# Continuum Mechanics

Lecture 2 - Tensor Algebra

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# schedule

- 20 Aug - Tensor Algebra
- 25 Aug - Tensor Calculus, HW1 Due
- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due

## outline

- symmetry
- transformation
- examples
- principal values
- invariants
- principal directions
- examples



## symmetry

- Symmetry can be a very powerful tool
- Here we define some useful forms of symmetry in index notation
- Symmetric
  - $a_{ij...z} = a_{z...ji}$
  - $a_{ij...m...n...z} = a_{ij...n...m...z}$
- Anti-symmetric (skew symmetric)
  - $a_{ij...z} = -a_{z...ji}$
  - $a_{ij...m...n...z} = -a_{ij...n...m...z}$

## symmetry

- Useful identity
  - If $a_{ij...m...n...k}$ is symmetric in $mn$ and $b_{pq...m...n...r}$ is antisymmetric in $mn$, then the product is zero $$a_{ij...m...n...k}b_{pq...m...n...r} = 0$$
- We can also write any tensor as the sum of its symmetric and anti-symmetric parts $$a_{ij} = \frac{1}{2} (a_{ij} + a_{ji}) + \frac{1}{2} (a_{ij} - a_{ji})$$
- This textbook uses a special shortcut notation (S and A superscript) for the symmetric and anti-symmetric portions of a tensor



## linear transformation

- Let us consider some transformation, $\textbf{T}$, which transforms any vector into another vector
- If we transform $\textbf{Ta} = c$ and $\textbf{Tb} = d$
- We call $\textbf{T}$ a linear transformation (and a tensor) if $$\begin{aligned} \textbf{T}(\textbf{a} + \textbf{b}) &= \textbf{Ta} + \textbf{Tb}\\ \textbf{T}(\alpha \textbf{a}) = \alpha\textbf{Ta} \end{aligned}$$
- Where $\alpha$ is any arbitrary scalar and $\textbf{a}$, $\textbf{b}$ are arbitrary vectors

# 2d coordinate transformation

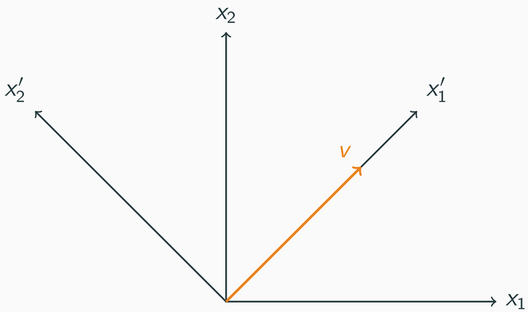


**Figure 1:** 2d coordinate transformation example with vector pointing from (0,0) to (1,1)



# 2d coordinate transformation

- The vector, `\(v\)`, remains fixed, but we transform our coordinate system
- In the new coordinate system, the `\(x_2^\prime\)` portion of `\(v\)` is zero.
- To transform the coordinate system, we first define some unit vectors.
- `\(\hat{e}_1\)` is a unit vector in the direction of `\(x_1\)`, while `\(\hat{e}_1^\prime\)` is a unit vector in the direction of `\(x_1^\prime\)`

## 2d coordinate transformation

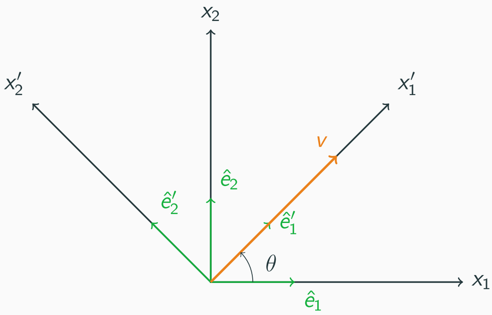


**Figure 2:** 2d coordinate transformation from previous figure with unit vectors drawn along the x and y axes



## 2d coordinate transformation

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45^\circ$
- We can write the transformed unit vectors, $\hat{e}_1^\prime$ and $\hat{e}_2^\prime$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, $\theta$. $$\begin{aligned} \hat{e}_1^\prime &= \langle \hat{e}_1 \cos \theta , \hat{e}_2 \sin \theta\rangle\\ \hat{e}_2^\prime &= \langle -\hat{e}_1 \sin \theta , \hat{e}_2 \cos \theta \rangle \end{aligned}$$

## 2d coordinate transformation

- We can write the vector, `\(v\)`, in terms of the unit vectors describing our axis system
- `\(v = v_1 \hat{e}_1 + v_2 \hat{e}_2\)`
- (note: `\(\hat{e}_1=\langle 1, 0 \rangle\)` and `\(\hat{e}_2 = \langle 0,1 \rangle\)`)
- `\(v = \langle 2, 2 \rangle = 2 \langle 1, 0 \rangle + 2 \langle 0,1 \rangle\)`



## 2d coordinate transformation

- When expressed in the transformed coordinate system, we refer to `\(v^\prime\)`
- `\(v^\prime = \langle v_1 \cos \theta + v_2 \sin \theta, -v_1 \sin \theta + v_2 \cos \theta \rangle\)`
- `\(v^\prime = \langle 2\sqrt{2}, 0 \rangle\)`
- We can recover the original vector from the transformed coordinates:
- `\(v = v_1^\prime \hat{e}_1^\prime + v_2^\prime \hat{e}_2^\prime\)`
- (note: `[\hat{e}_1^\prime=\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2} \rangle]` and `[\hat{e}_2^\prime = \langle`

## coordinate transformation

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- We define $Q_{ij}$ as the cosine of the angle between the $x_i^\prime$ axis and the $x_j$ axis.
- This is also referred to as the "direction cosine" $$Q_{ij} = \cos (x_i^\prime, x_j)$$
- *health warning* the direction cosine can also be defined inversely ($Q_{ij} =\cos (x_i, x_j^\prime)$), and the indexes are switched in the transformation law



## coordinate transformation

- We can use this form on our 2D transformation example $$\begin{aligned} Q_{ij} &= \cos (x_i^\prime, x_j)\\ &= \begin{bmatrix} \cos (x_1^\prime, x_1) & \cos (x_1^\prime, x_2)\\ \cos (x_2^\prime, x_1) & \cos (x_2^\prime, x_2) \end{bmatrix}\\ &= \begin{bmatrix} \cos \theta & \cos (90-\theta)\\ \cos (90+\theta) & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

## coordinate transformation

- We can transform any-order tensor using $Q_{ij}$
- Vectors (first-order tensors): $v^\prime_i = Q_{ij}v_j$
- Matrices (second-order tensors): $\sigma_{mn}^\prime =Q_{mi}Q_{nj}\sigma_{ij}$
- Fourth-order tensors: $C_{ijkl}^\prime = Q_{im}Q_{jn}Q_{ko}Q_{lp}C_{mnop}$



## coordinate transformation

- We can similarly use $Q_{ij}$ to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji}v_j^\prime$
- Matrices (second-order tensors): $\sigma_{mn} =Q_{im}Q_{jn}\sigma_{ij}^\prime$
- Fourth-order tensors: $C_{ijkl} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{mnop}^\prime$

## coordinate transformation

- We can derive some interesting properties of the transformation tensor, `\(Q_{ij}\)`
- We know that `\(v_i = Q_{ji}v_j^\prime\)` and that `\(v^\prime_i = Q_{ij}v_j\)`
- If we substitute (changing the appropriate indexes) we find: `[v_i = Q_{ji}Q_{jk}v_k]`
- We can now use the Kronecker Delta to substitute `\(v_i = \delta_{ik}v_k\)` which gives `[\delta_{ik}v_k = Q_{ji}Q_{jk}v_k]`



## example

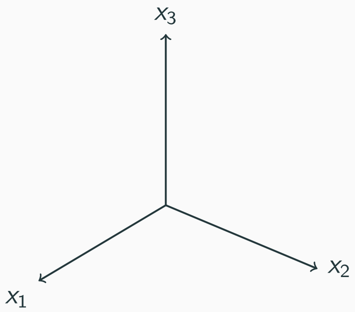


**Figure 3:** 3d coordinate system to start general transformation example

# example

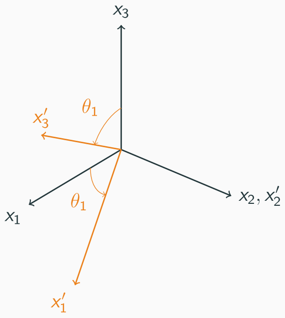


**Figure 4:** 3d illustration of first transformation



# example

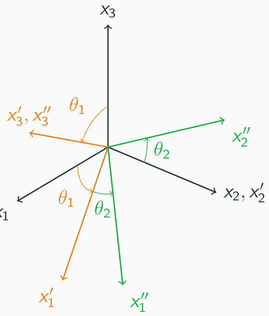


**Figure 5:** 3d illustration of second transformation (about the axes of the first)

# example

- `\(Q_{ij}^1 = \cos (x_i^\prime,x_j)\)`
- `\(Q_{ij}^2 = \cos (x_i^{\prime\prime},x_j^\prime)\) [Q_{ij}^1 = \begin{bmatrix}     \cos 60 & \cos 90 & \cos 150\\     \cos 90 & \cos 0 & \cos 90\\     \cos 30 & \cos 90 & \cos 60     \end{bmatrix}] [Q_{ij}^2 = \begin{bmatrix}     \cos 30 & \cos 60 & \cos 90\\     \cos 120 & \cos 30 & \cos 90\\     \cos 90 & \cos 90 & \cos 0 \end{bmatrix}]`



# example

- We now use `\(Q_{ij}\)` to find `\(\hat{e}_i^\prime\)` and `\(\hat{e}_i^{\prime \prime}\)`
- First, we need to write `\(\hat{e}_i\)` in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. `\(\hat{e}_1 = e_i^1\)`
- `\(e_i^\prime = Q^1_{ij} e_j\)`
- `\(e_i^{\prime\prime} = Q^2_{ij} e_j^\prime\)`

## example

- How do we find $e_i^{\prime\prime}$ in terms of $e_i$?
- $e_i^{\prime\prime} = Q^2_{ij} Q^1_{jk} e_k$



## principal values

- In the 2D coordinate transformation example, we were able to eliminate one value from a vector using coordinate transformation
- For second-order tensors, we desire to find the "principal values" where all non-diagonal terms are zero
- The direction determined by the unit vector, $n_j$, is said to be the *principal direction* or *eigenvector* of the symmetric second-order tensor, $a_{ij}$ if there exists a parameter, $\lambda$, such that [$a_{ij} n_j = \lambda n_i$]
- Where $\lambda$ is called the *principal value* or *eigenvalue* of the tensor

## principal values

- We can re-write the equation `[(a_{ij} - \lambda \delta_{ij})n_j = 0]`
- This system of equations has a non-trivial solution if and only if `\(\det [a_{ij} - \lambda \delta_{ij}] = 0\)`
- This equation is known as the characteristic equation, and we solve it to find the principal values of a tensor



## example

- Find the principal values of the tensor `[A_{ij} = \begin{bmatrix}      1 & 2\\      2 & 4 \end{bmatrix}]`
- From the characteristic equation, we know that `\(\det [A_{ij} - \lambda \delta_{ij}] = 0\)`, or `[\begin{vmatrix}      1-\lambda & 2\\      2 & 4 - \lambda \end{vmatrix} = 0]`

## example

- Calculating the determinant gives `[(1-\lambda)(4-\lambda) - 4 = 0]`
- Multiplying out and simplifying, we find `[\lambda^2 - 5\lambda = \lambda(\lambda-5) = 0]`
- This has the solution `\(\lambda = 0, 5\)`



## invariants

- Every tensor has some invariants which do not change with coordinate transformation
- These are known as *fundamental invariants*
- The characteristic equation for a tensor in 3D can be written in terms of the invariants `[\det [ a_{ij} - \lambda \delta_{ij}] = -\lambda^3 + I_\alpha \lambda^2 - II_\alpha \lambda + III_\alpha = 0]`

## invariants

- The invariants can be found by the following equations [\begin{aligned} I_\alpha &= a_{ii}\\ II_\alpha &= \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})\\ III_\alpha &= \det [ a_{ij}] \end{aligned}]



## invariants

- In the principal direction, \(a_{ij}^\prime\) will be [a_{ij}^\prime = \begin{bmatrix} \lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3 \end{bmatrix}]
- Since invariants do not change with coordinate systems, we can also write the invariants as [\begin{aligned} I_\alpha &= \lambda_1 + \lambda_2 + \lambda_3\\ II_\alpha &= \lambda_1\lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1\\ III_\alpha &= \lambda_1 \lambda_2 \lambda_3 \end{aligned}]

## principal directions

- We defined principal directions earlier `[(a_{ij} - \lambda \delta_{ij})n_j = 0]`
- `\(\lambda\)` are the principal values and `\(n_j\)` are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for `\(\lambda\)` back into this equation



## example

- Find the principal directions for the earlier principal values example
- Recall `\(\lambda = 0,5\)`, let us say `\(\lambda_1 = 5\)`, we find `\(n_j^{(1)}\)` by `[\begin{bmatrix}     1-\lambda_1 & 2\\ 2 & 4-\lambda_1 \end{bmatrix} \begin{Bmatrix}     n_1 \ n_2 \end{Bmatrix} = 0]`
- This gives `[\begin{bmatrix}     -4 & 2\\ 2 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \ n_2 \end{Bmatrix} = 0]`

## example

- We proceed to solve the equations using row-reduction, but we find `[\begin{bmatrix} 2 & -1\\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \ n_2 \end{Bmatrix} = 0]`
- This means we cannot uniquely solve for `\(n_j\)`
- We are only concerned with the direction, magnitude is not important
- Choose `\(n_2 = 1\)`, solve for `\(n_1\)`
- `\(n^{(1)} = \langle \frac{1}{2}, 1 \rangle\)`



## example

- Similarly, for `\(\lambda_2 =0\)`, we find `[\begin{bmatrix} 1 & 2\\ 2 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \ n_2 \end{Bmatrix} = 0]`
- Which, after row-reduction, becomes `[\begin{bmatrix} 1 & 2\\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \ n_2 \end{Bmatrix} = 0]`
- If we choose `\(n_2 = 1\)`, we find `\(n_1 = -2\)`
- This gives `\(n^{(2)} = \langle -2, 1 \rangle\)`

## example

- We can assemble a transformation matrix, `\(Q_{ij}\)`, from the principal directions
- First we need to normalize them to unit vectors
- `\(||n^{(1)}|| = \sqrt{\frac{5}{4}}\)`
- `\(\hat{n}^{(1)} = \frac{2}{\sqrt{5}} \langle \frac{1}{2}, 1 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle\)`
- `\(||n^{(2)}|| = \sqrt{5}\)`
- `\(\hat{n}^{(2)} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle\)`



## example

- This gives `[Q_{ij} = \frac{1}{\sqrt{5}}\begin{bmatrix}     1 & 2\\     -2 & 1 \end{bmatrix}]`
- And we find `[A_{mn}^\prime = Q_{mi}Q_{nj}A_{ij}]` `[A_{ij}^\prime = \begin{bmatrix}     5 & 0 \\     0 & 0 \end{bmatrix}]`

## example

- Find principal values, principal directions, and invariants for the tensor `[c_{ij} = \begin{bmatrix}     8 & 0 & 0\\     0 & 3 & 1\\     0 & 1 & 3 \end{bmatrix}]`



## example

- Characteristic equation simplifies to
- `\(-\lambda^3 + 14\lambda^2 -56 \lambda + 64 = 0\)`
- Which has the solutions `\(\lambda = 2, 4, 8\)`

## example

- To find the principal direction for $\lambda_1 = 8$ $$\begin{bmatrix} 8-8 & 0 & 0\\ 0 & 3-8 & 1\\ 0 & 1 & 3-8 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0$$



## example

- After row-reduction, we find $$\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -24\\ 0 & 1 & -5 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0$$
- This means that $n_3 = 0$ and, as a result, $n_2 = 0$.
- $n_1$ can be any value, we choose $n_1 = 1$ to give a unit vector.
- $n^{(1)} = \langle 1, 0, 0 \rangle$

## example

- To find the principal direction for `\(\lambda_2 = 4\)` `[\begin{bmatrix} 8-4 & 0 & 0\\ 0 & 3-4 & 1\\ 0 & 1 & 3-4 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]`



## example

- After row-reduction, we find `[\begin{bmatrix} 4 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]`
- This means that `\(n_1 = 0\)`
- We also know that `\(n_2 = n_3\)`, so we choose `\(n_2 = n_1 = 1\)`
- This gives `\(n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle\)` after normalization

# example

- To find the principal direction for `\(\lambda_3 = 2\)` `[\begin{bmatrix}     8-2 & 0 & 0\\     0 & 3-2 & 1\\     0 & 1 & 3-2     \end{bmatrix}\begin{Bmatrix}     n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]`



# example

- After row-reduction, we find `[\begin{bmatrix}     6 & 0 & 0\\     0 & 1 & 1\\     0 & 0 & 0 \end{bmatrix}\begin{Bmatrix}     n_1 \ n_2 \ n_3 \end{Bmatrix} = 0]`
- This means that `\(n_1 = 0\)`
- We also know that `\(n_2 = -n_3\)`, so we choose `\(n_2 = 1\)` and `\(n_1 = -1\)`
- This gives `\(n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1 \rangle\)` after normalization

## example

- In summary, for $c_{ij}$ we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0 \rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1 \rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1 \rangle$
- We can assemble $n^{(i)}$ into a transformation tensor $$Q_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & -1 \end{bmatrix}$$



## example

- What is $c_{ij}^\prime$?
- $c_{ij}^\prime = Q_{im}Q_{jn}c_{mn}$ $$c_{ij}^\prime = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## example

- We can also find the invariants for $$c_{ij} = \begin{bmatrix} 8 & 0 & 0\\ 0 & 3 & 1\\ 0 & 1 & 3 \end{bmatrix}$$
- Recall: $$\begin{aligned} I_\alpha &= a_{ii}\\ II_\alpha &= \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})\\ III_\alpha &= \det [ a_{ij}] \end{aligned}$$



## example

- First invariant $$I_\alpha = a_{ii} = 8 + 3 + 3 = 14$$
- Second invariant $$II_\alpha = \frac{1}{2}(a_{ii} a_{jj} - a_{ij}a_{ij})$$ $$a_{ii} a_{jj} = 14 \times 14$$ $$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + ... + a_{33}a_{33}$$ $$II_\alpha = \frac{1}{2}(196 - 84) = 56$$

# example

- Third invariant `[III_\alpha = \det [ a_{ij}]]` `[III_\alpha = 8 \times (3 \times 3 - 1 \times 1) = 64]`