Continuum Mechanics

Lecture 11 - Anisotropy

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schedule

- 8 Oct - Anisotropy
- 13 Oct - Large Deformation, HW 6 Due
- 15 Oct - Anisotropy and Large Deformation
- 20 Oct - Exam Review, HW 7 Due, HW 6 Self-Grade Due
- 22 Oct - Exam 2

outline

- anisotropic materials
- physical interpretation
- material symmetries
- experimental considerations

anisotropic materials

- Many materials exhibit different properties in different directions
- Composites are one example, but many polymers have anisotropy due to crystallinity
- Even some metals exhibit minor amounts of anisotropy due to grain boundaries and rolling
- Wood is a common anisotropic material

anisotropic materials

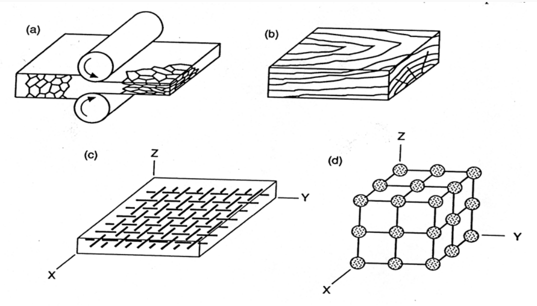


Figure 1: image

hooke's law

- Although it cannot be simplified as much as for isotropic materials, Hooke's Law still applies to linear anisotropic materials

$$T_{ij} = C_{ijkl}E_{kl}$$

hooke's law

- Or, written in "engineering" form

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{1322} & C_{1222} \\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & C_{1333} & C_{1233} \\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & C_{1323} & C_{1223} \\ C_{1113} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1213} \\ C_{1112} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}$$

hooke's law

- Note: while the most common order of terms is the one I have written, the shear terms can be written in a different order
- This will also change the order of the corresponding stiffness terms
- Also, the indexes are often written in contracted form ($11 = 1$, $22 = 2$, $33 = 3$, $23 = 4$, $13 = 5$, $12 = 6$) this makes it not immediately apparent which convention somebody is using
- Finally, the shear tensorial strains (which must be multiplied by 2) can often be expressed as engineering shear strains ($\gamma_{12} = 2E_{12}$)

transformation

- We know that

$$\sigma'_{mn} = Q_{mi} Q_{nj} \sigma_{ij}$$

- We can expand this to write in terms of engineering stress
- We will expand only two terms, as they show the general pattern for all 6

stress transformation

$$\begin{aligned}\sigma'_1 = \sigma'_{11} = & \ Q_{11}Q_{11}\sigma_{11} + Q_{11}Q_{12}\sigma_{12} + Q_{11}Q_{13}\sigma_{13} \\ & + Q_{12}Q_{11}\sigma_{21} + Q_{12}Q_{12}\sigma_{22} + Q_{12}Q_{13}\sigma_{23} \\ & + Q_{13}Q_{11}\sigma_{31} + Q_{13}Q_{12}\sigma_{32} + Q_{13}Q_{13}\sigma_{33}\end{aligned}$$

$$\begin{aligned}\sigma'_1 = & \ Q_{11}^2\sigma_1 + Q_{12}^2\sigma_2 + Q_{13}^2\sigma_3 \\ & + 2Q_{11}Q_{12}\sigma_6 + 2Q_{11}Q_{13}\sigma_5 + 2Q_{12}Q_{13}\sigma_4\end{aligned}$$

stress transformation

$$
\begin{aligned}
\sigma'_4 = \sigma'_{23} = Q_{21}Q_{31}\sigma_{11} + Q_{21}Q_{32}\sigma_{12} + Q_{21}Q_{33}\sigma_{13} \\
+Q_{22}Q_{31}\sigma_{21} + Q_{22}Q_{32}\sigma_{22} + Q_{22}Q_{33}\sigma_{23} \\
+Q_{23}Q_{31}\sigma_{31} + Q_{23}Q_{32}\sigma_{32} + Q_{23}Q_{33}\sigma_{33}
\end{aligned}
$$

$$
\begin{aligned}
\sigma'_4 = Q_{21}Q_{31}\sigma_1 + Q_{22}Q_{32}\sigma_{22} + Q_{23}Q_{33}\sigma_3 \\
+(Q_{21}Q_{32} + Q_{22}Q_{31})\sigma_6 + (Q_{21}Q_{33} + Q_{23}Q_{31})\sigma_5 \\
+(Q_{22}Q_{33} + Q_{23}Q_{32})\sigma_4
\end{aligned}
$$

stress transformation

- We often write $\sigma' = R_\sigma \sigma$ for engineering notation

$$
R_\sigma = \begin{bmatrix}
Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & 2Q_{12}Q_{13} & 2Q_{11}Q_{13} & 2Q_{11}Q_{12} \\
Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & 2Q_{22}Q_{23} & 2Q_{21}Q_{23} & 2Q_{21}Q_{22} \\
Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & 2Q_{32}Q_{33} & 2Q_{31}Q_{33} & 2Q_{31}Q_{32} \\
Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} & Q_{23}Q_{32} + Q_{22}Q_{33} & Q_{23}Q_{31} + Q_{21}Q_{33} & Q_{22}Q_{31} + Q_{21}Q_{32} \\
Q_{11}Q_{31} & Q_{12}Q_{32} & Q_{13}Q_{33} & Q_{13}Q_{32} + Q_{12}Q_{33} & Q_{13}Q_{31} + Q_{11}Q_{33} & Q_{12}Q_{31} + Q_{11}Q_{32} \\
Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} & Q_{13}Q_{22} + Q_{12}Q_{23} & Q_{13}Q_{21} + Q_{11}Q_{23} & Q_{12}Q_{21} + Q_{11}Q_{22}
\end{bmatrix}
$$

strain transformation

- We can follow the exact same procedure to transform strain
- The values are almost the same, notice the highlighted terms

$$R_\epsilon = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & Q_{12}Q_{13} & Q_{11}Q_{13} & Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & Q_{22}Q_{23} & Q_{21}Q_{23} & Q_{21}Q_{22} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & Q_{32}Q_{33} & Q_{31}Q_{33} & Q_{31}Q_{32} \\ 2Q_{21}Q_{31} & 2Q_{22}Q_{32} & 2Q_{23}Q_{33} & Q_{23}Q_{32}+Q_{22}Q_{33} & Q_{23}Q_{31}+Q_{21}Q_{33} & Q_{22}Q_{31}+Q_{21}Q_{32} \\ 2Q_{11}Q_{31} & 2Q_{12}Q_{32} & 2Q_{13}Q_{33} & Q_{13}Q_{32}+Q_{12}Q_{33} & Q_{13}Q_{31}+Q_{11}Q_{33} & Q_{12}Q_{31}+Q_{11}Q_{32} \\ 2Q_{11}Q_{21} & 2Q_{12}Q_{22} & 2Q_{13}Q_{23} & Q_{13}Q_{22}+Q_{12}Q_{23} & Q_{13}Q_{21}+Q_{11}Q_{23} & Q_{12}Q_{21}+Q_{11}Q_{22} \end{bmatrix}$$

stiffness transformation

- We can now formulate the transformation of the stiffness matrix. We know that

$$\sigma' = R_\sigma \sigma = C'E'$$

- And since $\sigma = CE$, we can say

$$R_\sigma CE = C'E'$$

- Now we know that $E' = R_E E$, so we substitute that to find

$$R_\sigma CE = C'R_E E$$

stiffness transformation

- We can right multiply both sides by E^{-1} to cancel E
- Then we can right multiply both sides by R_E^{-1} to get C' by itself

$$C' = R_\sigma C (R_E)^{-1}$$

- Note that $R_E^{-1} = R_\sigma^T$

conventions

- There are two things that can be very confusing when transforming engineering stiffness
- First, while I have used the most standard ordering of stress/strain terms, not everyone uses the same order
- Second, the equations used here are for engineering strain (which is the most common)
- However, tensorial strain may also be used, in which case $R_\sigma = R_E$, but that adds other complications

physical interpretation

- To find the physical interpretation of elastic constants in Hooke's Law, it is easiest to use the inverse form
- If we include the thermal effects, we have

$$T_{ij} = C_{ijkl}(E_{kl} - \alpha_{kl}\Delta T)$$

- And the inverse form, where the compliance tensor, $S_{ijkl} = C^{-1}_{ijkl}$ is

$$E_{ij} = S_{ijkl}T_{kl} + \alpha_{ij}\Delta T$$

compliance

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1113} & S_{1112} \\ S_{1122} & S_{2222} & S_{2233} & S_{2223} & S_{1322} & S_{1222} \\ S_{1133} & S_{2233} & S_{3333} & S_{2333} & S_{1333} & S_{1233} \\ S_{1123} & S_{2223} & S_{2333} & S_{2323} & S_{1323} & S_{1223} \\ S_{1113} & S_{1322} & S_{1333} & S_{1323} & S_{1313} & S_{1213} \\ S_{1112} & S_{1222} & S_{1233} & S_{1223} & S_{1213} & S_{1212} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} + \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{bmatrix} \Delta T$$

physical interpretation

- If we now consider the case of uniaxial tension, we see that

$$
\begin{aligned}
E_{11} &= S_{1111} T_{11} \\
E_{22} &= S_{1122} T_{11} \\
E_{33} &= S_{1133} T_{11} \\
2E_{23} &= S_{1123} T_{11} \\
2E_{13} &= S_{1113} T_{11} \\
2E_{12} &= S_{1112} T_{11}
\end{aligned}
$$

- S_{1111} is familiar, acting like $1/E_Y$

poisson's ratio

- For isotropic materials we defined Poisson's ratio as $\nu = -E_{22}/E_{11}$
- For anisotropic materials, we can have a different Poisson's ratio acting in different directions
- We define $\nu_{ij} = -E_{jj}/E_{ii}$ (with no summation), the ratio of the transverse strain in the j direction when stress is applied in the i direction

poisson's ratio

- For this example we can find ν_{12} and ν_{13} as

$$\nu_{12} = -E_{22}/E_{11} = -S_{1122}/S_{1111}$$
$$\nu_{13} = -E_{33}/E_{11} = -S_{1133}/S_{1111}$$

poisson's ratio

- Note that we cannot, in general, say that $\nu_{12} = \nu_{21}$
- However, due to the symmetry of the stiffness/compliance tensors, we know that

$$\nu_{21} E_x = \nu_{12} E_y$$
$$\nu_{31} E_x = \nu_{13} E_z$$
$$\nu_{32} E_y = \nu_{23} E_z$$

- Where E_x refer's to the Young's Modulus in the x-direction, etc.

shear coupling coefficients

- An unfamiliar effect is that shear strains are introduced from a normal stress
- We define shear coupling coefficients as $\eta_{1112} = \eta_{16} = -2E_{12}/E_{11}$ due to T_{11}
- These coupling terms can also effect shear strain in a different plane from the applied shear stress

shear coupling

- Like the Poisson's ratio, these are not entirely independent

$$\eta_{61} E_x = \eta_{16} G_6$$

- Where G_6 is the shear modulus in the 12 plane

shear coupling

- Shear coupling coefficients are sometimes placed in two groups
- Coefficients of mutual influence relate shear stress to normal strain and normal stress to shear strain
- Chentsov coefficients relate shear stress in one plane to shear strain in another plane
- In general we can say

$$\eta_{nm} E_m = \eta_{mn} G_n \qquad (m = 1, 2, 3) \qquad (n = 4, 5, 6)$$

and

$$\eta_{nm} G_m = \eta_{mn} G_n \qquad (m, n = 4, 5, 6) \qquad m \neq n$$

material symmetries

- Very few anisotropic materials are fully anisotropic
- Most have some degree of symmetry
- We will consider monoclinic, transversely isotropic, and orthotropic symmetries

symmetry

- Let S_1 be the plane with a normal in the 1-direction
- The transformation describing a reflection with respect to the plane S_1 is

$$Q = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If a material is symmetric about S_1, we know that

$$C_{ijkl} = C'_{ijkl} = Q_{mi} Q_{nj} Q_{ok} Q_{pl} C_{mnop}$$

monoclinic symmetry

- A monoclinic material is symmetric about one plane
- If we consider the 1-direction to be the plane of material symmetry, we can use the previous equation with the Q found earlier

monoclinic symmetry

- As an example, we find the $C_{1112} = Q_{m1} Q_{n1} Q_{o1} Q_{p2} C_{mnop}$, however when $i \neq j$, $Q_{ij} = 0$
- This means we have $C_{1112} = (-1)^3 (1) C_{1112}$, which can only be satisfied when $C_{1112} = 0$
- We similarly can show that $C_{1113} = C_{1222} = C_{1223} C_{1233} = C_{1322} = C_{1323} = C_{1333} = 0$

monoclinic symmetry

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & 0 & 0 \\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & C_{1213} \\ 0 & 0 & 0 & 0 & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}$$

orthotropic symmetry

- If a material has two mutually perpendicular planes of symmetry (for example, S_1 and S_2 with normals in the 1 and 2 directions), then S_3 plane will also automatically be plane of symmetry
- This state of symmetry is known orthotropy
- Orthotropic materials are more common in engineering use than monoclinic
- All shear coupling terms are zero for orthotropic materials

orthotropic symmetry

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}$$

transversely isotropic symmetry

- If there exists a plane, such as the S_3 plane, where any plane perpendicular to that plane is a plane of symmetry, we call this transverse isotropy
- The direction normal to that plane is the axis of transverse isotropy

transversely isotropic symmetry

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2(C_{1111} - C_{2222}) \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}$$

characterizing anisotropic materials

- Characterizing anisotropic materials is not as simple as isotropic materials
- Requires additional testing
- 2 unique properties for isotropic (can be found with one test)
- 5 unique properties for transversely isotropic
- 9 unique properties for orthotropic

characterizing anisotropic materials

- Also can be difficult to obtain state of pure shear/tension with traditional gripping
- If material is heterogeneous that can introduce other challenges
- Specimen alignment is much more important than for isotropic materials

tensile testing

- If we consider an orthotropic material (such as a composite lamina), no shear is introduced when the fibers are perfectly aligned in the load direction
- When orthotropic (or transversely isotropic) material is rotated, shear-coupling terms can be introduced
- This shear deformation can cause failure at the grips

tensile testing

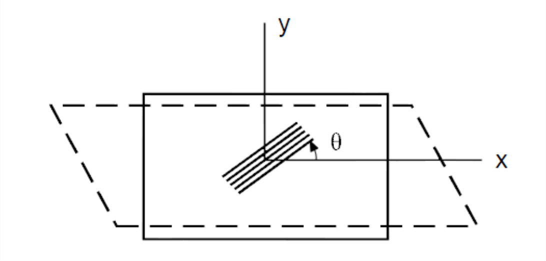


Figure 2: image

tensile testing

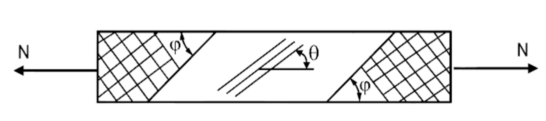
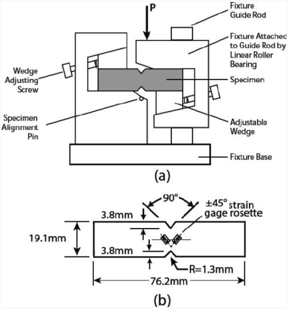


Figure 4: Oblique tabs developed by Sun and Chung in 1993

shear testing

- Just as grip constraints can make a state of pure tension difficult to obtain, it can be difficult to obtain a pure state of shear
- There are many different shear test methods for anisotropic materials, most involve specialized grips and specialized specimen geometry

iosipescu shear



two-rail shear

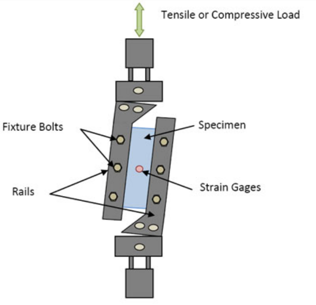


Figure 5: image

review

- Group 1: How many (theoretically) tests required to characterize a monoclinic material
- Group 2: How many (theoretically) tests to characterize an orthotropic material

reading for next class

- Analytic techniques for anisotropic elasticity: link
- Large Deformation - pp. 334 - 349