Continuum Mechanics

Lecture 4 - Material Derivative

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schedule

- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due
- 3 Sep - Polar Decomposition
- 8 Sep - Exam Review

outline

- motion of a continuum
- material derivative
- displacement field
- deformation

motion of a continuum

- For a particle we define the path as a function of time
$$r_i = r_i(t) = \langle x_1(t), x_2(t), x_3(t) \rangle$$
- For N particles there will be N path lines
- In a continuum, there are infinitely many particles

motion of a continuum

- Instead of identifying particles by some identifying number, we identify them by their position

$$x_i = x_i(X_1, X_2, X_3, t) \qquad \text{with} \qquad X_i = x_i(X_1, X_2, X_3, t_0)$$

- (X_1, X_2, X_3) are known as the material coordinates and are used to identify the different particles of a body, while equation 4.2 describes a motion

motion of a continuum

- Sketch the motion described by

$$x_i = X_i + kt\langle 2X_1, X_2, 0\rangle$$

material and spatial

- Common properties that are functions of time in a continuum are temperature, Θ, velocity, v, and stress, T
- The material (lagrangian) description tracks particles, i.e.

$$\Theta = \hat{\Theta}(X_1, X_2, X_3, t)$$
$$v = \hat{v}(X_1, X_2, X_3, t)$$
$$T = \hat{T}(X_1, X_2, X_3, t)$$

material and spatial

- The spatial description tracks these properties through fixed locations, i.e.

$$\Theta = \tilde{\Theta}(x_1, x_2, x_3, t)$$
$$v = \tilde{v}(x_1, x_2, x_3, t)$$
$$T = \tilde{T}(x_1, x_2, x_3, t)$$

material and spatial

- The motion of a continuum is defined as

$$x_i = \langle X_1 + ktX_2, (1 + kt)X_2, X_3 \rangle$$

- The temperature field is defined as

$$\Theta = \alpha(x_1 + x_2)$$

- Find the material description of the temperature field and find the velocity and rate of change of temperature (in both material and spatial descriptions)

material derivative

- The time rate of change of some quantity of a material particle is known as the material derivative
- The material derivative is generally denoted as D/Dt
- When using the material (Lagrangian) description, the material derivative is simply

$$\frac{D\Theta}{Dt} = \left(\frac{\partial \hat{\Theta}}{\partial t} \right)_{X_i - fixed}$$

material derivative

- For the spatial description, however, x_i are not constant for a given material with fixed X_i, hence

$$\frac{D\Theta}{Dt} = \left(\frac{D\tilde{\Theta}}{Dt}\right)_{X_i-fixed} = \left(\frac{\partial\tilde{\Theta}}{\partial x_i}\right)\frac{\partial x_i}{\partial t} + \left(\frac{\partial\tilde{\Theta}}{\partial t}\right)_{x_i-fixed}$$

material derivative

- Since $\frac{\partial x_i}{\partial t}$ for fixed X_i are the velocity components of a given particle, we can also write

$$\frac{D\Theta}{Dt} = \left(\frac{D\tilde{\Theta}}{Dt}\right)_{X_i-fixed} = \frac{\partial\tilde{\Theta}}{\partial t} + v_i\tilde{\Theta}_{,i}$$

material derivative

- The above equations are valid only for a Cartesian coordinate system
- A more general definition is

$$\frac{D\Theta}{Dt} = \left(\frac{D\tilde{\Theta}}{Dt}\right)_{X_i - fixed} = \frac{\partial \tilde{\Theta}}{\partial t} + v_i \cdot \nabla \tilde{\Theta}$$

material derivative

- Find the material derivative, $D\Theta/Dt$ for the temperature field in the previous example

acceleration of a particle

– The acceleration of a particle is the material derivative of the velocity of a particle

$$a_i = \frac{Dv_i}{Dt}$$

acceleration of a particle

– If we know the material description of a particle's velocity, this calculation is simple
– If we only know the spatial description of velocity, we must use (4.15)

$$a_i = \frac{\partial \tilde{v}_i}{\partial t} + v_i \cdot \nabla \tilde{v}_i$$

acceleration of a particle

- A particle is rotating with angular velocity $\omega_i = \langle 0, 0, \omega \rangle$
- Find the velocity field
- Use the velocity field to find the acceleration field

displacement field

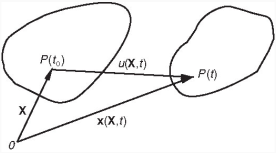

Figure 1: image

- The displacement vector for a particle in a continuum is defined as the vector from the reference position to the current position

$$u_i(X_i, t) = x_i(X_i, t) - X_i$$

rigid body motion

– Rigid translation is given by

$$x_i = X_i + c_i(t)$$

– Rigid rotation about a fixed point is given by

$$x_i - b_i = R_{ij}(t)(X_j - b_j)$$

Where $R_{ij}(t)$ is a proper orthogonal (rotation) tensor

rigid body motion

– When the rigid rotation is about the origin we have

$$x_i = R_{ij}(t)X_j$$

– General rigid body motion can include both translation and rotation

$$x_i = R_{ij}(t)(X_j - b_j) + c_i(t)$$

rigid body motion

Derive the relation between the velocity of a material point with the angular velocity of the body and velocity of the chosen point.

infinitesimal deformation

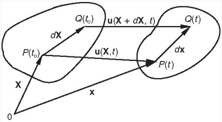

Figure 2: image

- We recall P, which undergoes some displacement, u
- A neighboring point, Q, at $X_i + dX_i$ arrives at $x_i + dx_i$

$$x_i + dx_i = X_i + dX_i + u_i(X_i + dX_i, t)$$

infinitesimal deformation

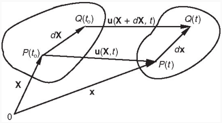

Figure 3: image

- Subtracting dx_i and using the definition of the gradient of a vector function, we have

$$dx_i = dX_i + u_{i,j}dX_j$$

infinitesimal deformation

- We can re-write (4.24)

$$dx_i = dX_i + u_{i,j}dX_j$$
$$dx_i = dX_j\delta_{ij} + u_{i,j}dX_j$$
$$dx_i = (u_{i,j} + \delta_{ij})dX_j$$

- We define the deformation gradient, F as $F = u_{i,j} + \delta_{ij}$

infinitesimal deformation

- We can find some interesting information by finding the length of dx_i to the undeformed length of dX_i

$$dx_i dx_i = F_{ij} dX_j F_{ik} dX_k$$

- We can rearrange this to

$$dx_i dx_i = dX_j F_{ij} F_{ik} dX_k$$

infinitesimal deformation

- We now define the right Cauchy-Green deformation tensor as $C_{jk} = F_{ij} F_{ik}$, and note that if $C_{jk} = \delta_{jk}$, then the deformed length is equal to the original length, corresponding to rigid body motion

lagrange strain tensor

- We can break down the right Cauchy-Green deformation tensor to derive the Lagrange strain tensor

$$C_{ij} = F_{ki}F_{kj} = F^T F = (I + \nabla u)^T (I + \nabla u) =$$
$$I + \nabla u + (\nabla u)^T + (\nabla u)^T (\nabla u)$$

lagrange strain tensor

- We recall that $C = I$ refers to rigid body motion, and thus define the Lagrange strain tensor as one-half of the deformation with no rigid body motion

$$E^* = \frac{1}{2}\left[\nabla u + (\nabla u)^T + (\nabla u)^T (\nabla u)\right]$$

lagrange strain tensor

- The Lagrange strain tensor is a finite deformation tensor
- For infinitesimal deformations, we assume that $(\nabla u)^T(\nabla u)$ is negligible when compared with ∇u
- In this case the Lagrange strain tensor would reduce to

$$E = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right]$$

lagrange strain tensor

- Which is simply the symmetric portion of ∇u
- In rectangular coordinates, we have

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

physical meaning

- If we consider two elements, $dX_i^{(1)}$ and $dX_i^{(2)}$
- Due to motion they become $dx_i^{(1)}$ and $dx_i^{(2)}$
- For small deformations, we know that

$$dx_i^{(1)}dx_i^{(2)} = F_{ij}dX_j^{(1)}F_{ik}dX_k^{(2)} = dX_j^{(1)}C_{jk}dX_k^{(2)} = dX_j^{(1)}(\delta_{jk} + 2E_{jk})dX_k^{(2)}$$

physical meaning

- Which we can expand to

$$dx_i^{(1)}dx_i^{(2)} = dX_i^{(1)}dX_i^{(2)} + 2E_{jk}dX_j^{(1)}dX_k^{(2)}$$

physical meaning

- If we look at the length of a single material element, $dX_i = dSdn_i$ we find the deformed length, ds to be

$$ds^2 = dS^2 + 2dS^2(n_i E_{ij} n_j)$$

- For small deformations, we make the assumption that

$$ds^2 - dS^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$$

physical meaning

- Which leads to

$$\frac{ds - dS}{dS} = n_i E_{ij} n_j$$

- This means that the diagonal terms of E_{ij} give the unit elongation for an element originally in the 1, 2 or 3 directions

physical meaning

- If we consider two unit vectors, m_i and n_i which are initially perpendicular, we have $dX_i^{(1)} = dS_1 m_i$ and $dX_i^{(2)} = dS_2 n_i$
- We can find the angle between the two deformed vectors, $dx_i^{(1)}$ and $dx_i^{(2)}$

$$dx_i^{(1)} dx_i^{(2)} = ds_1 ds_2 \cos\theta = 2E_{jk} dS_1 m_j dS_2 n_k$$

- Since the angle between the vectors was originally $\pi/2$, we define the change in angle as $\gamma = \pi/2 - \theta$

physical meaning

- We also note that $\cos\theta = \cos(\pi/2 - \gamma) = \sin\gamma$
- For small deformations (i.e. small γ) we have $\sin\gamma \approx \gamma$ and $\frac{ds_1}{dS_1} \approx 1$ and $\frac{ds_2}{dS_2} \approx 1$
- This gives

$$\gamma = 2E_{ij} m_i n_j$$

physical meaning

- We can isolate off-diagonal terms in E_{ij} by letting $m_i = \langle 1, 0, 0 \rangle$ and $n_j = \langle 0, 1, 0 \rangle$ (and other perpendicular directions)
- This means that $2E_{12}$ gives the change in angle between two elements initially in the x_1 and x_2 directions