Continuum Mechanics

Lecture 7 - Stress

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schedule

- 15 Sep - Stress
- 17 Sep - Linear Elasticity
- 22 Sep - Equations of Motion, HW 4 Due
- 24 Sep - Elastic Problems

outline

- traction vector and stress tensor
- linear momentum and static equilibrium
- piola kirchoff stress tensors

some words on notation

- This text uses a different notation from what is generally taught in elasticity, but the concepts are identical
- “stress vector” is equivalent to a “traction vector”
- The symbol T used for the stress tensor is equivalent to σ used for the stress tensor in Elasticity

traction

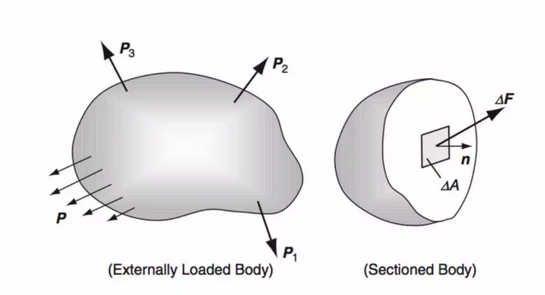


Figure 1: image

traction

- The traction vector is defined as

$$\hat{t}^n(x, \hat{n}) = \lim_{\Delta A \to 0} \frac{\Delta \hat{f}}{\Delta A}$$

- By Newton's third law (action-reaction principle)

$$\hat{t}^n(x, \hat{n}) = -\hat{t}^n(x, -\hat{n})$$

traction

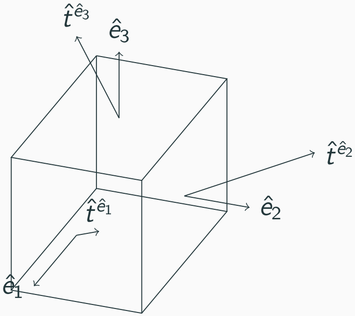


Figure 2: traction vector

traction

- If we consider the special case where the normal vectors, $\hat{n}$, align with the coordinate system ($\hat{e}_1$,$\hat{e}_2$,$\hat{e}_3$)
- On the 1-face:

$$\hat{n} = \hat{e}_1 : \qquad \hat{t}^n = t_i^{(\hat{e}_1)}\hat{e}_i = t_1^{(\hat{e}_1)}\hat{e}_1 + t_2^{(\hat{e}_1)}\hat{e}_2 + t_3^{(\hat{e}_1)}\hat{e}_3$$

- On the 2-face:

$$\hat{n} = \hat{e}_2 : \qquad \hat{t}^n = t_i^{(\hat{e}_2)}\hat{e}_i = t_1^{(\hat{e}_2)}\hat{e}_1 + t_2^{(\hat{e}_2)}\hat{e}_2 + t_3^{(\hat{e}_2)}\hat{e}_3$$

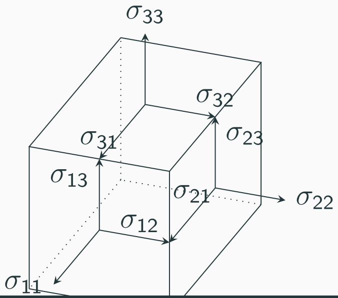
- And on the 3-face:

$$\hat{n} = \hat{e}_3 : \qquad \hat{t}^n = t_i^{(\hat{e}_3)}\hat{e}_i = t_1^{(\hat{e}_3)}\hat{e}_1 + t_2^{(\hat{e}_3)}\hat{e}_2 + t_3^{(\hat{e}_3)}\hat{e}_3$$

stress tensor

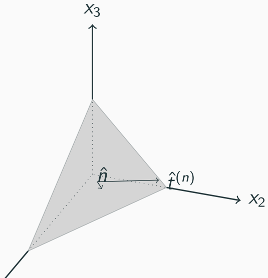
- To simplify the notation, we introduce the stress tensor

$$\sigma_{ij} = t_j^{(\hat{e}_i)}$$



traction

- We can find some interesting information about the traction vector by considering an arbitrary tetrahedron with some traction $\hat{t}^{(n)}$ applied to the surface



traction

- If we consider the balance of forces in the x_1-direction

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

- The area components are:

$$\begin{aligned} dA_1 &= n_1 dA \\ dA_2 &= n_2 dA \\ dA_3 &= n_3 dA \end{aligned}$$

- And $dV = \frac{1}{3} h dA$.

traction

$$t_1 dA - \sigma_{11} n_1 dA - \sigma_{21} n_2 dA - \sigma_{31} n_3 dA + b_1 \rho \frac{1}{3} h dA = 0$$

- If we let $h \to 0$ and divide by dA

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

- We can write this in index notation as

$$t_1 = \sigma_{i1} n_i$$

- We find, similarly

$$\begin{aligned} t_2 &= \sigma_{i2} n_i \\ t_3 &= \sigma_{i3} n_i \end{aligned}$$

traction

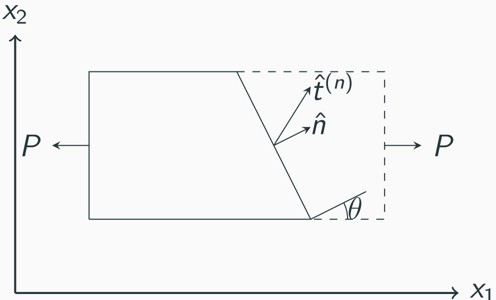
- We can further combine these results in index notation as

$$t_j = \sigma_{ij} n_i$$

- This means with knowledge of the nine components of σ_{ij}, we can find the traction vector at any point on any surface

example

- Consider a block of material with a uniformly distributed force acting on the 1-face. Find the tractions on an arbitrary interior plane



example

- First we consider a vertical cut on the interior 1-face ($n_i = \langle 1, 0, 0 \rangle$)
- Next we represent the force P as a vector, $p_i = \langle P, 0, 0 \rangle$
- Balancing forces yields

$$t_i A - p_i = 0$$

- We find $t_1 = \frac{P}{A} = \sigma_{11}$, $t_2 = 0 = \sigma_{12}$ and $t_3 = 0 = \sigma_{13}$
- No force is applied in the other directions, so it is trivial to find the rest of the stress tensor

$$\sigma_{ij} = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

example

- We can now consider any arbitrary angle of interior cut.
- The normal for a cut as shown in the diagram will be $n_i = \langle \cos\theta, \sin\theta, 0 \rangle$.
- We can again use $t_j = \sigma_{ij} n_i$ to find t_j for any angle θ.

$$t_1 = \frac{P}{A}\cos\theta$$
$$t_2 = 0$$
$$t_3 = 0$$

linear momentum

- From the principle of linear momentum, we know that $F = ma$
- If we consider some internal body force, B, and use the knowledge that tractions on opposing faces must be equal, we find (in Cartesian coordinates)

$$T_{ij,j} + \rho B_i = \rho a_i$$

- These are known as Cauchy's equations of motion
- For a body to be in static equilibrium $a_i = 0$

static equilibrium

- Most of the time, we will deal with bodies which are not in motion, and can use the condition of static equilibrium

$$T_{ij,j} + \rho B_i = 0$$

- In some cases, the body forces (usually gravity) are negligible compared with other forces acting on a body
- In invariant form, we can write this as

$$\text{div} T_{ij} + \rho B_i = \rho a_i$$

which is valid in any coordinate system

example

- Is the following stress distribution in static equilibrium?

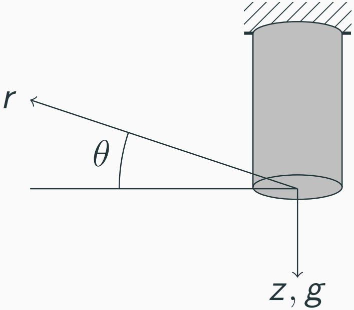
$$T_{ij} = \begin{bmatrix} x_2^2 + \nu(x_1^2 - x_2^2) & -2\nu x_1 x_2 & 0 \\ -2\nu x_1 x_2 & x_1^2 + \nu(x_2^2 - x_1^2) & 0 \\ 0 & 0 & \nu(x_1^2 + x_2^2) \end{bmatrix}$$

boundary conditions

- In most problems, we don't know anything about the internal stress state, but we do know what is applied on the surface
- We apply these as traction boundary conditions, which can be used to find the internal stress tensor
- If a surface is "free" with no boundary or force constraints, it is a traction-free boundary condition

example

- Suppose a cylinder has a variable density given by $\rho = r^2$
- Find the state of stress from gravity in these



conditions

example

- There is no traction along the outer surfaces
- Using Cauchy's stress theorem with the normal $n = \langle 1, 0, 0 \rangle$ we can find

$$
\begin{aligned}
t_j &= \sigma_{ij} n_i \\
&= \langle \sigma_{rr} n_r + \sigma_{r\theta} n_\theta + \sigma_{rz} n_z, \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta + \sigma_{\theta z} n_z, \sigma_{zr} n_r + \sigma_{z\theta} n_\theta + \sigma_{z} \\
&= \langle \sigma_{rr}, \sigma_{\theta r}, \sigma_{zr} \rangle
\end{aligned}
$$

- And choosing another normal, we find
- $\sigma_r = \sigma_{r\theta} = \sigma_\theta = \sigma_{rz} = \sigma_{\theta_z} = 0$
- We can also find that $\sigma_z = 0$ at the free surface

example

- Since gravity only acts in the z-direction, we make the assumption that all stress functions be functions of z only
- To find the stress in the z direction, we use the third equilibrium equation

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + \rho B_z = 0$$

- We can substitute known values to find that

$$\frac{\partial \sigma_z}{\partial z} + r^2 g = 0$$

example

- Since we desire to find the stress at any point, we introduce a variable to indicate the coordinate of our free body diagram cut

example

- We integrate over this free body to find

$$\sigma_z = -\int_L^z r^2 g dz$$
$$= r^2 g(L - z)$$

- In this case, the stress is a function of radial distance (just like the body force was)

piola kirchoff stress tensors

- The Cauchy stress tensor is based on the differential area at the current position (deformed state)
- The first and second Piola Kirchoff stress tensors are based on the undeformed area
- Equations of motion can be formulated in either the deformed or un-deformed configuration, based on which is more convenient for a given problem
- For large deformation problems, whether the rate of deformation tensor D_{ij}, DF_{ij}/Dt , or DE^*_{ij}/Dt is used facilitates the use of Cauchy or one of the Piola Kirchoff Stress tensors

first piola kirchoff stress tensor

- For the first Piola Kirchoff stress tensor (also known as the Lagrangian stress tensor), we let

$$df_i = t_i^0 dA^0$$

- Note that t_i^0 is a pseudo traction vector, and does not describe the actual intensity of df_i, which is acting on the deformed area
- Also note that since df_i is the acting on the deformed area, t_i^0 acts in the same direction as t_i
- We can now formulate the stress tensor in the same way as the Cauchy stress tensor

$$t_i^0 = T_{ij}^0 n_j^0$$

first piola kirchoff stress tensor

- We can also relate the first Piola-Kirchoff stress tensor to the Cauchy stress tensor

$$T_{ij}^0 = J T_{im} F_{jm}^{-1}$$

- And the inverse relationship is

$$T_{ij} = \frac{1}{J} T_{im}^0 F_{jm}$$

- In general, the first Piola-Kirchoff stress tensor is not symmetric

second piola kirchoff stress tensor

- If instead we consider a pseudo-differential force acting on the un-deformed area

$$d\tilde{f}_i = \tilde{t}_i dA^0$$

where

$$df_i = F_{ij} d\tilde{f}_j$$

- In general, the traction vector $\tilde{t}_i$ is in a different direction that t_i and t_i^0
- Once again, we can formulate the second Piola-Kirchoff stress tensor as the others

$$\tilde{t}_i = \tilde{T}_{ij} n_j^0$$

second piola kirchoff stress tensor

- We can easily relate the Second and First Piola Kirchoff stress tensors

$$\tilde{T}_{ij} = F_{im}^{-1} T_{mj}^0$$

- We can now substitute to relate the Second Piola Kirchoff stress tensor to the Cauchy stress tensor

$$\tilde{T}_{ij} = J F_{im}^{-1} T_{mn} F_{jn}^{-1}$$

- For a symmetric stress tensor, T_{ij}, the Second Piola Kirchoff stress tensor is symmetric