# Continuum Mechanics

Lecture 5 - Conservation and Compatibility

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## schedule

- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due
- 3 Sep - Polar Decomposition
- 8 Sep - Exam Review

# outline

- deformation
- review
- infinitesimal strain
- conservation of mass and compatibility
- finite deformation



# infinitesimal deformation

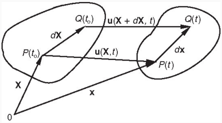


**Figure 1:** image

- We recall $P$, which undergoes some displacement, $u$
- A neighboring point, $Q$, at $X_i + dX_i$ arrives at $x_i + dx_i$

$$x_i + dx_i = X_i + dX_i + u_i(X_i + dX_i, t)$$

- Subtracting $dx_i$ and using the definition of the gradient of a vector function, we have

$$dx_i = dX_i + u_{i,j}dX_j$$

## infinitesimal deformation

- We can re-write (4.24)

$$dx_i = dX_i + u_{i,j} dX_j$$
$$dx_i = dX_j \delta_{ij} + u_{i,j} dX_j$$
$$dx_i = (u_{i,j} + \delta_{ij}) dX_j$$

- We define the deformation gradient, $F$ as $F = u_{i,j} + \delta_{ij}$



## infinitesimal deformation

- We can find some interesting information by finding the length of $dx_i$ relative to the undeformed length of $dX_i$
-

$$dx_i dx_i = F_{ij} dX_j F_{ik} dX_k$$

- We can rearrange this to

$$dx_i dx_i = dX_j F_{ij} F_{ik} dX_k$$

- We now define the right Cauchy-Green deformation tensor as $C_{jk} = F_{ij} F_{ik}$, and note that if $C_{jk} = \delta_{jk}$, then the deformed length is equal to the original length, corresponding to rigid body motion

# lagrange strain tensor

- We can break down the right Cauchy-Green deformation tensor to derive the Lagrange strain tensor

$$C_{ij} = F_{ki}F_{kj} = F^T F = (I + \nabla u)^T (I + \nabla u) = \\ I + \nabla u + (\nabla u)^T + (\nabla u)^T (\nabla u)$$

- We recall that $C = I$ refers to rigid body motion, and thus define the Lagrange strain tensor as one-half of the deformation with no rigid body motion

$$E^* = \frac{1}{2}\left[\nabla u + (\nabla u)^T + (\nabla u)^T(\nabla u)\right]$$



# lagrange strain tensor

- The Lagrange strain tensor is a finite deformation tensor
- For infinitesimal deformations, we assume that $(\nabla u)^T(\nabla u)$ is negligible when compared with $\nabla u$
- In this case the Lagrange strain tensor would reduce to

$$E = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$$

- Which is simply the symmetric portion of $\nabla u$
- In rectangular coordinates, we have

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

## physical meaning

- If we consider two elements, $dX_i^{(1)}$ and $dX_i^{(2)}$
- Due to motion they become $dx_i^{(1)}$ and $dx_i^{(2)}$
- For small deformations, we know that

$$dx_i^{(1)}dx_i^{(2)} = F_{ij}dX_j^{(1)}F_{ik}dX_k^{(2)} = dX_j^{(1)}C_{jk}dX_k^{(2)} = dX_j^{(1)}(\delta_{jk} + 2E_{jk})dX_k^{(2)}$$

- Which we can expand to

$$dx_i^{(1)}dx_i^{(2)} = dX_i^{(1)}dX_i^{(2)} + 2E_{jk}dX_j^{(1)}dX_k^{(2)}$$



## physical meaning

- If we look at the length of a single material element, $dX_i = dSdn_i$ we find the deformed length, $ds$ to be

$$ds^2 = dS^2 + 2dS^2(n_iE_{ij}n_j)$$

- For small deformations, we make the assumption that

$$ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$$

- Which leads to

$$\frac{ds - dS}{dS} = n_iE_{ij}n_j$$

- This means that the diagonal terms of $E_{ij}$ give the unit elongation for an element originally in the 1, 2 or 3 directions

## physical meaning

- If we consider two unit vectors, $m_i$ and $n_i$ which are initially perpendicular, we have $dX_i^{(1)} = dS_1 m_i$ and $dX_i^{(2)} = dS_2 n_i$
- We can find the angle between the two deformed vectors, $dx_i^{(1)}$ and $dx_i^{(2)}$

$$dx_i^{(1)} dx_i^{(2)} = ds_1 ds_2 \cos\theta = 2E_{jk} dS_1 m_j dS_2 n_k$$

- Since the angle between the vectors was originally $\pi/2$, we define the change in angle as $\gamma = \pi/2 - \theta$



## physical meaning

- We also note that $\cos\theta = \cos(\pi/2 - \gamma) = \sin\gamma$
- For small deformations (i.e. small $\gamma$) we have $\sin\gamma \approx \gamma$ and $\frac{ds_1}{dS_1} \approx 1$ and $\frac{ds_2}{dS_2} \approx 1$
- This gives

$$\gamma = 2E_{ij} m_i n_j$$

- We can isolate off-diagonal terms in $E_{ij}$ by letting $m_i = \langle 1, 0, 0 \rangle$ and $n_j = \langle 0, 1, 0 \rangle$ (and other perpendicular directions)
- This means that $2E_{12}$ gives the change in angle between two elements initially in the $x_1$ and $x_2$ directions

## Group 1

- Describe the difference between a material (Lagrangian) and a spatial (Eulerian) description
- Give some examples of situations where each would be more convenient



## Group 2

- What is the material derivative?
- Why is it calculated differently in material and spatial descriptions?

# Group 3

- What is the physical meaning of the components of the infinitesimal strain tensor?
- Describe (in general) how we can derive this physical meaning



# principal strains

- Principal strains and their corresponding directions can be calculated just as any other eigenvalues and vectors
- Since there is no shear in the principal directions, a unit cube in the principal directions will only undergo stretching
- From this idea, we can derive the dilatation (change in volume)

$$e = E_{ii}$$

## rotation tensor

- The strain tensor was found by taking the symmetric portion of the deformation tensor
- The anti-symmetric portion of the deformation tensor is known as the rotation tensor, $\Omega$

$$\nabla u = E + \Omega$$



## rate of deformation

- The change in a material element is given by

$$dx_i = x_i(X_i + dX_i, t) - x_i(X_i, t)$$

- We obtain the rate of change by taking the material derivative

$$\frac{D}{Dt} dx_i = \frac{D}{Dt} x_i(X_i + dX_i, t) - \frac{D}{Dt} x_i(X_i, t)$$

- Since $\frac{D}{Dt} x_i = v_i$, we have

$$\frac{D}{Dt} dx_i = v_i(X_i + dX_i, t) - v_i(X_i, t) = \nabla v_i = v_{i,j}$$

# rate of deformation

- As with the strain tensor, we can de-compose the velocity gradient into symmetric and anti-symmetric portions

$$v_{i,j} = D_{ij} + W_{ij}$$

- $D_{ij}$ is known as the rate of deformation tensor
- $W_{ij}$ is known as the spin tensor



# rate of deformation

- We can develop expressions for the physical meaning of $D_{ij}$ the same way we did for $E_{ij}$
- If we let $dx_i = dsn_i$ then we can express the magnitude of $dx_i$ as

$$dx_i dx_i = (ds)^2$$

- Since we are concerned with the rate of deformation, we take the material derivative of both sides

$$2dx_i \frac{D}{Dt} dx_i = 2ds \frac{D}{Dt} ds$$

## rate of deformation

- We also know that

$$dx_i \frac{D}{Dt} dx_i = dx_i v_{i,j} dx_j = dx_i D_{ij} dx_j + dx_i W_{ij} dx_j$$

- However since $W_{ij}$ is antisymmetric, we know that

$$dx_i W_{ij} dx_j = dx_i W_{ji} dx_j = -dx_i W_{ij} dx_j = 0$$

- Which means that

$$dx_i \frac{D}{Dt} dx_i = dx_i D_{ij} dx_j$$



## rate of deformation

- Substituting back into (5.8) we find

$$dx_i \frac{D}{Dt} dx_i = dx_i D_{ij} dx_j = ds \frac{D}{Dt} ds$$

- Since $dx_i = dsn_i$ we can re-write this as

$$dsn_i D_{ij} dsn_j = ds \frac{D}{Dt} ds$$

- $ds$ is a scalar, so we can divide both sides by $ds^2$ to give

$$n_i D_{ij} n_j = \frac{1}{ds} \frac{D}{Dt} ds$$

## physical interpretation

- Similar to the results we found for strain, $D_{11}$ gives rate of extension (stretch) in the $x_1$-direction
- We could also follow a similar development as used for the shear terms to see that $2D_{12}$ gives the rate of decrease in angle between elements in the $x_1$ and $x_2$ directions
- We can also find the rate of change of volume

$$D_{11} + D_{22} + D_{33} = \frac{1}{V}\frac{D}{Dt}dV = v_{i,i}$$



## spin tensor

- Any anti-symmetric tensor, $W_{ij}$ is equivalent to some vector, $\omega$ in that

$$W_{ij}a_j = \epsilon_{ijk}\omega_j a_k$$

- Since $a_j$ can be any vector, we can write

$$W_{ij}dx_j = \epsilon_{ijk}\omega_j dx_k$$

- Therefore

$$\frac{D}{Dt}dx_i = v_{i,j}dx_j = (D_{ij} + W_{ij})dx_j = D_{ij}dx_j + \epsilon_{ijk}\omega_j dx_k$$

## conservation of mass

- While an element's volume and density may change, its mass must remain constant
- Thus the material derivative of $\rho dV$ will be zero

$$\frac{D}{Dt}(\rho dV) = 0$$

$$\rho \frac{D}{Dt} dV + dV \frac{D}{Dt}\rho = 0$$

$$\rho v_{i,i} + \frac{D}{Dt}\rho = 0$$

- In the spatial description

$$\frac{D}{Dt}\rho = \frac{\partial \rho}{\partial t} + v_i \rho_{,i}$$

- In continuum mechanics, this is also referred to as the equation of continuity



## conservation of mass

- Some materials (large deformation rubbers and many liquids) are treated as incompressible
- For an incompressible material, the material derivative of density is zero, thus the conservation of mass reduces to

$$v_{i,i} = 0$$

## compatibility

- Conservation of mass gives one form of "continuity," but displacements must also be continuous
- It is possible to have a strain field for which no continuous displacement field can be found
- Consider the following strain tensor

$$E = \begin{bmatrix} kX_2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Since no displacement can satisfy this strain field, we say that the strain field is incompatible



## compatibility

- To satisfy compatibility a strain field must satisfy the following equations

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2}$$

$$\frac{\partial^2 E_{22}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 2\frac{\partial^2 E_{23}}{\partial X_2 \partial X_3}$$

$$\frac{\partial^2 E_{33}}{\partial X_1^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 2\frac{\partial^2 E_{31}}{\partial X_3 \partial X_1}$$

$$\frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = \frac{\partial}{\partial X_1}\left(-\frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3}\right)$$

$$\frac{\partial^2 E_{22}}{\partial X_3 \partial X_1} = \frac{\partial}{\partial X_2}\left(-\frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1}\right)$$

$$\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2} = \frac{\partial}{\partial X_3}\left(-\frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2}\right)$$

# example

- Is the following strain field compatible?

$$E = \begin{Bmatrix} -\frac{X_2}{X_1^2+X_2^2} & \frac{X_1}{2(X_1^2+X_2^2)} & 0 \\ \frac{X_1}{2(X_1^2+X_2^2)} & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$



# compatibility for rate of deformation

- As with strain, when the velocity functions exist, we can always determine the deformation components
- If we have only the deformation tensor, however, compatibility must be satisfied (they are identical to the strain compatibility equations)
- In fluid mechanics we usually deal directly with velocity functions, in which case compatibility is not a concern

## deformation gradient

- If we recall the definition of the deformation gradient, we have

$$dx_i = F_{ij} dX_j$$

$$F = I + \nabla u_i = \nabla x_i = x_{i,j}$$

- We will now consider a few physical requirements
- First, $dx_i$ can be zero only if $dX_i$ is zero, thus we know $F^{-1}$ exists

$$dX_i = F^{-1} dx_i$$

- We can also ensure no reflections occur in deformation by ensuring that $det(F_{ij}) > 0$



## finite deformation

- Let us use the notation $U_{ij} = F_{ij}$ for the special case when $U_{ij}$ is symmetric and positive definite
- This means that for any vector, $a_i$

$$a_i U_{ij} a_j = 0$$

- In this case all eigenvalues will be positive
- The eigenvalues of $U_{ij}$ are the principal stretches, they include the maximum and minimum stretches

## rigid body motion

- Another special case of $F$ is the case when $F_{ij}F_{ik} = \delta_{jk}$ and $detF_{ij} = 1$
- This gives a rigid body rotation, and can be denoted as a rotation tensor, $R_{ij}$



## polar decomposition

- It can be shown that for any real tensor $F_{ij}$ with a nonzero determinant

$$F_{ij} = R_{ik}U_{kj}$$

$$F_{ij} = V_{ik}R_{kj}$$

- Where $U_{ij}$ is known as the right stretch tensor and $V_{ij}$ is known as the left stretch tensor
- (5.35) and (5.36) are known as the Polar Decomposition Theorem

# polar decomposition

- If we recall that the product of any matrix with its transpose gives a symmetric matrix, and apply it to the deformation gradient, we see

$$F^T \cdot F = (R \cdot U)^T \cdot (R \cdot U) = U^T \cdot R^T \cdot R \cdot U$$

- But, since $R^T \cdot R = I$, we find that

$$F^T \cdot F = (R \cdot U)^T \cdot (R \cdot U) = U^T \cdot R^T \cdot R \cdot U = U^T \cdot U$$

- This is also equal to $C$, the Right Cauchy-Green Deformation Tensor
- We could use the same development using $F \cdot F^T$ and substituting $F = V \cdot R$ to find

$$F \cdot F^T = V \cdot V^T$$

- $V \cdot V^T$ is often called $B$, the Left Cauchy-Green



# polar decomposition

- The challenge now is to take the square root of $F \cdot F^T$
- To calculate the square root of a matrix, we must first diagonalize it
    1. Rotate matrix into principal direction
    2. Calculate square root
    3. Rotate back to original direction
- We can use this method to calculate $U$ or $V$, and then we use the inverse to find $R$

## polar decomposition

- An object deforms according to the motion

$$x_1 = X_1 + 2X_2 \sin t + 0.5X_3$$
$$x_2 = -\frac{1}{3}X_1 + X_2 - X_3 \sin t x_3 \quad = X_1^2 \sin 2t + 1.5X_3$$

- Find $U$ and $R$ at the point $X = (1, 1, 1)$ after $t = 0.25$



## right cauchy-green deformation tensor

- We can follow the same development as with small strain to extract physical meaning from the right Cauchy-Green deformation tensor

$$dx_i^{(1)} dx_i^{(2)} = F_{ji} dX_i^{(1)} F_{jk} dX_k^{(2)} = dX_i^{(1)} F_{ji} F_{jk} dX_k^{(2)} = dX_i^{(1)} C_{ik} dX_k^{(2)}$$

- We find that $C_{11} = \left(\frac{ds_1}{dS_1}\right)^2$
- Following a similar development for shear, we find

$$C_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(dx_i^{(1)}, dx_i^{(2)})$$