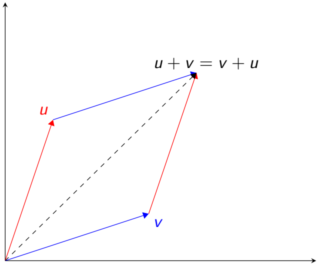
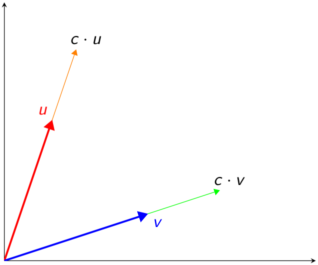
Recitation 1.1: Vector Spaces

TA: Nate Clause

Vector Spaces

- A *vector space* V over a field $\mathbb{F}$ consists of a set of elements, called *vectors* and two operations:





Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- (associativity of addition) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- (associativity of addition) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.
- (commutativity of addition) For all $u, v \in V$, $u + v = v + u$.

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- (associativity of addition) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.
- (commutativity of addition) For all $u, v \in V$, $u + v = v + u$.
- (additive identity) $\exists 0 \in V$ such that $0 + v = v + 0 = v$ for all $v \in V$.

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- (associativity of addition) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.
- (commutativity of addition) For all $u, v \in V$, $u + v = v + u$.
- (additive identity) $\exists 0 \in V$ such that $0 + v = v + 0 = v$ for all $v \in V$.
- (additive inverse) For all $v \in V$, $\exists(-v) \in V$ such that $v + (-v) = 0$.

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- For all $a, b \in \mathbb{F}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$.

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- For all $a, b \in \mathbb{F}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$.
- (multiplicative identity) $1 \in \mathbb{F}$ has $1 \cdot v = v$ for all $v \in V$.

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- For all $a, b \in \mathbb{F}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$.
- (multiplicative identity) $1 \in \mathbb{F}$ has $1 \cdot v = v$ for all $v \in V$.
- (multiplication distributes over vector addition) For all $a \in \mathbb{F}$ and $u, v \in V$:

$$a \cdot (u + v) = a \cdot u + a \cdot v$$

Vector Spaces

Definition

A *vector space* over a field $\mathbb{F}$ is a set V, along with operations $+ : V \times V \to V$ given by $(u, v) \mapsto u + v$ and $\cdot : \mathbb{F} \times V \to V$ given by $(c, v) \mapsto c \cdot v$, satisfying:

- For all $a, b \in \mathbb{F}$ and $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$.
- (multiplicative identity) $1 \in \mathbb{F}$ has $1 \cdot v = v$ for all $v \in V$.
- (multiplication distributes over vector addition) For all $a \in \mathbb{F}$ and $u, v \in V$:
$$a \cdot (u + v) = a \cdot u + a \cdot v$$
- (multiplication distributes over field addition) For all $a, b \in \mathbb{F}$ and $v \in V$:
$$(a + b) \cdot v = a \cdot v + b \cdot v$$

Vector Space Examples

Common examples of vector spaces include:

1.) $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$, $+$, $\cdot$ usual addition and multiplication of real numbers.

Vector Space Examples

Common examples of vector spaces include:

1.) $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$, $+$, $\cdot$ usual addition and multiplication of real numbers.

2.) $V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$, $+$ and $\cdot$ defined by:

$$\begin{aligned}(u_1, \ldots, u_n) + (v_1, \ldots, v_n) &:= (u_1 + v_1, \ldots, u_n + v_n) \\ c \cdot (v_1, \ldots, v_n) &:= (c \cdot v_1, \ldots, c \cdot v_n)\end{aligned}$$

Vector Space Examples

Common examples of vector spaces include:

1.) $V = \mathbb{R}$, $\mathbb{F} = \mathbb{R}$, $+$, $\cdot$ usual addition and multiplication of real numbers.

2.) $V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$, $+$ and $\cdot$ defined by:

$$\begin{aligned} (u_1, \dots, u_n) + (v_1, \dots, v_n) &:= (u_1 + v_1, \dots, u_n + v_n) \\ c \cdot (v_1, \dots, v_n) &:= (c \cdot v_1, \dots, c \cdot v_n) \end{aligned}$$

3.) Fix a field $\mathbb{F}$, $V = \mathbb{F}^n$, with $+$ and $\cdot$ defined coordinate-wise as in 2.)

- Common field: $\mathbb{Z}_2$, also denote $\mathbb{Z}/2\mathbb{Z}$. Set is $\{0, 1\}$ and operations are addition modulo 2 and multiplication modulo 2.

Direct Sums

Definition

Let V, W be vector spaces over a field $\mathbb{F}$. The (external) *direct sum* $V \oplus W$, is a vector space with set $V \times W := \{(v, w) \mid v \in V,\ w \in W\}$, and operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$c \cdot (v_1, w_1) = (c \cdot v_1, c \cdot w_1)$$

Direct Sums

Definition

Let V, W be vector spaces over a field $\mathbb{F}$. The (external) *direct sum* $V \oplus W$, is a vector space with set $V \times W := \{(v, w) \mid v \in V,\ w \in W\}$, and operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$c \cdot (v_1, w_1) = (c \cdot v_1, c \cdot w_1)$$

- Example: for $\mathbb{F}$ a field, $\mathbb{F}^n$ is a vector space, with $\mathbb{F}^n = \oplus_{i=1}^{n} \mathbb{F}$.

Basis of a Vector Space

Definition

A *basis* B of a vector space V over a field $\mathbb{F}$ is a subset $B \subseteq V$ satisfying the following:

1.) Linear independence: for every finite subset $\{v_1, \ldots, v_n\} \subseteq B$, if $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$ for some $c_1, c_2, \ldots, c_n \in \mathbb{F}$, then $c_1 = c_2 = \ldots = c_n = 0$.

Basis of a Vector Space

Definition

A *basis* B of a vector space V over a field $\mathbb{F}$ is a subset $B \subseteq V$ satisfying the following:

1.) Linear independence: for every finite subset $\{v_1, \ldots, v_n\} \subseteq B$, if $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$ for some $c_1, c_2, \ldots, c_n \in \mathbb{F}$, then $c_1 = c_2 = \ldots = c_n = 0$.

2.) Spanning: for every $v \in V$, there exists $c_1, c_2, \ldots, c_n \in \mathbb{F}$ and $v_1, v_2, \ldots, v_n \in B$ such that $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

Basis of a Vector Space

Definition

A *basis* B of a vector space V over a field $\mathbb{F}$ is a subset $B \subseteq V$ satisfying the following:

1.) Linear independence: for every finite subset $\{v_1, \dots, v_n\} \subseteq B$, if $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ for some $c_1, c_2, \dots, c_n \in \mathbb{F}$, then $c_1 = c_2 = \dots = c_n = 0$.

2.) Spanning: for every $v \in V$, there exists $c_1, c_2, \dots, c_n \in \mathbb{F}$ and $v_1, v_2, \dots, v_n \in B$ such that $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$.

Standard example: if $V = \mathbb{F}^n$, then $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)\}$ is a basis for V. We denote by e_i the basis element with 1 in coordinate i.

Basis of a Vector Space

Definition

A *basis* B of a vector space V over a field $\mathbb{F}$ is a subset $B \subseteq V$ satisfying the following:

1.) Linear independence: for every finite subset $\{v_1, \ldots, v_n\} \subseteq B$, if $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$ for some $c_1, c_2, \ldots, c_n \in \mathbb{F}$, then $c_1 = c_2 = \ldots = c_n = 0$.

2.) Spanning: for every $v \in V$, there exists $c_1, c_2, \ldots, c_n \in \mathbb{F}$ and $v_1, v_2, \ldots, v_n \in B$ such that $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

Standard example: if $V = \mathbb{F}^n$, then $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$ is a basis for V. We denote by e_i the basis element with 1 in coordinate i. If V has a basis B with n elements, then we say V is an *n-dimensional* vector space.

Linear Subspace

Definition

A subset W of a vector space V over $\mathbb{F}$ is a *linear subspace* if W with the operations of V restricted to W is a vector space over $\mathbb{F}$. Equivalently, $W \subseteq V$ is a linear subspace if for all $a, b \in \mathbb{F}$ and $u, v \in W$, $au + bv \in W$.

Linear Subspace

Definition

A subset W of a vector space V over $\mathbb{F}$ is a *linear subspace* if W with the operations of V restricted to W is a vector space over $\mathbb{F}$. Equivalently, $W \subseteq V$ is a linear subspace if for all $a, b \in \mathbb{F}$ and $u, v \in W$, $au + bv \in W$.

Examples:

- If $m \leq n$, then $\mathbb{R}^m$ can be viewed as a linear subspace of $\mathbb{R}^n$ by embedding $(v_1, \ldots, v_m) \mapsto (v_1, \ldots, v_m, 0, \ldots, 0)$.

Linear Subspace

Definition

A subset W of a vector space V over $\mathbb{F}$ is a *linear subspace* if W with the operations of V restricted to W is a vector space over $\mathbb{F}$. Equivalently, $W \subseteq V$ is a linear subspace if for all $a, b \in \mathbb{F}$ and $u, v \in W$, $au + bv \in W$.

Examples:

- If $m \leq n$, then $\mathbb{R}^m$ can be viewed as a linear subspace of $\mathbb{R}^n$ by embedding $(v_1, \ldots, v_m) \mapsto (v_1, \ldots, v_m, 0, \ldots, 0)$.
- For $W = \{w_1, \ldots, w_n\} \subset V$, define

$$\mathrm{span}(W) := \left\{ \sum_{i=1}^{n} c_i w_i \mid c_i \in \mathbb{F} \right\}.$$

  $\mathrm{span}(W)$ is a linear subspace of V.