Continuum Mechanics

Lecture 13 - Anisotropic Hyperelasticity

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schedule

- 15 Oct - Anisotropy and Large Deformation
- 20 Oct - Exam Review, HW7 Due, HW6 Self-Grade Due
- 22 Oct - Exam 2
- 27 Oct - Newtonian Fluids

outline

- large deformation
- simple deformation modes
- thermodynamics formulation
- anisotropic hyperelasticity

change of frame

- For small deformations, the current and deformed frame have negligible differences
- For large deformations, we need to ensure that our constitutive law is objective, or frame-indifferent
- Examples:
  - distance between two material points is a frame-indifferent scalar
  - speed of a material point is not frame-independent (non-objective)
  - position vector and velocity vector of a material point are non-objective
  - relative velocity between two material points is objective

change of frame

- In general, a "frame" can have its own time scale, origin, and directionality
- a change of frame would then be given by

$$x_i^* = c_i(t) + Q_{ij}(t)(x_j - x_j^0)$$

- If we consider the position vector of two material points, in the starred frame we have

$$x_1^* = c(t) + Q(t)(x_1 - x_0) \qquad x_2^* = c(t) + Q(t)(x_2 - x_0)$$

change of frame

- The relative position vector, $b = x_1 - x_2$ can also be found in the starred frame as

$$b^* = x_1^* - x_2^* = Q(t)(x_1 - x_2)$$

- Any vector which obeys this law is known as an objective vector

change of frame

- If we consider some tensor, T_{ij} which transforms an objective vector b_j into another objective vector, c_i

$$c_i = T_{ij} b_j$$

- Since both b and c are objective vectors, we can write

$$c_i^* = Q_{ik} c_k = Q_{ik} T_{kj} b_j = Q_{ik} T_{kj} Q_{lj} b_l$$

- This means that for T_{ij} to be objective it must satisfy

$$T_{ij}^* = Q_{ik} T_{kl} Q_{jl}$$

group problems

- Group 1: Is the first Piola-Kirchhoff stress tensor objective? How does it transform? Recall:

$$T_{ij}^0 = J T_{ik} F_{jk}^{-1}$$

- Group 2: Is the second Piola-Kirchhoff stress tensor objective? How does it transform? Recall:

$$\tilde{T}_{ij} = J F_{ik}^{-1} T_{kl} F_{jl}^{-1}$$

constitutive equations

- For large deformation, a constitutive law must be objective
- $T_{ij} = f(C_{ij})$ is not an acceptable form, but $T_{ij} = f(B_{ij})$ is
- $\tilde{T}_{ij} = f(C_{ij})$ is also an acceptable form of the constitutive equation
- If we assume our material to be isotropic, it can be shown that, with no loss of generality

$$f(B_{ij}) = a_0\delta_{ij} + a_1 B_{ij} + a_2 B_{ij}^2$$

constitutive equations

- A commonly used alternate form of the constitutive equation is

$$T_{ij} = \varphi_0\delta_{ij} + \varphi_1 B_{ij} + \varphi_2 B_{ij}^{-1}$$

- If the material is incompressible, the stress is indeterminate to some arbitrary hydrostatic pressure

$$T_{ij} = -p\delta_{ij} + \varphi_1 B_{ij} + \varphi_2 B_{ij}^{-1}$$

- This is known as the Mooney-Rivlin model, which can be written in different ways

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{1}{2} + \beta\right) B_{ij} - \mu\left(\frac{1}{2} - \beta\right) B_{ij}^{-1}$$

neo-hookean solid

- A simpler model, which is only accurate for strains less than 20%, is the neo-Hookean solid
- For incompressible materials, the neo-Hookean equation is

$$T_{ij} = -p\delta_{ij} + 2C_1 B_{ij}$$

- Where, for consistency with Hooke's Law, $2C_1 = \mu$
- When large tensile strains are not important, the neo-Hookean model is popular because it only needs one material constant, which has more physical meaning than Mooney-Rivlin constants.

incompressible uniaxial stretch

- In large deformation problems, the stretch ratio, λ is often used (instead of strain)
- λ_1 represents the ratio of deformed length in x_1 to undeformed length in x_1
- For uniaxial extension we have

$$x_1 = \lambda_1 X_1$$
$$x_2 = \lambda_2 X_2$$
$$x_3 = \lambda_2 X_3$$

- Also if the material is incompressible we know

$$\lambda_1 \lambda_2^2 = 1$$

simple shear

- For large shear deformation, we have

$$
\begin{aligned}
x_1 &= X_1 + KX_2 \\
x_2 &= X_2 \\
x_3 &= X_3
\end{aligned}
$$

- And we find B and B^{-1} as

$$
B_{ij} = \begin{bmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_{ij}^{-1} = \begin{bmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

helmholtz free energy

- Large deformation constitutive laws are generally formulated from an energy framework
- From thermodynamics, it can be shown that the second Piola-Kirchhoff stress is the partial derivative of the Helmholtz free energy with respect to the Green strain

$$
\tilde{T}_{ij} = \rho_0 \frac{\partial \Psi}{\partial E_{ij}}
$$

helmholtz free energy

- The Helmholtz free energy has both thermal and mechanical strain energy components, but in general we neglect the thermal portion, and use W to represent the mechanical work done (with density included)

$$\tilde{T}_{ij} = \frac{\partial W}{\partial E_{ij}}$$

hyperelastic materials

- Most commonly, the strain energy density, W is formulated as a function of the invariants of B_{ij}

$$\begin{aligned} I_B &= B_{kk} \\ II_B &= \frac{1}{2}(I_B^2 - B_{ik}B_{ki}) \\ III_B &= J^2 \end{aligned}$$

hyperelastic materials

- For nearly incompressible materials, it is often more convenient to write an alternate form of the invariants

$$\bar{I}_B = \frac{B_{kk}}{J^{2/3}}$$
$$\bar{II}_B = \frac{1}{2}(\bar{I}_B^2 - \frac{B_{ik}B_{ki}}{J^{4/3}})$$
$$\bar{III}_B = J = \sqrt{\det(B_{ij})}$$

hyperelastic models

- The algebra involved in taking the partial derivative to calculate stress can get tedious
- Cauchy stress for compressible materials (in terms of the invariants of B_{ij}) can be expressed as

$$\sigma_{ij} = \frac{2}{\sqrt{I_3}}\left[\left(\frac{\partial W}{\partial I_B} + I_B\frac{\partial W}{\partial II_B}\right)B_{ij} - \frac{\partial U}{\partial II_B}B_{ik}B_{kj}\right] + 2\sqrt{III_B}\frac{\partial U}{\partial III_B}\delta_{ij}$$

hyperelastic models

- For nearly incompressible materials (in terms of $\bar{I}$), we have

$$\sigma_{ij} = \frac{2}{J}\left[\frac{1}{J^{2/3}}\left(\frac{\partial W}{\partial \bar{I}_B} + \bar{I}_B \frac{\partial W}{\partial \bar{II}_B}\right) B_{ij} - \left(\bar{I}_B \frac{\partial W}{\partial \bar{I}_B} + 2\bar{II}_B \frac{\partial W}{\partial \bar{II}_B}\right)\frac{\delta_{ij}}{3} - \frac{1}{J^{4/\cdots}} \cdots\right.$$

- Sometimes W is expressed in terms of F_{ij}, which gives

$$\sigma_{ij} = \frac{1}{J} F_{ik} \frac{W}{\partial F_{kj}}$$

hyperelastic materials

- W can also be expressed in terms of the stretch ratios, λ_1, λ_2 and λ_3 and the unit eigenvectors of B_{ij} ($b_i^{(j)}$)

$$\sigma_{ij} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_1} b_i^{(1)} b_j^{(1)} + \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_2} b_i^{(2)} b_j^{(2)} + \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_3} b_i^{(3)} b_j^{(3)}$$

neo-hookean solid

- The neo-hookean solid we discussed earlier can be expressed in this form

$$W = \frac{\mu_1}{2}(\bar{I}_B - 3) + \frac{K_1}{2}(J-1)^2$$

- Where μ_1 and K_1 correspond to the shear modulus and bulk modulus, respectively, at small deformations
- Should be used for rubbers with very limited compressiblity ($K_1 >> \mu_1$)

mooney-rivlin

- The Mooney-Rivlin Model can also be expressed in this form

$$W = \frac{\mu_1}{2}(\bar{I}_B - 3) + \frac{\mu_2}{2}(\bar{II}_B - 3) + \frac{K_1}{2}(J-1)^2$$

- Once again, this applies best with limited compressibility
- $\mu = \mu_1 + \mu_2$

rivlin models

- While the text shows one simplified version of the Mooney-Rivlin material model the generalized Rivlin model has the form

$$W = \sum_{p,q=0}^{N} C_{pq}(\bar{I}_B - 3)^p(\bar{II}_B - 3)^q + \sum_{m=1}^{M} D_m(J-1)^{2m}$$

with $C_{00} = 0$

- For consistency with linear elasticity, we know that

$$K = 2D_1 \qquad \mu = 2(C_{01} + C_{10})$$

- This is also known as the generalized polynomial rubber elasticity potential

other models

- Many micro-features can effect the large deformation behavior of a material
- Several families of large-deformation materials
  - Foams
  - Rubbery foams
  - Thermoplastic polymers
  - Elastomers
  - Bio-tissue
  - There are many families of strain energy functions tailored to specific material responses

material constants

- For simple models, elastic constants can be found easily from uniaxial tension
- If you are working with a rubbery material, you can generally assume it is incompressible
- Least-squares fit of uniaxial test is often used, but can lead to error
- If multi-axial performance is desired, it is more accurate to fit the constants to a multi-axial test
- Most rubbers have coupled volumetric and shear responses under large hydrostatic stress, Rivlin-derived models do not account for this coupling

anisotropic hyperelasticity

- Most engineering materials are mostly isotropic under large deformation
- Hyperelasticity has other use cases
  - Biomechanics
  - Geophysics (soil compaction)
  - Foams, heterogeneous materials
  - Paper

decomposition

- Although the paper by Hozapfel, Gasser and Ogden was developed specifically for arterial walls, it can be applied elsewhere and uses common techniques in the development
- F_{ij} (and by extension, C_{ij} and B_{ij}) are decomposed into dilatational and distortional parts

$$
\begin{aligned}
F &= (J^{1/3} I)\bar{F} \\
C &= F^T F = J^{2/3}\bar{C} \\
\bar{C} &= \bar{F}^T \bar{F} \\
B &= F F^T = J^{2/3}\bar{B} \\
\bar{B} &= \bar{F}\bar{F}^T
\end{aligned}
$$

decomposition

- And the Green strain tensor

$$
\begin{aligned}
E &= \frac{1}{2}(C - I) = J^{2/3}\bar{E} + \frac{1}{2}(J^{2/3} - 1)I \\
\bar{E} &= \frac{1}{2}(\bar{C} - I)
\end{aligned}
$$

helmholtz free energy

- The Helmholtz free energy is normally formulated in terms of one tensor, A
- Here, the authors assume a set of n second-order tensors to characterize the anisotropic effects on energy
- They further assume that the energy effects of dilatational and distortional contributions are completely decoupled, giving

$$\Psi(E, A_1, ..., A_n) = U(J) + \bar{\Psi}(\bar{E}, A_1, ..., A_n)$$

anisotropic contributions

- The authors propose further separating the helmholtz free energy into isotropic and anisotropic parts, they proceed to use $\bar{C}$ instead of $\bar{E}$

$$\bar{\Psi}(\bar{C}, a_{01}, a_{02}) = \bar{\Psi}_{iso}(\bar{C}) + \bar{\Psi}_{aniso}(\bar{C}, a_{01}, a_{02})$$

- They proceed to define further invariants in terms of $\bar{C}$, a_{01} and a_{02}

$$\bar{I}_4 = \bar{C} \because A_1$$

$$\bar{I}_6 = \bar{C} \because A_2$$

stiffness model

- Now they choose a form which will give the anisotropic stiffness, since they observed a strong stiffening effect, they choose an exponential function

$$\bar{\Psi}_{aniso} = \frac{k_1}{2k_2} \sum_{i=4,6} \exp[k_2(\bar{I}_i - 1)^2] - 1$$

- Note that k_1 and k_2 are the only parameters to be found as a function of the material, as $\bar{I}_4$ and $\bar{I}_6$ are functions of the material orientation

reading material

- Thermodynamics formulation of Mooney-Rivlin models link
- Anisotropy in large deformation - “A New Constitutive Framework for Arterial Wall Mechanics and a Comparative Study of Material Models”
- Paper is interesting, but long, pp 10-21 are the most relevant.