Recitation 1.3: Quotient Spaces

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Quotient of a Vector Space

Definition

Let W be a linear subspace of the vector space V over $\mathbb{F}$. Define an equivalence relation $\sim$ on V by stating $u \sim v$ if $u - v \in W$. Then the *equivalence class* of $v \in V$ is denoted $[v]$. It is also sometimes referred to as a *coset*, denoted $v + W$, as we have $[v] = \{v + w \mid w \in W\}$.

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$$[u] + [v] = [u + v] \; \forall u, v \in V$$
$$c[v] = [cv] \; \forall c \in \mathbb{F}, v \in V$$

With these operations, V/W is a vector space over $\mathbb{F}$.

Quotient Space Examples

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- In general, if W has basis $\{w_1, \ldots, w_n\}$ and V has basis $\{w_1, \ldots, w_n, v_1, \ldots, v_m\}$, then $\{[v_1], \ldots, [v_m]\}$ is a basis for V/W.

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- Corollary: if V is an n dimensional vector space, and W is an m dimensional vector space with $m < n$, then V/W is an $n - m$ dimensional vector space.

Composition of Linear Transformations

- If $f : V \to W$ and $g : W \to X$ are linear transformations, then $g \circ f : V \to X$ is a linear transformation, with $(g \circ f)(v) = g(f(v))$.
- Later in this course, we will frequently encounter such f and g with $\operatorname{im}(f) \subseteq \ker(g)$. We will then compute $\ker(g)/\operatorname{im}(f)$. Let's work an example:

Computation Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T_f = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T_g = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{bmatrix}$. Compute $\operatorname{rref}(T_f)$ and $\operatorname{rref}(T_g)$:

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$$\mathrm{rref}(T_f) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathrm{rref}(T_g) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Computational Example

From $\text{rref}(T_f)$, we can see the image of f is spanned by $B_f := \{(1, 1, 1)\}$.

From $\text{rref}(T_g)$, we can compute the kernel of g is spanned by $\{(1, 1, 0), (0, 0, 1)\}$. We can rewrite the basis for $\ker(g)$ equivalently as $B_g := \{(1, 1, 1), (0, 0, 1)\}$.

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We get basis for $\ker(g)/\,\mathrm{im}(f)$ of $[(0,0,1)] = (0,0,1) + \mathrm{im}(f)$.