# Elliptic Curve Diffie-Hellman Key Exchange

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# What is an elliptic curve?

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- Projective transformation

# The Projective Plane

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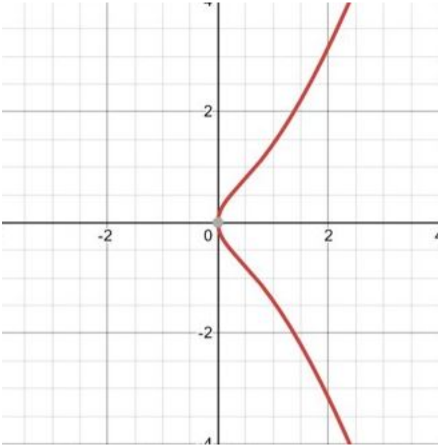
- Equivalent to $x^n + y^n = z^n, x, y \in \mathbb{Z}^+$
- $(tx)^n + (ty)^n = (tz)^n$
- Notion of a "point at infinity": $(1, -1, 0) : (1)^1 + (-1)^1 = 0^1$

# What do these curves look like?

$$y^2 = x(x^2 + 1)$$

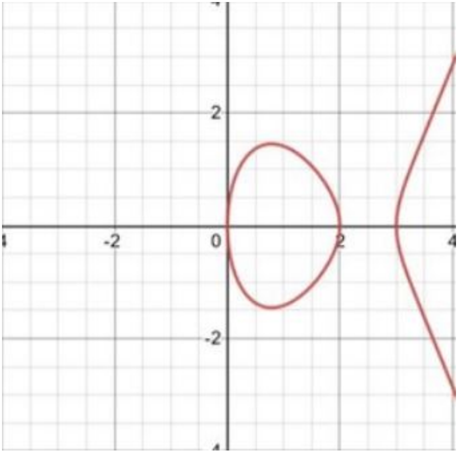



- One real root

# What do these curves look like?

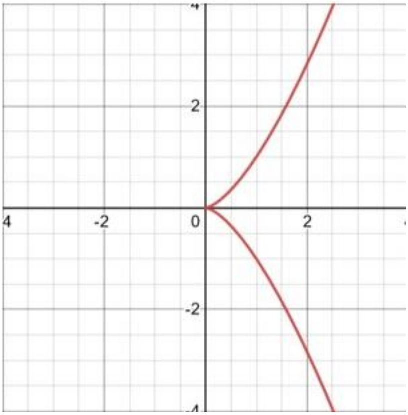
$$y^2 = x(x-2)(x-3)$$

- Three real roots



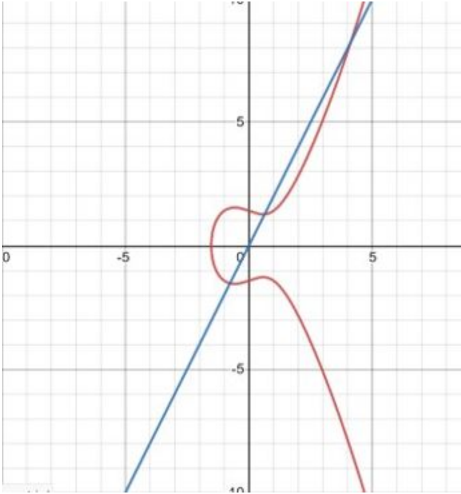
# What do these curves look like?

- Recall: roots must be distinct. Why?



# Point Composition

- Suppose we are given two points on a line.



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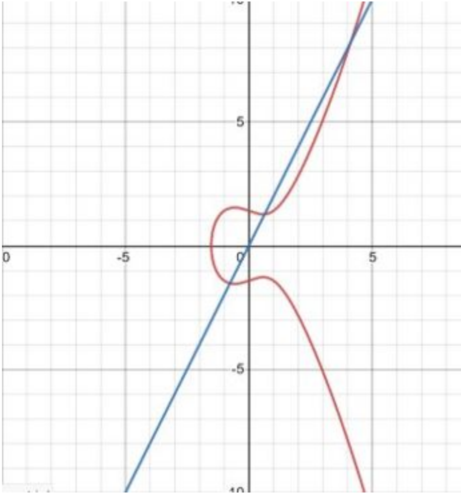

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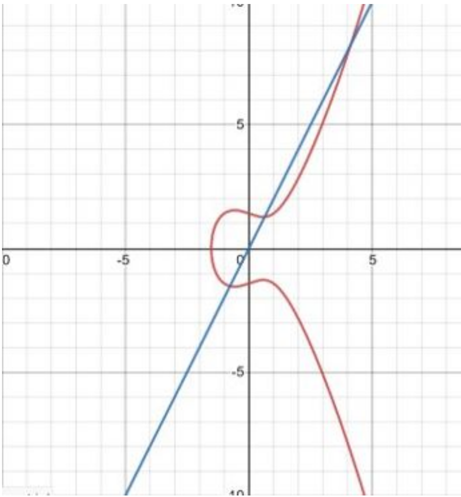

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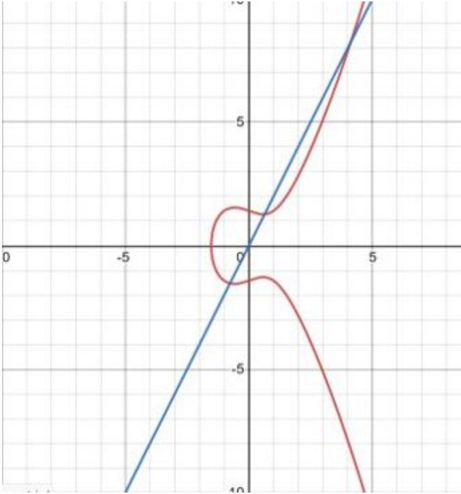

# Point Composition

- Suppose we are given two points on a line.
- We profit from the idea that a line intersects a cubic curve at three points.
- What about two points that construct a vertical line?
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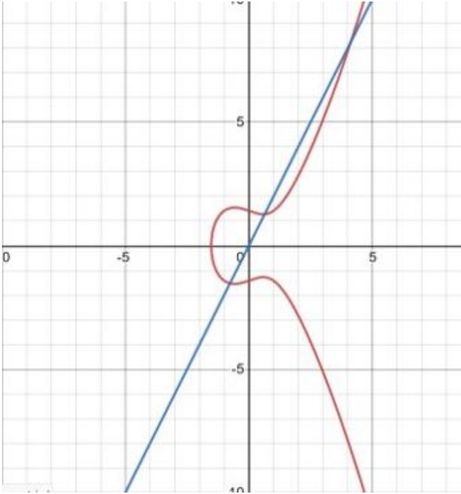
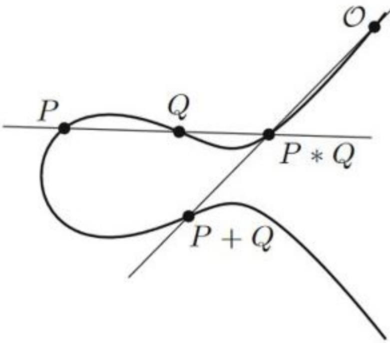
- If the first two points are rational, the third is rational.



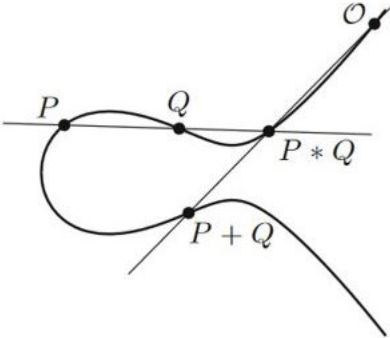
# Geometric Point Addition

- We can craft a new identity element, $O$.



# Geometric Point Addition

- We can craft a new identity element, $O$.
- $P + Q$ becomes a new operation defined by $O * (P * Q)$



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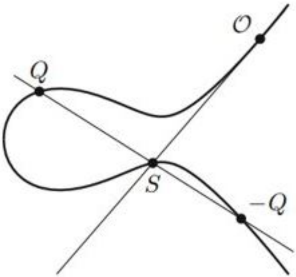

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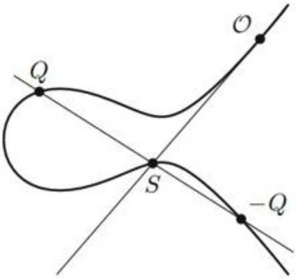

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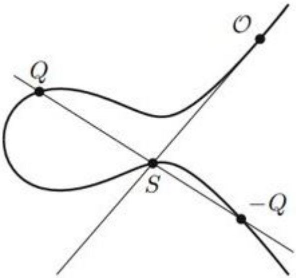

# Geometric Point Addition

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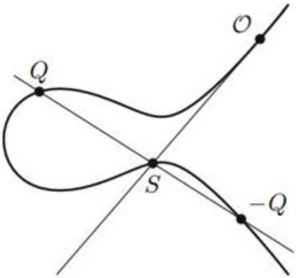
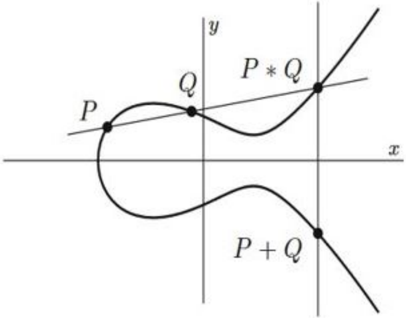
- Identity element: $O$
- Inverse: $Q + -Q = O$



# Moving the Identity Element



# Algebraic Point Addition

- $x_3 = m^2 - a - x_1 - x_2$

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- $x_3 = m^2 - a - x_1 - x_2$
- $y_3 = mx_3 + v$

# Finite Fields

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- $y = 2, 5$; points on curve in $F_7$ include $(1, 2), (1, 5)$
- How do we add these points? We use our previous algebraic method but compute all quantities mod p.

# Key Exchange Protocol

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- A and B take each other's $Q$ value and raise it to their own secret exponent.
- A then has $(Q_2)^{n_1} = (P^{n_2})^{n_1} = (P^{n_1})^{n_2} = (Q_1)^{n_2}$, the latter of which B has. Both parties arrive at the same decryption key, while an eavesdropper does not.

# Classical Discrete Logarithm Problem

- Consider a multiplicative cyclic group $G_p$, with elements $b_j$. Because any $b_j$ can be expressed as the power, $k$, of a generator, $g$, i.e., $b_j = g^k$, we may define $\log_g(b_j) = k (\mathrm{mod} p)$

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- If we have a generator of the group and an arbitrary element, can we compute k?

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- Given two points $Q, P$ of $E(F_P)$, we wish to find $n$ such that $Q = P + P + ... + P = nP$

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- In other words, we wish to find $\log_P Q = n$, the number of times $P$ operates on itself.
- Problems may arise where no $n$ exists such that $nP = Q$, or where multiple values $n$ exist that satisfy the equation.

# How difficult is this to compute?

- Naive way: Increment by $P$ each time (i.e., find $P$, then $2P$, ..., $tP$ and so on until n is found.

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- Fastest methods take $\sqrt{P}$ calculations in $E(F_p)$. $F_p$ should be large!

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- Numerous in-slide image and text citations coming soon

# References

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Questions?