Continuum Mechanics

Lecture 2 - Tensor Algebra

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schedule

- 20 Aug - Tensor Algebra
- 25 Aug - Tensor Calculus, HW1 Due
- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due

outline

symmetry

- Symmetry can be a very powerful tool
- Here we define some useful forms of symmetry in index notation
- Symmetric
  - $a_{ij...z} = a_{z...ji}$
  - $a_{ij...m...n...z} = a_{ij...n...m...z}$
- Anti-symmetric (skew symmetric)
  - $a_{ij...z} = -a_{z...ji}$
  - $a_{ij...m...n...z} = -a_{ij...n...m...z}$

symmetry

- Useful identity
  - If \(a_{ij...m...n...k}\) is symmetric in \(mn\) and \(b_{pq...m...n...r}\) is antisymmetric in \(mn\), then the product is zero [a_{ij...m...n...k}b_{pq...m...n...r} = 0]
- We can also write any tensor as the sum of its symmetric and anti-symmetric parts [a_{ij} = \frac{1}{2} (a_{ij} + a_{ji}) + \frac{1}{2} (a_{ij} - a_{ji})]
- This textbook uses a special shortcut notation (S and A superscript) for the symmetric and anti-symmetric portions of a tensor

linear transformation

- Let us consider some transformation, \(\textbf{T}\), which transforms any vector into another vector
- If we transform \(\textbf{Ta} = c\) and \(\textbf{Tb} = d\)
- We call \(\textbf{T}\) a linear transformation (and a tensor) if [\begin{aligned} \textbf{T}(\textbf{a} + \textbf{b}) &= \textbf{Ta} + \textbf{Tb}\ \textbf{T}(\alpha \textbf{a}) = \alpha\textbf{Ta} \end{aligned}]
- Where \(\alpha\) is any arbitrary scalar and \(\textbf{a}\), \(\textbf{b}\) are arbitrary vectors

coordinate transformation in two dimensions

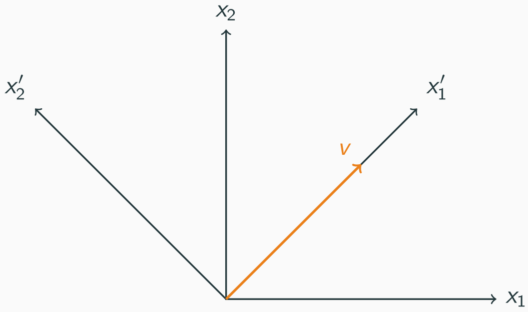

Figure 1: 2d coordinate transformation example with vector pointing from (0,0) to (1,1)

coordinate transformation in two dimensions

- The vector, `\(v\)`, remains fixed, but we transform our coordinate system
- In the new coordinate system, the `\(x_2^\prime\)` portion of `\(v\)` is zero.
- To transform the coordinate system, we first define some unit vectors.
- `\(\hat{e}_1\` is a unit vector in the direction of `\(x_1\)`, while `\(\hat{e}_1^\prime\)` is a unit vector in the direction of `\(x_1^\prime\)`

coordinate transformation in two dimensions

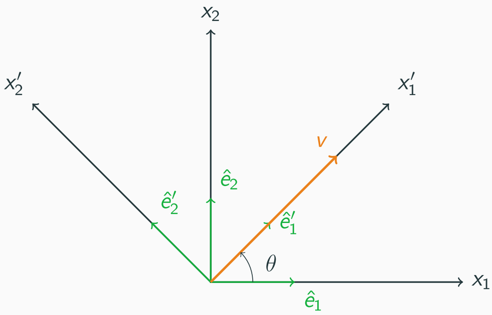

Figure 2: 2d coordinate transformation from previous figure with unit vectors drawn along the x and y axes

coordinate transformation in two dimensions

- For this example, let us assume `\(v = \langle 2, 2 \rangle\)` and `\(\theta = 45^\circ\)`
- We can write the transformed unit vectors, `\(\hat{e}_1^\prime\)` and `\(\hat{e}_2^\prime\)` in terms of `\(\hat{e}_1\)`, `\(\hat{e}_2\)` and the angle of rotation, `\(\theta\)`. `[\begin{aligned} \hat{e}_1^\prime &= \langle \hat{e}_1 \cos \theta , \hat{e}_2 \sin \theta\rangle\     \hat{e}_2^\prime &= \langle -\hat{e}_1 \sin \theta , \hat{e}_2 \cos \theta \rangle \end{aligned}]`

coordinate transformation in two dimensions

- We can write the vector, \(v\), in terms of the unit vectors describing our axis system
- \(v = v_1 \hat{e}_1 + v_2 \hat{e}_2\)
- (note: \(\hat{e}_1=\langle 1, 0 \rangle\) and \(\hat{e}_2 = \langle 0,1 \rangle\))
- \(v = \langle 2, 2 \rangle = 2 \langle 1, 0 \rangle + 2 \langle 0,1 \rangle\)

coordinate transformation in two dimensions

- When expressed in the transformed coordinate system, we refer to \(v^\prime\)
- \(v^\prime = \langle v_1 \cos \theta + v_2 \sin \theta, -v_1 \sin \theta + v_2 \cos \theta \rangle\)
- \(v^\prime = \langle 2\sqrt{2}, 0 \rangle\)
- We can recover the original vector from the transformed coordinates:
- \(v = v_1^\prime \hat{e}_1^\prime + v_2^\prime \hat{e}_2^\prime\)
- (note: [\hat{e}_1^\prime=\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2} \rangle] and [\hat{e}_2^\prime = \langle

general coordinate transformation

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- We define Q_{ij} as the cosine of the angle between the $x_i^\prime$ axis and the x_j axis.
- This is also referred to as the “direction cosine” $$Q_{ij} = \cos (x_i^\prime, x_j)$$
- *health warning* the direction cosine can also be defined inversely ($Q_{ij} =\cos (x_i, x_j^\prime)$), and the indexes are switched in the transformation law

general coordinate transformation

- We can use this form on our 2D transformation example $$\begin{aligned} Q_{ij} &= \cos (x_i^\prime, x_j)\\ &= \begin{bmatrix} \cos (x_1^\prime, x_1) & \cos (x_1^\prime, x_2)\\ \cos (x_2^\prime, x_1) & \cos (x_2^\prime, x_2) \end{bmatrix}\\ &= \begin{bmatrix} \cos \theta & \cos (90-\theta)\\ \cos (90+\theta) & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

general coordinate transformation

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v^\prime_i = Q_{ij}v_j$
- Matrices (second-order tensors): $$\sigma_{mn}^\prime =Q_{mi}Q_{nj}\sigma_{ij}$$
- Fourth-order tensors: $$C_{ijkl}^\prime = Q_{im}Q_{jn}Q_{ko}Q_{lp}C_{mnop}$$

general coordinate transformation

- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji}v_j^\prime$
- Matrices (second-order tensors): $\sigma_{mn} =Q_{im}Q_{jn}\sigma_{ij}^\prime$
- Fourth-order tensors: $C_{ijkl} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{mnop}^\prime$

general coordinate transformation

- We can derive some interesting properties of the transformation tensor, `\(Q_{ij}\)`
- We know that `\(v_i = Q_{ji}v_j^\prime\)` and that `\(v^\prime_i = Q_{ij}v_j\)`
- If we substitute (changing the appropriate indexes) we find: `[v_i = Q_{ji}Q_{jk}v_k]`
- We can now use the Kronecker Delta to substitute `\(v_i = \delta_{ik}v_k\)` which gives `[\delta_{ik}v_k = Q_{ji}Q_{jk}v_k]`

example

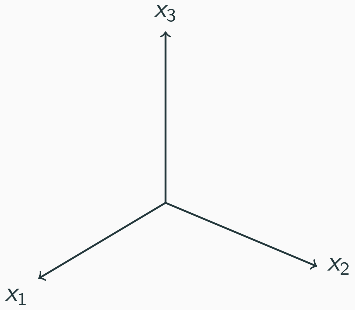

Figure 3: 3d coordinate system to start general transformation example

example

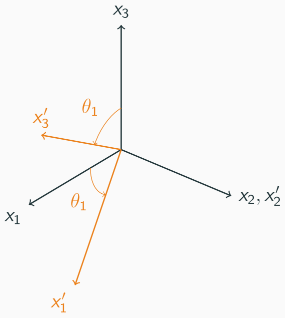

Figure 4: 3d illustration of first transformation

example

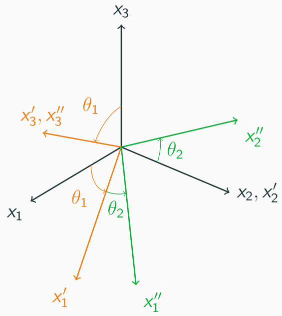

Figure 5: 3d illustration of second transformation (about the axes of the first)

example

- \(Q_{ij}^1 = \cos (x_i^\prime,x_j)\)
- \(Q_{ij}^2 = \cos (x_i^{\prime\prime},x_j^\prime)\) [Q_{ij}^1 = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150\ \cos 90 & \cos 0 & \cos 90\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}] [Q_{ij}^2 = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90\ \cos 120 & \cos 30 & \cos 90\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}]

example

- We now use \(Q_{ij}\) to find \(\hat{e}_i^\prime\) and \(\hat{e}_i^{\prime \prime}\)
- First, we need to write \(\hat{e}_i\) in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. \(\hat{e}_1 = e_i^1\)
- \(e_i^\prime = Q^1_{ij} e_j\)
- \(e_i^{\prime\prime} = Q^2_{ij} e_j^\prime\)

example

- How do we find $e_i^{\prime\prime}$ in terms of e_i?
- $e_i^{\prime\prime} = Q^2_{ij} Q^1_{jk} e_k$

principal values

- In the 2D coordinate transformation example, we were able to eliminate one value from a vector using coordinate transformation
- For second-order tensors, we desire to find the "principal values" where all non-diagonal terms are zero
- The direction determined by the unit vector, n_j, is said to be the *principal direction* or *eigenvector* of the symmetric second-order tensor, a_{ij} if there exists a parameter, λ, such that [$a_{ij} n_j = \lambda n_i$]
- Where λ is called the *principal value* or *eigenvalue* of the tensor

principal values

- We can re-write the equation `[(a_{ij} - \lambda \delta_{ij})n_j = 0]`
- This system of equations has a non-trivial solution if and only if `\(\det [a_{ij} - \lambda \delta_{ij}] = 0\)`
- This equation is known as the characteristic equation, and we solve it to find the principal values of a tensor

example

- Find the principal values of the tensor `[A_{ij} = \begin{bmatrix}     1 & 2\     2 & 4 \end{bmatrix}]`
- From the characteristic equation, we know that `\(\det [A_{ij} - \lambda \delta_{ij}] = 0\)`, or `[\begin{vmatrix}     1-\lambda & 2\     2 & 4 - \lambda \end{vmatrix} = 0]`

example

- Calculating the determinant gives `[(1-\lambda)(4-\lambda) - 4 = 0]`
- Multiplying out and simplifying, we find `[\lambda^2 - 5\lambda = \lambda(\lambda-5) = 0]`
- This has the solution `\(\lambda = 0, 5\)`

invariants

- Every tensor has some invariants which do not change with coordinate transformation
- These are known as *fundamental invariants*
- The characteristic equation for a tensor in 3D can be written in terms of the invariants `[\det [ a_{ij} - \lambda \delta_{ij}] = -\lambda^3 + I_\alpha \lambda^2 - II_\alpha \lambda + III_\alpha = 0]`

invariants

- The invariants can be found by the following equations [\begin{aligned}   I_\alpha &= a_{ii}\   II_\alpha &= \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})\   III_\alpha &= \det [a_{ij}] \end{aligned}]

invariants

- In the principal direction, \(a_{ij}^\prime\) will be [a_{ij}^\prime = \begin{bmatrix} \lambda_1 & 0 & 0\   0 & \lambda_2 & 0\   0 & 0 & \lambda_3 \end{bmatrix}]
- Since invariants do not change with coordinate systems, we can also write the invariants as [\begin{aligned}   I_\alpha &= \lambda_1 + \lambda_2 + \lambda_3\ II_\alpha &= \lambda_1\lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1\   III_\alpha &= \lambda_1 \lambda_2 \lambda_3 \end{aligned}]

principal directions

- We defined principal directions earlier $$(a_{ij} - \lambda \delta_{ij})n_j = 0$$
- λ are the principal values and n_j are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for λ back into this equation

example

- Find the principal directions for the earlier principal values example
- Recall $\lambda = 0,5$, let us say $\lambda_1 = 5$, we find $n_j^{(1)}$ by $$\begin{bmatrix} 1-\lambda_1 & 2\\ 2 & 4-\lambda_1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$
- This gives $$\begin{bmatrix} -4 & 2\\ 2 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

example

- We proceed to solve the equations using row-reduction, but we find $$\begin{bmatrix} 2 & -1\\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$
- This means we cannot uniquely solve for n_j
- We are only concerned with the direction, magnitude is not important
- Choose $n_2 = 1$, solve for n_1
- $n^{(1)} = \langle \frac{1}{2}, 1 \rangle$

example

- Similarly, for $\lambda_2 =0$, we find $$\begin{bmatrix} 1 & 2\\ 2 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$
- Which, after row-reduction, becomes $$\begin{bmatrix} 1 & 2\\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$
- If we choose $n_2 = 1$, we find $n_1 = -2$
- This gives $n^{(2)} = \langle -2, 1 \rangle$

example

- We can assemble a transformation matrix, Q_{ij}, from the principal directions
- First we need to normalize them to unit vectors
- $||n^{(1)}|| = \sqrt{\frac{5}{4}}$
- $\hat{n}^{(1)} = \frac{2}{\sqrt{5}} \langle \frac{1}{2}, 1 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$
- $||n^{(2)}|| = \sqrt{5}$
- $\hat{n}^{(2)} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$

example

- This gives $Q_{ij} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2\\ -2 & 1 \end{bmatrix}$
- And we find $A_{mn}^\prime = Q_{mi}Q_{nj}A_{ij}$ $A_{ij}^\prime = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

example

- Find principal values, principal directions, and invariants for the tensor $[c_{ij}] = \begin{bmatrix} 8 & 0 & 0\\ 0 & 3 & 1\\ 0 & 1 & 3 \end{bmatrix}$

example

- Characteristic equation simplifies to
- $-\lambda^3 + 14\lambda^2 -56 \lambda + 64 = 0$
- Which has the solutions $\lambda = 2, 4, 8$

example

- To find the principal direction for \(\lambda_1 = 8\) [\begin{bmatrix} 8-8 & 0 & 0\ 0 & 3-8 & 1\ 0 & 1 & 3-8 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]

example

- After row-reduction, we find [\begin{bmatrix} 0 & 0 & 0\ 0 & 0 & -24\ 0 & 1 & -5 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]
- This means that \(n_3 = 0\) and, as a result, \(n_2 = 0\).
- \(n_1\) can be any value, we choose \(n_1 = 1\) to give a unit vector.
- \(n^{(1)} = \langle 1, 0, 0 \rangle\)

example

- To find the principal direction for \(\lambda_2 = 4\) [\begin{bmatrix}   8-4 & 0 & 0\ 0 & 3-4 & 1\   0 & 1 & 3-4 \end{bmatrix}\begin{Bmatrix}   n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]

example

- After row-reduction, we find [\begin{bmatrix} 4 & 0 & 0\   0 & -1 & 1\   0 & 0 & 0 \end{bmatrix}\begin{Bmatrix}   n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]
- This means that \(n_1 = 0\)
- We also know that \(n_2 = n_3\), so we choose \(n_2 = n_1 = 1\)
- This gives \(n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle\) after normalization

example

- To find the principal direction for $\lambda_3 = 2$ $$\begin{bmatrix} 8-2 & 0 & 0\\ 0 & 3-2 & 1\\ 0 & 1 & 3-2 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

example

- After row-reduction, we find $$\begin{bmatrix} 6 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$
- This means that $n_1 = 0$
- We also know that $n_2 = -n_3$, so we choose $n_2 = 1$ and $n_1 = -1$
- This gives $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1 \rangle$ after normalization

example

- In summary, for c_{ij} we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0 \rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1 \rangle$
- We can assemble $n^{(i)}$ into a transformation tensor $$Q_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1 \end{bmatrix}$$

example

- What is $c_{ij}^\prime$?
- $c_{ij}^\prime = Q_{im}Q_{jn}c_{mn}$ $$c_{ij}^\prime = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

example

- We can also find the invariants for `[c_{ij} = \begin{bmatrix}     8 & 0 & 0\     0 & 3 & 1\     0 & 1 & 3 \end{bmatrix}]`
- Recall: `[\begin{aligned}     I_\alpha &= a_{ii}\     II_\alpha &= \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})\     III_\alpha &= \det [ a_{ij}] \end{aligned}]`

example

- First invariant `[I_\alpha = a_{ii} = 8 + 3 + 3 = 14]`
- Second invariant `[II_\alpha = \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})]` `[a_{ii} a_{jj} = 14 \times 14]` `[a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + ... + a_{33}a_{33}]` `[II_\alpha = \frac{1}{2}(196 - 84) = 56]`

example

- Third invariant `[III_\alpha = \det [ a_{ij}]]` `[III_\alpha = 8 \times (3 \times 3 - 1 \times 1) = 64]`