University of Minnesota Twin Cities

LIGO Group Meetings

Thermal Noise and Dissipation at LIGO

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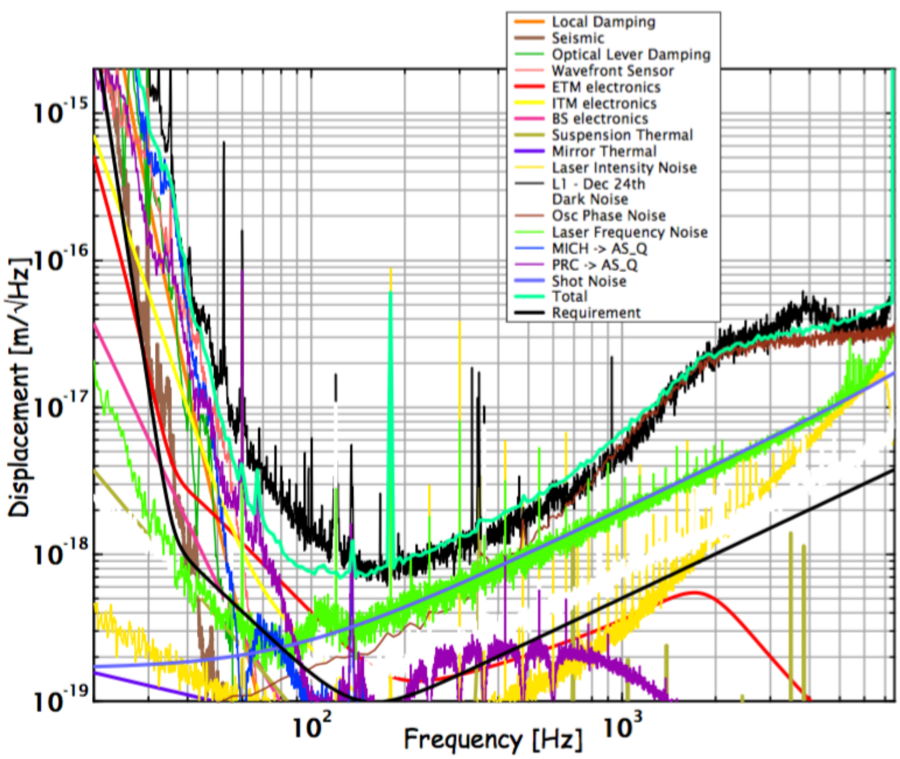

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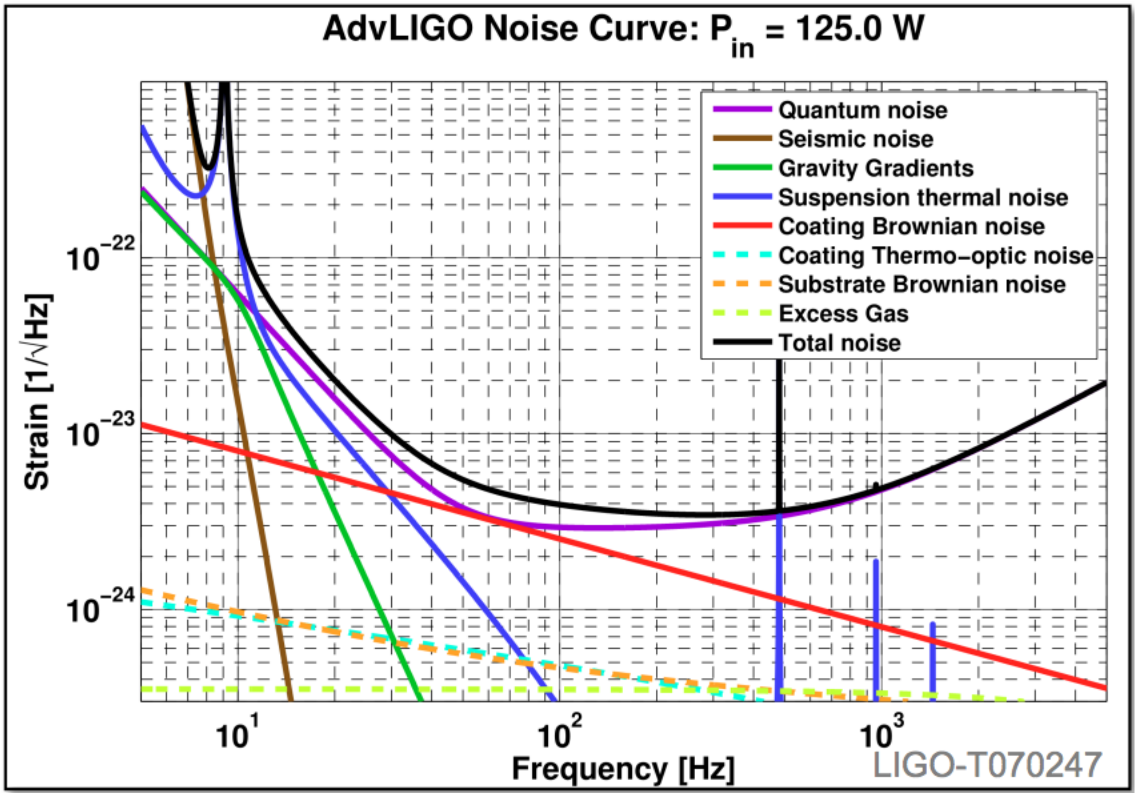
What is Thermal Noise?



Thermal noise is a type of displacement noise which causes changes in the length of the arm cavity (as opposed to sensing noises which appear in the readout signal but is not due to a GW). When we look at frequency bands in which we are not limited by the seismic noise ($\sim 10 - 200$Hz), the interferometer is then limited by thermal noise. There are a number of contributing factors to the total thermal noise, but I will primarily address the mechanical loss in the test masses which lead to fluctuations in the surface and itself dissipation in the the suspension wires which cause fluctuating forces on the test masses. The mirror coatings, for example, will not be discussed in any detail.

Back of the Envelope Calculations

How much movement do we naively expect? Well, consider the equipartition theorem and the potential energy of the moving mirror. As a back of the envelope calculation we can write

$$\frac{1}{2}kx^2 = \frac{1}{2}k_BT = \frac{1}{2}m(2\pi f)^2x^2 \rightarrow x^2(f) = \frac{4k_BT}{(2\pi f)^2}\frac{1}{4m} \tag{1}$$

Assuming that we have a 10kg mass (The Advanced LIGO test masses are 40 kg fused silica cylinders with a 17 cm radius) at 1Hz and at room temperature, then we find that the center of mass of the mirror moves about

$$x_{rms} = \sqrt{\frac{k_B T}{m(2\pi f)^2}} = \sqrt{\frac{300(1.38 \times 10^{-23})}{10(2\pi)^2}} \approx 3 \times 10^{-12}\,\mathrm{m} \tag{2}$$

(In 1905, Einstein showed that the random walk of a Brownian particle obeyed $\left\langle x^2(t)\right\rangle \sim 2k_B T t$). Needless to say, this is quite a lot. If this it true, how are we to detect GW's?

Langevin Approach

In 1908, Langevin published an interesting approach to Einstein's work on Brownian motion. Consider the following

$$m\ddot{x} + \gamma\dot{x} = F_{th}(t) \tag{3}$$

where $F_{th}(t)$ is the force due to thermal fluctuations. Now multiply the whole equation by x and consider an entire ensemble average,

$$\langle m x\ddot{x}\rangle + \langle \gamma x\dot{x}\rangle = \langle x F_{th}(t)\rangle \tag{4}$$

If x anf F_{th} are random and uncorrelated, then the term on the right is zero. The differential equation we are left with is simple to solve as is, but if we use the fact that

$$\frac{d^2}{dt^2}\left\langle x^2\right\rangle - 2\left\langle \dot{x}^2\right\rangle = x\ddot{x} \quad \text{and} \quad \frac{1}{2}m\left\langle \dot{x}^2\right\rangle = \frac{1}{2}k_B T \tag{5}$$

and then define

$$z \equiv \frac{d}{dt}\left\langle x^2\right\rangle \tag{6}$$

then we can write

$$\dot{z} = \frac{2k_B T}{\gamma} - \frac{\gamma z}{m} \tag{7}$$

After a large amount of time ($t \gg m/\gamma$) we find

$$z = \frac{2k_B T}{\gamma} \rightarrow \left\langle x^2\right\rangle = \frac{2k_B T t}{\gamma} \tag{8}$$

which is the average square displacement due to Brownian motion that Einstein showed in his paper. Two points here are that though the thermal force disappeared, we have still included it implicitly when we said that the system is at thermal equilibrium and second, that thermal noise is clearly related to dissipation which is seen by the presence of the damping term γ is the rms displacement.

The Fluctuation-Dissipation Theorem

When we study thermal noise, we generally mean either Brownian noise or internal friction. In either case, there is a single primary equation which is used to calculate responses to these types of thermal agitation: the fluctuation-dissipation theorem (FDT).

In the simplest words, the FDT states that fluctuations cause dissipation. When there is a process that dissipates energy, turning it into heat (e.g., friction), there is a reverse process related to thermal fluctuations. In fact these things are linearly related so lowering the fluctuations lowers the dissipation and vice versa. The relationship is remarkably simple and very powerful as it may be applied in a huge variety of ways. Said another way, if we prepare a system to be out of equilibrium, then the FDT can connect the linear relaxation response to statistical fluctuations in equilibrium.

Another notable property of the FDT is that the fluctuations are a microscopic event that we will be unlikely to measure individually, but the dissipation is the macroscopic event. Thus we can learn something about thermal fluctuations within materials by measuring the macroscopic dissipation.

When we say "dissipation," what we are really talking about is the "admittance" $\mathcal{Y}$ or the "impedance" $\mathcal{Z}$ and where $\mathcal{Y} = \mathcal{Z}^{-1}$

Qualitative examples from Wikipedia:

- Drag and Brownian motion

If an object is moving through a fluid, it experiences drag (air resistance or fluid resistance). Drag dissipates kinetic energy, turning it into heat. The corresponding fluctuation is Brownian motion. An object in a fluid does not sit still, but rather moves around with a small and rapidly-changing velocity, as molecules in the fluid bump into it. Brownian motion converts heat energy into kinetic energy - the reverse of drag.

- Resistance and Johnson noise

If electric current is running through a wire loop with a resistor in it, the current will rapidly go to zero because of the resistance. Resistance dissipates electrical energy, turning it into heat (Joule heating). The corresponding fluctuation is Johnson noise. A wire loop with a resistor in it does not actually have zero current, it has a small and rapidly-fluctuating current caused by the thermal fluctuations of the electrons and atoms in the resistor. Johnson noise converts heat energy into electrical energythe reverse of resistance. In LIGO, imagine an amplifier going through a resistor with a coil on the end which is facing the mirror was a magnet on it. The pendulum motion causes an emf and a fluctuating voltage in the resistor.

- Light absorption and thermal radiation

When light hits an object, some fraction of the light is absorbed, making the object hotter. In this way, light absorption turns light energy into heat. The corresponding fluctuation is thermal radiation (e.g., the glow of a "red hot" object). Thermal radiation turns heat energy into light energythe reverse of light absorption. Indeed, Kirchhoff's law of thermal radiation confirms that the more effectively an object absorbs light, the more thermal radiation it emits.

We can already apply these basic ideas directly to a situation at LIGO (more or less). Consider the act of driving the mirror. Somewhere there is a power source and resistive losses and eventually, some loop of wire that generates an emf and pushes the magnet on the mirror (In general, we try to keep lossy objects like magnets away from the mirror). But now, even when the power source is off, the resistor is warm and so there are small voltage fluctuations in the resistor. These fluctuations cause an emf in the loop and so drive the mirror. Therefore, we can think of the mirror as being in a heat bath with the resistor, even though they are not touching in any way. We can see a quick fix to this - dip your resistors in liquid nitrogen if you want to curb the Johnson noise driving the mirrors.

The actual equation for the FDT is somewhat easily derived but very simply stated as follows

$$S_x(f) = \frac{4k_BT}{(2\pi f)^2}\mathcal{R}[\mathcal{Y}(f)] \tag{9}$$

where the admittance is $\mathcal{Y} = v/F$, F is an applied (thermal) force and v is the time derivative of the readout. $\mathcal{R}[\mathcal{Y}(f)]$ is proportional to the dissipation. We immediately see two ways to lower the thermal noise: lower the temperature and/or lower the dissipation. The units work out to be m^2/Hz and so we can recover the rms noise by simply integrating the spectral density over the frequency band. In a mathematical statement,

$$S_x(f) = \lim_{T\to\infty}\frac{2}{T}\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t) - \langle x\rangle| e^{2\pi i f t}\, dt\right|^2 \tag{10}$$

and

$$\left\langle (x - \langle x\rangle)^2 \right\rangle = \int_{f_{min}}^{f_{max}} S_x(f)\, df \tag{11}$$

In practice, we have to multiply $S_x(f)$ by the bandwidth of the measurement. If we were to divide by the square the length of an arm cavity, then we have a LIGO interferometer strain as opposed to a test mass displacement spectral density.

Suspension Thermal Noise

Notice that we only take the real part of the admittance. This is crucial to understanding why we say "dissipation." Take the canonical example of a dampened, driven oscillator which has a mass m, a spring constant k, a quality factor $Q = 1/\phi$ (accounts for damping) and a driving frequency ω. We would like to find the spectral density of this dampened pendulum with a sinusoidally varying force. We know how to solve this for $x(t)$. The admittance is just $\dot{x}(t)/F$. I won't bore you with solving this system here. But the punchline is that

$$\mathcal{R}\left[\frac{v}{F}\right] = \frac{\omega k \phi}{(k - m\omega^2)^2 + (k\phi)^2} \tag{12}$$

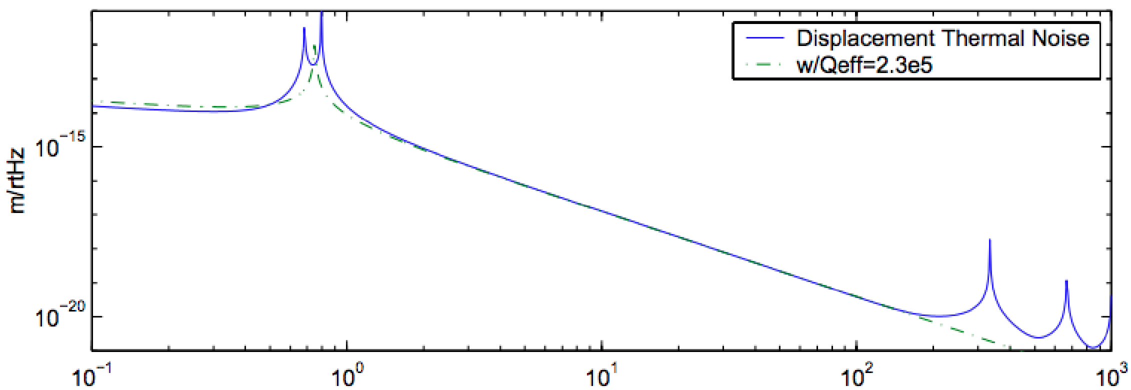
where ϕ is a actually $\phi(f)$. Saulson made some calculation using $\phi = \mathit{const}$ which seemed to be the case for the test masses and suspension wires, but there has been some reason to believe that there is indeed some dependency of the loss angle on the frequency. Away from resonance and assuming a small damping, we then have (using $\omega = 2\pi f$)

$$x^2(\omega) \approx \frac{4k_B T}{\omega^2} \frac{\omega_0^2 \phi}{m\omega^3} \tag{13}$$

which is valid for the frequencies on the order of 1-100Hz. We may use this then to predict the suspension thermal noise. Without the damping, the admittance would have been completely imaginary, to wit, $i\omega/(k - m\omega^2)$, and so the real part is zero. Thus it is only when we add the damping that we can get any thermal losses in the system. Sometimes the FDT for a dampened oscillator is written in terms of the frequency as

$$S_x(f) = \frac{4k_B T}{(2\pi)^2 m f} \frac{\phi}{f_0^2 \left[\left(1 - \frac{f^2}{f_0^2}\right)^2 + (f_0\phi)^2\right]} \tag{14}$$

The displacement thermal noise of a *single wire pendulum* without approximations (solid line) and approximated with a single mode with an effective Q (dashed line) is shown below in the log-log plot. The x-axis is the frequency.



A few words about the quality factor. Q is a dimensionless measure of the ratio of elastic restoring force to dissipative force. Measured by width of resonance or by ringdown time of vibration. It is related to "loss angle" by $\phi = 1/Q$. A typical is $Q \approx 10^3$, a good one is roughly $Q \approx 10^6$, and an excellent quality factor is $Q \approx 10^8$. From Eq. 7 we can see that the spectral density of the squared displacement is proportional to ϕ and so a large Q goes a long way in reducing thermal noise. The steel suspension wires have a rather bad quality factor, around 10^3. The fused silica has a reported quality factor as high as 10^8. Insofar as the mirror is concerned, we can limit the dissipation if we go to cryogenic temperatures or attain a higher Q. Sapphire is being considered which has a Q up to around 5×10^8.

The mirrors are suspended in an attempt to decouple them from the seismic noise. This of course makes the thermal noise dominant. The thermal noise in the suspension wires and in the mirror itself cause movement in the mirror in a few different ways:

- The internal forces in the mirror can distort the shape of the mirror (bend corners, normal modes, bulging, pitch yaw, roll, pringle)
- The mode in the wires couples the the mode in the mirror

We can even imagine a cosmic ray muon passing through the mirror. If there are any conductors nearby, this causes image charges to be created and these will rearrange themselves causing fluctuating currents. This is why we tend to move the conductors further away or isolate them.

Dissipation (Young's Modulus) & Loss Angle

The loss angle measured several times is easily incorporated into thermoelastic noise and random internal normal modes through Young's modulus, E, by adding a small imaginary phase: $E \to E(1 + i\phi)$. The real part of Young's modulus is called the elasticity and the imaginary part is the dissipation. Recall that if $W = \oint \mathbf{F} \cdot \mathbf{dl} \neq 0$, then the process is irreversible and there has been dissipation. The structure of the loss angle generally takes one of three forms:

- $\phi(\omega) \sim \omega$
- $\phi(\omega) = const$
- $\phi(\omega) \sim \frac{\omega\tau}{1+(\omega\tau)^2}$

For LIGO, we typically side with Saulson and assume that the loss angle is a constant. This relation is called "structural damping." This is a good approximation although there is reason to believe that there is some dependence of the loss angle on the frequency (ϕ seems to increase with the frequency). For example, the thermoelasticity in the suspension wires has a dissipation that is given by

$$\phi_{TE}(\omega) \propto \frac{E}{\rho C_V}\left(\frac{\omega\tau}{1+(\omega\tau)^2}\right) \tag{15}$$

where ω is the frequency and $\tau \sim d^2/D_{th}$ with a wire diameter d and a thermal diffusivity D_{th}. The thermoelastic damping in the test mass however may look very different,

$$S_{TE}(\omega) \sim \frac{k_B T^2 \kappa}{\rho C_V r_0^3 \omega^2} \tag{16}$$

where κ is the thermal conductivity and r_0 is the beam spot radius. Currently, fused silica suspension wires are used. These wires have a structural damping that dominates the thermoelastic damping. The desire to switch to sapphire wires may be curbed by the fact that the thermoelastic damping in the sapphire not only dominates over the structural damping, but may even be a bit worse at low frequencies than the fused silica.

Normal Modes

The normal modes inside the mirror can be a big problem, depending on the dynamics of that specific mode. Imagine the test mass as a cube. There could exist a mode where the corners of the mass are bent but distort the faces of the cubes very little. This could go mostly unnoticed. There could however be a normal mode in which the entire test mass contracts and expands along a particular direction. If this direction is along the path of the beam, then this is very well observed mode and can cause a great deal of noise. This calculation turns out to be rather hard to do. There is a way to cheat - supply a force to the test mass which has a profile identical to the beam (sinusoidally varying and Gaussian) and measure the response of the test mass. In doing so we find for homogeneous losses,

$$S_{resp.}(\omega) \sim \frac{4k_B T}{2\pi f}\frac{1-\sigma^2}{\sqrt{2\pi}E r_0}\phi \tag{17}$$

where σ is Poisson's ratio, E is Young's Modulus, and r_0 is the radius of the beam spot.

Good vs Good Enough

At this point we have some understanding of what the noise is, where it comes from and how to analyze it. The question remains, what do we do about it? And what about that giant $x_{rms} = 3 * 10^{-12}m$? Why does LIGO work and how can we improve on it?

The Test Mass

For the test mass, we have at least two options (ignoring the surface coatings)

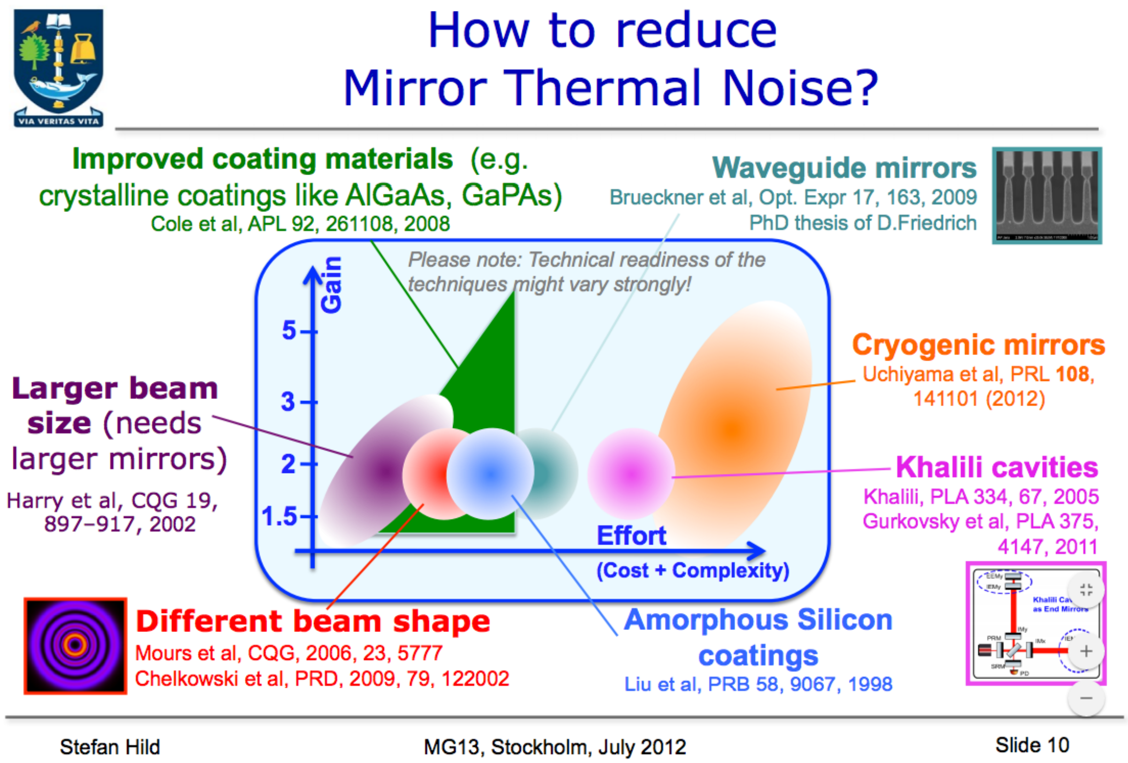
1. Pick a material with a high quality factor

2. Cool the mirror down

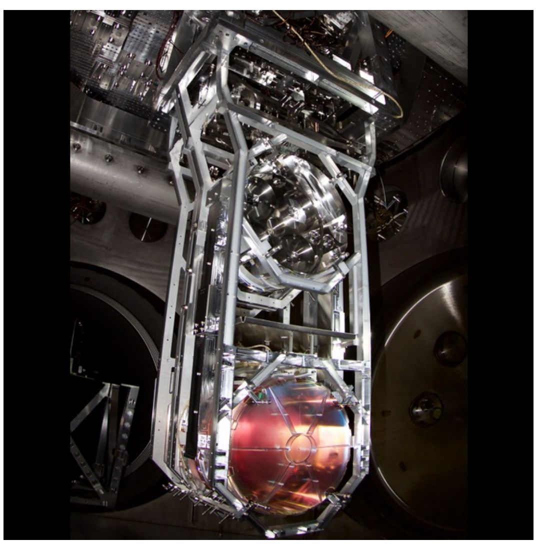
Initially, mirrors made of sapphire instead of fused silica were considered due to their excellent properties. They are expensive though and the silica was sufficient so we stuck with it. The quality factor of silica is roughly 2×10^8 on a good day and the sapphire is about 2×10^9. However, Braginsky noted that the thermoelastic noise may be higher in the GW frequency band than the thermo-mechanical dissipation (Brownian noise).

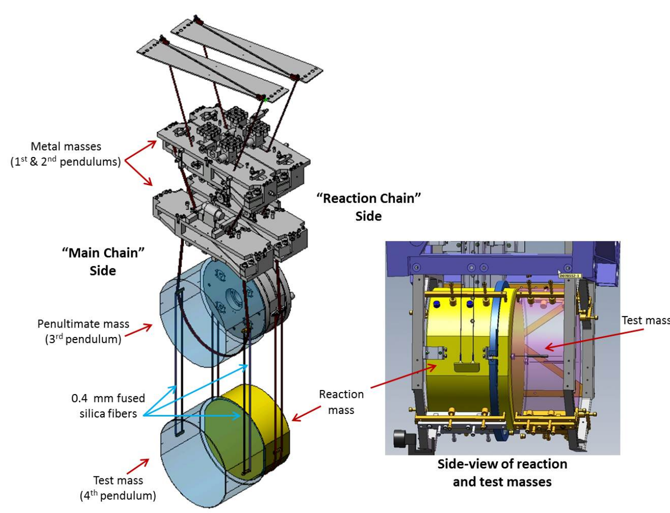
Cooling the mirror would be fantastically expensive and would require a change in material - likely to sapphire - but could have excellent results in terms of minimizing the test mass dissipation.

Here's a nice summary slide from Stefan Hild:



The Suspension System





For basic LIGO, steel piano wire was used to suspend the test masses. The obvious improvement here is to move to fused silica wires.

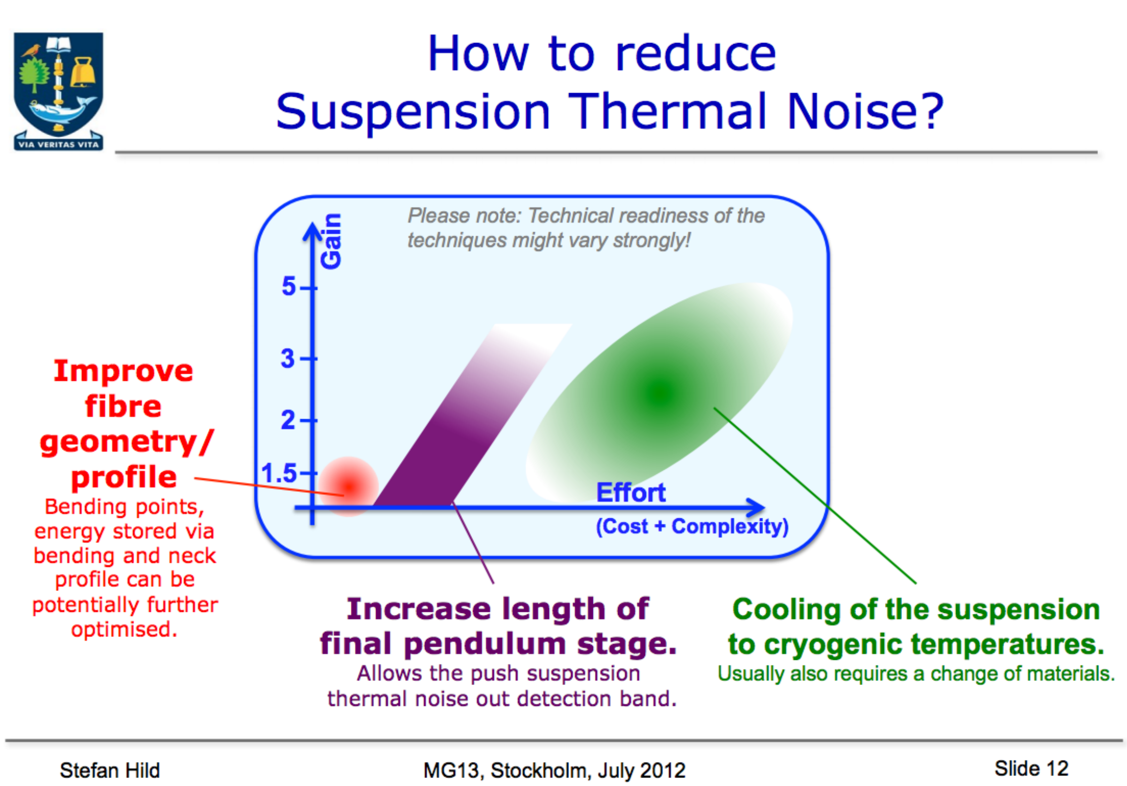
For Advanced LIGO, fused silica fibers will welded to the test masses and attached to an electrostatic drive (low force and low noise of course). Particularly "lossy" materials, like rubber and magnets, are kept away from the masses. As a result of these precautions, the fluctuations contributing to the noise in the strain can be brought down to roughly $3 \times 10^{-24} m/\sqrt{Hz}$.

Where the wires bend near the ends, the dissipation is greatest. To aid this problem, the silica wires are polished with a laser and have a larger diameter where they are fused with the test mass in order to curb flexing. Below ~ 30Hz, the suspension thermal noise still contributes at around $2 \times 10^{-21}/\sqrt{Hz}$

It has also been shown that cooling the silica wires to 40K can cause a factor of 10 improvement at 10Hz.

We could also cheat a little and change the pendulum length to push the resonance out of the frequency band of interest and therefore reduce the noise where re are interested in looking - make it a different frequency's problem.

Again, a nice summary slide was created by Stefan Hild:



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