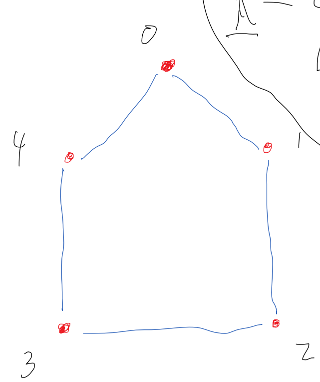
Simplicial Complexes:

- Non-example
- Vietoris-Rips complex
- Witness complex
- Sublevel set complex

- In practice, will consider abstract simplicial complexes
  - Key fact: all faces of every simplex must be in complex for it to be a simplicial complex:



$K = \{[0],[1],[2],[3],[4],[0,1],[1,2],[2,3],[3,4],[4,0],[0,1,4]\}$ is K a simplicial complex? No

$K_1 = K \cup \{[1,4]\}$ $\quad K_2 = K \setminus \{[0,1,4]\}$

Vietoris-Rips complex:

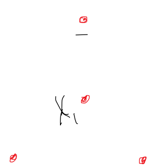
Vietoris-Rips complex on (X,d) at scale parameter $r \geq 0$ is $\underline{VR}^r(X) := \{\underline{\sigma} \subseteq X \mid \operatorname{diam}(\sigma) \leq r\}$

R^r

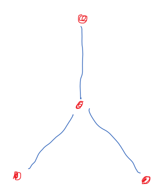
$$\sigma = (\sigma_1, \ldots, \sigma_n) \qquad \max_{1 \leq i,j \leq n} d(\sigma_i, \sigma_j)$$

ex) $d_X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$

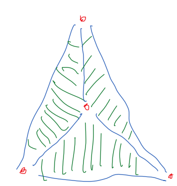
$VR^0(X)$:



$VR^1(X)$:



$VR^2(X)$



Exercise: for $\underline{K_1, K_2}$ from earlier, find a metric space $\underline{X}$ and scale parameters $\underline{r_1}$ and $\underline{r_2}$ such that $\underline{VR^{r_1}(X)} = \underline{K_1}$ and $VR^{r_2}(X) = \underline{K_2}$

Witness complex:

- Vietoris-Rips complex can be "too big". Idea: only consider some of the vertices.

<u>Def</u>: for (X, d_X), the witness complex $\underline{W(L, X, r)}$ at scale r has:

- $\underline{\text{0-simplices: } L \subseteq X}$ "landmarks"
- $\underline{[l_0, \ldots, l_k]} \in \underline{W(L, X, r)}$ if $\exists\, \underline{x} \in X$ such that:

$$\max(\underline{d(l_0, x)}, d(l_1, x), \ldots, d(\underline{l_k, x})) \leq \underline{r} + \underline{m_k(x)}$$

x is called the witness

where $m_k(x)$ is distance from x to its $k+1$-st closest landmark.

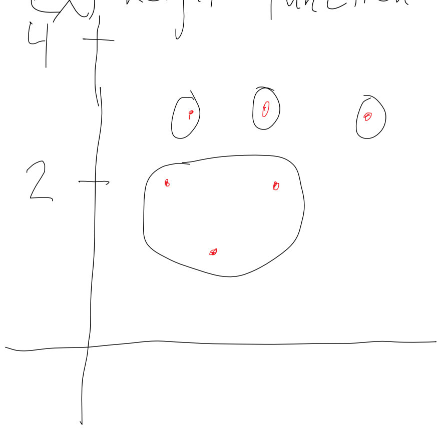
ex)

Sublevel set complex:

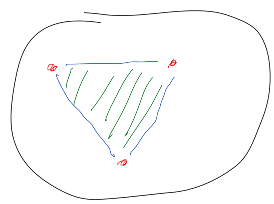
Let (X, d_X) be a metric space and $f: X \to \mathbb{R}$. Fix $r \in \mathbb{R}$.
Define the sublevel set complex at parameter r, $K^r(X,f)$ as:

$$K^r(X,f) := \left\{ [x_0, x_1, \ldots, x_n] \subseteq X : \max_{1 \le i \le n} f(x_i) \le r \right\}$$

ex) height function f:



$\to K^2(X,f)$:



$\vdots$

$K^4(X,f)$

Exercise: Suppose $K = pow(X)$. Let $f: K \to \mathbb{R}$ be a function, and set $K^r(X,f) = \{\sigma \in K : f(\sigma) \le r\}$. Is $K^r(X,f)$ a simplicial complex for all r, f, X? If not, what can we require about f to make this hold?