Persistent Homology and The Filling Radius

Sunhyuk Lim, Facundo Mémoli, and Osman Berat Okutan

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Abstract

In the applied algebraic topology community, the persistent homology induced by Vietoris-Rips complexes is a classical method to capture topological information from datasets. In this paper, we consider a new, more geometric way of generating persistent homology by embedding a given metric space into a larger space and considering thickening of the original space. Moreover, we construct appropriate category for this new persistent homology and will show that actually the classical persistent homology of the Vietoris-Rips filtration is isomorphic to our new geometric persistent homology in the category theoretic sense. Finally, as an application, we will connect the new persistent homology and filling radius of manifolds introduced by Gromov [22] and show some consequences related to the homotopy type of the Vietoris-Rips complexes of spheres that follows from work of Katz..

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1 Introduction

The simplicial complex nowadays referred to as the Vietoris-Rips complex was introduced by Leopold Vietoris in order to build homology theory for metric spaces [34]. Later, Elyahu Rips and Mikhail Gromov [7] both used the Vietoris-Rips complex to analyze hyperbolic groups.

Given a metric space (X, d_X) and $r \geq 0$ the r-Vietoris-Rips complex $\mathrm{VR}^r(X)$ has X as its vertex set, and simplices are all those finite subsets σ of X whose diameter does not exceed r. Since $\mathrm{VR}^r(X) \subset \mathrm{VR}^s(X)$ for all $0 < r \leq s$, this construction then naturally induces the so called Vietoris-Rips simplicial filtration of X, denoted $\mathrm{VR}^*(X) = \{\mathrm{VR}^r(X)\}_r$. By applying the simplicial homology functor (with coefficients in a given field) one obtains a *persistent module*: a directed system $V_* = \{V_r \to V_s\}_{r \leq s}$ of vector spaces and linear maps (induced by the simplicial inclusions).

Thus, the persistent homology of the Vietoris-Rips filtration of a metric space provides a functorial way[1] of assigning a persistence module to a metric space. In the areas of topological data analysis (TDA) and computational topology, this type of persistent homology is a widely used tool for capturing topological properties of a dataset (when this dataset is represented as a metric space).

The notion of *persistent homology* arose from work by Frosini, Ferri, and Landi [25, 26], Robins [29], and Edelsbrunner [27, 28] and collaborators. After that, applying persistent homology to the simplicial filtration induced from Vietoris-Rips complexes was a natural next step. For example, Carlsson, and de Silva [11] applied Vietoris-Rips persistent homology to topological estimation from point cloud data, and Ghrist and de Silva applied it to sensor networks [30]. A more detailed historical survery and review of general ideas can be found in [31, 23, 24].

Contributions. One main contribution of this paper is establishing a precise relationship (i.e. a filtered homotopy equivalence) between the Vietoris-Rips simplicial filtration of a metric space and a more geometric (or extrinsic) way of assigning a persistence module to a metric space, which consists of first embedding it into a larger space and then taking the persistence homology of the filtration obtained by considering the growing neighborhoods of the original space inside the ambient space. These neighborhoods are also metric spaces and we can benefit from this for example in obtaining a short proof of the Künneth formula for persistent homology.

A particularly nice ambient space inside which one can isometrically embed a compact metric space (X, d_X) is its Kuratowski space $\kappa(X)$ consisting of all real valued functions on X, together with the ℓ_∞ norm. That the Vietoris-Rips filtration of a metric space produces

[1]Where for metric spaces X and Y morphisms are given by 1-Lipschitz maps $\phi : X \to Y$, and for persistence modules V_* and W_* morphisms are systems of linear maps $\nu_* = (\nu_r : V_r \to W_r)_{r>0}$ making all squares commute.

persistence modules isomorphic to the sublevel set filtration of the distance function $\delta_X : \kappa(X) \to \mathbb{R}_+$, $\kappa(X) \ni f \mapsto \inf_{x\in X} \|d_X(x,\cdot) - f\|_\infty$ was used in [3] in order to prove the Gromov-Hausdorff stability of Vietoris-Rips persistence of finite metric spaces.

In this paper we generalize this point of view significantly by proving an isomorphism theorem between the Vietoris-Rips filtration of X and the filtration $\left(\delta_X^{-1}((0,r))\right)_{r>0}$. We do so by constructing a filtered homotopy equivalence between (the geometric realization of) the Vietoris-Rips filtration and the sublevel set filtration induced by δ_X. Furthermore, we prove that $\kappa(X)$ above can be replaced with any injective metric space admitting an isometric embedding of X.

A certain well known construction that uses the isometric embedding $X \hookrightarrow \kappa(X)$ is that of the *filling radius* of a Riemannian manifold [8] defined by Gromov. In that construction, given an n-dimensional Riemannian manifold M one studies for each $r > 0$ the inclusion $\iota_r : M \hookrightarrow \delta_M^{-1}((0,r))$, and seeks the infimal $r > 0$ such that the map induced by ι_r at n-th homology level maps the fundamental class $[M]$ to zero. In a series of papers [12, 17, 18, 19] M. Katz studied both the problem of computing the filling radius of spheres and complex projective spaces, and the problem of understanding the change in homotopy type of $\delta_X^{-1}((0,r))$ when $X \in \{\mathbb{S}^1, \mathbb{S}^2\}$.

Of central interest in topological data analysis has been the complete characterization of the persistence diagrams of spheres of different dimensions. Our isomorphism theorem permits applying Katz's results in order to provide partial answers to these questions.

Also, our isomorphism theorem permits providing succint independent proofs of theorems given Haussman [9] and Latschev [21].

Comparison to the work of Adamaszek, Adams, and Frick. The papers [14, 15] contain ideas similar to ours. The authors of those papers point out that even though Vietoris-Rips complexes induced from a subset X is an effective tool for capturing the topology of underlying manifold M, they cannot recover metric information of M. To remedy that situation, they embed X into the space of probability measures on M (with the 1-Wasserstein metric) and consider metric thickenings of X in that large ambient space. Here, their "thickening" is not exactly same thing as ours since we use the one arising from Kuratowski space of M (or any injective space inside which we can embed M). As a result, they could prove versions of Hausmann's theorem [10] and Latschev's theorem [21], and also computed the homotopy type of thickenings of spheres for some range of the thickening parameter.

Organization. In Section 2 we construct a category of metric pairs. This category will be the natural setting for our extrinsic persistence homology. Although being functorial is trivial in the case of Vietoris-Rips persistence, the type of functoriality which one is supposed to expect in the case of metric embeddings is a priori not obvious. We address this question in Section 2 by introducing a suitable category structure.

In Section 3, we show that the Vietoris-Rips filtration can be (categorically) seen as a special case of persistence obtained through metric embeddings.

In Section 4, we obtain new proofs of formulas about the Vietoris-Rips persistence of product and wedge sum of metric spaces.

In Section 5 we state a few results concerning the homotopy types of Vietoris-Rips filtrations of spheres and complex projective spaces.

In Section 6, we give some applications of our ideas to the filling radius of Riemannian manifolds.

In the Appendix, we provide some necessary definitions and results about Vietoris-Rips filtration, persistence and injective metric spaces.

2 Persistence via metric pairs

One of the insights leading to persistence associated to metric spaces was considering neighborhoods of a metric space in a nice (for example Euclidean) embedding. In this section we formalize this idea in a categorical way.

Definition 1 (Category of metric pairs).

- A *metric pair* is an ordered pair (X, E) of metric spaces such that X is a metric subspace of E.
- Let (X, E) and (Y, F) be metric pairs. A 1-Lipschitz map from (X, E) to (Y, F) is a 1-Lipschitz map from E to F mapping X into Y.
- Let (X, E) and (Y, F) be metric pairs and f and g be 1-Lipschitz maps from (X, E) to (Y, F). We say that f and g are *equivalent* if there exists a continuous family $(h_t)_{t\in[0,1]}$ of 1-Lipschitz maps from E to F and a 1-Lipschitz map $\phi : X \to Y$ such that $h_0 = f, h_1 = g$ and $h_t|_X = \phi$ for each t.
- We define PMet as the category whose objects are metric pairs and whose morphisms are defined as follows. Given metric pairs (X, E) and (Y, F), the morphisms from (X, E) to (Y, F) are equivalence classes of 1-Lipschitz maps from (X, E) to (Y, F).

Definition 2 (Persistence families). A *persistence family* is a collection $\{(U_r), f_{r,s}\}_{0<r\leq s}$ such that for each $0 < r \leq s \leq t$, U_r is a topological space, $f_{r,s} : U_r \to U_s$ is a continuous map, $f_{r,r} = Id_{U_r}$ and $f_{s,t} \circ f_{r,s} = f_{r,t}$.

Given two persistence families $(U_*, f_{*,*})$ and $(V_*, g_{*,*})$ a morphism from the first one to the second is a collection $(\phi_*)_{r>0}$ such that for each $0 < r \leq s$, ϕ_r is a homotopy class of maps from $U_r \to V_r$ and $\phi_s \circ f_{r,s}$ is homotopy equivalent to $g_{r,s} \circ \phi_r$.

We denote the category of persistence families with morphisms specified as above by hTop_*

Remark 2.1. *Let (X, E) and (Y, B) be persistent pairs and f be a 1-Lipschitz morphism between them. Then f maps $B_r(X, E)$ into $B_r(Y, F)$ for each $r > 0$. Furthermore, if g is equivalent to f, then they reduce to homotopy equivalent maps from $B_r(X, E)$ to $B_r(Y, F)$ for each $r > 0$.*

By the Remark above, we get the following functor from PMet to hTop_*.

Definition 3 (Persistence functor). Define the *persistence functor* $\mathrm{B}_* : \mathrm{PMet} \to \mathrm{hTop}_*$ sending (X, E) to the persistence family obtained by the filtration $(B_r(X, E))_{r>0}$ and sending a morphism between metric pairs to the homotopy classes of maps it induces between the filtrations.

Let Met be the category of metric spaces where morphisms are given by 1-Lipschitz maps. There is a forgetful functor from PMet to Met mapping (X, E) to X and mapping a morphism defined on (X, E) to its restriction to X. Although forgetful functors tend to have left adjoints, we are going to see that this one has a right adjoint.

Theorem 2.2. *The forgetful functor* PMet *to* Met *has a right adjoint.*

First we need prove a few results.

Lemma 2.3. *Let (X, E) and (Y, F) be metric pairs such that F is an injective metric space. Let f and g be 1-Lipschitz maps from (X, E) to (Y, F). Then f is equivalent to g if and only if $f|_X \equiv g|_X$.*

Proof. The only if part obvious. Now assume that $f|_X \equiv g|_X$. By [13, Proposition 3.8], there exists a geodesic bicombing $\gamma : F \times F \times [0, 1] \to F$ such that for each x, y, x', y' in F and t in $[0, 1]$,

$$d(\gamma(x, y, t), \gamma(x', y', t)) \leq (1 - t)\, d(x, x') + t\, d(y, y').$$

For t in $[0, 1]$, define $h : E \to F$ by $h_t(x) = \gamma(f(x), g(x), t)$. Note that $h_0 = f, h_1 = g$ and $(h_t)|_X$ is the same map for all t. The inequality above implies that h_t is 1-Lipschitz for all t. This completes the proof. □

See the appendix for background on injective metric spaces.

Lemma 2.4. *Let (X, E) and (Y, F) be metric pairs such that F is an injective metric space. Then for each $\phi : X \to Y$, there exist a unique 1-Lipschitz map from (X, E) to (Y, F) extending ϕ up to equivalence.*

Proof. The uniqueness part follows from Lemma 2.3. The existence part follows from the injectivity of F. □

Proof of Theorem 2.2. Let κ : Met $\to$ PMet be the functor sending X to $(X, \kappa(X))$ where $\kappa(X)$ is the Kuratowski space of X (see Appendix, Definition 13). A 1-Lipschitz maps $f : X \to Y$ is sent to the unique morphism (see Lemma 2.3) extending f. There is a natural morphsim

$$\mathrm{Hom}((X, E), (Y, \kappa(Y))) \to \mathrm{Hom}(X, Y),$$

sending a morphism to its restriction to X. By Lemma 2.4, this is a bijection. Hence κ is a right adjoint to the forgetful functor. □

Remark 2.5. *Note that in the above proof the Kuratowski space $\kappa(\cdot)$ can be replaced by the tight span $E(\cdot)$ (See Example 2.6 below and [5]) or any other construction which assigns an injective space containing a given space.*

Definition 4 (Metric homotopy pairing)**.** A functor η : Met $\to$ PMet is called a *metric homotopy pairing* if it is a right adjoint to the forgetful functor.

Example 2.6. Let (X, d_X) be a metric space. By $\kappa(X)$ denote the Kuratowski space associated to X, see Section A.2 for the precise definition. Consider also the following additional spaces associated to X:

$$\begin{aligned}
&\Delta(X) := \{f \in \ell_\infty(X) : f(x) + f(x') \geq d_X(x, x') \text{ for all } x, x' \in X\},\\
&E(X) := \{f \in \Delta(X) : \text{if } g \in \Delta(X) \textit{ and } g \leq f, \text{ then } g = f\},\\
&\Delta_1(X) := \Delta(X) \cap \mathrm{Lip}_1(X, \mathbb{R}),
\end{aligned}$$

with ℓ_∞ metrics for all of them (cf. [13, Section 3]). Here $E(X)$ is the *tight span* of X [5]. Then,

$$(X, \kappa(X)), (X, E(X)), (X, \Delta(X)), (X, \Delta_1(X))$$

are all metric homotopy pairings, since the second element in each pair is an injective metric space (see [13, Section 3]) into which X isometrically embeds.

3 Isomorphism

Recall that Met is the category of metric spaces with 1-Lipschitz maps as morphisms. We have a functor $\mathrm{VR}^* : \mathrm{Met} \to \mathrm{hTop}_*$ obtained by the (geometric realization of the) Vietoris-Rips filtration. The main theorem we prove in this section is the following:

Theorem 3.1 (Isomorphism Theorem)**.** *Let* $\eta : \mathrm{Met} \to \mathrm{PMet}$ *be a metric homotopy pairing (for example the Kuratowski functor). Then* $\mathrm{B}_* \circ \eta : \mathrm{Met} \to \mathrm{hTop}_*$ *is naturally isomorphic to* VR^{2*}.

Definition 5. Let (X, E) be a metric pair and $r > 0$. Let $\mathcal{U}_r$ denote the open cover of $B_r(X, E)$ consisting of open balls in E with center in X and radius r.

- We denote the geometric realization of the Čech complex of $\mathcal{U}_r$ by $S_r(X, E)$. Note that $S_*(X, E)$ is a filtration and we have an inclusion $S_*(X, E)$ into $\mathrm{VR}^{2*}(X)$ where by abuse of notation we denote the geometric realization of $\mathrm{VR}^{2*}(X)$ by the same notation.

- We denote the corresponding complex of spaces by $(X, E)_r$. We denote the amalgamation of $(X, E)_r$ by $\coprod_r (X, E)$ and realization of $(X, E)_r$ by $\Delta_r(X, E)$ (see [9, Chapter 4.G]).

- We have natural morphisms

$$B_r(X, E) \leftarrow \coprod_r (X, E) \leftarrow \Delta_r(X, E) \to S_r(X, E) \to \mathrm{VR}^{2r}(X).$$

(see [9, Chapter 4.G])

Proposition 3.2.

- *Given* $0 < r \leq s$, *the inclusion of open balls* $B_r(x, E) \subseteq B_s(x, E)$ *induces morphisms* $\coprod_r X \to \coprod_s X$ *and* $\Delta_r X \to \Delta_s X$ *such that the diagram in the definition above induces a diagram of filtered topological spaces:*

$$B_*(X, E) \leftarrow \coprod_* (X, E) \leftarrow \Delta_*(X, E) \to S_*(X, E) \to \mathrm{VR}^{2*}(X).$$

Furthermore, if E is an injective metric space, then all morphisms in the diagram are homotopy equivalences.

- *If (Y, F) is a metric pair and $f : (X, E) \to (Y, F)$ is 1-Lipschitz, then the following diagram commutes*

$$\begin{tikzcd} B_*(X,E) \arrow[d, "f_*"] & \coprod_*(X,E) \arrow[l] \arrow[d, "f_*"] & \Delta_*(X,E) \arrow[l] \arrow[r] \arrow[d, "f_*"] & S_*(X,E) \arrow[r] \arrow[d, "f_*"] & \mathrm{VR}^{2*}(X) \arrow[d, "f_*"] \\ B_*(Y,F) & \coprod_*(Y,F) \arrow[l] & \Delta_*(Y,F) \arrow[l] \arrow[r] & S_*(Y,F) \arrow[r] & \mathrm{VR}^{2*}(Y) \end{tikzcd}.$$

Furthermore, if we change f with an equivalent map, then the homotopy types of the vertical maps remain unchanged.

Proof. Commutativity of the diagrams in the statement follows easily from the definitions.

Note that $B_*(X, E) \leftarrow \coprod_*(X, E)$ is a homeomoprhism since the $\coprod_r(X, E)$ is an amalgamation corresponding to an open cover of $B_r(X, E)$. The morphism $\coprod_*(X, E) \leftarrow \Delta_*(X, E)$ is a homotopy equivalence by [9, Proposition 4G.2]. If E is an injective metric space, then the open cover $\mathcal{U}_r$ of $B_r(X, E)$ is a good cover by Lemma A.6, therefore $\Delta_*(X, E) \to S_*(X, E)$ is a homotopy equivalence by [9, Corollary 4G.3]. Finally, since E is hyperconvex, the morphism $S_*(X, E) \to \mathrm{VR}^{2*}(X)$ is a homeomorphsim (see the proof of Proposition A.5).

If f, g are equivalent, then the homotopy (h_t) between f, g induces homotopy $(h_t)_*$ for all horizontal maps (note that the two rightmost vertical maps does not change at all). $\square$

Proof of Theorem 3.1. Since all metric homotopy pairings are naturally isomorphic, without loss of generality we can assume that $\eta = \kappa$, the Kuratowski functor. By Proposition 3.2, $B_*, \coprod_*, \Delta_*, S_*$ and VR^{2*} are functors from PMet to the category of persistence families with certain natural transformations between them. By the homotopy equivalence part of Proposition 3.2, if we precompose these functors with the Kuratowski functor $\kappa : \mathrm{Met} \to \mathrm{PMet}$, they all become naturally isomorphic. $\square$

4 Application to Vietoris-Rips filtration

The following statements are obtained at the simplicial level in [20], [1, Proposition 4], [16, Proposition 13]. Here we give alternative proofs using neighborhoods in a hyperconvex embedding.

Theorem 4.1. *Let X, Y be metric spaces*

1. *(Persistent Künneth formula) Let $X \times Y$ denote the ℓ_∞ product of X, Y. If X has pointwise finite dimensional Vietoris-Rips persistent homology, then*

$$\mathrm{PH}(\mathrm{VR}(X \times Y)) \cong \mathrm{PH}(\mathrm{VR}(X)) \otimes \mathrm{PH}(\mathrm{VR}(Y)).$$

2. *Let p and q be points in X and Y respectively. Let $X \vee Y$ denote the wedge sum of metric spaces X and Y along p and q. Then*

$$\mathrm{PH}(\mathrm{VR}(X \vee Y)) \cong \mathrm{PH}(\mathrm{VR}(X)) \oplus \mathrm{PH}(\mathrm{VR}(Y)).$$

Remark 4.2. *Note that the tensor product of two simple persistence modules corresponding to intervals I, J is the simple persistence module corresponding to the interval $I \cap J$. Therefore, the first part of Theorem 4.1 implies that*

$$\mathrm{dgm}_n^{\mathrm{VR}}(X \times Y) := \{I \cap J : I \in \mathrm{dgm}_i^{\mathrm{VR}}(X), J \in \mathrm{dgm}_j^{\mathrm{VR}}(Y), i + j = n\}.$$

To be able to prove Theorem 4.1, we need the following lemmas:

Lemma 4.3. *If E and F are injective metric spaces, then so is their ℓ_∞ product.*

Proof. Let X be a metric space. Note that $(f, g) : X \to E \times F$ is 1-Lipschitz if and only if f and g are 1-Lipschitz. Given such f and g and a metric embedding X into Y, we have 1-Lipschitz extensions $\tilde{f}, \tilde{g}$ of f and g from Y to E and F respectively. Hence, $(\tilde{f}, \tilde{g}) : Y \to E \times F$ is a 1-Lipschitz extension of (f, g). Therefore $E \times F$ is injective. □

Lemma 4.4. *If E and F are injective metric spaces, then so is their wedge sum along any two points.*

Proof. Let p and q be points in E and F respectively and $E \vee F$ denote the wedge sum of E and F along p and q. We are going to show that $E \vee F$ is hyperconvex, hence injective.

Let $(x_i, r_i)_i, (y_j, r_j)_j$ be such that x_i is in E, y_j is in F, $r_i \geq 0$, $s_j \geq 0$, $d(x_i, x_{i'}) \leq r_i + r_{i'}$, $d(y_j, y_{j'}) \leq s_j + s_{j'}$ and $d(x_i, y_j) \leq r_i + s_j$ for each i, i', j, j'. Define ϵ by

$$\epsilon := \max(\inf_i r_i - d(x_i, p), \inf_j s_j - d(y_j, q)).$$

Let us show that $\epsilon \geq 0$. If the second element inside the maximum is negative, then there exists j_0 such that $d(y_{j_0}, q) - s_{j_0} > 0$. Since $d(x_i, y_{j_0}) = d(x_i, p) + d(q, y_{j_0})$ for all i, we have

$$r_i - d(x_i, p) \geq d(y_{j_0}, q) - s_{j_0} > 0.$$

Therefore the first element inside the maximum is positive. Hence $\epsilon > 0$.

Without loss of generality let us assume that

$$\epsilon := \inf_i r_i - d(x_i, p) > 0.$$

This implies that the non-empty closed ball $b_\epsilon(q, F)$ is contained in $b_{r_i}(x_i, E \vee F)$ for all i. Now, for each j, we have

$$\begin{aligned} \epsilon + s_j &= \inf_i r_i - d(x_i, p) + s_j \\ &\geq \inf_i d(x_i, y_j) - d(x_i, p) \\ &= d(y_j, q). \end{aligned}$$

Therefore,

$$(\cap_i b_{r_i}(x_i, E \vee F)) \cap (\cap_j b_{s_j}(y_j, E \vee F)) \supseteq b_\epsilon(q, F) \cap (\cap_j b_{s_j}(y_j, F))$$

which is non-empty by hyperconvexity of F. □

Proof of Theorem 4.1. "1)" Let E and F be injective metric spaces containing X and Y respectively. Let $E \times F$ denote the ℓ_∞ product of E and F. Note that for each $r > 0$,

$$B_r(X \times Y, E \times F) = B_r(X, E) \times B_r(Y, F).$$

Hence, by the Künneth formula,

$$\mathrm{H}_*(B_r(X \times Y, E \times F)) \cong \mathrm{H}_*(B_r(X, E)) \otimes \mathrm{H}_*(B_r(Y, F)).$$

Now, the result follows from Lemma 4.3 and Proposition 3.2.

"2)" Let E and F be as above and $E \vee F$ denote wedge product of E and F along p and q. Note that

$$B_r(X \vee Y, E \vee F) = B_r(X, E) \vee B_r(Y, F).$$

Hence, by [9, Corollary 2.25],

$$\mathrm{H}_*(B_r(X \vee Y, E \vee F)) \cong \mathrm{H}_*(B_r(X, E)) \oplus \mathrm{H}_*(B_r(Y, F)).$$

Now, the result follows from Lemma 4.4 and Proposition 3.2. $\square$

5 Application to homotopy types of $\mathrm{VR}^r(X)$, for $X \in \{\mathbb{S}^1, \mathbb{S}^2, \mathbb{CP}^n\}$

In a series of papers [12, 17, 18, 19] M. Katz studied the filling radius of spheres and complex projective space, and also obtained results for the critical points of the diameter function on each of these spaces. This allowed him to provide some results about the different homotopy types of $B_r(X, \kappa(X))$ for $X \in \{\mathbb{S}^1, \mathbb{S}^2, \mathbb{CP}^n\}$ as r varies.

Here we obtain some corollaries that follow from the work of M. Katz [17, 18] and Theorem 3.1.

5.1 The case of spheres

Katz found the first r where the homotopy type of $B_r(\mathbb{S}^n, \kappa(\mathbb{S}^n))$ changes for general $n \in \mathbb{Z}_{>0}$. Therefore we have the following result.

Corollary 5.1. *For any $n \in \mathbb{Z}_{>0}$, we have $\mathrm{VR}^r(\mathbb{S}^n) \cong \mathbb{S}^n$ for any $r \in \left(0, \arccos\left(-\frac{1}{n+1}\right)\right]$.*

Proof. By the remark right after the proof of Theorem 2 of [12], we know that $B_r(\mathbb{S}^n, \kappa(\mathbb{S}^n))$ is homotopy equivalent to $\mathbb{S}^n$ for $r \in \left(0, \frac{1}{2}\arccos\left(-\frac{1}{n+1}\right)\right]$. Hence, if we apply Theorem 3.1, we have the required result. $\square$

Moreover, especially for $\mathbb{S}^1$ and $\mathbb{S}^2$, we have more information.

5.1.1 The case of $\mathbb{S}^1$

The complete characterization of the different homotopy types of $\mathrm{VR}^r(\mathbb{S}^1)$ as $r > 0$ grows was obtained by Adamaszek and Adams in [35]. Their proof is combinatorial in nature and takes place at the simplicial level.

Below, by invoking Theorem 3.1, we show how partial results can be obtained from the work of Katz who directly analyzed the filtration $\big(B_r(\mathbb{S}^1, \kappa(\mathbb{S}^1))\big)_r$ via a Morse theoretic argument.

For each non negative integer $k \geq 1$ let $\lambda_k = \frac{2\pi k}{2k+1}$. Katz proved in [17] that $B_r(\mathbb{S}^1, \kappa(\mathbb{S}^1))$ changes homotopy type only when $r = \frac{1}{2}\lambda_k$ for some k. In particular, he proved:

Corollary 5.2. *For $r \in \left(\frac{2\pi}{3}, \frac{4\pi}{5}\right)$, $\mathrm{VR}^r(\mathbb{S}^1) \simeq \mathbb{S}^3$.*

Proof. $B_r(\mathbb{S}^1, \kappa(\mathbb{S}^1))$ is homotopy equivalent to $\mathbb{S}^3$ for $r \in \left(\frac{1}{2} \cdot \frac{2\pi}{3}, \frac{1}{2} \cdot \frac{4\pi}{5}\right)$ by [17, Theorem 1.1]. Hence, if we apply Theorem 3.1, we have the required result. □

5.1.2 The case of $\mathbb{S}^2$

Similar arguments hold for the case of $\mathbb{S}^2$.

Corollary 5.3. *For $r \in \left(\arccos\left(-\frac{1}{3}\right), \arccos\left(-\frac{1}{\sqrt{5}}\right)\right)$, $\mathrm{VR}^r(\mathbb{S}^2) \simeq \mathbb{S}^2 * \mathbb{S}^3/E_6$.*

Proof. $B_r(\mathbb{S}^2, \kappa(\mathbb{S}^2))$ is homotopy equivalent to the topological join of $\mathbb{S}^2$ and $\mathbb{S}^3/E_6$ where E_6 is the binary tetrahedral group for $r \in \left(\frac{1}{2} \cdot \arccos\left(-\frac{1}{3}\right), \frac{1}{2} \cdot \arccos\left(-\frac{1}{\sqrt{5}}\right)\right)$ by [17, Theorem 7.1]. Hence, if we apply Theorem 3.1, we have the required result. □

5.2 The case of $\mathbb{CP}^n$

Some partial information can be provided for the case of $\mathbb{CP}^n$ as well. First of all, recall that the complex projective line $\mathbb{CP}^1$ with its canonical metric is actually the sphere $\mathbb{S}^2$. Hence, one can apply Corollary 5.1 and Corollary 5.3 to $\mathbb{CP}^1$, also. For general $\mathbb{CP}^n$, not much is known. Nevertheless one has the following results.

Corollary 5.4. *Let $\mathbb{CP}^n$ be the complex projective space with curvature in between $1/4$ and 1 with canonical metric. Then,*

1. *There exist $\alpha_n \in \left(0, \arccos\left(-\frac{1}{3}\right)\right]$ such that $\mathrm{VR}^r(\mathbb{CP}^n)$ is homotopy equivalent to $\mathbb{CP}^n$) for any $r \in (0, \alpha_n]$.*

2. *Let A be the space of equilateral 4-tuples in projective lines of $\mathbb{CP}^n$. Let X be the partial join of $\mathbb{CP}^n$ and $\mathbb{CP}^n$ where $x \in \mathbb{CP}^n$ is joined to a tuple $a \in A$ by a line segment if x is contained in the projective line determined by a. There exists a constant $\beta_n > 0$ such that if*
$$\arccos\left(-\frac{1}{3}\right) < r < \arccos\left(-\frac{1}{3}\right) + \beta_n,$$
then $\mathrm{VR}^r(\mathbb{CP}^n)$ is homotopy equivalent to X.

Proof. By Hausmann's theorem [10], there exist $\alpha_n > 0$ such that $\mathrm{VR}^r(\mathbb{CP}^n)$ is homotopy equivalent to $\mathbb{CP}^n$) for any $r \in (0, \alpha_n]$. Also, [17, Theorem 8.1], α_n cannot be greater than $\arccos(-1/3)$. The second claim is the direct result from Theorem 3.1 and [17, Theorem 8.1]. $\square$

6 Applications to the filling radius

Here we prove a few statements about the filling radius of a Riemannian manifold [12]. We also define a strong notion of filling radius which is akin to the so called *maximal persistence* in the realm of topological data analysis.

6.1 Spread

We recall a metric concept called *spread*. The following definition is a variant of the one given in [12].

Definition 6 (k-spread)**.** For every non-negative integer k, the *k-th spread* $\mathrm{spread}_k(X)$ of a metric space (X, d_X) is the infimal $r > 0$ such that there exists a subset A of X with cardinality at most k such that

- $\mathrm{diam}(A) < r$
- $\sup_{x\in X} \inf_{a\in A} d_X(x, a) < r$.

Finally, the *spread* of X is defined to be $\mathrm{spread}(X) := \inf_k \mathrm{spread}_k(X)$, i.e. the set A is allowed to have arbitrary cardinality.

Remark 6.1. *Recall that the* radius *of a compact metric space* (X, d_X) *is* $\mathrm{rad}(X) := \inf_{p\in X} \max_{x\in X} d_X(p, x)$. *Thus* $\mathrm{rad}(X) = \mathrm{spread}_1(X)$.

6.2 Bounding barcode length via spread

Let (X, d_X) be a metric space. Then, for each integer $k \geq 0$ we define $\mathrm{dgm}_k^{\mathrm{VR}}(X)$ as the persistence barcode associated to $\mathrm{H}_k(\mathrm{VR}^r(X))$, the k-th persistence module induced by the Vietoris-Rips filtration of X. For the definition of persistence module and barcode, see Section A.1.

Proposition 6.2. *Let X be a metric space, $k \geq 1$, and $I \in \mathrm{dgm}_k^{\mathrm{VR}}(X)$. Then*

$$\mathrm{length}(I) \leq \mathrm{spread}(X).$$

Proof. Let $r > \mathrm{spread}(X)$. It is enough to show that for each $s > 0$, the map

$$\mathrm{H}_*(\mathrm{VR}^s(X)) \to \mathrm{H}_*(\mathrm{VR}^{r+s}(X))$$

induced by the inclusion is zero. Let A be a subset realizing r in the definition of spread. Let $\pi : X \to A$ be a map sending x to a closest point in A. Let $\iota : A \to X$ be the inclusion map. These induces the following maps

$$\mathrm{VR}^s(X) \to \mathrm{VR}^r(A) \to \mathrm{VR}^{r+s}(X).$$

Note that it is enough to show that composition of these maps is contiguous to the inclusion, since then the inclusion maps splits through $\mathrm{H}(\mathrm{VR}^r(A)) = 0$ in the homology level. Let $\{x_1, \ldots, x_k\}$ be a subset of X with diameter less than s. Let $a_i := \pi(x_i)$. Then we have

$$d(x_i, a_j) \leq d(x_i, x_j) + d(x_j, a_j) < r + s.$$

Hence the diameter of the subspace $\{x_1, \ldots, x_k, a_1, \ldots, a_k\}$ is strictly less than $r + s$. This shows the desired contiguity and completes the proof. □

6.3 Bounding the filling radius

For a given compact space (X, d_X) consider the the Kuratowski embedding $\iota : X \to \kappa(X)$ (see Appendix, Definition 13). Recall that given a compact n-dimensional manifold M one defines a the filling radius [12] of M as follows:

$$\mathrm{FillRad}(M) := \inf\{\epsilon > 0 | \, \mathrm{H}_n(\iota_\epsilon, A)([M]) = 0\},$$

where $\iota_\epsilon : \iota(M) \hookrightarrow B_\epsilon(\iota(M), \kappa(M))$, $[M]$ is the fundamental class of M, and $A = \mathbb{Z}$ if M is orientable and $A = \mathbb{Z}_2$ otherwise.

Remark 6.3. *Note that relative filling radius can be defined for every metric pair (M, E) by considering ϵ-neighborhoods of X in E — it is denoted by* $\mathrm{FillRad}(M, E)$. *Gromov [8] showed that we obtain the minimal possible relative filling radius through the Kuratowski embedding. This also follows from our work but in greater generality in the context of of embeddings into injective metric spaces. If M can be isometrically embedded into an injective metric space I, then this embedding can be extended to a 1-Lipschitz map $f : E \to I$, which induces a map of filtrations $f : B_\epsilon(M, E) \to B_\epsilon(M, I)$. Hence, if the fundamental class of M vanishes in $B_\epsilon(M, E)$, then it also vanishes in $B_\epsilon(M, I)$. Therefore,* $\mathrm{FillRad}(M, I) \leq \mathrm{FillRad}(M, E)$.

Remark 6.4. *In [12, Theorem 2] Katz proved that* $\mathrm{FillRad}(\mathbb{S}^n) = \frac{1}{2}\arccos\left(-\frac{1}{n+1}\right)$. *Moreover, in a remark right after the proof of Theorem 2 he shows that $B_r(\mathbb{S}^n, \kappa(\mathbb{S}^n))$ is homotopy equivalent to $\mathbb{S}^n$ if $r \in (0, \mathrm{FillRad}(\mathbb{S}^n)]$. One might then conjecture that* $\mathrm{FillRad}(M)$ *is the first point where the homotopy type of $B_r(M, \kappa(M))$ changes. In general, however, this is not true as the following two examples show.*

1. *It is known [19] that* $\mathrm{FillRad}(\mathbb{CP}^3) > \mathrm{FillRad}(\mathbb{CP}^1) = \frac{1}{2}\arccos\left(-\frac{1}{3}\right)$. *Also, by Theorem [17, Theorem 8.1], $B_r(\mathbb{CP}^3, \kappa(\mathbb{CP}^3))$ is not homotopy equivalent to $\mathbb{CP}^3$ for $r \in \left(\frac{1}{2}\arccos\left(-\frac{1}{3}\right), \frac{1}{2}\arccos\left(-\frac{1}{3}\right) + \varepsilon_0\right)$ where $\varepsilon_0 > 0$ is unknown positive constant. In other words, the homotopy type of $B_r(\mathbb{CP}^3, \kappa(\mathbb{CP}^3))$ changed already before* $\mathrm{FillRad}(\mathbb{CP}^3)$.

2. *The following example provides a geometric intuition for how homotopy type of Kuratowski neighborhoods may change before the filling radius. Consider a big sphere with a small handle attached (see Figure 1). Since the top dimensional hole in this space is big, we expect the filling radius to be big. On the other hand, 1-dimensional homology class coming from the small handle dies in a small Kuratowski neighborhood, hence the homotopy type changes.*

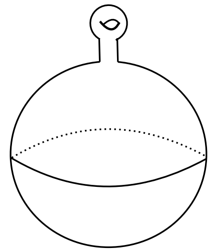

Figure 1: A big sphere X with a small handle. In this case, as $r > 0$ increases, $B_r(X, \kappa(X))$ changes homotopy type from that of X to that of $\mathbb{S}^2$ as soon as $r > r_0$ for some $r_0 < \mathrm{FillRad}(X)$.

Proposition 6.5. *Let M be a compact n-dimensional Riemannian manifold and let B be the n-th persistence barcode of the open Vietoris-Rips filtration of X. Then B contains a unique interval I with left endpoint 0. Furthermore, the right endpoint of I is $2\,\mathrm{FillRad}(M)$.*

Proof. By Haussmann [10], the persistent homology of the Vietoris-Rips filtration of M contains a unique interval with left endpoint 0. Now the result follows from the definition of the filling radius and Theorem 3.1. □

The following proposition is proved in [12]. Here we present different proof which easily follows from what we have done until now from the persistence homology perspective.

Proposition 6.6. *Let M be a compact Riemannian manifold. Then,*

$$\mathrm{FillRad}(M) \leq \mathrm{spread}(X)/2.$$

Proof. Follows from Proposition 6.5 and Proposition 6.2. □

A Appendix

A.1 Vietoris-Rips filtration and persistence

The definitions and results in this subsection can be found in [2]. We include them here for completeness.

Definition 7 (Vietoris-Rips filtration)**.** Let X be a metric space and $r > 0$. The *open Vietoris-Rips complex* $\mathrm{VR}^r(X)$ of X is the simplicial complex whose vertices are the points of X and whose simplices are finite subsets of X with diameter strictly less then r. Note that if $r < s$, then $\mathrm{VR}^r(X)$ is included in $\mathrm{VR}^s(X)$. Hence, the family $\mathrm{VR}^*(X)$ is a filtration, which is called the *open Vietoris-Rips filtration* of X.

Definition 8 (Persistence module)**.** A persistence module over $\mathbb{R}_{>0}$ is a family of vector spaces $V_{r>0}$ with morphisms $f_{r,s} : V_r \to V_s$ for each $r \leq s$ such that

- $f_{r,r} = \mathrm{Id}_{V_r}$,

- $f_{s,t} \circ f_{r,s} = f_{r,t}$ for each $r \leq s \leq t$.

Remark A.1. $\mathrm{H}_*(\mathrm{VR}^r(X))$ *forms a persistence module where the morphisms are induced by inclusions. As a persistence module, it is denoted by* $\mathrm{PH}(\mathrm{VR}^*(X))$.

Definition 9 (Irreducible persistence modules)**.** Given an interval I in $\mathbb{R}_{>0}$, the persistence module $I_{\mathbb{R}}$ is defined as follows. The vector space at r is $\mathbb{R}$ if r is in I and zero otherwise. Given $r \leq s$, the morphism from corresponding to (r, s) is identity if r, s are in I and zero otherwise.

Theorem A.2. *If V_* is a persistence module such that each V_r is finite dimensional, then there is a family $(I_\lambda)_{\lambda \in \Lambda}$, unique up to reordering, such that V_* is isomorphic to $\oplus_\lambda (I_\lambda)_{\mathbb{R}}$.*

Definition 10 (Barcode)**.** A *barcode* is a multiset of intervals. By Theorem A.2, there exists a barcode associated to each pointwise finite dimensional persistence module. It is called the *persistence barcode* associated to the persistence module.

If X is a metric space such that $\mathrm{PH}(\mathrm{VR}*(X))$ has a persistence barcode, then we denote the barcode corresponding to $\mathrm{PH}_k(\mathrm{VR}^*(X))$ by $\mathrm{dgm}_k^{\mathrm{VR}}(X)$.

A.2 Injective/Hyperconvex metric spaces

References for this section are [13, 5, 6].

Definition 11 (Hyperconvex space)**.** A metric space is called *hyperconvex* if for every family $(x_i, r_i)_{i \in I}$ of x_i in X and $r_i \geq 0$ such that $d(x_i, x_j) \leq r_i + r_j$ for each i, j in I, there exists a point x such that $d(x_i, x) \leq r_i$ for each i in I.

Definition 12. A metric space E is called injective if for each 1-Lipschitz map $f : X \to E$ and isometric embedding of X into $\tilde{X}$, there exists a 1-Lipschitz map $\tilde{f} : \tilde{X} \to E$ extending f.

For a proof of the following proposition, see [13, Proposition 2.2].

Proposition A.3. *A metric space is hyperconvex if and only if it is injective.*

Example A.4. For any set S, the space $\ell_\infty(S)$ of real valued functions on S with ℓ_∞ norm is a hyperconvex space.

Definition 13. (X, d) is a metric space, the space $\ell_\infty(X)$ is called the *Kuratowski space* and is denoted by $\kappa(X)$. The map $X \to \kappa(X)$, $p \mapsto d(p, \cdot)$ is an isometric embedding and it is called the *Kuratowski embedding.* Hence every metric space can be isometrically embedded in a hyperconvex space.

Proposition A.5. *Let X be a subspace of a hyperconvex space E. For $\epsilon \geq 0$, let $B_\epsilon(X)$ denote the open ϵ-neighborhood of X in E. Then the open Vietoris-Rips complex* $\mathrm{VR}^{2\,\epsilon}(X)$ *is homotopy equivalent to $B_\epsilon(X)$.*

Lemma A.6. *In a hyperconvex space, any non-empty intersection of open balls is contractible.*

Proof. A geodesic bicombing of a geodesic space S is a map $\gamma : S \times S \times [0,1] \to S$ such that for each x, y in S, $\gamma(x, y, \cdot) : [0,1] \to S$ is a constant speed length minimizing geodesic from x to y. Note given a point p in S, restricting γ to $S \times \{p\} \times [0,1]$ gives a deformation retraction of S onto p. By [13, Proposition 3.8], E has a geodesic bicombing γ such that for each x, y, x', y' in E and t in $[0,1]$,

$$d(\gamma(x,y,t), \gamma(x',y',t)) \leq (1-t)\, d(x,x') + t\, d(y,y').$$

In particular, by letting $x' = y' = z$, we get

$$d(\gamma(x,y,t), z) \leq \max(d(x,z), d(y,z)).$$

Hence, if x, y are contained in an open ball of z, then $\gamma(x,y,t)$ is contained in the same ball for each t in $[0,1]$. Therefore, if U is a non-empty intersection of open balls in E, then γ restricts to $U \times U \times [0,1] \to U$, which implies that U is contractible. □

Proof of Proposition A.5. Let

$$\mathcal{U} := \{B_\epsilon(x) \subseteq E : x \in X\}.$$

By Lemma A.6, $\mathcal{U}$ is a good cover of $B_\epsilon(X)$. By nerve lemma, $B_\epsilon(X)$ is homotopy equivalent to the Čech complex of $\mathcal{U}$. By the definition of hyperconvexity, given a finite set $x_1, \ldots, x_n$ in X, $B_\epsilon(x_i) \subseteq E$ has non-empty intersection if and only if $d(x_i, x_j) < 2\epsilon$ for each i, j. Hence the Čech complex of U is same with $\mathrm{VR}^{2\epsilon}(X)$. □

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