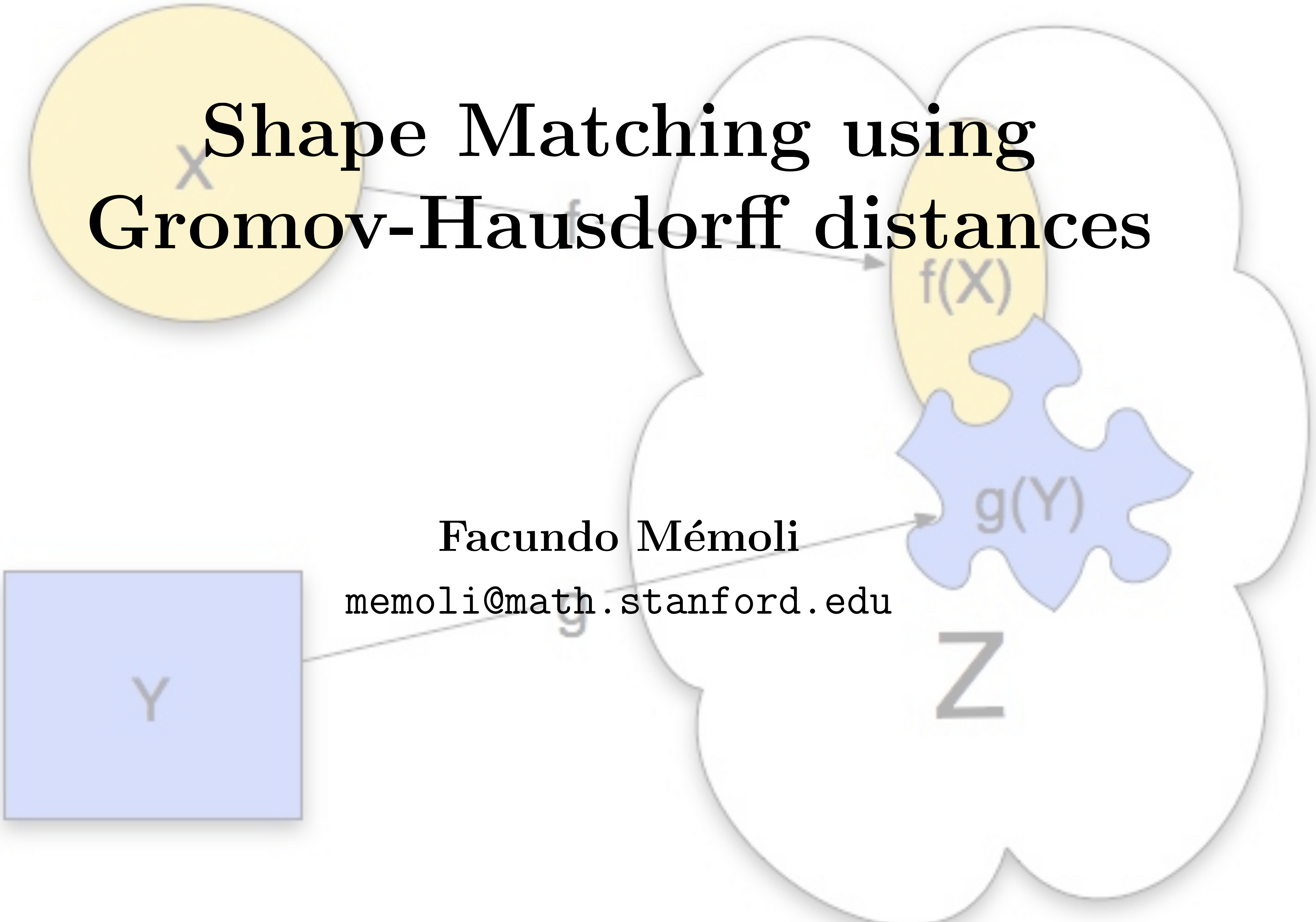
Shape Matching using Gromov-Hausdorff distances



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Background concepts

- **Metric Space**. A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}^+$ s.t.

  1. For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.
  2. For all $x, y \in X$, $d(x, y) = d(y, x)$.
  3. $d(x, y) = 0$ <u>if and only if</u> $x = y$.

- **Folklore Lemma**. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\}$ and $\mathbb{Y}_n = \{y_1, \ldots, y_n\}$ be points in $\mathbb{R}^k$. If

$$\|x_i - x_j\| = \|y_i - y_j\|$$

  for all i, j, then there exists a *rigid isometry* $T : \mathbb{R}^k \to \mathbb{R}^k$ s.t.

$$T(x_i) = y_i, \text{ for all } i$$

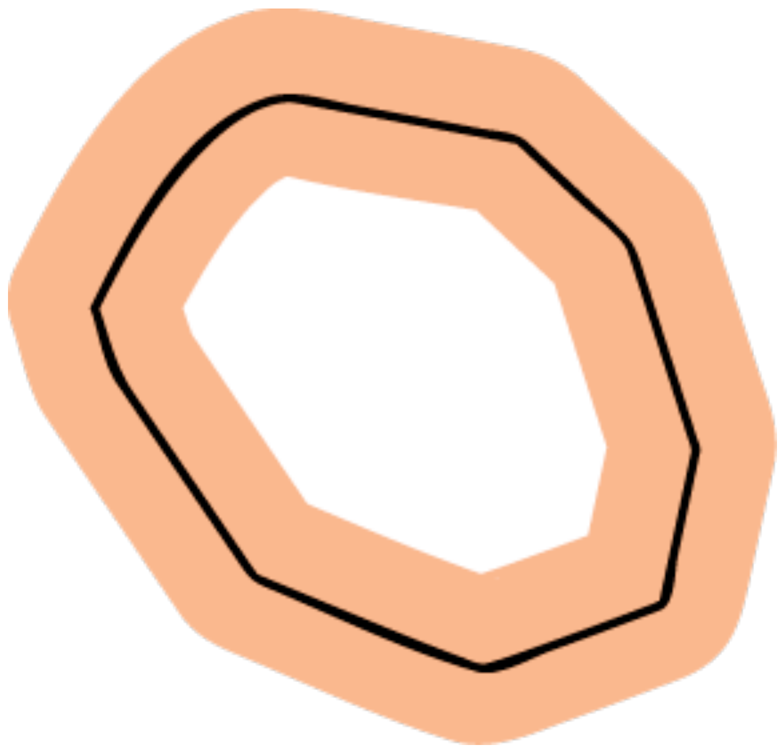
- **Hausdorff distance**. For (compact) subsets A, B of a (compact) metric space (Z, d), the *Hausdorff distance* between them, $d_{\mathcal{H}}^{Z}(A, B)$, is defined to be the infimal $\varepsilon > 0$ s.t.

$$A \subset B^{\varepsilon}$$

and

$$B \subset A^{\varepsilon}$$

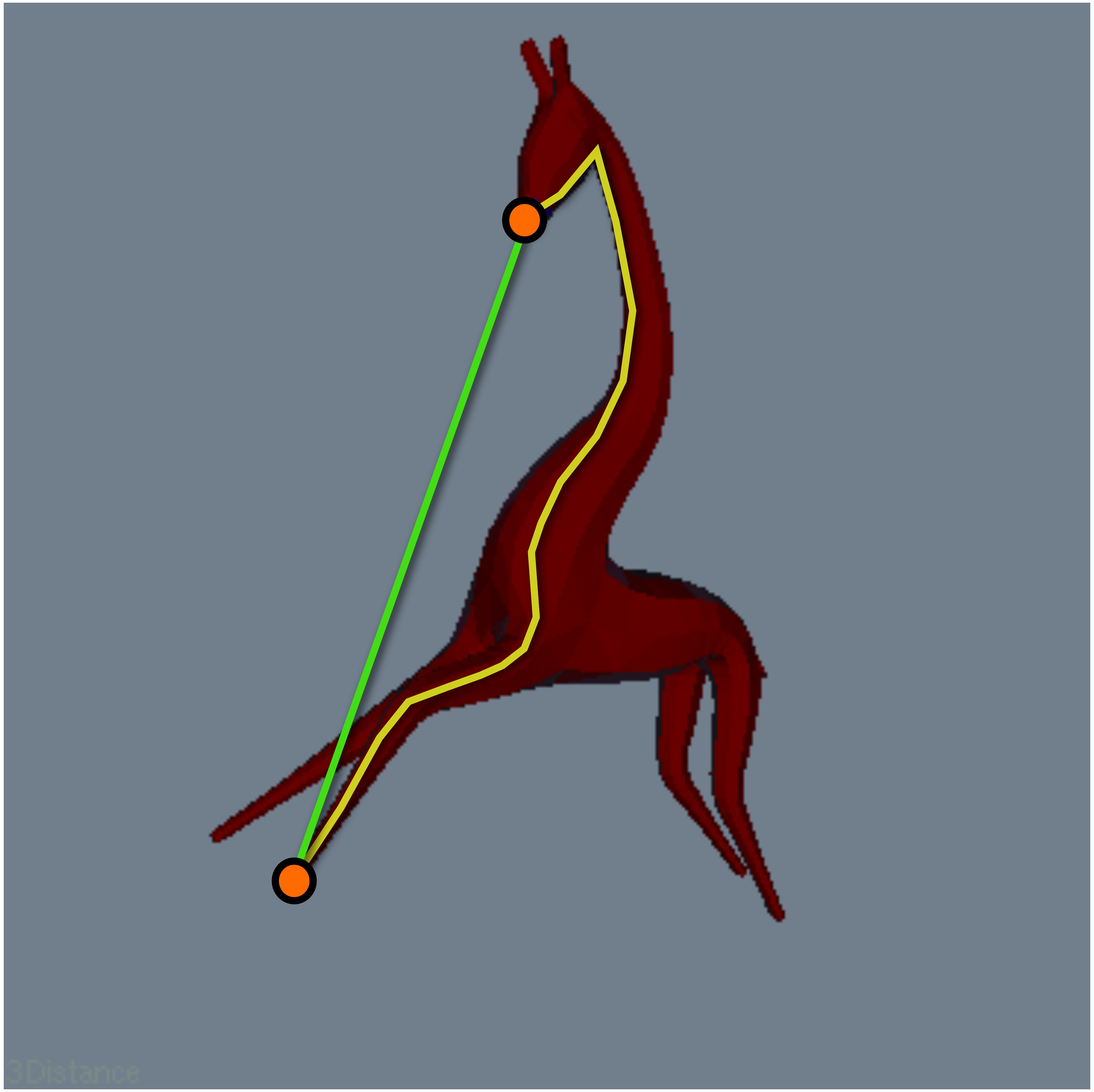
where $A^{\varepsilon} = \{z \in Z |\, d(z, A) < \varepsilon\}$.



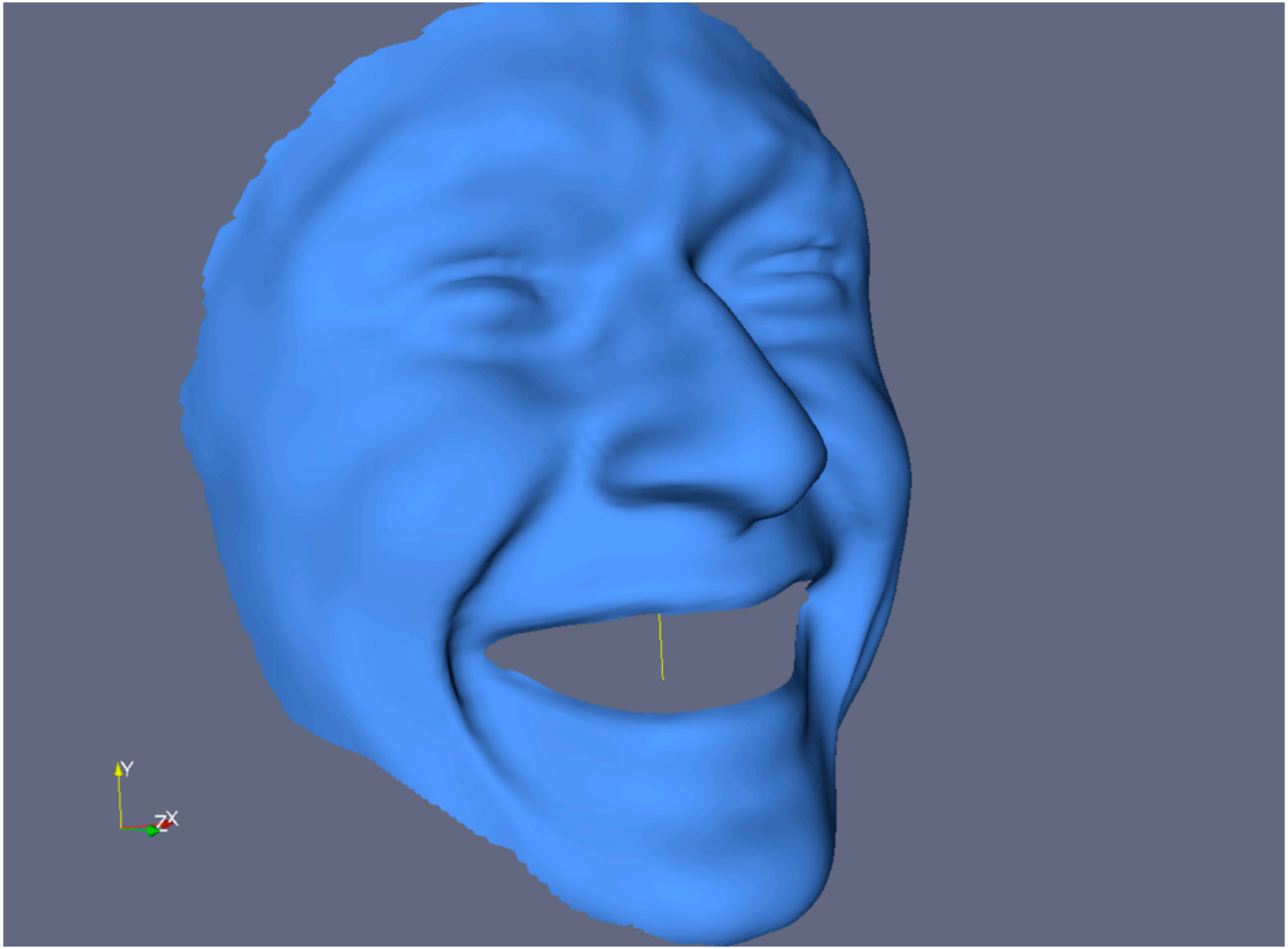
Equivalently,

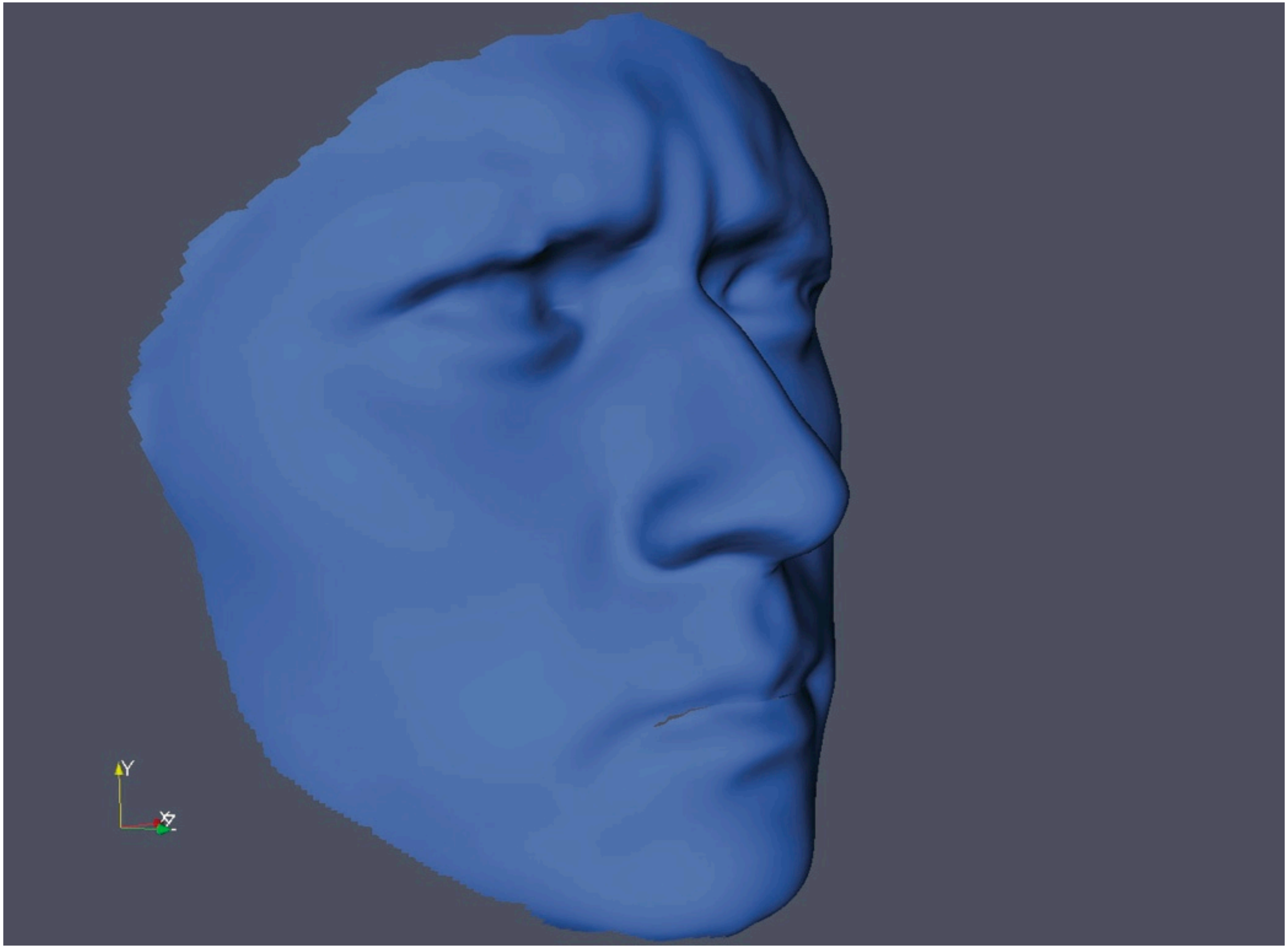
$$d_{\mathcal{H}}^{Z}(A, B) = \max(\max_{b \in B} \min_{a \in A} d(a, b), \max_{a \in A} \min_{b \in B} d(a, b))$$

Geodesic distance vs Euclidean distance



Geodesic distance: invariance to ‘bends’

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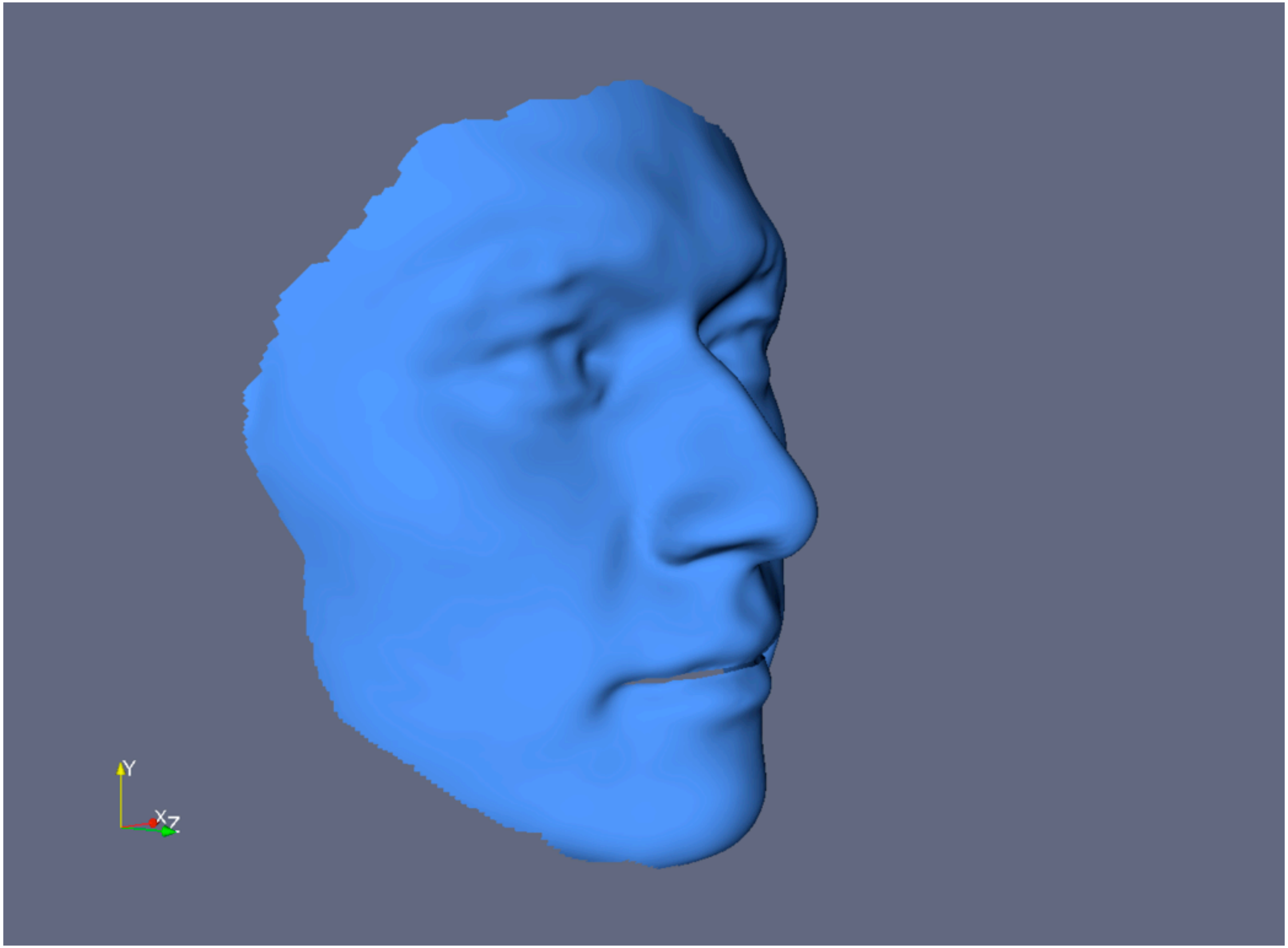
Y
X
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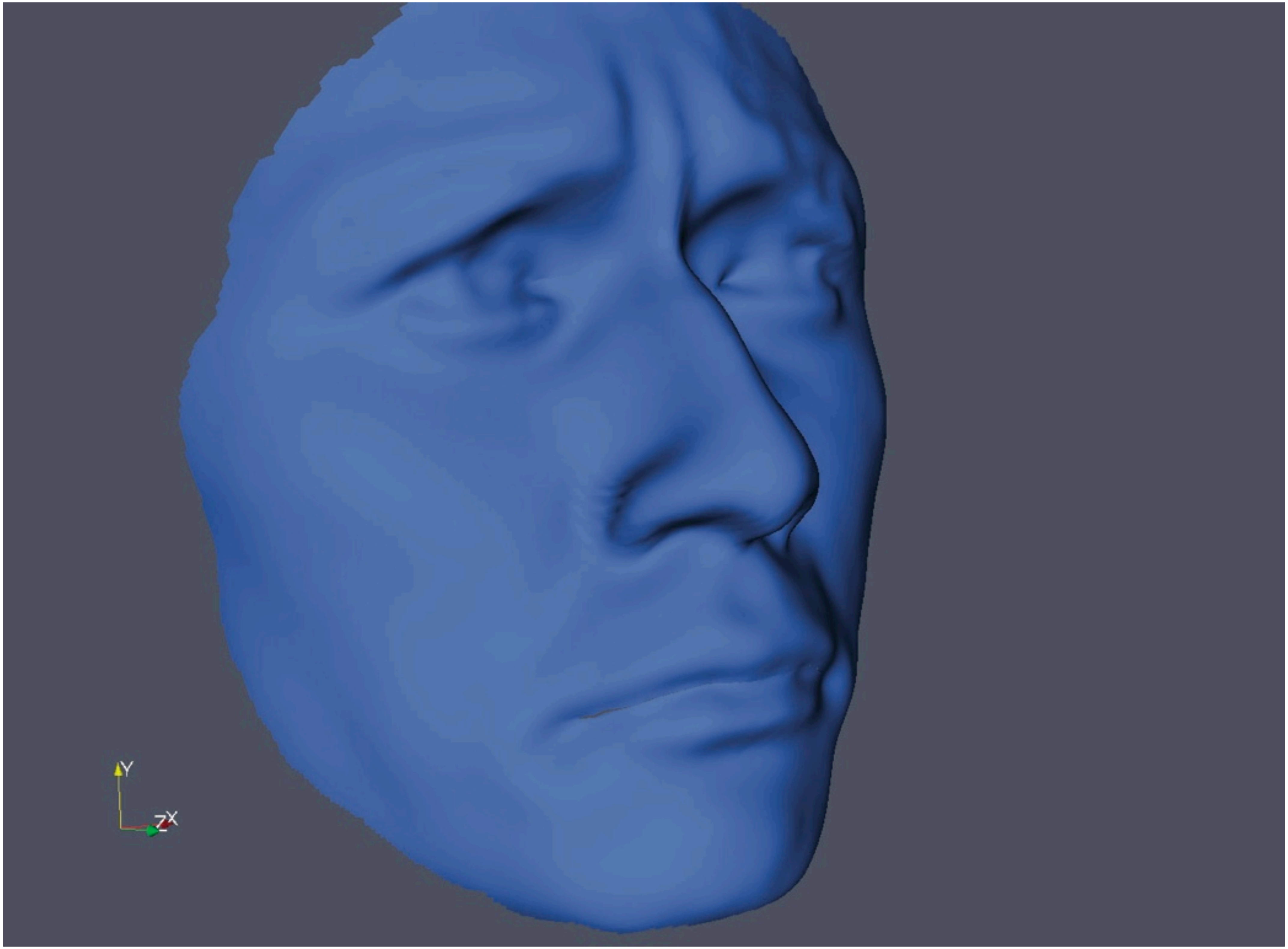
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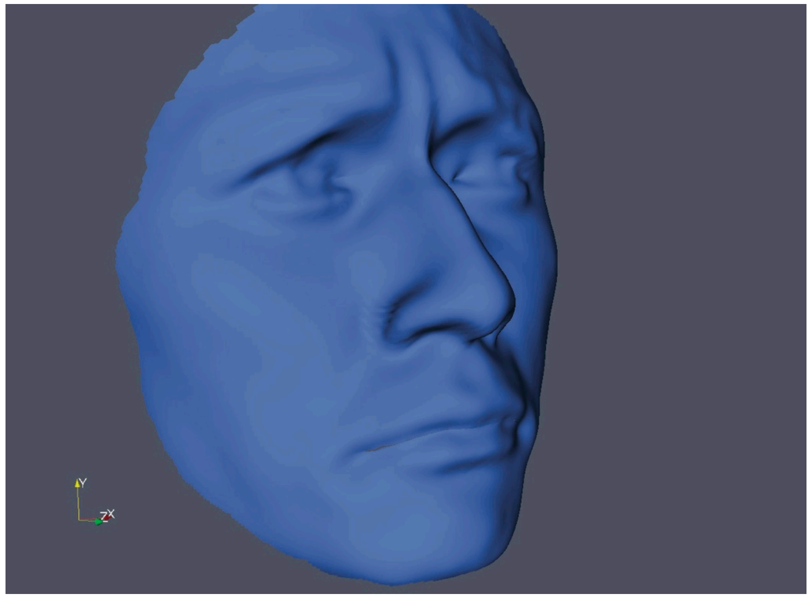
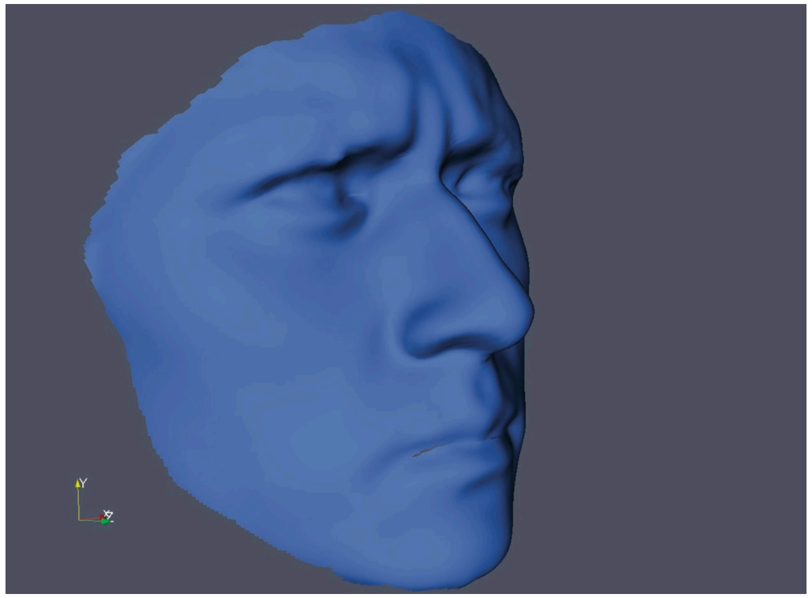
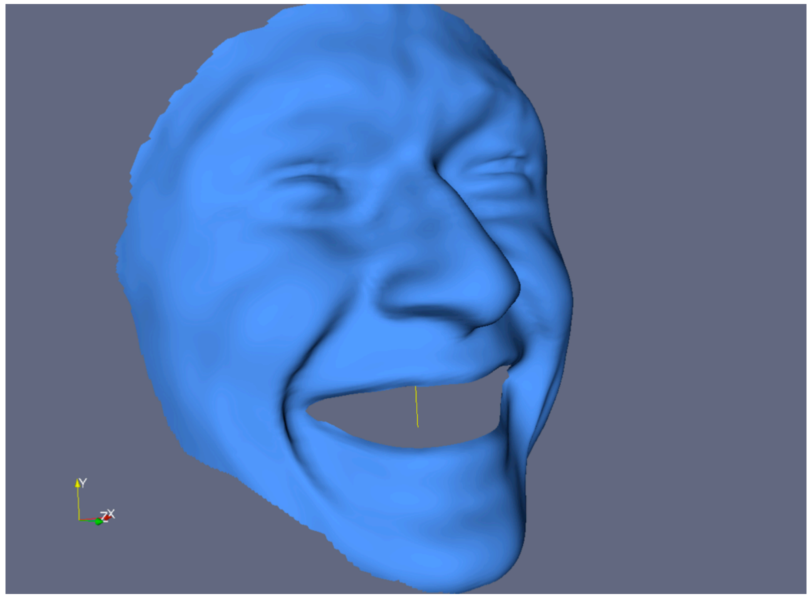
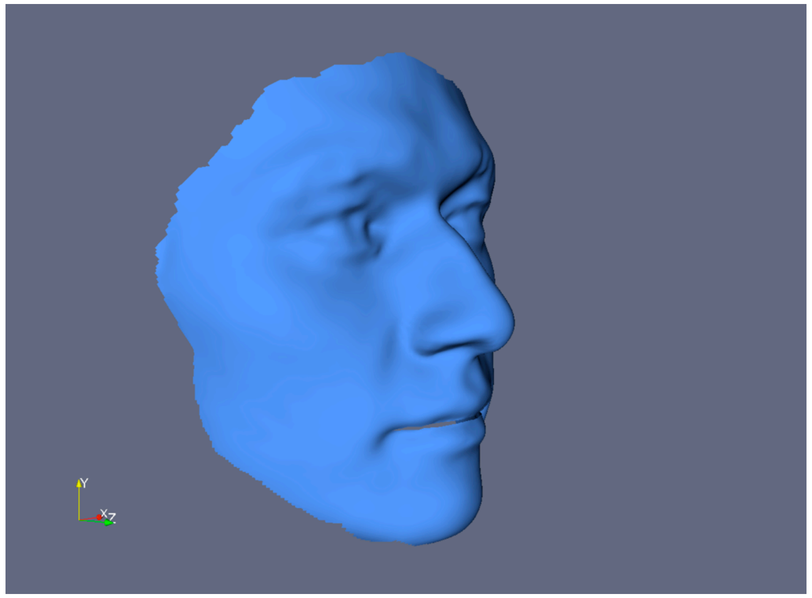
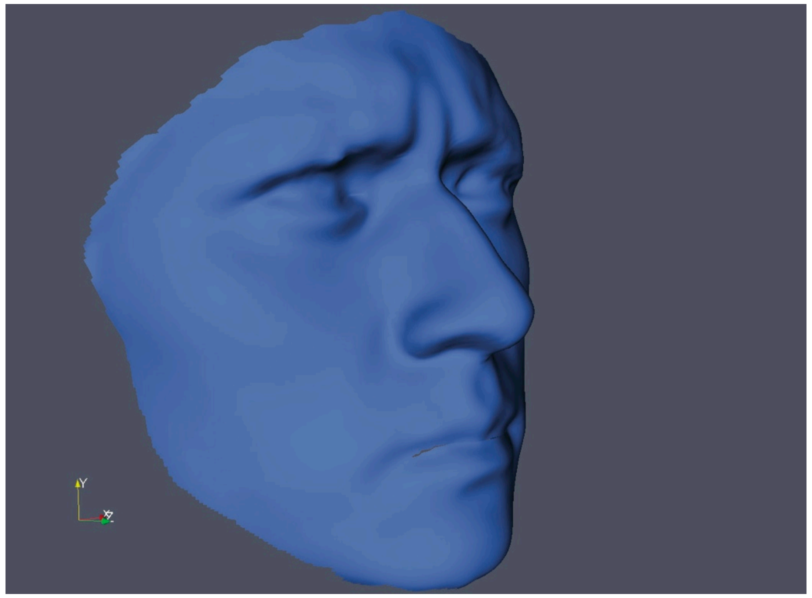
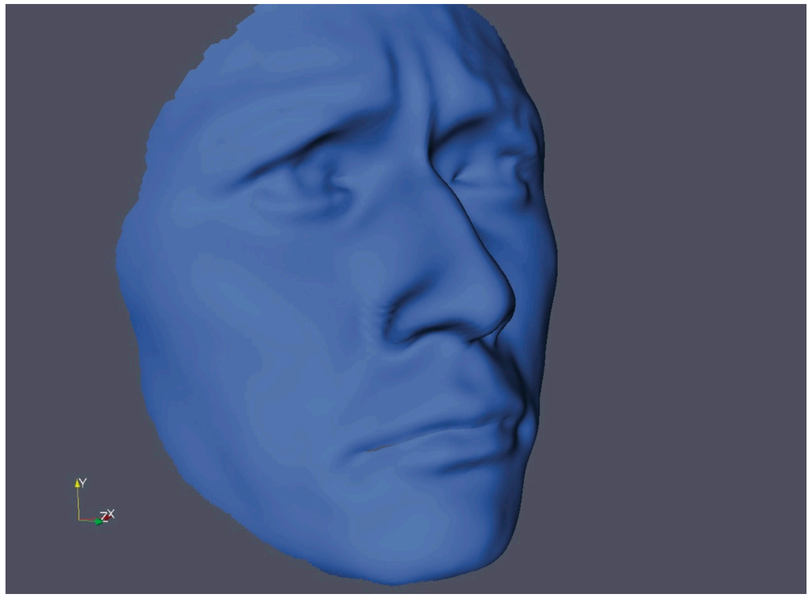


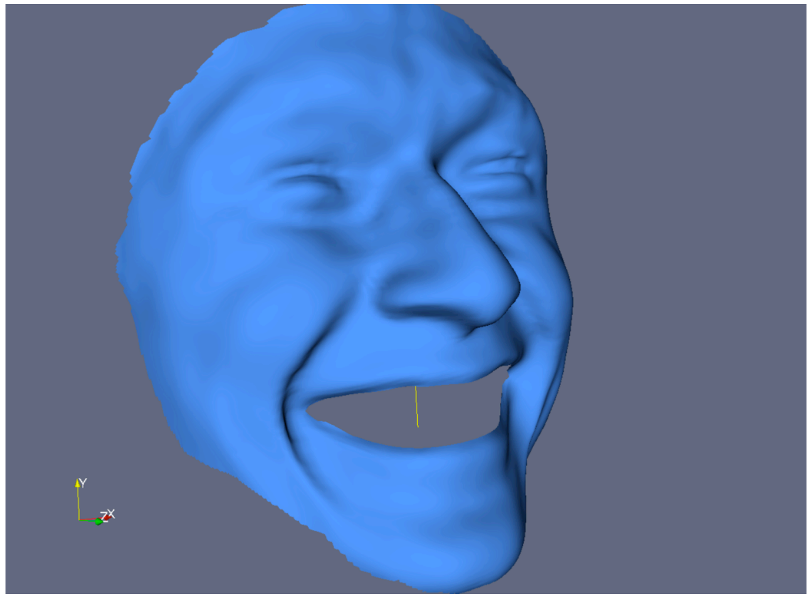


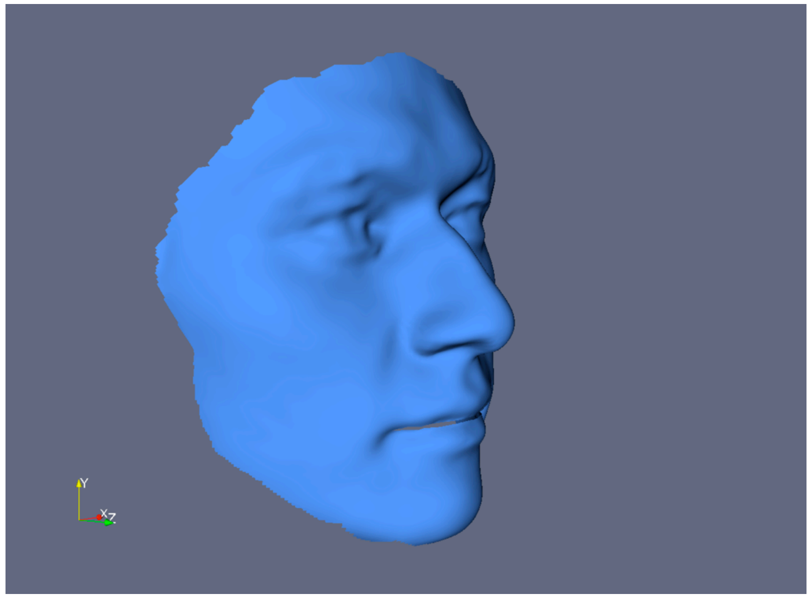


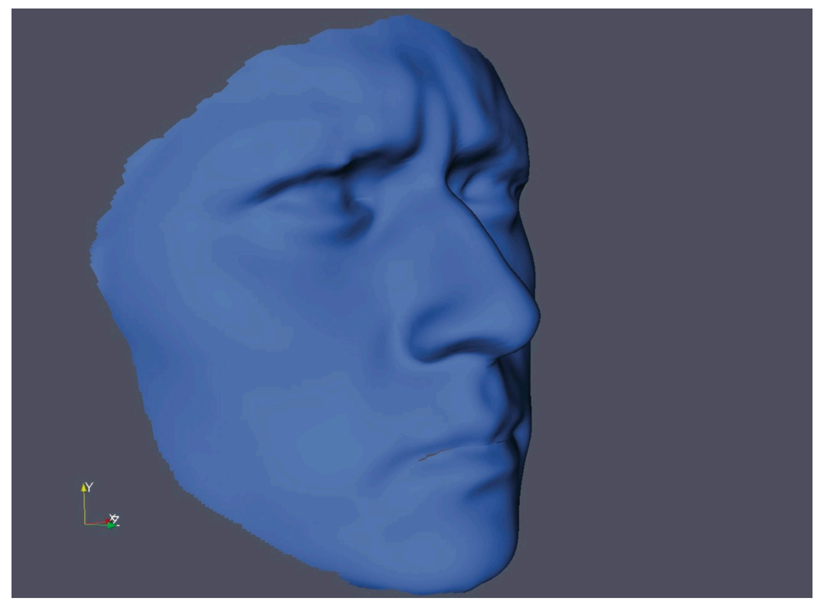


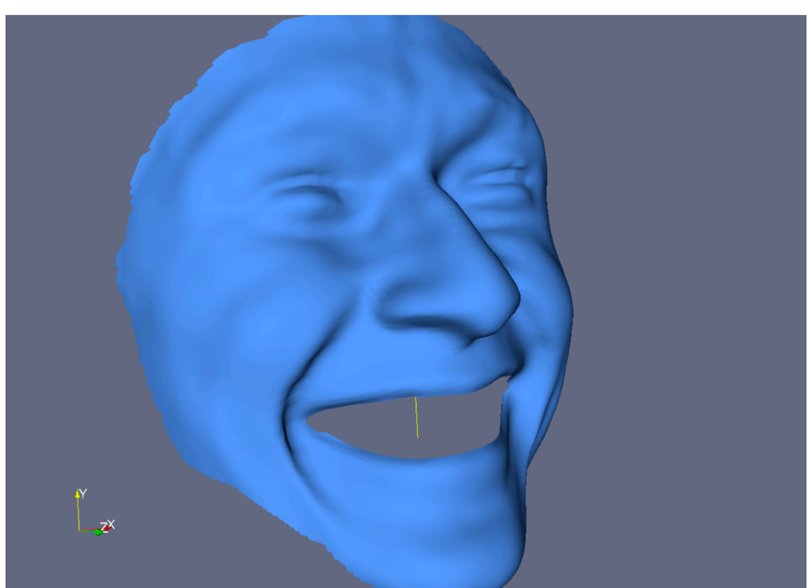


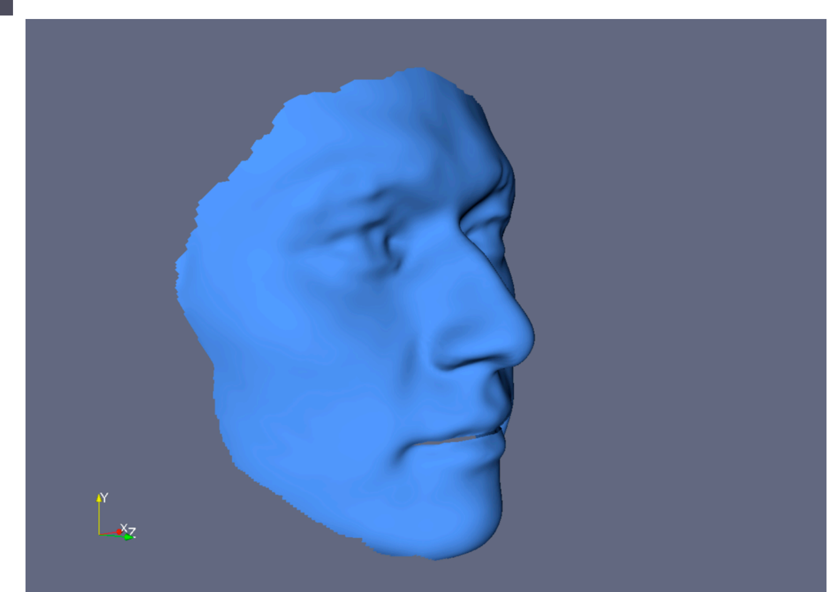


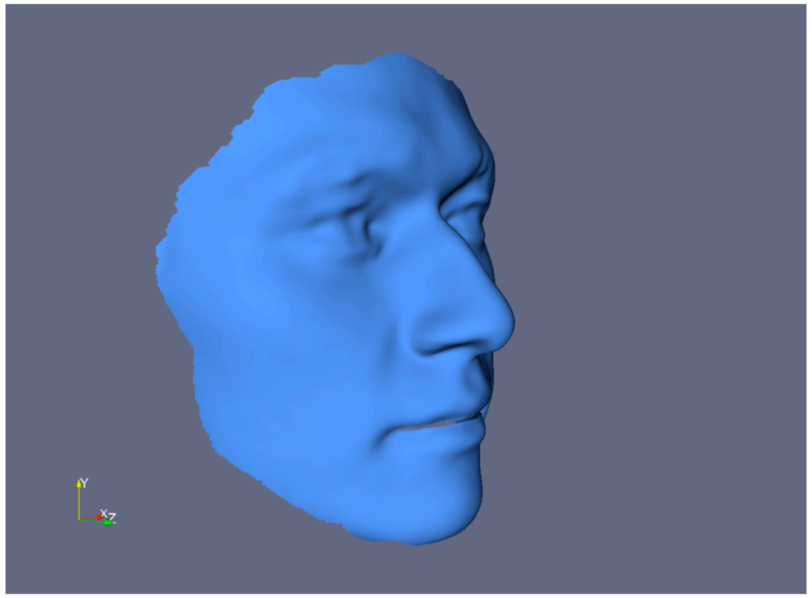


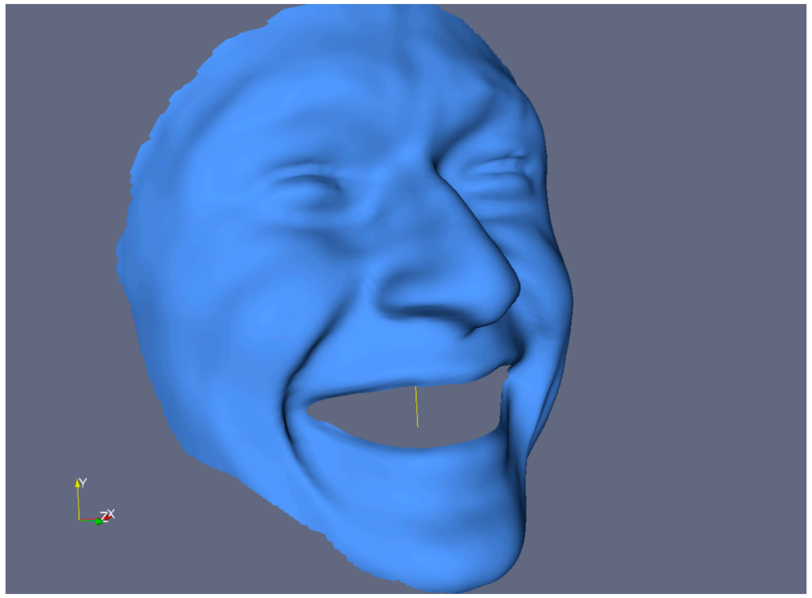


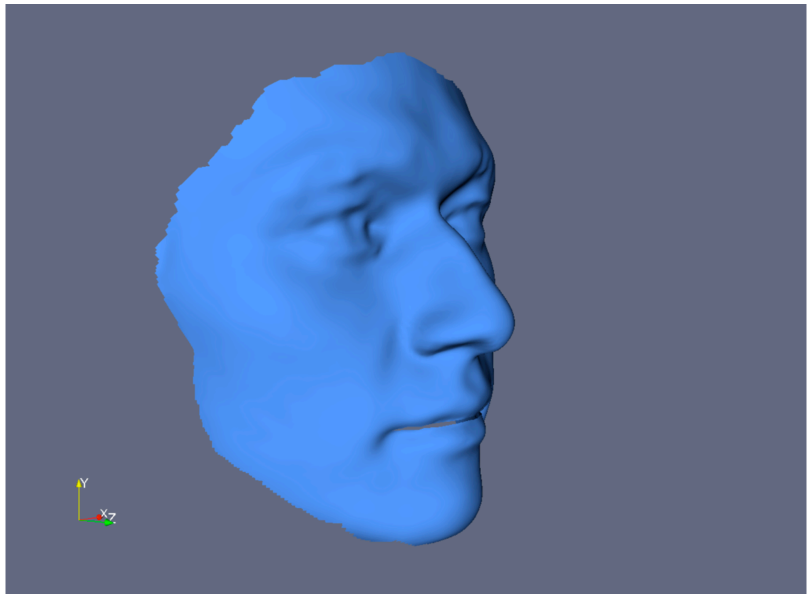








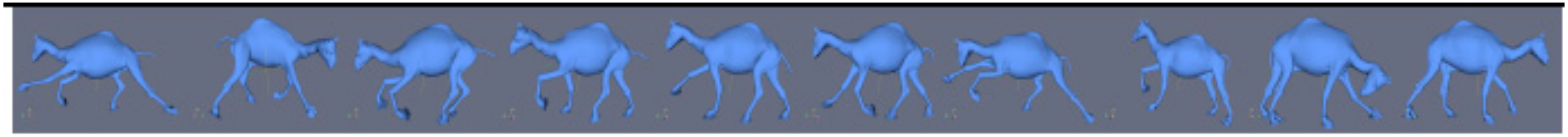




Y
X
Z

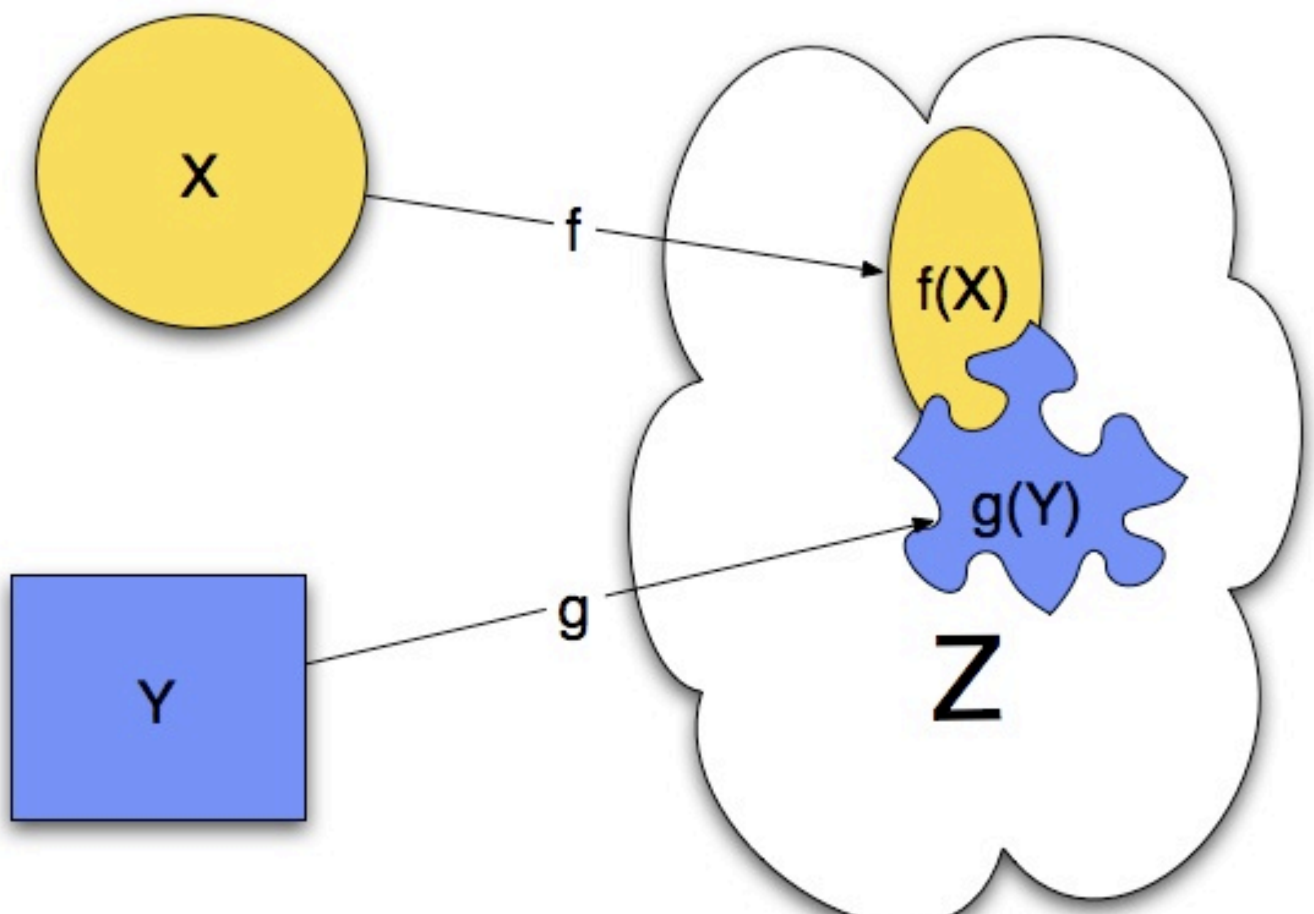
The GH distance for Shape Comparison

- Regard shapes as (compact) metric spaces, [**MS04**], [**MS05**].
- The metric with which one endows the shapes depends on the desired invariance. For example, if invariance to
  - *rigid isometries* is desired, use Euclidean distance (remember Folklore Lemma).
  - *bends* is desired, use "geodesic" distance.
- Let $\mathcal{X}$ denote set of all compact metric spaces. Define GH distance (metric) on $\mathcal{X}$, then $(\mathcal{X}, d_{\mathcal{GH}})$ is itself a metric space.
- GH distance provides reasonable framework for Shape Comparison: good theoretical properties.
- However, it leads to difficult optimization problems.



GH: definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



It would be much more intuitive to compare the metrics d_X and d_Y directly..

For maps $f : X \to Y$, and $g : Y \to X$ compute

$$\mathrm{dist}(f) = \max_{x,x'} |d_X(x, x') - d_Y(f(x), f(x'))|$$

and

$$\mathrm{dist}(g) = \max_{y,y'} |d_Y(y, y') - d_X(g(y), g(y'))|$$

and then minimize $\max(\mathrm{dist}(f), \mathrm{dist}(g))$ over all choices of f and g.

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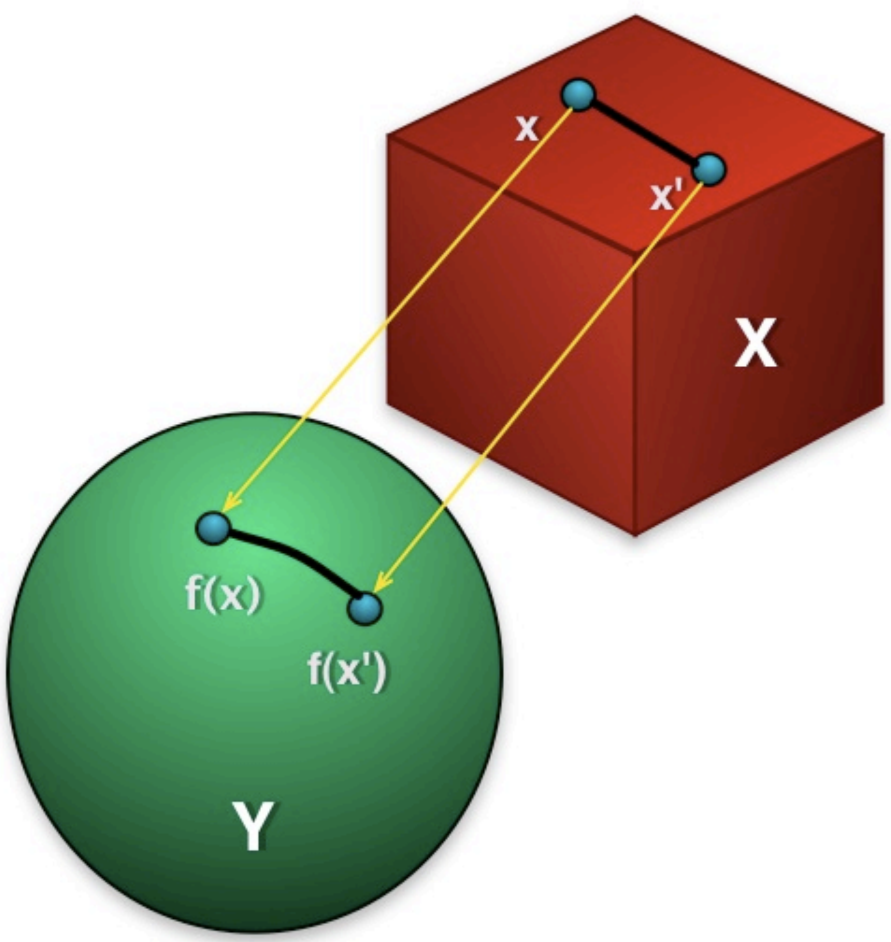

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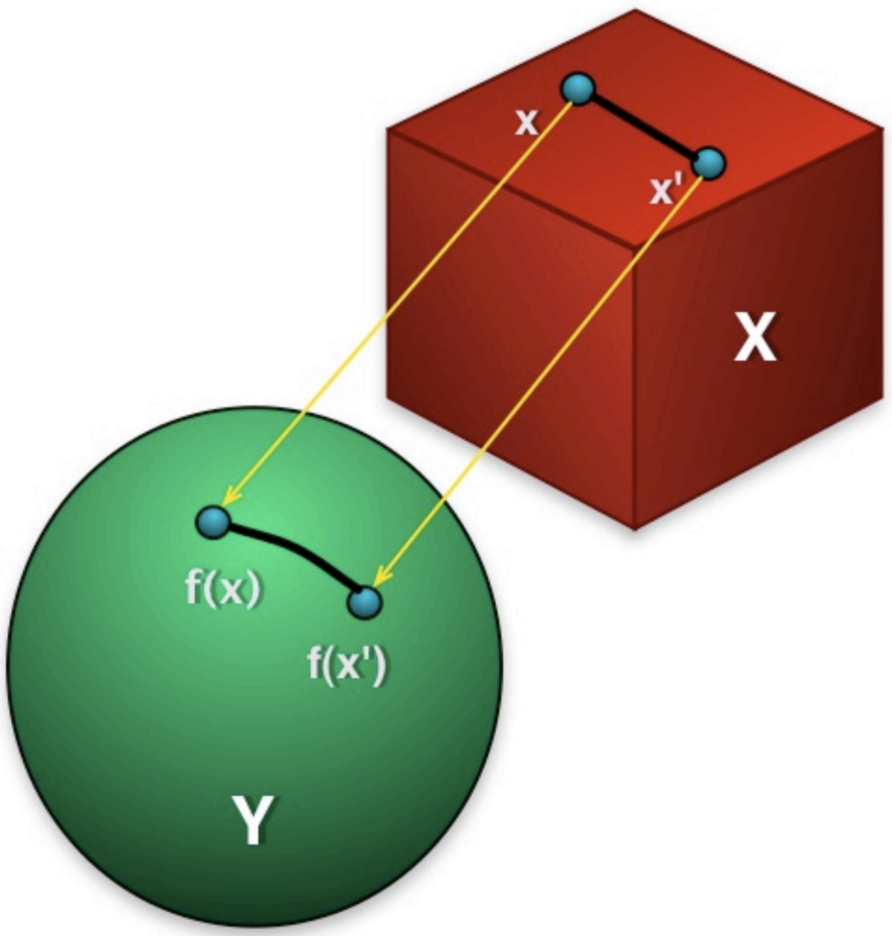

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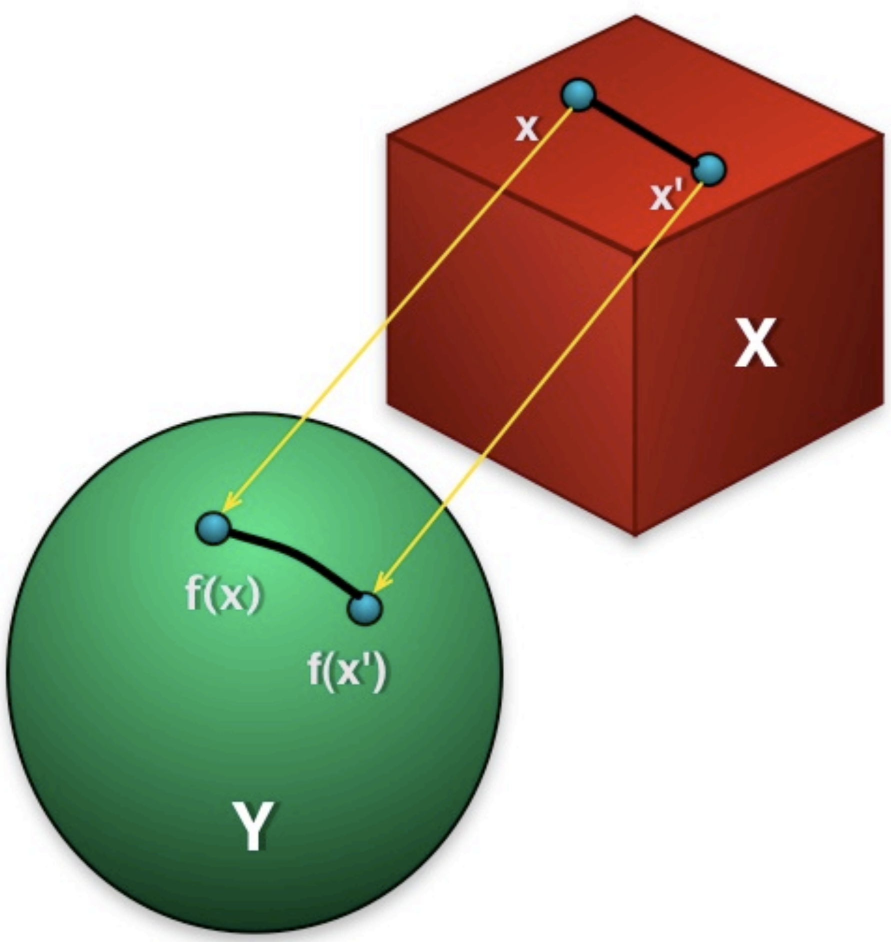

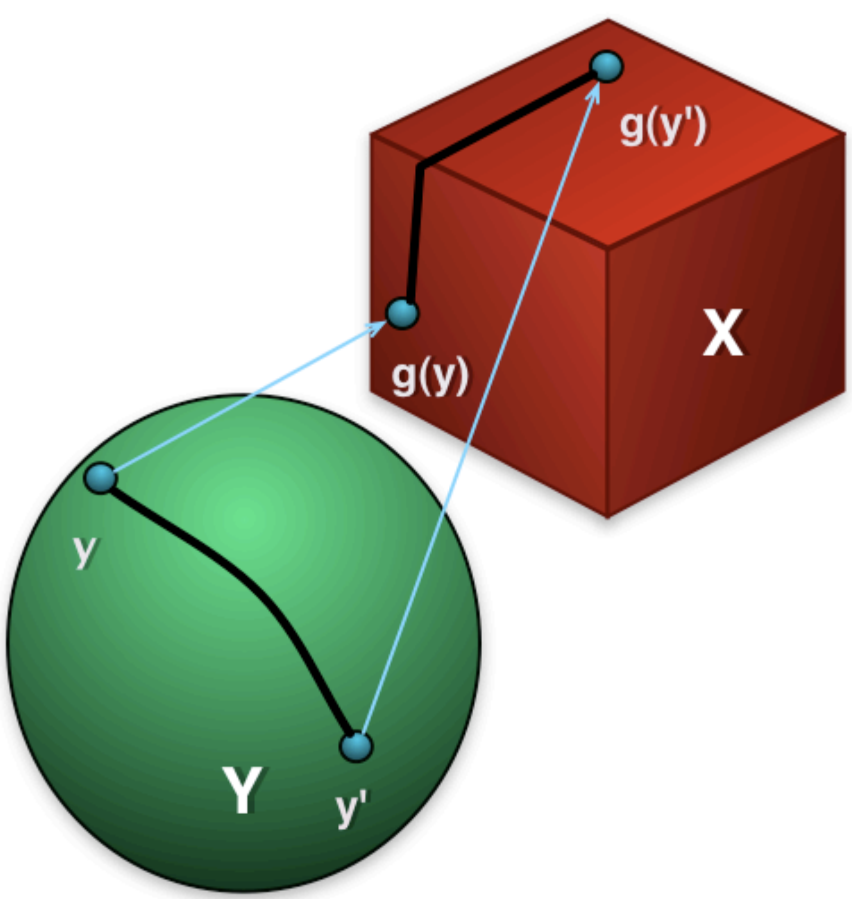

correspondences

Definition [Correspondences]

For sets A and B, a subset $R \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\ a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\ b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B.

Note that in the case $n_A = n_B$, correspondences are larger than bijections.

correspondences

Note that when A and B are finite, $R \in \mathcal{R}(A,B)$ can be represented by a matrix $((r_{a,b})) \in \{0,1\}^{n_A \times n_B}$ s.t.

$$\sum_{a \in A} r_{ab} \geq 1 \;\; \forall b \in B$$

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$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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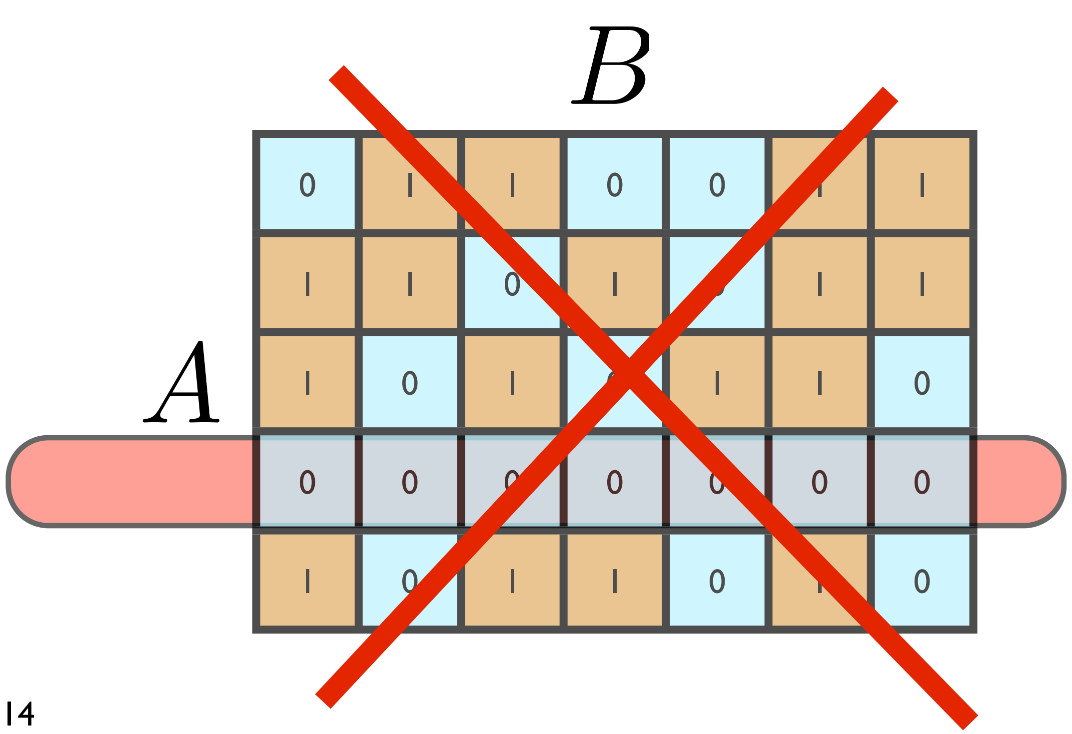

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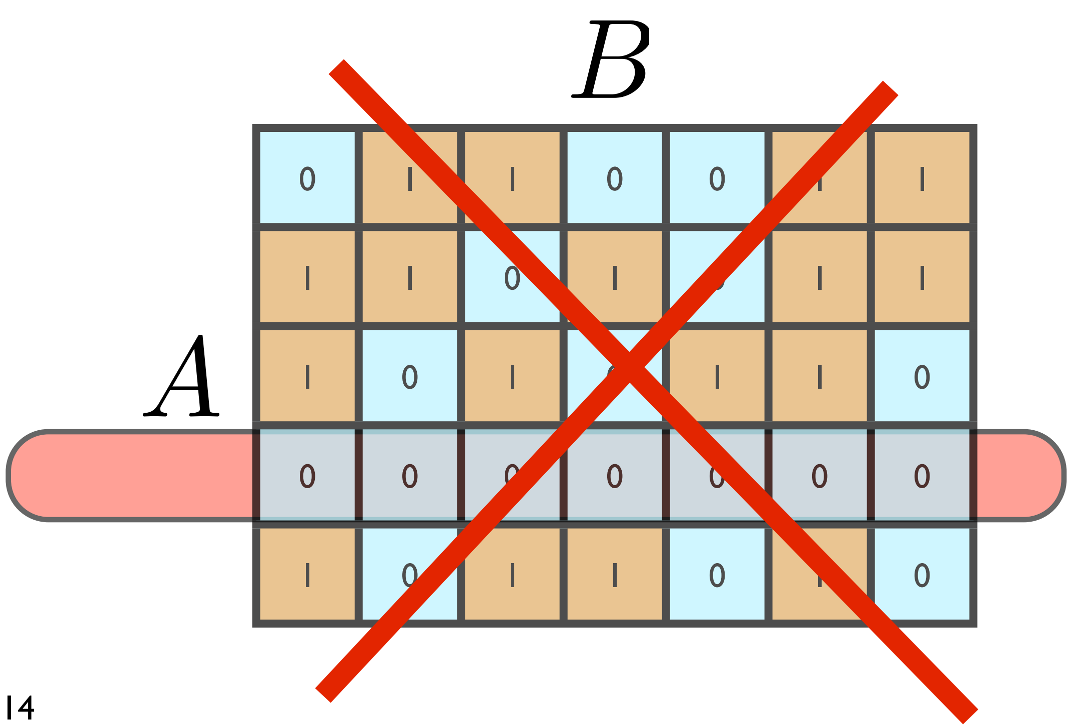

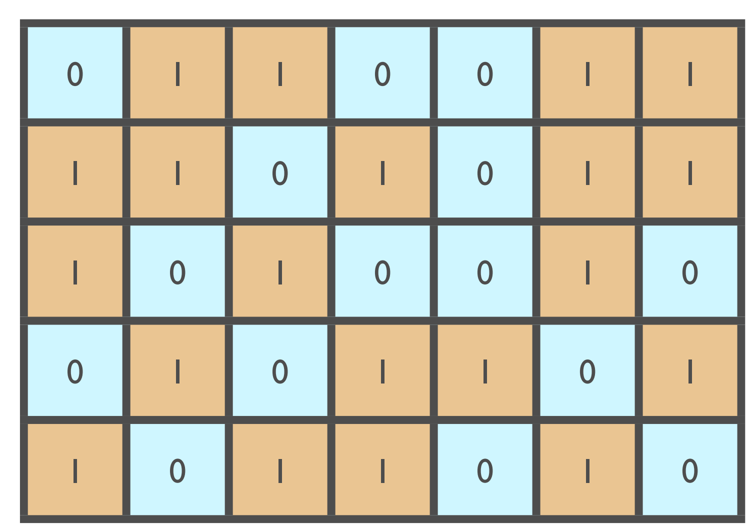

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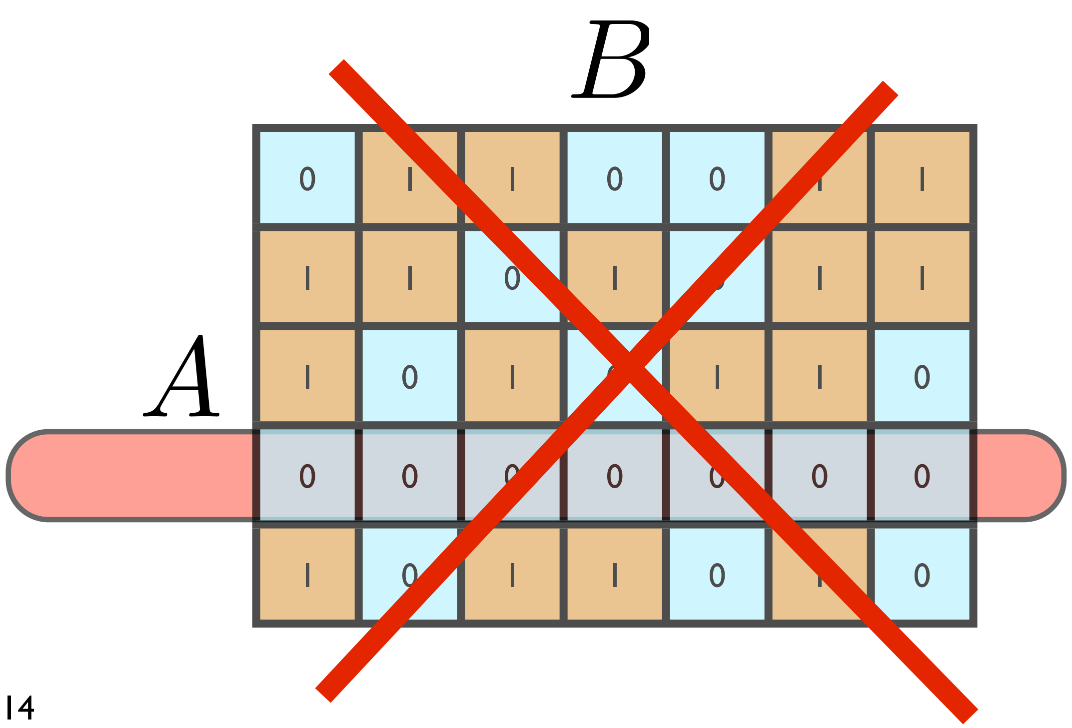
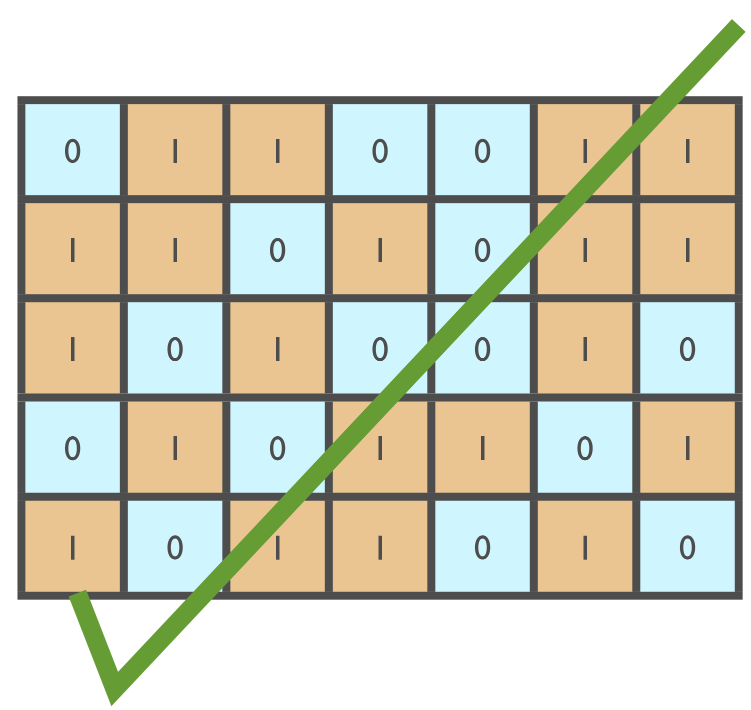


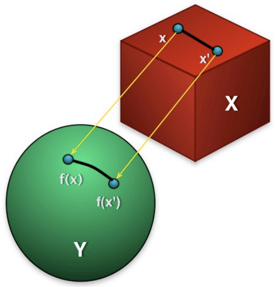


Another expression for the GH distance

A theorem, [BuBuIv]

For compact metric spaces (X, d_X) and (Y, d_Y),

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_{R} \max_{(x,y),(x',y')\in R} |d_X(x,x') - d_Y(y,y')|$$



Main Properties

1. Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces then
$$d_{\mathcal{GH}}(X, Y) \leq d_{\mathcal{GH}}(X, Z) + d_{\mathcal{GH}}(Y, Z).$$

2. If $d_{\mathcal{GH}}(X, Y) = 0$ and (X, d_X), (Y, d_Y) are compact metric spaces, then (X, d_X) and (Y, d_Y) are isometric.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a finite subset of the compact metric space (X, d_X). Then,
$$d_{\mathcal{GH}}(X, \mathbb{X}_n) \leq d_{\mathcal{H}}(X, \mathbb{X}_n).$$

4. For compact metric spaces (X, d_X) and (Y, d_Y):
$$\begin{aligned} \frac{1}{2}\left|\mathrm{diam}(X) - \mathrm{diam}(Y)\right| &\leq d_{\mathcal{GH}}(X, Y) \\ &\leq \frac{1}{2}\max\left(\mathrm{diam}(X), \mathrm{diam}(Y)\right) \end{aligned}$$

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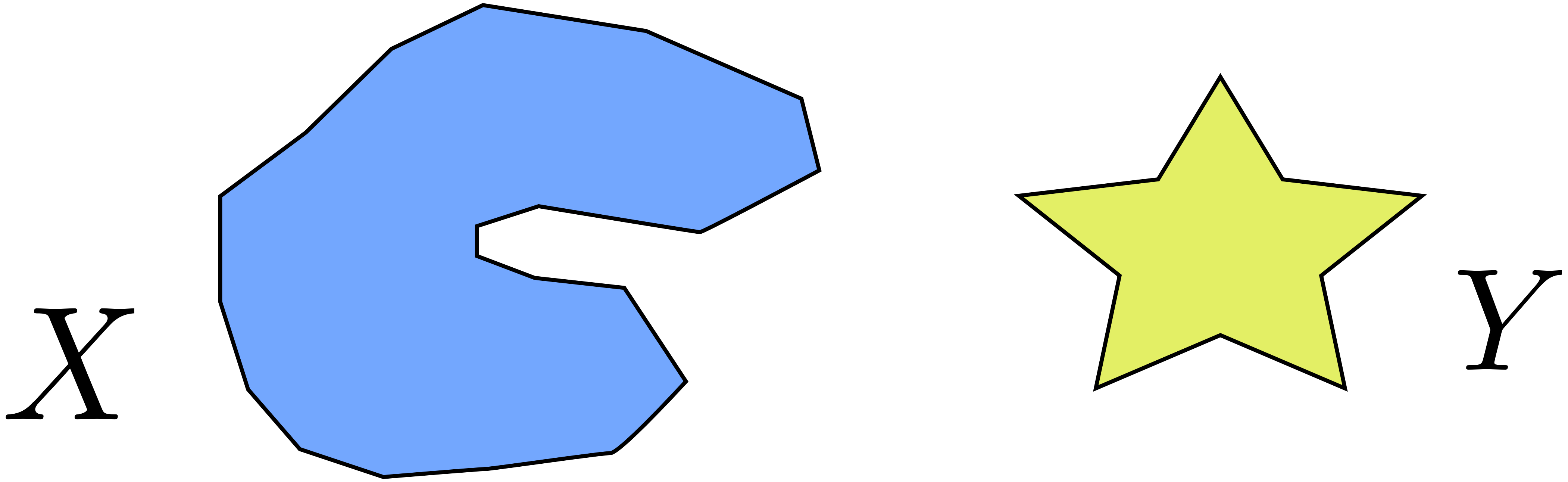
Stability, [MS05]

$$|d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(\mathbb{X}_n, \mathbb{Y}_m)| \leq r(\mathbb{X}_n) + r(\mathbb{Y}_m)$$

for finite samplings $\mathbb{X}_n \subset X$ and $\mathbb{Y}_m \subset Y$, where $r(\mathbb{X}_n)$ and $r(\mathbb{Y}_m)$ are the covering radii.



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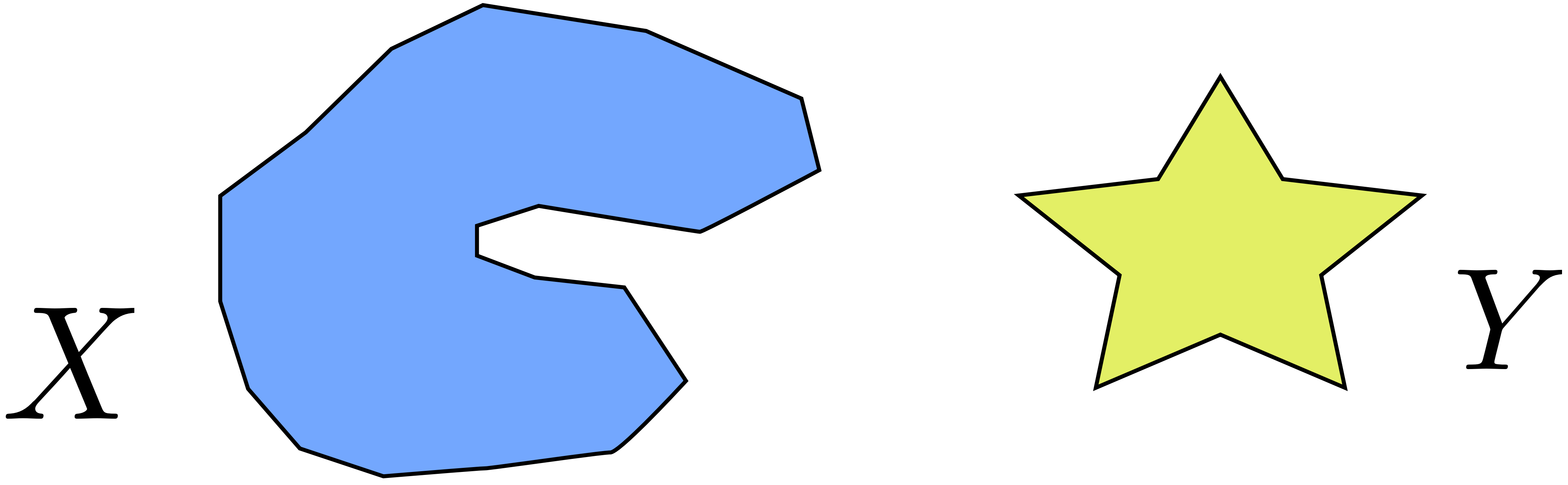

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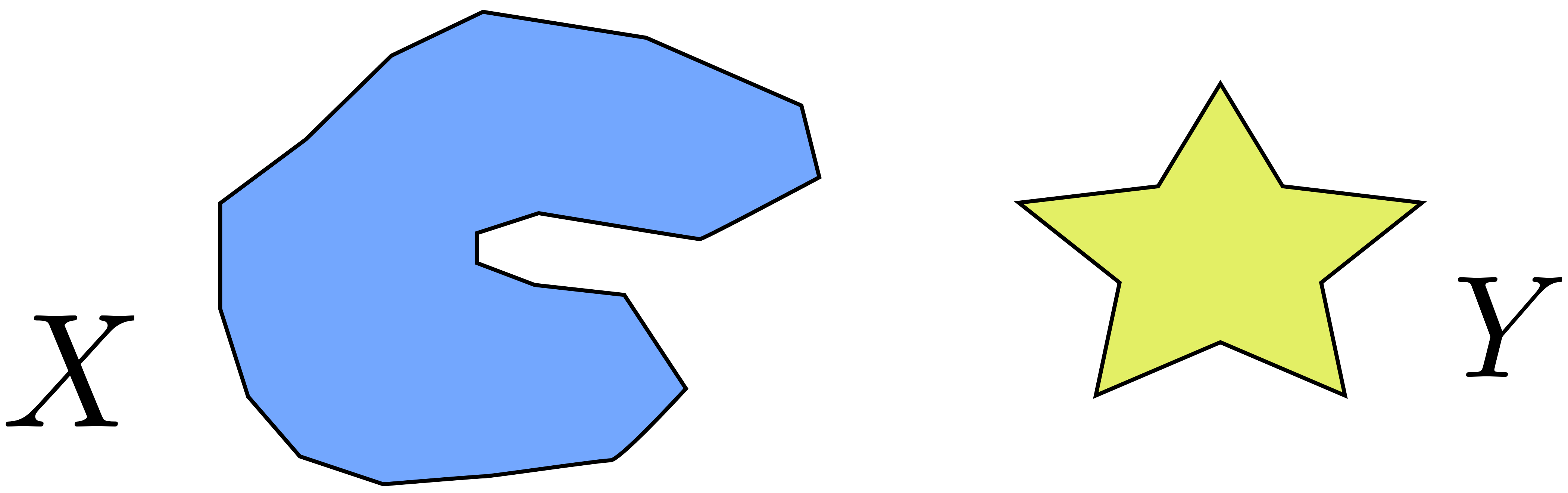

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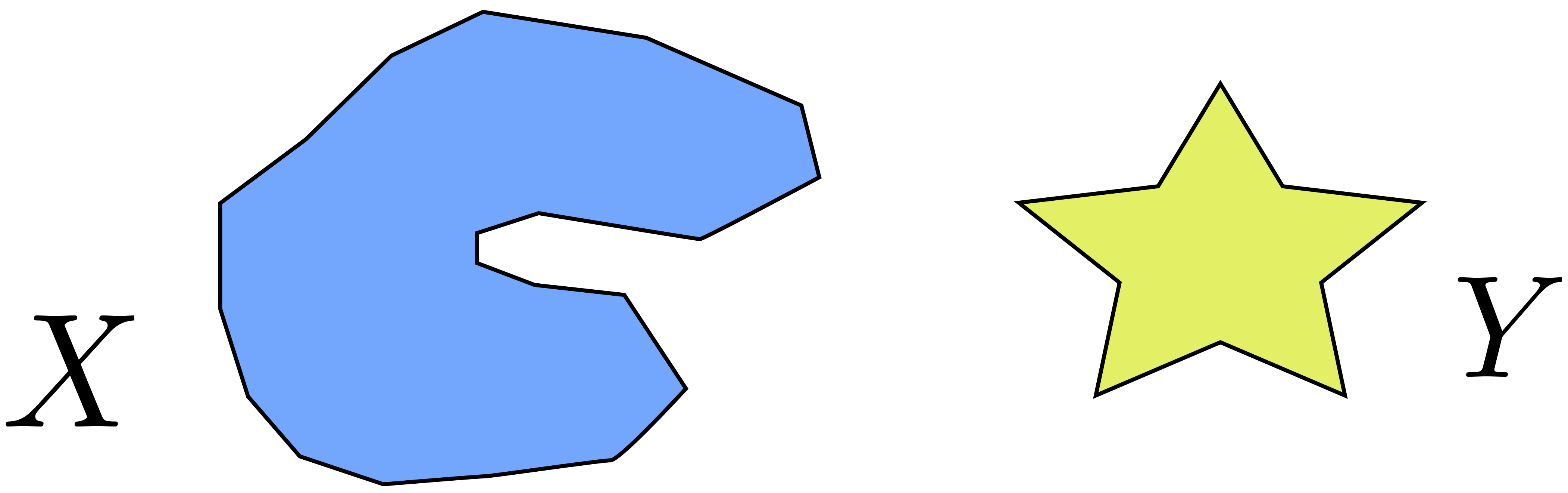

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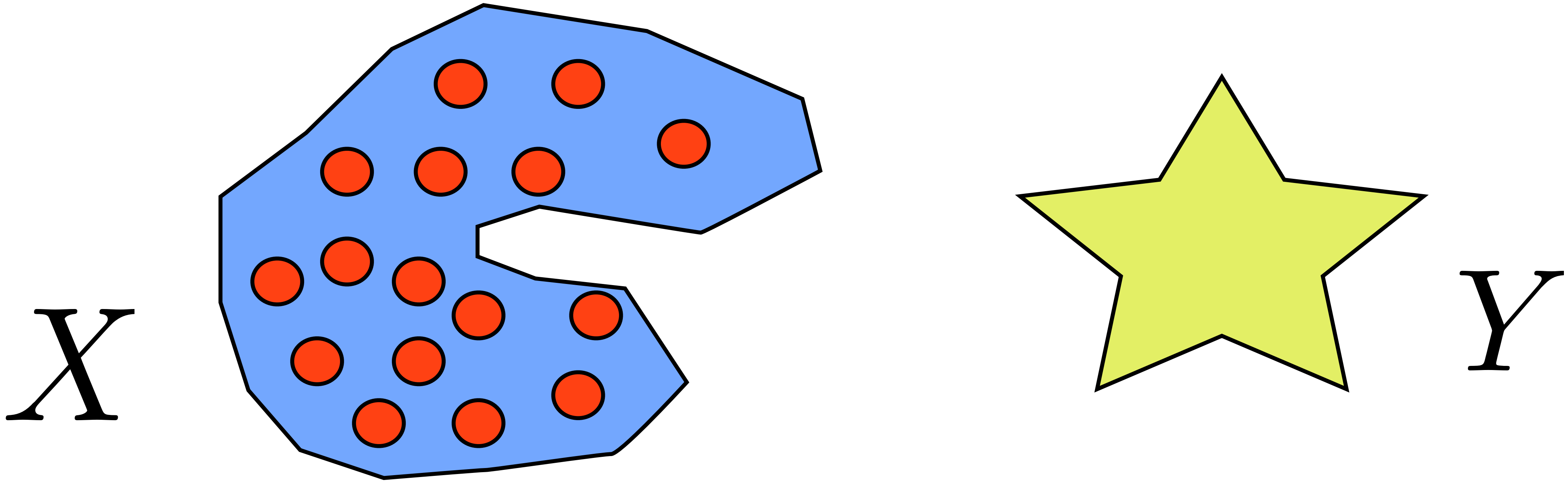

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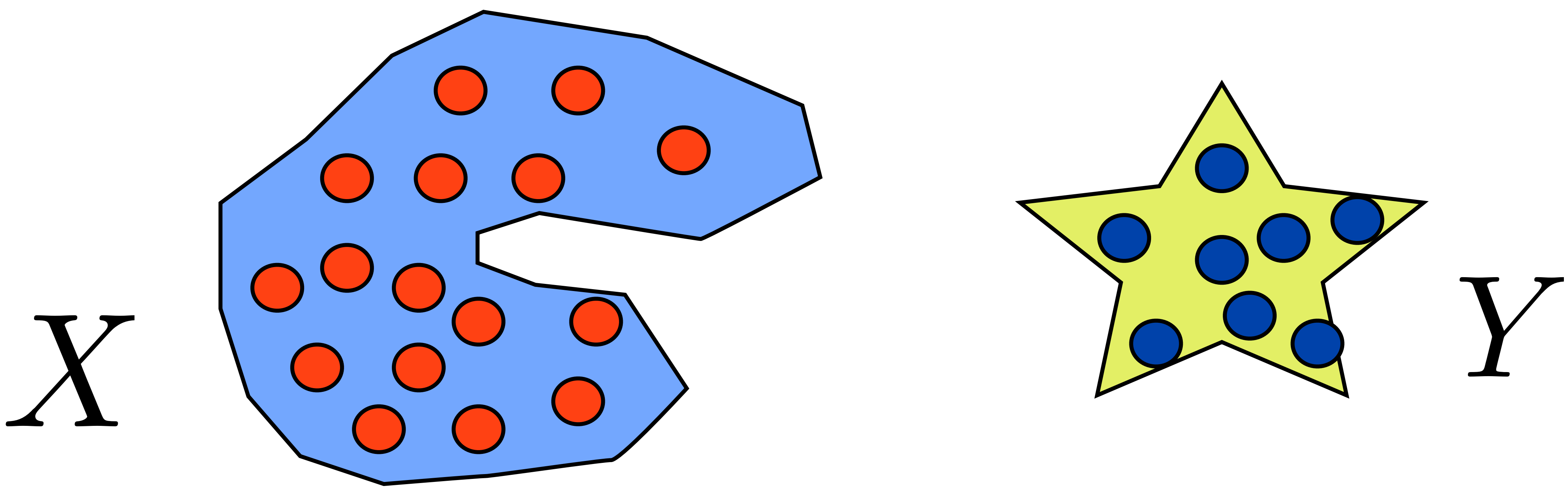

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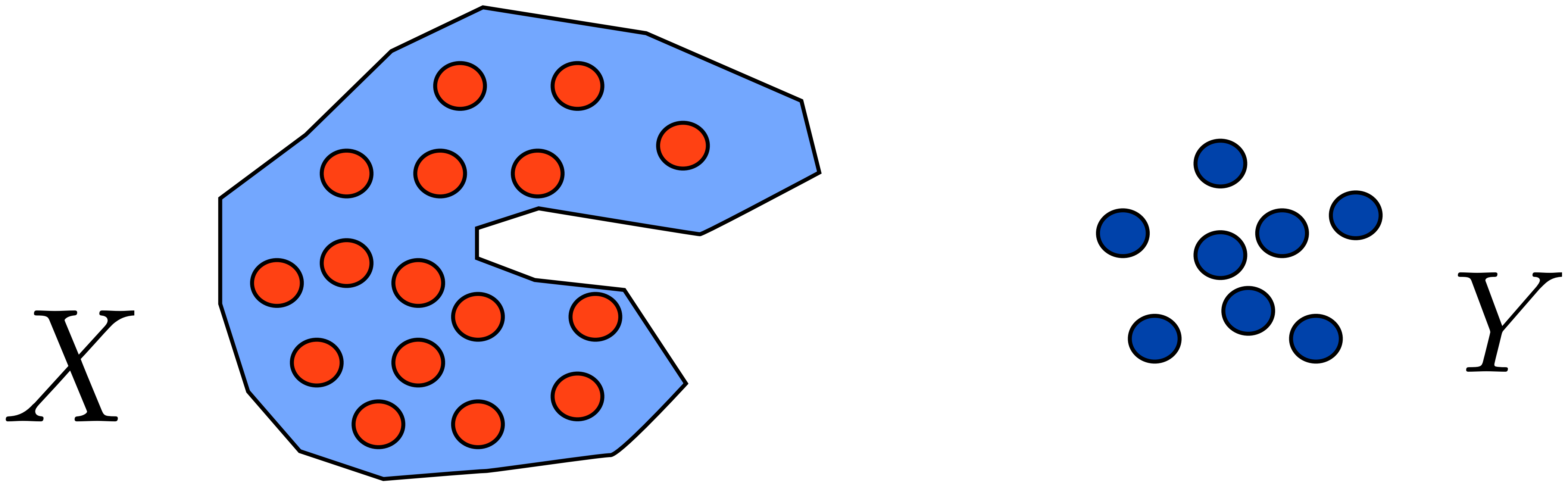

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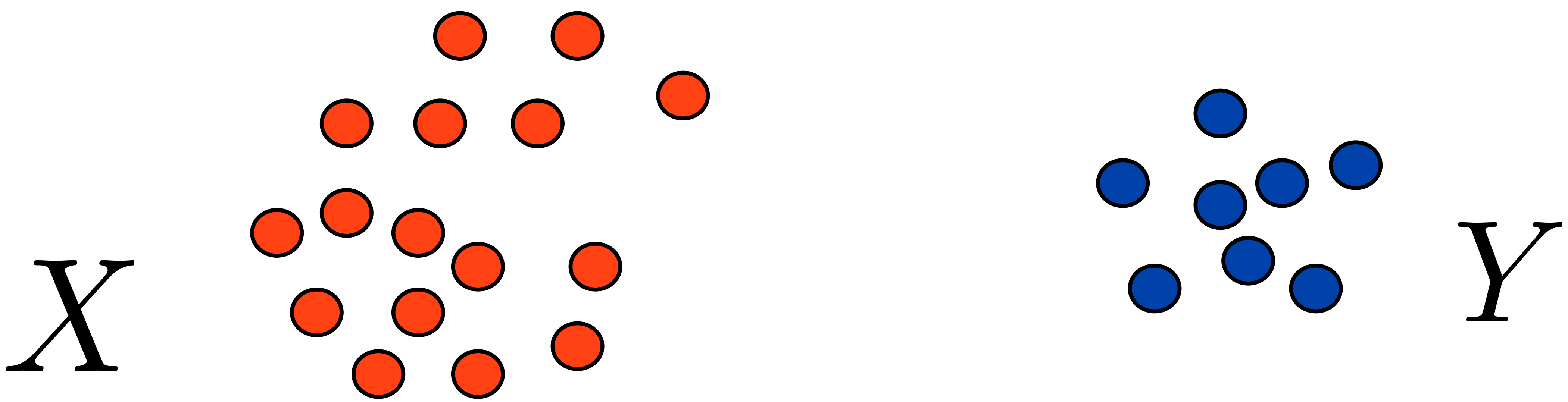

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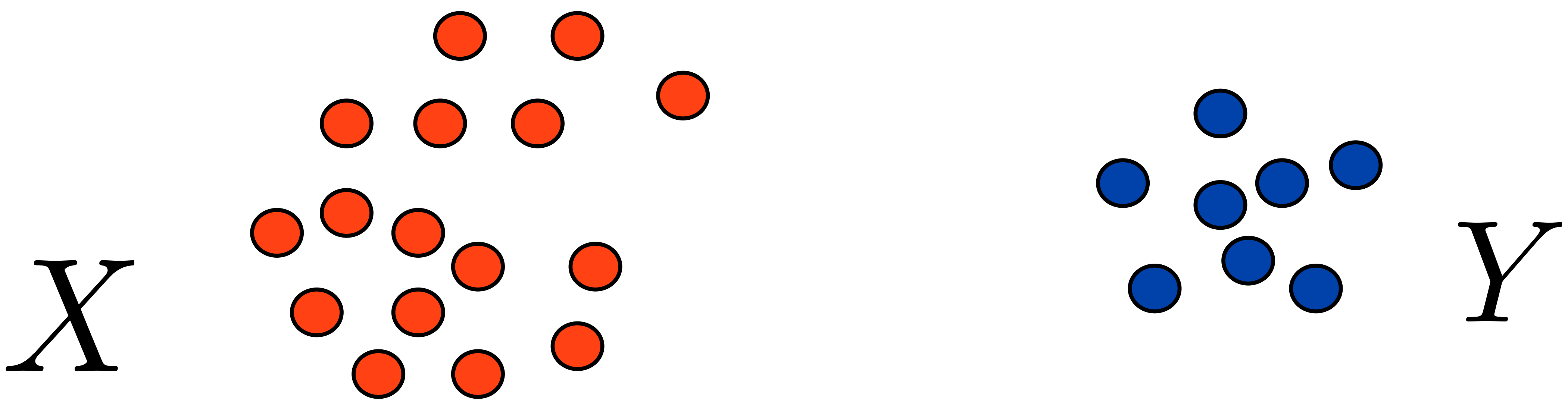

Critique

- Was not able to show connections with (sufficiently many) pre-existing approaches
- Computationally hard: currently only two attempts have been made:
  - [MS04,MS05] and [BBK06] only for surfaces.
  - [MS05] gives probabilistic guarantees for estimator based on sampling parameters.
  - Full generality leads to a hard **combinatorial optimization problem**: QAP.

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Desiderata

- Obtain an L^p version of the GH distance that:
  - retains theoretical underpinnings
  - its implementation leads to easier (continuous, quadratic, with linear constraints) optimization problems
  - can be related to pre-existing approaches (shape contexts, shape distributions, Hamza-Krim,..) via lower/upper bounds.

First attempt: naive relaxation

Remember that

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_{R} \max_{(x,y),(x',y') \in R} |d_X(x,x') - d_Y(y,y')|$$

where $R \in \mathcal{R}(X,Y)$. Using the matricial representation of R one can write

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2} \inf_{R} \max_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')|\, r_{x,y}\, r_{x',y'}$$

where $R = ((r_{x,y})) \in \{0,1\}^{n_X \times n_B}$ s.t.

$$\sum_{x \in X} r_{xy} \geq 1 \;\; \forall y \in Y$$

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First attempt: naive relaxation (continued)

- The idea would be to use L^p norm instead of L^∞ $(\max\max)$
- relax $r_{x,y}$ to be in $[0,1]$ (!)

Then, the idea would be to compute (for some $p \geq 1$):

$$\widehat{d}_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_R \left(\sum_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')|^{\mathbf{p}} \, r_{x,y} \, r_{x',y'} \right)^{1/\mathbf{p}}$$

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- The resulting problem is a continuous variable QOP with linear constraints, but..
- there is no limit problem.. this discretization cannot be connected to the GH distance..

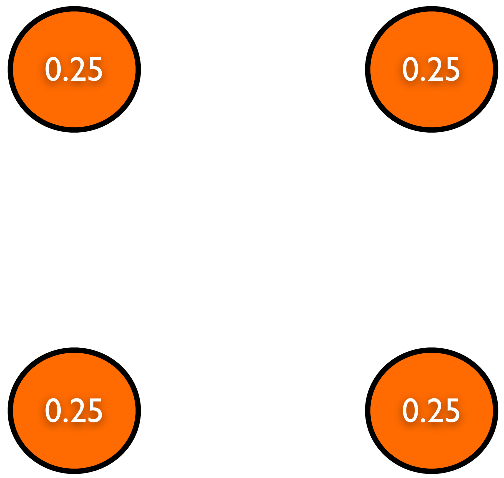
we need to identify the **correct** relaxation of the GH distance. More precisely, the correct notion of *relaxed correspondence*.

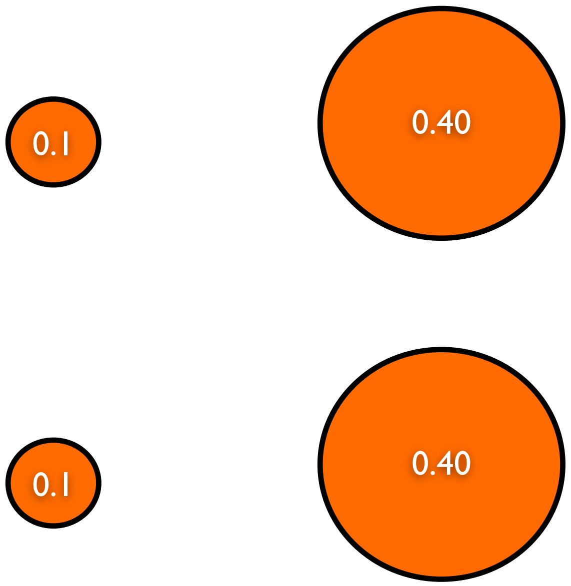
More background

Consider a finite set $A = \{a_1, \ldots, a_n\}$. A set of *weights*, $W = \{w_1, \ldots, w_n\}$ on A is called a *probability measure* on A if $w_i \geq 0$ and $\sum_i w_1 = 1$.

Probability measures can be interpreted as a way of assigning (relative) importance to different points.

There is a more general definition that we do not need.





correspondences and measure couplings

Let A and B be compact subsets of the compact metric space (X, d) and μ_A and μ_B be **probability measures** supported in A and B respectively.

Definition [Measure coupling] Is a probability measure μ on $A \times B$ s.t. (in the finite case this means $((\mu_{a,b})) \in [0,1]^{n_A \times n_B}$)

- $\sum_{a \in A} \mu_{ab} = \mu_B(b) \;\; \forall b \in B$
- $\sum_{b \in B} \mu_{ab} = \mu_A(a) \;\; \forall a \in A$

Let $\mathcal{M}(\mu_A, \mu_B)$ be the set of all couplings of μ_A and μ_B.
Notice that in the finite case, $((\mu_{a,b}))$ must satisfy $n_A + n_B$ *linear* constraints.

Y

	0.30	0.30	0.40
0.25	0.25	0	0
0.25	0	0.15	0.10
0.25	0	0.1	0.15
0.25	0.05	0.05	0.15

X

The support of the coupling consists of the non-zero entries.

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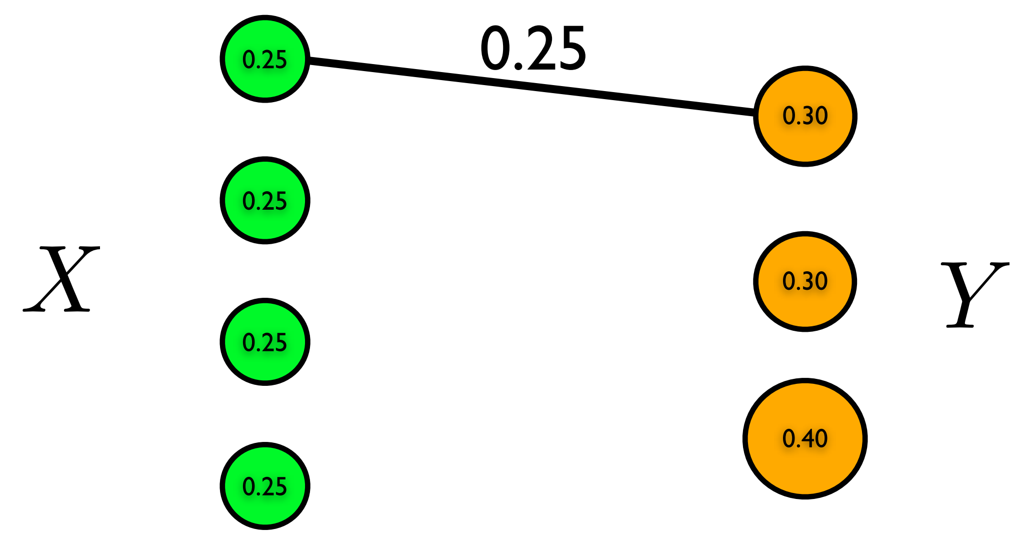

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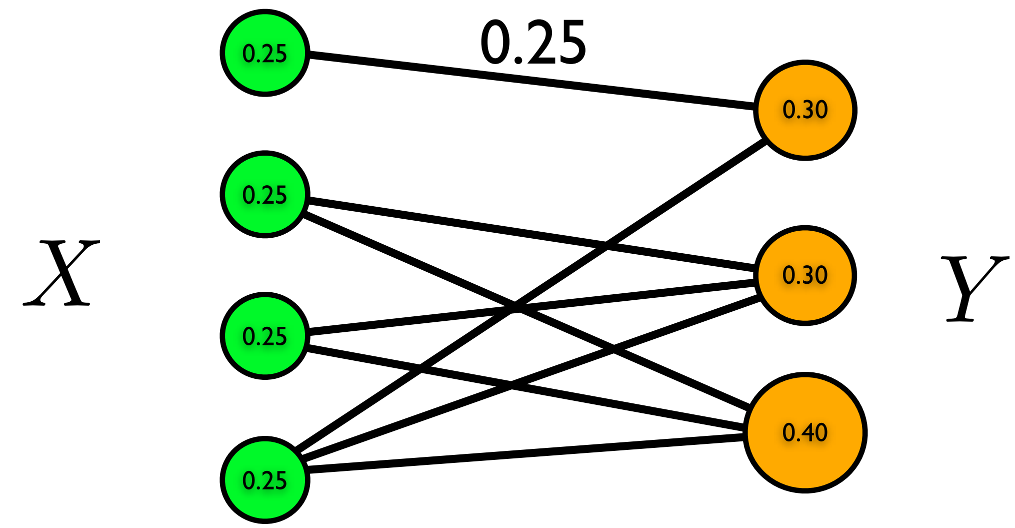

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L^p Gromov-Hausdor distances [M07]

Compute (for some $p \geq 1$):

$$\mathbf{D}_p(X,Y) = \frac{1}{2} \inf_{\mu} \left(\sum_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')|^p \, \mu_{x,y} \, \mu_{x',y'} \right)^{1/p}$$

where $\mu = ((\mu_{x,y})) \in [\mathbf{0},\mathbf{1}]^{n_X \times n_Y}$ s.t.

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Numerical Implementation

- The numerical implementation of the second option leads to solving a continuous variable **QOP** with linear constraints:

$$\begin{gathered}\min_U \tfrac{1}{2} U^T \boldsymbol{\Gamma} U \\ \text{s.t. } U_{ij} \in [0,1],\, U\mathbf{A} = \mathbf{b}\end{gathered}$$

  where $U \in \mathbb{R}^{n_X \times n_Y}$ is the *unrolled* version of μ, $\boldsymbol{\Gamma} \in \mathbb{R}^{n_X \times n_Y \times n_X \times n_Y}$ is the unrolled version of $\Gamma_{X,Y}$ and $\mathbf{A}$ and $\mathbf{b}$ encode the <u>linear</u> constrains $\mu \in \mathcal{M}(\mu_X, \mu_Y)$.

- This can be approached for example via gradient descent. The QOP is non-convex in general!

- Initialization is done via solving one of the several *lower bounds* (discussed ahead). All these lower bounds lead to solving **LOP**s.

Shapes as mm-spaces, [M07]

- Now we are talking of triples (X, d_X, μ_X) where X is a set, d_X a metric on X and μ_X a probability measure on X.
- These objects are called *measure metric spaces*, or mm-spaces for short.
- two mm-spaces X and Y are deemed *equal* or *isomorphic* whenever there exists an isometry $\Phi : X \to Y$ s.t. $\mu_Y(B) = \mu_X(\Phi^{-1}(B)$ for all (measurable) sets $B \subset Y$.

$$(X, d_X, \mu_X)$$

Properties of $\mathbf{D}_p$, [M07]

1. Let X, Y and Z mm-spaces then

$$\mathbf{D}_p(X,Y) \leq \mathbf{D}_p(X,Z) + \mathbf{D}_p(Y,Z).$$

2. If $\mathrm{D}_p(X,Y) = 0$ if and only if X and Y are isomorphic.

3. Let $\mathbb{X}_n = \{x_1, \ldots, x_n\} \subset X$ be a subset of the mm-space (X, d, ν). Endow $\mathbb{X}_n$ with the metric d and a prob. measure ν_n, then

$$\mathbf{D}_p(X, \mathbb{X}_n) \leq d_{\mathcal{W},p}(\nu, \nu_n).$$

The parameter p is not superfluous

The simplest lower bound one has is based on the triangle inequality plus

$$2 \cdot \mathbf{D}_p(X, \{q\}) = \left(\int_{X\times X} d_X(x,x')\,\nu(dx)\nu(dx')\right)^{1/p} := \mathbf{diam}_p(X)$$

That is

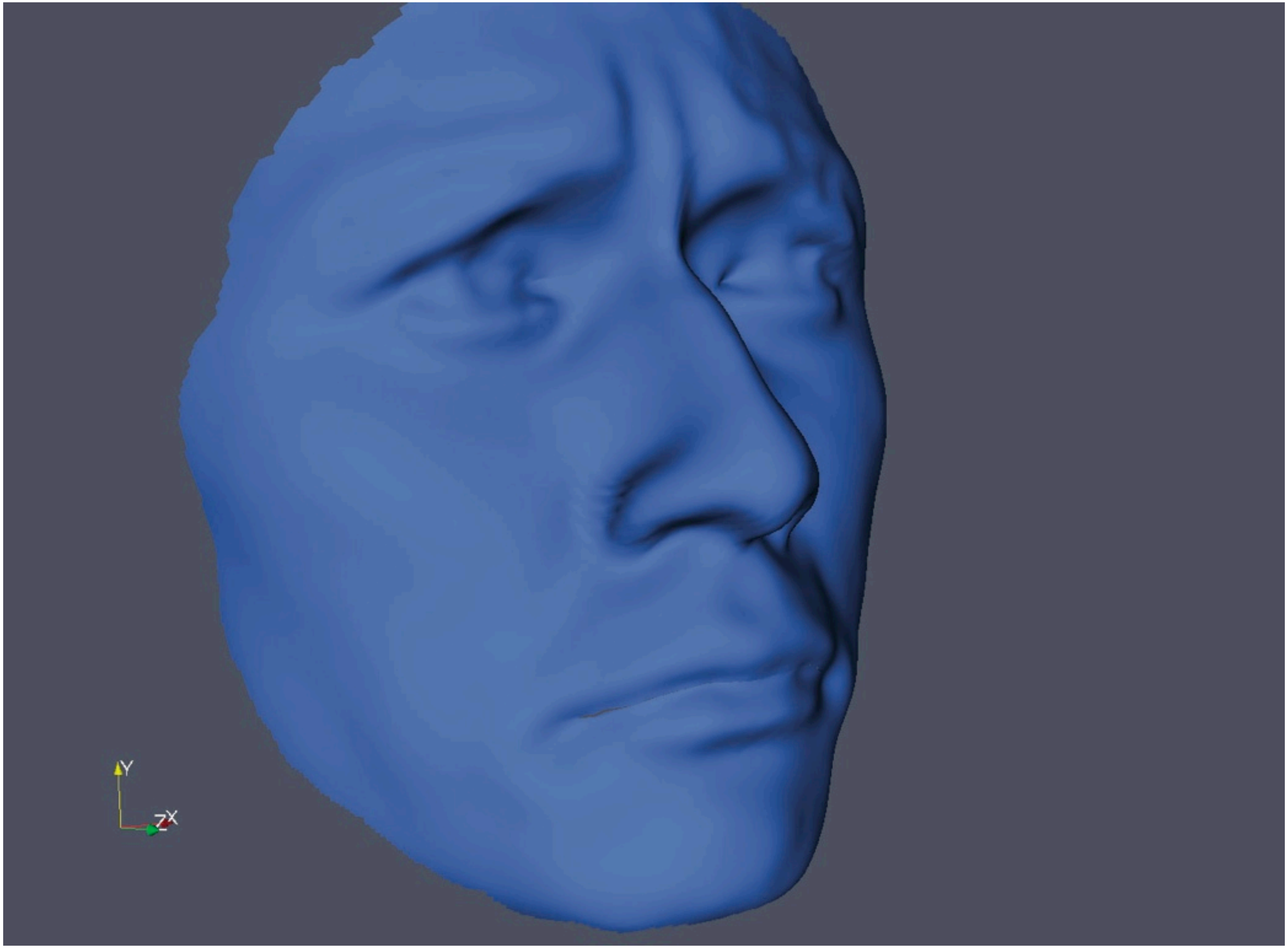
$$\mathbf{D}_p(X,Y) \geq \frac{1}{2}|\mathbf{diam}_p(X) - \mathbf{diam}_p(Y)|$$

For example, when $X = S^n$ (spheres with uniform measure and usual intrinsic metric):

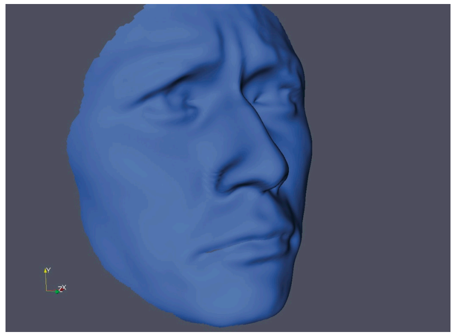
- $p = \infty$ gives $\mathbf{diam}_\infty(S^n) = \pi$ for all $n \in \mathbb{N}$
- $p = 1$ gives $\mathbf{diam}_1(S^n) = \pi/2$ for all $n \in \mathbb{N}$
- $p = 2$ gives $\mathbf{diam}_2(S^1) = \pi/\sqrt{3}$ and $\mathbf{diam}_2(S^2) = \sqrt{\pi^2/2 - 2}$

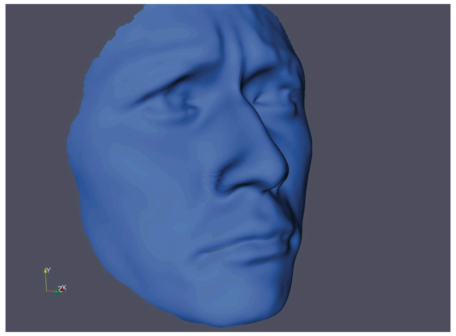
Connections with other approaches

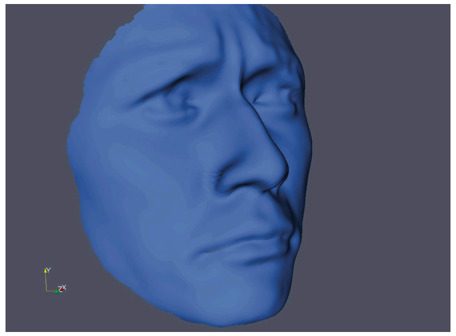
- Shape Distributions [**Osada-et-al**]
- Shape contexts [**SC**]
- Hamza-Krim, Hilaga et al approach [**HK**]
- Rigid isometries invariant Hausdorff [**Goodrich**]
- Gromov-Hausdorff distance [**MS04**] [**MS05**]
- Elad-Kimmel idea [**EK**]
- Topology based methods



Y
Z
X







$$
\begin{pmatrix}
0 & d_{12} & d_{13} & d_{14} & \cdots \\
d_{12} & 0 & d_{23} & d_{24} & \cdots \\
d_{13} & d_{23} & 0 & d_{34} & \cdots \\
d_{14} & d_{24} & d_{34} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Shape Distributions [Osada-et-al]

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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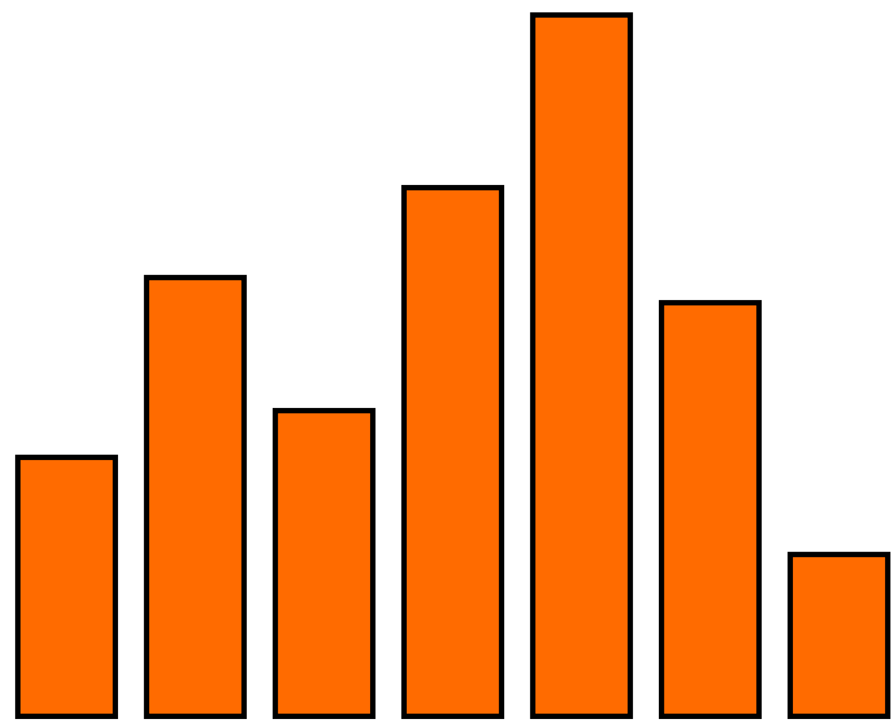

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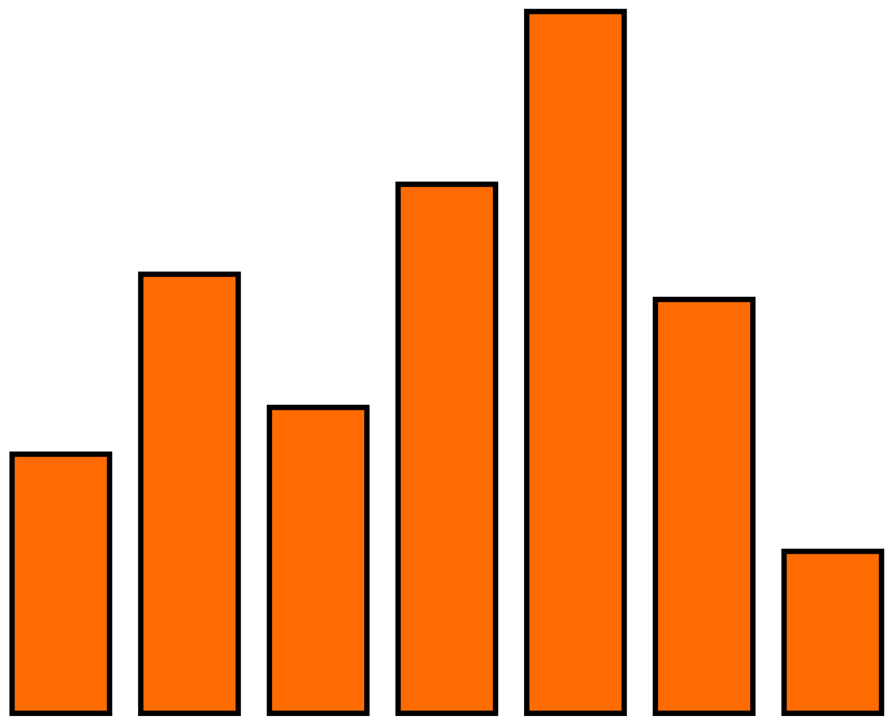

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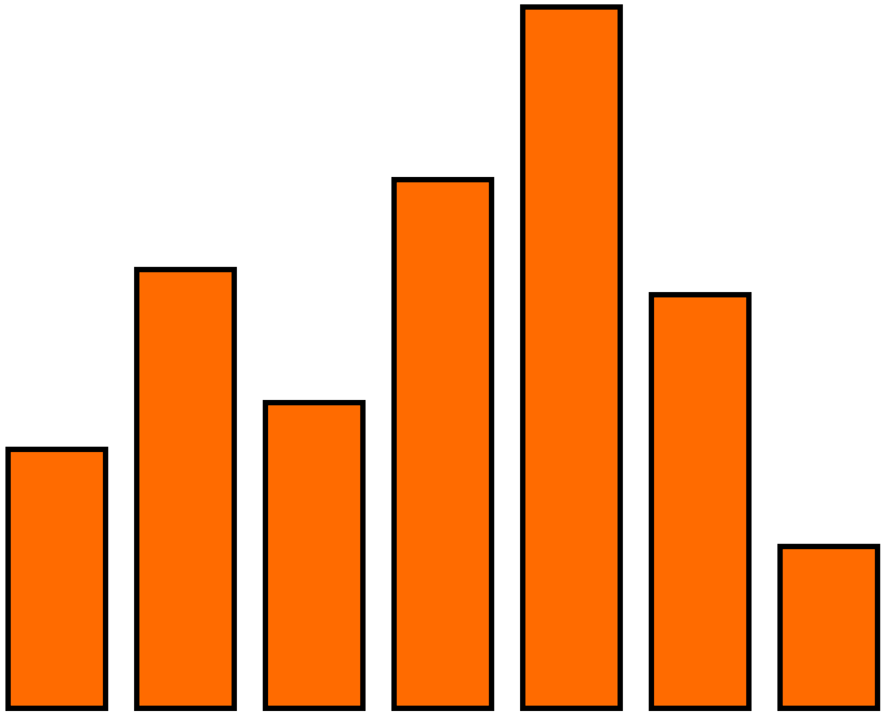

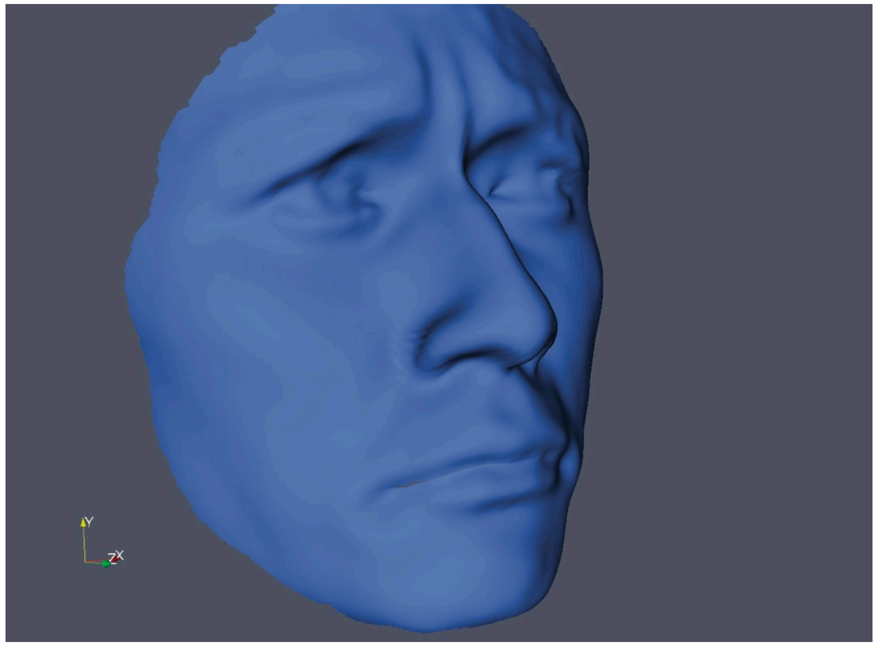

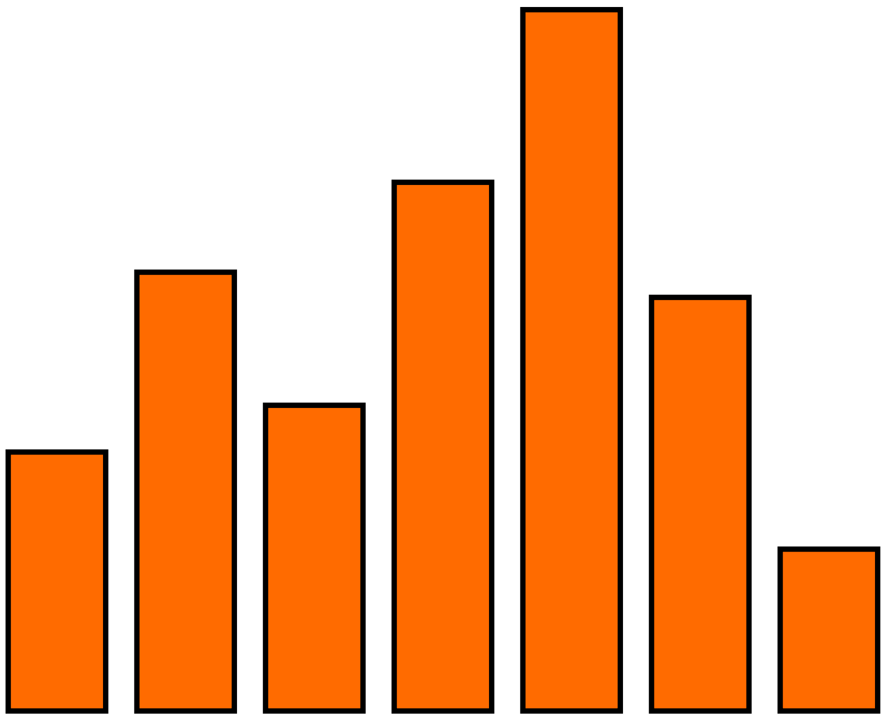

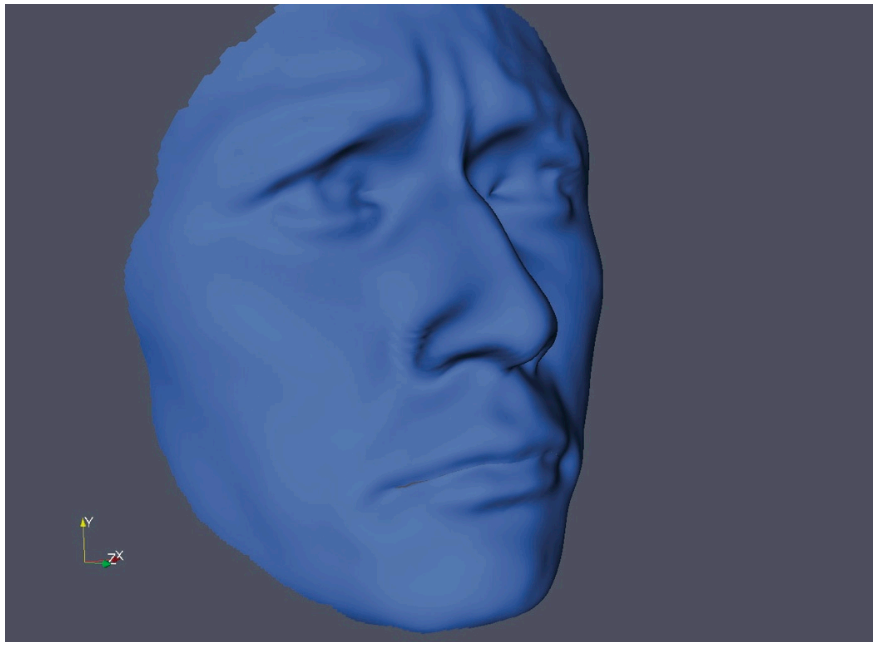

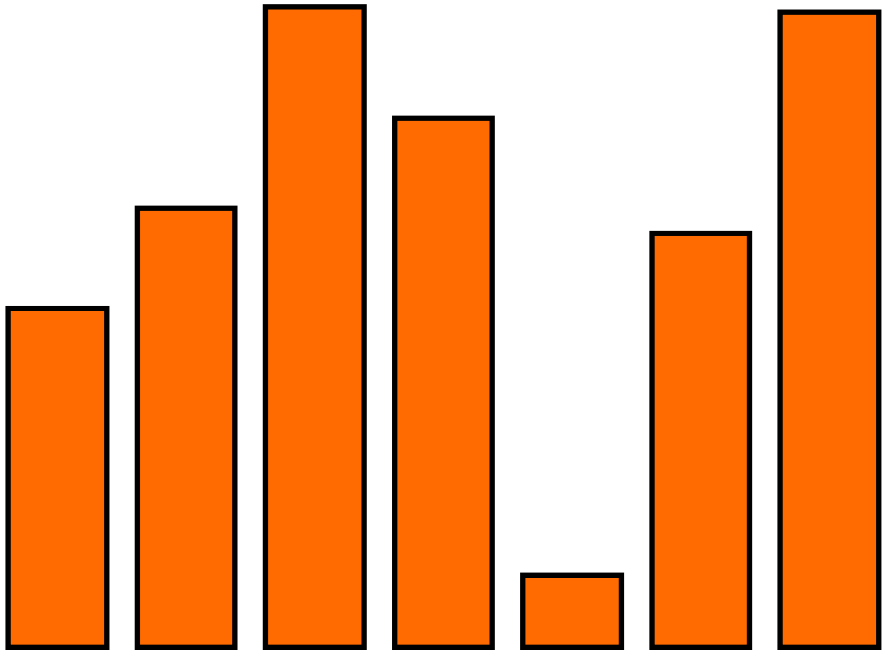

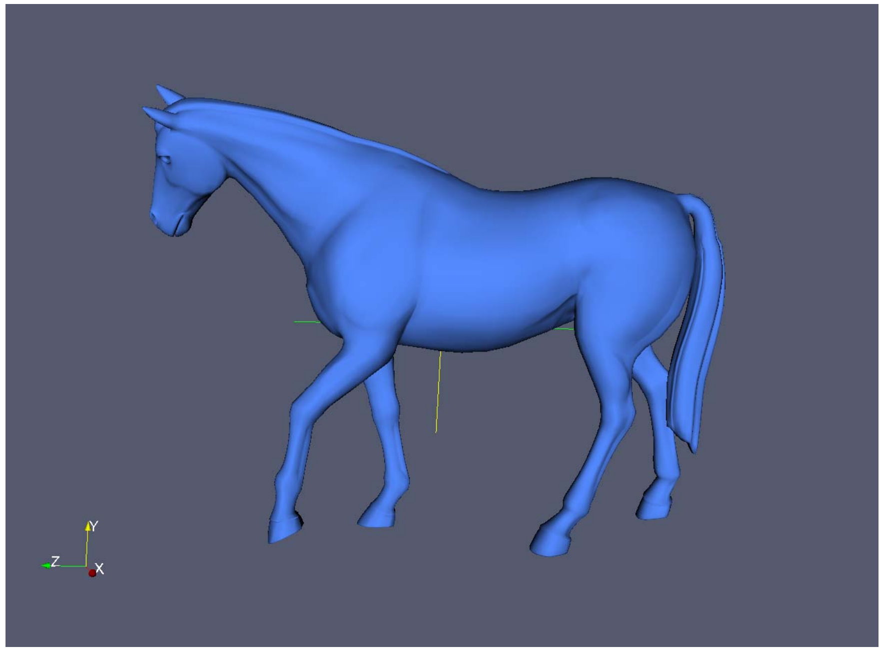

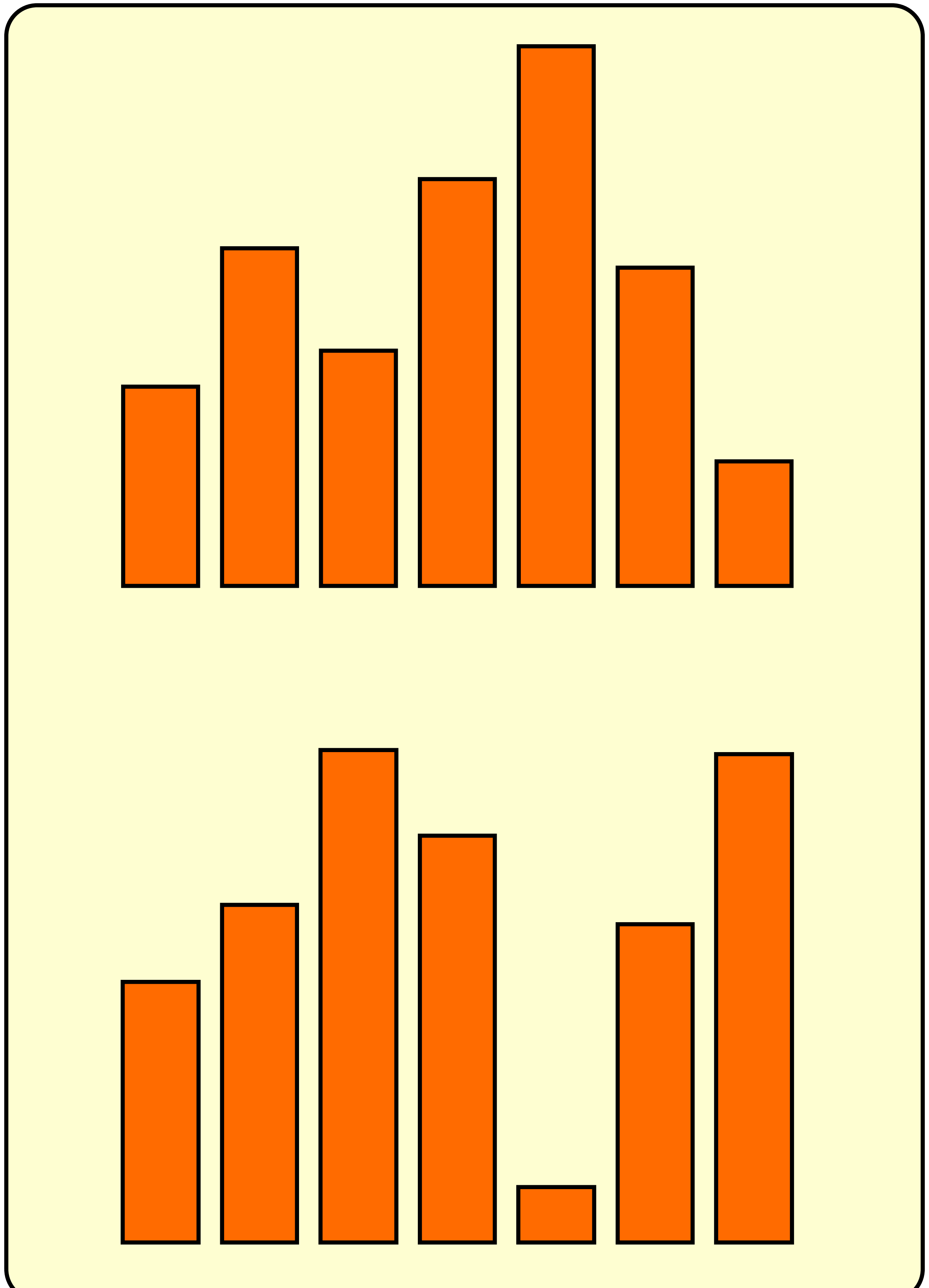

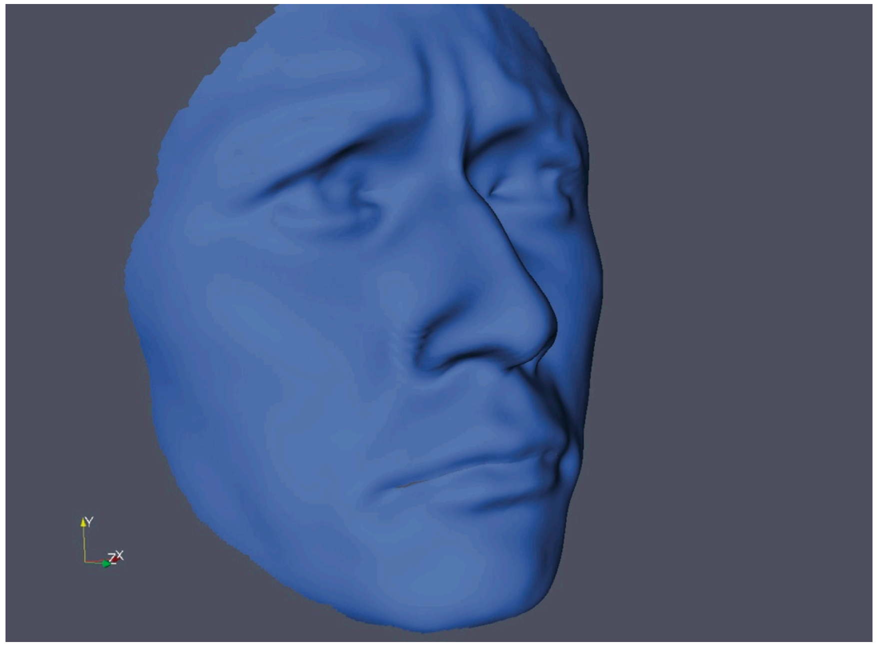

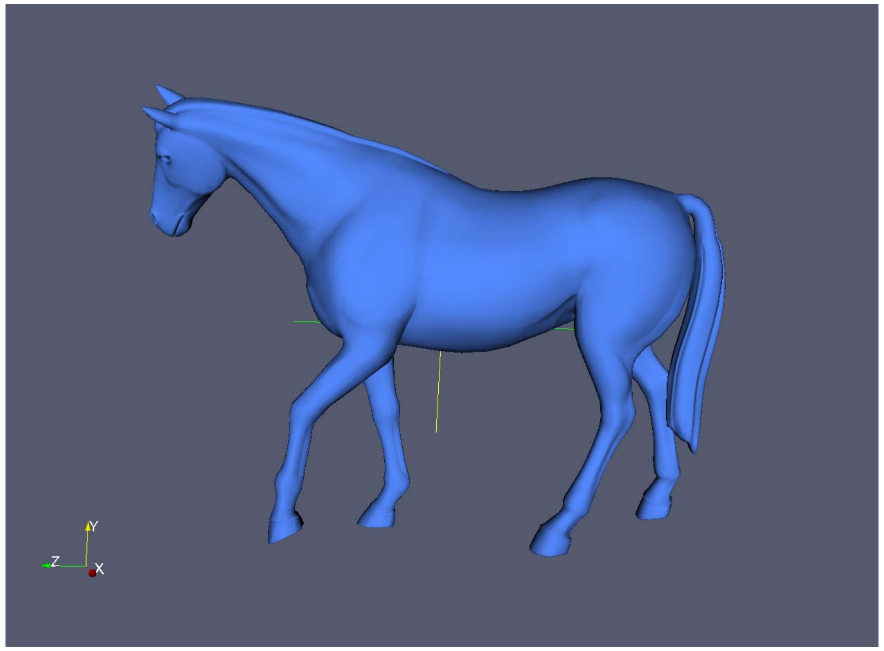

Upper and Lower bounds Let $(X, d, \ )$ be an mm-space.

- **Shape Distributions** [Osada-et-al]: construct histogram of interpoint distances, $F_X : \mathbb{R} \to [0,1]$ given by

$$\boxed{t \mapsto \ \otimes \ (\{(x,x') \,|\, d(x,x') \leq t\})}$$

- **Shape Contexts** [SC]: at each $x \in X$, construct histogram of $d(x,\cdot)$, $C_X : X \times \mathbb{R} \to [0,1]$ given by

$$(x,t) \mapsto \ (\{x' \,|\, d(x,x') \leq t\})$$

- **Hamza-Krim** [HK]: at each $x \in X$ compute mean distance to rest of points, $H_X : X \to \mathbb{R}$

$$x \mapsto \left(\int_X d^p(x,x') \ \ (dx')\right)^{1/p}$$

- **Wasserstein under Euclidean isometries**: consider $X, Y \subset \mathbb{R}^d$ and compute

$$d^{iso}_{\mathcal{W},p}(X,Y) = \inf_T d_{\mathcal{W},p}(X, T(Y))$$

- **Gromov-Hausdorff distance** [MS04][MS05][BBK06]

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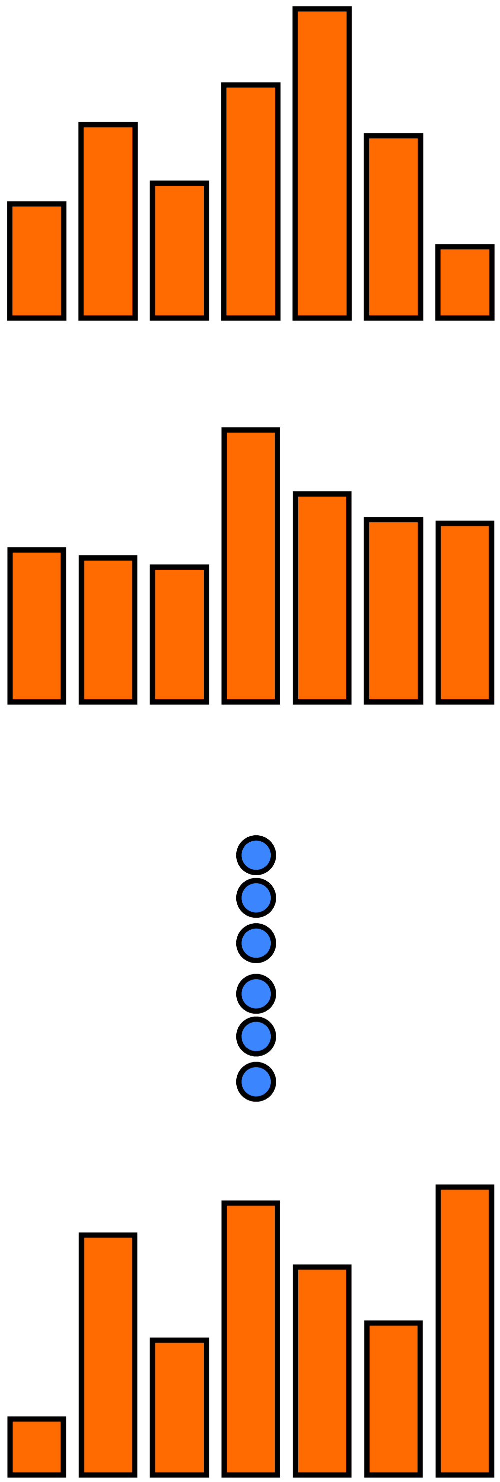
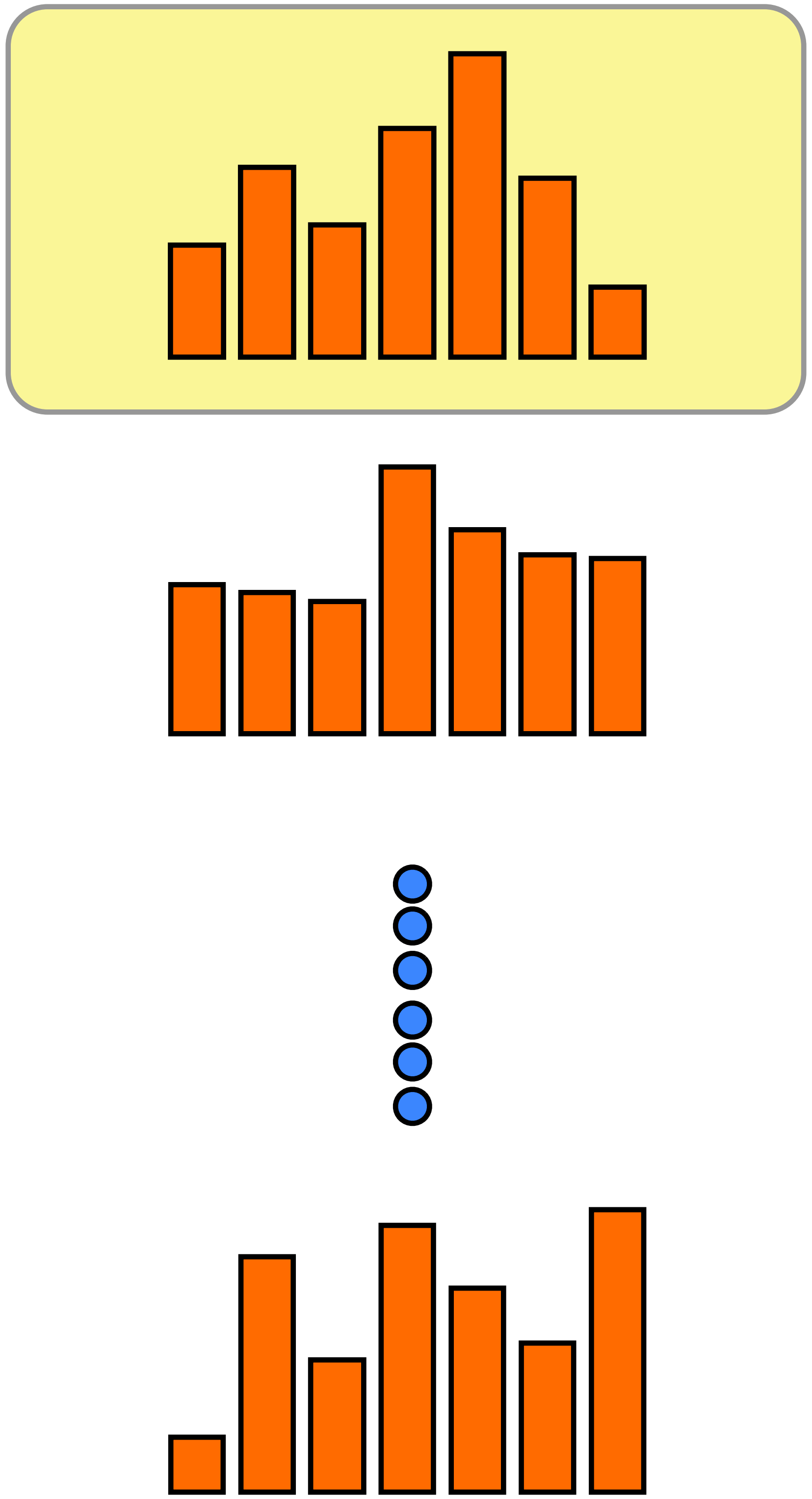
Shape Contexts

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Shape Contexts

$$
\begin{pmatrix}
0 & d_{12} & d_{13} & d_{14} & \dots \\
d_{12} & 0 & d_{23} & d_{24} & \dots \\
d_{13} & d_{23} & 0 & d_{34} & \dots \\
d_{14} & d_{24} & d_{34} & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$



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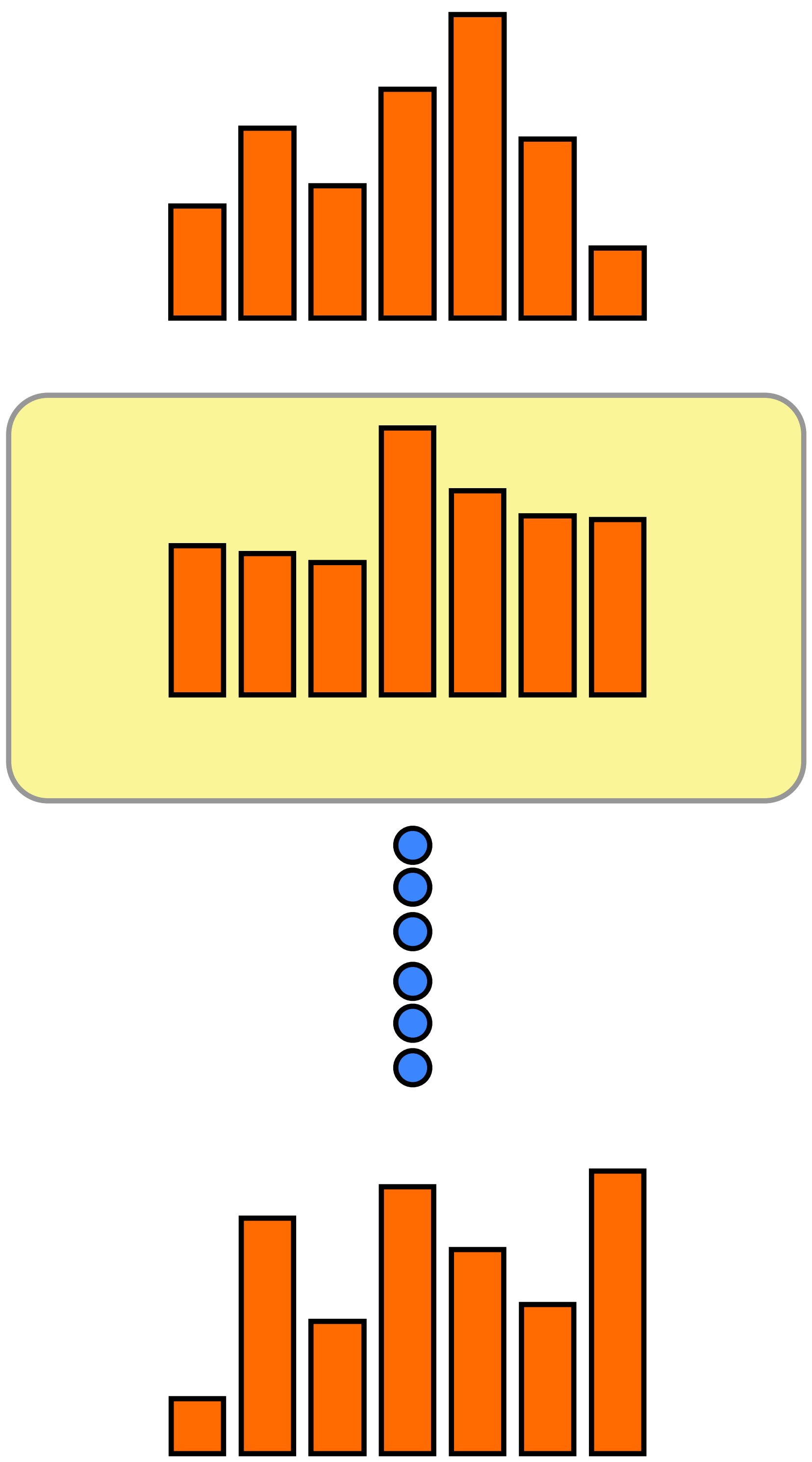

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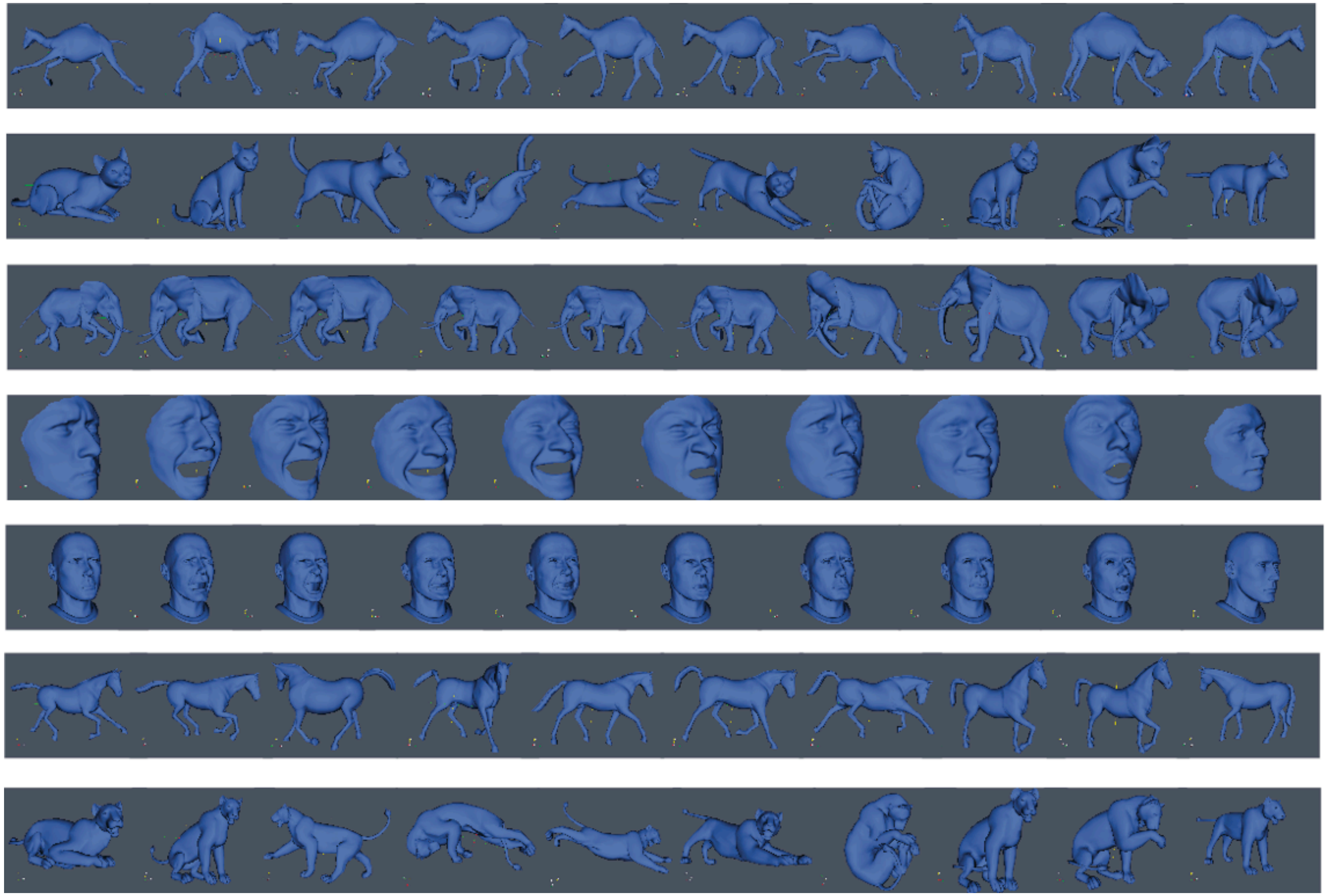
Some Experiments



Some experimentation: ~ 70 models in 7 classes. Classification using 1-nn: $P_e \sim 2\%$. Hamza-Krim gave $\sim 15\%$ on same db with all same parameters etc.

Discussion

Identifying a notion of distance between shapes is important.

- When will you say that two shapes are the same? This is the zero of your distance between shapes.
- Having a true metric on the space of shapes permits proving *stability* and having a *sampling theory*.
- Understand hierarchy of lower/upper bounds. When is a particular LB better than another? study highly symmetrical shapes.

Discussion

- Implementation is easy: Gradient descent or alternate opt.
- Solving lower bounds yields a seed for the gradient descent. These lower bounds are compatible with the metric in the sense that a layered recognition system is possible: given two shapes, (1) solve for a LB (this gives you a μ), if value small enough, then (2) solve for GW using the μ as seed for your favorite iterative algorithm.
- Easy extension to **partial matching**– preprint available from my webpage soon.
- Interest in relating GH/GW ideas to other methods in the literature. Interrelating methods is important also for applications: when confronted with N methods, how do they compare to each other? which one is better for the situation at hand?
  - Euclidean case.
  - Persistent Topology based methods (Frosini et al., Carlsson et al.)
- No difference between continuous and discrete. Probability measures take care of the 'transition'.

`http://math.stanford.edu/~memoli/ShapeComp/sc.html`

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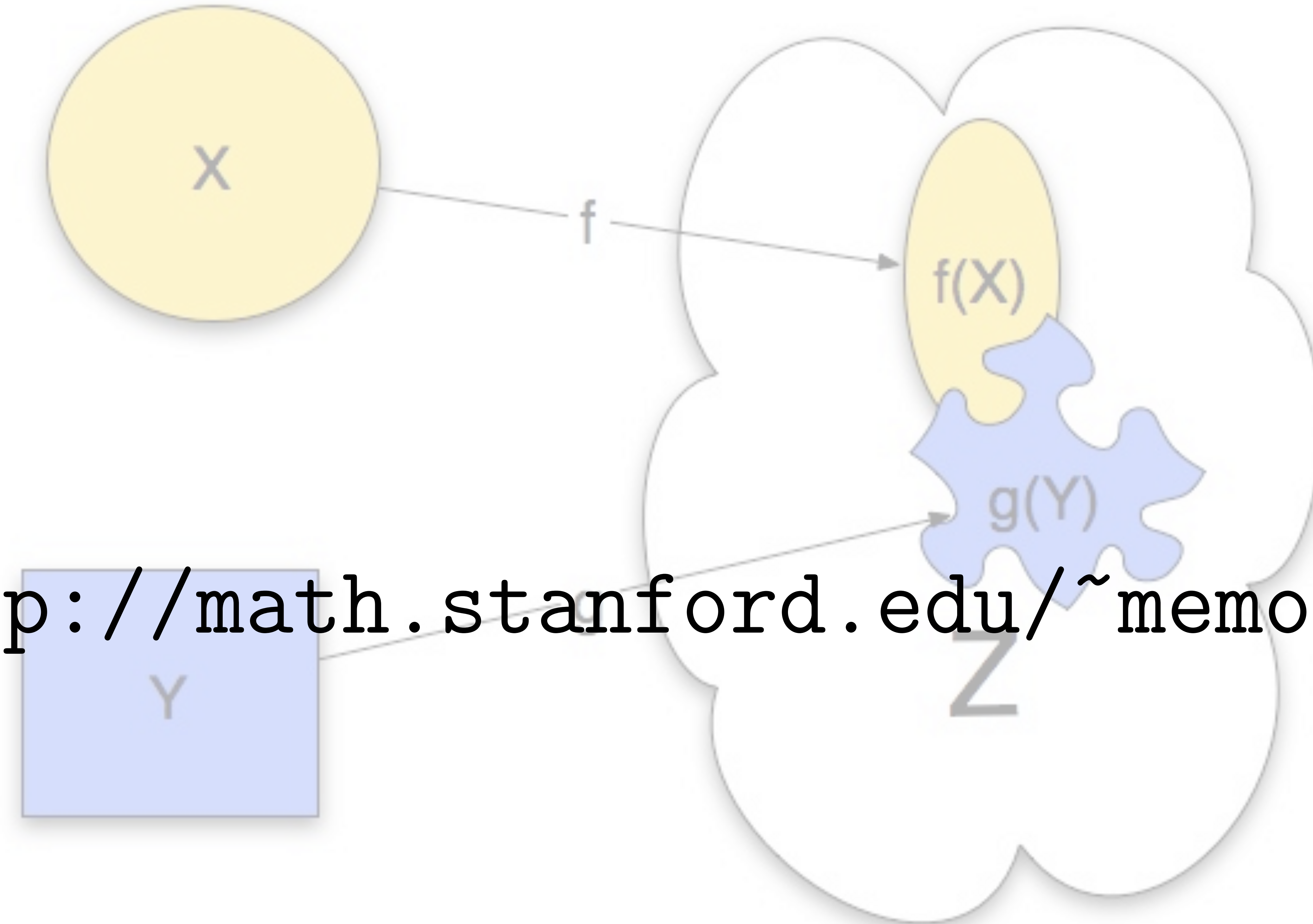

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