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Competence

RTQF Level: 6

Credits: 10

Sector: ICT

Sub-sector: Information technology

Learning hours

 100

Issue date: August, 2020

By the end of the module, the trainee will be able to:

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- ❑ Apply introduction to Mathematical Logic
- ❑ Apply Counting theory and Discrete probability
- ❑ Apply Number theory and Cryptograph
- ❑ Apply Introduction to Graph theory and Finite automata
- ❑ Apply Sequences and Series
- ❑ References

Wednesday,
July 12,
2023

Introduction to Mathematical Logic

1. Propositions
2. Basic connectives
3. Truth tables
4. Logical equivalences
5. Validity of arguments

1. Propositions

Introduction to Propositions

- ▶ **Propositions:** A proposition is a statement which can be either **true** or **false** but **not both**.
- ▶ **Example:**
 - ✓ Kigali is capital of Rwanda
 - ✓ Burundi is our country
 - ✓ $2+3=5$
 - ✓ $1+0=0$
- Propositions are represented by small letters : p, q, r, s, t, \dots
- The truth or falsity of a proposition is called its **truth value**
- If a proposition is true its **truth** value is **1** and if it is **false** its truth value is **0**

2.Basic connectives

- ▶ **Logical connectives:** words or phrases like **not, and, or ,ifthen, if and only if** , that are used to form a new proposition from new propositions are known as **logical connectives**.
- ▶ New propositions are called **compound propositions**
- ▶ Original propositions are called **primitives** or **components** of **compound propositions**.
- ▶ Propositions which do not contain logical connectives are called **simple propositions**.

➤ **Negation:**

- ✓ A proposition obtained by inserting word “**NOT**” is called negation, the negation of a proposition p is denoted by $\sim p$ or $\neg p$ (read as **not p**).

➤ **Conjunction:**

- ✓ Conjunction is a compound statement obtained by combining two simple statements by inserting word “**AND**” between them.
- ✓ The conjunction of p and q is denoted by $p \wedge q$,

Note that: $p \wedge q$ is **true or 1** only when **both p and q are true or 1** , otherwise $p \wedge q$ is **false or 0**.

Disjunction:

- ✓ Disjunction is a compound statement obtained by combining two simple statements by inserting word “**OR**” between them.
- ✓ The disjunction of p and q is denoted by $p \vee q$,

Note that: $p \vee q$ is **false or 0** only when **both p and q are false or 0** , otherwise $p \vee q$ is **true of 1**.

➤ **Implication (Conditional):**

- ✓ A compound statement obtained by combining two propositions by words "if" and "then" is called Implication or Conditional. If p then q is denoted by $p \rightarrow q$.
- ✓ $p \rightarrow q$ is **false** only when **p is true and q is false**, otherwise **it is true**.

➤ **Biconditional:**

- ✓ consider two statements p and q , then two conditionals $p \rightarrow q$ and $q \rightarrow p$ is called biconditional of p and q denoted by $p \leftrightarrow q$ (if p then q and if q then p).

➤ **Exclusive disjunction:**

- ✓ The compound proposition $P \underline{\vee} q$ (either p or q but not both) is called Exclusive disjunction of p and q.
- ✓ Exclusive disjunction is true when either p is true or q is true but not both.

3. Truth tables

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$P \underline{\vee} q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Tautology

- ▶ A compound proposition which is true for all possible truth value is called a **tautology**.
- ▶ Example: Prove that for any 3 propositions p, q, r the compound proposition given by $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ is a tautology.

Solution: Identify all possible combinations of truth value of the 3 propositions. And construct the truth table.

Truth table

P	q	r	$X = p \vee q$	$Y = p \rightarrow r$	$Z = q \rightarrow r$	$W = x \wedge y \wedge z$	r	$w \rightarrow r$
0	0	0	0	1	1	0	0	1
0	0	1	0	1	1	0	1	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Contradiction

- ▶ A compound proposition which has false for all truth value is called a contradiction(or **absurdity**)
- ▶ **Example:** Show that the statement $p \wedge \sim p$ is a contradiction.

Truth table:

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Contingency

- ▶ A compound statement that can be either true or false depending on its truth value is called **contingency**.
- ▶ **Example:** Show that the statement $(p \rightarrow q) \rightarrow (p \wedge q)$ is a contingency.

Truth table:

p	q	$p \rightarrow q$	$p \wedge q$	$(p \rightarrow q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

4. Logical equivalence

- ▶ Logical equivalence is the condition of equality that exists between two statements or sentences in propositional logic.
- ▶ **Example:** Use a truth table to show that the statements $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$ are not logically equivalent.

▶ **Solution:**

P	Q	R	$Q \rightarrow R$	$P \rightarrow Q$	$P \rightarrow (Q \rightarrow R)$	$(P \rightarrow Q) \rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

From the truth table, the truth values of column 6 and 7 are not the same, Hence these statements are **not** logically equivalent

Below is a list of important equivalences laws

Equivalence	Law
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation Laws

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Logical Equivalences Involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

5. Valid and Invalid logical arguments

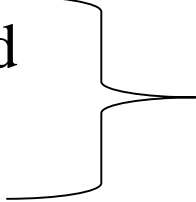
Definitions:

- ✓ An **argument** consists of premises and a conclusion. You must have at least one premise but you have exactly one conclusion.
- ✓ An argument is **valid** if whenever the premises are true, the conclusion must be true.
- ✓ An argument is **invalid** if it is possible for the premises to be true and the conclusion false.
- ✓ To decide if an argument is valid, we construct a truth table for the premises and conclusion. Then we check for whether there is a case where the premises are true and the conclusion false.

Using a truth table to analyze the validity of an arguments.

- Turn the arguments into one single statement with general form given below:
 $[premise1 \wedge premise1 \wedge premise1 \wedge \dots] \rightarrow conclusion$
- If the truth value of $[premise1 \wedge premise2 \wedge premise3 \wedge \dots] \rightarrow conclusion$ is true always, then we conclude that the argument is valid.

Example:1. Determine whether the following arguments are valid or invalid.

1. If you wear a tinfoil hat , then aliens can not read your mind
 2. you wear a tinfoil hat
- 
- Premises.

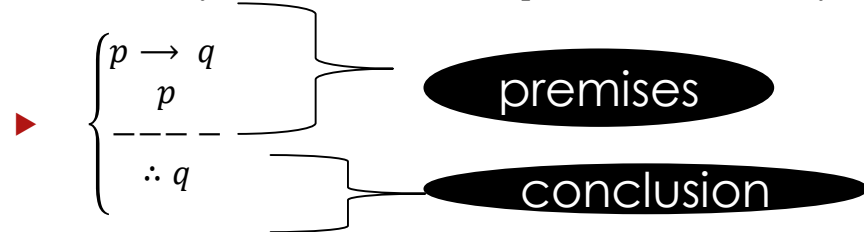
1. Therefore aliens can not read your mind
- 
- Conclusion

Valid or invalid?

Solution:

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- ▶ Let p = you wear a tinfoil hat, q = aliens can not read your mind,



- ▶ Compound statement using symbols: $[(p \rightarrow q) \wedge p] \rightarrow q$

- ▶ Truth table:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The last column is always true
we conclude that $[(p \rightarrow q) \wedge$

Example:2. Determine whether the following arguments are valid or invalid.

1. If you take the medicine, then you will feel better
 2. you will feel better
- Premises.

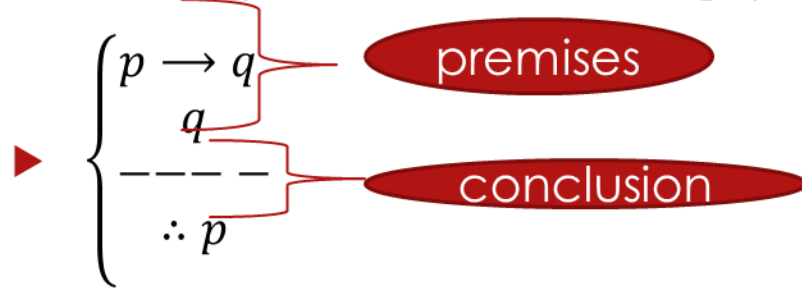
1. Therefore take the medicine
- Conclusion

Valid or invalid?

Solution:

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- ▶ Let p = you take the medicine, q = you will feel better,



- ▶ Compound statement using symbols: $[(p \rightarrow q) \wedge q] \rightarrow p$

- ▶ Truth table:

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The last column has truth value which is false, therefore the argument is invalid.

Four important valid forms (Classical method of reasoning)

- **Direct reasoning or Modulus Ponens(Direct method):**

$$\left\{ \begin{array}{c} p \longrightarrow q \\ q \\ \hline \therefore p \end{array} \right.$$

- **Contrapositive reasoning or Modulus Tollens(Reversed method):**

$$\left\{ \begin{array}{c} p \longrightarrow q \\ \sim p \\ \hline \therefore \sim p \end{array} \right.$$

Four important valid forms (Classical method of reasoning)

➤ **Transitive reasoning:**

$$\left\{ \begin{array}{l} p \longrightarrow q \\ q \longrightarrow r \\ \hline \therefore p \longrightarrow r \end{array} \right.$$

➤ **Valid disjunctive reasoning:**

$$\left\{ \begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array} \right.$$

Four important invalid forms:

➤ **Fallacy of inverse:**

$$\left\{ \begin{array}{c} p \rightarrow q \\ \sim p \\ \hline \therefore \sim q \end{array} \right.$$

➤ **Fallacy of converse**

$$\left\{ \begin{array}{c} p \rightarrow q \\ \sim p \\ \hline \therefore \sim q \end{array} \right.$$

Four important invalid forms

➤ Invalid transitive reasoning:

$$\left\{ \begin{array}{l} p \longrightarrow q \\ q \longrightarrow r \\ \hline \therefore r \longrightarrow p \end{array} \right.$$

➤ Invalid disjunctive reasoning:

$$\left\{ \begin{array}{l} p \vee q \\ p \\ \hline \therefore \sim q \end{array} \right.$$