- Review simplicial homology w/ $\mathbb{Z}_2$ coefficients + examples
- Discuss other coefficients
- Examples in Java Plex

Simplicial homology w/ $\mathbb{Z}_2$ coefficients:

Recall: C_k: group of k-chains: elements are a sum of simplicies

boundary map: $\partial_k : C_k \to C_{k-1}$ $\qquad \partial_k(\sigma) = \sum_{i=0}^{k} (v_0, \ldots, \hat{v}_i, \ldots, v_k)$

$\boxed{\sigma = (v_0, v_1, \ldots, v_k)}$

ex) $\sigma = [v_0, v_1, v_2, v_3]$, then $\partial_3(\sigma) = [0,1,2] + [0,1,3] + [0,2,3] + [1,2,3]$

Linear algebra format:

Can put boundary maps into matrix form:

if C_2 has basis $\{[0,1,2], [0,1,3], [1,2,3]\}$ and

C_1 has basis $\{[0,1], [0,2], [0,3], [1,2], [1,3], [2,3]\}$ then:

$\partial_2 : C_2 \to C_1$

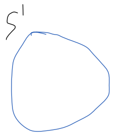
$$
D_2 = \begin{array}{c|ccc}
 & [0,1,2] & [0,1,3] & [1,2,3] \\
\hline
[0,1] & 1 & 1 & 0 \\
[0,2] & 1 & 0 & 0 \\
[0,3] & 0 & 1 & 0 \\
[1,2] & 1 & 0 & 1 \\
[1,3] & 0 & 1 & 1 \\
[2,3] & 0 & 0 & 1
\end{array}
$$

Recall: $H_k(X) = \ker \partial_k / \operatorname{im} \partial_{k+1}$

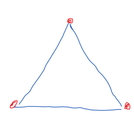
ex) let $X = S^n$ (n-sphere) for $n \geq 1$. We know S^n is homeomorphic to all faces of $(n+1)$-simplex:

S^1

ex: S^1



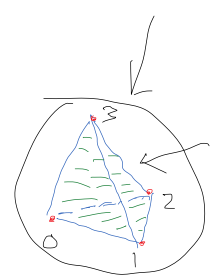
$\cong$



when $n=1$

S^2

Lets compute $\boxed{H_k(S^2; \mathbb{Z}_2)}$ with linear algebra approach.



basis for: $C_3(S^2)$: $\emptyset$ 0

basis

$C_2(S^2)$: $\{[0,1,2], [0,1,3], [0,2,3], [1,2,3]\}$ shorthand 012

$C_1(S^2)$: $\{[0,1], [0,2], [0,3], [1,2], [1,3], [2,3]\}$

$C_1(S^2): \{[0,1],[0,2],[0,3],[1,2],[1,3],[2,3]\}$

$C_0(S^2): \{[0],[1],[2],[3]\}$

$H_2(S^2;\mathbb{Z}_2) = \ker \partial_2 / \underbrace{\operatorname{im}\partial_3}_{=0}$

$= \ker \partial_2$

$\mathbb{Z}_2$

$2 \to 0$

$0 \quad 0$

$D_3 = 0$

$$D_2 = \begin{array}{c|cccc} & 012 & 013 & 023 & 123 \\ \hline 01 & 1 & 1 & 0 & 0 \\ 02 & 1 & 0 & 1 & 0 \\ 03 & 0 & 1 & 1 & 0 \\ 12 & 1 & 0 & 0 & 1 \\ 13 & 0 & 1 & 0 & 1 \\ 23 & 0 & 0 & 1 & 1 \end{array}$$

$$D_1 = \begin{array}{c|cccccc} & 01 & 02 & 03 & 12 & 13 & 23 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

$D_0 = 0 \left(\Rightarrow \ker(\overset{\partial_0}{D_0}) = \operatorname{span}(0,1,2,3) \right)$

perform column reduction:

$$D_2': \begin{array}{c|cccc} & \boxed{012} & \boxed{013+012} & \boxed{023+013+012} & \boxed{123+023+013+012} \\ \hline \to 01 & 1 & 0 & 0 & 0 \\ \to 02 & 1 & 1 & 0 & 0 \\ 03 & 0 & 1 & 0 & 0 \\ \to 12 & 1 & 1 & 1 & 0 \\ 13 & 0 & 1 & 1 & 0 \\ 23 & 0 & 0 & 1 & 0 \end{array}$$

$\ker \partial_2$

$\operatorname{im}(\partial_2) \subseteq C_1$

$\ker(\partial_2) = \operatorname{span}(012+013+023+123)$

$\ker(\partial_2) = \mathrm{span}(012 + 013 + 023 + 123)$

$\mathrm{im}(\partial_2) = \mathrm{span}(\{01+02+12,\ 02+03+12+13,\ 12+13+23\})$

$$\ker(\partial_2) / \mathrm{im}(\partial_3) = \boxed{\mathbb{Z}_2} = H_2(S^2; \mathbb{Z}_2)$$

Exercise: find ∂_1, use to compute $H_1(S^2; \mathbb{Z}_2)$, $H_0(S^2; \mathbb{Z}_2)$

Impact of Coefficients

- benefits of $\mathbb{Z}_2$ coefficients:
  - Simplicity of operations
  - geometric interpretation: boundary of a simplex is sum of its proper faces

- if homology uses non-$\mathbb{Z}_2$ coefficients, we must orient (order) simplices, and this affects boundary maps. $-1 \neq 1$

$[0,1,2] \xrightarrow{-1} [0,2,1]$

Now:

$$\boxed{\partial_k([v_0, \dots, v_k]) = \sum_{i=0}^{k} \underbrace{(-1)^i}_{\neq 1} [v_0, \dots, \hat{v_i}, \dots, v_k]}$$

//

Now:

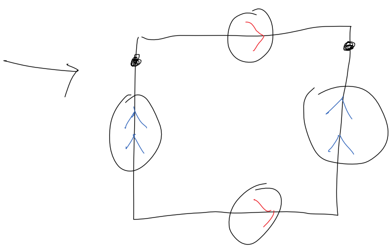
$$\boxed{\partial_k([v_0, \dots, v_k]) = \sum_{i=0} (-1)^i [v_0, \dots, \hat{v_i}, \dots, v_k]}$$

$\mathbb{Z}$

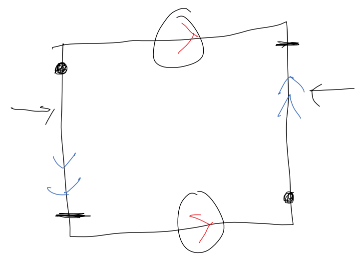
ex) $\partial_2([0,1,2]) = [1,2] - [0,2] + [0,1]$

Universal coefficient thm

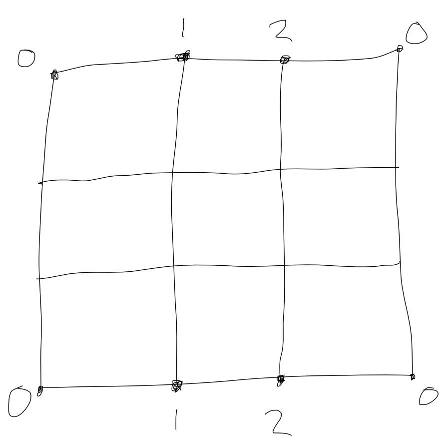
ex) torus:

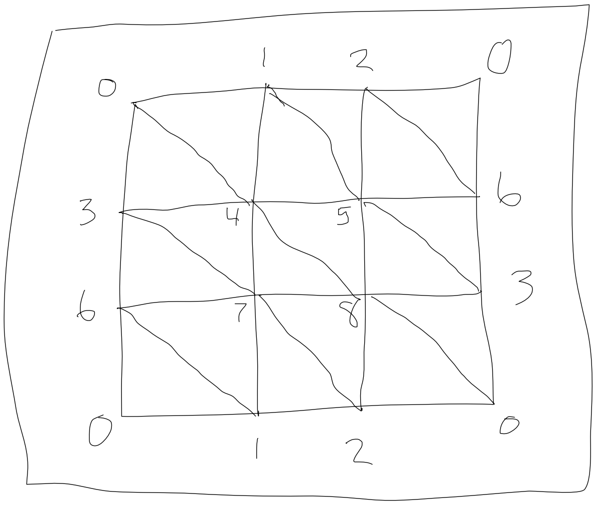


Klein bottle:



Find triangulation (simplicial complex) for klein bottle:





basis for $\underline{C_2(K)}$: $\{[0,1,4], [0,3,4], [1,2,5], [1,4,5], [0,2,6], [2,5,6],$
$[3,4,7], [3,6,7], [4,5,8], [4,7,8], [5,6,3], [5,8,3],$
$[0,1,6], [1,6,7], [2,7,8], [1,2,7], [0,3,8], [0,2,8]\}$

basis for $C_1(K)$: $\{[0,1], [0,2], [0,3], [0,4], [0,6],$
$[1,2], [1,4], [1,5], [1,6], [1,7],$
$[2,5], [2,6], [2,7], [2,8],$
$[3,4], [3,5], [3,6], [3,7], [3,8],$
$[4,5], [4,7], [4,8],$
$[5,6], [5,8], [6,7], [7,8]\}$

D_2

- with $\mathbb{Z}_2$ vs. $\mathbb{Z}_3$ coefficients, computations change.

ex) in $\mathbb{Z}_2$ coefficients: $\partial_2(\underline{014+034}) = 01+04+14+03+34+04 = \underline{01+14+03+34}$

in $\mathbb{Z}_3$ " : $\partial_2(\underline{014+034}) = 01+14+03+34+2(04)$

$= \underline{01+14+03+34-04}$

Can compute simplicial homology to get:

$\underline{\dim(H_2(K;\mathbb{Z}_2)) = 1}$ but $\underline{\dim(H_2(K;\mathbb{Z}_3)) = 0}$

$\dim(H_1(K;\mathbb{Z}_2)) = 2$ but $\dim(H_1(K;\mathbb{Z}_3)) = 1$

$\dim(H_2(S^2;\mathbb{Z}_2)) = 1$ $\dim(H_2(S^2;\mathbb{Z}_3))$