RWANDA POLYTECHNIC

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Competence

RTQF Level: 6

Learning hours 100

Credits: 10

Sector: ICT

Sub-sector: Information technology

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By the end of the module, the trainee will be able to:

- Apply introduction to Mathematical Logic
- Apply Counting theory and Discrete probability
- Apply Number theory and Cryptograph
- Apply Introduction to Graph theory and Finite automata
- Apply Sequences and Series
- References

Introduction to Mathematical Logic

1. Propositions
2. Basic connectives
3. Truth tables
4. Logical equivalences
5. Validity of arguments

1. Propositions

Introduction to Propositions

- **Propositions:** A proposition is a statement which can be either **true** or **false** but **not both.**
- **Example:**
  - Kigali is capital of Rwanda
  - Burundi is our country
  - 2+3=5
  - 1+0=0
- Propositions are represented by small letters : p,q,r,s,t,…..
- The truth or falsity of a proposition is called its **truth value**
- If a proposition is true its **truth** value is **1** and if it is **false** its truth value is **0**

2.Basic connectives

- **Logical connectives:** words or phrases like **not**, **and**, **or** ,**if …..then**, **if and only if** , that are used to form a new proposition from new propositions are known as **logical connectives**.
- New propositions are called **compound propositions**
- Original propositions are called **primitives** or **components** of **compound propositions.**
- Propositions which do not contain logical connectives are called **simple propositions.**

- **Negation:**
  - ✓A proposition obtained by inserting word “**NOT**” is called negation, the negation of a proposition p is denoted by **~p** or **¬p** (read as **not p**).

- **Conjunction:**
  - ✓Conjunction is a compound statement obtained by combining two simple statements by inserting word ”**AND**” between them.
  - ✓The conjunction of p and q is denoted by **p^q**,

Note that: p^q is **true or 1** only when **both p and q are true or 1** , otherwise **p^q is false or 0**.

Disjunction:

- ✓Disjunction is a compound statement obtained by combining two simple statements by inserting word ”**OR**” between them.
- ✓The disjunction of p and q is denoted by **pVq**,

Note that: pVq is **false or 0** only when **both p and q are false or 0** , otherwise **pVq is true of 1**.

- **Implication (Conditional):**
  - ✓A compound statement obtained by combining two propositions by words ”**if**” and “**then**” is called Implication or Conditional. If p then q is denoted by **$p \rightarrow q$.**
  - ✓$p \rightarrow q$ is **false** only when **p is true and q is false**, otherwise **it is true**.

- **Biconditional:**
  - ✓consider two statements p and q , then two conditionals p $\rightarrow$q and q $\rightarrow$p is called biconditional of p and q denoted by $p \leftrightarrow q$ (if p then q and if q then p).

- **Exclusive disjunction:**
  - ✓The compound proposition $P\underline{\vee}q$ (either p or q but not both) is called Exclusive disjunction of p and q.
  - ✓Exclusive disjunction is true when either p is true or q is true but not both.

3. Truth tables

p	q	~p	p^q	pvq	p→q	p↔q	P$\underline{V}$q
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Tautology

- A compound proposition which is true for all possible truth value is called a **tautology**.

- Example: Prove that for any 3 propositions p, q, r the compound proposition given by $[(p \vee q) \wedge (p \longrightarrow r) \wedge (q \longrightarrow r)] \longrightarrow r$ is a tautology.

Solution: Identify all possible combinations of truth value of the 3 propositions. And construct the truth table.

Truth table

P	q	r	X=$p \vee q$	Y=$p \longrightarrow r$	Z=$q \longrightarrow r$	**W=x∧y∧z**	**r**	**w⟶r**
0	0	0	0	1	1	0	0	1
0	0	1	0	1	1	0	1	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Contradiction

- A compound proposition which has false for all truth value is called a contradiction(or **absurdity**)
- **Example:** Show that the statement p $\wedge$~p is a contradiction.

Truth table:

p	~p	p $\wedge$~p
T	F	F
F	T	F

Contingency

- A compound statement that can be either true or false depending on its truth value is called **contingency**.
- **Example:** Show that the statement $(p \rightarrow q) \longrightarrow (p \wedge q)$ is a contingency.

Truth table:

p	q	$p \rightarrow q$	$p \wedge q$	$(p \rightarrow q) \longrightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

4. Logical equivalence

- Logical equivalence is the condition of equality that exists between two statements or sentences in propositional logic.
- **Example:** Use a truth table to show that the statements $p \to (q \to r)$ and $(p \to q) \to r$ are not logically equivalent.
- **Solution:**

P	Q	R	$Q \to R$	$P \to Q$	$P \to (Q \to R)$	$(P \to Q) \to R$
T	T	T	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

From the truth table, the truth values of column 6 and 7 are not the same,
Hence these statements are **not** logically equivalent

Below is a list of important equivalences laws

Equivalence	Law
$p \wedge T \equiv p$ $p \vee F \equiv p$	**Identity Laws**
$p \vee T \equiv T$ $p \wedge F \equiv F$	**Domination Laws**
$p \vee p \equiv p$ $p \wedge p \equiv p$	**Idempotent Laws**
$\neg(\neg p) \equiv p$	**Double Negation Law**
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	**Commutative Laws**

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	**Associative Laws**
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	**Distributive Laws**
$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	**De Morgan's Laws**
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee p) \equiv p$	**Absorption Laws**
$\mathrm{p} \vee \neg \mathrm{p} \equiv \mathrm{T}$ $\mathrm{p} \wedge \neg \mathrm{p} \equiv \mathrm{F}$	**Negation Laws**

Logical Equivalences Involving Conditional Statements

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg\left(p \to \neg q\right)$$

$$\neg\left(p \to q\right) \equiv p \wedge \neg q$$

$$\left(p \to q\right) \wedge \left(p \to r\right) \equiv p \to \left(q \wedge r\right)$$

$$\left(p \to r\right) \wedge \left(q \to r\right) \equiv \left(p \vee q\right) \to r$$

$$\left(p \to q\right) \vee \left(p \to r\right) \equiv p \to \left(q \vee r\right)$$

$$\left(p \to r\right) \vee \left(q \to r\right) \equiv \left(p \wedge q\right) \to r$$

Logical Equivalences Involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

5. Valid and Invalid logical arguments

Definitions:

- ✓ An **argument** consists of premises and a conclusion. You must have at least one premise but you have exactly one conclusion.
- ✓ An argument is **valid** if whenever the premises are true, the conclusion must be true.
- ✓ An argument is **invalid** if it is possible for the premises to be true and the conclusion false.
- ✓ To decide if an argument is valid, we construct a truth table for the premises and conclusion. Then we check for whether there is a case where the premises are true and the conclusion false.

Using a truth table to analyze the validity of an arguments.

- Turn the arguments into one single statement with general form given below: $[premise1 \wedge premise1 \wedge \cdots \ldots.] \longrightarrow conlusion$

- If the truth value of $[premise1 \wedge premise2 \wedge premise3 \wedge \cdots \ldots.] \longrightarrow conlusion$ is true always, then we conclude that the argument is valid.

Example:1. Determine whether the following arguments are valid or invalid.

1. If you wear a tinfoil hat , then aliens can not read your mind
2. you wear a tinfoil hat

Premises.

1. Therefore aliens can not read your mind

Conclusion

Valid or invalid?

Solution:

- Let p= you wear a tinfoil hat, q=aliens can not read your mind,
- $$\begin{cases} \left.\begin{array}{c} p \longrightarrow q \\ p \end{array}\right\} \text{premises} \\ \text{-----} \\ \therefore q \quad \} \text{conclusion} \end{cases}$$
- Compound statement using symbols: $[(p \longrightarrow q) \wedge p] \longrightarrow q$
- Truth table:

p	**q**	$p \longrightarrow q$	$(p \longrightarrow q) \wedge p$	$[(p \longrightarrow q) \wedge p] \longrightarrow q$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

The last column is always true we conclude that $[(p \longrightarrow q) \wedge$

Example:2. Determine whether the following arguments are valid or invalid.

1. If you take the medicine, then you will feel better
2. you will feel better

Premises.

1. Therefore take the medicine

Conclusion

Valid or invalid?

Solution:

- Let p= you take the medicine, q= you will feel better,
- $$\begin{cases} p \longrightarrow q \\ \quad q \\ ---- - \\ \therefore p \end{cases}$$
  premises
  conclusion
- Compound statement using symbols: $[(p \longrightarrow q) \land q] \longrightarrow p$
- Truth table:

p	**q**	$p \longrightarrow q$	$(p \longrightarrow q) \land q$	$[(p \longrightarrow q) \land q] \longrightarrow p$
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	T	**F**
F	F	T	F	**T**

The last column has truth value which is false, therefore the argument is invalid.

Four important valid forms
(Classical method of reasoning)

- **Direct reasoning or Modulus Ponens(Direct method):**

$$\begin{cases} p \rightarrow q \\ \quad q \\ \text{-----} \\ \therefore p \end{cases}$$

- **Contrapositive reasoning or Modulus Tollens(Reversed method):**

$$\begin{cases} p \rightarrow q \\ \quad \sim p \\ \text{------} \\ \therefore \sim p \end{cases}$$

Four important valid forms (Classical method of reasoning)

- **Transitive reasoning:**

$$\begin{cases} p \longrightarrow q \\ q \longrightarrow r \\ \text{---- --} \\ \therefore p \longrightarrow r \end{cases}$$

- **Valid disjunctive reasoning:**

$$\begin{cases} p \vee q \\ \sim p \\ \text{---- --} \\ \therefore q \end{cases}$$

Four important invalid forms:

- **Fallacy of inverse:**

$$\begin{cases} p \longrightarrow q \\ \quad \sim p \\ \hline \therefore \sim q \end{cases}$$

- **Fallacy of converse**

$$\begin{cases} p \longrightarrow q \\ \quad \sim p \\ \hline \therefore \sim q \end{cases}$$

Four important invalid forms

- **Invalid transitive reasoning:**

$$\begin{cases} p \longrightarrow q \\ q \longrightarrow r \\ \text{--- --- --- ---} \\ \therefore r \longrightarrow p \end{cases}$$

- **Invalid disjunctive reasoning:**

$$\begin{cases} p \vee q \\ p \\ \text{--- --- --- ---} \\ \therefore \sim q \end{cases}$$