

Lecture 10 - Airy Stress

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1

schedule

- 1 Oct - Airy Stress
- 6 Oct - Anisotropy
- 8 Oct - Large Deformation
- 13 Oct - Anisotropy and Large Deformation

2

- planar problems
- airy stress function
- examples

plane strain

- If motion in any one direction (for example x_3) is restricted, such that $u_3 = 0$, a body is said to be in a state of plane strain
- The plane strain is often used for very thick materials, where $u_3 \ll u_1, u_2$, but is also applicable any time u_3 is restricted

- Under plane strain conditions we have

$$E_{13} = E_{23} = E_{33} = 0$$

$$E_{11} = E_{11}(x_1, x_2)$$

$$E_{22} = E_{22}(x_1, x_2)$$

$$E_{12} = E_{12}(x_1, x_2)$$

- Using Hooke's Law, we find that $T_{13} = T_{23} = 0$, but

$$T_{33} = \nu(T_{11} + T_{22})$$

- While the other stress components, T_{11} , T_{12} and T_{22} are all functions of x_1 and x_2

- In the absence of body forces, we find the equilibrium equations as

$$\begin{aligned}\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} &= 0 \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} &= 0 \\ \frac{\partial T_{33}}{\partial x_3} &= 0\end{aligned}$$

7

- For very thin bodies, we often make the assumption that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Since we have started from an assumption in stress instead of displacement, it is not yet apparent whether this stress field is allowable

8

- To simplify calculations for equilibrium and compatibility, we define a stress function, φ such that

$$\sigma_{11} = \varphi_{,22}$$

$$\sigma_{22} = \varphi_{,11}$$

$$\sigma_{12} = -\varphi_{,12}$$

- We can quickly verify that this satisfies equilibrium

$$\varphi_{,221} - \varphi_{,122} = 0$$

$$\varphi_{,121} - \varphi_{,112} = 0$$

- We will not follow all the details, but from compatibility we find

$$\varphi = \varphi_0(x_1, x_2) - \frac{\nu}{1 + \nu} \Psi(x_1, x_2) \frac{1}{2} x_3^2$$

- For plane stress problems, we generally consider the case where $x_3 \ll x_1, x_2$, and thus we can neglect the second term, giving our stresses as functions of x_1 and x_2 only
- Note that in general plane stress solutions only approximately satisfy compatibility

11

beltrami-mitchell

- It is convenient to write the compatibility equations in terms of stress
- This is known as the Beltrami-Mitchell equations
- For planar problems, the Beltrami-Mitchell equations are

$$\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{4\rho}{1 + \kappa} (b_{1,1} + b_{2,2})$$

12

- κ is used to differentiate between plane strain and plane stress

$$\kappa = \begin{cases} 3 - 4\nu & \text{Plane Strain} \\ \frac{3-\nu}{1+\nu} & \text{Plane Stress} \end{cases}$$

13

polar coordinates

- Many planar problems are more conveniently expressed in polar coordinates
- For planar problems in polar coordinates the Beltrami-Mitchell equations become

$$\nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = -\frac{4\rho}{1 + \kappa} \left(b_{r,r} + \frac{1}{r} b_{\theta,\theta} \right)$$

- There is a good summary of in-plane field equations in the supplemental elasticity text on page 127 (Cartesian) and 131-132 (Polar)

14

- Body forces are often neglected, but they can be included in the Airy stress formulation by defining a potential function as follows

$$\begin{aligned}\rho b_1 &= -\frac{\partial \mathcal{V}}{\partial x_1} \\ \rho b_2 &= -\frac{\partial \mathcal{V}}{\partial x_2}\end{aligned}$$

15

airy stress

- Let us now define the Airy stress function, $\varphi(x_1, x_2)$ such that

$$\begin{aligned}T_{11} &= \frac{\partial^2 \varphi}{\partial x_2^2} + \mathcal{V} \\ T_{12} &= -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \\ T_{22} &= \frac{\partial^2 \varphi}{\partial x_1^2} + \mathcal{V}\end{aligned}$$

16

- We can verify that any stress function obtained in this fashion will satisfy the equilibrium equations

$$\varphi_{,221} + \mathcal{V}_{,1} - \varphi_{,122} - \mathcal{V}_{,1} = 0$$

$$\varphi_{,121} + \mathcal{V}_{,2} - \varphi_{,112} - \mathcal{V}_{,2} = 0$$

- We use the Beltrami-Mitchell equation for compatibility

$$\nabla^2 \nabla^2 \varphi = -\frac{2(\kappa - 1)}{1 + \kappa} \nabla^2 \mathcal{V}$$

17

- The general solution to the Beltrami-Mitchell equation can be written in the form

$$\varphi = \varphi_c + \varphi_p$$

- Where the complementary solution is a bi-harmonic function, while the particular solution depends on the body force

18

- Note: written in this manner the airy stress function is equally valid in polar or Cartesian coordinates, but note that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

19

airy solutions

- To solve a problem using Airy stress functions, we need to solve this biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

- One solution to this is the power series

$$\phi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n$$

20

- Note that terms for $m + n \leq 1$ do not contribute to the stress, and can be neglected
- Also note that for $m + n \leq 3$ compatibility is automatically satisfied
- For $m + n \geq 4$ the coefficients must be related for compatibility to be satisfied

21

fourier methods

- While polynomial methods are very useful, they are somewhat ad-hoc
- Fourier methods provide a more direct approach
- Consider a separable Airy Stress Function of the form

$$\phi(x, y) = X(x)Y(y)$$

- Where $X = e^{\alpha x}$ and $Y = e^{\beta y}$

22

- Substituting these values into the compatibility equation gives the requirement

$$(\alpha^4 + 2\alpha^2\beta^2 + \beta^4)e^{\alpha x}e^{\beta y} = 0$$

- For this equation to be satisfied,

$$(\alpha^2 + \beta^2)^2 = 0$$

- Which means $\alpha = \pm i\beta$

23

fourier methods

- Substituting this result back into the general expression for ϕ and converting to trigonometric and hyperbolic forms gives

$$\begin{aligned}\phi = & \sin \beta x [(A + C\beta y) \sinh \beta y + (B + D\beta y) \cosh \beta y] \\ & + \cos \beta x [(A' + C'\beta y) \sinh \beta y + (B' + D'\beta y) \cosh \beta y] \\ & + \sin \alpha x [(E + G\alpha x) \sinh \alpha x + (F + H\alpha x) \cosh \alpha x] \\ & + \cos \alpha x [(E' + G'\alpha x) \sinh \alpha x + (F' + H'\alpha x) \cosh \alpha x] \\ & + \phi_{\alpha=0} + \phi_{\beta=0}\end{aligned}$$

24

- Some problems with sinusoidal boundary conditions can be solved using the form developed, but a more general solution method incorporates a Fourier series
- Any periodic function with period $2L$ can be represented as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

25

- Where

$$a_n = \frac{1}{L} \int_{-L}^L f(\xi) \cos \frac{n\pi \xi}{L} d\xi \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(\xi) \sin \frac{n\pi \xi}{L} d\xi \quad n = 1, 2, 3, \dots$$

26

- There are many computational advantages to using a Fourier series (Fast Fourier Transform algorithms can be used)
- Analytically, Fourier Methods are particularly convenient when a boundary is known to either be an even or an odd function
- sin terms drop from even functions, while cos terms drop from odd functions

27

polar coordinates

- All solutions to the Beltrami-Mitchell equations in polar coordinates which are periodic in θ can be summarized as

$$\begin{aligned}
 \phi = & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
 & + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r) \theta \\
 & + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r \theta + a_{16} r \theta \log r \right) \cos \theta \\
 & + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r \theta + b_{16} r \theta \log r \right) \sin \theta \\
 & + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
 & + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
 \end{aligned}$$

28

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

29

polar coordinates

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2 \log r + 1$	0	$2 \log r + 3$
$r^2 \theta$	2θ	-1	2θ
$r^3 \cos \theta$	$2r \cos \theta$	$2r \sin \theta$	$6r \cos \theta$
$r^3 \sin \theta$	$2r \sin \theta$	$-2r \cos \theta$	$6r \sin \theta$
$r\theta \sin \theta$	$2 \cos \theta / r$	0	0
$r\theta \cos \theta$	$-2 \sin \theta / r$	0	0
$r \log r \cos \theta$	$\cos \theta / r$	$\sin \theta / r$	$\cos \theta / r$
$r \log r \sin \theta$	$\sin \theta / r$	$-\cos \theta / r$	$\sin \theta / r$
$\cos \theta / r$	$-2 \cos \theta / r^3$	$-2 \sin \theta / r^3$	$2 \cos \theta / r^3$
$\sin \theta / r$	$-2 \sin \theta / r^3$	$2 \cos \theta / r^3$	$2 \sin \theta / r^3$

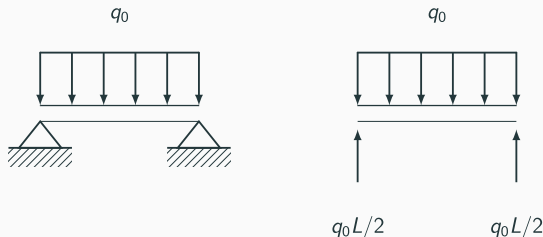
30

polar coordinates

$r^4 \cos 2\theta$	0	$6r^2 \sin 2\theta$	$12r^2 \cos 2\theta$
$r^4 \sin 2\theta$	0	$-6r^2 \cos 2\theta$	$12r^2 \sin 2\theta$
$r^2 \cos 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$	$2 \cos 2\theta$
$r^2 \sin 2\theta$	$-2 \sin 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$
$\cos 2\theta$	$-4 \cos 2\theta / r^2$	$-2 \sin 2\theta / r^2$	0
$\sin 2\theta$	$-4 \sin 2\theta / r^2$	$2 \cos 2\theta / r^2$	0
$\cos 2\theta / r^2$	$-6 \cos 2\theta / r^4$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$
$\sin 2\theta / r^2$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$	$6 \sin 2\theta / r^4$
$r^n \cos n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$	$n(n-1)r^{n-2} \cos n\theta$
$r^n \sin n\theta$	$-n(n-1)r^{n-2} \sin n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$
$r^{n+2} \cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$(n+1)nr^n \sin n\theta$	$(n+2)(n+1)r^n \cos n\theta$
$r^{n+2} \sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$-(n+1)nr^n \cos n\theta$	$(n+2)(n+1)r^n \sin n\theta$
$\cos n\theta / r^n$	$-(n+1)n \cos n\theta / r^{n+2}$	$-(n+1)n \sin n\theta / r^{n+2}$	$(n+1)n \cos n\theta / r^{n+2}$
$\sin n\theta / r^n$	$-(n+1)n \sin n\theta / r^{n+2}$	$(n+1)n \cos n\theta / r^{n+2}$	$(n+1)n \sin n\theta / r^{n+2}$
$\cos n\theta / r^{n-2}$	$-(n+2)(n-1) \cos n\theta / r^n$	$-n(n-1) \sin n\theta / r^n$	$(n-1)(n-2) \cos n\theta / r^n$
$\sin n\theta / r^{n-2}$	$-(n+2)(n-1) \sin n\theta / r^n$	$n(n-1) \cos n\theta / r^n$	$(n-1)(n-2) \sin n\theta / r^n$

31

pinned beam



32

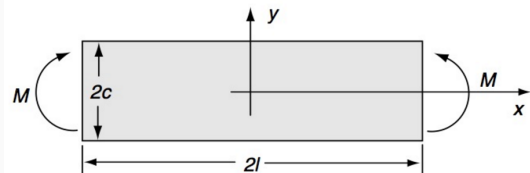


Figure 1: image

33

example

- Locally along the ends, there will be some tractions in order to apply the bending moment
- These tractions will cancel out, however, so we can use Saint Venant's principle to avoid modeling them explicitly

34

$$\sigma_y(x, \pm c) = 0$$

$$\tau_{xy}(x, \pm c) = \tau_{xy}(\pm L, y) = 0$$

$$\int_{-c}^c \sigma_x(\pm L, y) dy = 0$$

$$\int_{-c}^c \sigma_x(\pm L, y) y dy = -M$$

35

example

- What is the simplest form of polynomial stress function that would satisfy these boundary conditions?

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

36

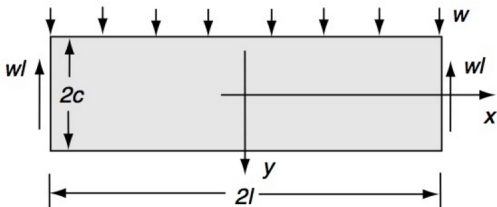


Figure 2: image

37

- In this case we can write the boundary conditions as

$$\tau_{xy}(x, \pm c) = 0$$

$$\sigma_y(x, c) = 0$$

$$\sigma_y(x, -c) = -w$$

$$\int_{-c}^c \sigma_x(\pm l, y) dy = 0$$

$$\int_{-c}^c \sigma_x(\pm l, y) y dy = 0$$

$$\int_{-c}^c \tau_{xy}(\pm l, y) dy = \mp wl$$

38

example

- And find that the stress function

$$\phi = Ax^2 + Bx^2y + Cx^2y^3 + Dy^3 - \frac{1}{5}Cy^5$$

- Can satisfy the boundary conditions as well as compatibility

39

example

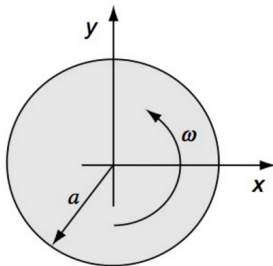


Figure 3: image

40

- Anisotropic elasticity - pp. 319 - 333