Multiparameter Persistent Homology

Topological Data Analysis and Nueroscience

https://github.com/ndag/TDA-and-Neuro

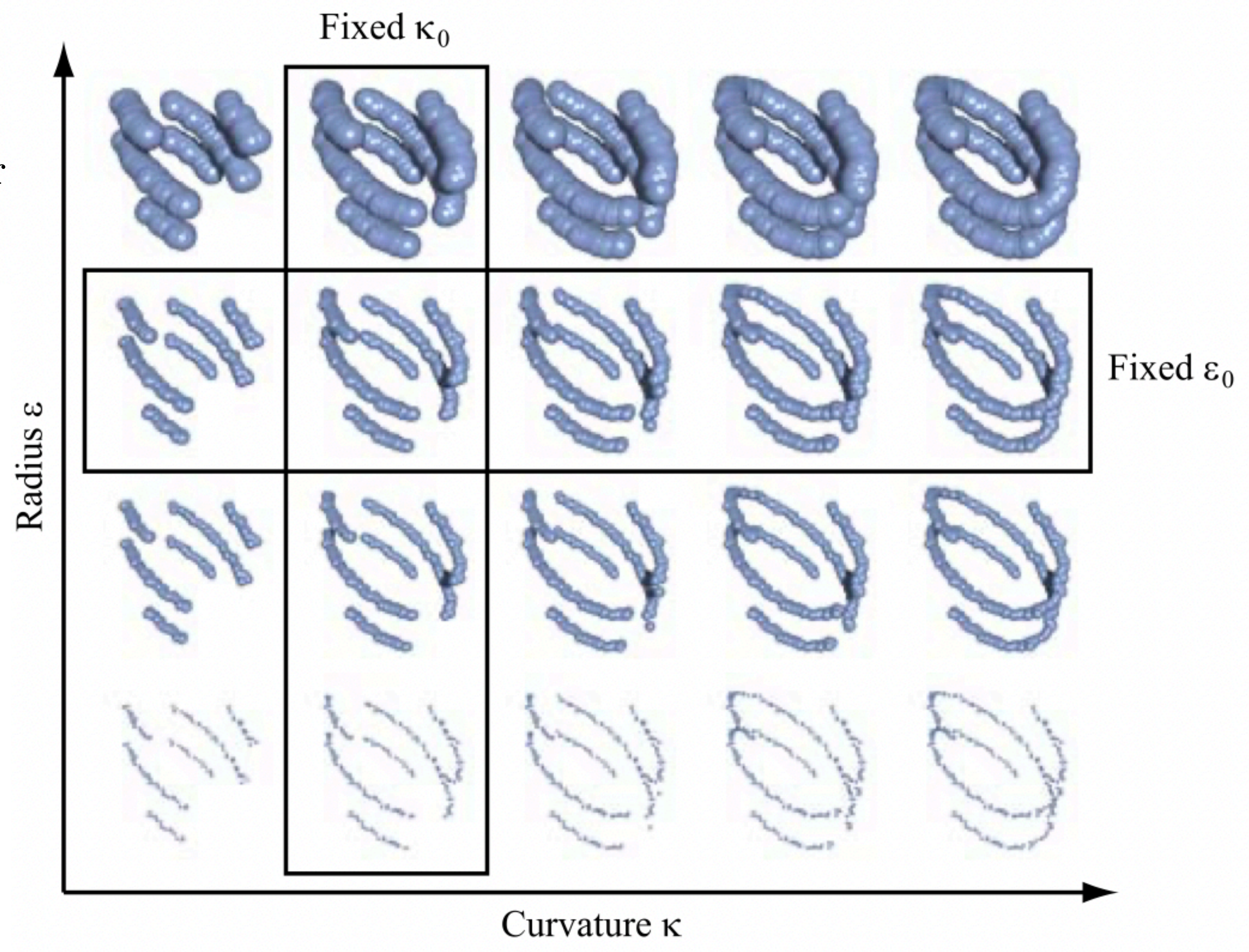
Alex McCleary

The Main Idea

- The goal of 1-parameter persistence is to break down filtrations into their fundamental parts, called indecomposables.
- Unfortunately, this goal has not been realized in a practical way. Either the indecomposables used are so complicated they are impossible to work with or they lose information.
- While this situation is not ideal, we can still learn a lot from many of the invariants proposed.

Multiparameter Filtrations

- A multiparameter filtration F consists of a topological space $\mathsf{F}(\vec{a})$ for each $\vec{a} \in \mathbb{R}^n$ and an inclusion $\mathsf{F}(\vec{a}) \hookrightarrow \mathsf{F}(\vec{b})$ whenever $\vec{a} \leq \vec{b}$.



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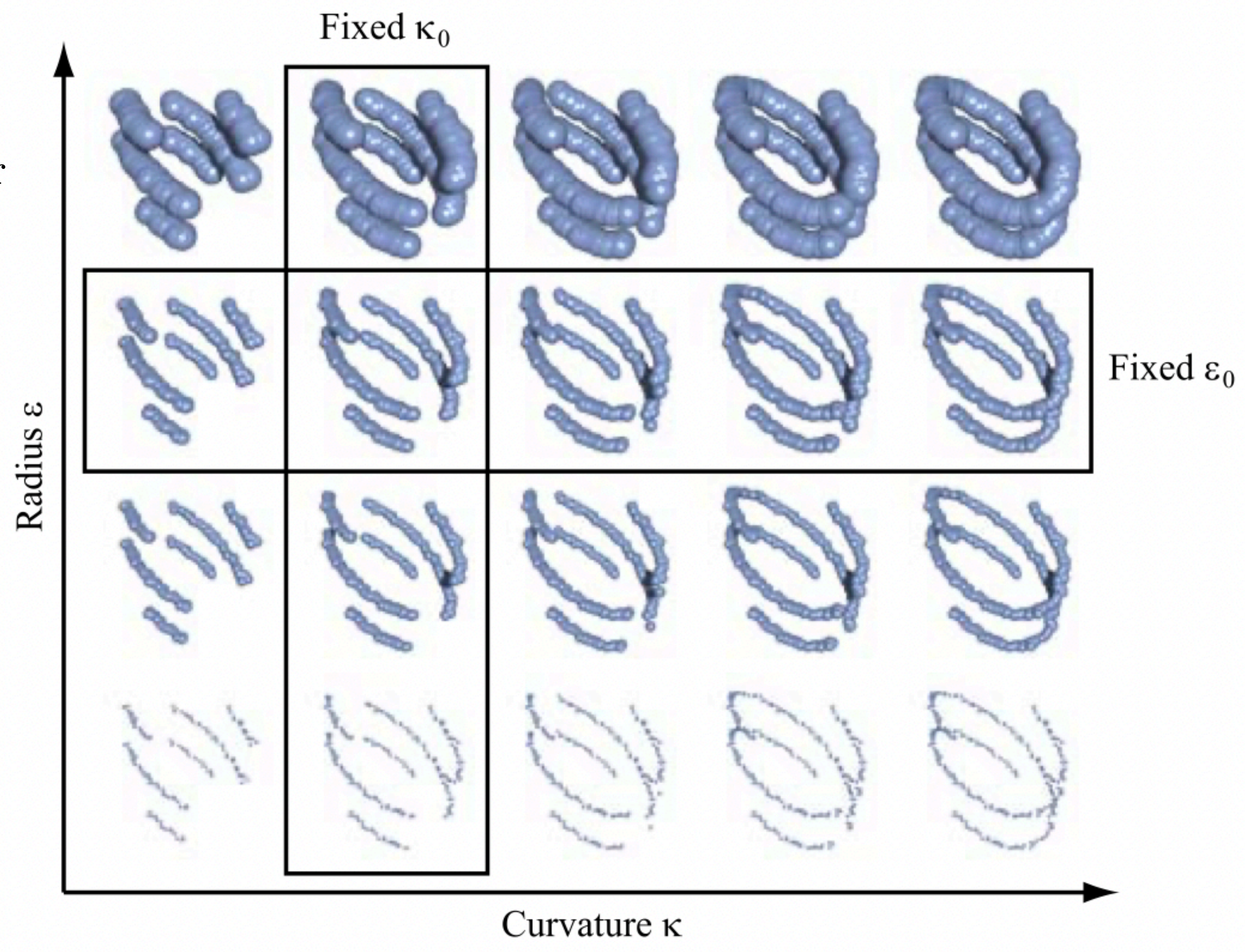
- The main sources of examples for multiparameter filtrations are:
  - Augmented metric spaces. That is, metric spaces with real valued functions on them.
  - Sublevel set filtrations.
  - Level set filtrations.



Example: Sublevel Set Filtrations

Let X be a topological space (or simplicial complex). Every function $f : X \to \mathbb{R}^n$ induces an n-parameter filtration M by $\mathsf{M}(\vec{a}) = f^{-1}((-\infty, \vec{a}])$

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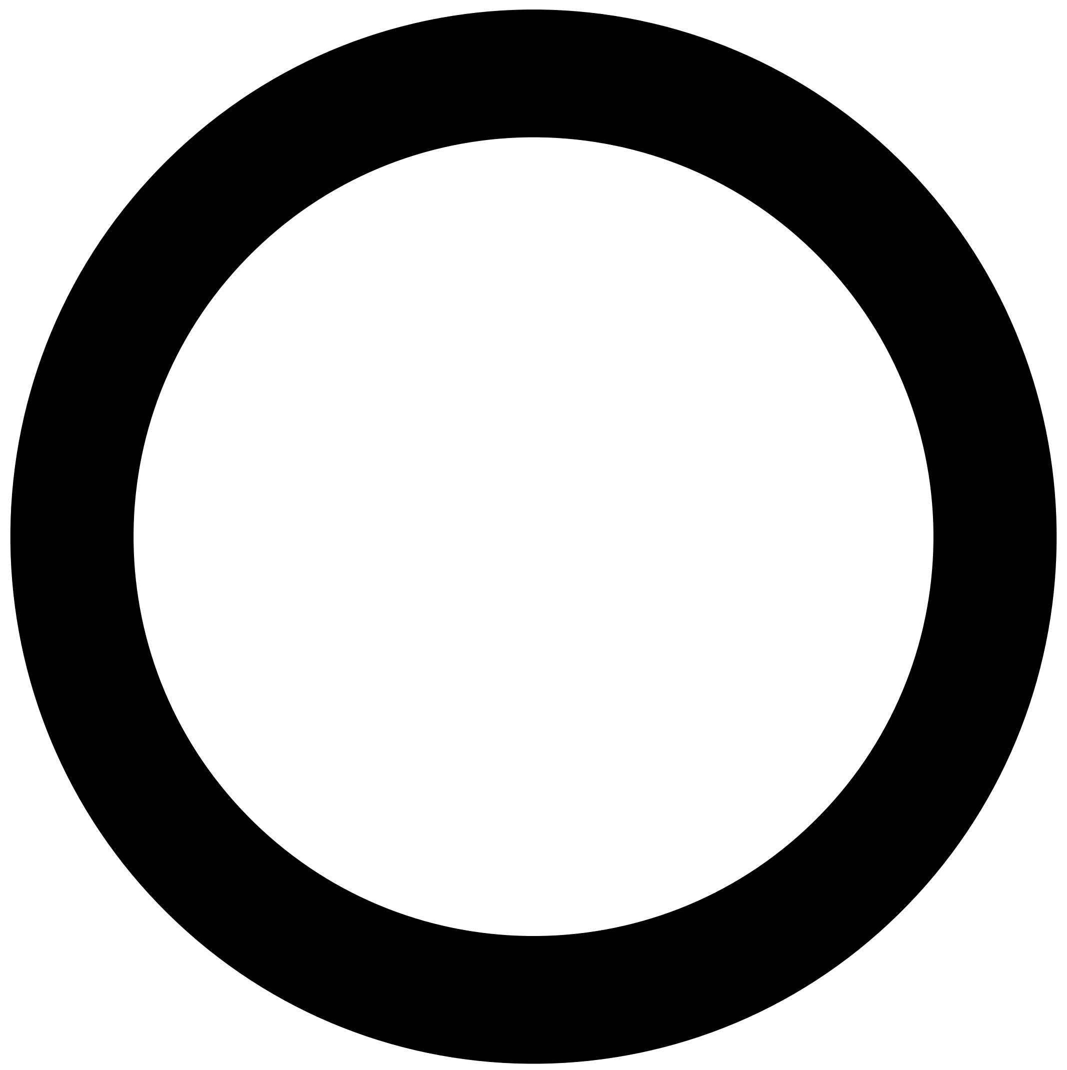

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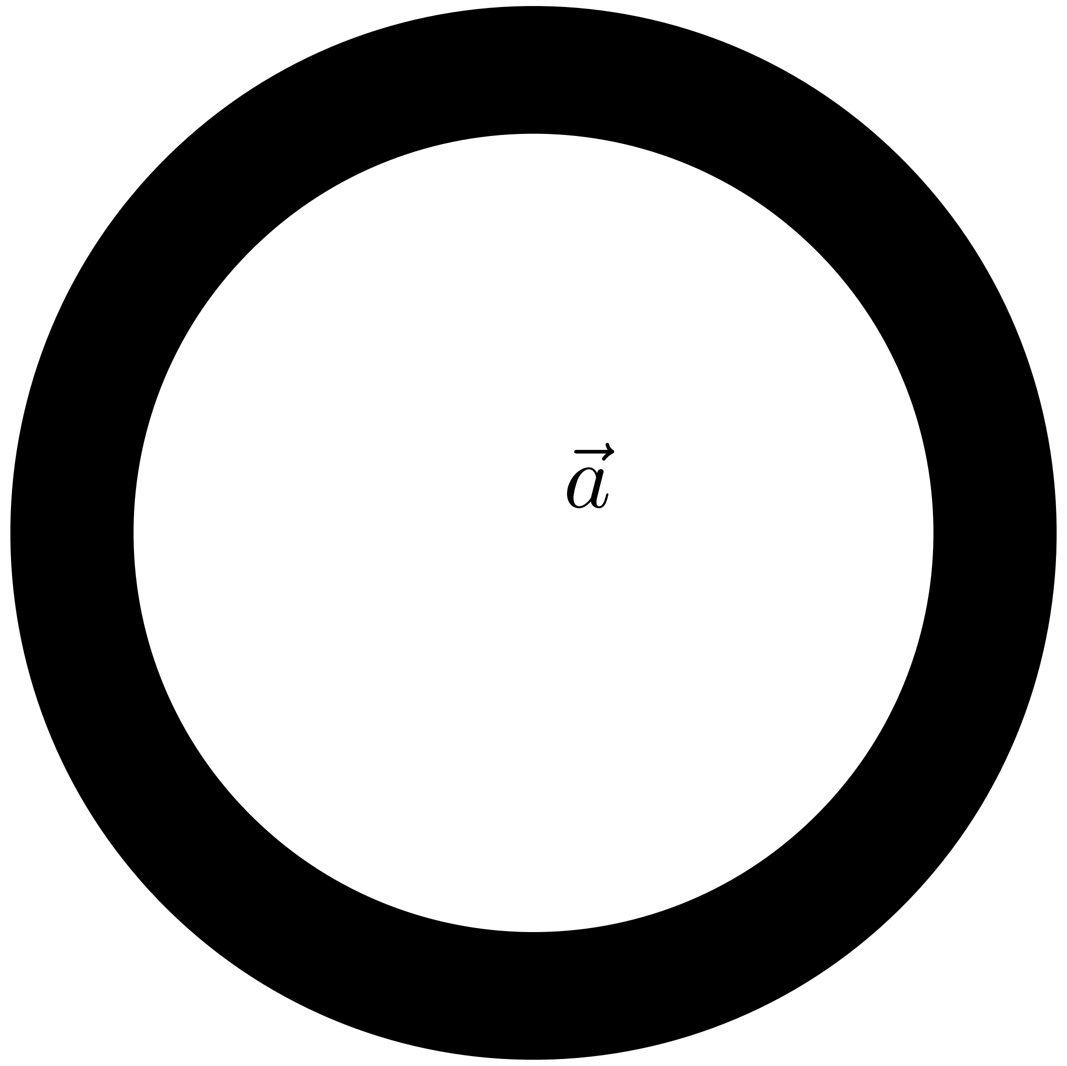

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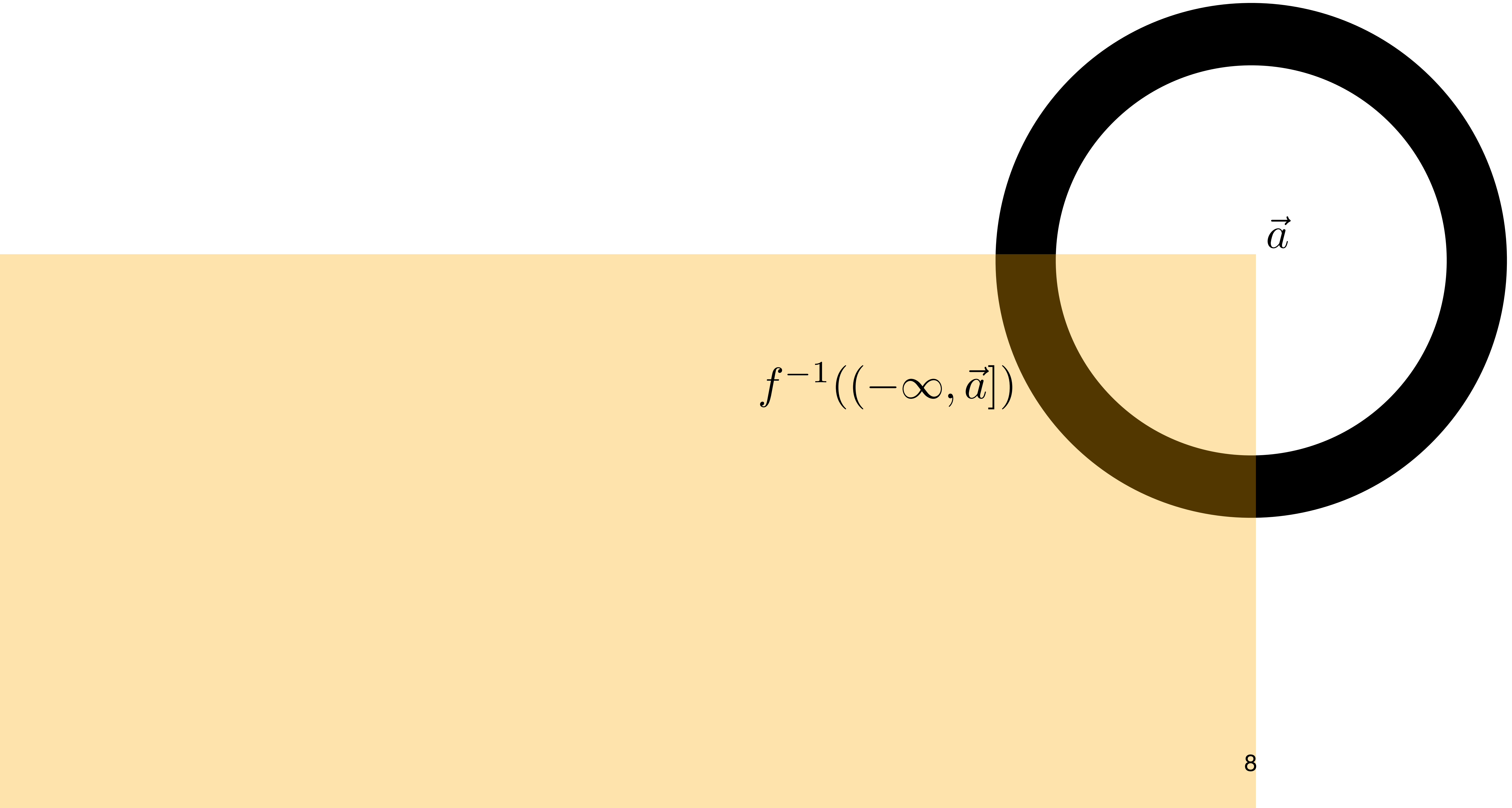

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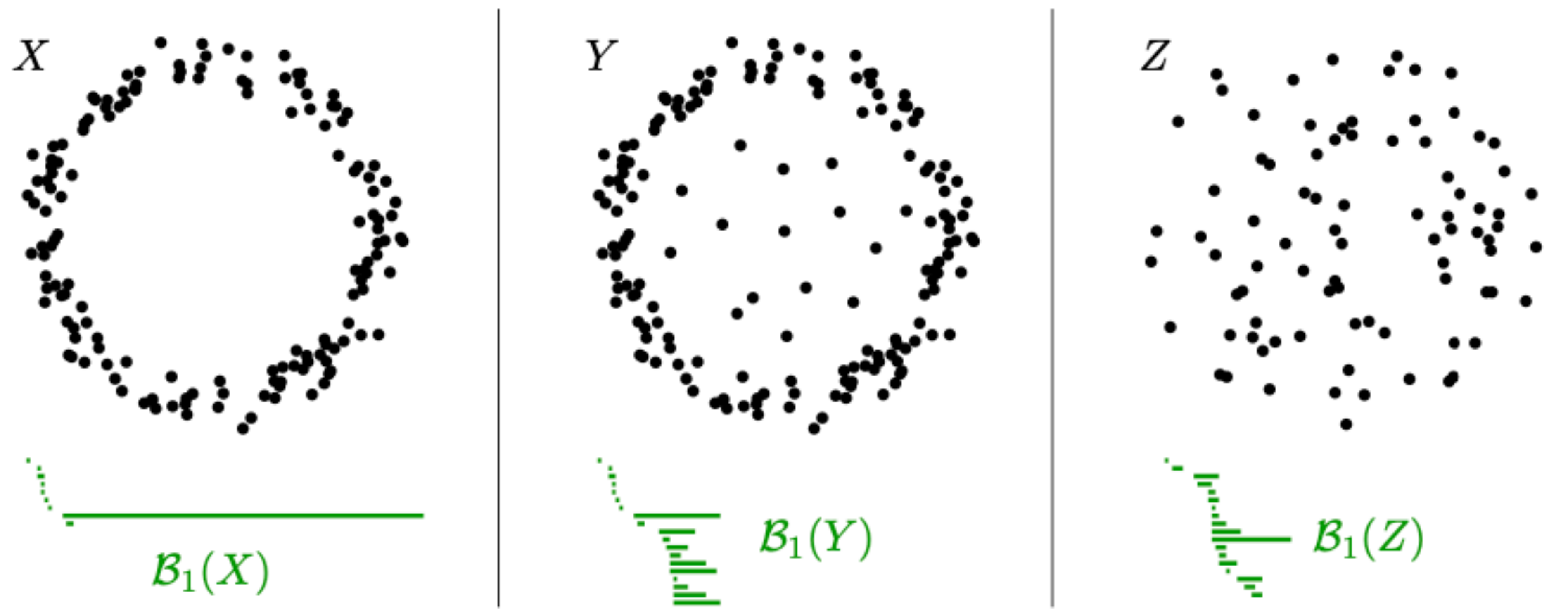

Persistence Modules

Multiparameter Persistence Modules

- A multiparameter **persistence module** consists of a vector space $\mathsf{M}(a)$ for each $a \in \mathbb{R}^n$ and a linear transformation $\mathsf{M}(a \leq b) : \mathsf{M}(a) \to \mathsf{M}(b)$ for each $a \leq b$.

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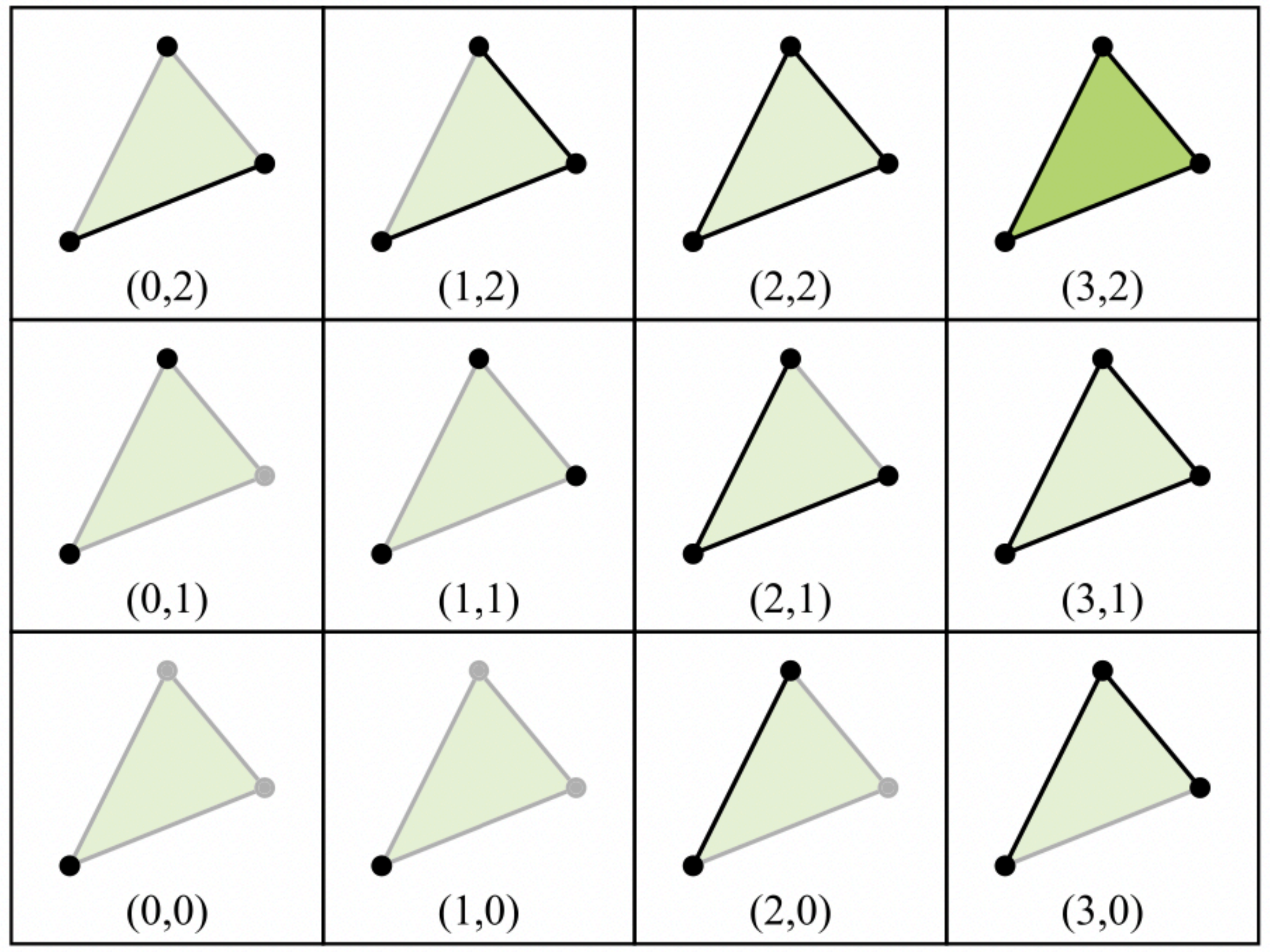

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- An ε-interleaving between M and N consists of linear maps $\varphi_{\vec{a}} : M(\vec{a}) \to N(\vec{a} + \vec{\varepsilon})$ and $\psi_{\vec{a}} : N(\vec{a}) \to M(\vec{a} + \vec{\varepsilon})$ for each $a \in \mathbb{R}^n$ such that the following diagrams commute

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$$\begin{tikzcd} M(\vec{b}) \arrow[r, "\varphi_{\vec{b}}"] & N(\vec{b}+\vec{\varepsilon}) \\ M(\vec{a}) \arrow[u] \arrow[r, "\varphi_{\vec{a}}"] & N(\vec{a}+\vec{\varepsilon}) \arrow[u] \end{tikzcd}$$

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Interval Decomposable Persistence Modules

- A subset $I \subseteq \mathbb{R}^n$ is an interval if:
  - For any $a, c \in I$ and any $b \in \mathbb{R}^n$ with $a \leq b \leq c$, $b \in I$
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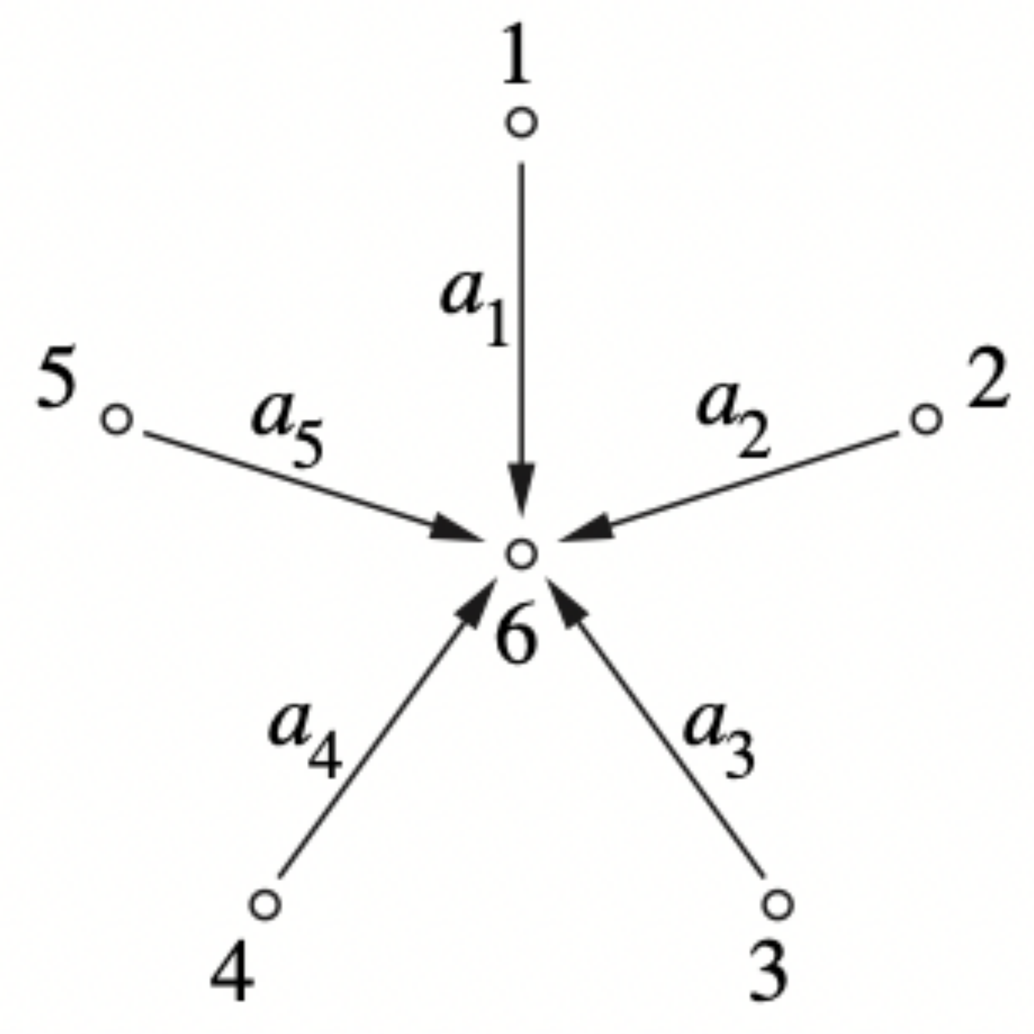
Quivers

- A quiver Q is a directed multigraph. That is, a quiver is a (finite) set of vertices Q_0 and a (finite) set of edges Q_1 with two functions $h, t : Q_1 \to Q_0$.



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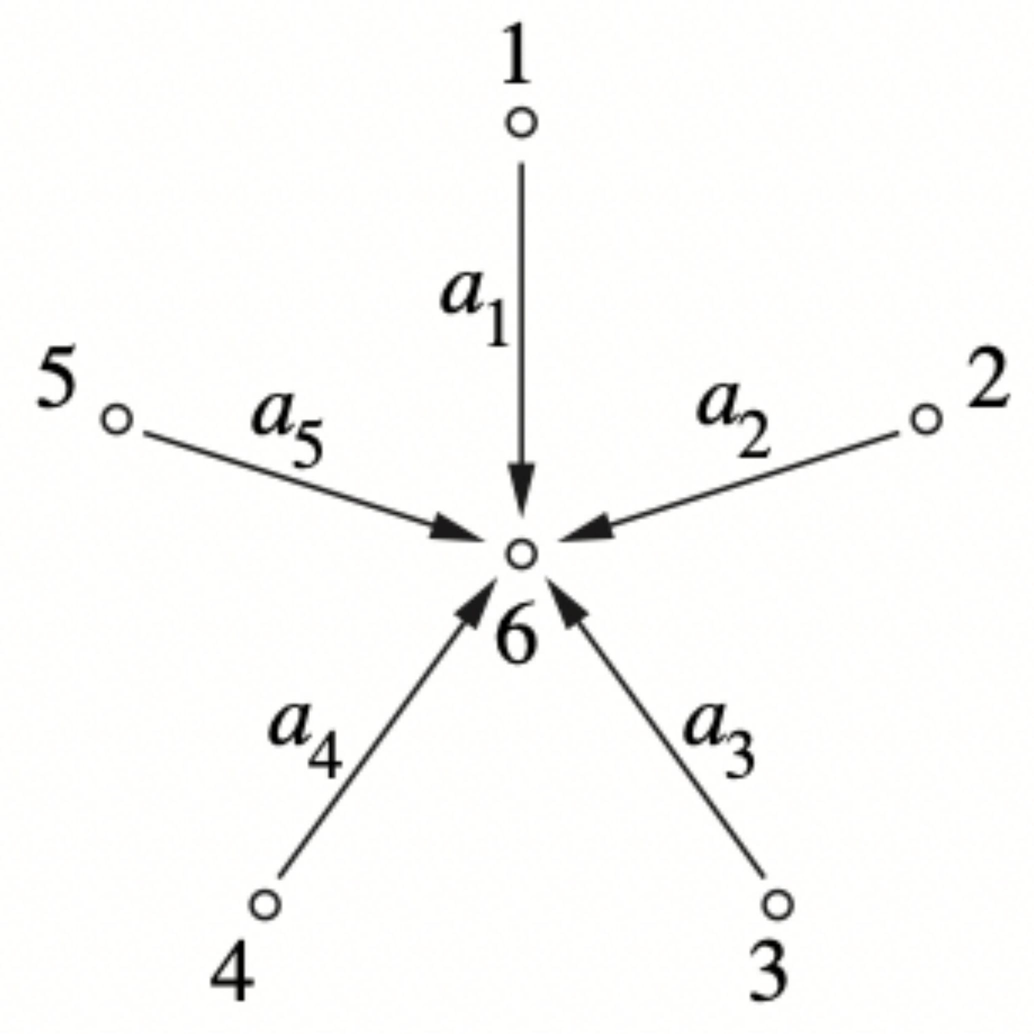

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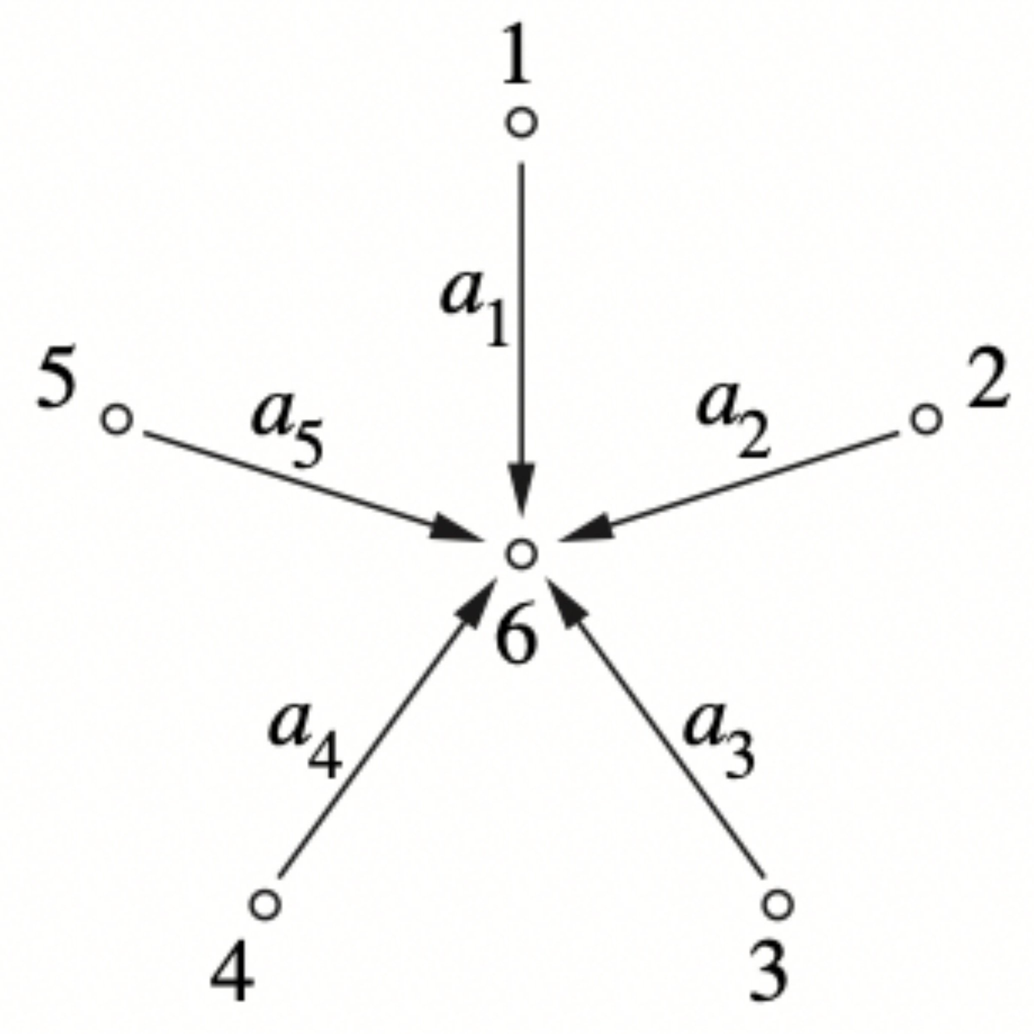
- Let V and W be representations of Q. A morphism from V to W consists of linear maps $\varphi_x : V_x \to W_x$ for each $x \in Q_0$ such that the following diagram commutes for any $a \in Q_1$

$$\begin{tikzcd} V_{t(a)} \arrow[r, "V_a"] \arrow[d, "\varphi_{t(a)}"'] & V_{h(a)} \arrow[d, "\varphi_{h(a)}"] \\ W_{t(a)} \arrow[r, "W_a"] & W_{h(a)} \end{tikzcd}$$



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- For some quivers, this is easy, for others, this is nearly impossible.

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Indecomposable Representations

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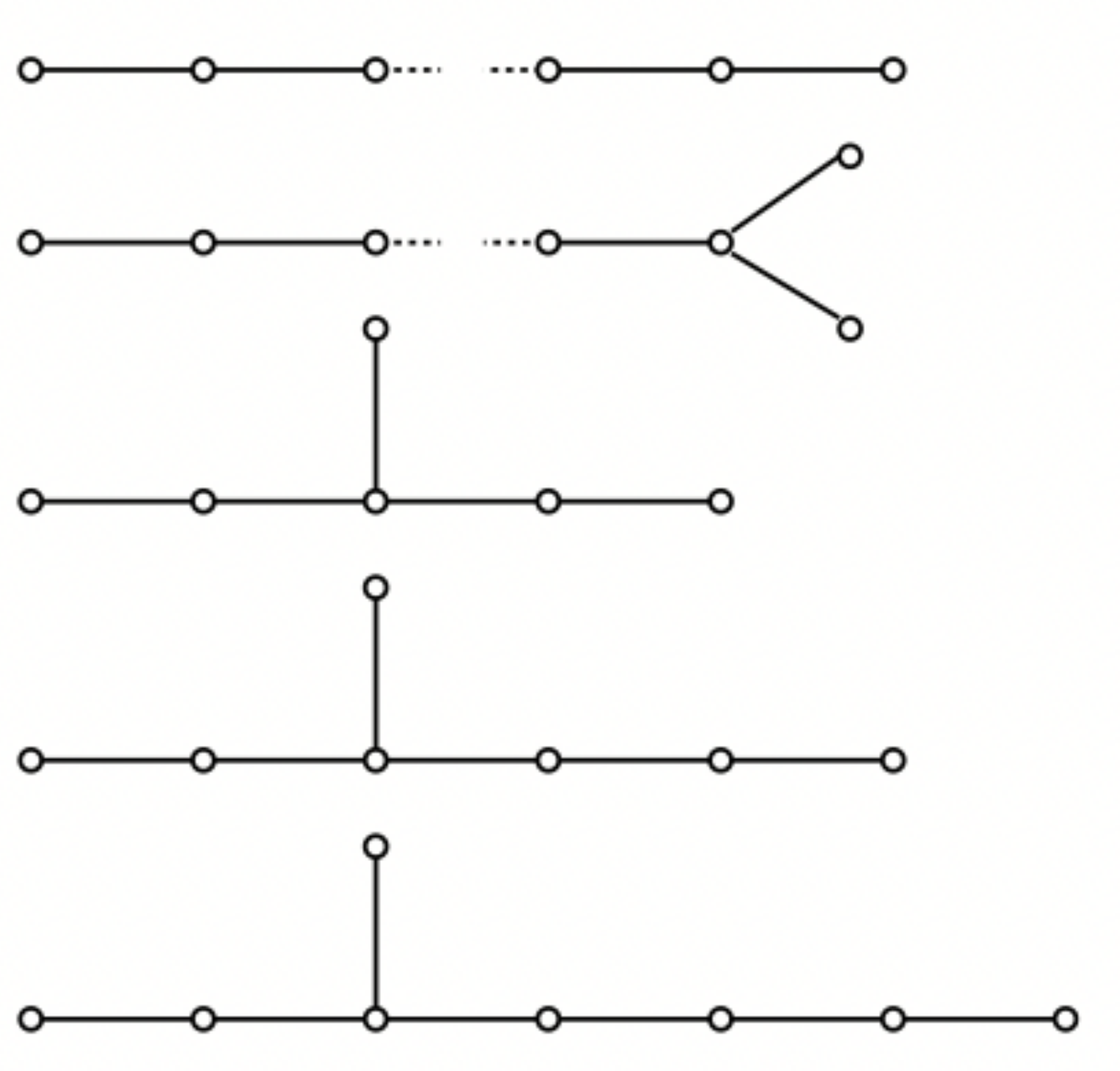
- A nice classification of indecomposable quiver representations would provide an answer to multiparameter persistence.

Gabriel's Theorem

A quiver Q is of finite type if it has only finitely many indecomposable representations.

Theorem[Gabriel, 1972] A quiver Q is of finite type if and only if the underlying graph is a union of the following graphs:



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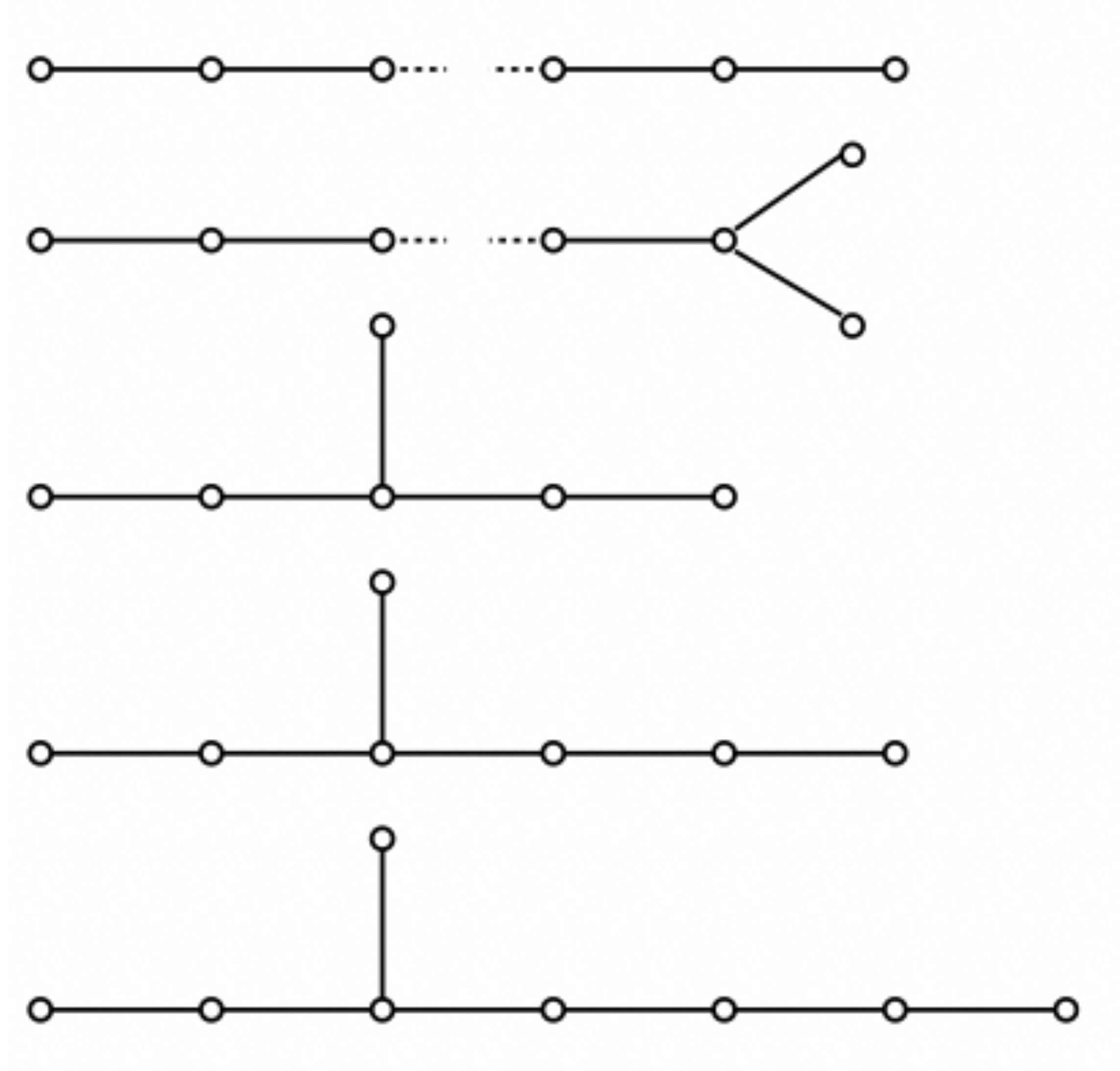

Gabriel's Theorem implies that there are infinitely many indecomposables for most multiparameter persistence modules.

Zigzag Persistence Modules

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- The indecomposables of a zigzag persistence module are of the form $k^{[b,d]}$ where $k_a^{[b,d]} \cong k$ if $a \in [b,d]$ and $k^{[b,d]} \cong 0$ otherwise. The maps between adjacent elements in $[b,d]$ are identity maps and all other maps are 0.

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- This means we can decompose any zigzag persistence module M as

$$\bigoplus_{I \in \mathcal{I}} k^I$$

where $\mathcal{I}$ is some multiset of intervals.

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V_1 \arrow[r] & V_2 & & \\
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- We call such a decomposition a barcode of M

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& & V_5 \arrow[u] \arrow[r] & \cdots
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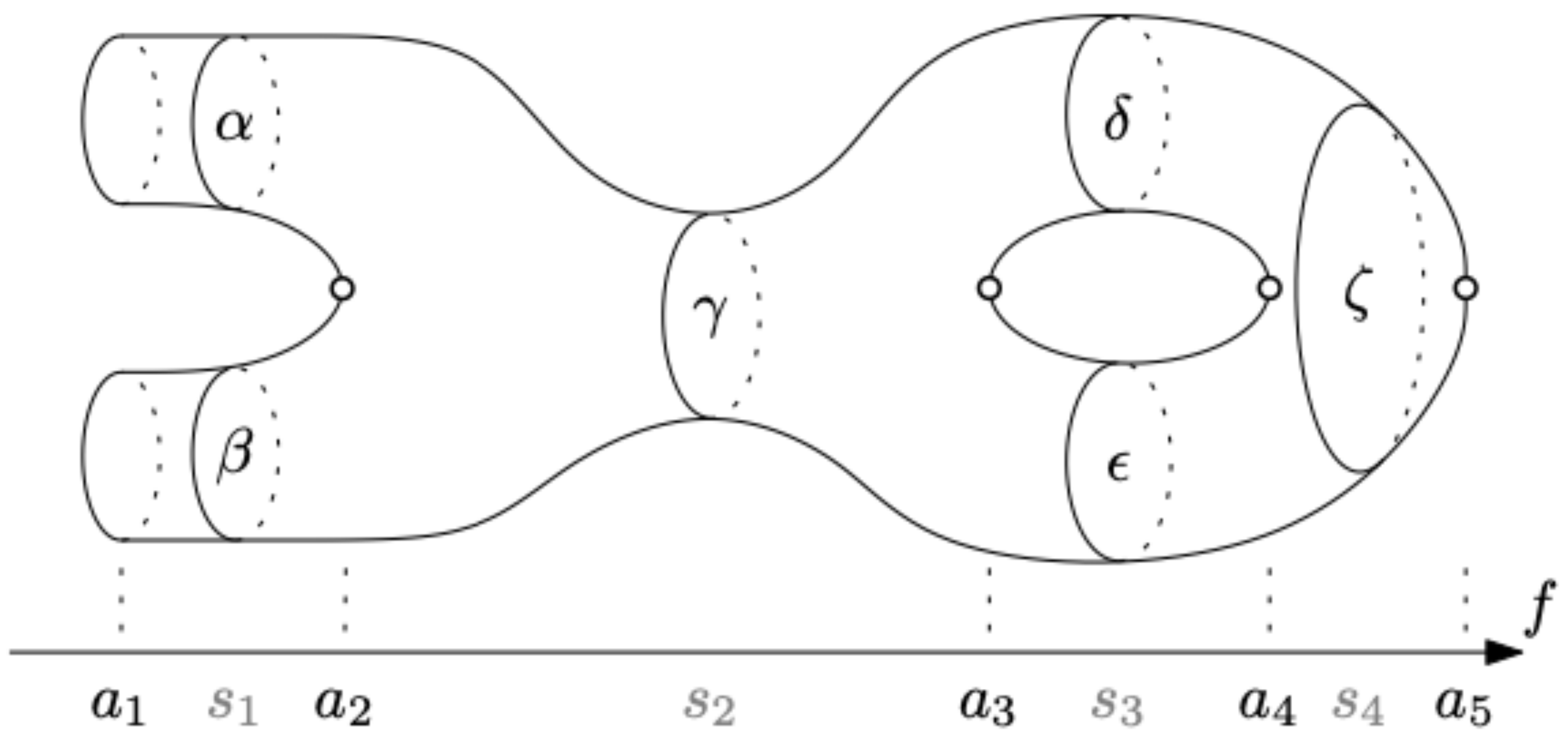
Level Set Persistence

- Let $f : X \to \mathbb{R}$ be a "nice" continuous map from a topological space X to $\mathbb{R}$. The level set of f at $t \in \mathbb{R}$ is $f^{-1}(t) \subseteq X$.



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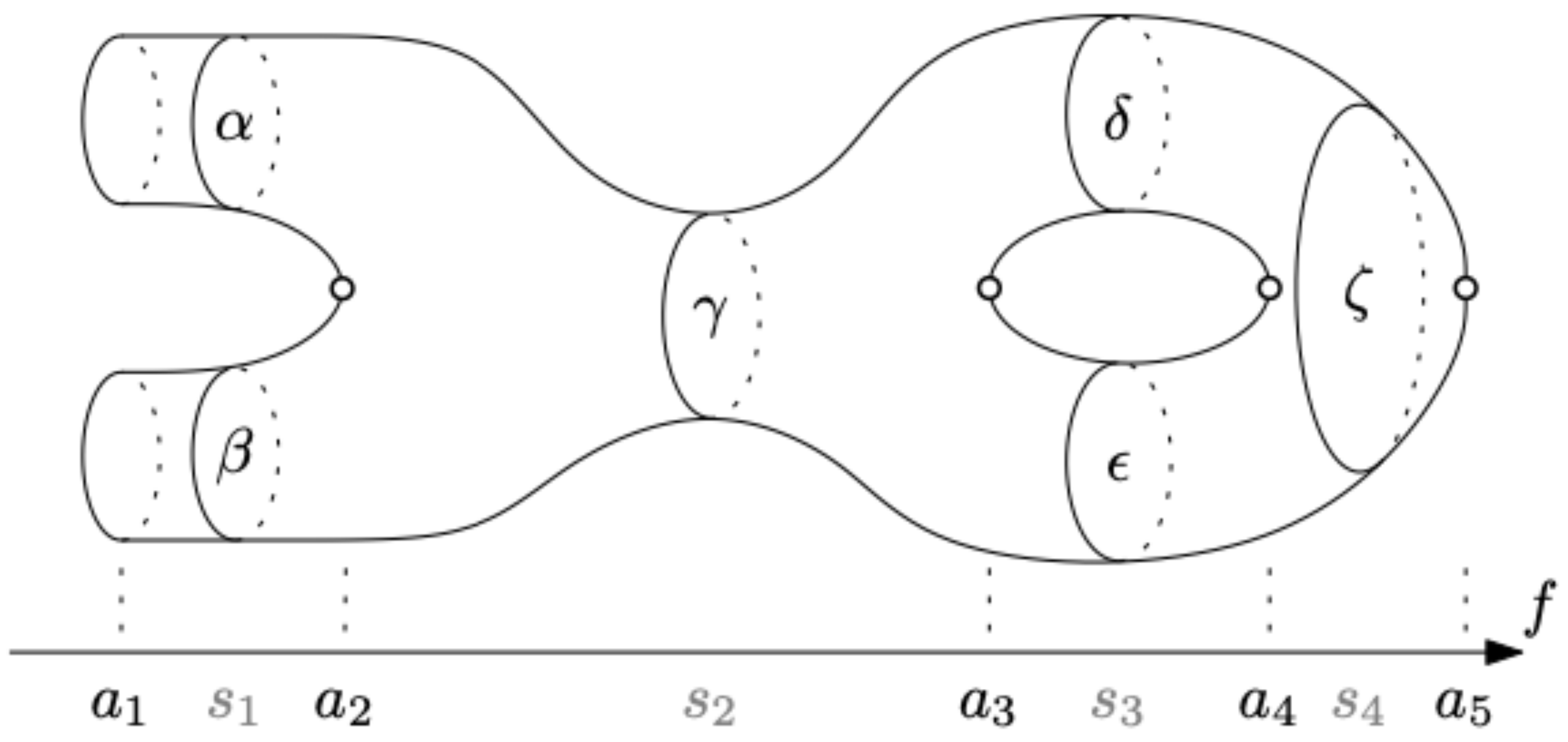
- As t varies, there are only finitely many places where $f^{-1}(t)$ changes homotopy type. We call these critical points.



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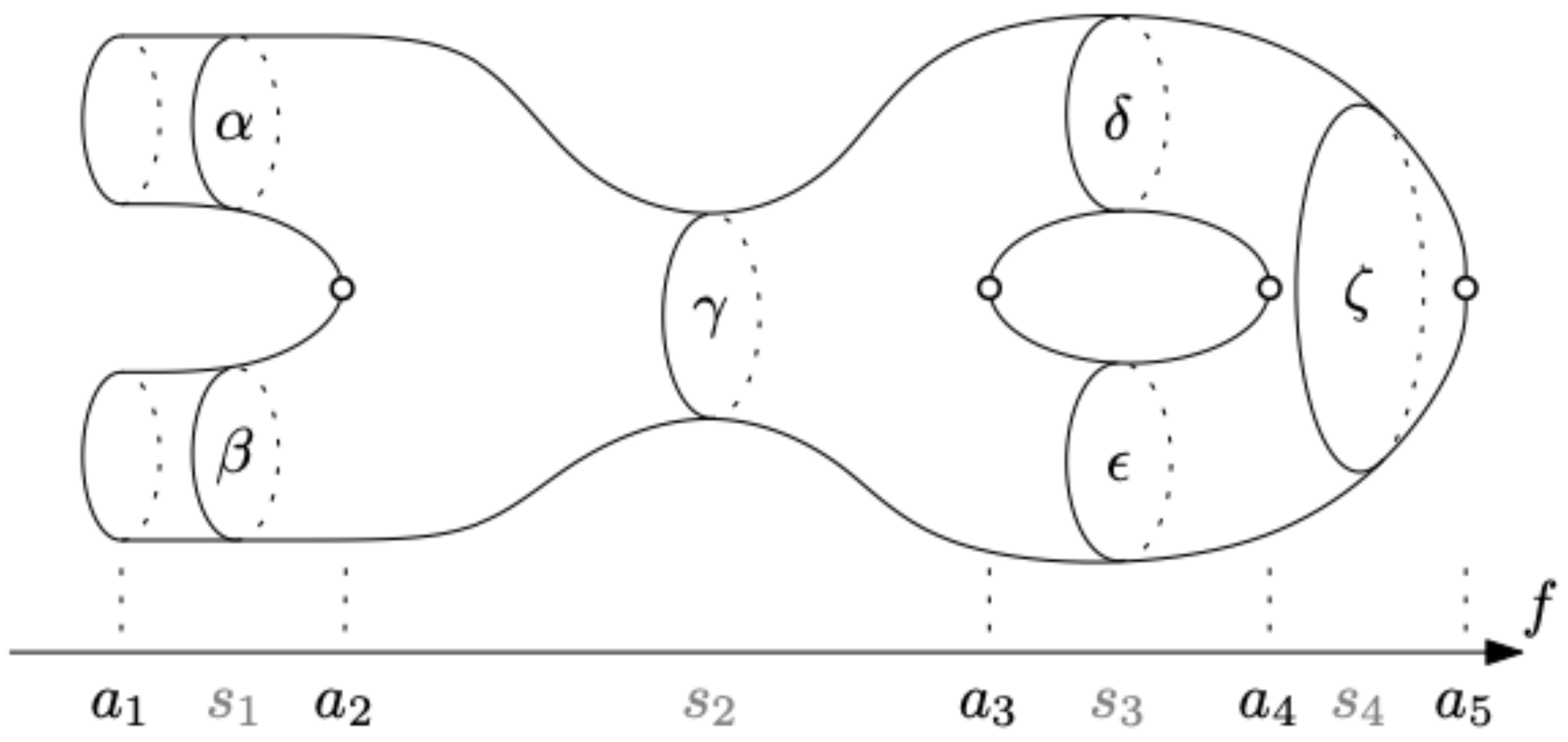

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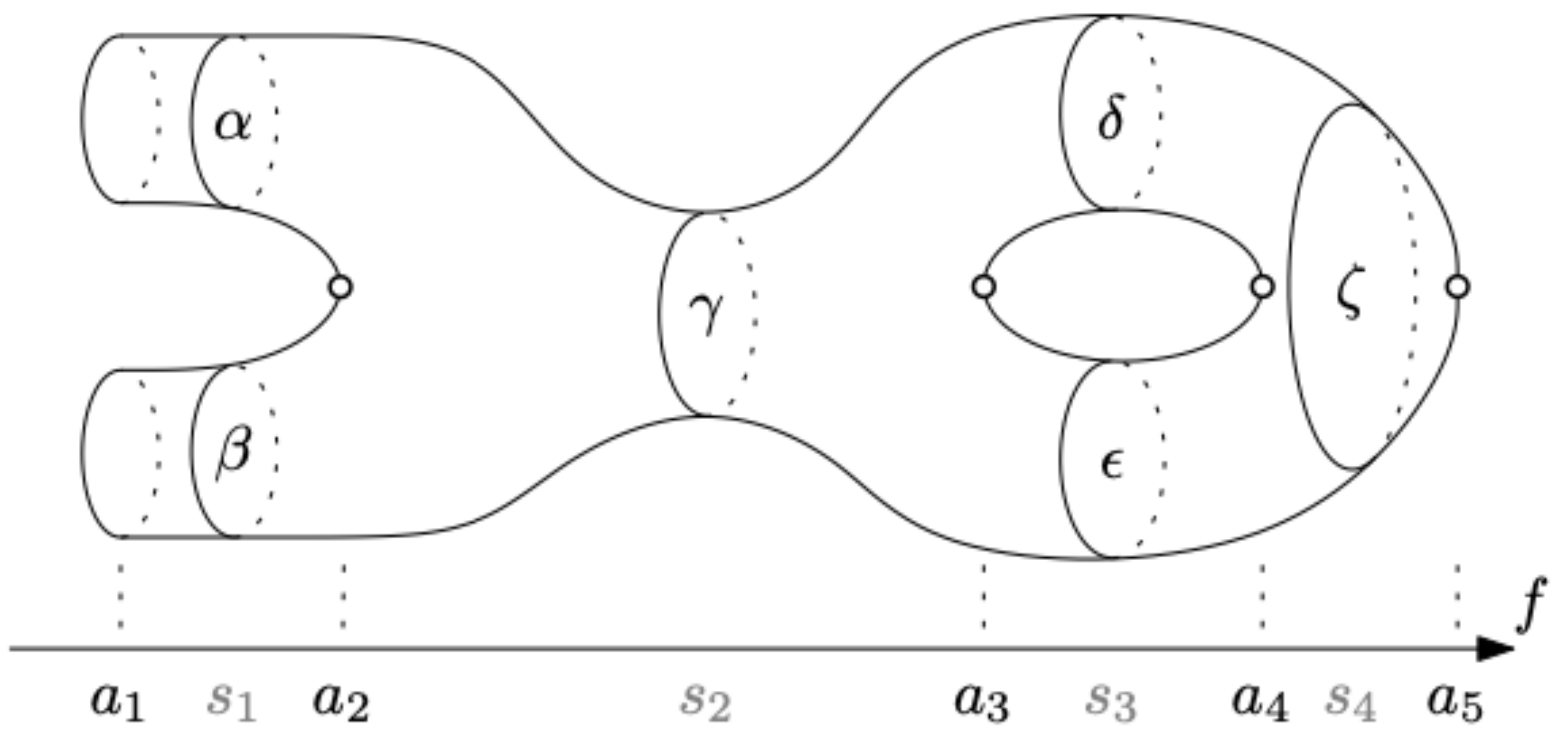

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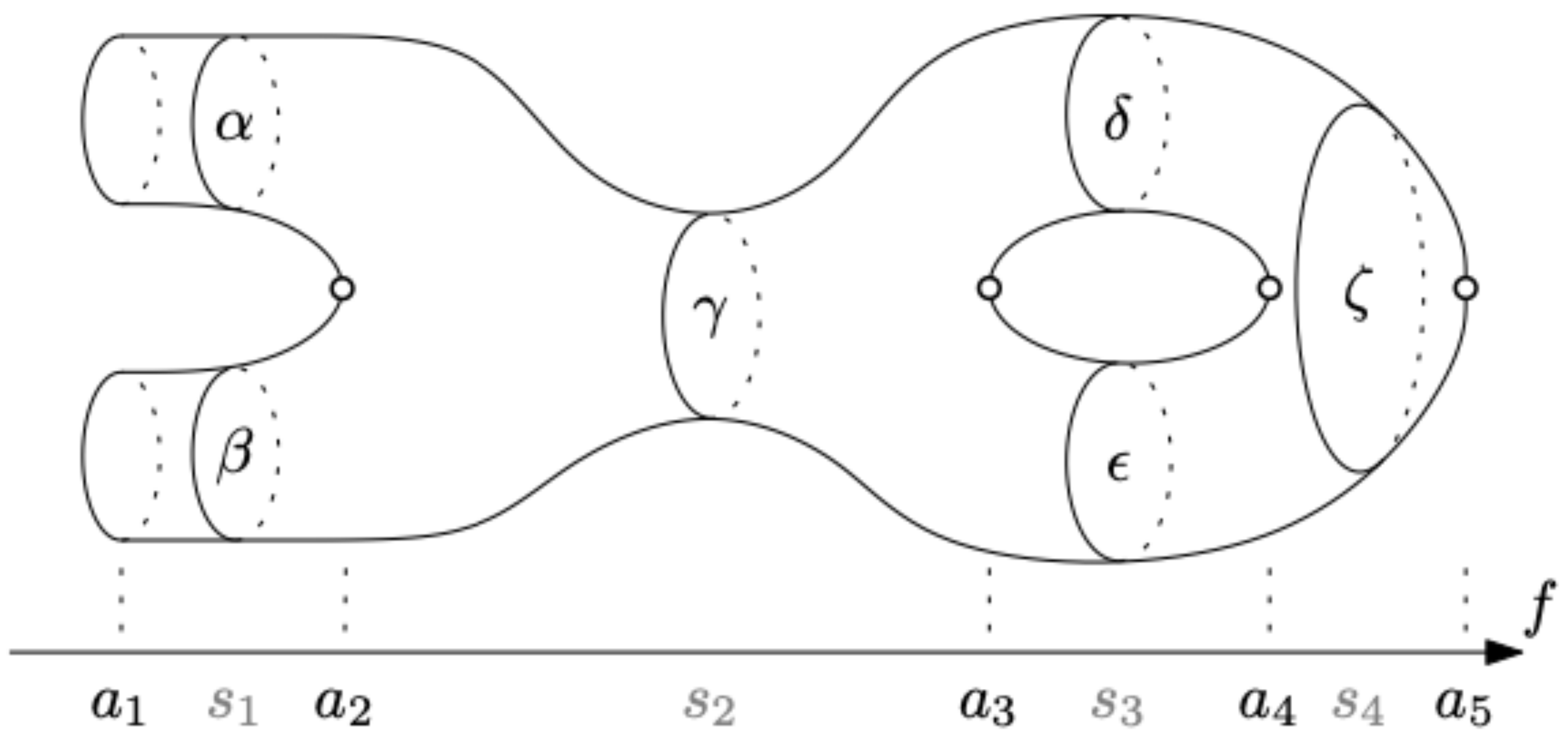
- Applying homology to this sequence of topological spaces induces a zigzag persistence module

$$H_i(f^{-1}(s_0)) \longrightarrow H_i(f^{-1}([s_0, s_1])) \longleftarrow H_i(f^{-1}(s_1)) \longrightarrow H_i(f^{-1}([s_1, s_2])) \longleftarrow \cdots$$



Invariants

The Rank Function

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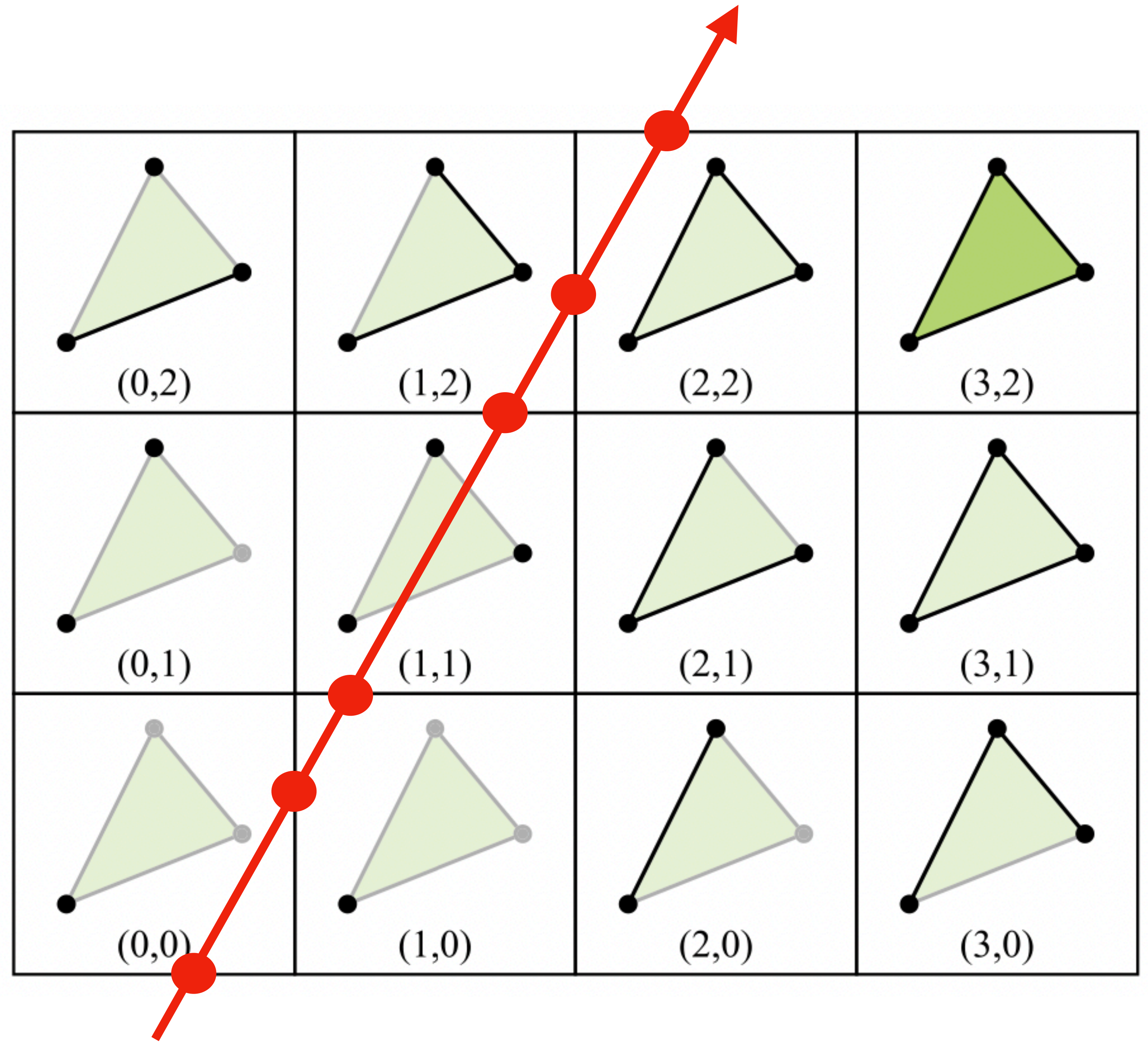
- The rank function of a persistence module M is the function that assigns to any interval $[a, b] \subseteq \mathbb{R}^n$ the value $\mathsf{rank}\,\mathsf{M}(a \leq b)$.
- In the 1-d setting, the rank function is equivalent to the persistence diagram.

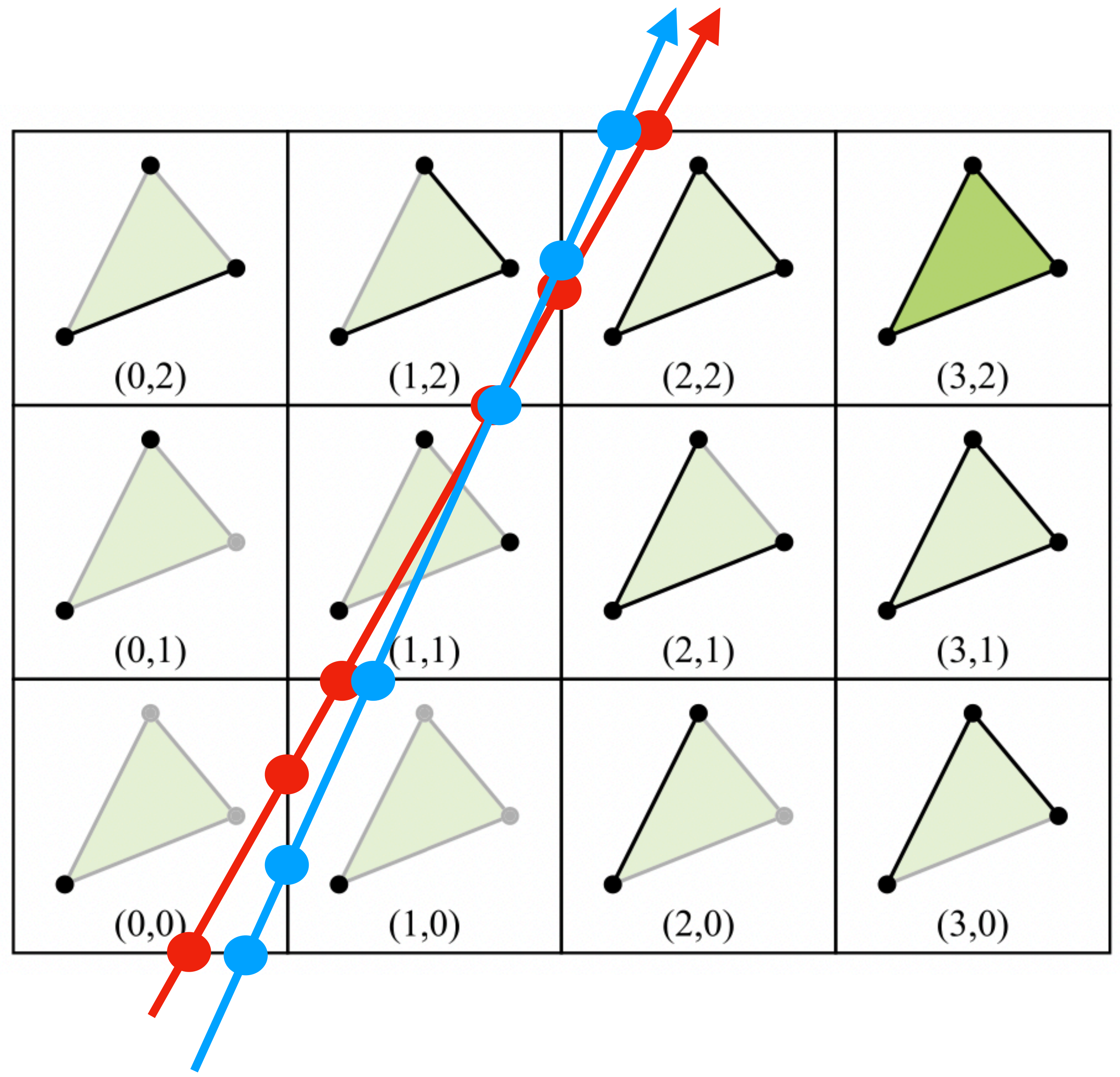
Slicing Multiparameter Persistence Modules

- Given a multiparameter persistence module, we can slice it to obtain 1-parameter persistence modules.
- This allows us to use the ordinary persistent homology algorithm to obtain a barcode for each slice.
- The collection of all these barcodes is known as the fibered barcode.
- This idea was developed in [4]. They prove that the fibered barcode is equivalent to the rank invariant.



Slicing Continued

- The persistence diagrams for each slice do not need to be computed independently.
- Nearby lines will have similar barcodes.
- The vineyard algorithm can be used to compute the barcode of a nearby line from the barcode of a fixed line in linear time.
- The fibered barcode is a stable invariant [4].



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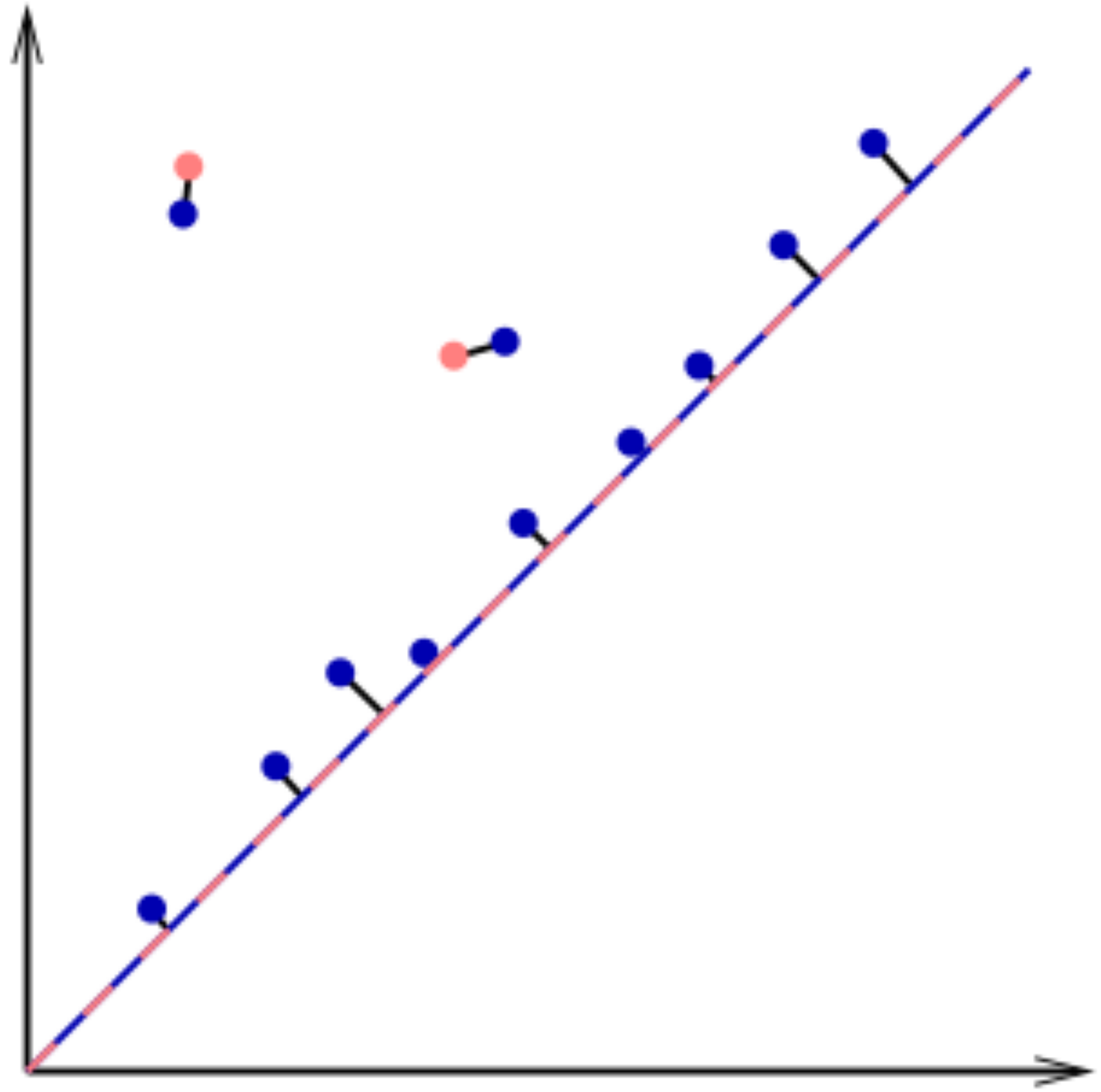

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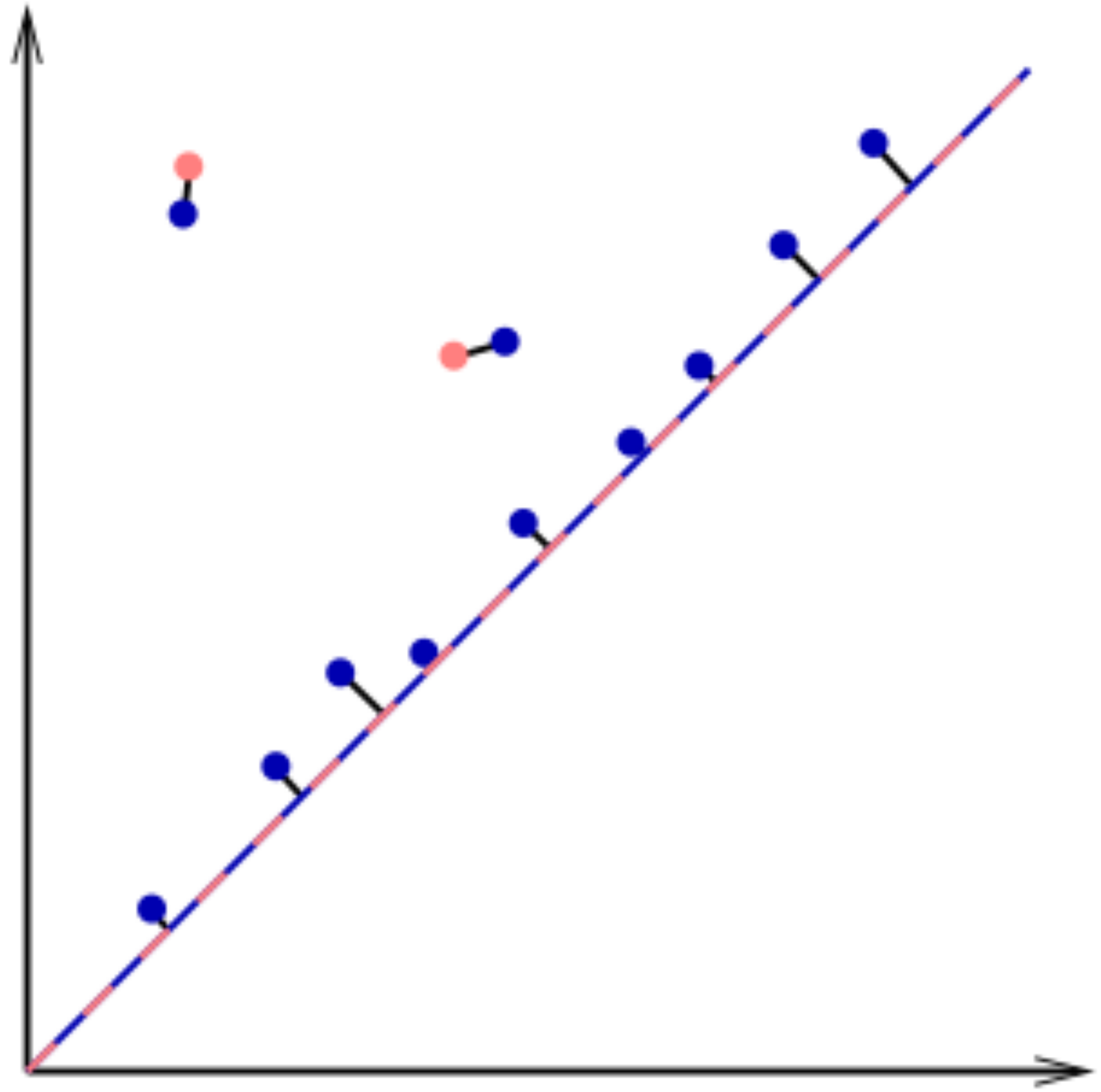

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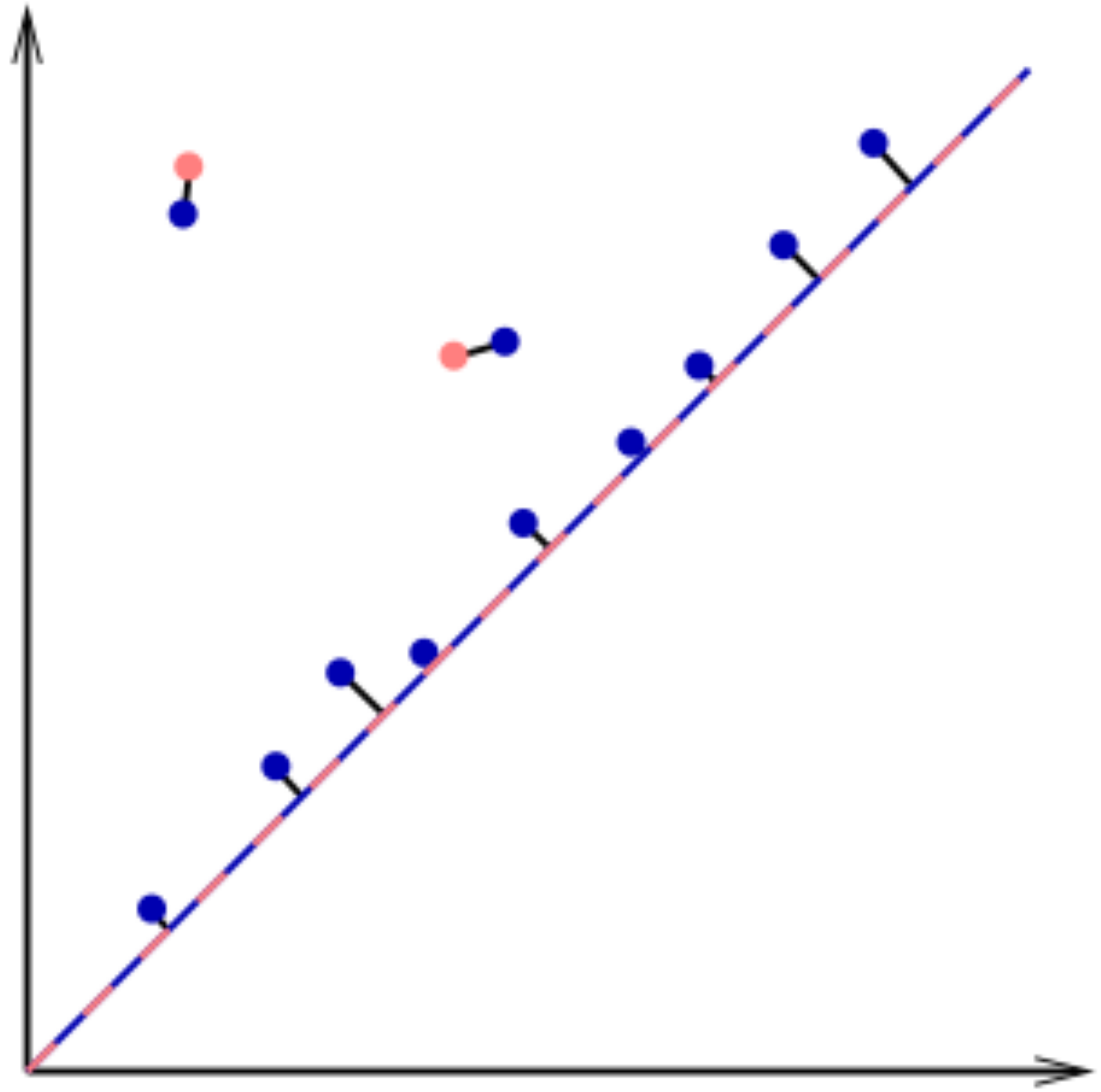

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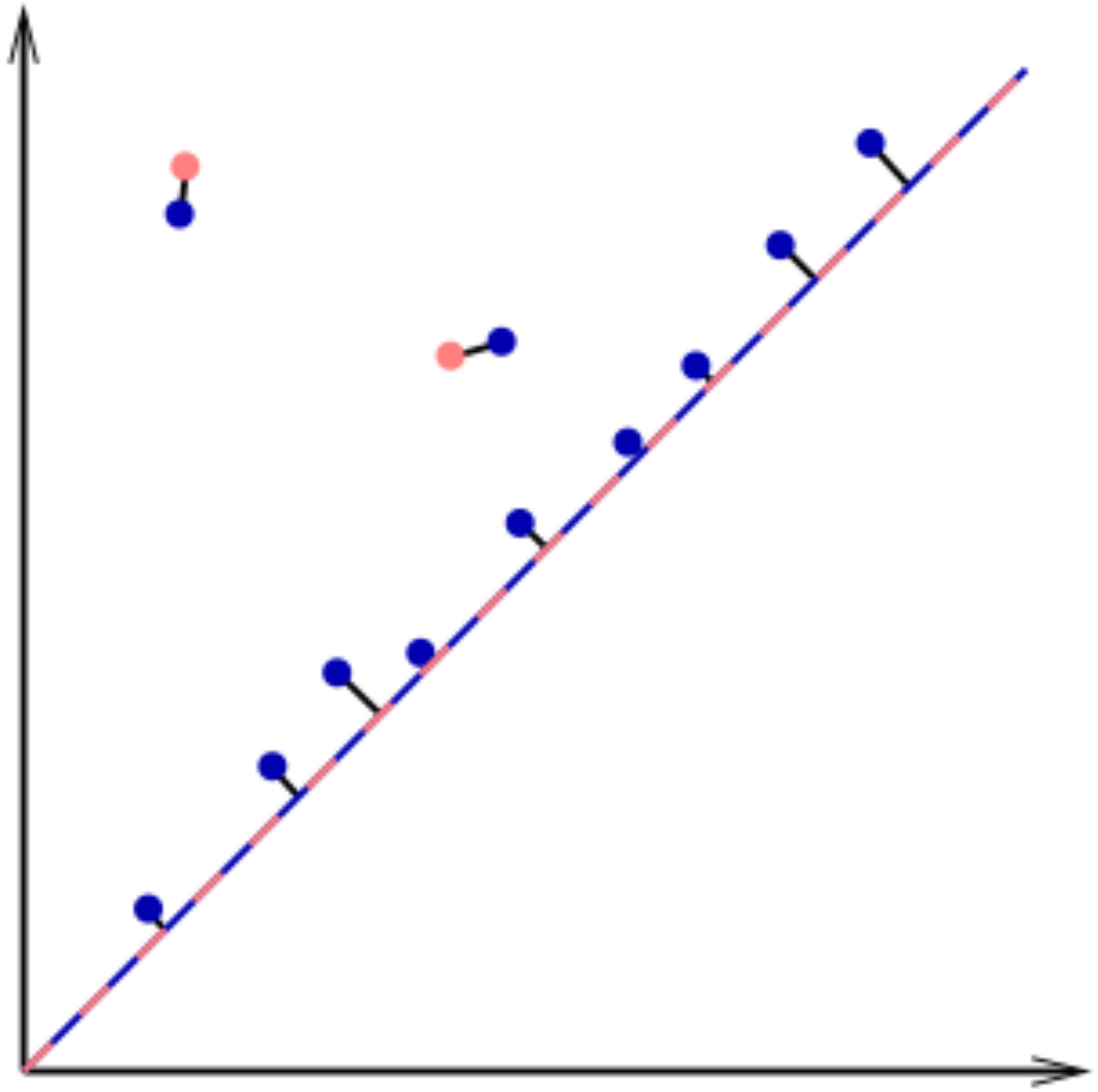
- The bottleneck distance extends to a distance between fibered barcodes by maximizing the bottleneck distance between each sliced barcode.



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- The matching distance between the fibered barcodes of M and N is

$$\sup_{\ell} \ \omega_\ell \, d_B(BM_\ell, BN_\ell)$$

where $\omega_\ell = \frac{1}{\sqrt{1+m^2}}$ when $\ell = mx + b$

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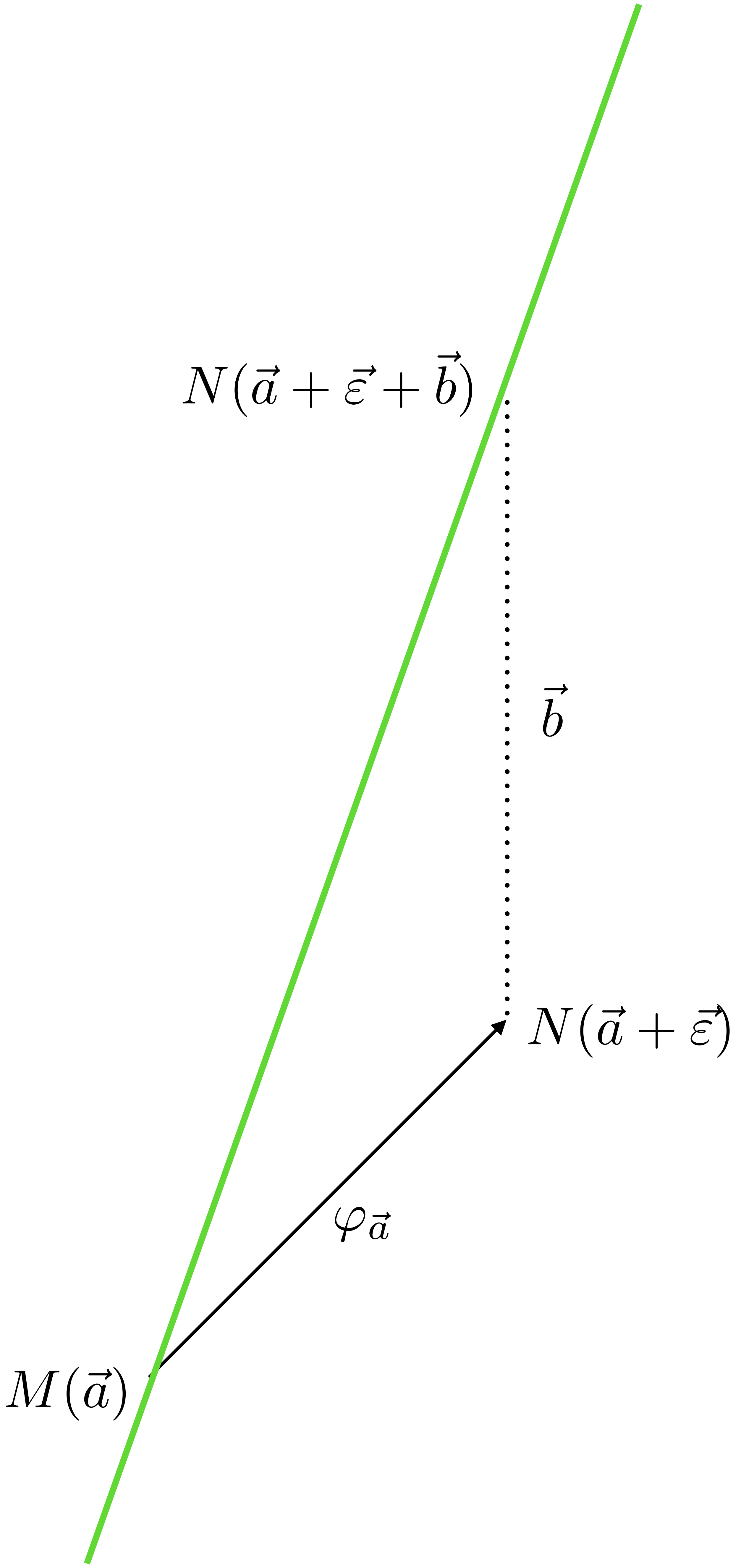

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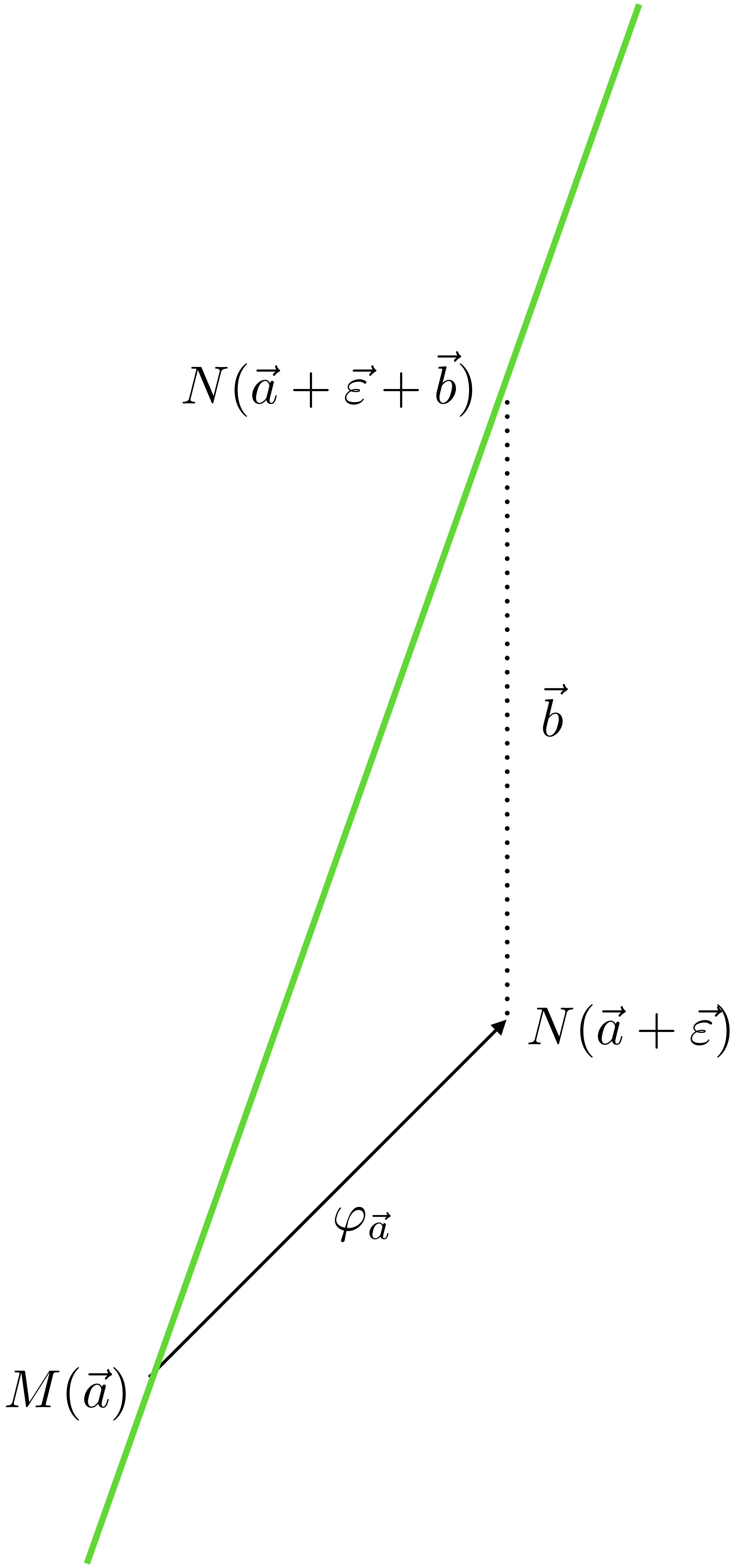

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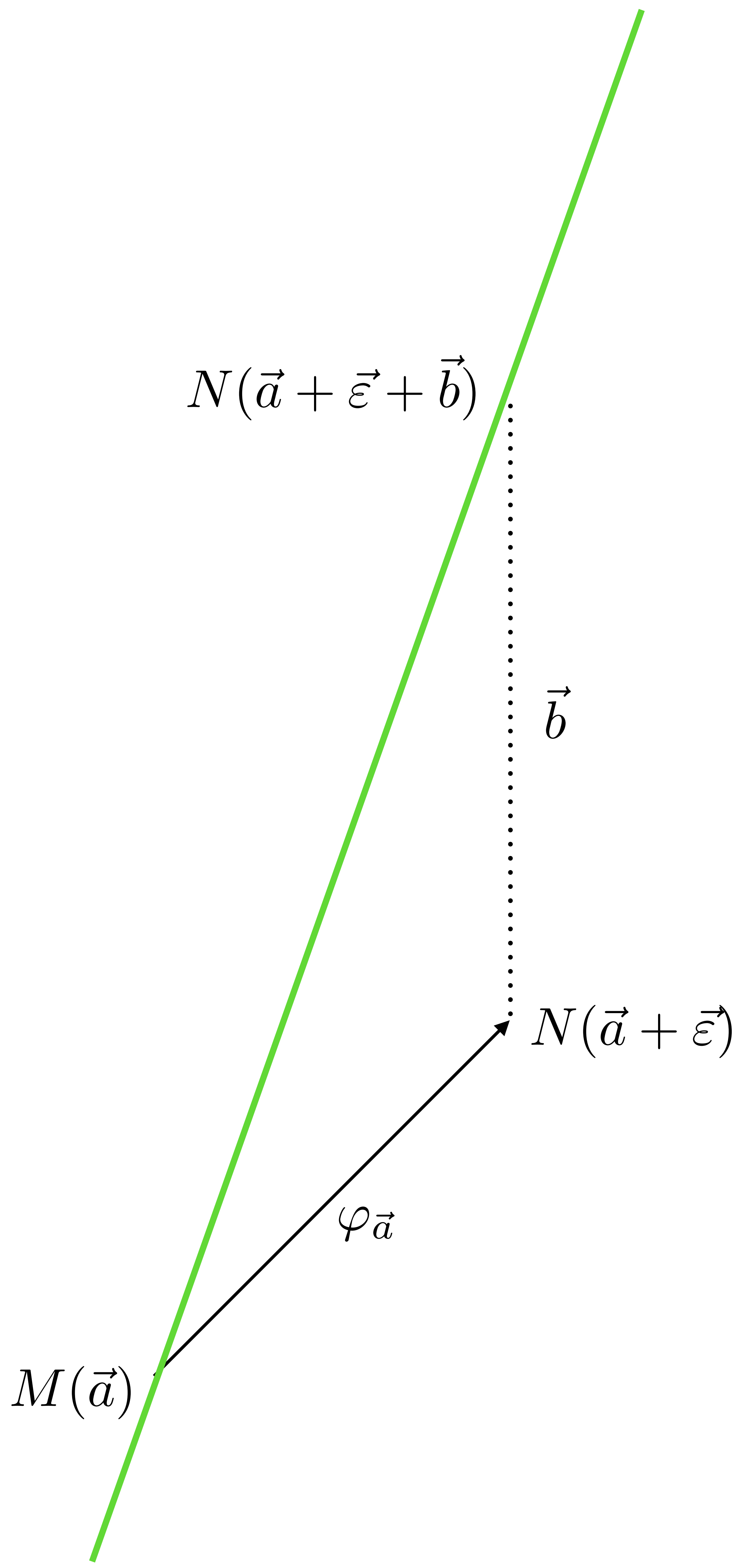
- It follows that the matching distance between the fibered barcodes of M and N is at most ε.



Algorithms

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- Essentially, RIVET allows you to input a multiparameter persistence module, select a line, and it will compute the persistence diagram of the module restricted to that line.
- Unfortunately, RIVET is too slow for use with large data sets, running in time $O(n^8)$ [6].

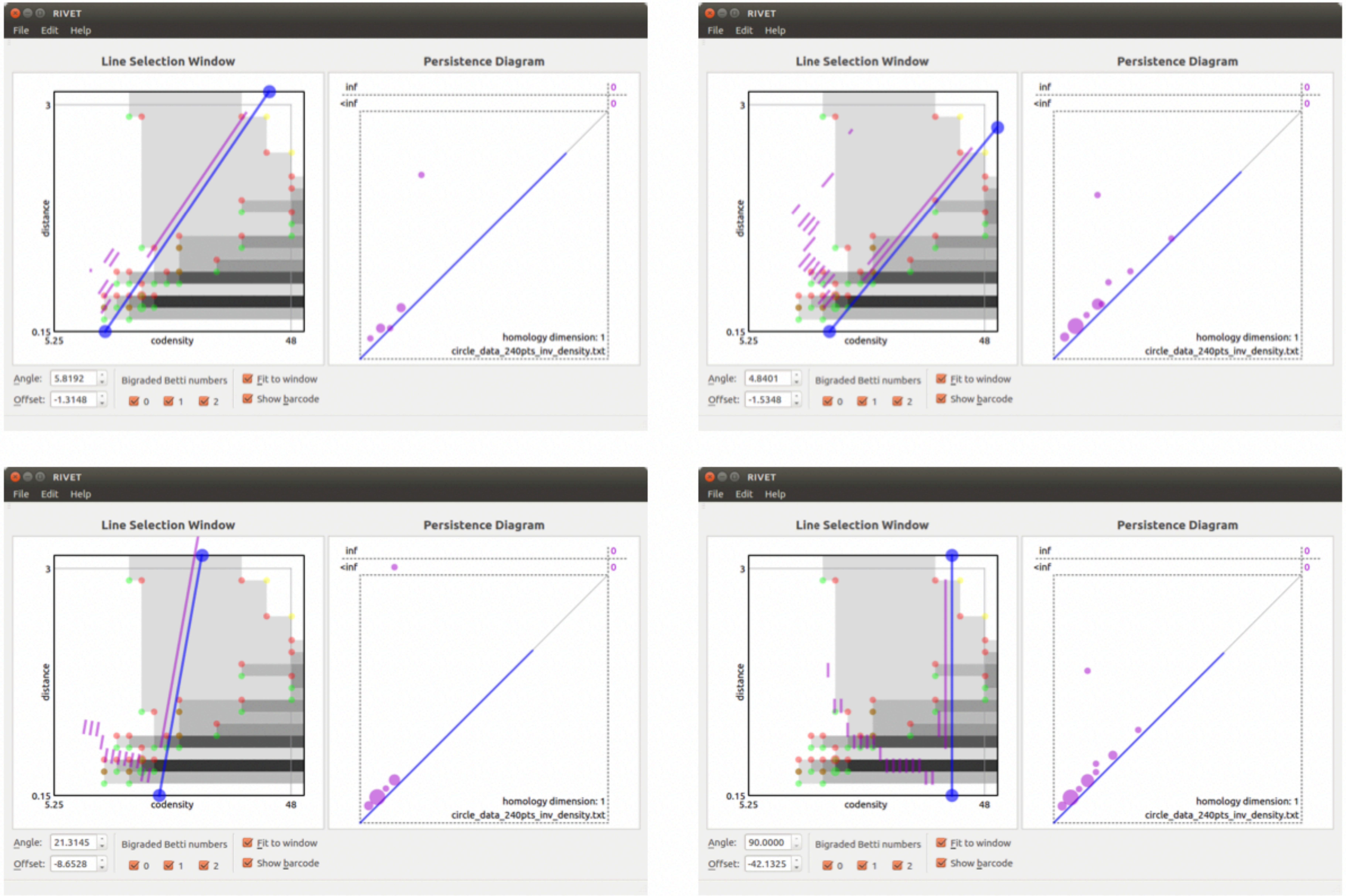

Figure 3: Screenshots of RIVET for a single choice of 2-D persistence module M and four different lines L. RIVET provides visualizations of the dimension of each vector space in M (greyscale shading); the 0^{th}, 1^{st}, and 2^{nd} bi-graded Betti numbers of M (green, red, and yellow dots); and the barcodes of the 1-D slices M^L, for each L (in purple).

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$$\mathbb{R} \xrightarrow{x\mapsto(x,0)} \mathbb{R}^2 \xrightarrow{\mathrm{id}} \mathbb{R}^2 \xrightarrow{(x,y)\mapsto x+y} \mathbb{R} \longrightarrow 0$$

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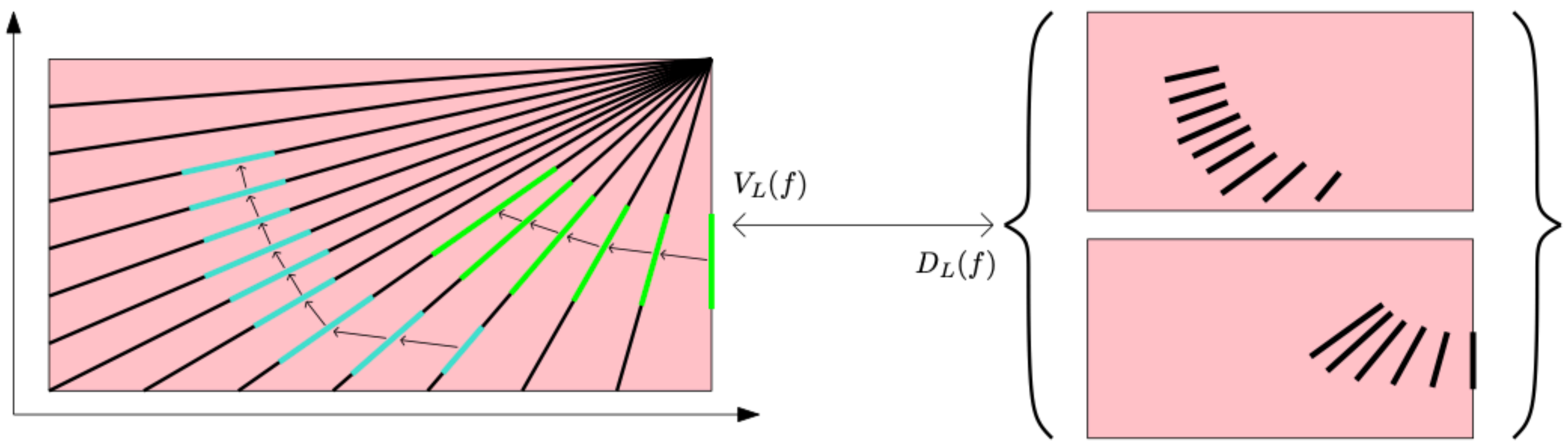

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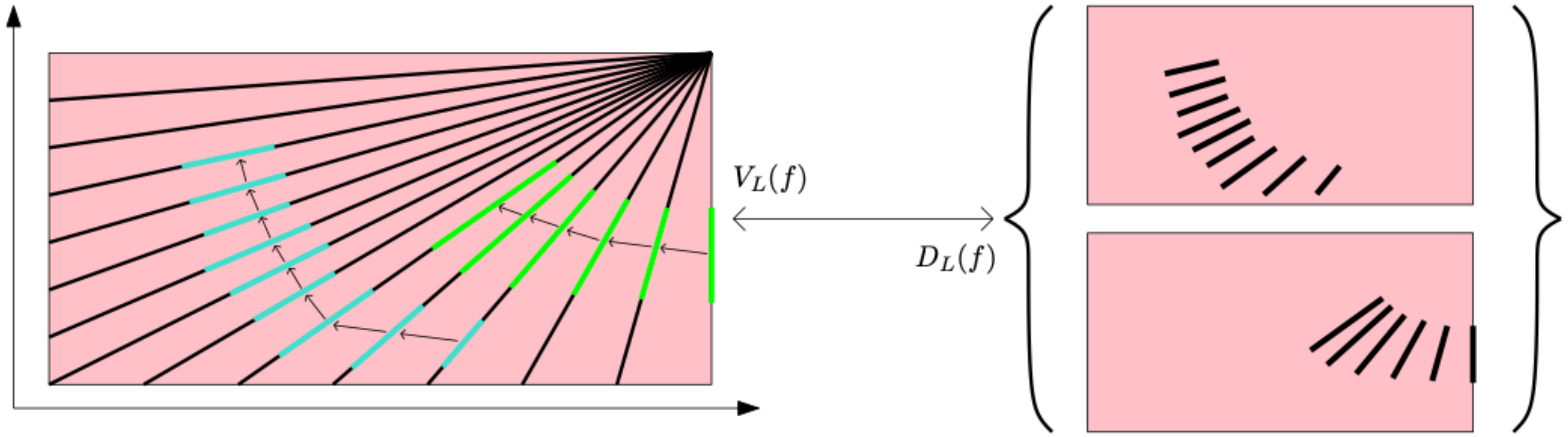

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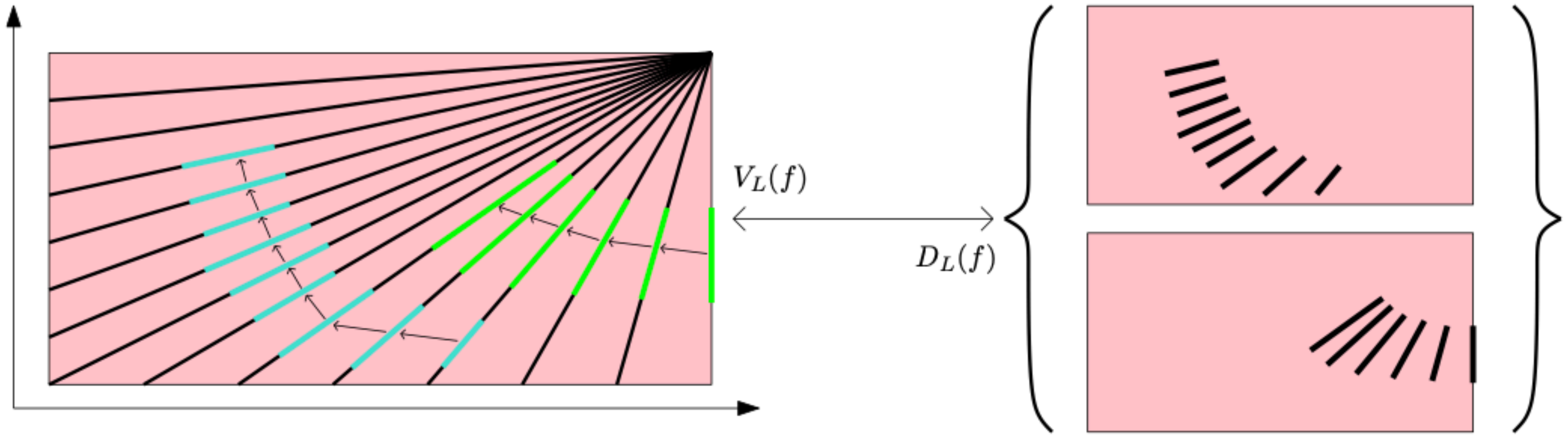
- Vineyard Decompositions are stable [3].



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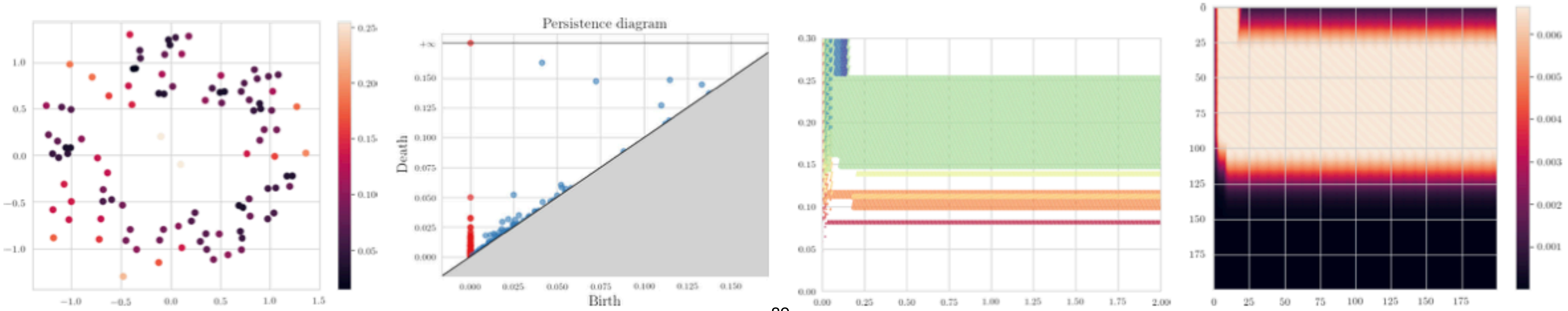

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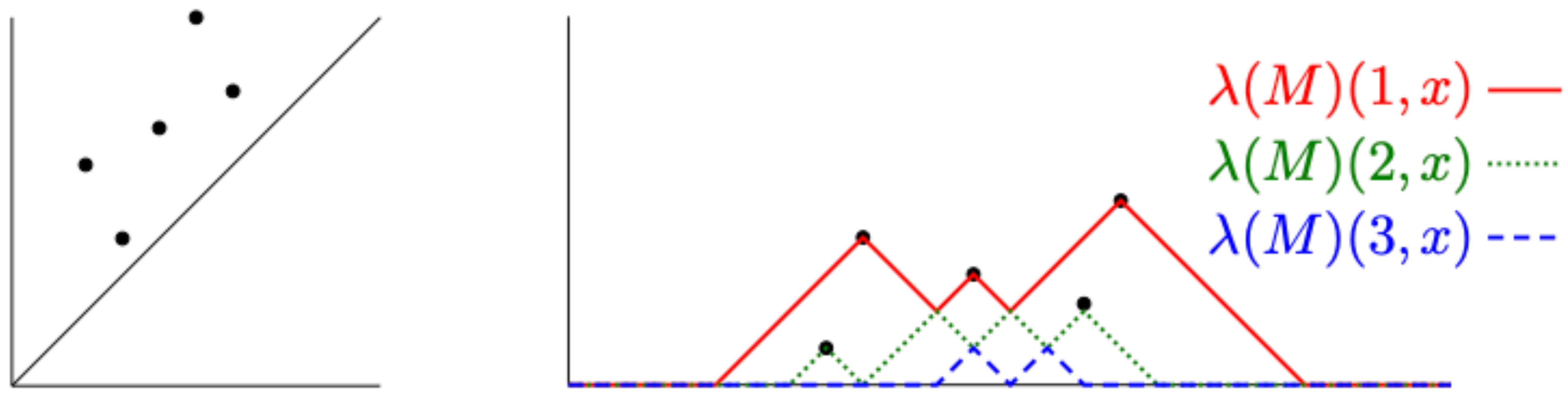

Statistics

- Even in the 1-parameter setting, barcodes do not necessarily have well defined means.
- Persistence landscapes and persistence images fix this issue, producing vectors in euclidean space where an arsenal of statistics techniques can be applied.
- Multiparameter persistence landscapes in particular obey a strong law of large numbers and a central limit theorem [7].

Complexity

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- Multiparameter persistence landscapes are built off of the fibered barcode and thus require a program such as RIVET first.
- Multiparameter persistence images are generally faster to compute than fibered barcodes, with a complexity depending on the resolution one wishes to calculate with.

Summary

- Multiparameter persistence is hard. There's no clear answer unlike the 1-parameter case.
- Even though we don't have a complete invariant, we can still compute invariants that lose information.
- The most widely used invariant for multiparameter persistence is the fibered barcode (AKA the rank invariant).
- The fibered barcode is stable to noise.
- Persistence images and landscapes are useful for machine learning applications and statistics.
- The elder rule staircode is the best bet for 0-dimensional homology with augmented metric spaces.

Coming Weeks

- Next week Nate Clause will go more in depth into the algorithms for multiparameter persistence including zigzag persistence and a demo of RIVET.
- The following week Brantley Vose will cover “The Importance of Forgetting” and show how zigzag persistence appears in neuroscience.

References

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