Continuum Mechanics

Lecture 3 - Tensor Calculus

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August 25, 2020

schedule

- 25 Aug - Tensor Calculus, HW1 Due
- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due
- 3 Sep - Polar Decomposition

outline

- tensor calculus
- other coordinate systems
- examples
- tensor review

dyadic notation

- There is an antiquated notation that you may encounter reading older papers and texts
- Now known as "dyadic notation" (or sometimes "tensor product notation")
- Dyadic product: $C_{ij} = a_i b_j$ is written as $C = a \otimes b$
- Double dot product: $A_{ij} B_{ji} = c$ is written as $A : B = c$

tensor valued functions

- If we consider some scalar variable, such as time, t
- A tensor function can be a function of this scalar variable, $T_{ij}(t)$
- The formal definition for the derivative of T_{ij} with respect to t is

$$\frac{dT_{ij}}{dt} = \lim_{\Delta t \to t} \frac{T_{ij}(t + \Delta t) - T_{ij}(t)}{\Delta t}$$

- In this case, all the rules apply, such as the chain rule

scalar fields

- A scalar-valued function of a position vector is known as a scalar field
- Density, temperature, electric potential are all examples of scalar fields
- The gradient, ∇, of a scalar field, ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x_i} = \langle \frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3} \rangle$$

directional derivative

- The gradient can be used to find the directional derivative, or the rate of change of ϕ in a certain direction
- If r_i is a vector in a direction, then the directional derivative of the scalar field ϕ in the r_i direction is

$$\nabla\phi \cdot r = \phi_{,i} r_i$$

- The vector produced by the gradient will be perpendicular to a surface of constant ϕ

vector fields

- A vector-valued function of a position vector is known as a vector field
- Velocity and displacement are common examples of vector fields
- The gradient of a vector field is a second-order tensor

$$\nabla v_i = v_{i,j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

divergence and curl

- The divergence of a vector field is a scalar

$$\operatorname{div}(v) = \operatorname{tr}(\nabla v) = \frac{\partial v_i}{\partial x_i}$$

- We can also find the divergence of a tensor field, for a second-order tensor we find

$$\operatorname{div}(T) = \frac{\partial T_{ij}}{\partial x_i}$$

- The curl is defined as two times the dual vector of the antisymmetric portion of ∇v

$$\operatorname{curl}(v) = 2t^A = -\epsilon_{ijk} v_{j,k}$$

laplacian

- The Laplacian of a scalar field is defined as

$$\nabla^2 f = \operatorname{div}(\nabla f)$$

- Which in rectangular coordinates is

$$\nabla^2 f = f_{i,i} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

laplacian

- The Laplacian of a vector field is defined as

$$\nabla^2 v = \nabla(\mathrm{div}(v)) - \mathrm{curl}(\mathrm{curl}(v))$$

- Which in rectangular coordinates is

$$\nabla^2 v = v_{i,jj}$$

other coordinate systems

- Many times it is beneficial to use another coordinate system
- Certain geometries and symmetries can be handled much more easily in polar coordinates, cylindrical coordinates, or spherical coordinates
- Let us first consider the 2D case, polar coordinates

polar coordinates

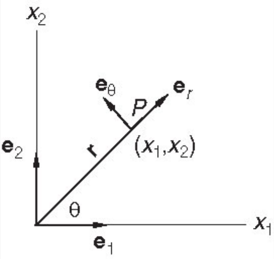


Figure 1: image

- We can readily see that

$$e_r = \langle \cos\theta, \sin\theta \rangle$$
$$e_\theta = \langle -\sin\theta, \cos\theta \rangle$$

polar coordinates

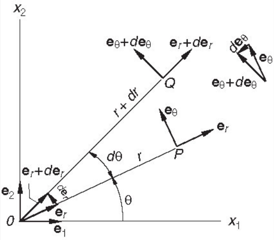


Figure 2: image

- We see that as the angle θ changes, so do the polar coordinate unit base vectors

$$de_r = \langle -\sin\theta, \cos\theta \rangle = d\theta e_\theta$$
$$de_\theta = \langle -\cos\theta, -\sin\theta \rangle = -d\theta e_r$$

polar coordinates

– We can now find the components of ∇f, ∇v_i, $\text{div}(v_i)$, $\text{div}(T_{ij})$, $\nabla^2 f$ and $\nabla^2 v_i$.

$$\nabla f = \langle f_{,r}, \frac{1}{r} f_{,\theta} \rangle$$

$$= \begin{bmatrix} v_{r,r} & \frac{1}{r}(v_{r,\theta} - v_\theta) \\ v_{\theta,r} & \frac{1}{r}(v_{\theta,\theta} - v_r) \end{bmatrix}$$

$$\text{div}(v_i) = \text{tr}(\nabla v_i) = v_{r,r} + \frac{1}{r}(v_{\theta,\theta} + v_r)$$

polar coordinates

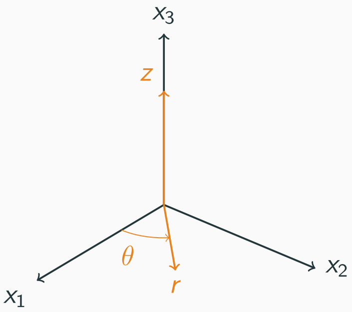
$$\text{curl}(v_i) = \langle 0, 0, v_{\theta,r} + \frac{v_\theta}{r} - \frac{1}{r} v_{r,\theta} \rangle$$

$$\text{div}(T_{ij}) = \langle T_{rr,r} + \frac{1}{r} T_{r\theta,\theta} + \frac{T_{rr} - T_{\theta\theta}}{r}, T_{r\theta,r} + \frac{1}{r} T_{\theta\theta,\theta} + \frac{T_{r\theta} + T_{\theta r}}{r} \rangle$$

$$\nabla^2 f = f_{,rr} + \frac{1}{r^2} f_{,\theta\theta} + \frac{1}{r} f_{,r}$$

$$\begin{aligned} \nabla^2 v_i =& \langle v_{r,rr} + \frac{1}{r^2} v_{r,\theta\theta} + v_{r,zz} + \frac{1}{r} v_{r,r} - \frac{2}{r^2} v_{\theta,\theta} - \frac{v_r}{r^2}, \\ & v_{\theta,rr} + \frac{1}{r^2} v_{\theta,\theta\theta} + \frac{1}{r} v_{\theta,r} + \frac{2}{r^2} v_{r,\theta} - \frac{v_\theta}{r^2} \rangle \end{aligned}$$

cylindrical coordinates



cylindrical coordinates

- Calculus in cylindrical coordinates is nearly identical to polar coordinates

$$\nabla f = \langle f_{,r}, \frac{1}{r} f_{,\theta}, f_{,z} \rangle$$

$$= \begin{bmatrix} v_{r,r} & \frac{1}{r}(v_{r,\theta} - v_\theta) & v_{r,z} \\ v_{\theta,r} & \frac{1}{r}(v_{\theta,\theta} + v_r) & v_{\theta,z} \\ v_{z,r} & \frac{v_{z,\theta}}{r} & v_{z,z} \end{bmatrix}$$

$$\mathrm{div}(v_i) = \mathrm{tr}(\nabla v_i) = v_{r,r} + \frac{1}{r}(v_{\theta,\theta} + v_r) + v_{z,z}$$

cylindrical coordinates

$$\begin{aligned}\mathrm{curl}(v_i) = \langle &\frac{v_{z,\theta}}{r} - v_{\theta,z},\\ &v_{r,z} - v_{z,r},\\ &v_{\theta,r} + \frac{v_\theta}{r} - \frac{v_{r,\theta}}{r}\rangle\end{aligned}$$

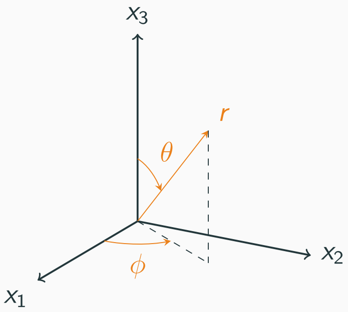
$$\begin{aligned}\mathrm{div}(T_{ij}) = \langle &T_{rr,r} + \frac{T_{r\theta,\theta}}{r} + \frac{T_{rr} - T_{\theta\theta}}{r} + T_{rz,z},\\ &T_{r\theta,r} + \frac{T_{\theta\theta,\theta}}{r} + \frac{T_{r\theta} + T_{\theta r}}{r} + T_{\theta z,z},\\ &T_{zr,r} + \frac{T_{z\theta,\theta}}{r} + T_{zz,z} + T_{zr,r}\rangle\end{aligned}$$

$$\nabla^2 f = f_{,rr} + \frac{1}{r^2}f_{,\theta\theta} + \frac{1}{r}f_{,r} + f_{,zz}$$

cylindrical coordinates

$$\begin{aligned}\nabla^2 v_i = \langle &v_{r,rr} + \frac{1}{r^2}v_{r,\theta\theta} + v_{r,zz} + \frac{1}{r}v_{r,r} - \frac{2}{r^2}v_{\theta,\theta} - \frac{v_r}{r^2},\\ &v_{\theta,rr} + \frac{1}{r^2}v_{\theta,\theta\theta} + v_{\theta,zz} + \frac{1}{r}v_{\theta,r} + \frac{2}{r^2}v_{r,\theta} - \frac{v_\theta}{r^2},\\ &v_{z,rr} + \frac{v_{z,\theta\theta}}{r^2} + \frac{v_{z,r}}{r} + v_{z,zz}\rangle\end{aligned}$$

spherical coordinates



spherical coordinates

- For calculus in spherical coordinates, we have

$$\nabla f = \langle f_{,r}, \frac{f_{,\theta}}{r}, \frac{f_{,\phi}}{r \sin\theta} \rangle$$

$$\nabla v_i = \begin{bmatrix} v_{r,r} & \frac{v_{r,\theta} - v_\theta}{r} & \frac{v_{r,\phi}}{r\sin\theta} - \frac{v_\phi}{r} \\ v_{\theta,r} & \frac{v_{\theta,\theta} + v_r}{r} & \frac{v_{\theta,\phi}}{r\sin\theta} - \frac{v_\phi \cot\theta}{r} \\ v_{\phi,r} & \frac{v_{\phi,\theta}}{r} & \frac{v_{\phi,\phi}}{r\sin\theta} + \frac{v_r + v_\theta\cot\theta}{r} \end{bmatrix}$$

$$\mathrm{div}(v_i) = \frac{(r^2 v_r)_{,r}}{r^2} + \frac{(v_\theta \sin\theta)_{,\theta}}{r\sin\theta} + \frac{v_{\phi,\phi}}{r\sin\theta}$$

spherical coordinates

$$\operatorname{curl}(v_i) = \langle \frac{v_\phi cot\theta + v_{\phi,\theta}}{r} - \frac{v_{\theta,\phi}}{r\sin\theta}, \frac{v_{r,\phi}}{r\sin\theta} - \frac{(rv_\phi)_{,r}}{r}, \frac{(rv_\theta)_{,r} - v_{r,\theta}}{r} \rangle$$

$$\operatorname{div}(T_{ij}) = \begin{matrix} \langle \dfrac{(r^2 T_{rr})_{,r}}{r^2} + \dfrac{(T_{r\theta}\sin\theta)_{,\theta} + T_{r\phi,\phi}}{r\sin\theta} - \dfrac{T_{\theta\theta} + T_{\phi\phi}}{r} \\ \dfrac{(r^3 T_{\theta r})_{,r}}{r^3} + \dfrac{(T_{\theta\theta}\sin\theta)_{,\theta} + T_{\theta\phi,\phi}}{r\sin\theta} + \dfrac{T_{r\theta} - T_{\theta r} - T_{\phi\phi}\cot\theta}{r} \\ \dfrac{(r^3 T_{\phi r})_{,r}}{r^3} + \dfrac{(T_{\phi\theta}\sin\theta)_{,\theta} + T_{\phi\phi,\phi}}{r\sin\theta} + \dfrac{T_{r\phi} - T_{\phi r} + T_{\theta\phi}\cot\theta}{r} \rangle \end{matrix}$$

spherical coordinates

$$\nabla^2 f = f_{,rr} + \frac{2f_{,r}}{r} + \frac{f_{,\theta\theta} + f_{,\theta}\cot\theta}{r^2} + \frac{f_{,\phi\phi}}{r^2\sin^2\theta}$$

$$\nabla^2 v_i = \begin{matrix} \langle \dfrac{(r^2 v_r)_{,rr} + v_{r,\theta\theta} + v_{r,\theta}\cot\theta}{r^2} - \dfrac{2(r^2 v_r)_{,r}}{r^3} + \dfrac{v_{r,\phi\phi}}{r^2\sin^2\theta} \\ - \dfrac{2(v_\theta\sin\theta)_{,\theta} + 2v_{\phi,\phi}}{r^2\sin\theta} \\ \dfrac{(r^2 v_{\theta,r})_{,r}\left(\frac{(v_\theta\sin\theta)_{,\theta}}{\sin\theta}\right)_{,\theta} + 2v_{r,\theta}}{r^2} + \dfrac{v_{\theta,\phi\phi}}{r^2\sin^2\theta} - \dfrac{2\cot\theta v_{\phi,\phi}}{r^2\sin\theta} \\ \dfrac{(r^2 v_{\phi,r})_{,r}\left(\frac{(v_\phi\sin\theta)_{,\theta}}{\sin\theta}\right)_{,\theta}}{r^2} + \dfrac{v_{\phi,\phi\phi}}{r^2\sin^2\theta} + \dfrac{2v_{r,\phi} + 2\cot\theta v_{\theta,\phi}}{r^2\sin\theta} \rangle \end{matrix}$$

spherical coordinates

- Calculate div(u_i) in spherical coordinates for the vector field

$$u_i = \langle Ar + \frac{B}{r^2}, 0, 0\rangle$$

group one

- Solve the following expression for T_{ij}

$$E_{ij} = \frac{1}{2\mu}\left[T_{ij} - \frac{\lambda}{3\lambda + 2\mu} T_{kk}\delta_{ij}\right]$$

- Hint: First solve for T_{kk}, and then substitute that result to complete the solution

group two

- For the tensor T_{ij},

$$T_{ji} = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

  find T'_{11} where $e'_1 = -e_2 + 2e_3$ and $e'_2 = e_1$
- Hint: unless you have an advanced calculator, this problem will be easier in tensor notation, since you need only find one term of the transformed tensor (let $i = j = 1$).

group three

- Consider the ellipsoidal surface defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

- Find the unit vector normal to the surface at some point, (x, y, z)

group four

– Calculate div(u_i) in cylindrical coordinates for the vector field

$$u_i = \langle \frac{\sin\theta}{r}, 0, 0 \rangle$$