Continuum Mechanics

Lecture 16 - Energy, Rotation, Vorticity

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schedule

- 12 Nov - Energy, Rotation, Vorticity, HW8 Due
- 17 Nov - Non-Newtonian Fluids
- 1 Dec - Final Review, Research Project, HW9 Due
- 3 Dec - Final Review

energy dissipation

- Fluids can be used to dissipate energy
- sloshing video
- We can use the concept of stress power to find the energy dissipated in a fluid

incompressible newtonian fluid

- For an incompressible Newtonian fluid, we have

$$T_{ij} = -p\delta_{ij} + T'_{ij}$$

- If we multiply both sides by the velocity gradient, $v_{i,j}$, we find

$$T_{ij}v_{i,j} = -pv_{i,i} + T'_{ij}v_{i,j}$$

- But for an incompressible fluid, we know that $v_{i,i} = 0$, therefore

$$T_{ij}v_{i,j} = T'_{ij}v_{i,j}$$

incompressible newtonian fluid

- Note that $T_{ij}v_{i,j} = T_{ij}(D_{ij} + W_{ij})$
- Since T_{ij} is symmetric and W_{ij} is antisymmetric, we find that $T_{ij}v_{i,j} = T_{ij}D_{ij}$, which is an expression for stress power
- The right sight can be written as

$$T'_{ij}v_{i,j} = 2\mu D_{ij}v_{i,j} = 2\mu D_{ij}(D_{ij} + W_{ij})$$

- But $D_{ij}W_{ij} = 0$, since D_{ij} is symmetric and W_{ij} is antisymmetric

incompressible newtonian fluid

- Thus the stress power is given as

$$P_s = 2\mu D_{ij}D_{ij} = 2\mu(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2)$$

- This represents the work done per unit volume to change the shape
- Is also known as the dissipation function for an incompressible Newtonian fluid

compressible newtonian fluid

- For a compressible Newtonian fluid, the continuity equation does not reduce as conveniently
- Recall that for a compressible Newtonian fluid

$$T'_{ij} = \lambda D_{kk}\delta_{ij} + 2\mu D_{ij}$$

- Where $D_{kk} = v_{i,i}$
- Multiplying both sides of the constitutive equation as before, we find the stress power to be

$$P_s = -pv_{i,i} + \lambda v_{i,i}^2 + 2\mu D_{ij} D_{ij}$$

compressible newtonian fluid

- From the equation for P_s, $-pv_{i,i}$ represents the work done to change the volume of the fluid
- $\lambda v_{i,i}^2 + 2\mu D_{ij} D_{ij}$ represents the rate at which work is converted to heat
- This is known as the dissipation function for compressible Newtonian fluids

energy equation

- Recall that the energy equation for a continuum is

$$\rho \frac{Du}{Dt} = T_{ij} v_{i,j} - q_{i,i} + \rho q_s$$

- We see that the change in energy is related to the stress power, plus heat which enters the continuum from internal or external sources

energy equation

- In fluids the internal energy is often express as $u = c\Theta$, where c is the specific heat and Θ is the temperature
- If we assume that the only heat transfer follows Fourier's law (and there is no internal heat source) we find

$$\rho c \frac{D\Theta}{Dt} = P_s + \kappa \frac{\partial^2 \Theta}{\partial x_j \partial x_j}$$

example

- As an example, we can consider plane Couette flow with prescribed temperatures at the boundaries
- Use the energy equation to find the steady-state temperature distribution

vorticity vector

- We know that W_{ij} is the antisymmetric part of the velocity gradient
- We also know that antisymmetric tensors can be expressed as some dual vector such that

$$Wx = \omega \times x$$

- Where the angular velocity vector, ω is given by

$$\omega_i = -\langle W_{23}, W_{31}, W_{12} \rangle$$

vorticity vector

- The angular velocity vector can be expressed directly in terms of components of the velocity gradient

$$\omega = \frac{1}{2}\langle v_{3,2} - v_{2,3}, v_{1,3} - v_{3,1}, v_{2,1} - v_{1,2}\rangle$$

- The vorticity vector is defined to eliminate the 1/2 from the angular velocity vector

$$\zeta = 2\omega = \langle v_{3,2} - v_{2,3}, v_{1,3} - v_{3,1}, v_{2,1} - v_{1,2}\rangle$$

- In index notation, we can write

$$\zeta = \epsilon_{ijk} v_{k,j}$$

- Or in direct notation $\zeta = \text{curl } v$

irrotational flow

- If the vorticity vector is zero, the flow is considered irrotational
- If we consider some function $\varphi(x_1, x_2, x_3, t)$ and derive the velocity field from that function such that

$$v_i = -\frac{\partial \varphi}{\partial x_i}$$

- We find that the vorticity vector will be zero

irrotational flow

- For an incompressible fluid, we know that $v_{k,k} = 0$
- Substituting φ into this equation gives the Laplacian equation

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = 0$$

inviscid incompressible flow

- If viscosity is 0 (or negligible), the equations of motion reduce to

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \rho B_i$$

- These are known as Euler's equations of motion
- If *rho* is constant and body forces are express as

$$B_i = -\frac{\partial \Omega}{\partial x_i}$$

then we can write

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i} \left(\frac{p}{\rho} + \Omega \right)$$

inviscid incompressible flow

- For irrotational flow, the velocity gradient will be symmetric

$$v_j \frac{\partial v_i}{\partial x_j} = v_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial x_i}(v_j v_j)$$

- We now let $v^2 = v_i v_i$ and write

$$\frac{\partial}{\partial x_i}\left(-\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \Omega\right) = 0$$

inviscid incompressible flow

- The portion inside the parenthesis must be a function of time only
- For steady flow, it will be equal to a constant and the φ term will vanish

$$\frac{v^2}{2} + \frac{p}{\rho} + \Omega = C$$

- This is Bernoulli's equation, and shows that irrotational flows are always possible when density is constant in an inviscid, incompressible flow

compressible flow

- For a compressible fluid, we still need the pressure p to not be a function of deformation rate
- The pressure will be some function of density and temperature
- For example, ideal gases will follow the ideal gas law $p = R\rho\Theta$
- The constitutive equation for a compressible Newtonian fluid are

$$T_{ij} = -p(\rho,\Theta)\delta_{ij} + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij} + 2\mu D_{ij}$$

- And the contraction of that is

$$T_{ii}/3 = -p(\rho,\Theta) + k\frac{\partial v_k}{\partial x_k}$$

compressible flow

- Some gases have zero bulk viscosity
- The assumption that $k = 0$ is known as the Stokes assumption, valid for monatomic gases
- We can also write the constitutive equation in terms of μ and k

$$T_{ij} = -p(\rho,\Theta)\delta_{ij} + \lambda\frac{\partial v_k}{\partial x_k}\delta_{ij} + 2\mu D_{ij}$$

compressible flow

- The Navier-Stokes equations are

$$\rho\frac{Dv_i}{Dt} = -\rho B_i - \frac{\partial P}{\partial x_i} + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + k\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)$$

- And the energy equation is

$$\rho\frac{Du}{Dt} = T_{ij}\frac{\partial v_i}{\partial x_j} + \kappa\frac{\partial^2 \Theta}{\partial x_j \partial x_j}$$

- We also recall the continuity equation

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_j}{\partial x_j} = 0$$

- Together with some equation $p = p(\rho, \Theta)$ and $u = u(\rho, \Theta)$ gives 7 equations

- The 7 unknowns are v_i, p, ρ, Θ, and u

acoustic waves

- Let us consider the propagation of sound in an inviscid fluid with negligible body forces

- The equations of motion reduce to

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}$$

- If the fluid is initially at rest ($v_i = 0, \rho = \rho_0, p = p_0$) with some perturbation such that

$$v_i = v_i'(x_i, t) \qquad \rho = \rho_0 + \rho'(x_i, t) \qquad p = p_0 + p'(x_i, t)$$

- Substituting gives

$$\frac{\partial v_i'}{\partial t} + v_j'\frac{\partial v_i'}{\partial x_j} = -\frac{1}{\rho_0(1 + \rho'/\rho_0)}\frac{\partial p'}{\partial x_i}$$

acoustic waves

- If the disturbance is infinitesimal, then $v_j' \frac{\partial v_i'}{\partial x_j}$ will be negligible compared to $\frac{\partial v_i'}{\partial t}$ and ρ'/ρ_0 will be negligible compared to 1

$$\frac{\partial v_i'}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i}$$

- Similarly the continuity equation

$$\frac{\partial \rho'}{\partial t} + v_j' \frac{\partial \rho'}{\partial x_j} + \rho_0(1 + \rho'/\rho_0)\frac{\partial v_i'}{\partial x_i} = 0$$

- Which reduces to

$$\frac{\partial v_i'}{\partial x_i} = -\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t}$$

acoustic waves

- We can differentiate with respect to x_i and with respect to t to eliminate velocity from both equations

$$\frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2 p'}{\partial t^2}$$

- If pressure depends only on density (barotropic flow) then we can expand the pressure function as a Taylor series

$$p = p_0 + \left(\frac{dp}{d\rho}\right)_{\rho_0}(\rho - \rho_0) + \ldots$$

- Neglecting higher-order terms and re-arranging gives

$$p - p_0 = \left(\frac{dp}{d\rho}\right)_{\rho_0}(\rho - \rho_0)$$

- if we let $c_0^2 = \left(\frac{dp}{d\rho}\right)$ then we can write

acoustic waves

- For barotropic flow this gives

$$c_0^2 \frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2 p'}{\partial t^2}$$

- c_0 is the speed of sound at stagnation

reynolds transport theorem

- Reynolds transport theorem is the general framework we use to convert conservation equations from "control mass" to "control volume"
- Arbitrary control volume can be fixed or moving
- "Control mass" equations are things like
  - Conservation of mass
  - $F = ma$
  - First and second laws of thermodynamics

reynolds transport theorem

- In general, Reynolds Transport Theorem states that for some scalar or tensor function of spatial coordinates and time, $T(x_i, t)$
- Where V_m is a material volume which consists of the same material particles at all times and S_m is the boundary of this volume

$$\frac{D}{Dt}\int_{V_m(t)} T(x_i, t)dV = \int_{V_c} \frac{\partial T(x_i, t)}{\partial t} dV + \int_{S_c} T(v_i n_i) dS$$

or

$$\frac{D}{Dt}\int_{V_m(t)} T(x_i, t)dV = \int_{V_c} \left(\frac{DT}{Dt} + T\mathrm{div}v\right) dV$$

- Mathematically, this gives us a framework for moving derivatives inside an integral

conservation of mass

- The conservation of mass states that the total mass of a material must remain constant
- We can write this as

$$\frac{D}{Dt}\int_{V_m} \rho(x_i, t)dV = 0$$

linear momentum

- We can write the global form of $F = ma$ as

$$\int_{S_c} t_i dS + \int_{V_c} \rho B_i dV = \frac{D}{Dt} \int_{V_m} \rho v_i dV$$

examples

- rope sliding from table
- jet of water on a curved pane

moving frames

- If we define two frames, F_1 and F_2
- The position vector r_i denotes the position of some differential mass, dm relative to F_1
- x_i denotes position relative to F_2, we define velocity in the respective frames as

$$(dr_i/dt)_{F_1} = v_{F_1}$$
$$(dx_i/dt)_{F_2} = v_{F_2}$$

moving frames

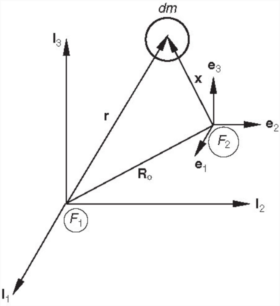

Figure 1: image

moving frames

- Since we can write $r_i = R_i + x_i$, we can say that

$$\left(\frac{dr_i}{dt}\right)_{F_1} = \left(\frac{dR_i}{dt}\right)_{F_1} + \left(\frac{dx_i}{dt}\right)_{F_1}$$

- Which we can define as

$$\left(\frac{dr_i}{dt}\right)_{F_1} = (v_0)_{F_1} + \left(\frac{dx_i}{dt}\right)_{F_1}$$

- We also know that for any vector b_i

$$\left(\frac{db_i}{dt}\right)_{F_1} = \left(\frac{db_i}{dt}\right)_{F_2} + \omega_i \times b_i$$

where ω_i is the angular velocity of F_2 relative to F_1

moving frames

- We find

$$v_{F_1} = (v_0)_{F_1} + v_{F_2} + \omega_i \times x_i$$

- Taking the global form we find

$$\left(\frac{D}{Dt}\right)_{F_1} \int v_{F_1} dm = \left(\frac{D}{Dt}\right)_{F_1} \left[\int (v_0)_{F_1} dm + \int v_{F_2} dm + \omega_i \times x_i dm\right]$$

moving frames

- If F_1 is the inertial frame, then we have

$$\left(\frac{D}{Dt}\right)_{F_1}\int v_{F_1} dm = \int t_i dS + \int \rho B_i dV$$

- And in the moving frame

$$\left(\frac{D}{Dt}\right)_{F_2}\int v_{F_2} dm = \int t_i dS + \int \rho B_i dV -$$
$$\left[m(a_0) + 2\omega \times \int v_{F_2} + \dot{\omega} \times \int x_i dm + \omega \times \left(\omega \times \int x_i dm\right)\right]$$

- Where a_0 is the acceleration of the moving frame with respect to the inertial frame

reading

- Linear Viscoelasticity - 443-456