Continuum Mechanics

Lecture 10 - Airy Stress

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schedule

- 1 Oct - Airy Stress
- 6 Oct - Elastic Solid
- 8 Oct - Elastic Solid
- 13 Oct - Exam Review

outline

- planar problems
- airy stress function
- examples

plane strain

- If motion in any one direction (for example x_3) is restricted, such that $u_3 = 0$, a body is said to be in a state of plane strain
- The plane strain is often used for very thick materials, where $u_3 \ll u_1, u_2$, but is also applicable any time u_3 is restricted
- Under plane strain conditions we have

$$
\begin{aligned}
E_{13} &= E_{23} = E_{33} = 0 \\
E_{11} &= E_{11}(x_1, x_2) \\
E_{22} &= E_{22}(x_1, x_2) \\
E_{12} &= E_{12}(x_1, x_2)
\end{aligned}
$$

plane strain

- Using Hooke's Law, we find that $T_{13} = T_{23} = 0$, but

$$T_{33} = \nu(T_{11} + T_{22})$$

- While the other stress components, T_{11}, T_{12} and T_{22} are all functions of x_1 and x_2
- In the absence of body forces, we find the equilibrium equations as

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = 0$$
$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} = 0$$
$$\frac{\partial T_{33}}{\partial x_3} = 0$$

plane stress

- For very thin bodies, we often make the assumption that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$
- Since we have started from an assumption in stress instead of displacement, it is not yet apparent whether this stress field is allowable
- To simplify calculations for equilibrium and compatibility, we define a stress function, φ such that

$$\sigma_{11} = \varphi_{,22}$$
$$\sigma_{22} = \varphi_{,11}$$
$$\sigma_{12} = -\varphi_{,12}$$

plane stress

- We can quickly verify that this satisfies equilibrium

$$\varphi_{,221} - \varphi_{,122} = 0$$
$$\varphi_{,121} - \varphi_{,112} = 0$$

- We will not follow all the details, but from compatibility we find

$$\varphi = \varphi_0(x_1, x_2) - \frac{\nu}{1+\nu}\Psi(x_1, x_2)\frac{1}{2}x_3^2$$

- For plane stress problems, we generally consider the case where $x_3 \ll x_1, x_2$, and thus we can neglect the second term, giving our stresses as functions of x_1 and x_2 only
- Note that in general plane stress solutions only approximately satisfy compatibility

beltrami-mitchell

- It is convenient to write the compatibility equations in terms of stress
- This is known as the Beltrami-Mitchell equations
- For planar problems, the Beltrami-Mitchell equations are

$$\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{4\rho}{1+\kappa}\left(b_{1,1} + b_{2,2}\right)$$

- κ is used to differentiate between plane strain and plane stress

$$\kappa = \begin{cases} 3 - 4\nu & \text{Plane Strain} \\ \frac{3-\nu}{1+\nu} & \text{Plane Stress} \end{cases}$$

polar coordinates

- Many planar problems are more conveniently expressed in polar coordinates
- For planar problems in polar coordinates the Beltrami-Mitchell equations become

$$\nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = -\frac{4\rho}{1+\kappa}\left(b_{r,r} + \frac{1}{r}b_{\theta,\theta}\right)$$

- There is a good summary of in-plane field equations in the supplemental elasticity text on page 127 (Cartesian) and 131-132 (Polar)

body forces

- Body forces are often neglected, but they can be included in the Airy stress formulation by defining a potential function as follows

$$\rho b_1 = -\frac{\partial \mathcal{V}}{\partial x_1}$$
$$\rho b_2 = -\frac{\partial \mathcal{V}}{\partial x_2}$$

airy stress

- Let us now define the Airy stress function, $\varphi(x_1, x_2)$ such that

$$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2} + \mathcal{V}$$
$$T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$$
$$T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2} + \mathcal{V}$$

airy stress

- We can verify that any stress function obtained in this fashion will satisfy the equilibrium equations

$$\varphi_{,221} + \mathcal{V}_{,1} - \varphi_{,122} - \mathcal{V}_{,1} = 0$$
$$\varphi_{,121} + \mathcal{V}_{,2} - \varphi_{,112} - \mathcal{V}_{,2} = 0$$

- We use the Beltrami-Mitchell equation for compatibility

$$\nabla^2 \nabla^2 \varphi = -\frac{2(\kappa - 1)}{1 + \kappa} \nabla^2 \mathcal{V}$$

airy stress

- The general solution to the Beltrami-Mitchell equation can be written in the form

$$\varphi = \varphi_c + \varphi_p$$

- Where the complementary solution is a bi-harmonic function, while the particular solution depends on the body force
- Note: written in this manner the airy stress function is equally valid in polar or Cartesian coordinates, but note that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

airy stress function solutions

- To solve a problem using Airy stress functions, we need to solve this biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

- One solution to this is the power series

$$\phi(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} x^m y^n$$

power series solution

- Note that terms for $m + n \leq 1$ do not contribute to the stress, and can be neglected
- Also note that for $m + n \leq 3$ compatibility is automatically satisfied
- For $m + n \geq 4$ the coefficients must be related for compatibility to be satisfied

fourier methods

- While polynomial methods are very useful, they are somewhat ad-hoc
- Fourier methods provide a more direct approach
- Consider a separable Airy Stress Function of the form

$$\phi(x, y) = X(x)Y(y)$$

- Where $X = e^{\alpha x}$ and $Y = e^{\beta y}$

fourier methods

- Substituting these values into the compatibility equation gives the requirement

$$(\alpha^4 + 2\alpha^2\beta^2 + \beta^4)e^{\alpha x}e^{\beta y} = 0$$

- For this equation to be satisfied,

$$(\alpha^2 + \beta^2)^2 = 0$$

- Which means $\alpha = \pm i\beta$

fourier methods

- Substituting this result back into the general expression for ϕ and converting to trigonometric and hyperbolic forms gives

$$\begin{aligned}
\phi = &\sin\beta x\,[(A + C\beta y)\sinh\beta y + (B + D\beta y)\cosh\beta y] \\
&+ \cos\beta x\,[(A' + C'\beta y)\sinh\beta y + (B' + D'\beta y)\cosh\beta y] \\
&+ \sin\alpha y\,[(E + G\alpha x)\sinh\alpha x + (F + H\alpha x)\cosh\alpha x] \\
&+ \cos\alpha y\,[(E' + G'\alpha x)\sinh\alpha x + (F' + H'\alpha x)\cosh\alpha x] \\
&+ \phi_{\alpha=0} + \phi_{\beta=0}
\end{aligned}$$

fourier methods

- Some problems with sinusoidal boundary conditions can be solved using the form developed, but a more general solution method incorporates a Fourier series
- Any periodic function with period $2L$ can be represented as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

- Where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}d\xi \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}d\xi \quad n = 1, 2, 3, \ldots$$

fourier methods

- There are many computational advantages to using a Fourier series (Fast Fourier Transform algorithms can be used)
- Analytically, Fourier Methods are particularly convenient when a boundary is known to either be an even or an odd function
- sin terms drop from even functions, while cos terms drop from odd functions

polar coordinates

- All solutions to the Beltrami-Mitchell equations in polar coordinates which are periodic in θ can be summarized as

$$
\begin{aligned}
\phi = {} & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
& + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r \right) \cos\theta \\
& + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r \right) \sin\theta \\
& + \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
& + \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}
$$

polar coordinates

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to

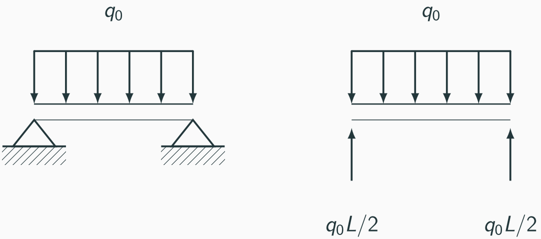
$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

polar coordinates

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$

Figure 1: image

pinned beam



example

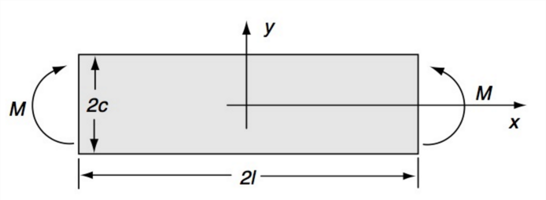


Figure 2: image

example

- Locally along the ends, there will be some tractions in order to apply the bending moment
- These tractions will cancel out, however, so we can use Saint Venant's principle to avoid modeling them explicitly

$$\sigma_y(x, \pm c) = 0$$
$$\tau_{xy}(x, \pm c) = \tau_{xy}(\pm L, y) = 0$$
$$\int_{-c}^{c} \sigma_x(\pm l, y) dy = 0$$
$$\int_{-c}^{c} \sigma_x(\pm l, y) y dy = -M$$

example

- What is the simplest form of polynomial stress function that would satisfy these boundary conditions?

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

example

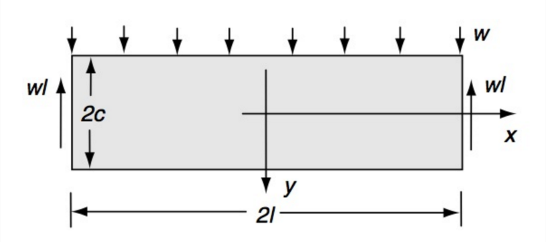


Figure 3: image

example

- In this case we can write the boundary conditions as

$$\tau_{xy}(x, \pm c) = 0$$
$$\sigma_y(x, c) = 0$$
$$\sigma_y(x, -c) = -w$$
$$\int_{-c}^{c} \sigma_x(\pm l, y)dy = 0$$
$$\int_{-c}^{c} \sigma_x(\pm l, y)ydy = 0$$
$$\int_{-c}^{c} \tau_{xy}(\pm l, y)dy = \mp wl$$

example

- And find that the stress function

$$\phi = Ax^2 + Bx^2y + Cx^2y^3 + Dy^3 - \frac{1}{5}Cy^5$$

- Can satisfy the boundary conditions as well as compatibility

example

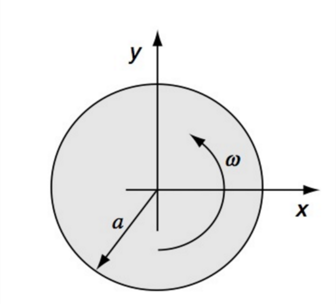


Figure 4: image

reading for next class

- Anisotropic elasticity - pp. 319 - 333