Continuum Mechanics

Lecture 2 - Tensor Algebra

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schedule

- 20 Aug - Tensor Algebra
- 25 Aug - Tensor Calculus, HW1 Due
- 27 Aug - Material Derivative
- 1 Sep - Conservation and Compatibility, HW2 Due

outline

- symmetry
- transformation
- examples
- principal values
- invariants
- principal directions
- examples

symmetry

- Symmetry can be a very powerful tool
- Here we define some useful forms of symmetry in index notation
- Symmetric
  - $a_{ij\ldots z} = a_{z\ldots ji}$
  - $a_{ij\ldots m\ldots n\ldots z} = a_{ij\ldots n\ldots m\ldots z}$
- Anti-symmetric (skew symmetric)
  - $a_{ij\ldots z} = -a_{z\ldots ji}$
  - $a_{ij\ldots m\ldots n\ldots z} = -a_{ij\ldots n\ldots m\ldots z}$

symmetry

- Useful identity
  - If $a_{ij\ldots m\ldots n\ldots k}$ is symmetric in mn and $b_{pq\ldots m\ldots n\ldots r}$ is antisymmetric in mn, then the product is zero

$$a_{ij\ldots m\ldots n\ldots k}b_{pq\ldots m\ldots n\ldots r} = 0$$

- We can also write any tensor as the sum of its symmetric and anti-symmetric parts

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$$

- This textbook uses a special shortcut notation (S and A superscript) for the symmetric and anti-symmetric portions of a tensor

linear transformation

- Let us consider some transformation, $\mathbf{T}$, which transforms any vector into another vector
- If we transform $\mathbf{Ta} = c$ and $\mathbf{Tb} = d$
- We call $\mathbf{T}$ a linear transformation (and a tensor) if

$$\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{Ta} + \mathbf{Tb}$$
$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta}$$

- Where α is any arbitrary scalar and $\mathbf{a}$, $\mathbf{b}$ are arbitrary vectors

2d coordinate transformation

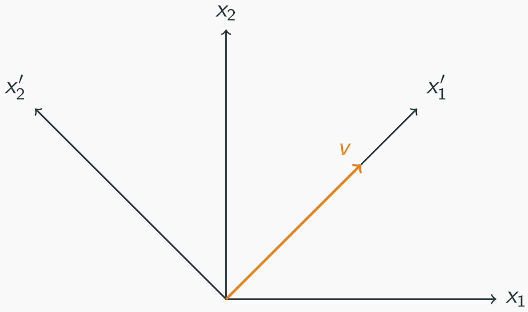

Figure 1: 2d coordinate transformation example with vector pointing from (0,0) to (1,1)

2d coordinate transformation

- The vector, v, remains fixed, but we transform our coordinate system
- In the new coordinate system, the x_2' portion of v is zero.
- To transform the coordinate system, we first define some unit vectors.
- $\hat{e}_1$ is a unit vector in the direction of x_1, while $\hat{e}_1'$ is a unit vector in the direction of x_1'

2d coordinate transformation

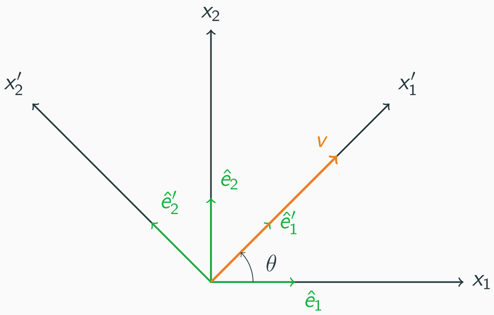

Figure 2: 2d coordinate transformation from previous figure with unit vectors drawn along the x and y axes

2d coordinate transformation

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45°$
- We can write the transformed unit vectors, $\hat{e}_1'$ and $\hat{e}_2'$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, θ.

$$\hat{e}_1' = \langle \hat{e}_1 \cos\theta, \hat{e}_2 \sin\theta \rangle$$
$$\hat{e}_2' = \langle -\hat{e}_1 \sin\theta, \hat{e}_2 \cos\theta \rangle$$

2d coordinate transformation

- We can write the vector, v, in terms of the unit vectors describing our axis system
- $v = v_1\hat{e}_1 + v_2\hat{e}_2$
- (note: $\hat{e}_1 = \langle 1, 0\rangle$ and $\hat{e}_2 = \langle 0, 1\rangle$)
- $v = \langle 2, 2\rangle = 2\langle 1, 0\rangle + 2\langle 0, 1\rangle$

2d coordinate transformation

- When expressed in the transformed coordinate system, we refer to v'
- $v' = \langle v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta\rangle$
- $v' = \langle 2\sqrt{2}, 0\rangle$
- We can recover the original vector from the transformed coordinates:
- $v = v'_1\hat{e}'_1 + v'_2\hat{e}'_2$

2d coordinate transformation

- note:

$$\hat{e}'_1 = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$

  and

$$\hat{e}'_2 = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$$

- therefore

$$v = 2\sqrt{2}\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, 0\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle 2, 2 \rangle$$

coordinate transformation

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- We define Q_{ij} as the cosine of the angle between the x'_i axis and the x_j axis.

coordinate transformation

- This is also referred to as the “direction cosine”

$$Q_{ij} = \cos(x'_i, x_j)$$

- *health warning* the direction cosine can also be defined inversely ($Q_{ij} = \cos(x_i, x'_j)$), and the indexes are switched in the transformation law

coordinate transformation

- We can use this form on our 2D transformation example

$$Q_{ij} = \cos(x'_i, x_j) \quad = \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90 - \\ \cos(90 + \theta) & \cos\theta \end{bmatrix}$$

coordinate transformation

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v'_i = Q_{ij} v_j$
- Matrices (second-order tensors):

$$\sigma'_{mn} = Q_{mi} Q_{nj} \sigma_{ij}$$

- Fourth-order tensors:

$$C'_{ijkl} = Q_{im} Q_{jn} Q_{ko} Q_{lp} C_{mnop}$$

coordinate transformation

- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji} v'_j$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im} Q_{jn} \sigma'_{ij}$
- Fourth-order tensors: $C_{ijkl} = Q_{mi} Q_{nj} Q_{ok} Q_{pl} C'_{mnop}$

coordinate transformation

- We can derive some interesting properties of the transformation tensor, Q_{ij}
- We know that $v_i = Q_{ji}v'_j$ and that $v'_i = Q_{ij}v_j$
- If we substitute (changing the appropriate indexes) we find:

$$v_i = Q_{ji}Q_{jk}v_k$$

- We can now use the Kronecker Delta to substitute $v_i = \delta_{ik}v_k$ which gives

$$\delta_{ik}v_k = Q_{ji}Q_{jk}v_k$$

example

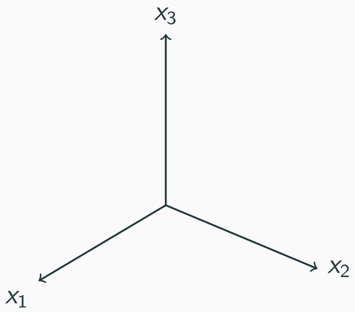

Figure 3: 3d coordinate system to start general transformation example

example

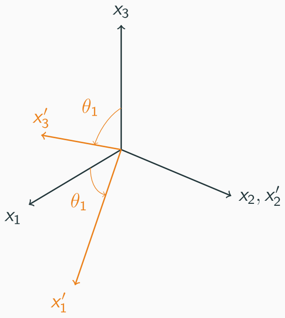

Figure 4: 3d illustration of first transformation

example

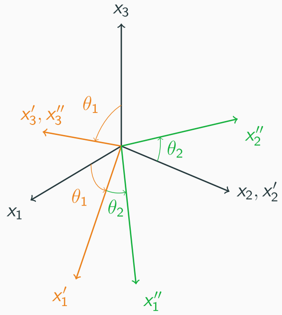

Figure 5: 3d illustration of second transformation (about the axes of the first)

example

- $Q^1_{ij} = \cos(x'_i, x_j)$
- $Q^2_{ij} = \cos(x''_i, x'_j)$

$$Q^1_{ij} = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150 \\ \cos 90 & \cos 0 & \cos 90 \\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}$$

$$Q^2_{ij} = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90 \\ \cos 120 & \cos 30 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}$$

example

- We now use Q_{ij} to find $\hat{e}'_i$ and $\hat{e}''_i$
- First, we need to write $\hat{e}_i$ in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. $\hat{e}_1 = e^1_i$
- $e'_i = Q^1_{ij} e_j$
- $e''_i = Q^2_{ij} e'_j$

example

- How do we find e_i'' in terms of e_i?
- $e_i'' = Q_{ij}^2 Q_{jk}^1 e_k$

principal values

- In the 2D coordinate transformation example, we were able to eliminate one value from a vector using coordinate transformation
- For second-order tensors, we desire to find the "principal values" where all non-diagonal terms are zero
- The direction determined by the unit vector, n_j, is said to be the *principal direction* or *eigenvector* of the symmetric second-order tensor, a_{ij} if there exists a parameter, λ, such that

$$a_{ij} n_j = \lambda n_i$$

- Where λ is called the *principal value* or *eigenvalue* of the tensor

principal values

- We can re-write the equation

$$(a_{ij} - \lambda\delta_{ij})n_j = 0$$

- This system of equations has a non-trivial solution if and only if $\det[a_{ij} - \lambda\delta_{ij}] = 0$
- This equation is known as the characteristic equation, and we solve it to find the principal values of a tensor

example

- Find the principal values of the tensor

$$A_{ij} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- From the characteristic equation, we know that $\det[A_{ij} - \lambda\delta_{ij}] = 0$, or

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

example

– Calculating the determinant gives

$$(1-\lambda)(4-\lambda)-4=0$$

– Multiplying out and simplifying, we find

$$\lambda^2-5\lambda=\lambda(\lambda-5)=0$$

– This has the solution $\lambda=0,5$

invariants

– Every tensor has some invariants which do not change with coordinate transformation
– These are known as *fundamental invariants*
– The characteristic equation for a tensor in 3D can be written in terms of the invariants

$$\det[a_{ij}-\lambda\delta_{ij}]=-\lambda^3+I_\alpha\lambda^2-II_\alpha\lambda+III_\alpha=0$$

invariants

– The invariants can be found by the following equations

$$I_\alpha = a_{ii}$$
$$II_\alpha = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$$
$$III_\alpha = \det[a_{ij}]$$

invariants

– In the principal direction, a'_{ij} will be

$$a'_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

– Since invariants do not change with coordinate systems, we can also write the invariants as

$$I_\alpha = \lambda_1 + \lambda_2 + \lambda_3$$
$$II_\alpha = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$
$$III_\alpha = \lambda_1\lambda_2\lambda_3$$

principal directions

- We defined principal directions earlier

$$(a_{ij} - \lambda\delta_{ij})n_j = 0$$

- λ are the principal values and n_j are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for λ back into this equation

example

- Find the principal directions for the earlier principal values example
- Recall $\lambda = 0, 5$, let us say $\lambda_1 = 5$, we find $n_j^{(1)}$ by

$$\begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 4-\lambda_1 \end{bmatrix} \left\{ n_1 \; n_2 \right\} = 0$$

- This gives

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \left\{ n_1 \; n_2 \right\} = 0$$

example

- We proceed to solve the equations using row-reduction, but we find

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \left\{ n_1 \; n_2 \right\} = 0$$

- This means we cannot uniquely solve for n_j
- We are only concerned with the direction, magnitude is not important
- Choose $n_2 = 1$, solve for n_1
- $n^{(1)} = \langle \frac{1}{2}, 1 \rangle$

example

- Similarly, for $\lambda_2 = 0$, we find

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \left\{ n_1 \; n_2 \right\} = 0$$

- Which, after row-reduction, becomes

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \left\{ n_1 \; n_2 \right\} = 0$$

- If we choose $n_2 = 1$, we find $n_1 = -2$
- This gives $n^{(2)} = \langle -2, 1 \rangle$

example

- We can assemble a transformation matrix, Q_{ij}, from the principal directions
- First we need to normalize them to unit vectors
- $||n^{(1)}|| = \sqrt{\frac{5}{4}}$
- $\hat{n}^{(1)} = \frac{2}{\sqrt{5}}\langle\frac{1}{2}, 1\rangle = \langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$
- $||n^{(2)}|| = \sqrt{5}$
- $\hat{n}^{(2)} = \langle\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

example

- This gives

$$Q_{ij} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- And we find

$$A'_{mn} = Q_{mi}Q_{nj}A_{ij}$$

$$A'_{ij} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

example

- Find principal values, principal directions, and invariants for the tensor

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

example

- Characteristic equation simplifies to
- $-\lambda^3 + 14\lambda^2 - 56\lambda + 64 = 0$
- Which has the solutions $\lambda = 2, 4, 8$

example

- To find the principal direction for $\lambda_1 = 8$

$$\begin{bmatrix} 8-8 & 0 & 0 \\ 0 & 3-8 & 1 \\ 0 & 1 & 3-8 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & 1 & -5 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

- This means that $n_3 = 0$ and, as a result, $n_2 = 0$.
- n_1 can be any value, we choose $n_1 = 1$ to give a unit vector.
- $n^{(1)} = \langle 1, 0, 0 \rangle$

example

– To find the principal direction for $\lambda_2 = 4$

$$\begin{bmatrix} 8-4 & 0 & 0 \\ 0 & 3-4 & 1 \\ 0 & 1 & 3-4 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

example

– After row-reduction, we find

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

– This means that $n_1 = 0$

– We also know that $n_2 = n_3$, so we choose $n_2 = n_1 = 1$

– This gives $n^{(2)} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$ after normalization

example

- To find the principal direction for $\lambda_3 = 2$

$$\begin{bmatrix} 8-2 & 0 & 0 \\ 0 & 3-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

example

- After row-reduction, we find

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ n_1 \; n_2 \; n_3 \right\} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = -n_3$, so we choose $n_2 = 1$ and $n_1 = -1$
- This gives $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1 \rangle$ after normalization

example

- In summary, for c_{ij} we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0 \rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \frac{1}{\sqrt{2}} \langle 0, 1, -1 \rangle$
- We can assemble $n^{(i)}$ into a transformation tensor

$$Q_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

example

- What is c'_{ij}?
- $c'_{ij} = Q_{im} Q_{jn} c_{mn}$

$$c'_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

example

- We can also find the invariants for

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

- Recall:

$$\begin{aligned} I_\alpha &= a_{ii} \\ II_\alpha &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) \\ III_\alpha &= \det[a_{ij}] \end{aligned}$$

example

- First invariant

$$I_\alpha = a_{ii} = 8 + 3 + 3 = 14$$

- Second invariant

$$II_\alpha = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$$

$$a_{ii}a_{jj} = 14 \times 14$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + ... + a_{33}a_{33}$$

$$II_\alpha = \frac{1}{2}(196 - 84) = 56$$

example

– Third invariant

$$III_\alpha = \det[a_{ij}]$$

$$III_\alpha = 8 \times (3 \times 3 - 1 \times 1) = 64$$