Continuum Mechanics

Lecture 13 - Anisotropic Hyperelasticity

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schedule

- 29 Oct - Newtonian Fluids
- 3 Nov - Newtonian Fluids
- 5 Nov - Reynolds Transport Theorem
- 10 Nov - Viscoelastic Materials
- 12 Nov - Viscoelastic Materials

outline

- newtonian fluids
- flow conditions

fluids in rigid motion

- We define a fluid as a material which is unable to resist shear stress at rest
- For a fluid in rigid body motion, the stress vector on any plane will be normal to that plane

$$T_{ij} n_j = \lambda n_j$$

- The symmetry of the stress tensor leads us to find that

$$T_{ij} = -p\delta_{ij}$$

compressible and incompressible fluids

- Most liquids can be treated as incompressible in many fluid problems
- Their change in density is negligible under a wide range of pressures
- Most gases, however, must be treated as compressible
- Recall the conservation of mass

$$\frac{D}{Dt}\rho + \rho\frac{\partial v_k}{\partial x_k} = 0$$

- Which for an incompressible material becomes

$$\frac{\partial v_k}{\partial x_k} = 0$$

- Density of an incompressible material can vary in space, as long as it does not vary in time

hydrostatics

- If we substitute $T_{ij} = -p\delta_{ij}$ into the equilibrium equations, we find

$$\frac{\partial p}{\partial x_i} = \rho B_i$$

- If gravity is the only body force and acts in x_3, then pressure will only be a function of x_3 (for static fluid)
- If the fluid is in rigid body motion then we have

$$-\frac{\partial p}{\partial x_i} + \rho B_i = \rho a_i$$

example

- You are planning to load your fish tank into your friend's car for transportation
- Your friend brags that he can accelerate from 0 to 60 in 5 seconds
- Assuming this is true, and your tank is 2'x4' and 2' deep, how deep can you fill the tank without allowing any spilling due to acceleration?

general motion of fluids

- For a fluid in general motion, we de-compose the stress tensor into two portions

$$T_{ij} = -p\delta_{ij} + T'_{ij}$$

- Where T'_{ij} depends only on the rate of deformation and p is a scalar which does not depend on the rate of deformation

newtonian fluids

- For a fluid to be Newtonian, we make two assumptions
- First, we assume that T'_{ij} is linearly dependent on D_{ij} and nothing else
- Second, we assume the fluid is isotropic
- This gives

$$T'_{ij} = \lambda D_{kk}\delta_{ij} + 2\mu D_{ij}$$

physical interpretation

- If we consider a shear flow given by the velocity field

$$v_1 = f(x_2) \qquad v_2 = v_3 = 0$$

- We have a rate of deformation tensor with

$$D_{12} = \frac{1}{2}\frac{dv_1}{dx_2}$$

- With all other $D_{ij} = 0$
- Thus we find $T_{12} = \mu\frac{dv_1}{dx_2}$
- μ relates shear stress to the rate of change of the angle, is known as viscosity

physical interpretation

- For a general velocity field, if we take 1/3 of the contraction of the viscous stress tensor, we find

$$\frac{1}{3}T'_{ii} = \left(\lambda + \frac{2\mu}{3}\right) D_{ii}$$

- The quantity $\left(\lambda + \frac{2\mu}{3}\right)$ relates the mean viscous normal stress to the change in volume
- It is often referred to as the bulk viscosity

incompressible fluid

- If a fluid is considered to be incompressible, then $D_{ii} = 0$
- This gives the constitutive equation

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij}$$

- It is convenient to write it in terms of the velocity vector

$$T_{ij} = -p\delta_{ij} + 2\mu(v_{i,j} + v_{j,i})$$

navier-stokes

- If we recall Navier-Stokes equations of motion

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=\frac{\partial T_{ij}}{\partial x_j}+\rho B_i$$

- We can substitute the constitutive equation for newtonian fluids to find

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=\rho B_i-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

- This gives three equations with four unknowns, we use the continuity equation to find the fourth unknown

$$\frac{\partial v_i}{\partial x_i}=0$$

cylindrical and spherical coordinates

- Navier-Stokes equations in cylindrical and spherical coordinates are found on p. 364-365 of the text
- There is a typo in 6.8.1, should read

$$\frac{\partial v_r}{\partial r}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta}-v_\theta\right)+v_z\frac{\partial v_r}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+B_r$$
$$+\frac{\mu}{\rho}\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r}{r^2}\right]$$

nonslip

- A common assumption is that of *nonslip* boundaries
- Agrees well with experiments
- Both Newtonian and non-Newtonian fluids
- Fluid moves with boundary, for rigid boundaries the velocity at the boundary is 0

streamline

- In general, fluid flow is characterized by a velocity field
- As a vector field, there are different ways in which to visualize the field
- Streamlines, pathlines, streaklines and timelines are common ways we talk about fluids

steady and unsteady flow

- A flow is called *steady* if it is fixed in time (at a fixed location)
- Otherwise it is called unsteady
- Steady flow does not mean the material derivative is zero ($D\Psi/Dt \neq 0$)
- But it does mean that the partial derivative with respect to time is zero ($\partial\Psi/\partial t = 0$)
- For steady flow, streamlines, streaklines, and pathlines are the same

streamline

- A streamline is a curve which is instantaneously tangent to the velocity vector
- Experimentally, streamlines can be found on the surface of a fluid by sprinkling reflective particles and making a short-time exposure photograph
- Mathematically, streamlines can be found by considering a parametric equation for a curve $x_i = x_i(s)$
- We choose s so that $dx_i/ds = v_i$ and $s = 0$ corresponds to the point x_0, which is the originating point of our streamline

streamline example

– Given the velocity field

$$v_i = \langle \frac{kx_1}{1+\alpha t}, kx_2, 0 \rangle$$

find the streamline passing through (a_1, a_2, a_3) at time t

pathline

– A pathline is the path traversed by a fluid particle
– Experimentally, pathlines can be found by using one reflective particle and a long-time exposure photograph
– Mathematically, the pathline can be obtained from the velocity field as follows

$$\frac{dx_i}{dt} = v_i(x_i, t)$$
$$x_i(t_0) = X_i$$

pathline example

- Given the velocity field

$$v_i = \langle \frac{kx_1}{1+\alpha t}, kx_2, 0 \rangle$$

find the pathline passing through (a_1, a_2, a_3) at time t

streakline

- Streaklines are commonly found experimentally, but are difficult to express mathematically
- A streakline is formed when dye is steadily injected into a fluid from a fixed point
- The path that the very first point of dye follows is a pathline
- But the dye following behind is altered by the changing flow field, which makes the streakline left by the continuously injected dye different from a pathline

timeline

- The final common method for visualizing fluid flows is known as a timeline
- Fluid particles are marked at a given instance of time (often forming a line at t_0)
- After set intervals of time, lines are drawn between these particles
- These lines are called timelines

laminar flow

- Laminar flow is very orderly
- Fluid particles move in smooth layers (*laminae*)
- Occurs when fluid flow is relatively slow

reynolds number

- Dimensionless parameter to compare how “fast” or “slow” a fluid is moving
- For experiments under otherwise identical conditions, reynolds number is used to determine whether flow will be laminar
- Ratio of inertial forces to viscous forces
- In a tube, Reynolds number is

$$N_R = \frac{v_m \rho d}{\mu}$$

- For water in a tube, $N_R < 2100$ gives laminar flow

turbulent flow

- In laminar flow, small perturbations are quickly overcome
- For turbulent flow, unsteady vortices appear and interact with each other
- Turbulent flows are highly irregular and chaotic
- Turbulence increases diffusivity, causing fluids to mix more quickly
- High Reynolds numbers correspond to turbulence, but how high depends on the specific experiment
- There is often a large transition range between laminar and turbulent flow

reading

- pp 365-375