Recitation 1.2: Linear Transformations

TA: Nate Clause

Linear Transformations

Definition

Let V and W be vector spaces over the same field $\mathbb{F}$. A *linear transformation*, sometimes called a *linear map* is a function $f : V \to W$ such that:

Linear Transformations

Definition

Let V and W be vector spaces over the same field $\mathbb{F}$. A *linear transformation*, sometimes called a *linear map* is a function $f : V \to W$ such that:

(i) For all $u, v \in V$: $f(u + v) = f(u) + f(v)$.

Linear Transformations

Definition

Let V and W be vector spaces over the same field $\mathbb{F}$. A *linear transformation*, sometimes called a *linear map* is a function $f : V \to W$ such that:

(i) For all $u, v \in V$: $f(u + v) = f(u) + f(v)$.

(ii) For all $c \in \mathbb{F}$ and $v \in V$: $f(c \cdot v) = c \cdot f(v)$.

Linear Transformations

Definition

Let V and W be vector spaces over the same field $\mathbb{F}$. A *linear transformation*, sometimes called a *linear map* is a function $f : V \to W$ such that:

(i) For all $u, v \in V$: $f(u + v) = f(u) + f(v)$.

(ii) For all $c \in \mathbb{F}$ and $v \in V$: $f(c \cdot v) = c \cdot f(v)$.

- If $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ for some $m, n \geq 1$, then we can represent a linear transformation $f : V \to W$ by an $m \times n$ matrix T, with entries in $\mathbb{F}$, such that for all $v \in V$, $f(v) = Tv$.

Linear Transformation Examples

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(v) = c \cdot v$ for some fixed $c \in \mathbb{R}$. Then T is given by the diagonal matrix with entries c along the diagonal.

Linear Transformation Examples

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(v) = c \cdot v$ for some fixed $c \in \mathbb{R}$. Then T is given by the diagonal matrix with entries c along the diagonal.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be given by $f((x, y)) = x + y$. Then T is given by $T = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

Linear Transformation Examples

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(v) = c \cdot v$ for some fixed $c \in \mathbb{R}$. Then T is given by the diagonal matrix with entries c along the diagonal.
- Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be given by $f((x, y)) = x + y$. Then T is given by $T = \begin{bmatrix} 1 & 1 \end{bmatrix}$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(v) = v + 1$. Then f is not a linear transformation, as $f(0) = 1$, but $f(0) = f(0 + 0) = f(0) + f(0) = 2$, and $1 \neq 2$.

Injective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *injective*, or also *one to one*, if for all $v_1, v_2 \in V$, $f(v_1) = f(v_2)$ implies $v_1 = v_2$.

Injective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *injective*, or also *one to one*, if for all $v_1, v_2 \in V$, $f(v_1) = f(v_2)$ implies $v_1 = v_2$.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$.

Injective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *injective*, or also *one to one*, if for all $v_1, v_2 \in V$, $f(v_1) = f(v_2)$ implies $v_1 = v_2$.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$.
- $f : \mathbb{R}^n \to \mathbb{R}^m$ when $n \leq m$, with $f(x) = (x, 0)$.

Surjective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *surjective*, or also *onto*, if for all $w \in W$, there exists a $v \in V$ such that $f(v) = w$.

Surjective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *surjective*, or also *onto*, if for all $w \in W$, there exists a $v \in V$ such that $f(v) = w$.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$.

Surjective Linear Transformations

Definition

A linear transformation $f : V \to W$ is called *surjective*, or also *onto*, if for all $w \in W$, there exists a $v \in V$ such that $f(v) = w$.

Examples:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$.
- $f : \mathbb{R}^n \to \mathbb{R}^m$ when $n \geq m$, with $f((x_1, x_2, \ldots, x_n)) = (x_1, x_2, \ldots, x_m)$.

Isomorphism of Vector spaces

Definition

If $f : V \to W$ is injective and surjective, we say it is an *isomorphism* of vector spaces.

Isomorphism of Vector spaces

Definition

If $f : V \to W$ is injective and surjective, we say it is an *isomorphism* of vector spaces. If V and W are vector spaces and there exists an isomorphism $f : V \to W$, we say V and W are *isomorphic*, which we denote by $V \cong W$.

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$ is an isomorphism.

Isomorphism of Vector spaces

Definition

If $f : V \to W$ is injective and surjective, we say it is an *isomorphism* of vector spaces. If V and W are vector spaces and there exists an isomorphism $f : V \to W$, we say V and W are *isomorphic*, which we denote by $V \cong W$.

- $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(v) = c \cdot v$, for any $c \in \mathbb{R}$ with $c \neq 0$ is an isomorphism.
- Exercise: if n and m are finite, then $\mathbb{F}^n \cong \mathbb{F}^m$ if and only if $n = m$.
- Exercise: if V is an n-dimensional vector space over $\mathbb{F}$, then $V \cong \mathbb{F}^n$.

Kernel of a Linear Transformation

Definition

Let $f : V \to W$ be a linear transformation. The *kernel* of f, denoted $\ker(f)$ is given by:

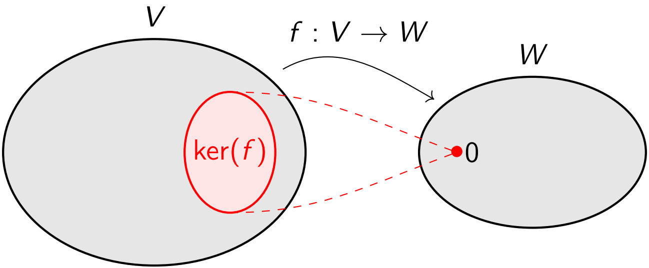
$$\ker(f) := \{v \in V \mid f(v) = 0\}$$

Kernel of a Linear Transformation

Definition

Let $f : V \to W$ be a linear transformation. The *kernel* of f, denoted $\ker(f)$ is given by:

$$\ker(f) := \{v \in V \mid f(v) = 0\}$$



Kernel Examples

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f((x, y)) = x + y$. $\ker(f)$ is the one-dimensional subspace consisting of all elements $(x, y) \in \mathbb{R}^2$ with $x = -y$. This subspace has basis $\{(1, -1)\}$.

Kernel Examples

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f((x, y)) = x + y$. $\ker(f)$ is the one-dimensional subspace consisting of all elements $(x, y) \in \mathbb{R}^2$ with $x = -y$. This subspace has basis $\{(1, -1)\}$.
- Suppose $n > m$ and $f : \mathbb{F}^n \to \mathbb{F}^m$ is projection onto the first m coordinates. Then the basis for $\ker(f)$ is $\{e_i\}_{m+1 \leq i \leq n}$, where e_i is the standard i-th basis element for $\mathbb{F}^n$.

Calculating Kernel of a Linear Transformation

- Computing kernel of a linear transformation:
  - Step 1: put linear transformation into matrix form.

Calculating Kernel of a Linear Transformation

- Computing kernel of a linear transformation:
  - Step 1: put linear transformation into matrix form.
  - Step 2: put the matrix in reduced row-echelon form (rref).

Calculating Kernel of a Linear Transformation

- Computing kernel of a linear transformation:
  - Step 1: put linear transformation into matrix form.
  - Step 2: put the matrix in reduced row-echelon form (rref).
  - Step 3: convert rref matrix to system of equations.

Calculating Kernel of a Linear Transformation

- Computing kernel of a linear transformation:
  - Step 1: put linear transformation into matrix form.
  - Step 2: put the matrix in reduced row-echelon form (rref).
  - Step 3: convert rref matrix to system of equations.
  - Step 4: find solution to system of equations.

Calculating Kernel of a Linear Transformation

- Computing kernel of a linear transformation:
  - Step 1: put linear transformation into matrix form.
  - Step 2: put the matrix in reduced row-echelon form (rref).
  - Step 3: convert rref matrix to system of equations.
  - Step 4: find solution to system of equations.
  - Step 5: parameterize the solution to form a basis for the kernel.

Kernel Calculation Example

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T = \begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix}$.

Kernel Calculation Example

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T = \begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix}$.

Step 2: We convert T to $\mathrm{rref}(T)$:

$$\begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{r1 \to r1 - r3} \begin{bmatrix} 1 & 2 & 0 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[r3 \mapsto r3 - 2r1]{r2 \mapsto r2 - 4r1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 7 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{r2 \mapsto r2 - 8r3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[r1 \mapsto r1 - 2r2]{r3 \mapsto r3 + r2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Kernel Calculation Example

Step 3: This rref matrix gives the following system of equations:

$$1x + 0y + 2z = 0$$
$$0x + 1y - 1z = 0$$

Step 4: We can solve, yielding $y = z$ and $x = -2z$.
Step 5: We can let z be the free variable. Setting $z = 1$, we get a basis for $\ker(T)$, hence $\ker(f)$, is given by $\{(-2, 1, 1)\}$.

Image of a Linear Transformation

Definition

Let $f : V \to W$ be a linear transformation. The *image* of f, denoted $\mathrm{im}(f)$, is given by:

$$\mathrm{im}(f) := \{w \in W \mid \exists v \in V \text{ with } f(v) = w\}$$

Image of a Linear Transformation

Definition

Let $f : V \to W$ be a linear transformation. The *image* of f, denoted $\mathrm{im}(f)$, is given by:

$$\mathrm{im}(f) := \{w \in W \mid \exists v \in V \text{ with } f(v) = w\}$$

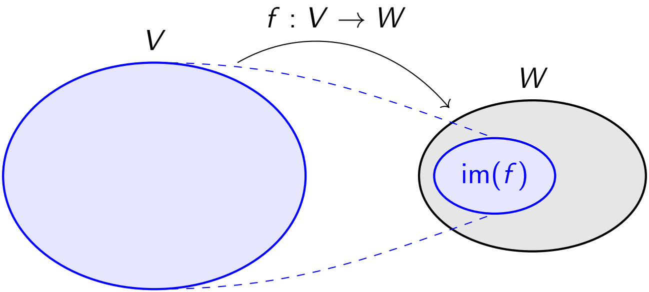


Image Examples

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = cx$ for some constant $c \neq 0$. Then $\text{im}(f) = \mathbb{R}$ as $f(0) = 0$ and for any $y \in \mathbb{R} \setminus \{0\}$, $f(\frac{y}{c}) = y$.

Image Examples

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = cx$ for some constant $c \neq 0$. Then $\text{im}(f) = \mathbb{R}$ as $f(0) = 0$ and for any $y \in \mathbb{R} \setminus \{0\}$, $f(\frac{y}{c}) = y$.
- Suppose $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, 2x)$. Then the image of f is the subspace of $\mathbb{R}^2$ spanned by $\{(1, 2)\}$.

Image Examples

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = cx$ for some constant $c \neq 0$. Then $\text{im}(f) = \mathbb{R}$ as $f(0) = 0$ and for any $y \in \mathbb{R} \setminus \{0\}$, $f(\frac{y}{c}) = y$.
- Suppose $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, 2x)$. Then the image of f is the subspace of $\mathbb{R}^2$ spanned by $\{(1, 2)\}$.
- Suppose $m < n$ and $f : \mathbb{F}^m \to \mathbb{F}^n$ is the inclusion into the first m coordinates. Then the basis for $\text{im}(f)$ is $\{e_i\}_{1 \leq i \leq m}$, where e_i is the i-th standard basis element in $\mathbb{F}^n$.

Computing the Image of a Linear Transformation

- Computing image of a linear transformation:
  - Step 1: put linear transformation in matrix form.
  - Step 2: put matrix in rref.

Computing the Image of a Linear Transformation

- Computing image of a linear transformation:
  - Step 1: put linear transformation in matrix form.
  - Step 2: put matrix in rref.
  - Step 3: use pivot columns to extract basis vectors from columns of the original matrix.

Image Calculation Example

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T = \begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix}$. The column vectors always span the image, but may not form a basis, as they are not necessarily linearly independent.

Image Calculation Example

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T = \begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix}$. The column vectors always span the image, but may not form a basis, as they are not necessarily linearly independent.

Step 1,2: From before, we computed the rref for T as:

$$\mathrm{rref}(T) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Image Calculation Example

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T = \begin{bmatrix} 3 & 5 & 1 \\ 4 & 1 & 7 \\ 2 & 3 & 1 \end{bmatrix}$. The column vectors always span the image, but may not form a basis, as they are not necessarily linearly independent.

Step 1,2: From before, we computed the rref for T as:

$$\mathrm{rref}(T) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: The pivot columns are columns 1 and 2. These columns in the original matrix give a basis for T. Hence, a basis for $\mathrm{im}(f)$ is given by $\{(3,4,2),(5,1,3)\}$.