Persistent cohomology and an application in neuroscience

TDA+Neuro (CSE 5339), Lecture 6

Ling Zhou

Brain GPS system

Place cells

Grid cells

1971

2005

The 2014 Nobel Prize in Physiology or Medicine:

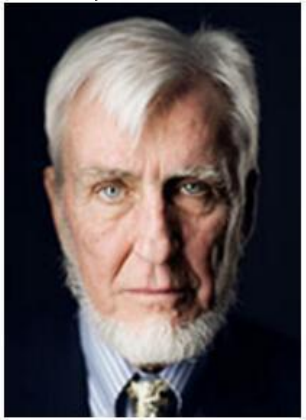

Photo: A. Mahmoud
John O'Keefe
Prize share: 1/2

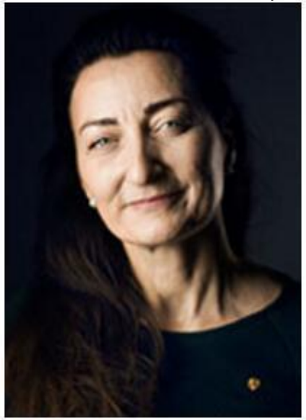

Photo: A. Mahmoud
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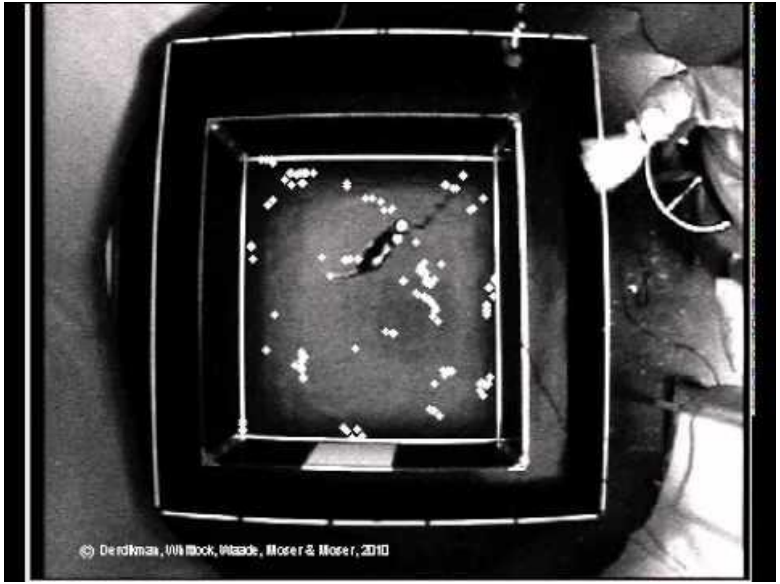
Place cells → 'You are here'

Place cells → **'You are here'**

Grid cells → **Map**



Place cells → 'You are here'

Grid cells → Map

Head-direction cells → Compass

Place cells → 'You are here'

Grid cells → Map

Head-direction cells → Compass

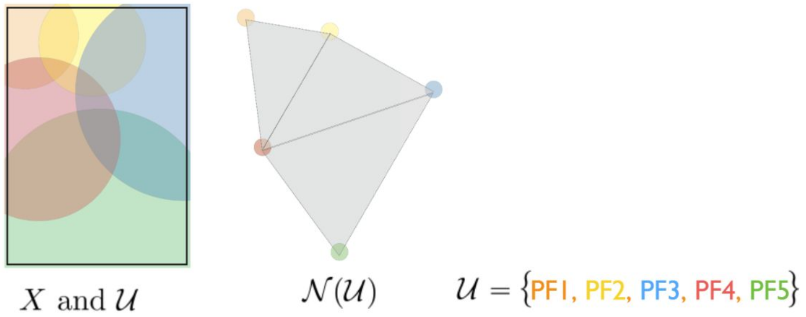
An online lecture for head direction cells, grid cells, and others:
https://www.youtube.com/watch?v=CQPswbIuCkk&ab_channel=MITCBMM

Firing fields for place cells (place fields)

Recall from Lecture 1:

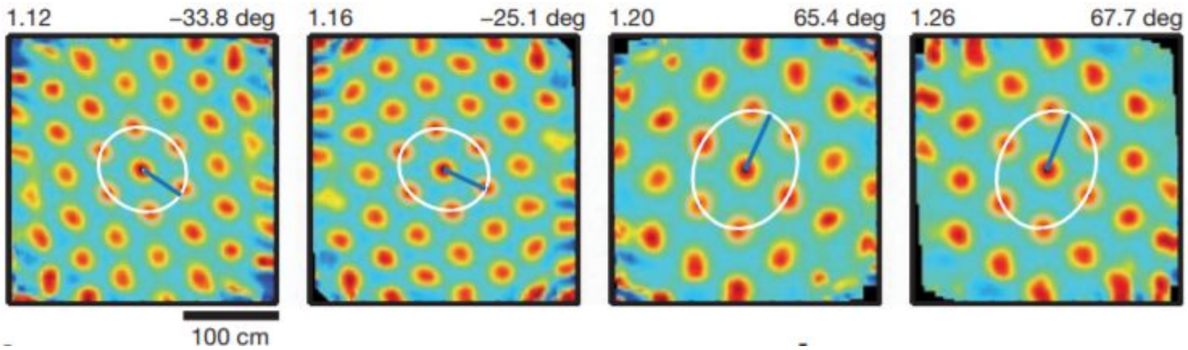
- Place fields form a good cover of the arena
- Nerve lemma => the topology of the nerve complex associated to place fields is the same as that of the arena

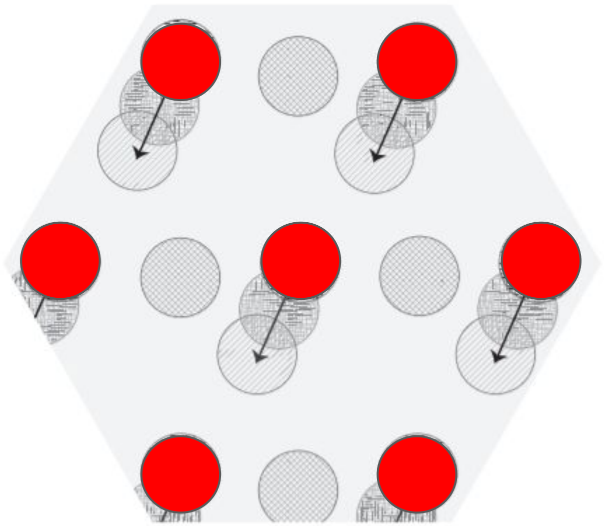
Theorem. *If $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of a compact space X such that every non-empty intersection of finitely many sets in $\mathcal{U}$ is contractible, then X and (the geometric realization of) $\mathcal{N}(\mathcal{U})$ are homotopy equivalent.*



Firing fields for grid cells

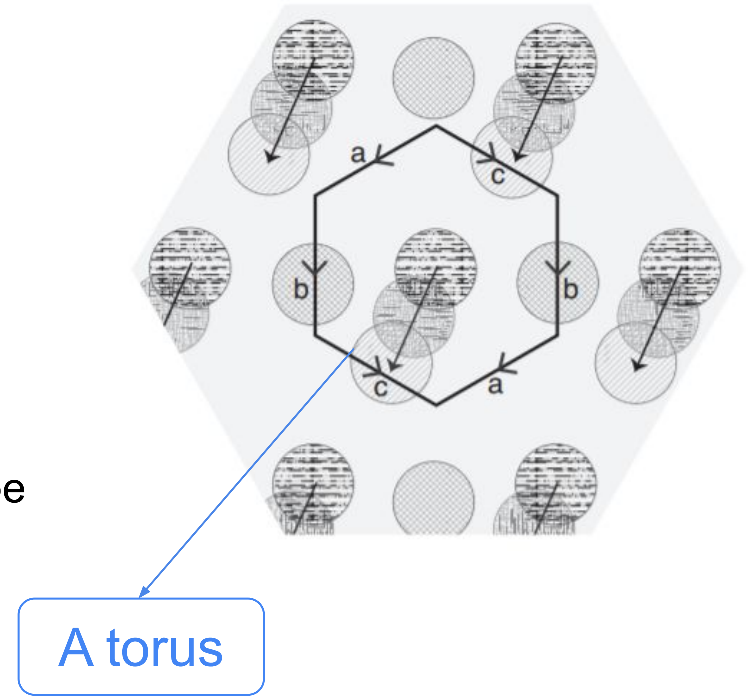
- Grid cell fires at multiple locations, typically in a grid-like triangular lattice
- Firing fields have different modules (depending on scales and orientations), [Stensola et al., 2012].





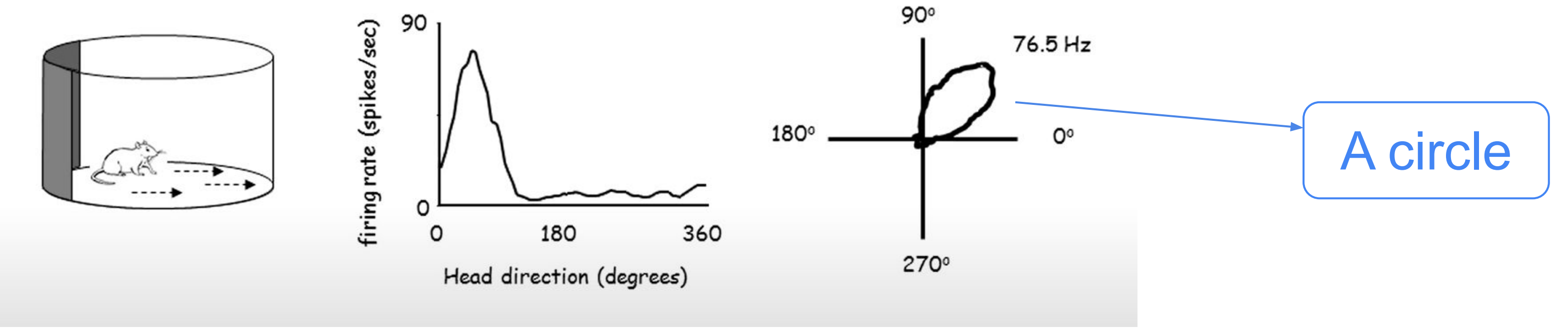
Firing fields for grid cells

- Grid cell fires at multiple locations, typically in a grid-like triangular lattice
- Firing fields have different modules (depending on scales and orientations), [Stensola et al., 2012].
- Firing fields do NOT form a good cover for the whole arena (intersections are not always contractible)
- Firing fields form a good cover for a hexagon-shape fundamental domain, [Curto, 2017].



Firing fields for head-direction cells

- Head direction cell fires when the head is pointing to a certain direction (independent of the location).
- Firing field for head-direction cells is a circle.



Reference for the above figure: https://www.youtube.com/watch?v=0oM_ueYPqDs&ab_channel=Spatialcognition

Main reference:

[Kang et al.,2021] Kang, L., Xu, B., & Morozov, D. (2021). Evaluating state space discovery by persistent cohomology in the spatial representation system. *Frontiers in computational neuroscience*, *15*, 28.

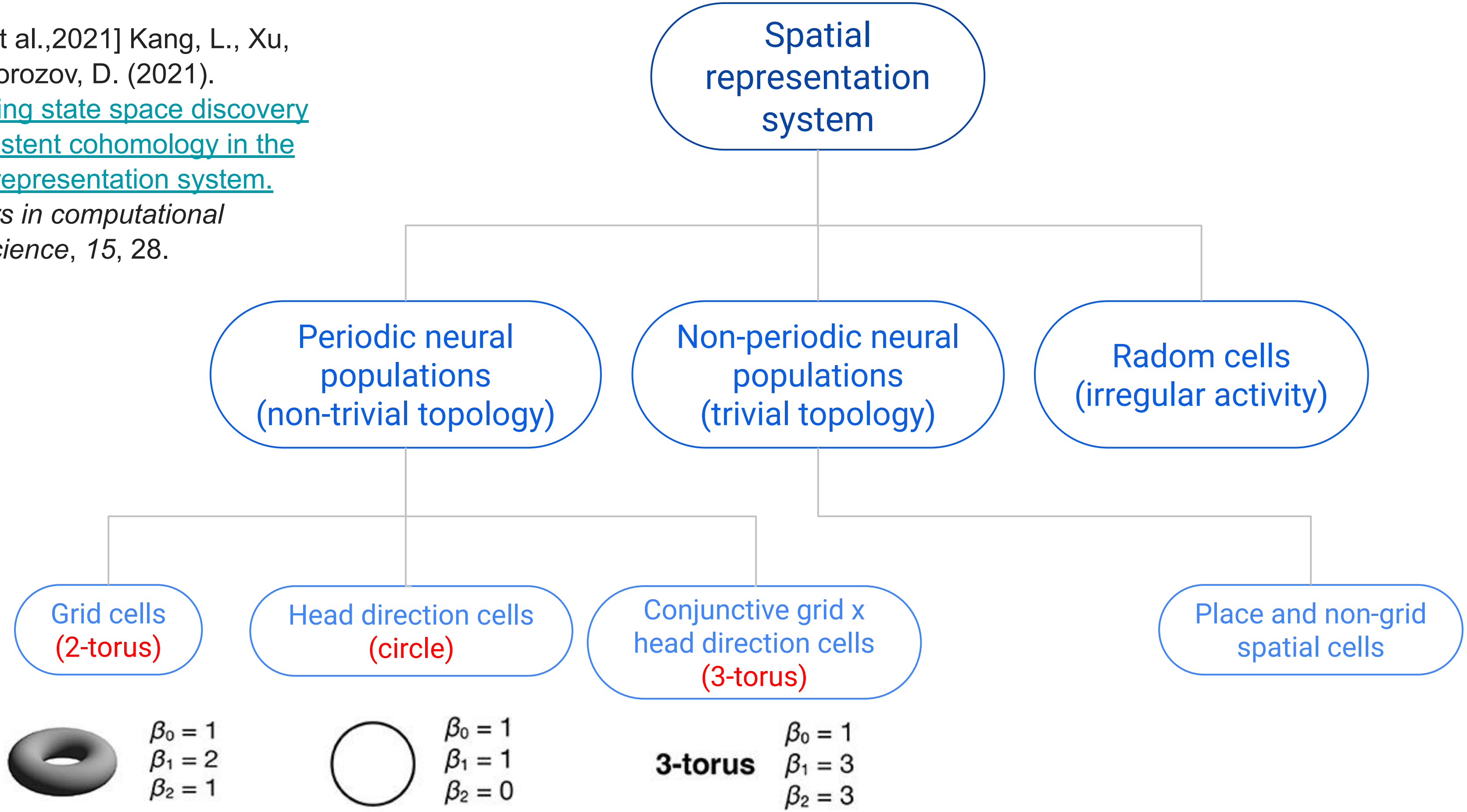
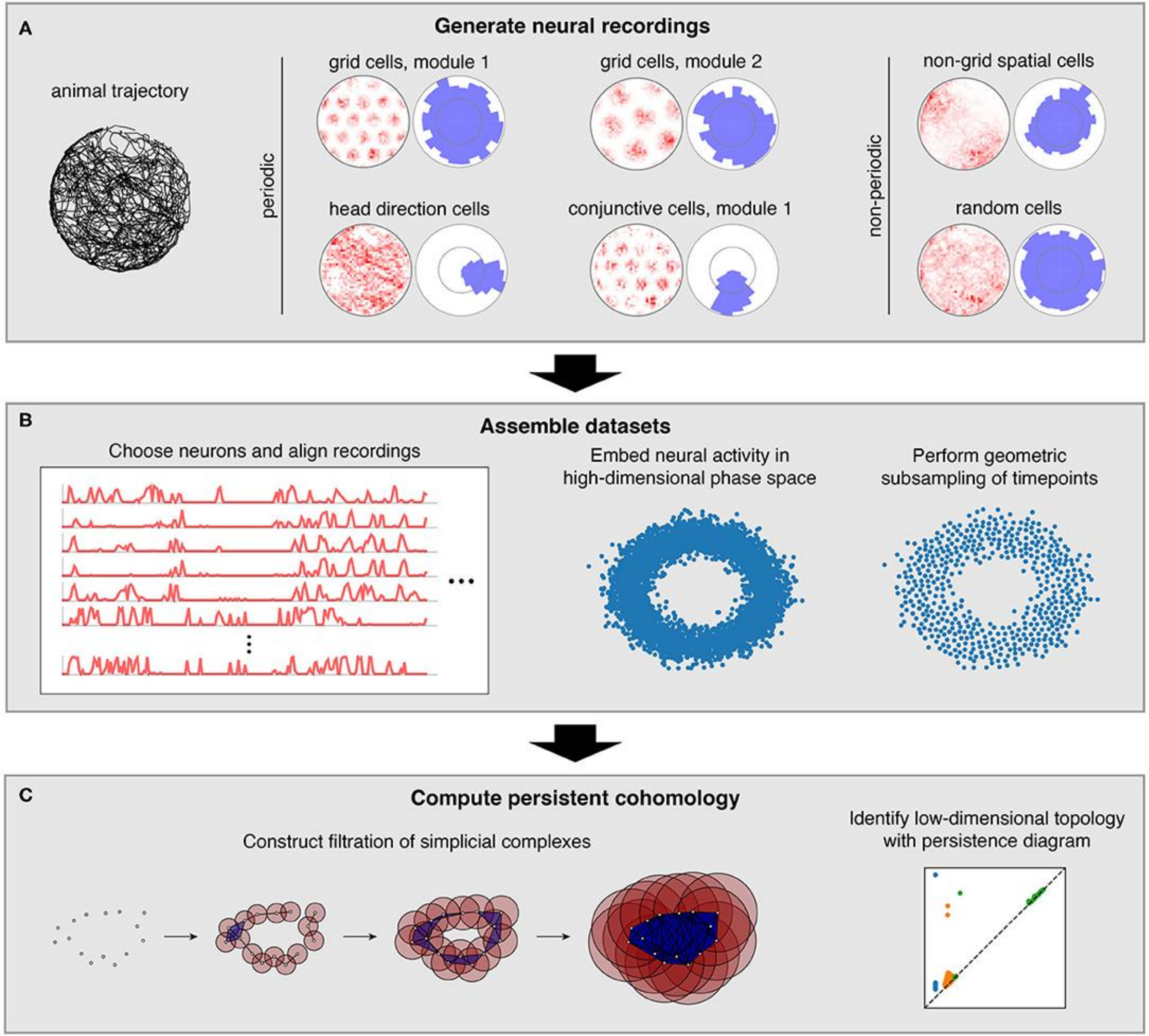


Figure 1

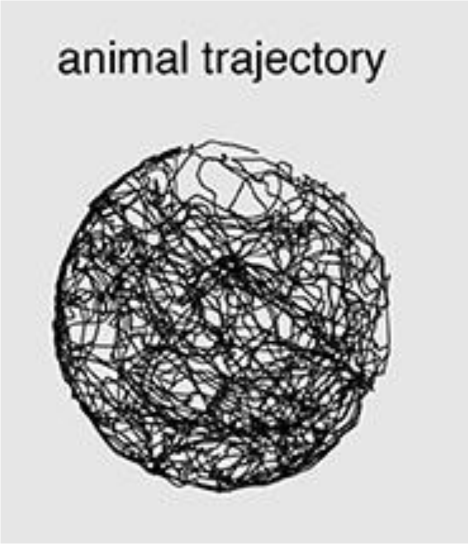


From now on, figures are labeled in the same way as reference [3].

Generate neural recordings and assemble datasets

Animal trajectory

- Arena: a circular enclosure of diameter 1.8 m
- Animal trajectory: use 1,000s of data from a trajectory extracted from an experimental animal, [Hafting et al. 2005, Burak & Fiete, 2009]
- Time bin: 0.2s (=> 5000 timepoints)
- Obtain data for positions and directions by taking average over each time bin.



The neural recordings will be simulated from the positions and the directions obtained from this trajectory.

Generate neural activities for periodic neural populations

$\boxed{\text{Tuning curve}}$: a function of position and/or direction for each neuron

- For grid cell: $s_{\text{grid}}(\mathrm{x};\mathrm{b}) = f\left(\frac{1}{0.45l}\left\|\mathsf{A}\left\langle \mathsf{A}^{-1}\mathrm{x}-\mathrm{b}\right\rangle_{1/2}\right\|\right)$

position

offsets

- For head direction cell: $s_{\text{dir}}(\theta; c) = f\left(4\langle\theta - c\rangle_{\pi}\right)$.

direction

- For conjunctive cell: $s_{\text{conj}}(\mathrm{x},\theta;\mathrm{b},c) = s_{\text{grid}}(\mathrm{x};\mathrm{b})\, s_{\text{dir}}(\theta;c)$.

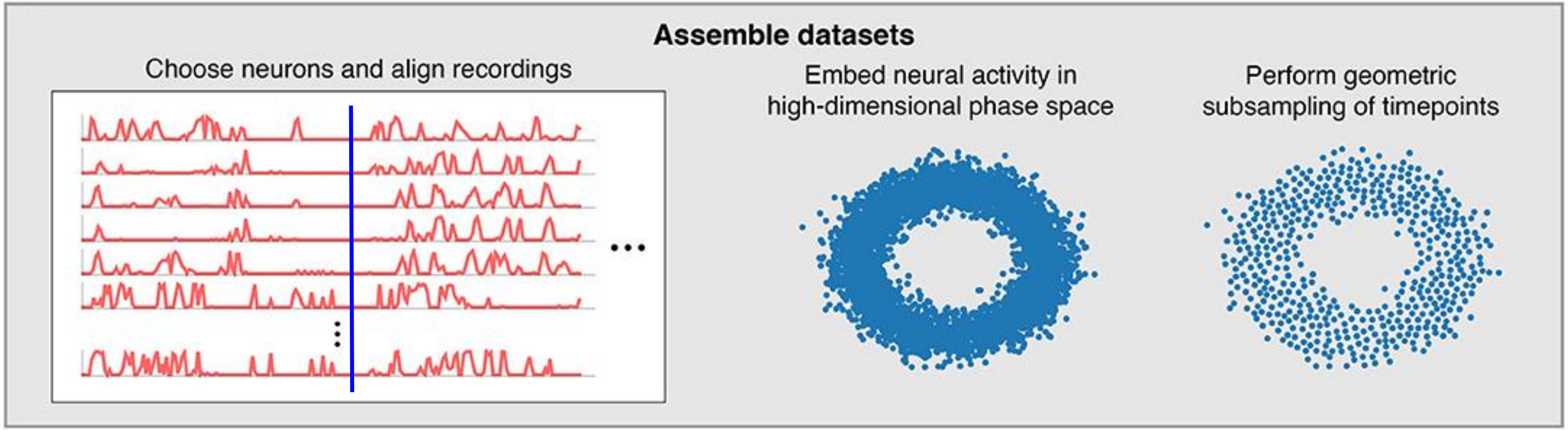
$$f(z) = \begin{cases} \frac{1+\cos\pi z}{2} & |z| < 1 \\ 0 & |z| \geq 1. \end{cases}$$

$$\mathsf{A} = l\begin{pmatrix} \cos\phi & \cos\left(\phi+\frac{\pi}{3}\right) \\ \sin\phi & \sin\left(\phi+\frac{\pi}{3}\right)\end{pmatrix}$$

$l = 40$ cm and $\phi = 0$

$\langle a\rangle_m \equiv (a + m \bmod 2m) - m$

Assemble datasets & reduce size of data (5000 -> 1000)



1. Remove timepoints when all neurons are not active enough:
2. Geometric subsampling:
    - pick the first point at random
    - iteratively add a point that is the furthest from the chosen points

The above steps do not lose important information (due to stability).

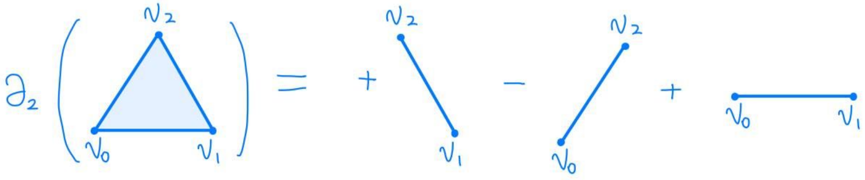
Persistent cohomology and circular coordinates

Homology

p-chain: linear combination of p-simplices

$C_p(X)$: set of p-chains

∂_p: boundary map



$$0 \leftarrow C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \leftarrow \cdots$$

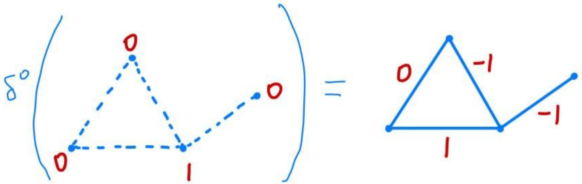
$$\boxed{H_p = \ker \partial_p \,/\, \mathrm{Im}\, \partial_{p+1}}$$

Cohomology

p-cochain: map from $C_p(X)$ to the coefficients

$C^p(X)$: set of p-cochains

δ^p: coboundary map



$$0 \rightarrow C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} C^2(X) \rightarrow \cdots$$

$$\boxed{H^p = \ker \delta^p \,/\, \mathrm{Im}\, \delta^{p-1}}$$

Persistent cohomology

For a point cloud P, its Vietoris-Rips complex at scale r is

$$\mathrm{VR}(P,r) = \{\sigma \subseteq P \mid \|p-q\| \leq r \forall p,q \in \sigma\}.$$

Apply cohomology to the Vietoris-Rips filtration, we obtain

$$\mathrm{H}^k(\mathrm{VR}(P,r_1)) \leftarrow \mathrm{H}^k(\mathrm{VR}(P,r_2) \leftarrow \mathrm{H}^k(\mathrm{VR}(P,r_3)) \leftarrow \ldots$$

- Persistent homology and persistent cohomology have the same barcode.
- Persistent cohomology is faster to compute than persistent homology.

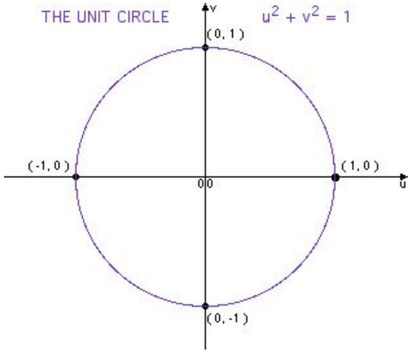
Circular coordinates

Dimension reduction algorithm: for a dataset X sampled from a high-dimensional manifold $M \subset R^N$, find coordinate mappings $(f_1, \ldots, f_n): X \to R$ for $n<N$, which faithfully preserves the 'intrinsic' structure of X.

Popular methods:

- Linear dimension reduction: principal components analysis, linear regression, etc;
- Non-linear dimension reduction: Isomap, Locally Linear Embedding, etc.

Problem: the above methods can lose significant information in some case. For example, think about embedding a circle (a 1-dimensional manifold) to R.



Reference [de Silva et al., 2011a] proposes to enlarge the class of coordinate functions to include circle-valued coordinates

$$\theta: X \to S^1.$$

Circular coordinates

The first cohomology provides the choices of circular coordinates:

$$[X, S^1] = \mathrm{H}^1(X; \mathbb{Z}).$$

The circular coordinate pipeline:

- Construct a filtered Vietoris-Rips complex to approximate the underlying manifold.
- Compute persistent cohomology over a finite field to identify significant 1-cocycles.
- Lift the 1-cocycles to obtain integer-valued 1-cocycles.
- Replace the integer-valued 1-cocycles by some smoothed real-valued 1-cocycles.
- Integrate the smoothed 1-cocycles to obtain circle-valued functions $\theta: X \to S^1$.

Choices of circular coordinates are not unique.

Apply persistent cohomology

Detecting grid cells

Goal: verify that the given cells are grid cells.

Method:

- Compute persistent cohomology of the Vietoris-Rips filtration of datasets and pick cocycles above a threshold.
- ‘Success’, if there are two 1-cocycles for grid cells.
- Experiment 100 times and compute ‘success rate’.

Conclusion:

longer memory
more neurons
} higher success rates

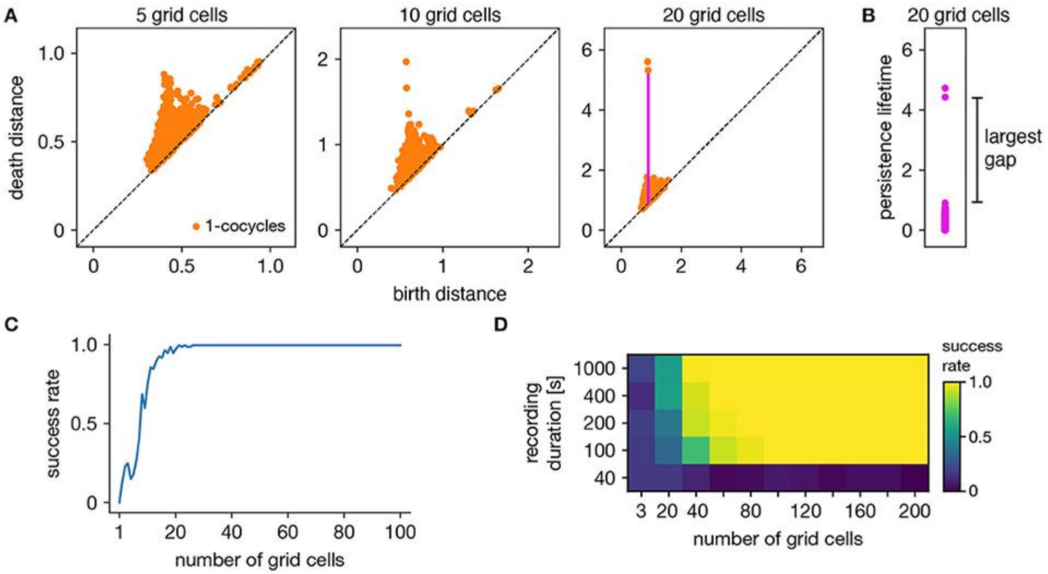


Figure 3

Detecting grid cells with mixed signal

Goal: verify that the given cells are grid cells.

Variant: assuming multi-neuron units, obtained by

- linearly combining spike trains of grid cells;
- coefficients are drawn from a uniform random distribution and tl
  normalized

Method: compute the success rate of having two significant 1-cocycle

Conclusion:

- Figure 4.A: multi-neuron units have different behavior from sing
  neurons
- Figure 4.B: the toroidal topology can still be discovered and
  success rate is independent of the number of grid cells in each
  unit.

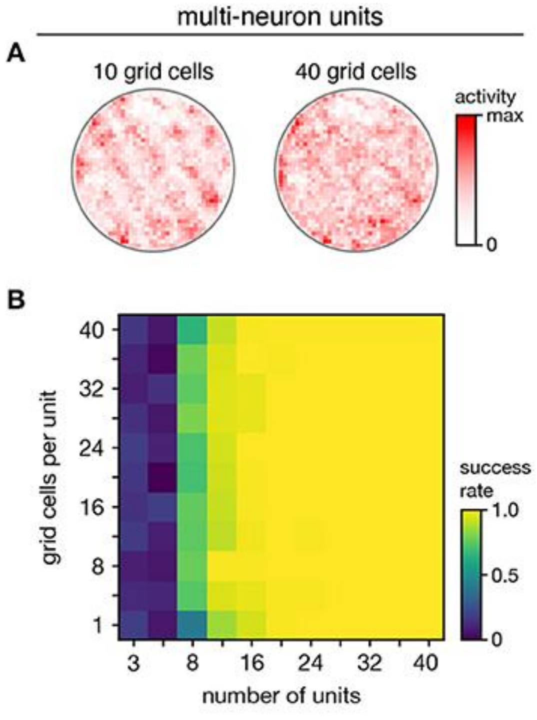


Figure 4

Detecting grid cells with spiking noise

Goal: verify that the given cells are grid cells.

Variant: presence of spiking noise

$$s_{\text{noisy}} = F \cdot X, \quad X \sim \text{Pois}\,(\lambda / F)\,.$$

Rescaled firing rate

Fano factor of the random process

Method: compute the success rate of having two significant 1-cocycles.

Conclusion: more neurons are required for higher Fano factors.

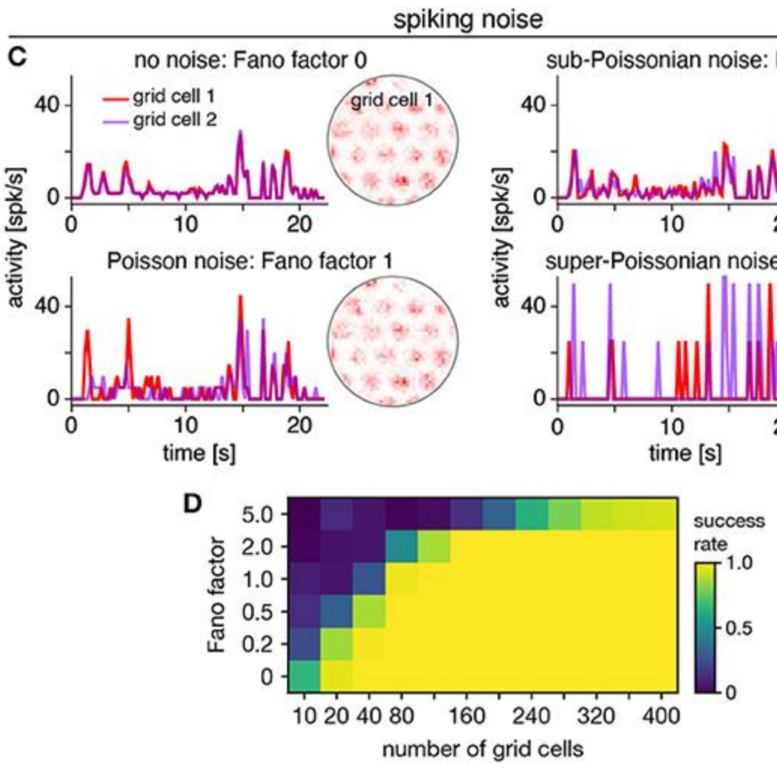


Figure 4

Cell detection with mixtures of neural populations

Goal: detect the types of grid cells or conjunctive cells, when mixed with other types of neurons.

Method: compute the success rate of having two significant 1-cocycles for grid cells, or having three significant 1-cocycles for conjunctive cells.

Conclusion:

$$\frac{\#\text{ other cells}}{\#\text{ grid cells}} < 2 \Rightarrow \text{reliable discovery of grid cells;}$$

Detection of conjunctive cells requires more neurons.

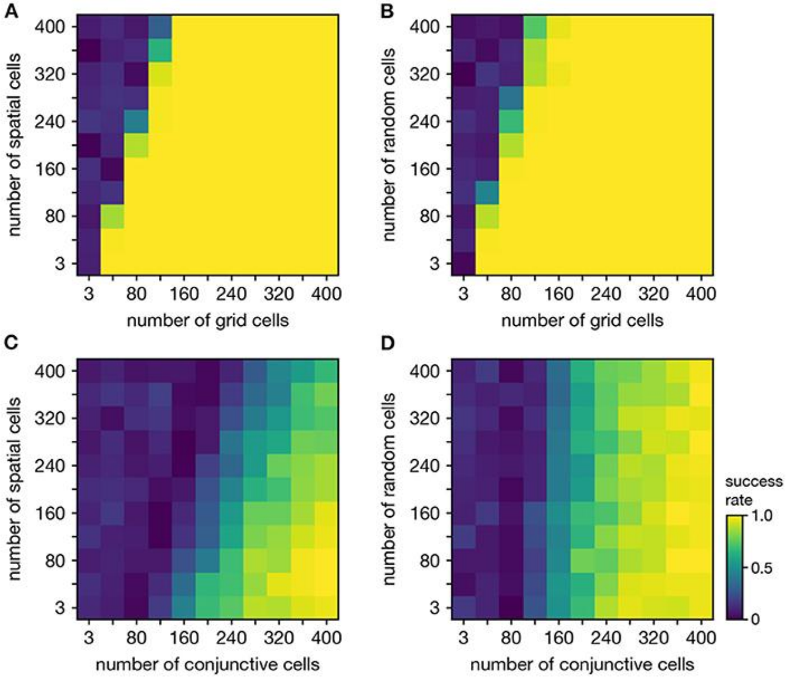


Figure 6

Cell detection with mixtures of neural populations

Goal: detect conjunctive cells under different ways of mixture.

Conclusion: detection is the best when there is no dominating population.

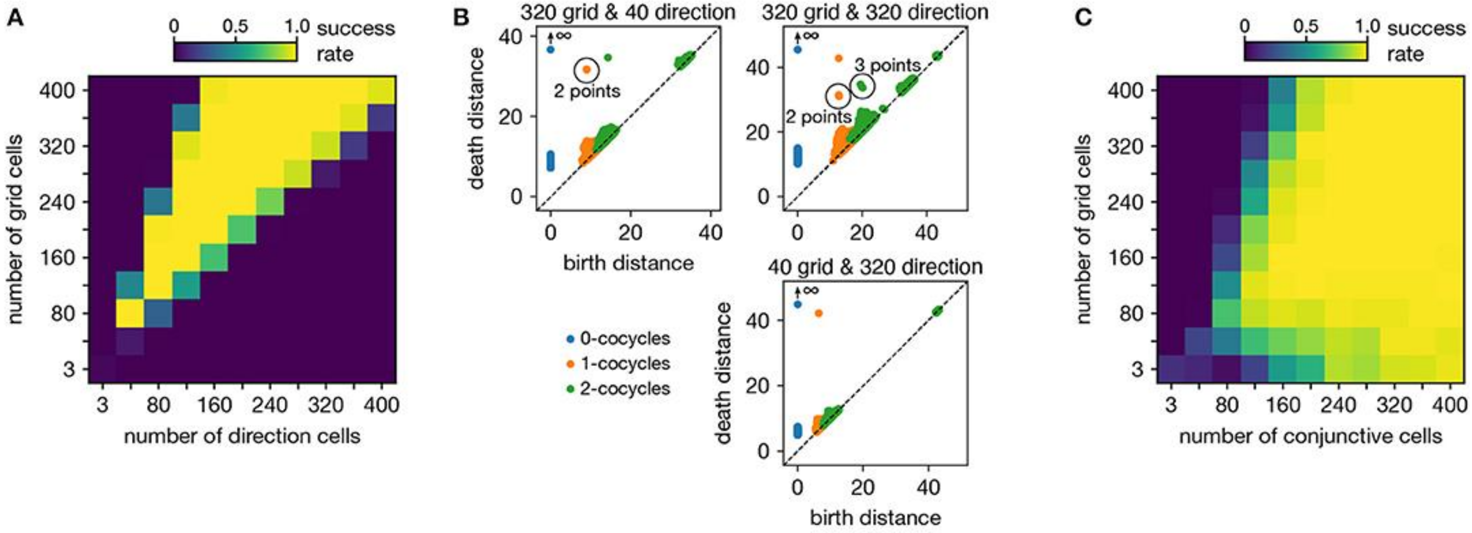


Figure 7

Cell detection with mixtures of neural populations

Goal: detect mixed grid cells from multiple modules (e.g. mixing 2 modules results into a 4-torus).

Conclusion: larger environments are needed to fully sample the 4-torus structure.

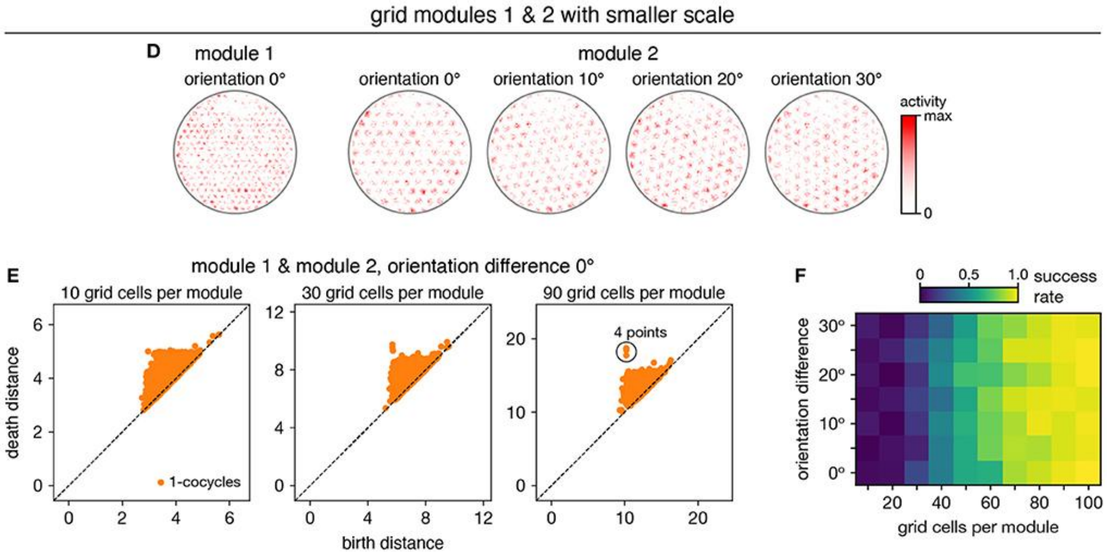


Figure 7

Reconstruct animal trajectory

Generate neural recordings for grid cells.

Compute persistent cohomology, and assign circular coordinates (for each significant 1-cocycle) to data points.

For each neuron, find data points such that this neuron is the most active.

Topological distance (C) v.s. Physical distance (A) of grid cells.

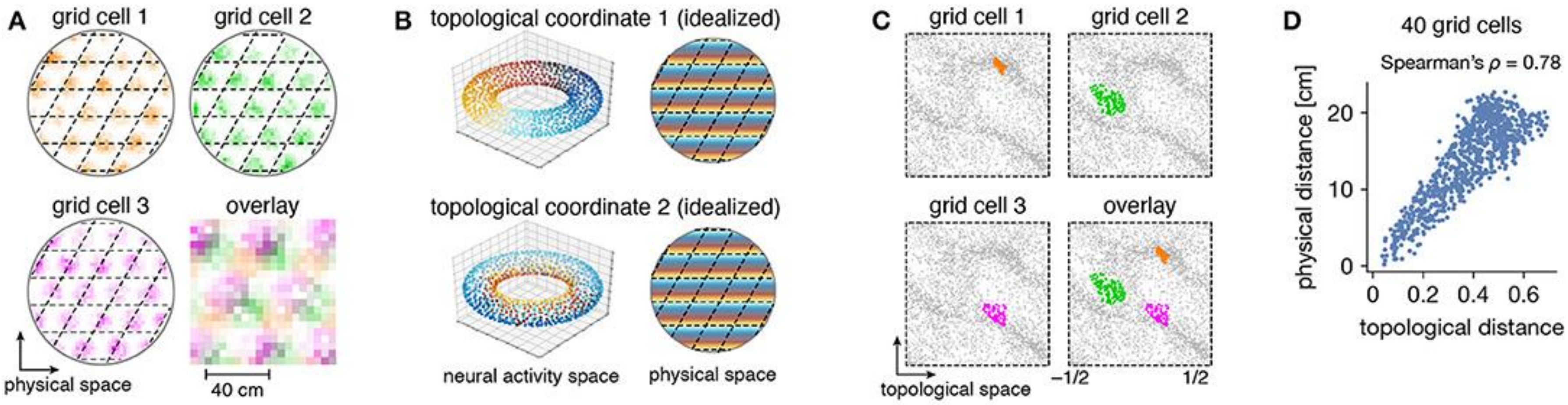


Figure 5

Reconstruct animal trajectory

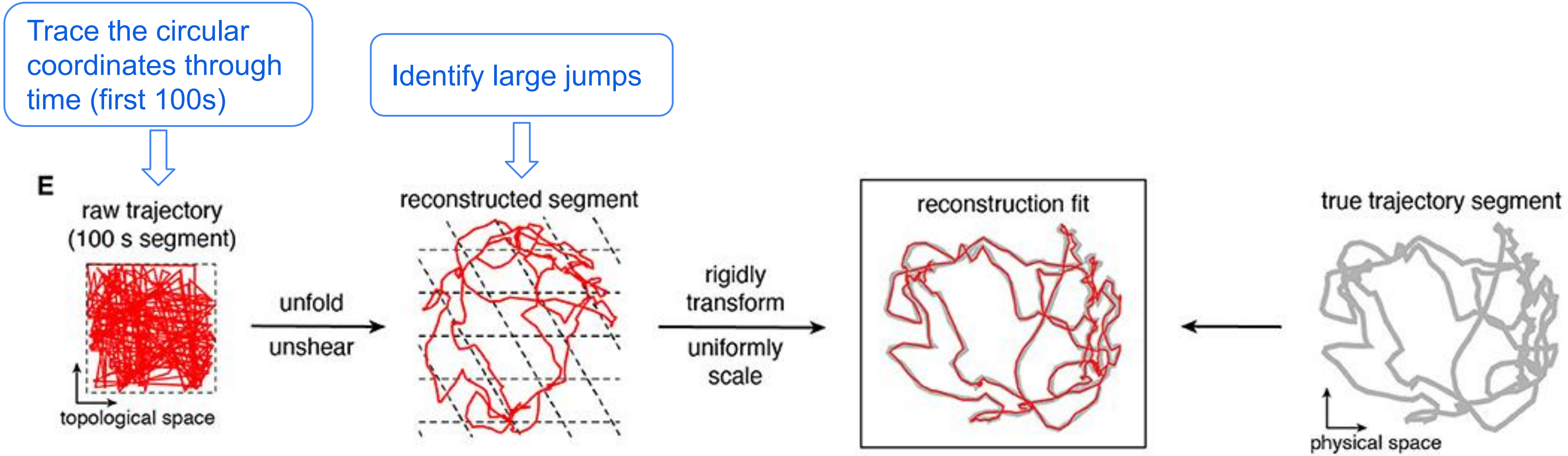


Figure 5

Conclusion:

- Figure 5.D: the organization of grid cells in topological space recovers the organization of them in physical space;
- Figure 5.E: persistent cohomology decode the trajectories in the physical space.

Summary

- Firing field itself can have non-trivial topology! For example, the grid cells, head direction cells and conjunctive cells.
- Persistent cohomology detects types of cells with non-trivial firing fields successfully.
- Dataset parameters (such as, noise or mixture with other cells) affect the performance of persistent cohomology.

References

[Burak & Fiete, 2009] Burak, Y., & Fiete, I. R. (2009). Accurate path integration in continuous attractor network models of grid cells. *PLoS computational biology*, *5*(2), e1000291.

[Curto, 2017] Curto, C. (2017). What can topology tell us about the neural code?. *Bulletin of the American Mathematical Society*, *54*(1), 63-78.

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[Hafting et al., 2005] Hafting, T., Fyhn, M., Molden, S., Moser, M. B., & Moser, E. I. (2005). Microstructure of a spatial map in the entorhinal cortex. *Nature*, *436*(7052), 801-806.

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[Stensola et al., 2012] Stensola, H., Stensola, T., Solstad, T., Frøland, K., Moser, M. B., & Moser, E. I. (2012). The entorhinal grid map is discretized. *Nature*, *492*(7427), 72-78.

Thank you!