Continuum Mechanics

Lecture 12 - Large Deformation

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering

8 October, 2020

schedule

- 8 Oct - Large Deformation
- 13 Oct - Anisotropy and Large Deformation
- 15 Oct - Exam Review
- 20 Oct - Exam 2

outline

- anisotropic solution techniques
- large deformation
- simple deformation modes

planar problems

- Many of our usual solution techniques are more difficult to apply to anisotropic materials
- For a general anisotropic material under the assumption of plane strain ($u_i = \langle u_1(x_1, x_2), u_2(x_1, x_2)\rangle$) the equilibrium equations (in terms of displacement) are

$$\begin{aligned}
C_{11}u_{1,11} + C_{66}u_{1,22} + C_{16}(2u_{1,12} + u_{2,11}) + C_{26}u_{2,22} + (C_{12} + C_{66})u_{2,12} &= 0 \\
C_{16}u_{1,11} + C_{26}u_{1,22} + (C_{66} + C_{12})u_{1,12} + C_{66}u_{2,11} + C_{22}u_{2,22} + 2C_{26}u_{2,12} &= 0 \\
C_{15}u_{1,11} + C_{46}u_{1,22} + (C_{56} + C_{14})u_{1,12} + C_{56}u_{2,11} + C_{24}u_{2,22} + (C_{25} + C_{46})u_{2,12} &= 0
\end{aligned}$$

planar problems

- Only two of these three equations can be solved with a plane strain displacement assumption
- This means that a general anisotropic body cannot be solved in plane strain
- However if a material possesses at least monoclinic symmetry, enough terms vanish that plane strain is an acceptable solution
- Alternatively, a generalized plane strain solution can be used for any anisotropic body

$$u = \langle u_1(x_1, x_2), u_2(x_1, x_2), u_3(x_1, x_2) \rangle$$

stroh representation

- Stroh developed a complex variable representation for generalized plane strain solutions in anisotropic materials
- If we assume a displacement field in the form $u_i = a_i f(z)$ where $z = x_1 + px_2$ the equilibrium equations in terms of displacement then become

$$C_{ijkl} u_{k,il} = 0$$

stroh representation

- Since $\partial z/\partial x_i = \delta_{i1} + p\delta_{i2}$, we find that

$$u_{k,l} = a_k(\delta_{l1}+p\delta_{l2})f'(z) \qquad u_{k,il} = a_k(\delta_{l1+p\delta_{l2}})(\delta_{j1}+p\delta_{j2})f''(z)$$

stroh representation

- The equilibrium equations with a generalized plane strain displacement function now become

$$C_{ijkl}(\delta_{l1} + p\delta_{l2})(\delta_{j1} + p\delta_{j2})a_k f''(z) = 0$$

- For non-trivial solutions (when $f''(z) \neq 0$), we can re-write the equations as

$$a_k = 0$$

stroh representation

- If we define the following quantities in terms of the stiffness tensor

$$Q = \begin{bmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{bmatrix} \qquad R = \begin{bmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{bmatrix}$$

$$T = \begin{bmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{bmatrix}$$

stroh representation

- we can now re-write the equilibrium equations in matrix form as

$$a = 0$$

- This is a type of eigenvalue problem

stroh representation

- Mathematically, it can be shown that for a material with physically admissible elastic constants, p will always be complex
- There will be three pairs of solutions $(p, \bar{p})$, and three pairs of complex-valued eigenvectors, $(a_i, \bar{a}_i)$

stroh representation

- Stress solutions can be calculated by defining b_i vectors

$$a^{(i)} = b^{(i)}$$

- Which gives stress solutions as

$$\sigma_{i1} = -p b_i f'(z) \qquad \sigma_{i2} = b_i f'(z)$$

change of frame

- For small deformations, the current and deformed frame have negligible differences
- For large deformations, we need to ensure that our constitutive law is objective, or frame-indifferent

change of frame

- Examples:
  - distance between two material points is a frame-indifferent scalar
  - speed of a material point is not frame-independent (non-objective)
  - position vector and velocity vector of a material point are non-objective
  - relative velocity between two material points is objective

change of frame

- In general, a "frame" can have its own time scale, origin, and directionality
- a change of frame would then be given by

$$x_i^* = c_i(t) + Q_{ij}(t)(x_j - x_j^0)$$

change of frame

- If we consider the position vector of two material points, in the starred frame we have

$$x_1^* = c(t) + Q(t)(x_1 - x_0) \qquad x_2^* = c(t) + Q(t)(x_2 - x_0)$$

- The relative position vector, $b = x_1 - x_2$ can also be found in the starred frame as

$$b^* = x_1^* - x_2^* = Q(t)(x_1 - x_2)$$

- Any vector which obeys this law is known as an objective vector

change of frame

- If we consider some tensor, T_{ij} which transforms an objective vector b_j into another objective vector, c_i

$$c_i = T_{ij} b_j$$

- Since both b and c are objective vectors, we can write

$$c_i^* = Q_{ik} c_k = Q_{ik} T_{kj} b_j = Q_{ik} T_{kj} Q_{lj} b_l$$

change of frame

- This means that for T_{ij} to be objective it must satisfy

$$T_{ij}^* = Q_{ik} T_{kl} Q_{jl}$$

examples

- dx_i
- ds
- v_i
- F_{ij}
- C_{ij}
- B_{ij}

group problems

- Group 1: Is the first Piola-Kirchhoff stress tensor objective? How does it transform? Recall:

$$T^0_{ij} = J T_{ik} F^{-1}_{jk}$$

- Group 2: Is the second Piola-Kirchhoff stress tensor objective? How does it transform? Recall:

$$\tilde{T}_{ij} = J F^{-1}_{ik} T_{kl} F^{-1}_{jl}$$

constitutive equations

- For large deformation, a constitutive law must be objective
- $T_{ij} = f(C_{ij})$ is not an acceptable form, but $T_{ij} = f(B_{ij})$ is
- $\tilde{T}_{ij} = f(C_{ij})$ is also an acceptable form of the constitutive equation
- If we assume our material to be isotropic, it can be shown that, with no loss of generality

$$f(B_{ij}) = a_0\delta_{ij} + a_1 B_{ij} + a_2 B_{ij}^2$$

constitutive equations

- A commonly used alternate form of the constitutive equation is

$$T_{ij} = \varphi_0\delta_{ij} + \varphi_1 B_{ij} + \varphi_2 B_{ij}^{-1}$$

- If the material is incompressible, the stress is indeterminate to some arbitrary hydrostatic pressure

$$T_{ij} = -p\delta_{ij} + \varphi_1 B_{ij} + \varphi_2 B_{ij}^{-1}$$

money rivlin

- This is known as the Mooney-Rivlin model, which can be written in different ways

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{1}{2} + \beta\right)B_{ij} - \mu\left(\frac{1}{2} - \beta\right)B_{ij}^{-1}$$

neo-hookean solid

- A simpler model, which is only accurate for strains less than 20%, is the neo-Hookean solid
- For incompressible materials, the neo-Hookean equation is

$$T_{ij} = -p\delta_{ij} + 2C_1 B_{ij}$$

neo-hookean solid

- Where, for consistency with Hooke's Law, $2C_1 = \mu$
- When large tensile strains are not important, the neo-Hookean model is popular because it only needs one material constant, which has more physical meaning than Mooney-Rivlin constants.

incompressible stretch

- In large deformation problems, the stretch ratio, λ is often used (instead of strain)
- λ_1 represents the ratio of deformed length in x_1 to undeformed length in x_1
- For uniaxial extension we have

$$x_1 = \lambda_1 X_1$$
$$x_2 = \lambda_2 X_2$$
$$x_3 = \lambda_2 X_3$$

- Also if the material is incompressible we know

$$\lambda_1 \lambda_2^2 = 1$$

simple shear

- For large shear deformation, we have

$$x_1 = X_1 + KX_2$$
$$x_2 = X_2$$
$$x_3 = X_3$$

- And we find B and B^{-1} as

$$B_{ij} = \begin{bmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_{ij}^{-1} = \begin{bmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reading material

- Thermodynamics formulation of Mooney-Rivlin models - link
- Anisotropy in large deformation - "A New Constitutive Framework for Arterial Wall Mechanics and a Comparative Study of Material Models"
- Paper is interesting, but long, pp 10-21 are the most relevant.