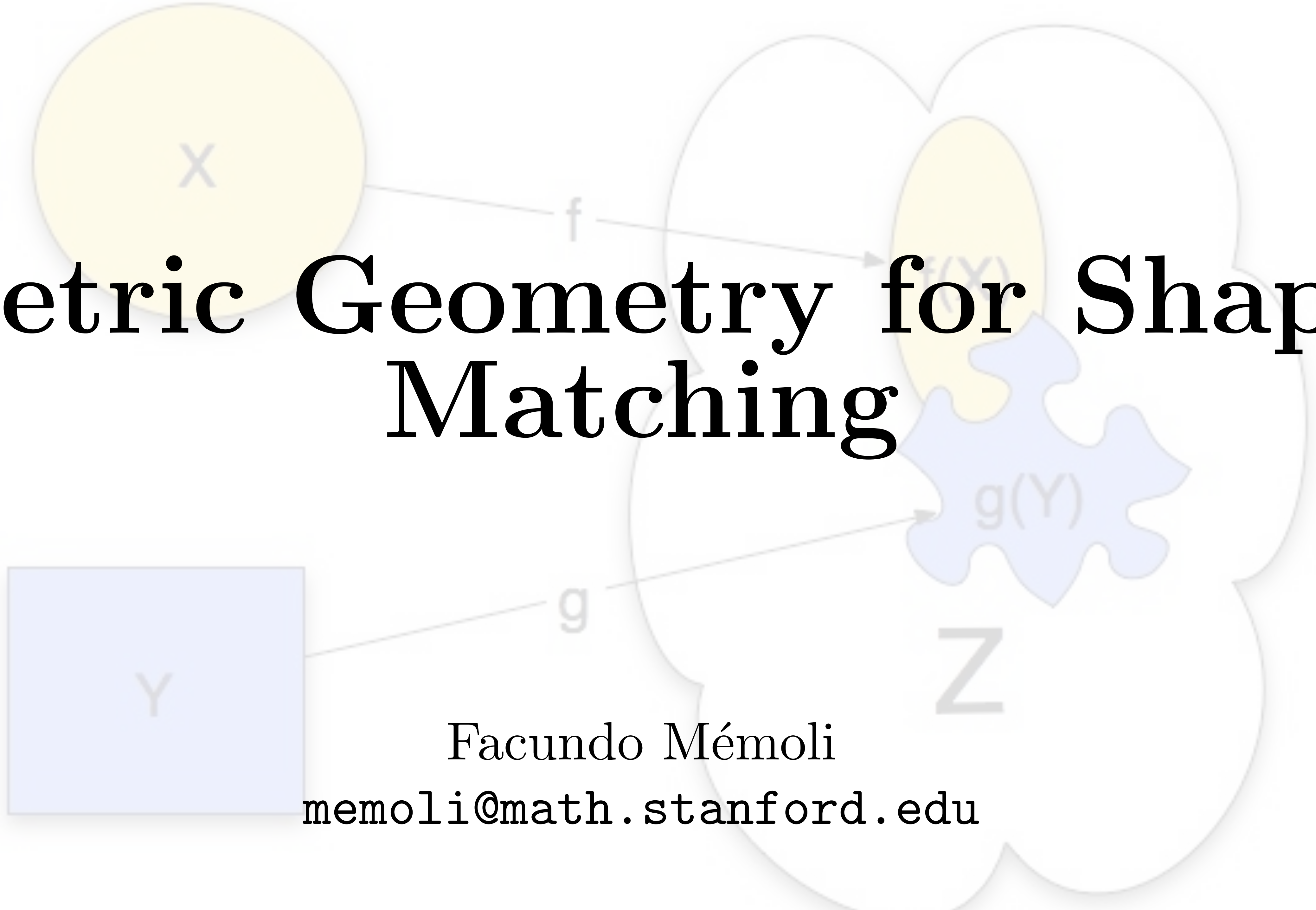
Metric Geometry for Shape Matching



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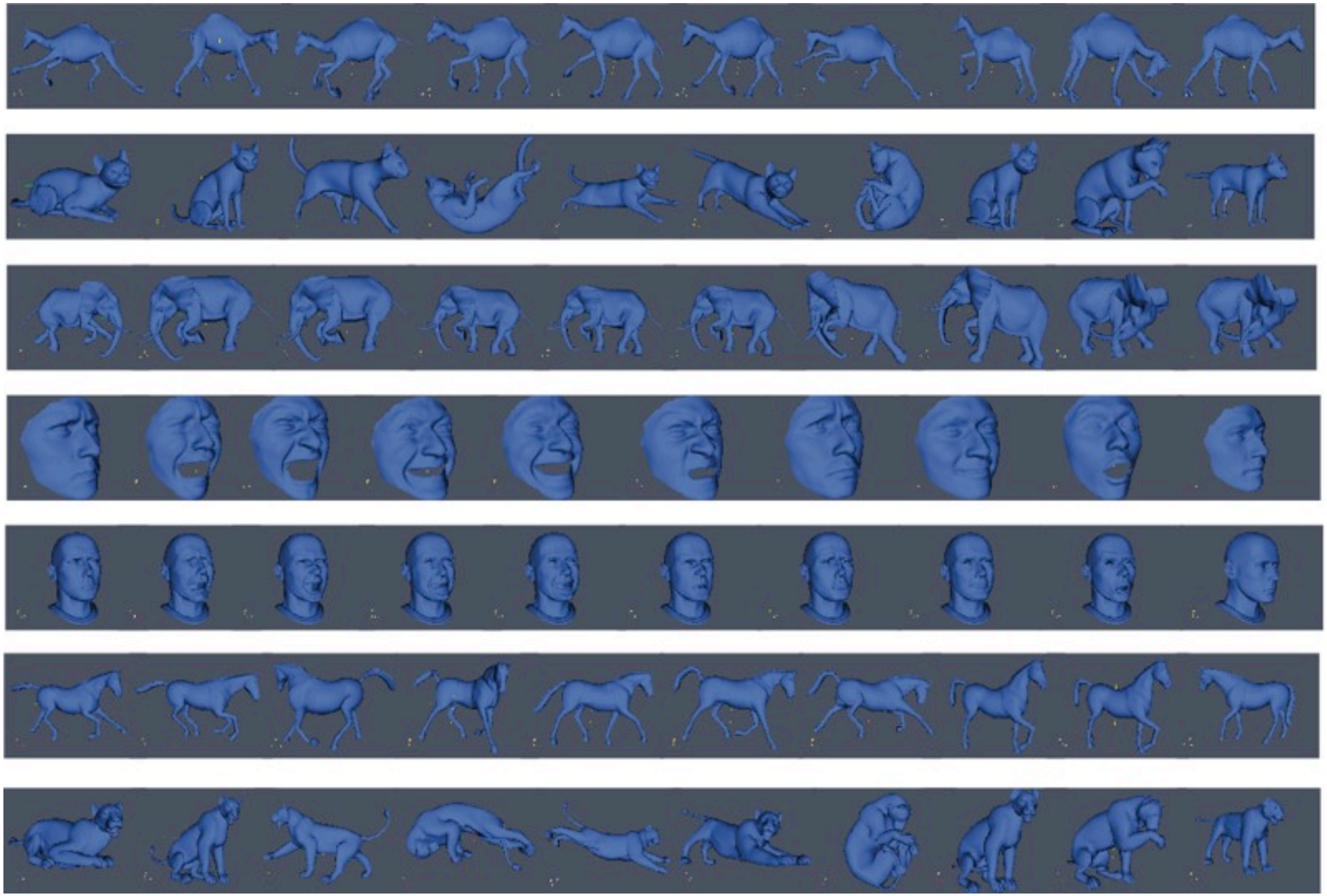
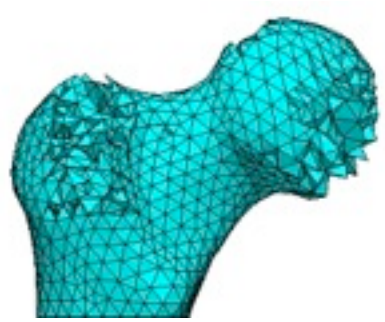
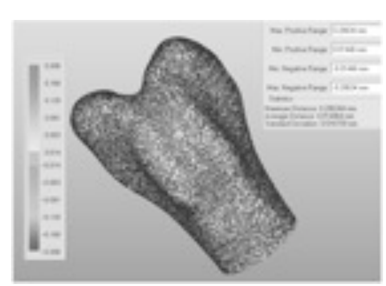
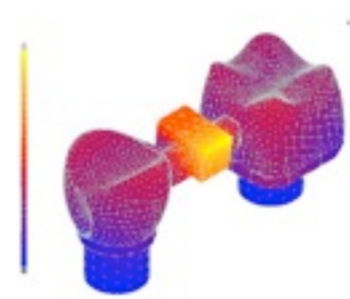
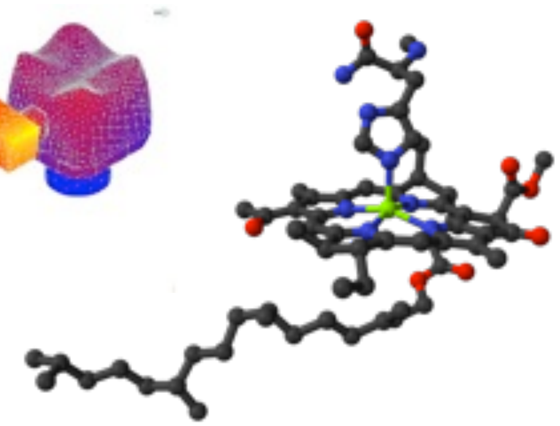
The Problem of Shape/Object Matching

- databases of *objects*
- objects can be many things:
  - proteins
  - molecules
  - 2D objects (imaging)
  - 3D shapes: as obtained via a 3D scanner
  - 3D shapes: modeled with CAD software
  - 3D shapes: coming from design of bone protheses
  - text documents
  - more complicated structures present in datasets (things you can't visualize)



3D objects: examples

- cultural heritage (Michelangelo project: `http://www-graphics.stanford.edu/projects/mich/`)
- search of parts in a factory of, say, cars
- face recognition: the face of an individual is a 3D shape...
- proteins: the *shape* of a protein reflects its function.. protein data bank: `http://www.rcsb.org`


Typical situation: classification

- assume you have database $\mathcal{D}$ of objects.
- assume $\mathcal{D}$ is composed by several objects, and that each of these objects belongs to one of n classes $C_1, \ldots, C_n$.
- imagine you are given a new object o, not in your database, and you are asked to determine whether o belongs to one of the classes. If yes, you also need to point to the class.
- One simple procedure is to say that you will assign object o the class of the *closest* object in $\mathcal{D}$:

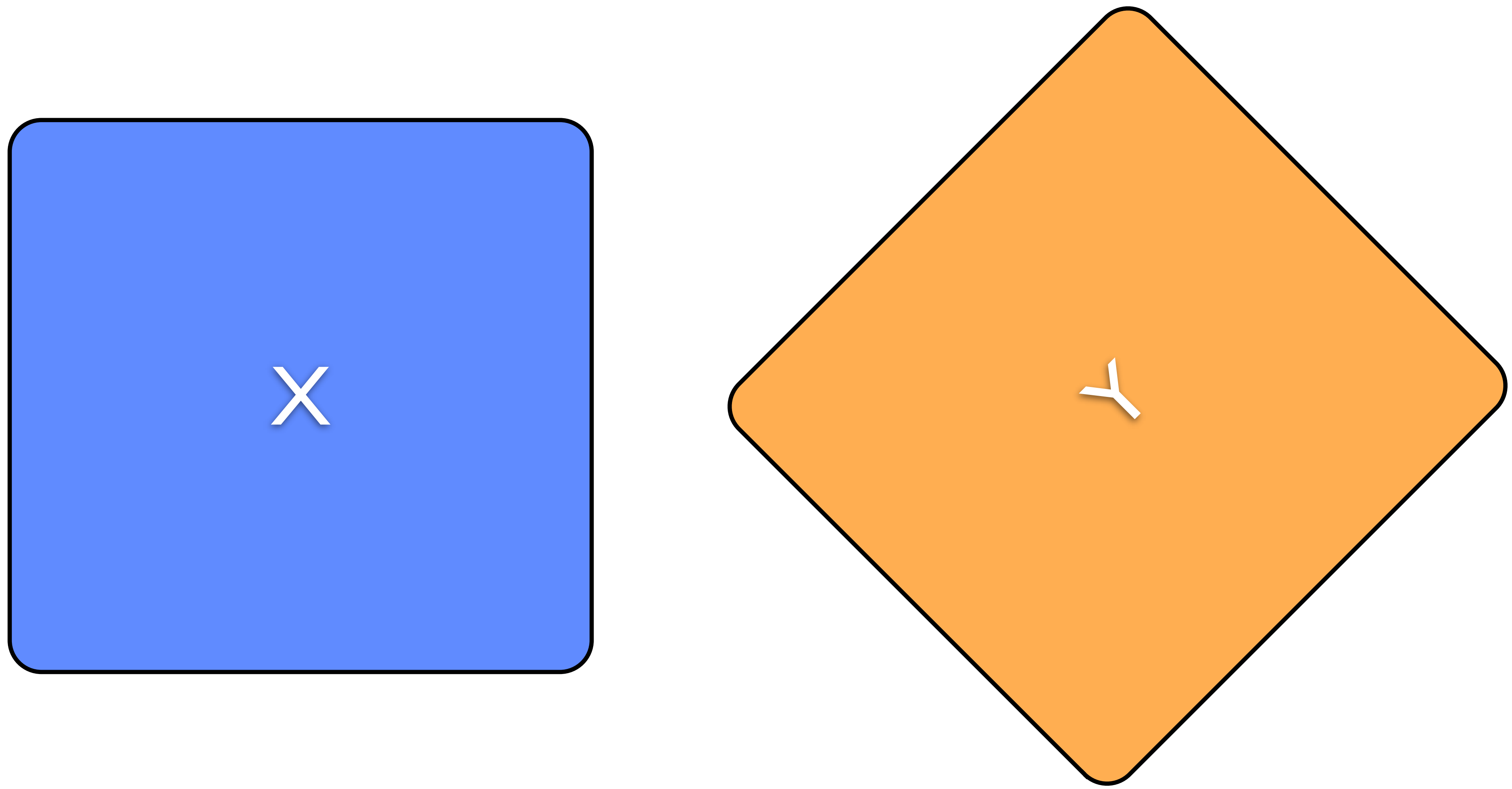
$$\text{class}(o) = \text{class}(z)$$

$$\text{where } z \in \mathcal{D} \text{ minimizes } \mathbf{dist}(o, z)$$

- in order to do this, one first needs to define a notion **dist** of *distance* or *dis-similarity between objects.*

Another important point: invariances

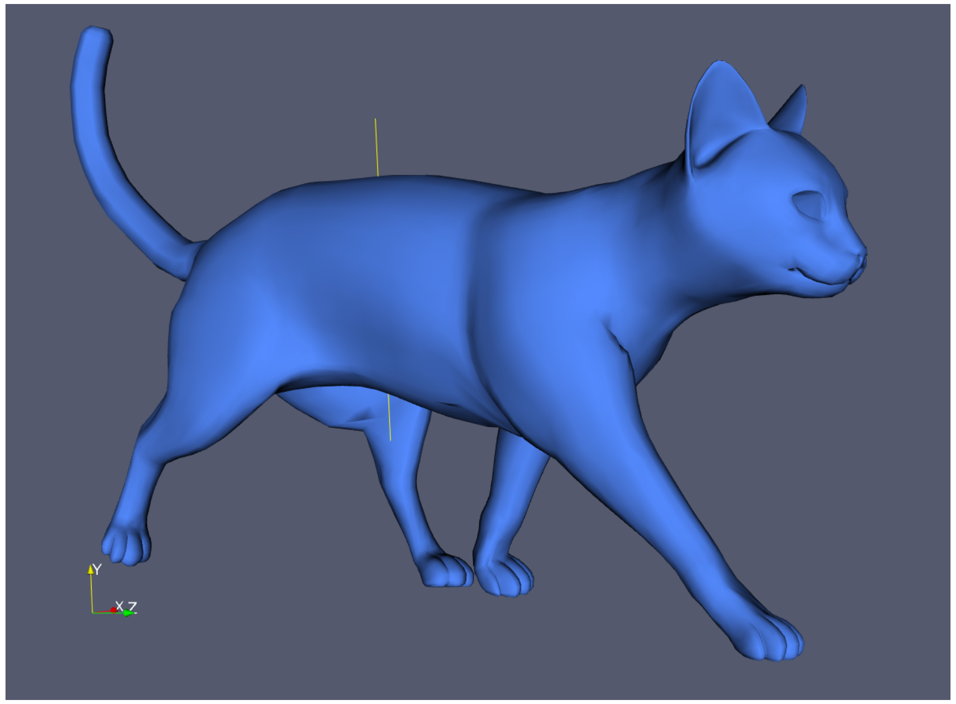
Are these two objects the same?

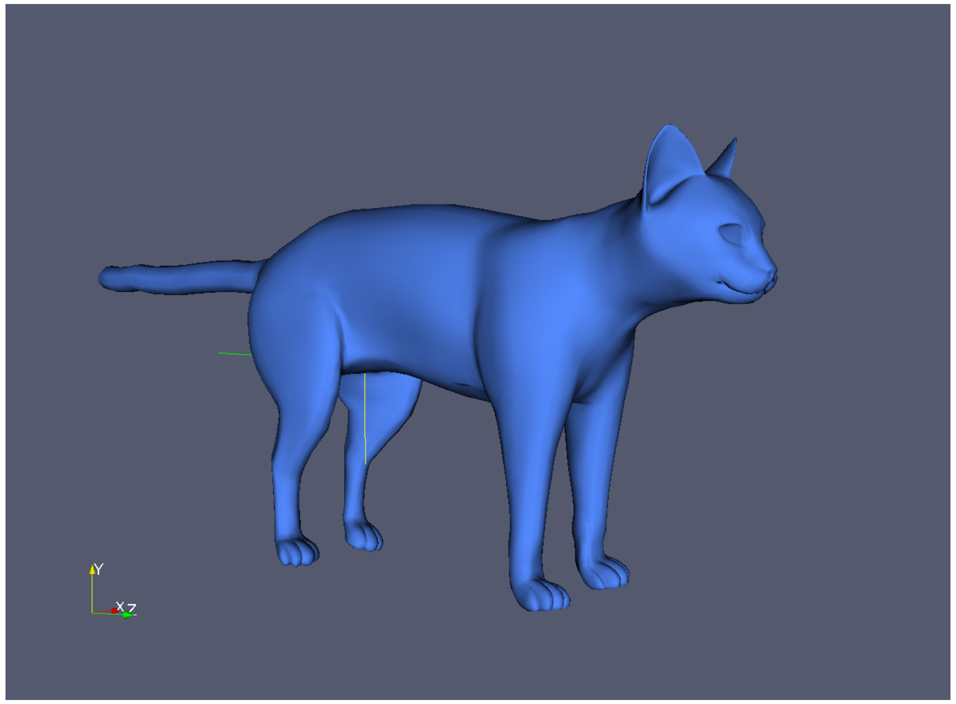


this is called invariance to *rigid transformations*

Another important points: invariances

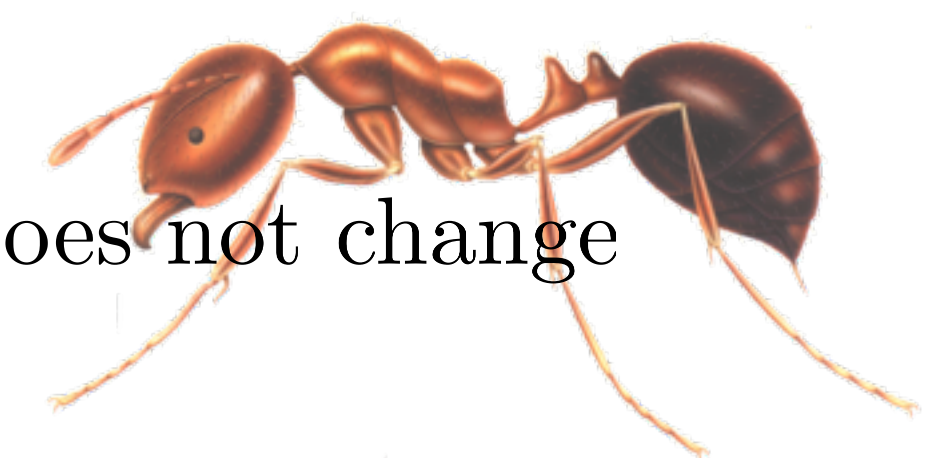
what about these two?



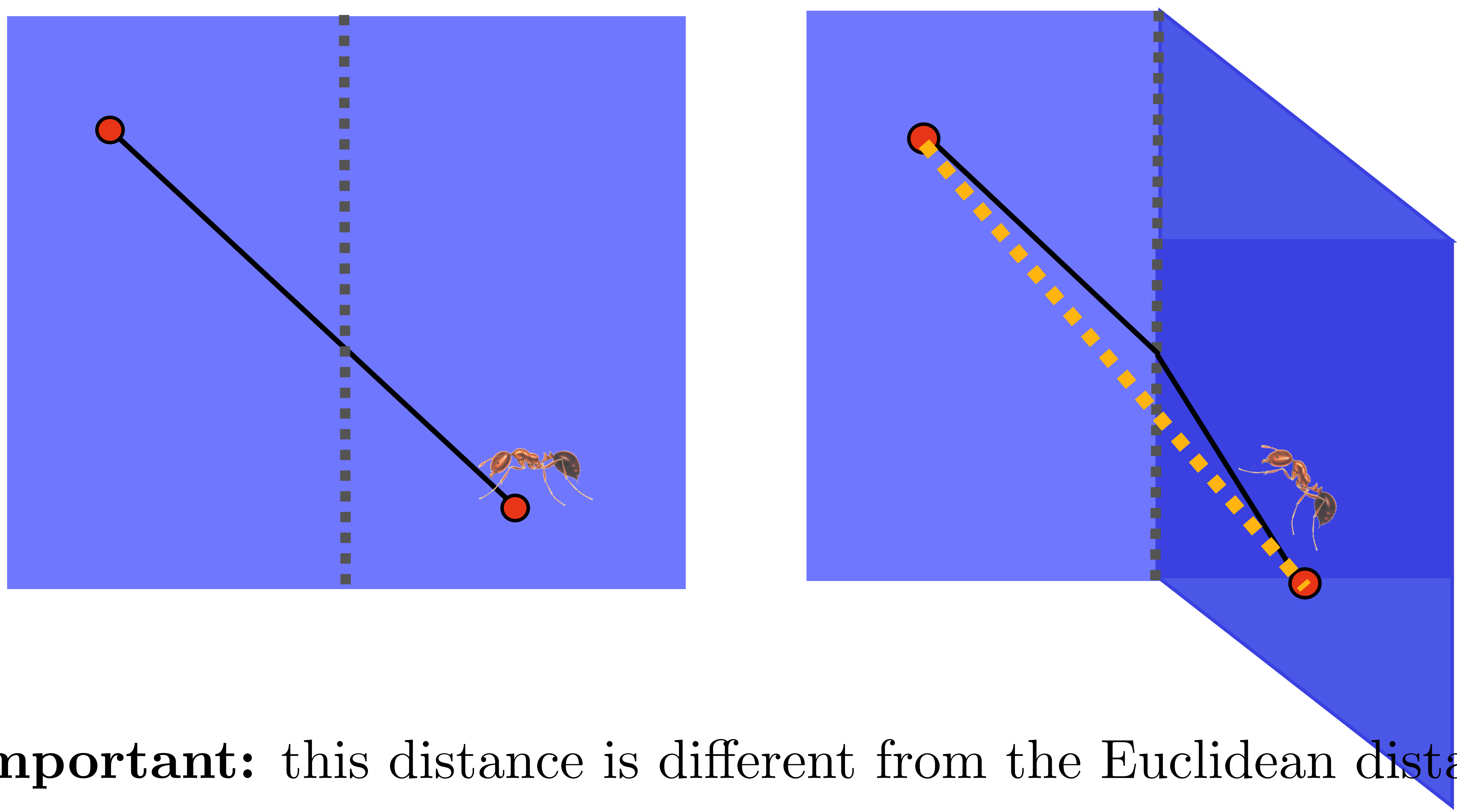


roughly speaking, this corresponds to invariance to *bending transformations*..

Bending transformations

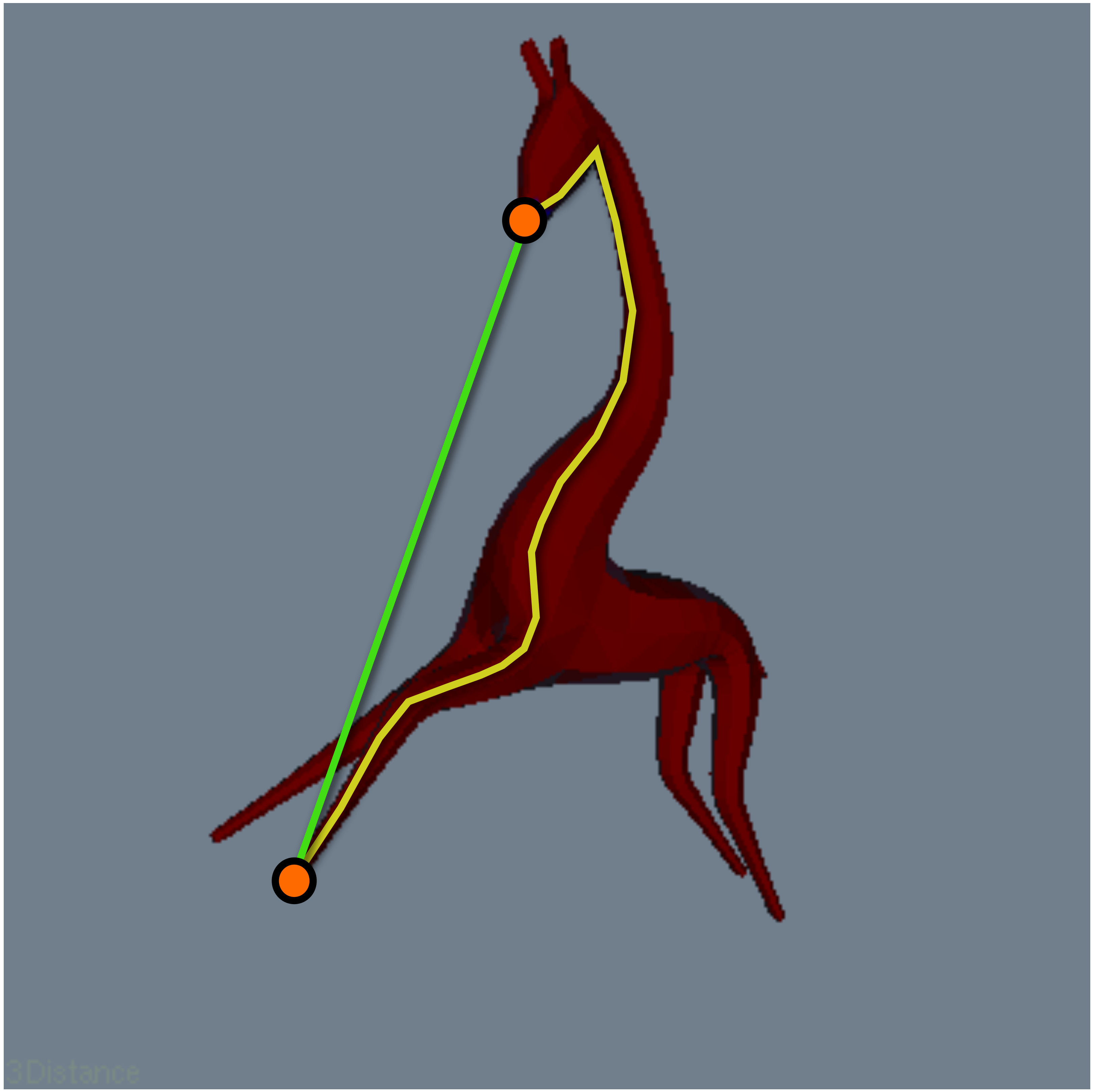


the distance, as measured by an ant, does not change

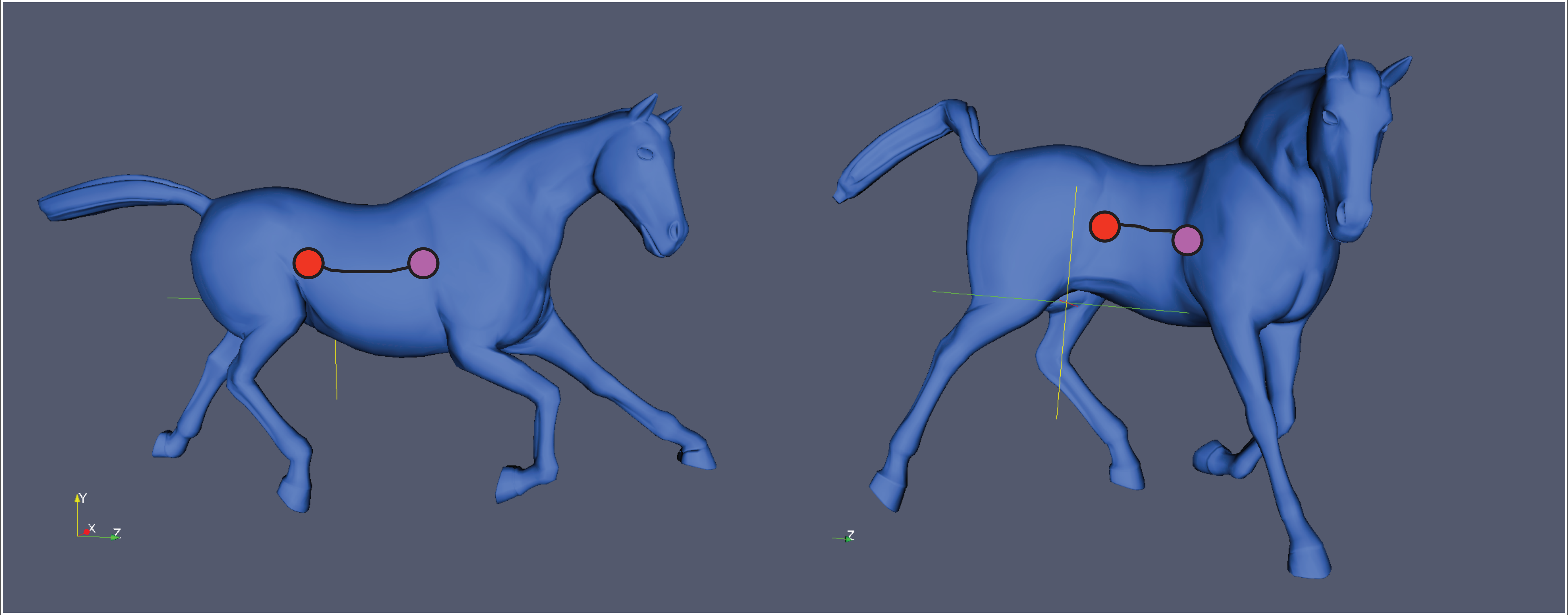


Important: this distance is different from the Euclidean distance!!

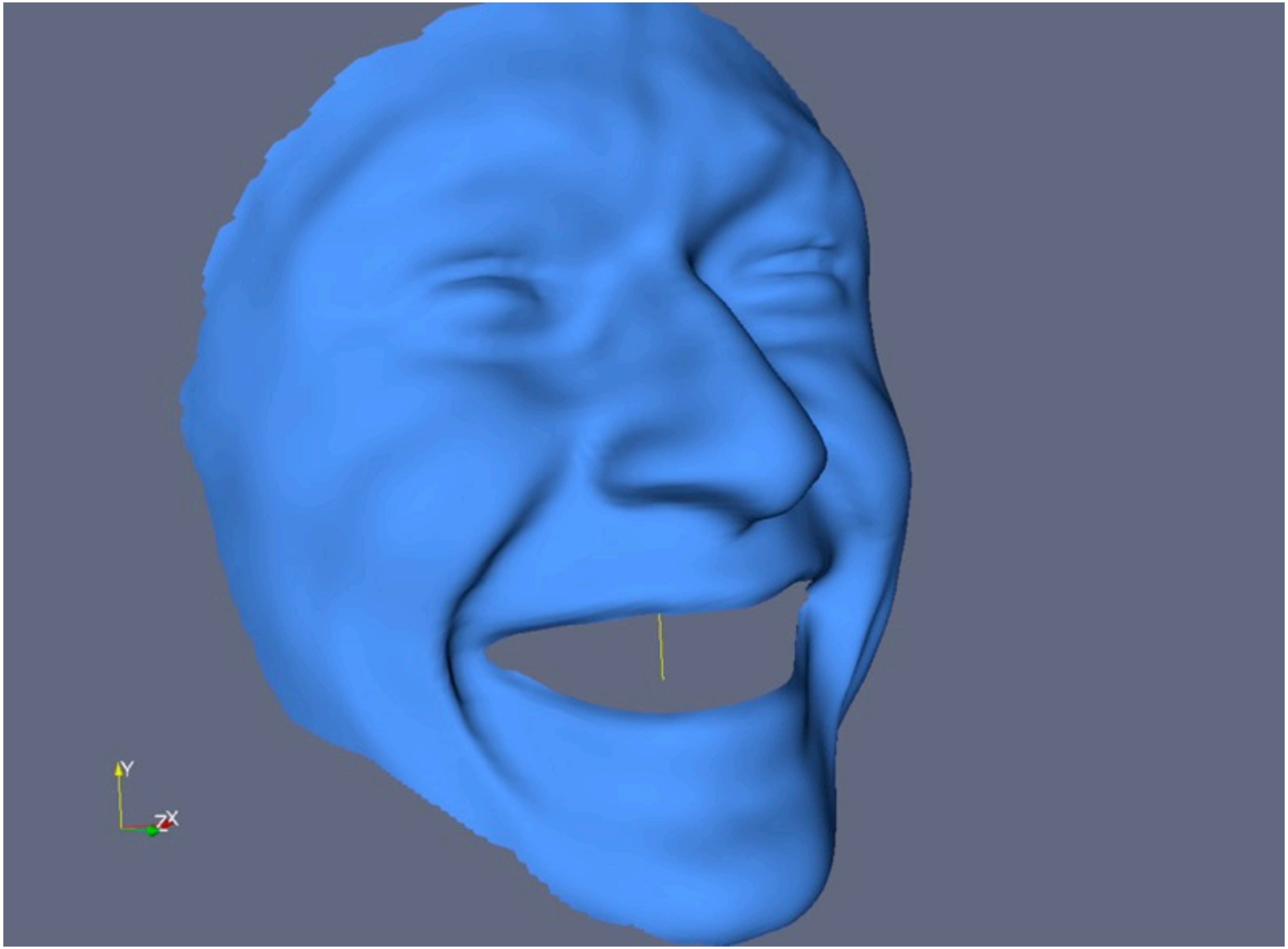
Geodesic distance vs Euclidean distance

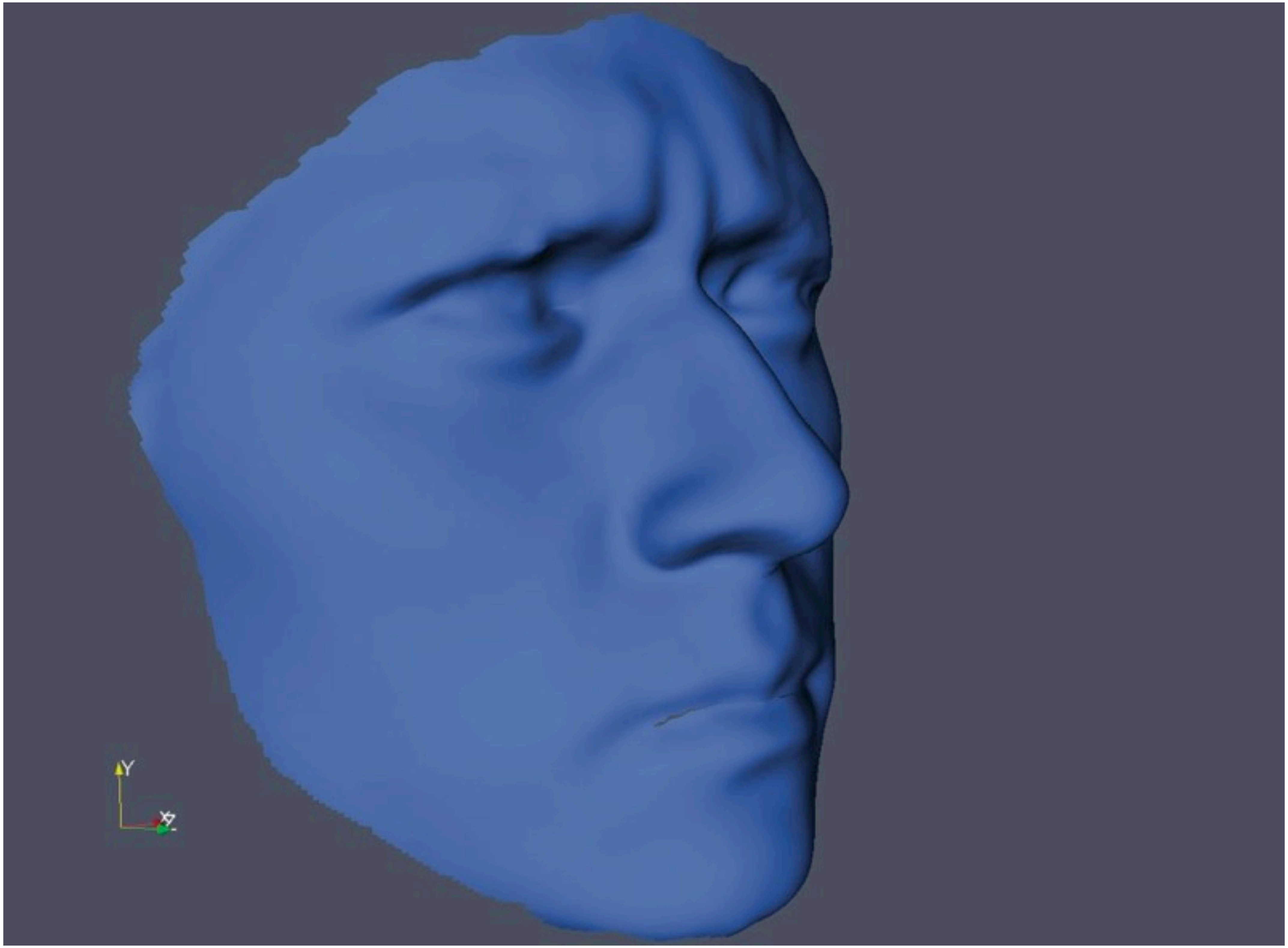


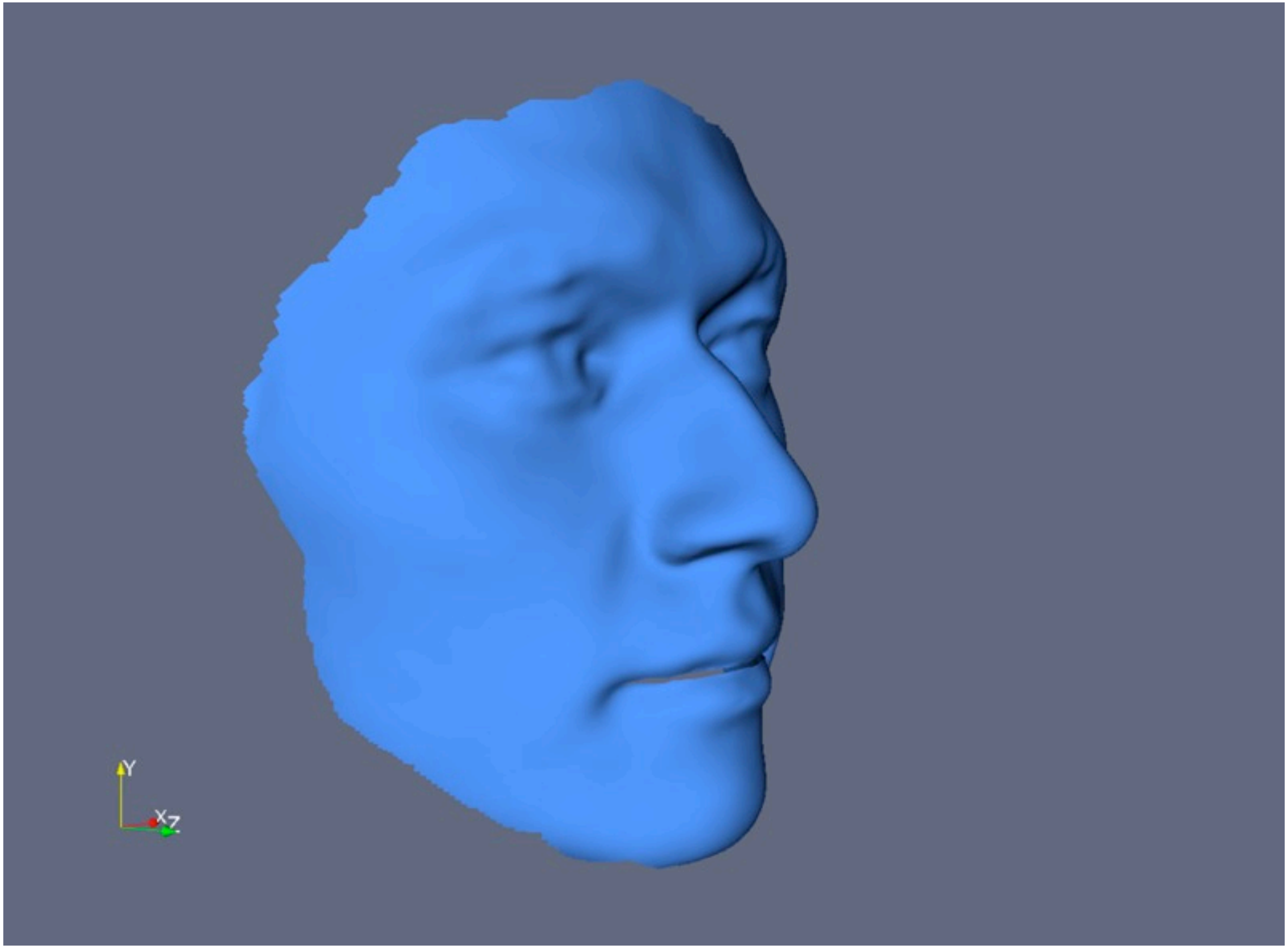
Geodesic distance: invariance to ‘bends’

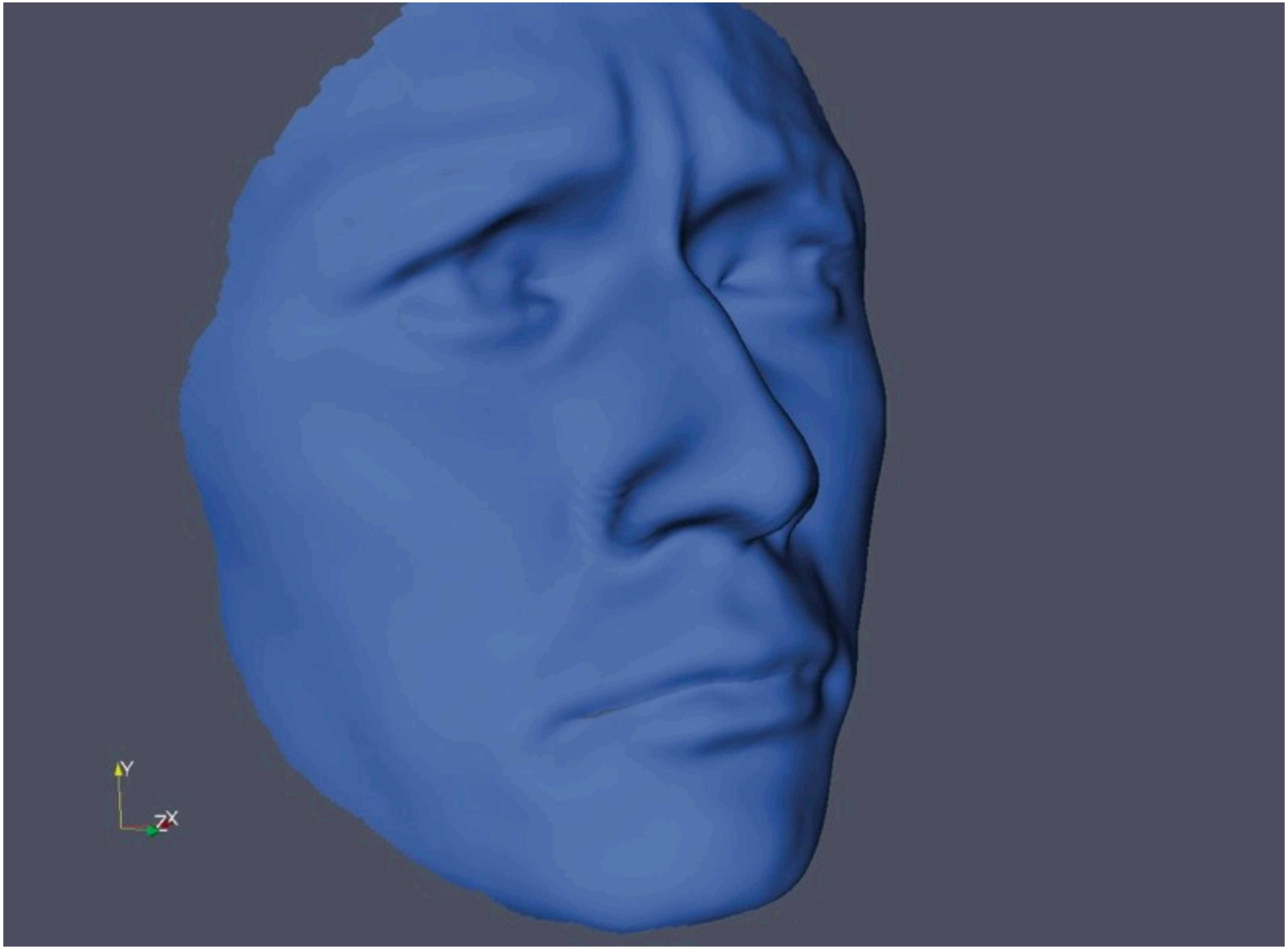


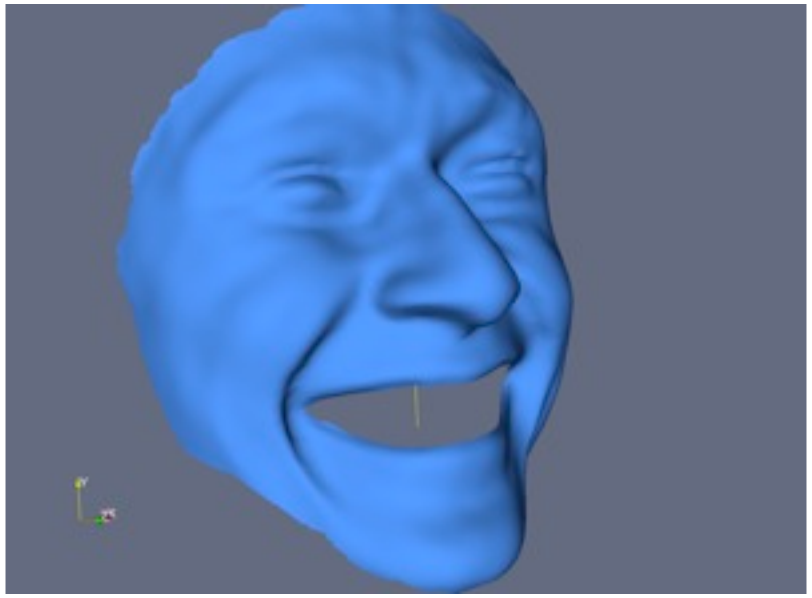
Geodesic distance -- invariance to 'bends' face recognition

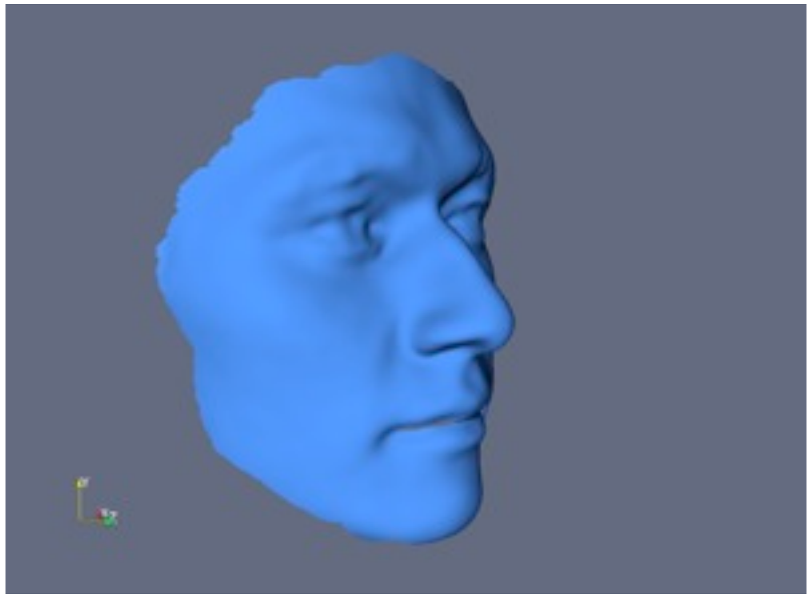



Y


Y
X
Z


Y
ZX





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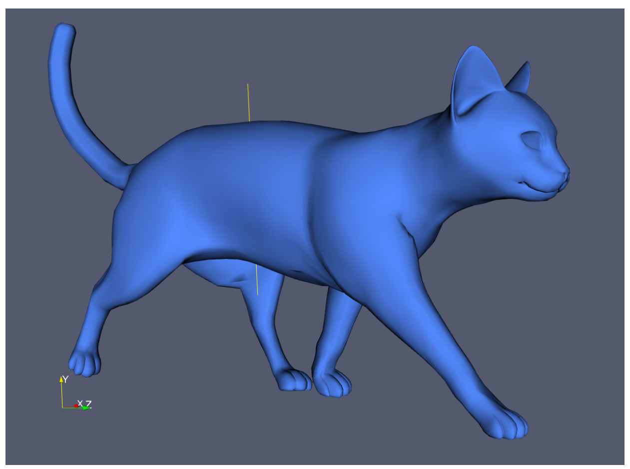
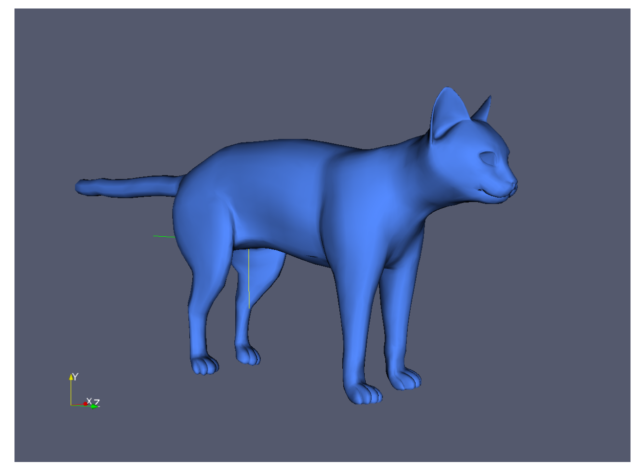
invariances...

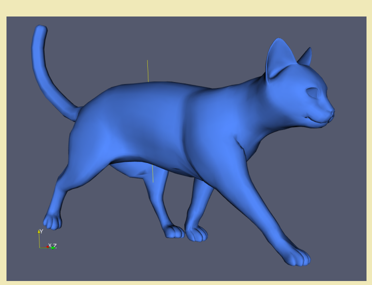
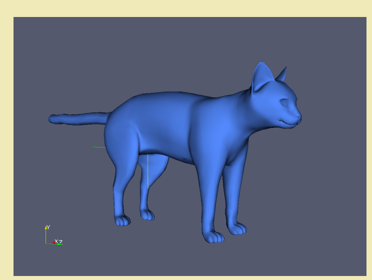
The measure of dis-similarity **dist** must capture the type of invariance you want to encode in your classification system.



$=$



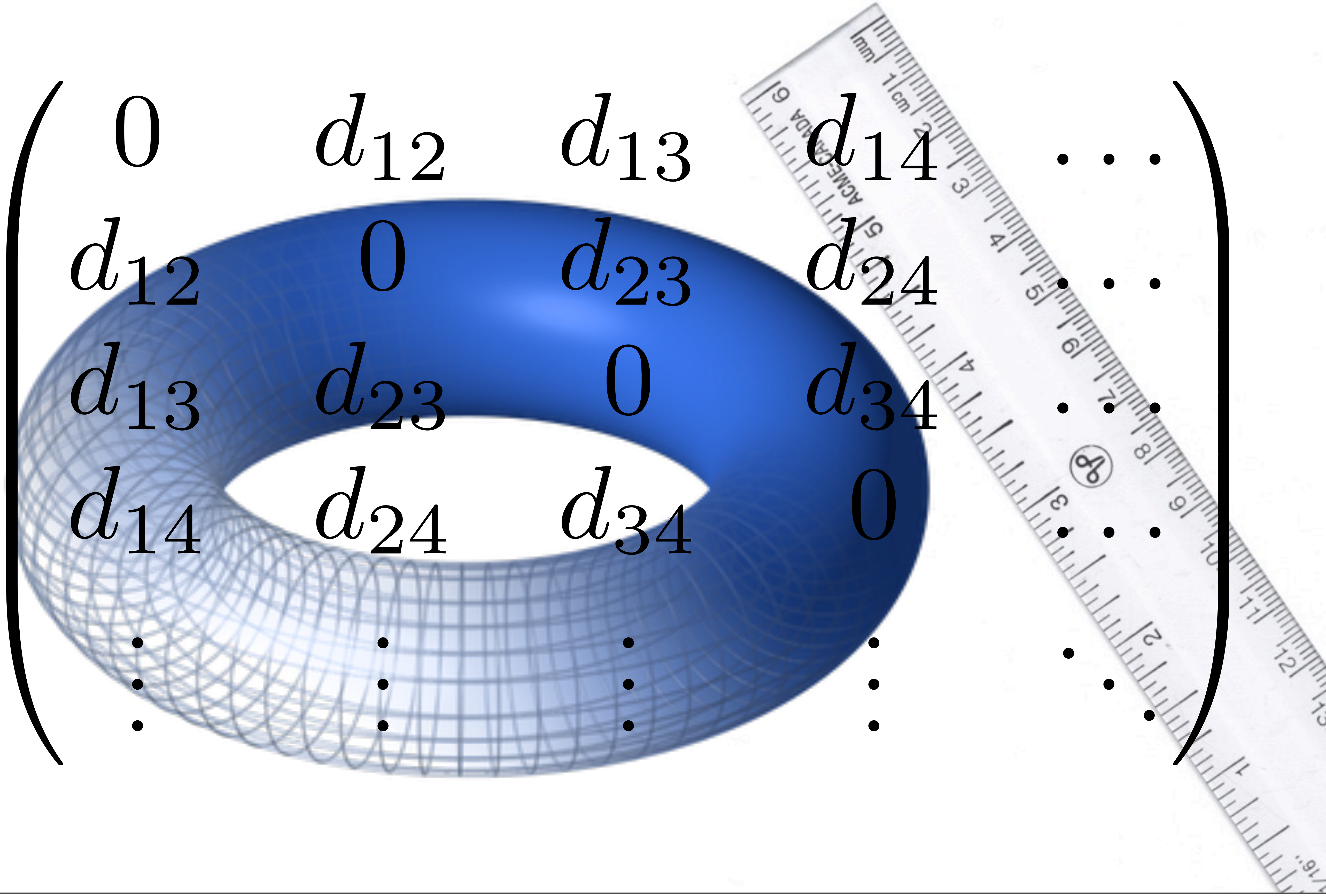
?

dist( , ) $= 0$?

Metric geometry?

- Metric Geometry studies the shape of things, where "things" are modeled as **metric spaces**.
- Several concepts from classical differential and Riemannian geometry admit counterparts in the metric geometry setting.
- One views surfaces, manifolds, etc as metric spaces by endowing the surfaces with the geodesic (or other) metric.
- One does not invoke smoothness of the underlying space.
- An intermediate level of specialization/generality between metric and Riemannian is given by **length spaces**.

$$\begin{pmatrix} \text{Riemannian} \\ \text{structures} \end{pmatrix} \subseteq \begin{pmatrix} \text{Length} \\ \text{structures} \end{pmatrix} \subseteq \begin{pmatrix} \text{Metric} \\ \text{structures} \end{pmatrix}$$



$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background concepts

- **Metric spaces**: A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}^+$ is called the **metric** or **distance** and satisfies
  - (strictness): for all $x, y \in X$, $d(x, y) = 0$ **if and only if** $x = y$.
  - (symmetry): for all $x, y \in X$, $d(x, y) = d(y, x)$.
  - (triangle inequality): for all $x, y, z \in X$, $d(x, y) \leqslant d(x, z) + d(z, y)$.

An ensuing question..

Assume that $\mathbb{X}_n$ and $\mathbb{Y}_n$ are such that there exists a permutation $\pi \in \Pi_n$ s.t. for some $\varepsilon > 0$

$$\|x_i - x_j\| - \varepsilon \leqslant \|y_{\pi_i} - y_{\pi_j}\| \leqslant \|x_i - x_j\| + \varepsilon \text{ for all } i, j.$$

Then, is it true that there exists a Euclidean isometry Ψ_ε s.t.

$$\|\Psi_\varepsilon(x_i) - y_i\| \leqslant \eta(\varepsilon) \text{ for all } i$$

for some $\eta(\cdot)$ non-decreasing, positive s.t. $\eta(0) = 0$?

This question can be answered with the metric geometry tools [**M08-euclidean**]: interestingly the only possibility is $\eta(\varepsilon) \simeq \sqrt{\varepsilon}$.

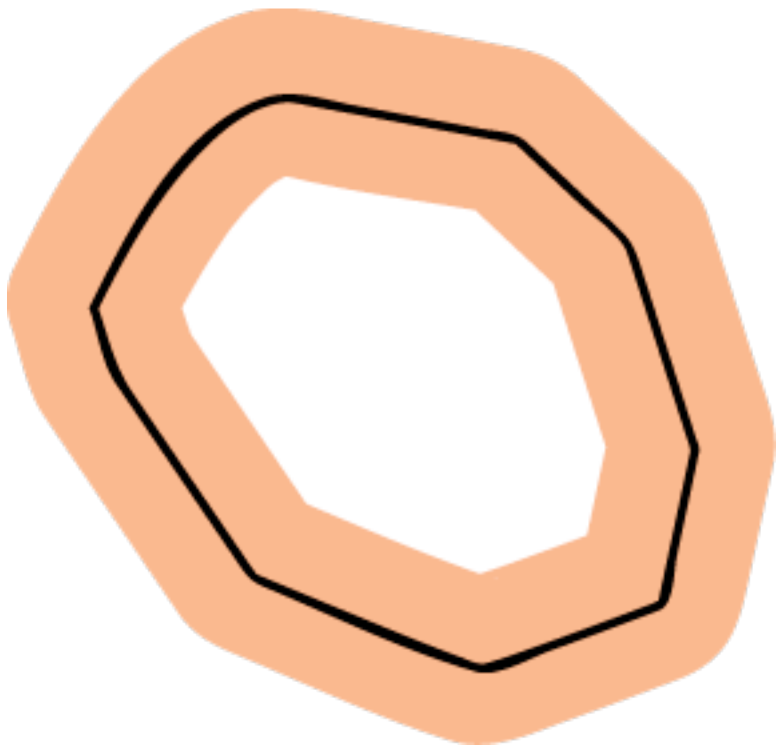
- **Hausdorff distance**. For (compact) subsets A, B of a (compact) metric space (Z, d), the *Hausdorff distance* between them, $d_{\mathcal{H}}^Z(A, B)$, is defined to be the infimal $\varepsilon > 0$ s.t.

$$A \subset B^\varepsilon$$

and

$$B \subset A^\varepsilon$$

where $A^\varepsilon = \{z \in Z | \, d(z, A) < \varepsilon\}$.



Abstraction of shape matching methods

- $\mathcal{M}$: collection of all datasets/shapes (metric spaces).
- $\mathcal{S}$: collection of all invariants/signatures.

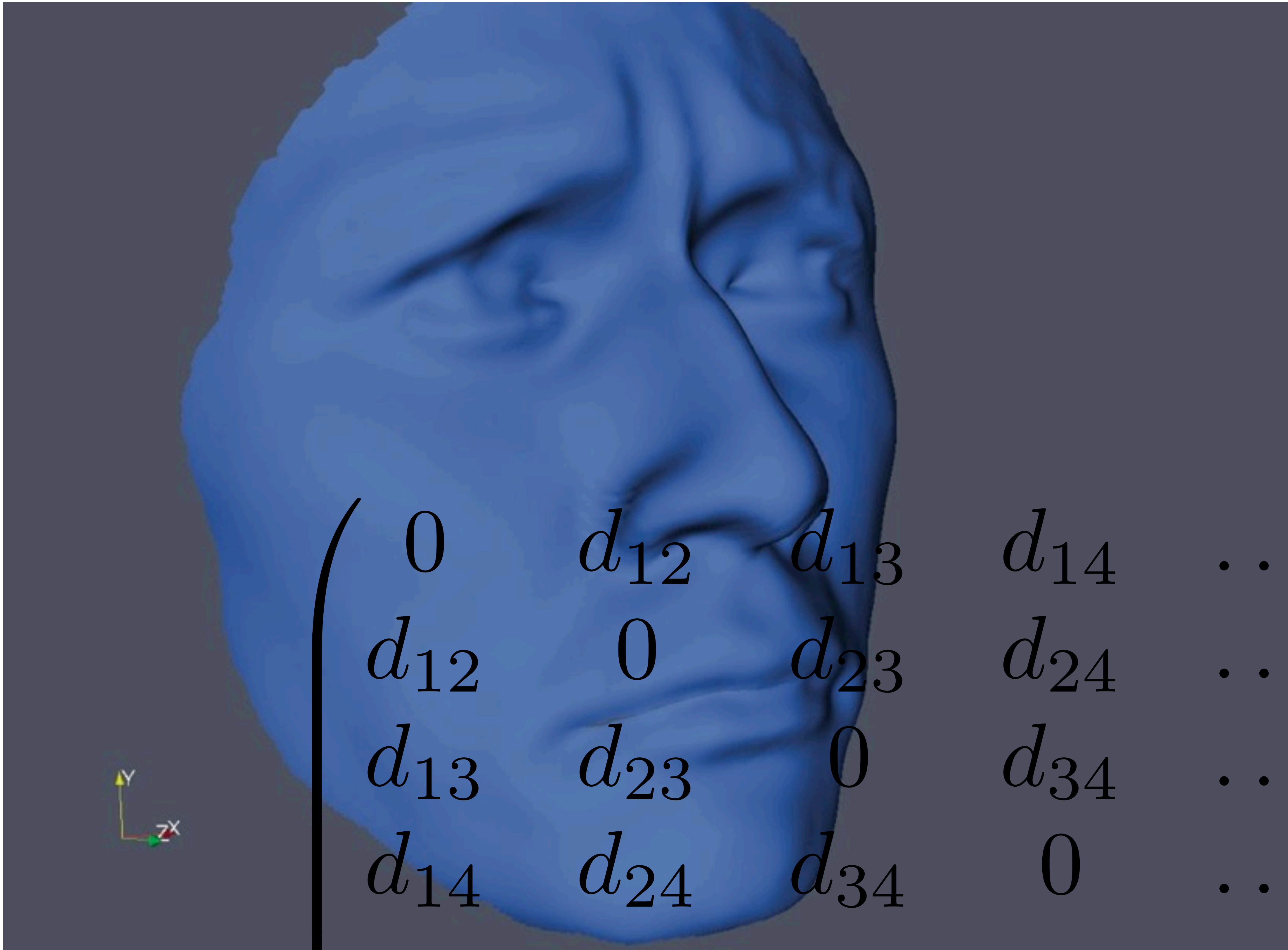
$$
\begin{array}{ccc}
\textbf{datasets/shapes} & & \textbf{signatures} \\
\left(\mathcal{M}, d_{\mathcal{M}}\right) & \xrightarrow{S_h,\ h \in \mathcal{H}} & \left(\mathcal{S}, d_{\mathcal{S}}\right)
\end{array}
$$

$$X, Y \qquad\qquad S_h(X),\ S_h(Y) \qquad h \in \mathcal{H}$$

$$\cancel{d_{\mathcal{M}}(X, Y)} \qquad \geqslant \qquad \boxed{d_{\mathcal{S}}\big(S_h(X),\ S_h(Y)\big)}$$

signatures

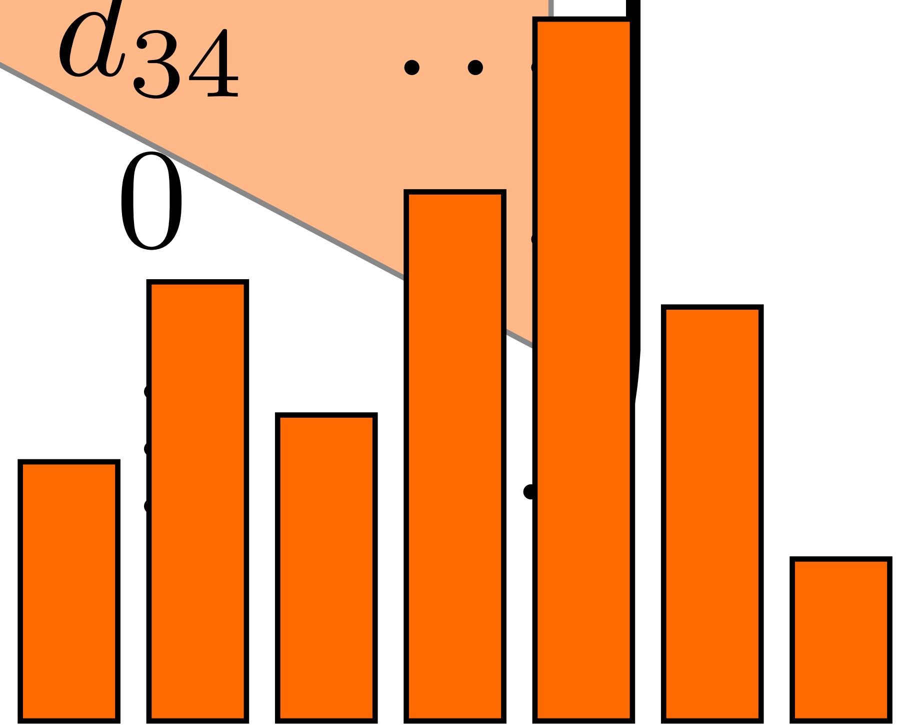
$$\left(\mathcal{S}, d_S\right)$$

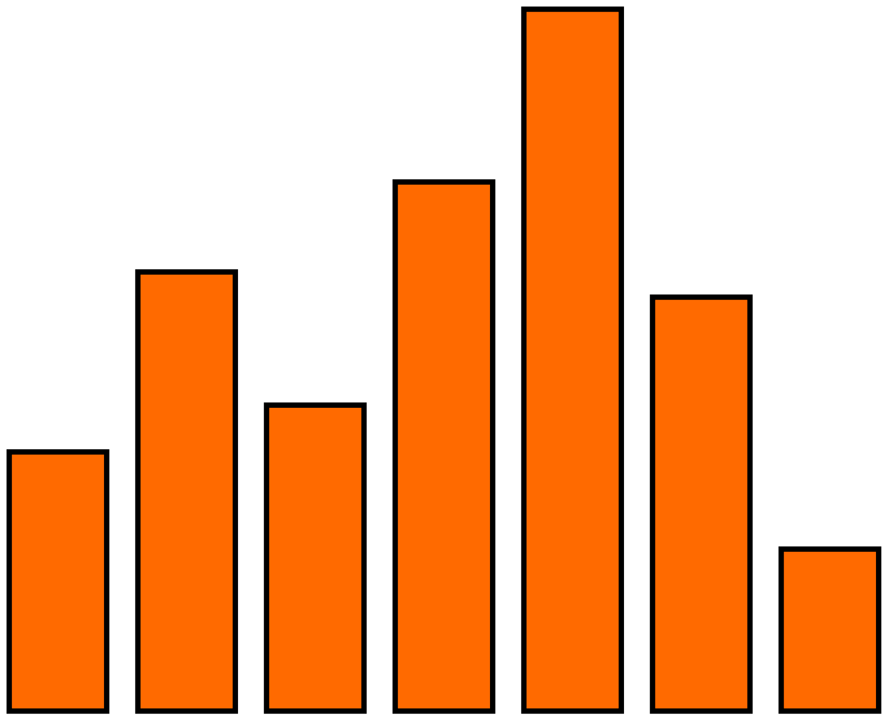


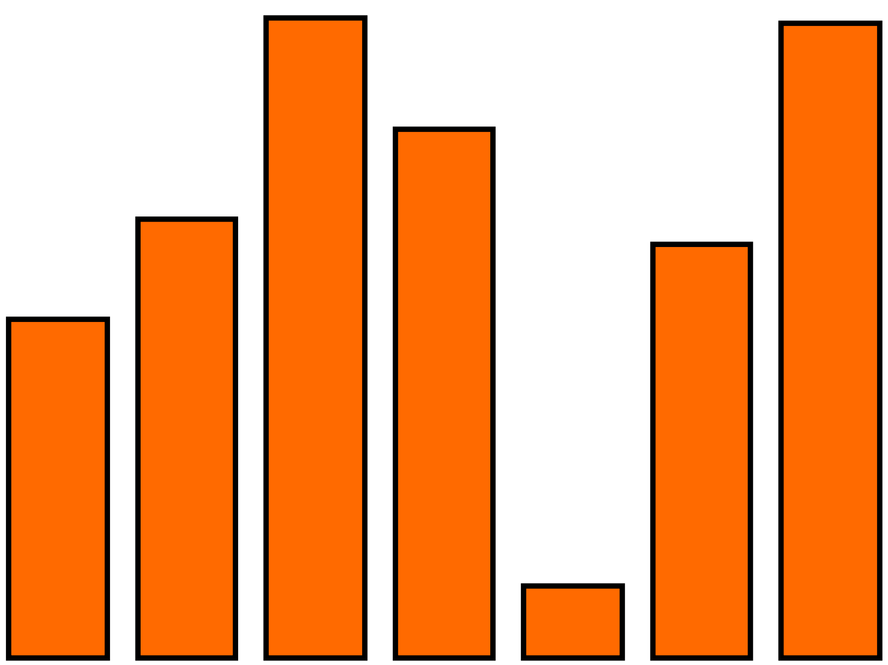
$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

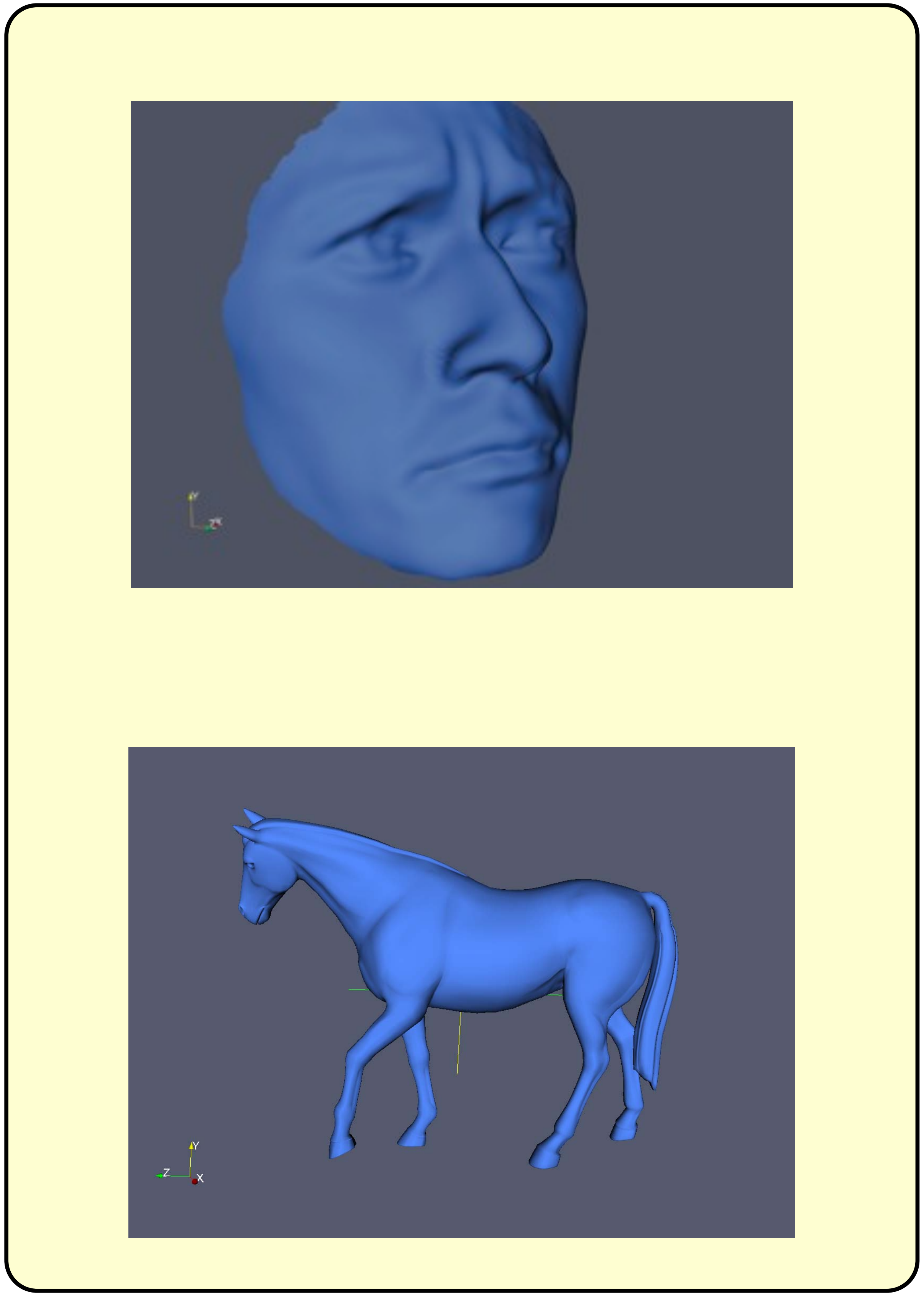
Shape Distributions [Osada-et-al]

$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \cdots \\ d_{12} & 0 & d_{23} & d_{24} & \cdots \\ d_{13} & d_{23} & 0 & d_{34} & \cdots \\ d_{14} & d_{24} & d_{34} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



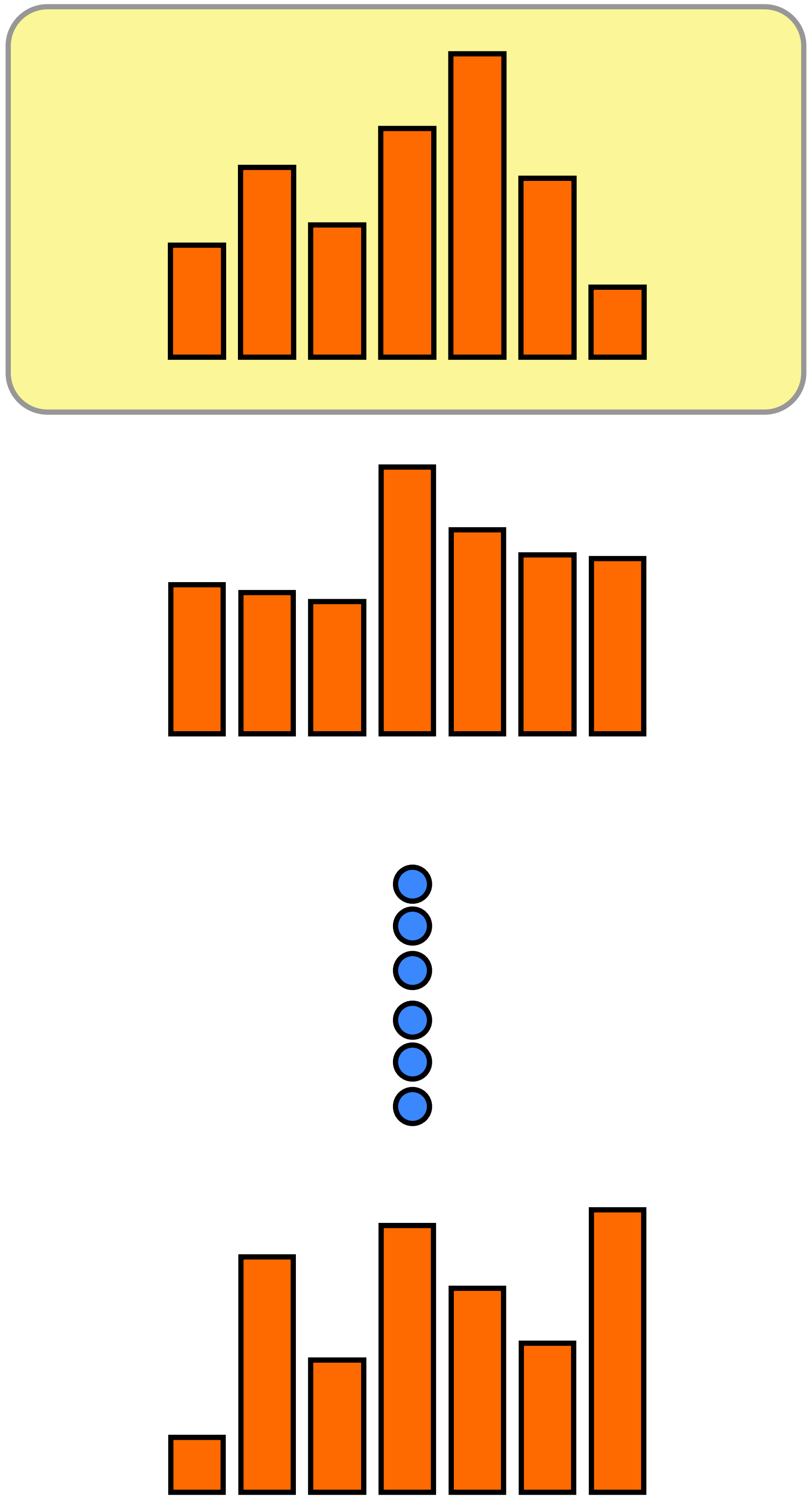






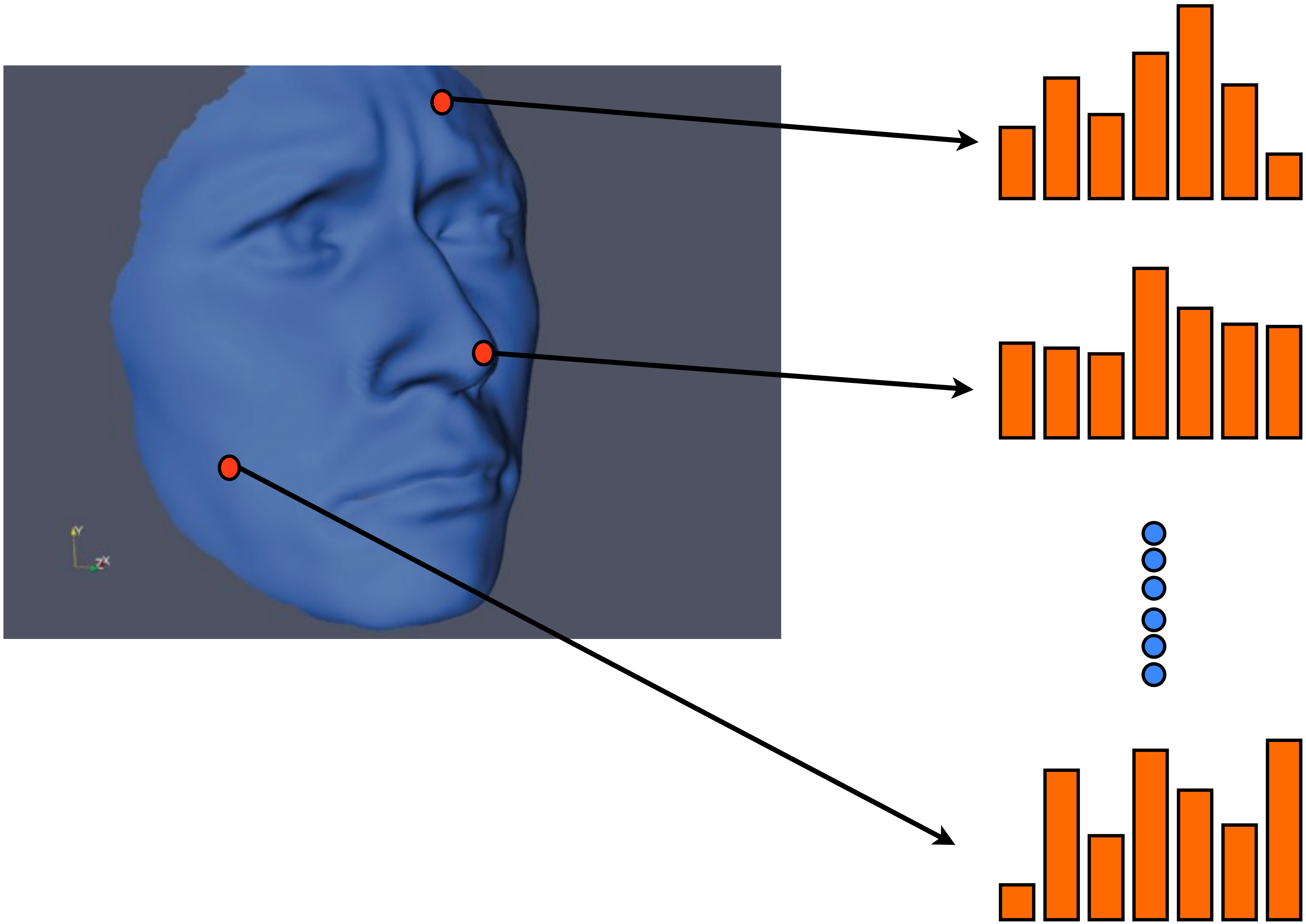
Shape Contexts/ integral invariants

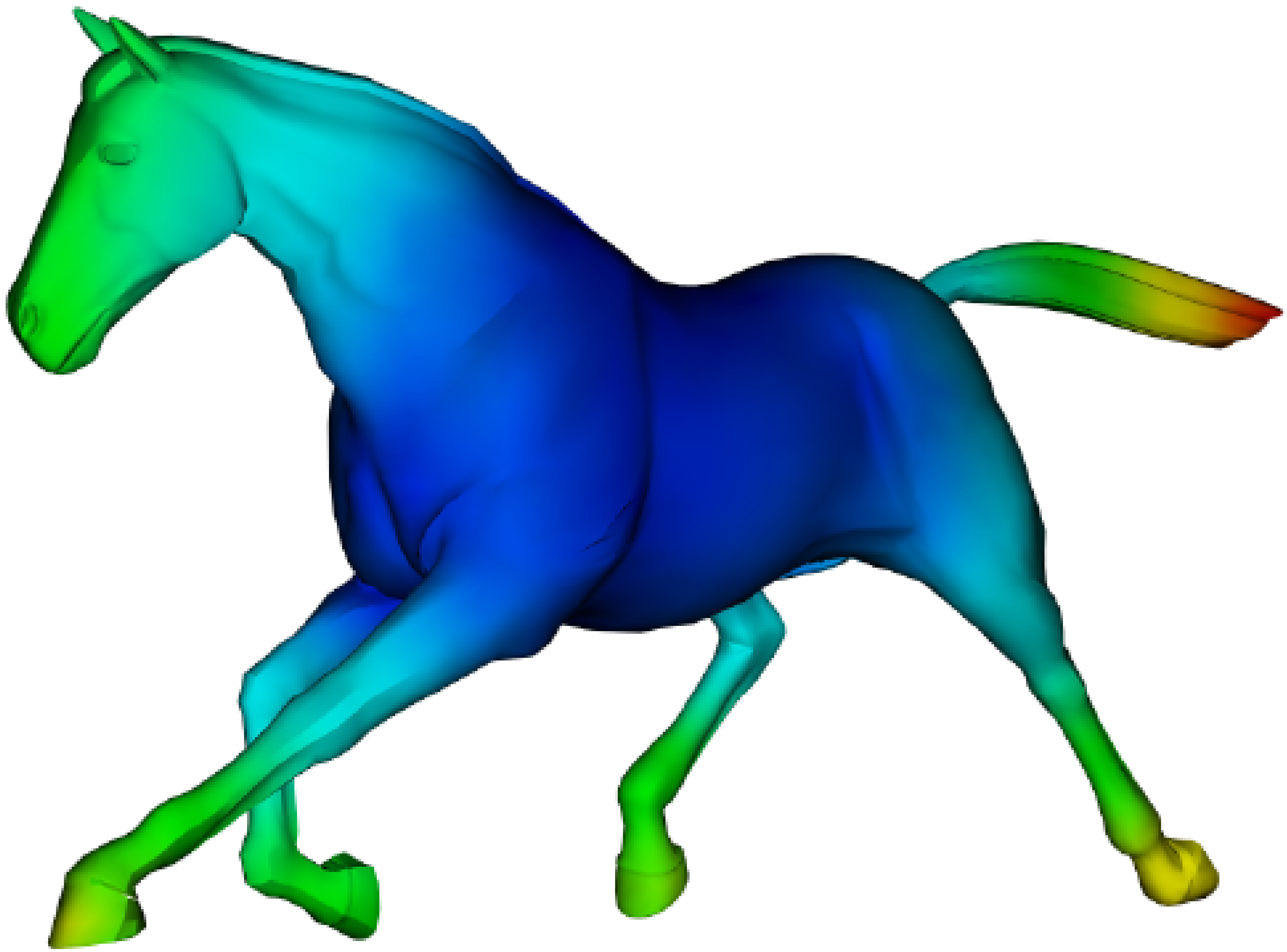
$$\begin{pmatrix} 0 & d_{12} & d_{13} & d_{14} & \dots \\ d_{12} & 0 & d_{23} & d_{24} & \dots \\ d_{13} & d_{23} & 0 & d_{34} & \dots \\ d_{14} & d_{24} & d_{34} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



about Local distributions of distances

$$h_X(x,t) = \mu_X\left(\overline{B(x,t)}\right)$$





datasets/shapes

$$\left(\mathcal{M}, d_{\mathcal{M}}\right)$$

The GH distance for Shape Comparison [MS04], [MS05]

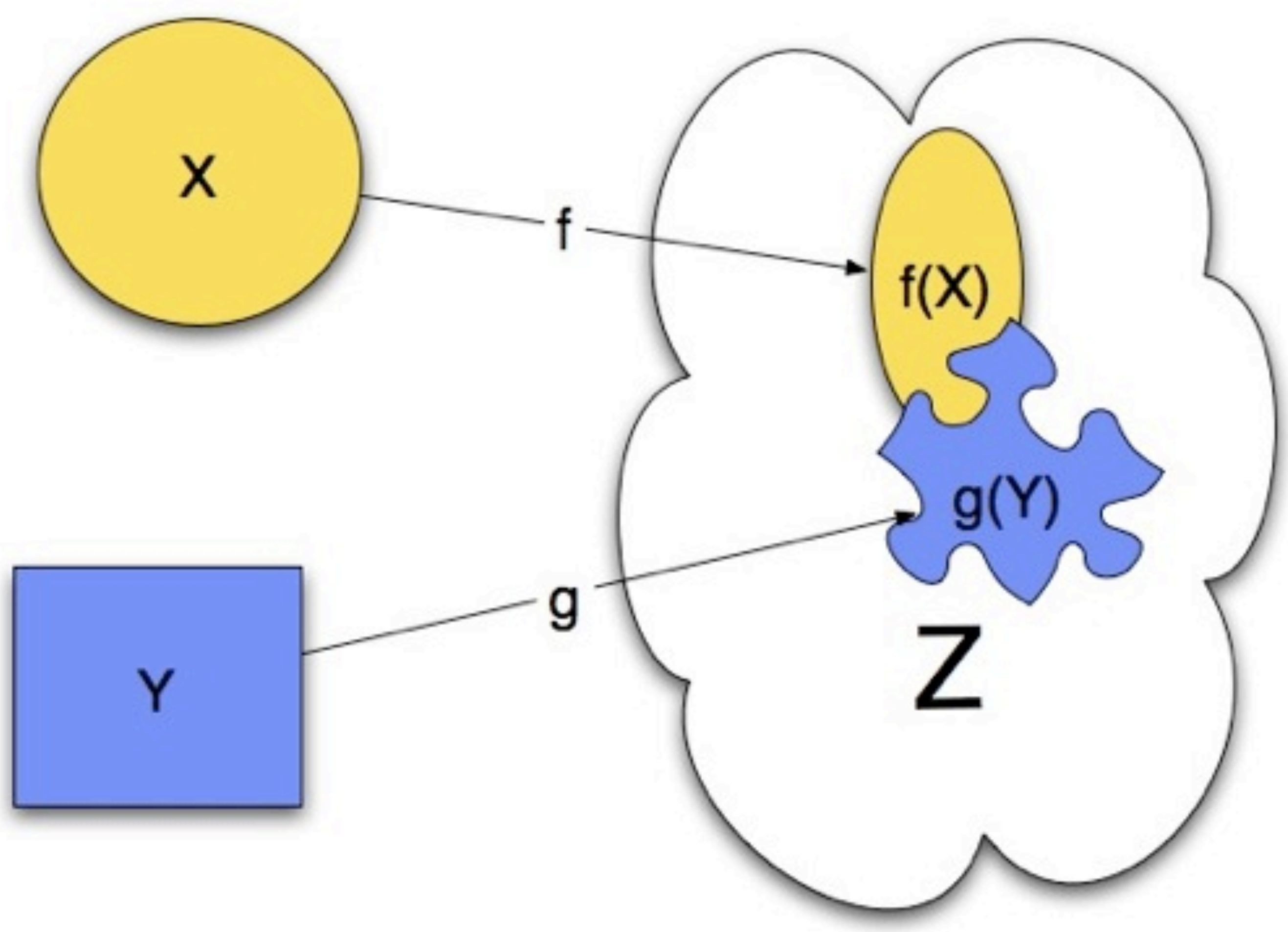
A metric on $\mathcal{M}$ (collection of all metric spaces):

$$(\mathcal{M}, d_{\mathcal{GH}}) \text{ is a metric space in itself!}$$

- Regard shapes as (compact) metric spaces, That is, a shape is going to be represented as pair (X, d_X).
- The metric d_X with which one endows the shapes depends on the desired invariance. For example, if invariance to
  - *rigid isometries* is desired, use Euclidean distance.
  - *bends* is desired, use "geodesic" distance.
- GH distance provides reasonable framework for Shape Comparison: good theoretical properties.
- However, it leads to difficult optimization problems.

GH: definition

$$d_{\mathcal{GH}}(X,Y) = \inf_{Z,f,g} d_{\mathcal{H}}^{Z}(f(X), g(Y))$$



It would be much more intuitive to compare the metrics d_X and d_Y directly..

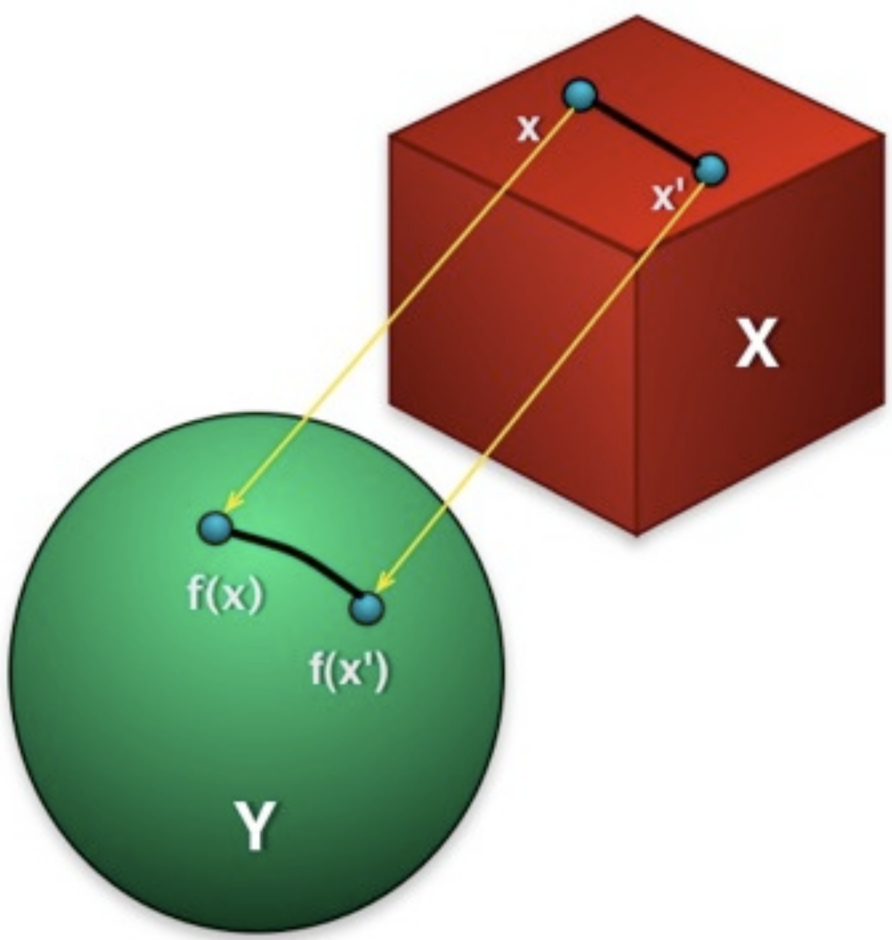
For maps $f : X \to Y$, and $g : Y \to X$ compute

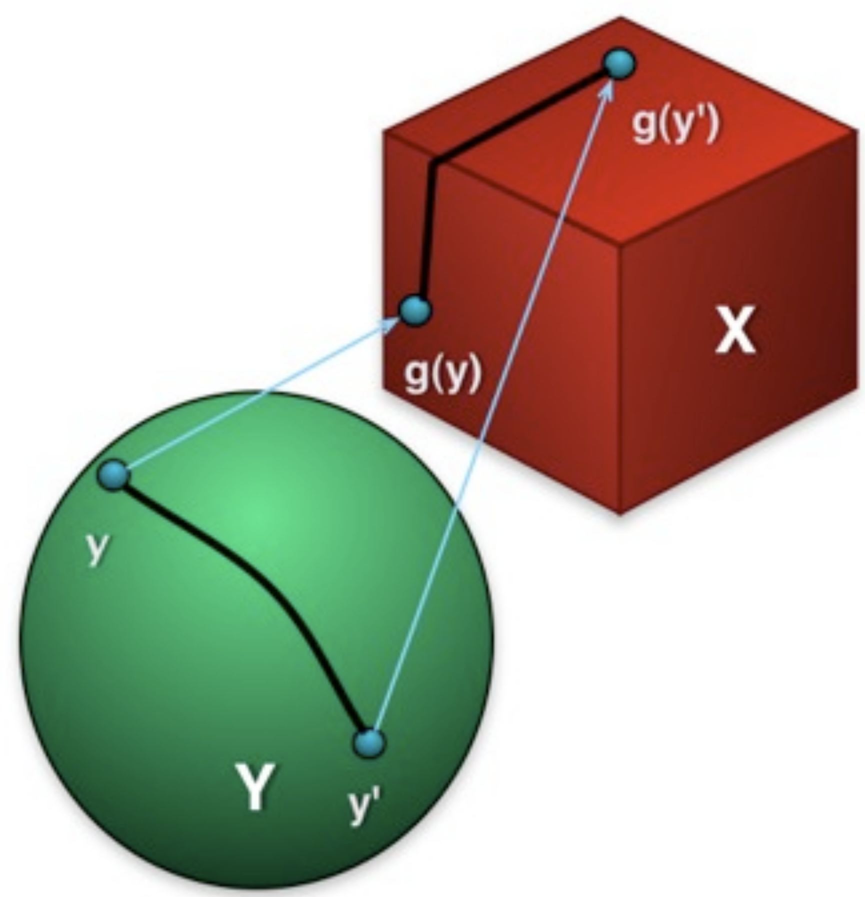
$$\mathrm{dist}(f) = \max_{x,x'} |d_X(x,x') - d_Y(f(x), f(x'))|$$

and

$$\mathrm{dist}(g) = \max_{y,y'} |d_Y(y,y') - d_X(g(y), g(y'))|$$

and then minimize $\boxed{\max(\mathrm{dist}(f), \mathrm{dist}(g))}$ over all choices of f and g.





correspondences

Definition [Correspondences]

For sets A and B, a subset $R \subset A \times B$ is a *correspondence* (between A and B) if and and only if

- $\forall\, a \in A$, there exists $b \in B$ s.t. $(a, b) \in R$
- $\forall\, b \in B$, there exists $a \in A$ s.t. $(a, b) \in R$

Let $\mathcal{R}(A, B)$ denote the set of all possible correspondences between sets A and B.

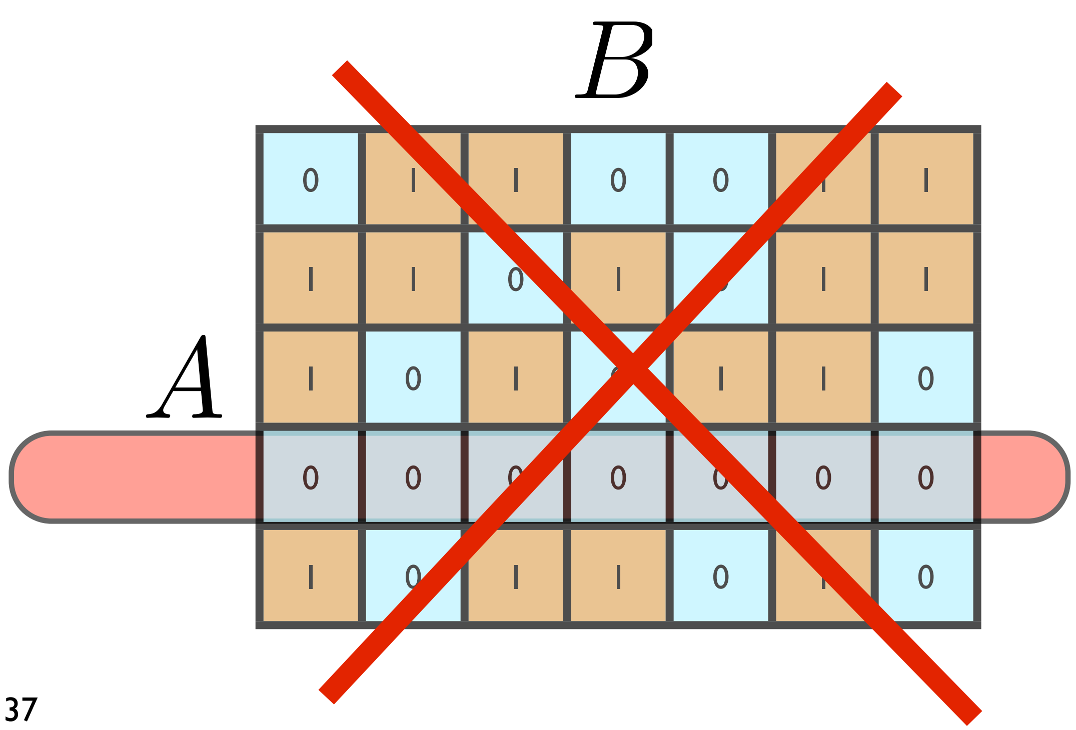
Note that in the case $n_A = n_B$, correspondences are larger than bijections.

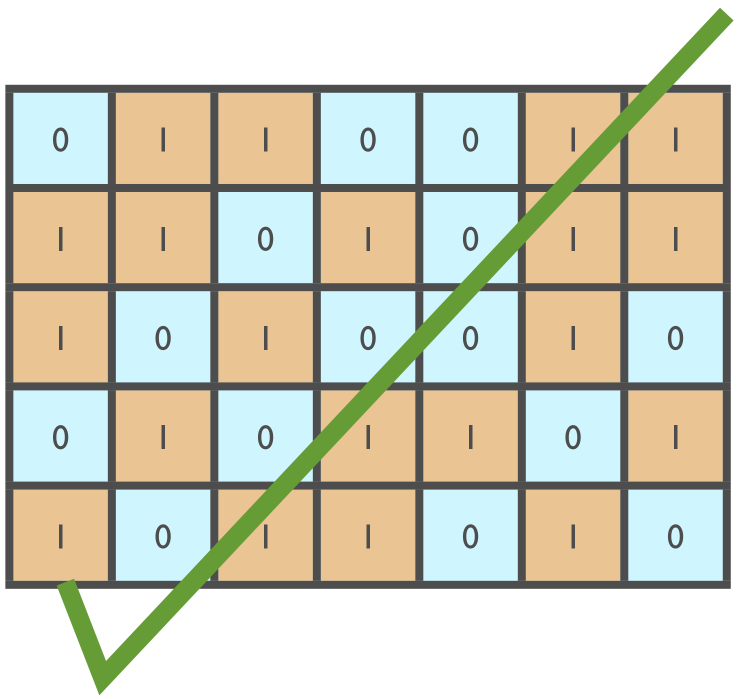
Correspondences

Given finite sets A and B, a **correspondence** between A and B is a matrix $R = ((r_{a,b})) \in \{0,1\}^{n_A \times n_B}$ s.t.

$$\sum_{a \in A} r_{ab} \geq 1 \;\; \forall b \in B$$

$$\sum_{b \in B} r_{ab} \geq 1 \;\; \forall a \in A$$

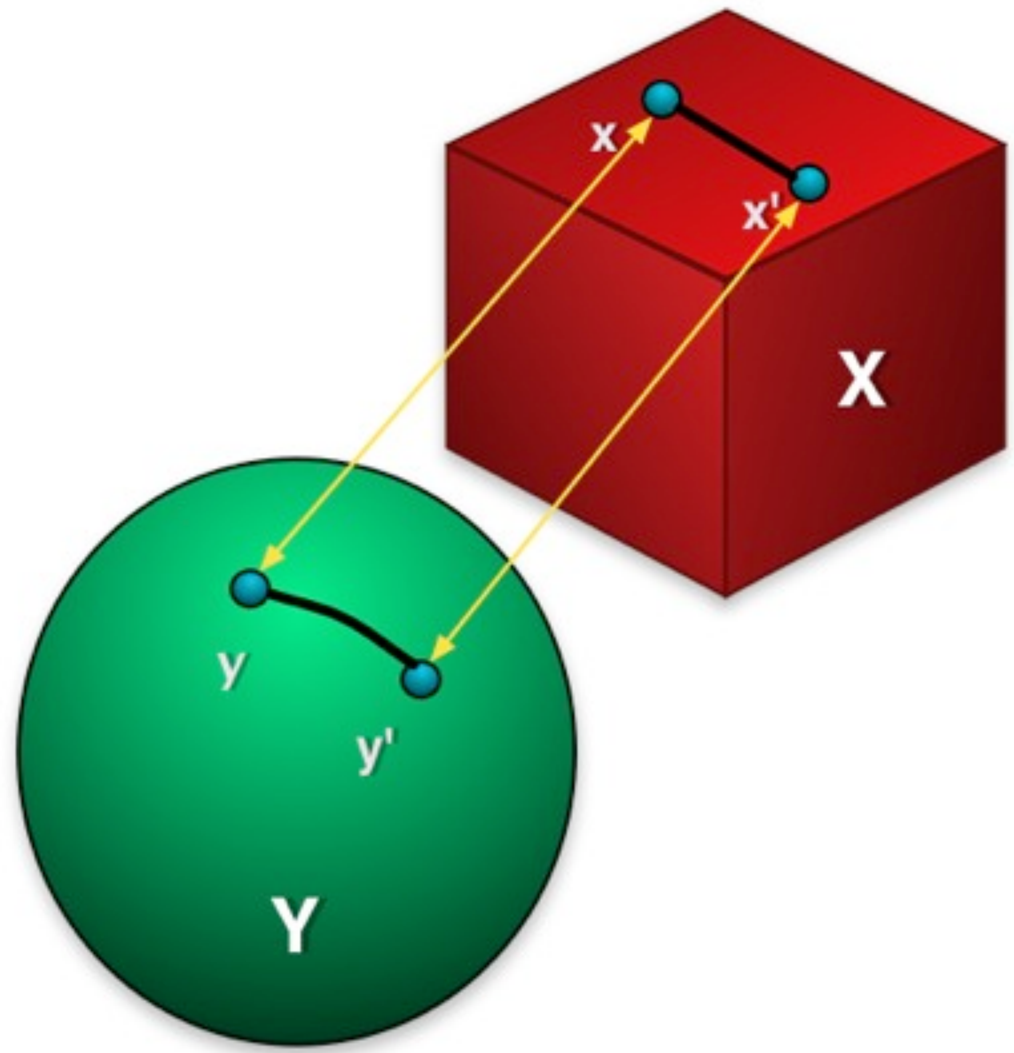




Theorem. *[Kalton-Ostrovskii] For finite metric spaces (X, d_X) and (Y, d_Y), the **Gromov-Hausdorff distance** between them is given by*

$$d_{\mathcal{GH}}(X,Y) = \frac{1}{2}\min_{R} \max_{x,y,x',y'} |d_X(x,x') - d_Y(y,y')|\, r_{x,y}\cdot r_{x',y'}$$

$$R = \begin{array}{c|ccccccc}
 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
x & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
x' & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
 & & y & & & y' & &
\end{array}$$



Critique

- Was not able to show connections with (sufficiently many) pre-existing approaches such as Shape Distributions, Shape Contexts, Hamza-Krim, Frosini et al.
- Computationally hard: currently only two attempts have been made:
  - [MS04,MS05] and [BBK06] only for surfaces.
  - [MS05] gives probabilistic guarantees for estimator based on sampling parameters.
  - Full generality leads to a hard **combinatorial optimization problem**: QAP.

Desiderata

- Obtain an L^p version of the GH distance that:
  - retains theoretical underpinnings
  - its implementation leads to easier (continuous, quadratic, with linear constraints) optimization problems
  - can be related to pre-existing approaches (shape contexts, shape distributions, Hamza-Krim,..) via lower/upper bounds.

First attempt: naive relaxation

Recall that

$$d_{\mathcal{GH}}\big(X,Y\big) = \min_R \boxed{\max_{x,x',y,y'}} \left|d_X(x,x') - d_Y(y,y')\right| \boxed{r_{x,y}} \cdot \boxed{r_{x',y'}}$$

where $R = \big(\!\big(r_{x,y}\big)\!\big) \in \{0,1\}^{n_X \times n_Y}$ s.t.

$$\sum_{x \in X} r_{x,y} \geqslant 1 \text{ for all } y \in Y$$

$$\sum_{y \in Y} r_{x,y} \geqslant 1 \text{ for all } x \in X.$$

First attempt: naive relaxation (continued)

- The idea would be to use L^p norm instead of L^∞ (max max)
- relax $r_{x,y}$ to be in $[0,1]$ (!)

Then, the idea would be to compute (for some $p \geq 1$):

$$\widehat{d}_{\mathcal{GH}}(X,Y) = \frac{1}{2}\inf_{R}\left(\sum_{x,x',y,y'} |d_X(x,x') - d_Y(y,y')|^{\mathbf{p}} \boxed{r_{x,y}}\boxed{r_{x',y'}}\right)^{1/\mathbf{p}}$$

where $R = \boxed{((r_{x,y})) \in [\mathbf{0},\mathbf{1}]^{n_X \times n_B}}$ s.t.

$$\sum_{x \in X} r_{xy} \geq 1 \;\; \forall y \in Y$$

$$\sum_{y \in Y} r_{xy} \geq 1 \;\; \forall x \in X$$

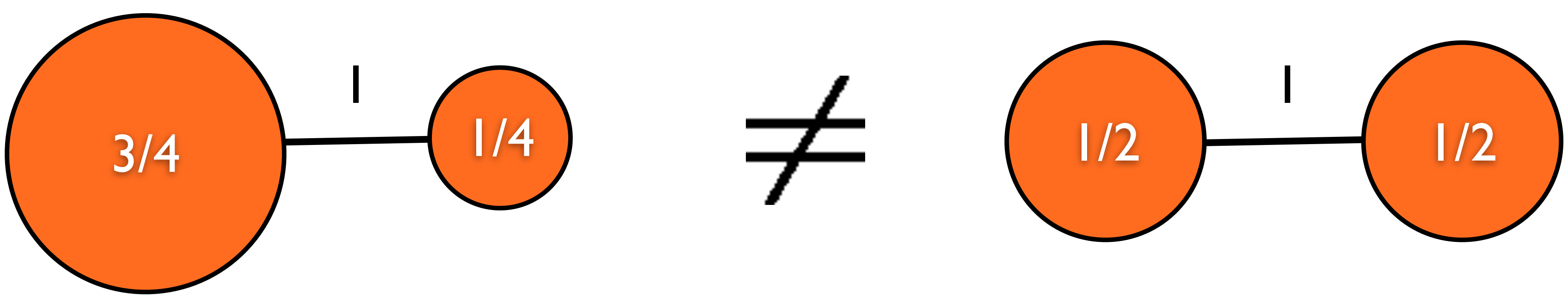
- The resulting problem is a continuous variable QOP with linear constraints, but..
- there is no limit problem.. this discretization cannot be connected to the GH distance.. do those sums converge to sth as number of samples increases?

we need to identify the **correct** relaxation of the GH distance. More precisely, the correct notion of *relaxed correspondence*.

Shapes as mm-spaces, [M07]

- Now we are talking of triples (X, d_X, μ_X) where X is a set, d_X a metric on X and μ_X a probability measure on X.
- These objects are called *measure metric spaces*, or mm-spaces for short.
- two mm-spaces X and Y are deemed *equal* or *isomorphic* whenever there exists an isometry $\Phi : X \to Y$ s.t. $\mu_Y \circ \Phi = \mu_X$.

$$(X, d_X, \mu_X)$$



We need a new concept Let A and B be sets and μ_A and μ_B be **probability measures** supported in A and B respectively.

Definition [Measure coupling] Is a probability measure μ on $A \times B$ s.t. (in the finite case this means $((\mu_{a,b})) \in [0,1]^{n_A \times n_B}$) such that:

- $\sum_{a \in A} \mu_{ab} = \mu_B(b) \;\; \forall b \in B$
- $\sum_{b \in B} \mu_{ab} = \mu_A(a) \;\; \forall a \in A$

Let $\mathcal{M}(\mu_A, \mu_B)$ be the set of all couplings of μ_A and μ_B.
Notice that in the finite case, $((\mu_{a,b}))$ must satisfy $n_A + n_B$ *linear* constraints.

Correspondence

B

A

0	1	1	0	0	1	1
1	1	0	1	0	1	1
1	0	1	0	0	1	0
0	1	0	1	1	0	1
1	0	1	1	0	1	0

Measure Coupling

μ_B

0.1	0.1	0.1	0.2	0.2	0.1	0.2

μ_A

μ_A							
0.2	0	0	0.01	0	0	x	x
0.2	0.03	x	0	x	0	x	x
0.2	x	0	x	0	x	x	0
0.2	0.03	0	0.08	0	0	0.09	0
0.2	0.04	0	0.01	x	0	x	0

L^p Gromov-Hausdorff distances [M07]

Compute (for some $p \geqslant 1$):

$$d_{\mathcal{GW},p}(X,Y) = \frac{1}{2} \inf_{\mu} \left(\sum_{x,x',y,y'} \overbrace{|d_X(x,x') - d_Y(y,y')|^p}^{\Gamma_{X,Y}} \boxed{\mu_{x,y}} \boxed{\mu_{x',y'}} \right)^{1/p}$$

where $\mu = ((\mu_{x,y})) \in [\mathbf{0},\mathbf{1}]^{n_X \times n_Y}$ s.t.

$$\sum_{x \in X} \mu_{x,y} = \mu_Y(y) \;\; \forall y \in Y$$

$$\sum_{y \in Y} \mu_{x,y} = \mu_X(x) \;\; \forall x \in X$$

This is a QOP with linear constrains! Also, thanks to concepts from measure theory, there is a continuous conterpart (sampling theory)

Numerical Implementation

- The numerical implementation of the second option leads to solving a continuous variable **QOP** with linear constraints:

$$\begin{gathered}\min_U \tfrac{1}{2} U^T \mathbf{\Gamma} U \\ \text{s.t. } U_{ij} \in [0,1],\, U\mathbf{A} = \mathbf{b}\end{gathered}$$

  where $U \in \mathbb{R}^{n_X \times n_Y}$ is the *unrolled* version of μ, $\mathbf{\Gamma} \in \mathbb{R}^{n_X \times n_Y \times n_X \times n_Y}$ is the unrolled version of $\Gamma_{X,Y}$ and $\mathbf{A}$ and $\mathbf{b}$ encode the <u>linear</u> constrains $\mu \in \mathcal{M}(\mu_X, \mu_Y)$.

- This can be approached for example via gradient descent. The QOP is non-convex in general!

- Initialization is done via solving one of the several *lower bounds* (discussed ahead). All these lower bounds lead to solving **LOP**s.

Gromov-Wasserstein distances [M07]

$$d_{\mathcal{GH}}$$

$$\Vert$$

(X, d_X) and (Y, d_Y)

$$\boxed{\frac{1}{2}\inf_R \|d_X - d_Y\|_{L^\infty(R\times R)}}$$

$$\downarrow$$

(X, d_X, μ_X) and (Y, d_Y, μ_Y)

$$\frac{1}{2}\inf_{\mu} \|d_X - d_Y\|_{L^p(\mu\otimes\mu)}$$

$$\Vert$$

$$\boxed{d_{\mathcal{GW},p}}$$

An analogy (why "Gromov-Wasserstein")

$$\frac{\mathbf{GH}}{\mathbf{H}} = \frac{\mathbf{GW}}{\color{red}{\mathbf{W}}}$$

The whole construction of the L^p-GH distance can be seen like in this diagram. It arises by analogy. It uses the **Wasserstein distance** instead of the Hausdorff distance. That's why it is better called **Gromov-Wasserstein** distance.

Underlying ideas are **mass transportation+metric geometry**, see [**M07**].

Theorem (Properties of the GW distance, **[M07]**).

1. $d_{\mathcal{GW},p}$ *defines a metric on the collection of (isomorphism classes of) mm-spaces.*

2. *If* (Z,d) *is a given metric compact space, and* ν, ν' *two probability measures on* Z*, then*
$$d_{\mathcal{GW},p}\big((Z,d,\nu),(Z,d,\nu')\big) \leqslant d^Z_{\mathcal{W},p}(\nu,\nu').$$

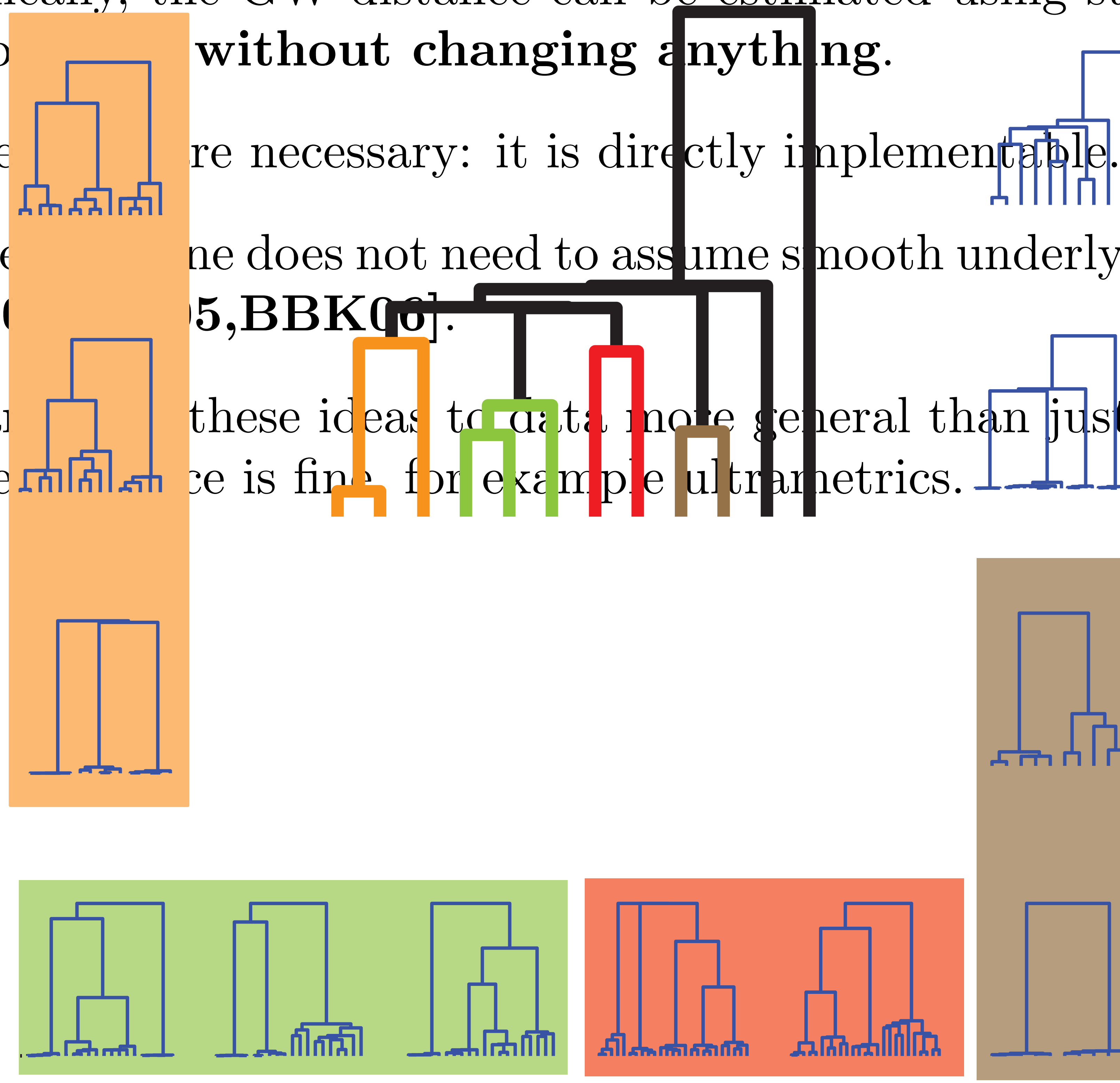
3. *If* d, d' *are two metrics on* (Z,ν)*, then*
$$d_{\mathcal{GW},p}\big((Z,d,\nu),(Z,d',\nu)\big) \leqslant \frac{1}{2}\|d-d'\|_{L^p(\nu\otimes\nu)}.$$

4. *Let* $Y=\{y\}$*, then* $d_{\mathcal{GW},p}(X,Y) = \mathbf{diam}_p(X)$. *From this and the triangle inequality:*
$$\frac{1}{2}\Big(\mathbf{diam}_p(X)+\mathbf{diam}_p(Y)\Big) \geqslant d_{\mathcal{GW},p}(X,Y) \geqslant \frac{1}{2}\Big|\mathbf{diam}_p(X)-\mathbf{diam}_p(Y)\Big|.$$

5. *Let* $\{\mathbf{x}_i\}\subset X$ *be a set of* m *random variables* $\mathbf{x}_i:\Omega\to X$ *defined on some probability space* Ω *with law* μ_X. *Let* $\mu_m(\omega,\cdot) := \frac{1}{m}\sum_{i=1}^m \delta^X_{\mathbf{x}_i(\omega)}$ *denote the empirical measure. For each* $\omega\in\Omega$ *consider the mm-spaces* (X,d_X,μ_X) *and* $(\{\mathbf{x}_i\},d_X,\mu_m)$*, then, for* μ_X*-almost all* $\omega\in\Omega$*,*
$$(\{\mathbf{x}_i\},d_X,\mu_m) \xrightarrow{d_{\mathcal{GW},p}} (X,d_X,\mu_X) \text{ as } m\uparrow\infty.$$

Remarks

- Algorithmically, the GW distance can be estimated using standard optimization p… **without changing anything**.
- No adjuste… re necessary: it is directly implementable.
- This implie… ne does not need to assume smooth underlying structure as in [**MS0… 5,BBK06**].
- So, one ca… these ideas to data more general than just shapes, any type of me… ce is fine, for example ultrametrics.



Connections with other approaches

- Shape Distributions [**Osada-et-al**]
- Shape contexts [**SC**]
- Hamza-Krim, Hilaga et al approach [**HK**]
- Rigid isometries invariant Hausdorff [**Goodrich**]
- Gromov-Hausdorff distance [**MS04**] [**MS05**]
- Elad-Kimmel idea [**EK**]
- Topology based methods

- $\mathcal{M}$: collection of all datasets/shapes (metric spaces).
- $\mathcal{S}$: collection of all invariants/signatures.

$$\begin{tikzcd} \boxed{\begin{array}{c}\textbf{datasets/shapes}\\ \left(\mathcal{M}, d_{\mathcal{M}}\right)\end{array}} \arrow[r, "S_h,\ h \in \mathcal{H}"] & \boxed{\begin{array}{c}\textbf{signatures}\\ \left(\mathcal{S}, d_{\mathcal{S}}\right)\end{array}} \end{tikzcd}$$

X, Y $\qquad$ $S_h(X),\ S_h(Y)$ $\qquad$ $h \in \mathcal{H}$

$$\cancel{d_{\mathcal{M}}(X, Y)} \qquad \geqslant \qquad \boxed{d_{\mathcal{S}}\big(S_h(X),\ S_h(Y)\big)}$$

Why am I interested in lower/upper bounds?

- We want to inter-connect different approaches proposed in the literature.
- Knowing interconnections allows for a better understanding of the landscape of different techniques.
- Most methods rely on comparison of signatures.
- Will show that many of the signatures known in the literature are **stable** with respect to the Gromov-Waserstein distance.
- these lower bounds provide ways of accelerating a potential shape comparison task: layered comparison.

$$
\begin{tikzcd}
 & \mathrm{LB}_{0,1}(X,Y) \arrow[r] & \mathrm{LB}_{0,2}(X,Y) & & \\
d_{\mathcal{GW},p}(X,Y) \arrow[ur] \arrow[r] & \mathrm{LB}_{1,1}(X,Y) \arrow[r] \arrow[dr] & \mathrm{LB}_{1,2}(X,Y) \arrow[r] & {} \arrow[r] & \mathrm{LB}_{1,n}(X,Y) \\
 & & \mathrm{LB}_{2,2}(X,Y) & &
\end{tikzcd}
$$

Theorem. *Let X and Y be two mm-spaces, then, $d_{\mathcal{GW},p}(X,Y)$ admits as lower bounds:*

- *Wasserstein distance between distributions of distances of X and Y.*
- *A solution of an LP that depends on the eccentricities of X and Y.*
- *A solution of an LP that depends on the local distribution of distances (cumulative shape contexts) of X and Y.*

The bound for the H-K approach

Let $p = 1$ for simplicity. For a mm-space (X, d_X, μ_X) let $s_X : X \to \mathbb{R}^+$ be given by

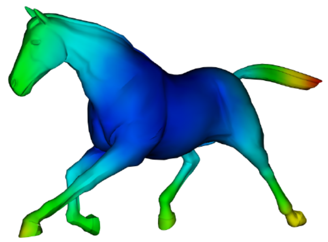
$$x \mapsto \sum_{x' \in X} \mu_X(x')\, d_X(x, x') \quad \text{(average distance to all other points).}$$

The HK lower bound, denoted by $LB_{HK}(X,Y)$ is defined to be (the mass transportation problem)

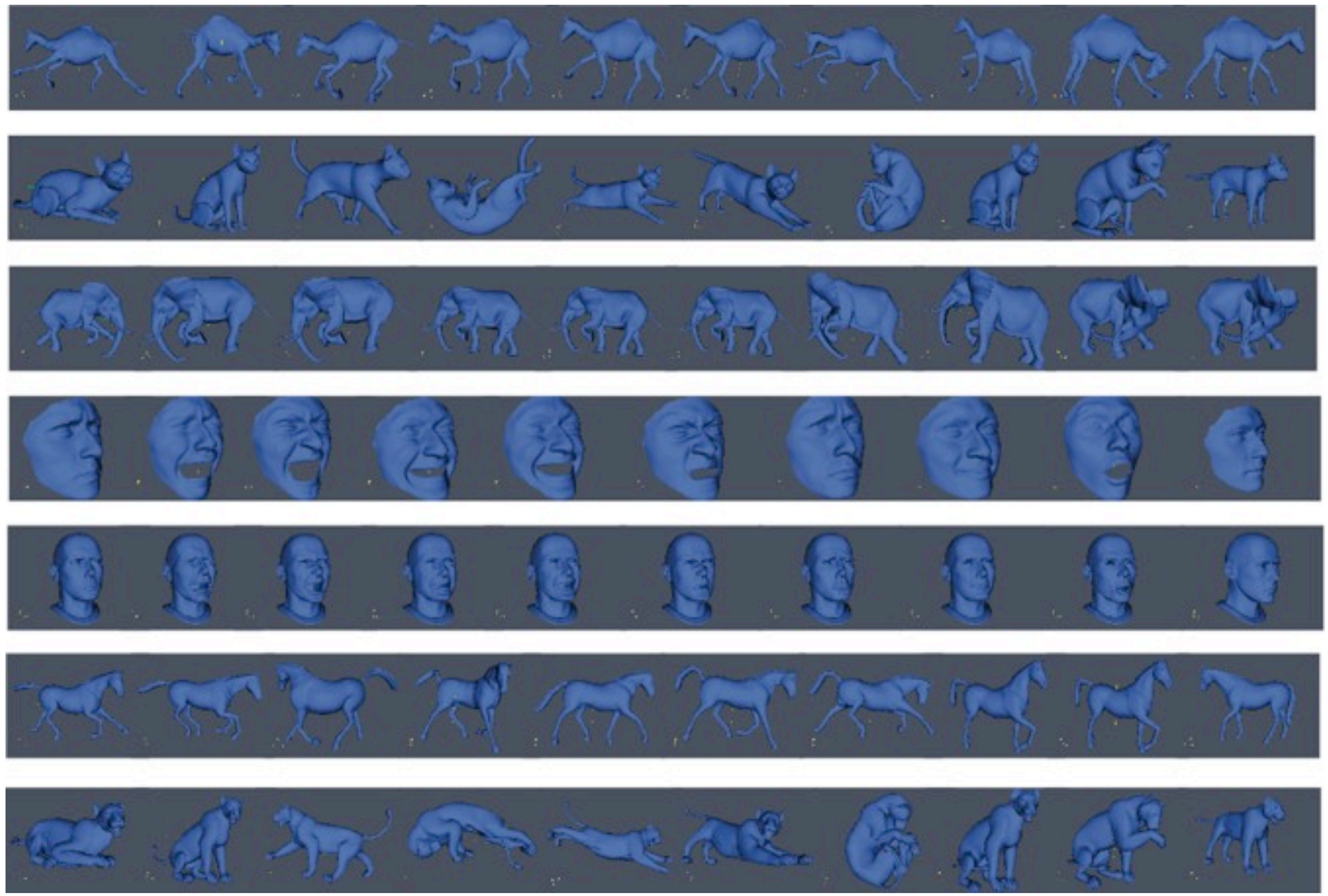
$$LB_{HK}(X,Y) := \min_{\mu \in \mathcal{M}(\mu_X,\mu_Y)} \sum_{x,y} \mu(x,y)\, |s_X(x) - s_Y(y)|.$$

Then, for all mm-spaces X and Y,

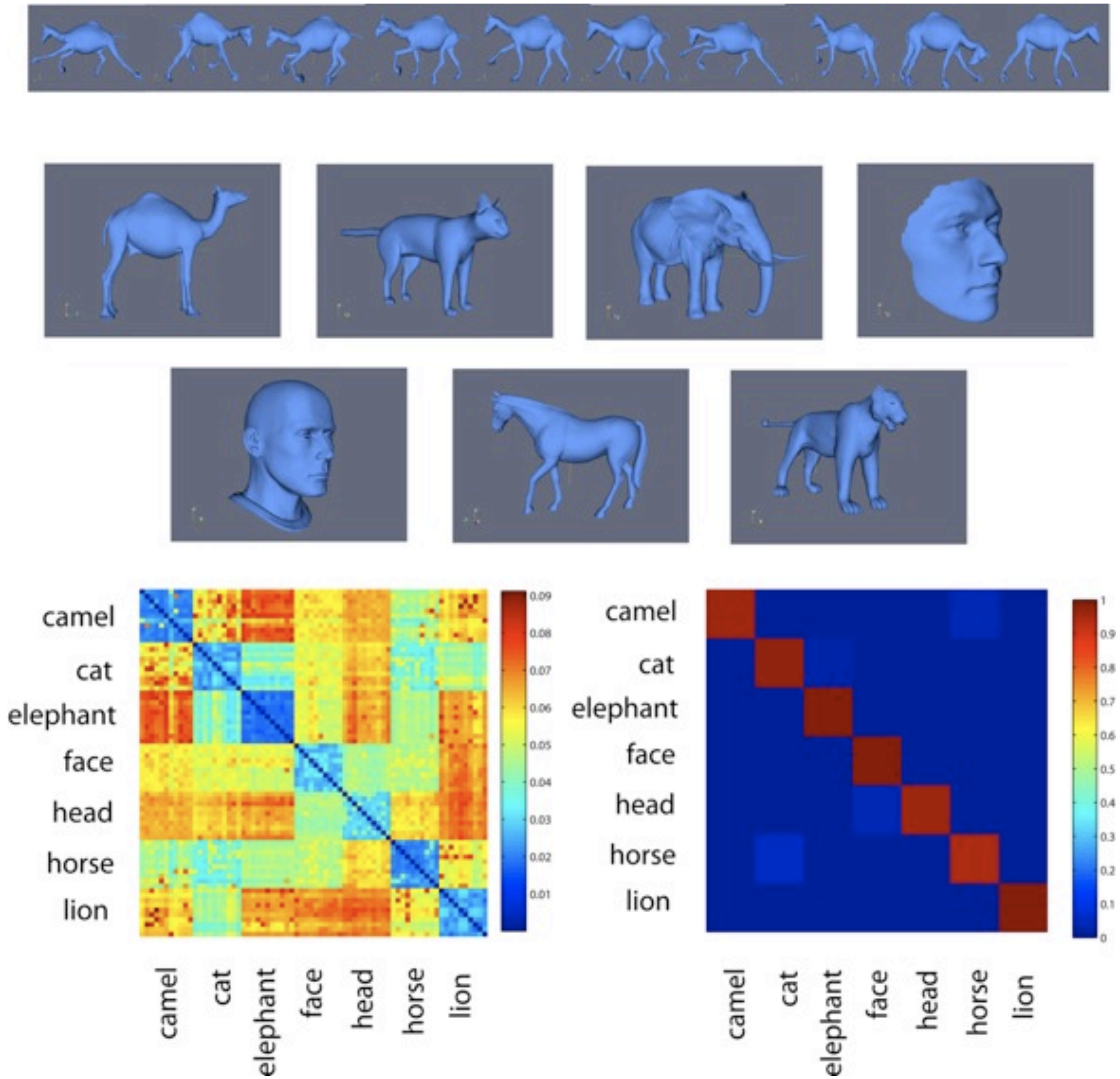
$$\frac{1}{2} LB_{HK}(X,Y) \leqslant d_{\mathcal{GW},1}(X,Y)$$



Some Experiments



Some experimentation: ~ 70 models in 7 classes. Classification using 1-nn: $P_e \sim 2\%$. Hamza-Krim gave $\sim 15\%$ on same db with all same parameters etc.



Some experimentation: ~ 70 models in 7 classes. Classification using 1-nn: $P_e \sim 2\%$. Hamza-Krim gave $\sim 15\%$ on same db with all same parameters etc.

Discussion

Identifying a notion of ***distance/metric*** *between shapes is useful/important.*

- When will you say that two shapes are the same? This is the zero of your distance between shapes.
- Having a true metric on the space of shapes permits proving *stability* and having a *sampling theory*.
- Understand hierarchy of lower/upper bounds. When is a particular LB better than another? study highly symmetrical shapes.
- Other developments: analysis of spectral methods, persistent topology invariants (Frosini et al).

Discussion

- Implementation is easy: gradient descent or alternate optimization.
- Solving lower bounds yields seed for gradient descent. Lower bounds are compatible with the metric in the sense that a layered recognition system is possible: given two shapes (1) solve for a LB (this gives you a μ), if LB value is small enough, then (2) solve for GW using the μ as seed.
- natural extension to partial matching.
- Interest in relating GH/GW ideas to other methods in the literature. Inter-relating methods is important for applications: when confronted with N methods, how do they compare to each other? which one is better for the situation at hand?
- Latest developments:
  - Partial matching [**M10-partial**].
  - Euclidean case [**M08-Euclidean**].
  - Persistent topology [**dgh-topo-pers-09**].
  - Spectral GW distance [**M09-dgw-spec**].
- No difference between continuous and discrete. Probability measures handle the "transition".

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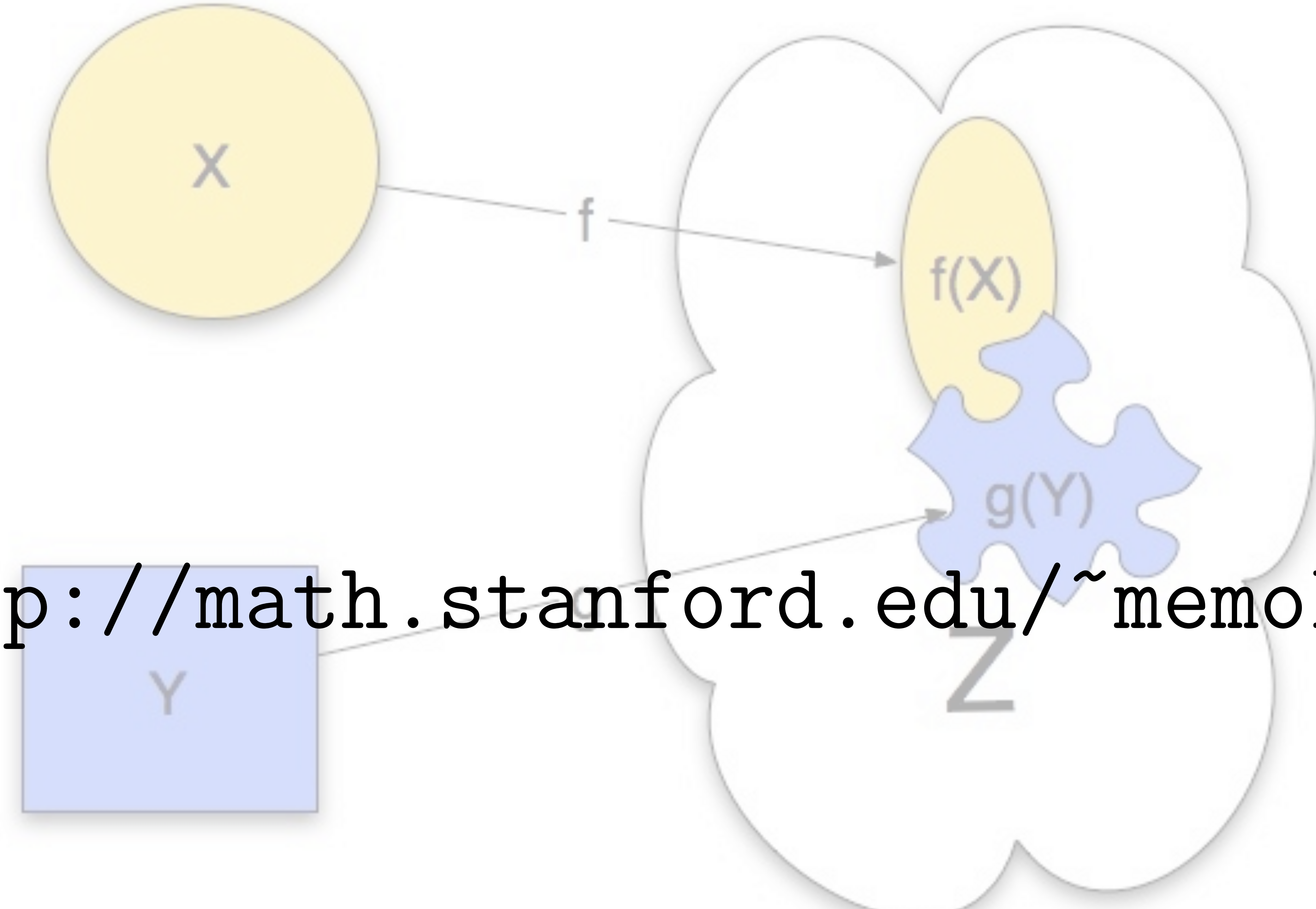

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