Continuum Mechanics

Lecture 1 - Tensors

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering

August 18, 2020

schedule

- 18 Aug - Introduction, Tensors
- 20 Aug - Tensor Algebra
- 25 Aug - Tensor Calculus, HW1 Due
- 27 Aug - Material Derivative

outline

- introduction
- syllabus
- course overview
- index notation
- example

family

education

- B.S. in Mechanical Engineering from Brigham Young University
  - Worked with ATK to develop tab-less gripping system for tensile testing composite tow specimens
  - Needed to align the specimen, as well as grip it without causing a stress concentration

education

- M.S. and Ph.D. from School of Aeronautics and Astronautics at Purdue University
  - Worked with Boeing to simulate mold flows
  - First ever mold simulation with anisotropic viscosity

research

Figure 1: picture of chopped carbon fiber prepreg

research

Figure 2: picture of lamborghini symbol made from compression molded chopped carbon fiber

research

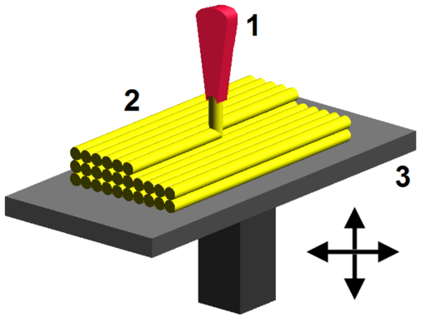

Figure 3: picture illustrating the fused deposition modeling 3D printing process, where plastic filament is melted and deposited next to other filament, and fuses together

introductions

- Name
- Student status (Undergrad, Masters, Ph.D)
- Full time or part time student?
- One interesting thing to remember you by
- What are you hoping to learn in Continuum Mechanics?

course textbook

- The textbook used in this class is: *Introduction to Continuum Mechanics*, W. Lai, 4th Ed.
- Homework problems will be posted on Blackboard, so other textbook editions may be used
- For additional reference in continuum mechanics, see
  - A.J.M. Spencer, *Continuum Mechanics*
  - G.E. Mase, *Schaum's Outline of Continuum Mechanics*
  - Y.C. Fung, *A First Course in Continuum Mechanics*

office hours

- I will e-mail everyone in the course a Doodle link we can use to find the optimal office hours
- Let me know if you do not receive the e-mail, you may need to update your information in Blackboard
- Take advantage of office hours, this is time that I have already set aside for you
- If the regular office hours do not work for your schedule, send me an e-mail and we can work out a time to meet
- While in person visits are often the most helpful, I will always try to answer questions as best as I can via e-mail

tentative course outline

- Tensors, Deformation, and Strain
  - Tensor Algebra (18 Aug)
  - Tensor Calculus (25 Aug)
  - Kinematics (3 Sep)
  - Exam 1 (15 Sep)

tentative course outline

- Behavior of Solids
  - Stress (22 Sep)
  - Linear Elasticity (29 Sep)
  - Airy Stress Functions (6 Oct)
  - Anisotropy (13 Oct)
  - Large Deformation (20 Oct)
  - Exam 2 (27 Oct)

tentative course outline

- Fluids and viscous solids
  - Newtonian Fluids (3 Nov)
  - Non-Newtonian Fluids (10 Nov)
  - Viscoelasticity (17 Nov)

grades

- Grade breakdown
- Homework 10%
- Exam 1 20%
- Exam 2 20%
- Final Exam 30%
- Research Project 20%

grades

- Follow a traditional grading scale
- (80% B-, 83% B, 87% B+, etc.)

curve

- I do NOT curve final grades
- Instead, each individual exam is curved on a best-fit linear scale
- This scale is somewhat subjective, best score is mapped to 100, I pick one other score to map that I feel is representative of a C or a B
- The end goal of this curve is to get a standard deviation close to 10% and a class average representative of the performance on the exam, usually between a C and a B

class expectations

- Consider the cost (to you or others) of your being in class
- I ask that you refrain from distracting behaviors during class
- When you have something more important than class to take care of, please take care of it outside of class

academic honesty

- While you are welcome to participate in study groups, every student must submit their own work
- As an example, every student should create their own Excel spreadsheet or MATLAB code for a problem, using a group spreadsheet with slight modifications is not acceptable
- All parties involved homework cheating will receive a zero for the assignment for a first offense
- Cheating on exams and repeat offenses will be handled on a case-by-case basis, but can lead to a failing grade in the course and expulsion from the university

self-grade

- Your homework will be self-graded, your self-grading will generally be due the week after the original assignment
- Homework solutions will be posted to Blackboard, and the remaining half of the homework credit will be assigned after you complete (and submit) your self-grade.
- You do not lose credit for incorrect answers, but your self-grade should explain the differences between your answer and the correct solution.
- Some problems will be somewhat open-ended and there may not be a "correct" answer, so consider that when looking at what is different between your solution and mine

what is continuum mechanics

- Study of the response of materials to different loading conditions
- Previous courses (mechanics of materials, theory of elasticity) focus on special cases (2D problems, small deformation, linear elastic materials, isotropy)
- In this course we will consider more general cases, such as large deformation, anisotropy, fluid response, and viscoelastic materials

index notation

- Consider the following

$$s = a_1 x_1 + a_2 x_2 + ... + a_n x_n$$

- Which we could also write as

$$s = \sum_{i=1}^{n} a_i x_i$$

index notation

- Using index notation, and Einstein's summation convention, we can also write this as

$$s = a_i x_i$$

dummy index

- In index notation, a repeated index implies summation
- This index is also referred to as a dummy index
- It is called a “dummy index” because the expression would have the same meaning with any index in its place

dummy index

- i.e. i, j, k, etc. would all have the same meaning when repeated
- Note, no index may be repeated more than once, thus the expression

$$s = \sum_{i=1}^{n} a_i b_i x_i$$

could not be directly written in index notation

free index

- Any index which is not repeated in an index notation expression is referred to as a free index
- The number of free indexes in an expression indicate the tensor order of that expression
- No free indexes = scalar expression (0-order tensor)
- One free index = vector expression (1st-order tensor)
- Two free indexes = matrix expression (2nd-order tensor)

index notation

Free Index - NOT repeated (on any term) - takes all values (1,2,3) - e.g. $u_i = \langle u_1, u_2, u_3 \rangle$ - must match across terms in an expression or equation

Dummy Index - IS repeated on at least one term - indicates summation over all values - e.g. $\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ - can not be used more than twice in the same term ($A_{ij}B_{jk}C_{kl}$ is good, $A_{ij}B_{ij}C_{ij}$ is not)

dummy index

- The dummy index can be triggered by any repeated index in a **term**
- Summation or not?
  - $a_i + b_{ij}c_j$
  - $a_{ij} + b_{ij}$
  - $a_{ij} + b_{ij}c_j$

matrix multiplication

- How can we write matrix multiplication in index notation?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

- $c_{11} = a_{11}b_{11} + a_{12}b_{21}$
- $c_{12} = a_{11}b_{21} + a_{12}b_{22}$

special symbols

- For convenience we define two symbols in index notation
- *Kronecker delta* is a general tensor form of the Identity Matrix

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is also used for higher order tensors

special symbols

- $\delta_{ij} = \delta_{ji}$
- $\delta_{ii} = 3$
- $\delta_{ij} a_j = a_i$
- $\delta_{ij} a_{ij} = a_{ii}$

special symbols

- *alternating symbol* or *permutation symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

- This symbol is not used as frequently as the *Kronecker delta*

special symbols

- For our uses in this course, it is enough to know that 123, 231, and 312 are even permutations
- 321, 132, 213 are odd permutations
- all other indexes are zero
- $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$

substitution

- When solving tensor equations, we often need to manipulate expressions
- We need to make sure the correct indexes are used when substituting, for example

$$a_i = U_{im} b_m \tag{1}$$

$$b_i = V_{im} c_m \tag{2}$$

- To substitute (2) into (1), we first need to change indexes

substitution

- We need to change the free index, i, to m in
- Since m is already used as the dummy index, we need to change that too

$$b_m = V_{mj} c_j \tag{3}$$

- We can now make the substitution

$$a_i = U_{im} V_{mj} c_j \tag{4}$$

multiplication

- We need to be careful with indexes when multiplying expressions

$$p = a_m b_m \qquad \text{and} \qquad q = c_m d_m$$

- We can express, pq, but remember the dummy index cannot be repeated more than once

$$pq \neq a_m b_m c_m d_m$$

multiplication

- Instead we must change the dummy index in one of the expressions first

$$pq = a_m b_m c_n d_n$$

factoring

- In the following expression, we would like to factor out n, but it has different indexes

$$T_{ij}n_j - \lambda n_i = 0$$

- Recall $\delta_{ij}a_j = a_i$, we can rewrite $n_i = \delta_{ij}n_j$

$$T_{ij}n_j - \lambda \delta_{ij}n_j = 0$$

$$(T_{ij} - \lambda \delta_{ij})n_j = 0$$

contraction

- T_{ii} is the contraction of T_{ij}
- This can often be a useful tool in solving tensor equations

$$T_{ij} = \lambda \Delta \delta_{ij} + 2\mu E_{ij}$$

$$T_{ii} = \lambda \Delta \delta_{ii} + 2\mu E_{ii}$$

partial derivative

- We indicate (partial) derivatives using a comma
- In three dimensions, we take the partial derivative with respect to each variable (x, y, z or x_1, x_2, x_3)
- For example a scalar property, such as density, can have a different value at any point in space
- $\rho = \rho(x_1, x_2, x_3)$

$$\rho_{,i} = \frac{\partial}{\partial x_i}\rho = \left\langle \frac{\partial \rho}{\partial x_1}, \frac{\partial \rho}{\partial x_2}, \frac{\partial \rho}{\partial x_3} \right\rangle$$

partial derivative

- Similarly, if we take the partial derivative of a vector, it produces a matrix

$$u_{i,j} = \frac{\partial}{\partial x_j}u_i = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

example

- Solve the equation below for U_k in terms of P_i and a_i.

$$\mu \left[\delta_{kj} a_i a_i + \frac{1}{1-2\nu} a_k a_j\right] U_k = P_j$$

- Multiply both sides by a_j

$$\mu \left[a_j \delta_{kj} a_i a_i + \frac{1}{1-2\nu} a_k a_j a_j\right] U_k = P_j a_j$$

example

- The dummy indexes can be changed

$$\mu \left[a_j \delta_{kj} a_i a_i + \frac{1}{1-2\nu} a_k a_i a_i\right] U_k = P_j a_j$$

- $a_j \delta_{kj} = a_k$

$$\mu U_k \left[a_k a_i a_i + \frac{1}{1-2\nu} a_k a_i a_i\right] = P_j a_j$$

example

- Factoring

$$\mu U_k a_k a_i a_i \left[1 + \frac{1}{1-2\nu}\right] = P_j a_j$$

- Simplifying

$$\mu U_k a_k a_i a_i \left[\frac{2(1-\nu)}{1-2\nu}\right] = P_j a_j$$

example

- Solve for $U_k a_k$

$$U_k a_k = \frac{P_j a_j (1-2\nu)}{2\mu a_i a_i (1-\nu)}$$

- This is a scalar equation, we need to find U_j, but we substitute this back into the original equation.

example

– First, expand the original equation

$$\mu U_k \delta_{kj} a_i a_i + \mu U_k \frac{1}{1-2\nu} a_k a_j = P_j$$

– After substitution, we find

$$\mu U_j a_i a_i + \mu \frac{1}{1-2\nu} \frac{P_j a_j (1-2\nu)}{2\mu a_i a_i (1-\nu)} a_j = P_j$$

example

– The index j is repeated too many times, so we need to change $P_j a_j$ to a different index

$$\mu U_j a_i a_i + \frac{P_k a_k}{2 a_i a_i (1-\nu)} a_j = P_j$$

– We can now solve for U_j

$$U_j = \frac{1}{\mu a_i a_i} \left[P_j - \frac{P_k a_k}{2 a_i a_i (1-\nu)} a_j \right]$$