \frac{1}{x^5}\,dx \sim \frac{1}{x^4} \tag{12}$$

$$\int x^n\,dx \sim x^{n+1} \tag{13}$$

$$\int x^{2.5}\,dx \sim x^{3.5} \tag{14}$$

$$\int x^{210}\,dx \sim x^{211} \tag{15}$$

$$\int \frac{1}{x^{88}}\,dx \sim \frac{1}{x^{87}} \tag{16}$$

Not so bad, yeah? The only time this doesn't work is when you have

$$\int \frac{1}{x}\,dx \sim ? \tag{17}$$

We know that the answer is a logarithm, but how can we guess that based on counting powers? Well, you can't, but we can still get some insight into those kinds of things and into integrating other kinds of functions and even uglier integrals.

4 Standard Functions

There is a rule for standard functions (sin, cos, tan, exp, log, sinh, *etc.*) that the argument of the function *must be dimensionless.* So suppose we have $f(x) = \sin(x)$. In order to get the argument of sine to be dimensionless (just a regular number) then x must just be some number with no units. Imagine we have some function $g(y) = y$. We could say that y is measured in inches. But it would make no sense to take the sine of 8 inches or whatever. It may be that y is dimensionless, but we can assume that is carries some dimensions in order to make the integrals or derivatives easier by just canceling out powers. But when the argument is stuffed inside some standard function, you have to be a little bit more careful and ensure that the argument is dimensionless.

Suppose now that we have $h(k) = \cos(ak)$. Now we can let k have some dimensions as long as a has the inverse of that, which means that the product ak is dimensionless, and that's all that matters. For example, imagine that k is in meters. Then a must be in inverse meters. The same rule applies for exponents. Consider $r(p) = e^{\beta p}$. We can let p have any dimension that we wish so long as β has the inverse dimensions so that the product βp is dimensionless. Let's put this to work.

We'll start easy.

$$\int e^x \, dx \sim ? \tag{18}$$

Since the argument must be dimensionless, then x (and also dx) must be dimensionless. Therefore the left side has no units and so the right side must also have no units. Well, what is laying around that has no units? We have arbitrary constants we can stuff in there (but we don't care about those) and the function itself. Thus we can guess:

$$\int e^x \, dx \sim \; e^x \tag{19}$$

It turns out that this is exactly right, so that's good for us, but we wouldn't have known that it is right unless we did it "properly." Try this one,

$$\int e^{ax} \, dx \sim ? \tag{20}$$

There is a little bit of a subtlety here. First, let's proceed as usual. We can let x have some dimension so long as a has the inverse of that. A more mathematical way to say that would be to write that $x \sim 1/a$ so then $x \cdot x \sim a \cdot 1/a = 1$ which is dimensionless like we need. Alright so we're allowing x have some dimensions, therefore, because of the dx, we have one power of x on the left hand side in total. BUT, since we are integrating over x, it makes more sense to say that the answer should have $1/a$ rather than x. Imagine that the integral were definite, then we wouldn't have any x's around anymore, only the numbers they were evaluated at. Remember too that exponents integrate to exponents, so we should still have the exponent in the answer. Therefore,

$$\int e^{ax} \, dx \sim \; \frac{1}{a} e^{ax} \tag{21}$$

Let's back up for a second and consider

$$\int \sin(ax) \, dx \sim ? \tag{22}$$

By the same reasoning as above, we can let x have some dimension and a have the inverse of that, so because of the dx, the left side has one power of x. BUT, since we're integrating over x, let's say that the left side has minus one power of a, or $1/a$. So the right side goes like $1/a$, but what about the sine function? Unfortunately, you can't know based on this method alone. Really, $\sin(ax)$ is just some number though, so if this were a definite integral we'd be in great shape. For now, the best we can do is

$$\int \sin(ax)\, dx \sim \frac{1}{a} \tag{23}$$

There is a way out of this pickle. We had great luck in dealing with exponential functions, so let's write $\sin(ax)$ as an exponential function. The trick is to remember the following:

$$e^{\pm iax} = \cos(ax) \pm i \sin(ax) \tag{24}$$

If you pick the plus sign on the left, pick the plus sign on the right, and vice versa. Also remember that $\mathcal{I}m$ means that you only consider the imaginary part and $\mathcal{R}e$ means that you only consider the real part. So if we had $f(x) = 3 + 2i$, then $\mathcal{I}m\,[3 + 2i] = 2$. Similarly $\mathcal{R}e\,[3 + 2i] = 3$. Since

$$\mathcal{I}m\left[e^{iax}\right] = \sin(ax) \tag{25}$$

then,

$$\int \sin(ax)\, dx = \mathcal{I}m\left[\int e^{iax}\, dx\right] \tag{26}$$

Don't worry too much about the extra brackets. Just do the integral like normal and then take the imaginary part. We can let x have some dimension and this time let ia have the inverse dimension. Since the integral goes like x, or rather $1/ia$, and since the exponent will still appear on the other side (due to the nature of exponents and since it is dimensionless), we can write

$$\begin{aligned}\int \sin(ax)\, dx &= \mathcal{I}m\left[\int e^{iax}\, dx\right] \sim \mathcal{I}m\left[\frac{1}{ia}e^{iax}\right] = \mathcal{I}m\left[\frac{-i}{a}e^{iax}\right] \\ &= \mathcal{I}m\left[\frac{-i}{a}\Big(\cos(ax) + i\sin(ax)\Big)\right] = -\frac{\cos(ax)}{a}\end{aligned} \tag{27}$$

Therefore,

$$\int \sin(ax)\, dx \sim -\frac{\cos(ax)}{a} \tag{28}$$

Again, we are exactly right. Exponents are awesome. One last thing, we can do multiple integrals too. If we write

$$\int d^3x \, x^3 \sim x^6 \tag{29}$$

So $d^n x$ counts as n powers of x. It's the same as writing

$$\int dx \int dx \int dx \, x^3 \sim x^6 \tag{30}$$

if that makes things a little more clear.

5 Recap

Now you can integrate and takes derivatives of an enormous number of functions all through multi-variable calc without breaking a sweat. This is how physicists do things. First, find a down n' dirty way to get an approximate answer so you know what you're aiming for when you do it again to try and figure out all of the extra factors exactly. With a little practice, you'll be solving triple integrals in your head faster than you can write them down.

I use this all the time. In fact, I'll walk you through my latest integral I solved in this way. It is common in quantum field theory, and you'll be able to solve it now too. It is no different than what you've seen here, it just lumps some ideas together. Here is the integral (essentially)

$$\int_0^\infty \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2} \tag{31}$$

The first thing we're going to do is ignore the $(2\pi)^3$. Next, let's set $m = 0$ as that will simplify things. Notice that $\vec{k}$ can have some dimension so long as $\vec{r}$ has the inverse of that. When $m = 0$, we have

$$\int_0^\infty d^3\vec{k} \, \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} \tag{32}$$

Looking better. The integral has one power of $\vec{k}$ in the numerator overall, or rather, one power of $1/\vec{r}$. This integral is definite, so there had better not be any $\vec{k}$'s lying around on the right hand side. So the exponent cannot be on the right, it has been reduced to a number and thus we do not care about it. Therefore,

$$\int_0^\infty d^3\vec{k} \, \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} \sim \frac{1}{\vec{r}} \tag{33}$$

So far so good. Let's go back to Eqn. (31). Since m and k are summed, they must have the same dimensions (You can only add apples to apples). In other words, $m \sim \vec{k} \sim 1/\vec{r}$.

We know what we would get when $m = 0$, but when m is not zero, the exponent can still be around, only we swap in m for k. So,

$$\int_0^\infty \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{ik\cdot\vec{r}}}{\vec{k}^2 + m^2} \sim \frac{1}{\vec{r}} e^{m\vec{r}} \tag{34}$$

Remember, the exponent is just a number now that isn't oscillating, so there shouldn't be an i in it anymore. What about adding an m in the numerator? Nope, because when we set it to zero the whole thing would be zero, but Eqn. (33) tells us that it is nonzero. Plus the dimensions would be wrong. What about adding an m in the denominator like

$$\frac{1}{\vec{r} + m} e^{m\vec{r}} \tag{35}$$

Nope. $\vec{r}$ and m don't have the same dimensions even though when m is zero this would work. Can't do it. What about then

$$\frac{1}{\vec{r} + 1/m} e^{m\vec{r}} \tag{36}$$

Also no because when $m \to 0$ the whole thing becomes zero. There's only one more part to consider; this is a physics problem, not a math problem. So when we evaluated the exponent at zero and infinity, we should have paid more attention. What happens as $\vec{r}$ gets arbitrarily large? Right now (in Eqn. (34)), the answer goes to infinity, and that's nonsense. The denominator cannot tame the exponent. The only sensible numbers that we should get are either 1 or 0. We can correct this with a minus sign and some absolute value brackets.

$$\int_0^\infty \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{ik\cdot\vec{r}}}{\vec{k}^2 + m^2} \sim \frac{1}{\vec{r}} e^{-m|\vec{r}|} \tag{37}$$

This makes physical sense now too. We have exhausted all of the possibilities now. And even better, we are correct. Eqn. (37) is damn close. There is a $1/4\pi$ that should be there but that can be worked out later. The important thing is that we know how the function behaves, and have solved the integral to a sufficient extent as to be able to fully understand the problem. A factor of 4π wouldn't help your understanding (but if you can work out the contour integrals, good for you!).

6 The End

Well that's all I've got for you. Like I said, the most important thing is to know the answer before you start calculating. This is how physicists do it. With minimal practice, you'll be flying through gnarly looking calc problems of all shapes and sizes like it's nothing. Good luck!