Continuum Mechanics

Lecture 8 - Linear Elasticity

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schedule

- 17 Sep - Linear Elasticity
- 22 Sep - Equations of Motion, HW 4 Due
- 24 Sep - Elastic Problems
- 29 Sep - Elastic Problems, HW 5 Due

outline

- equations of motion
- energetic conjugates
- corotational derivative
- heat, energy, and entropy
- integral formulation

reference configuration

- At this point it is desirable to formulate equations of motion in terms of the first and second Piola-Kirchhoff stress tensors
- Recall for Cauchy stress

$$T_{ij,j} + \rho B_i = \rho a_i$$

- Now we substitute $T_{ij} = \frac{1}{J} T^0_{im} F_{jm}$ to get

$$T_{ij,j} = \frac{\partial}{\partial x_j}\left(\frac{1}{J} T^0_{im} F_{jm}\right) = \frac{F_{jm}}{J}\frac{\partial T^0_{im}}{\partial x_j} + T^0_{im}\frac{\partial}{\partial x_j}\frac{F_{im}}{J} = \frac{F_{jm}}{J}\frac{\partial T^0_{im}}{\partial x_j}$$

reference configuration

- Since T^0_{ij} is expressed in terms of the reference configuration, we desire to change the partial derivative from x_j to X_j which we can do as follows

$$\frac{\partial T_{ij}}{\partial x_j}\frac{F_{jm}}{J}\frac{\partial T^0_{im}}{\partial x_j} = \frac{1}{J}\frac{\partial x_j}{\partial X_m}\frac{\partial T^0_{im}}{\partial X_n}\frac{\partial X_n}{\partial x_j} = \frac{1}{J}\delta_{mn}\frac{\partial T^0_{im}}{\partial X_n}$$

- Substituting this into the equation of motion, and multiplying both sides by J gives

$$\frac{\partial T^0_{ij}}{\partial X_j} + J\rho B_i = J\rho a_i$$

- The quantity $J\rho$ is sometimes written as ρ^0

work and power

- Stress and strains are considered *energetically conjugate* if their double-dot product reflects the strain energy
- Work is defined as force times distance, and power is force time velocity

$$P = \int F_i dv_i$$

- If we multiply and divide by the differential volume, we find

$$P = \int \frac{F_i}{A_j}\frac{dv_i}{dx_j} dV$$

power

- Since F_i is the force in the deformed configuration and A_j is the area in the deformed configuration, $\frac{F_i}{A_j}$ is the true stress (Cauchy stress)
- We also see that $\frac{dv_i}{dx_j}$ is the velocity gradient, so we can re-write as

$$P = \int \sigma_{ij} v_{i,j} dV$$

- We also recall that $v_{i,j} = D_{ij} + W_{ij}$

$$P = \int \sigma_{ij}(D_{ij} + W_{ij}) dV$$

- But since W_{ij} is anti-symmetric and D_{ij} is symmetric, we have

$$P = \int \sigma_{ij} D_{ij} dV$$

- This means that D_{ij} is the energetic conjugate for σ_{ij}

deformation gradient

- The velocity gradient is an Eulerian property, but we can convert it to Lagrangian using the deformation gradient
- If we take the time derivative of the deformation gradient we find

$$\dot{F}_{ij} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_j}\right) = \frac{\partial}{\partial X_j}\left(\frac{dx}{dt}\right) = \frac{\partial v_i}{\partial X_j}$$

- We can now apply the chain rule to find

$$\dot{F}_{ij} = \frac{\partial v_i}{\partial X_j} = \left(\frac{\partial v_i}{\partial x_k}\right)\left(\frac{\partial x_k}{\partial X_j}\right)$$

- We can re-arrange to write the velocity gradient in terms of the deformation gradient

$$v_{i,j} = \dot{F}_{ik} F^{-1}_{kj}$$

first piola-kirchhoff

- If we return to power in terms of stress and the velocity gradient, we can now re-write in terms of the deformation gradient

$$P = \int \sigma_{ij} v_{i,j} dV = \int \sigma_{ij} \dot{F}_{ik} F^{-1}_{kj} dV$$

- We can also integrate over the reference volume by using $dV = JdV_0$

$$P = \int \sigma_{ij} \dot{F}_{ik} F^{-1}_{kj} J dV_0$$

first piola-kirchhoff

- Recall that the first Piola-Kirchhoff stress tensor is given as

$$\sigma^0_{ij} = J \sigma_{im} F^{-1}_{jm}$$

- Changing indexes, m to j and j to k, we can substitute

$$P = \int \sigma^0_{ik} \dot{F}_{ik} dV_0$$

- Thus the material derivative of the deformation gradient is the energetic conjugate for the first piola-kirchhoff stress tensor

second piola-kirchhoff

- To find the energetic conjugate for the second Piola-Kirchhoff stress tensor, we return to power in terms of Cauchy stress and the deformation rate tensor

$$P = \int \sigma_{ij} D_{ij} dV$$

- This time we will replace D_{ij} with E^*_{ij}, Consider

$$ds^2 = dS^2 + 2dX_i E^*_{ij} dX_j$$

- Taking the material derivative of both sides we find

$$\frac{D}{Dt} ds^2 = 2dX_i \frac{DE^*_{ij}}{Dt} dX_j$$

- But we also know that

$$\frac{D}{Dt} ds^2 = 2dx_i D_{ij} dx_j = 2F_{im} dX_m D_{ij} F_{jn} dX_n$$

- Re-arranging terms, we can see that

$$\frac{DE^*_{ij}}{Dt} = F_{mi} D_{mn} F_{nj}$$

- Solving for D_{mn} gives

$$D_{ij} = F^{-1}_{mi} \dot{E}^*_{mn} F^{-1}_{nj}$$

second piola-kirchhoff

- Substituting gives

$$P = \int \sigma_{ij} F^{-1}_{mi} \dot{E}^{*}_{mn} F^{-1}_{nj} J dV_0$$

- And the second Piola-Kirchhoff tensor in terms of Cauchy stress is

$$\tilde{\sigma}_{ij} = J F^{-1}_{im} \sigma_{mn} F^{-1}_{jn}$$

- But since the Cauchy stress tensor is symmetric, we can re-write it as

$$\tilde{\sigma}_{ij} = J F^{-1}_{im} \sigma_{nm} F^{-1}_{jn}$$

- Which we can now substitute to find

$$P = \int \tilde{\sigma}_{ij} \dot{E}^{*}_{ij} dV_0$$

examples

- First let us consider an incompressible rubber specimen in tension
- Since it is incompressible, the volume must remain constant

$$L_0 A_0 = L_f A_f$$

examples

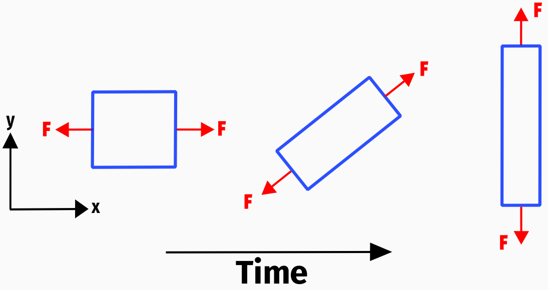

Figure 1: corotational example

corotational derivative

- Note: textbook addresses co-rotational derivative on pp. 483-486
- Rigid body rotations can cause problems when taking derivatives
- In our last example, the stress rotated from the 1 direction to the 2 direction, thus we can see that $\dot{\sigma}_{ij} \neq 0$
- However, the rate of deformation tensor, D_{ij} is zero because there is no deformation

material indifference

- We have many different stress (Cauchy, Piola-Kirchhoff) and strain (right and left Cauchy, Lagrangian, Eulierian) tensors
- A proper constitutive equation should be invariant under transformation

$$T^*_{ij} = Q_{im}(t) T_{mn} Q_{jn}(t)$$

- This dictates which stress and strain tensors can be related in a constitutive equation
- We can show that the Right Cauchy-Green strain tensor should not be used with the Cauchy Stress tensor

corotational derivative

- In general, the material derivative of a tensor which is material indifferent (also called an objective tensor) is not objective
- This motivates a new derivative to find the objective rate tensor for an objective tensor
- We can derive a corotational derivative for stress and strain by considering the most general form of linear materials

$$\sigma^0_{ij} = C_{ijkl} E^*_{kl}$$

- Now we substitute the Cauchy stress and solve to find

$$\sigma_{ij} = \frac{1}{J} F_{im} C_{mnop} E_{op} F_{jn}$$

corotational derivative

- If we now take the material derivative we find

$$\dot{\sigma}_{ij} = -\frac{\dot{J}}{J^2} F_{im} C_{mnop} E_{op} F_{jn} + \frac{1}{J} \dot{F}_{im} C_{mnop} E_{op} F_{jn} + \frac{1}{J} F_{im} C_{mnop} \dot{E}_{op} F_{jn} + \frac{1}{J} F_{im} C$$

- We can now substitute several identities, $\frac{\dot{J}}{J} = D_{ii}$, $\dot{F}_{ij} = v_{i,m} F_{mj}$, and $\dot{E}^*_{ij} = F_{mi} D_{mn} F_{nj}$ to find

$$\dot{\sigma}_{ij} = \frac{1}{J} [-D_{kk} F_{im} C_{mnop} E_{op} F_{jn} + v_{i,k} F_{km} C_{mnop} E_{op} F_{jn} + F_{im} C_{mnop} F_{ko} D_{kl} F_{lp} F_{jn} + F_{im} C_{mnop} E_{op} F_{kj} v_{n,k}]$$

corotational derivative

- Now we recall to simplify things somewhat

$$\dot{\sigma}_{ij} = -D_{kk}\sigma_{ij} + v_{i,k}\sigma_{kj} + \sigma_{ik} v_{j,k} + \frac{1}{J} F_{im} C_{mnop} F_{ko} D_{kl} F_{lp} F_{jn}$$

- This is often written as

$$\dot{\sigma}_{ij} - v_{i,k}\sigma_{kj} - \sigma_{ik} v_{j,k} = -D_{kk}\sigma_{ij} + \frac{1}{J} F_{im} C_{mnop} F_{ko} D_{kl} F_{lp} F_{jn}$$

- The left hand side is called the Lie Derivative, and is usually written as $\overset{\triangledown}{\sigma}_{ij}$
- $D_{kk} \approx 0$ in almost all cases, and is usually neglected
- We also define $C'_{ijkl} \equiv \frac{1}{J} F_{im} F_{jn} C_{mnop} F_{ko} F_{lp}$ to find

$$\overset{\triangledown}{\sigma}_{ij} = C'_{ijkl} D_{kl}$$

jaumann derivative

- For the special case when an object is rotating, but not deforming, we have $D_{ij} = 0$ which gives

$$\overset{\nabla}{\sigma}_{ij} = \dot{\sigma}_{ij} - v_{i,k}\sigma_{kj} - \sigma_{ik}v_{j,k} = 0$$

- And we can more clearly see the terms which account for $\dot{\sigma}_{ij} \neq 0$
- Since $D_{ij} = 0$, we can also re-write $v_{i,j} = D_{ij} + W_{ij} = W_{ij}$
- We also know that $W_{ij} = -W_{ji}$ which leads to the Jaumann derivative

$$\mathring{\sigma}_{ij} = \dot{\sigma}_{ij} - W_{ik}\sigma_{kj} + \sigma_{ik}W_{kj}$$

example

- Calculate $\dot{\sigma}_{ij}$ and $\mathring{\sigma}_{ij}$ for an object under constant stress

$$\sigma = \left[20 \quad 0\;0 \quad 0\right]$$

- With 2D rotation of

$$R = \left[\cos\theta \quad -\sin\theta\;\;\sin\theta \quad \cos\theta\right]$$

and

$$\dot{R} = \omega\left[-\sin\theta \quad -\cos\theta\;\;\cos\theta \quad -\sin\theta\right]$$

heat

- Let q_i be the vector whose magnitude gives the rate of heat flow across a unit area and whose direction indicates the direction of heat flow
- The net flow of heat into a differential element is

$$Q = -q_{i,i} dV$$

- Using the Fourier heat conduction law in steady state conditions we find

$$q_i = -\kappa \nabla \Theta$$

- In steady state conditions, no should be no net rate of heat flow, which produces the governing Laplace equation

$$\nabla^2 \Theta = 0$$

energy

- If we consider only the energy contributions from strain energy, kinetic energy, and heat
- by the conservation of energy, the rate of increase in energy for a particle equals the rate of work done plus the heat added

$$\frac{D}{Dt}(U + KE) = P + Q_c + Q_s$$

- Where $P = \frac{D}{Dt}(KE) + T_{ij} v_{i,j} dV$ and $Q_c = -q_{i,j} dV$

$$\frac{DU}{Dt} = T_{ij} v_{i,j} dV - q_{i,j} dV + Q_s$$

- This is also sometimes written as energy per unit mass as

$$\rho \frac{Du}{Dt} = T_{ij} v_{i,j} - q_{i,j} + \rho q_s$$

entropy inequality

- Let $\eta(x_i, t)$ denote the entropy per unit mass
- The rate of entropy following a particle is

$$\frac{D}{Dt}(\rho\eta dV) = \rho dV\frac{D\eta}{Dt} + \eta\frac{D}{Dt}(\rho dV) = \rho dV\frac{D\eta}{Dt}$$

- The entropy inequality states that the rate of increase of entropy is always greater than or equal to the entropy inflow plus the entropy supply

$$\rho\frac{D\eta}{Dt} \geq -\mathrm{div}\left(\frac{q}{\Theta}\right) + \frac{\rho q_s}{\Theta}$$

helmholtz energy function

- The Helmholtz energy function is defined as

$$A = u - \Theta\eta$$

- We can use this relationship to re-write the entropy inequality as

$$-\left(\rho\frac{DA}{Dt} + \rho\eta\frac{D\Theta}{Dt}\right) + T_{ij}D_{ij} - \frac{q_i}{\Theta}\frac{\partial\Theta}{\partial x_i} \geq 0$$

integral formulation

- To this point, we have derived field equations using the differential element approach (this is sometimes referred to as local principles)
- When can also formulate these principles globally by integrating over the volume. If the functions are smooth, these two methods will be equivalent
- In certain problems the integral formulation may be more convenient, or more numerically accurate when solving problems numerically

conservation of mass

- The conservation of mass states that the rate of increase of mass in a fixed part of a material is zero

$$\frac{D}{Dt}\int_{V_m} \rho dV = 0$$

$$\int_{V_m} \frac{D}{Dt}(\rho dV) = 0$$

$$\int_{V_m} dV\frac{D}{Dt}\rho + \rho\frac{D}{Dt}dV = 0$$

conservation of mass

- We previously found that D_{ii} is related to the rate of change of the volume, which we can write in terms of the velocity gradient as

$$\frac{DdV}{Dt} = v_{i,i} dV$$

- We can substitute this to find

$$\int_{V_m} dV \frac{D}{Dt}\rho + \rho v_{i,i} dV = 0$$

- Since this must be true for any volume, we find

$$\frac{D}{Dt}\rho + \rho v_{i,i} = 0$$