Introduction to Metric Spaces

TA: Nate Clause

What is a Metric Space

Definition

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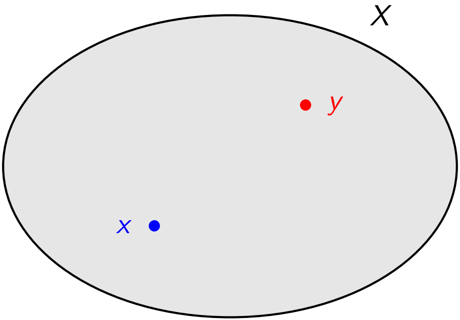
- If X is understood, sometimes the subscript is dropped and we just write d.

What is a Metric Space



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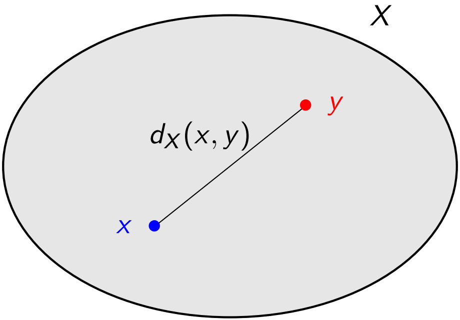

Non-negativity

Claim: If (X, d_X) is a metric space, then for all $x, y \in X$, $d(x, y) \geq 0$.

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Proof.

Let $x, y \in X$. Then we have:

$$2d(x,y) \overset{(ii)}{=} d(x,y) + d(y,x) \overset{(iii)}{\geq} d(x,x) \overset{(i)}{=} 0$$

□

Metric Space Example

- $X = \mathbb{R}$, $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(x, y) = |x - y|$. We show that this is a metric space:

Proof.

To see (i): $|x - y| = 0 \iff x - y = 0 \iff x = y$.

To see (ii): $x - y = -(y - x) \implies |x - y| = |y - x|$.

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To see (ii): $x - y = -(y - x) \implies |x - y| = |y - x|$. As an exercise, show that for all $x, y \in \mathbb{R}$, $|x + y| \overset{*}{\leq} |x| + |y|$. Using this fact, for all $x, y, z \in \mathbb{R}$, we have:

$$\begin{aligned} d(x, z) = |x - z| &= |(x - y) + (y - z)| \\ &\overset{*}{\leq} |x - y| + |y - z| = d(x, y) + d(y, z) \end{aligned}$$

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(iii) For all $x \in X$, $\|x\| = 0$ if and only if $x = 0$.

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- The function $\| \cdot \| : \mathbb{R} \to \mathbb{R}$ given by $\|x\| = |x|$.
- The Euclidean norm: $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ given by $\|x\| = \sqrt{x \cdot x}$, where $x \cdot x$ refers to the *dot product*.

p-norms

An important class of norms are the p-norms:

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For $n \in \mathbb{N}$, $p \in \mathbb{R}$ with $p \in [1, \infty)$, the p-norm is a norm on $\mathbb{R}^n$ given by:

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Examples:

- If $n = 1$, $p = 1$, we get the absolute value norm on $\mathbb{R}$.
- If $n \in \mathbb{N}$, $p = 2$, we get the Euclidean norm.

p-norms

Definition

For $n \in \mathbb{N}$, $p = \infty$, the p-norm is a norm on $\mathbb{R}^n$ given by:

$$\|x\|_\infty := \left(\max_{1 \leq i \leq n} |x_i| \right)$$

Norms to Metrics

Proposition

Let X be a vector space over $\mathbb{R}$ and $\|\cdot\|$ a norm on X. Define $d_X : X \times X \to \mathbb{R}$ by $d_X(x, y) := \|x - y\|$. Then (X, d_X) is a metric space.

Proof: **Exercise**

Minkowski distances

Definition

For $n \in \mathbb{N}$, $p \in [1, \infty]$, the *Minkowski distance of order p* is given by $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where

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Example

If $n = 2, p = 2$, then we have:

$$d_2((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

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Corollary

For $p \in [1, \infty]$, $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a metric.

Unit spheres in Minkowski metric spaces

- For $n \in \mathbb{N}$, $p \in [1, \infty]$, let $(\mathbb{R}^n, d_p)$ be the Minkowski metric space of order p. The *unit sphere* is defined as:

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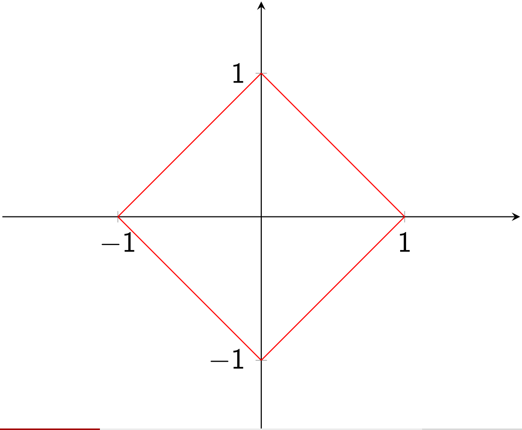
- Unit sphere with $n = 2,\ p = 1$:



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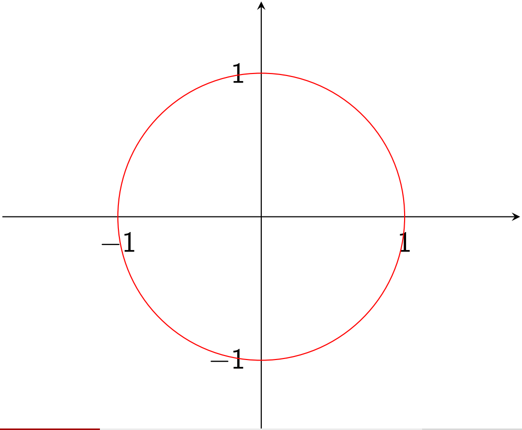

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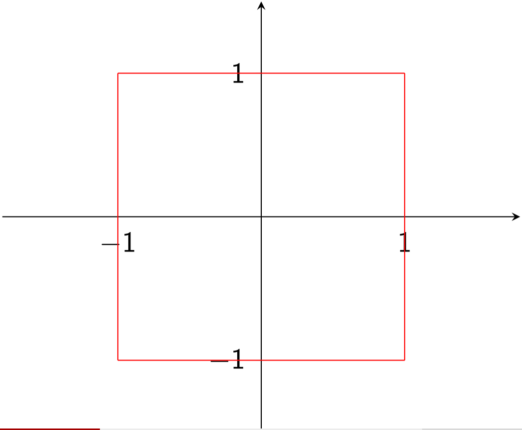

Modified definitions

There are some special types of spaces similar to metric spaces:

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A *pseudometric space* has d satisfy symmetry and the triangle inequality, but allows for $x \neq y$, with $d(x, y) = 0$.

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Definition

An *extended metric space* allows d to take on the value $d(x, y) = \infty$. All other properties of a metric remain, under the conventions $\infty = \infty$, $\infty \leq \infty$, $\infty + \infty = \infty$, and for $c \in \mathbb{R}$, $c < \infty$ and $c + \infty = \infty$.

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An *ultrametric space* satisfies conditions (i) and (ii) but replaces the triangle inequality with the *strong triangle inequality*:

$$\forall x, y, z \in X, \;\; d(x,z) \leq \max(d(x,y), d(y,z))$$

Part 2:

Part 2 video: Metric Spaces in Practice!