Theory of Elasticity

a first course on
fundamental principles
and methods of analysis

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Contents

Notation

a	radius
A	cross-sectional area
b	thickness, depth, radius
$\hat{b}$, b_i	body forces per unit mass
B	bulk modulus
D	plate stiffness, $Eh^3/12(1-\nu^2)$
$\hat{e}_i$	unit or base vectors
e_{ij}	Eulerian strain tensor
E, $\hat{E}$	Young's modulus, viscoelastic modulus
EI	beam flexural stiffness
E_{ij}	Lagrangian strain tensor
G	shear modulus
h	beam or rod height, plate thickness
$h_i(r, st)$	interpolation functions
I	second moment of area, $I = bh^3/12$ for rectangle
I_1, I_2, I_3	invariants of strain
J, J	general functional
J^o, J	Jacobians
J_e	Jacobian of the element transformation
K	stiffness, thermal conduction
$[\ k\ ]$, $[\ K\ ]$	stiffness matrix
L	length of element, distance to boundary
M, M_x	moment
$\hat{n}$, n_i	direction vector
$P(t), \hat{P}$	applied force history
q	distributed load, heat flux
r	radial coordinate
R	radius
S, S_o	arc length
t	time
$\hat{t}$, t_i	tractions
T	time window, temperature
$u(t)$	response; velocity, strain, etc.
u, v, w	displacements
$\mathcal{U}$, $\overline{\mathcal{U}}$	strain energy, strain energy density
$\mathcal{V}$	potential of loads
cW	work
x^o, y^o, z^o	original rectilinear coordinates
x, y, z	deformed rectilinear coordinates

Greek letters:

α	coefficient of thermal expansion
β_{ij}	matrix of direction cosines
ϵ_{ijk}	permutation symbol

δ	small quantity, variation
δ_{ij}	Kronecker delta
Δ	determinant, increment
η	viscosity, damping
κ	plate curvature
κ	material property: $(3-4\nu)$ or $(3-\nu)/(1+\nu)$
θ	angular coordinate
ν	Poisson's ratio
μ	Shear modulus
λ	Lamê constant, eigenvalue
ρ^o, ρ	mass density
σ, ϵ	stress, strain
ξ	space transform variable
ϕ, ϕ_x, ϕ_y	rotation
Φ, H_z	Helmholtz functions
ψ	lateral contraction

Special Symbols:

∇^2	differential operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
$\times$	vector cross-product
$[\quad]$	square matrix
$\{\ \}$	vector

Subscripts:

,	comma, partial differentiation

Superscripts:

K	Kirchhoff stress
o	original configuration
$*$	complex conjugate
$^-$	bar, local coordinates
$\cdot$	dot, time derivative
$\prime$	prime, derivative with respect to argument

Chapter 1

Cartesian Tensors

The theory of elasticity deals with quantities known as *tensors*. The purpose of this chapter is to introduce the basic concepts of tensor analysis and the associate notations. The reward of the effort is that concepts and derivations (especially in the three-dimensional theory) are handled conveniently and neatly. Since theorems proven in one coordinate system are proven for all coordinate systems, then it is sufficient (and convenient) to deal only with Cartesian (or rectilinear) coordinate systems.

1.1 Indicial Notation

In the theory of elasticity, we deal with groups of things such as u, v, w representing displacements, x, y, z representing coordinates, or $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \cdots$ representing stresses. We would like a notation that handles such groups conveniently; we will use the subscript or indicial notation to achieve this.

Let the symbol x_i with the range $i = 1, \cdots, n$ be used to denote any one of the variables in the set $\{x_1, x_2, \cdots, x_n\}$. The symbol i is called an *index*. Similar notations with multiple indices such as t_{ij}, $i, j = 1, \cdots, n$, are also used to represent individual components in the set of $[n \times n]$ elements $\{t_{11}, t_{12}, \cdots, t_{nn}\}$. We will restrict ourselves to having the subscript range always go from 1 to 3.

Summation Convention

Consider the equation

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = a$$

which can be interpreted as the scalar product of two vectors whose components are $\hat{x} = \{x_1, x_2, x_3\}$ and $\hat{y} = \{y_1, y_2, y_3\}$ in the 3-dimensional space. This equation can be written as

$$\sum_{i=1}^{3} x_i y_i = a$$

With the aid of the summation convention, the equation above is written in the simple, shorthand, form

$$x_i y_i = a$$

The summation convention states that the repetition of an index in a product term denotes a summation with respect to that index over its range. The repeated index is called a *dummy* index as opposed to one that is not summed, which is referred to as a *free* index. Since a dummy index indicates a summation operation, any index can be used without changing the result. For example, $x_i y_i$ and $x_j y_j$ have the identical meaning.

When there are more than two summation operations to be performed, caution must be exercised in the use of the summation convention. The following rules can therefore help:

- If a subscript occurs twice in a term in an equation, then it must be summed over its range. These are the *repeated* or *dummy* indices, e.g.,

$$C_{ikk} = C_{i11} + C_{i22} + C_{i33}$$

- If a subscript occurs once in a term then it must occur once in every other term in the equation. These are the *free* or *live* indices, e.g.,

$$F_i = ma_i \quad \Rightarrow \quad \begin{aligned} F_1 &= ma_1 \\ F_2 &= ma_2 \\ F_3 &= ma_3 \end{aligned}$$

- If a subscript occurs more than twice in a term, then it is a *mistake*, e.g.,

$$A_{iij} B_{ij}$$

An example of a proper equation is

$$\sigma_{ij} = S_{ij} + H_{ijkl} E_{kl}$$

where the free indices are i and j and the dummy indices are k and l.

A shared repeated index is a *contraction*. Contraction is a special summing operation performed on quantities with indices. It is done by equating two indices and summing over the range of that index. For example,

$$A_{ijkl} \qquad \text{(contraction over } i\text{)} \qquad A_{iikl}$$

After contraction, the original two free indices become a pair of dummy indices and the summation convention applies.

When contraction is applied to a pair of indices of a tensor of order n, a new tensor results. The order of this new tensor is $n - 2$.

Sometimes the summation convention can be ambiguous; in those cases we will make the summation explicit by using the $\sum$ sign.

Special Symbols

A number of special symbols have been introduced as a convenience in using the tensor notation. Two especially useful symbols are the Kronecker delta and the Permutation symbol. Some of their properties are summarized here.

A *base vector* is a unit vector parallel to a coordinate axis. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the base vectors for the (x_1, x_2, x_3) coordinate system then, a typical vector can be written as

$$\hat{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 = V_i\hat{e}_i$$

The V_i are referred to as the components of the vector $\hat{V}$.

The *Kronecker delta* is denoted by δ_{ij} and is defined as follows

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \quad\;\; i \neq j \end{cases} \qquad \text{or} \qquad [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \lceil\, I \,\rfloor$$

Some of its properties are:

- Symmetrical $\delta_{ps} = \delta_{sp}$
- Trace $\delta_{pp} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
- Contraction $A_{ik}\delta_{ij} = A_{1k}\delta_{1j} + A_{2k}\delta_{2j} + A_{3k}\delta_{3j} = A_{jk}$
- Dot Product $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ e.g., $\hat{e}_1 \cdot \hat{e}_2 = 0\,,$ $\hat{e}_3 \cdot \hat{e}_3 = 1$

Note that written in matrix form, it is the same as the identity matrix.

The *permutation symbol* is defined as:

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ form an even permutation of 123, e.g., 312} \\ -1 & ijk \text{ form an odd permutation of 123, e.g., 321} \\ 0 & \quad\text{otherwise, e.g., 122} \end{cases}$$

Some of its properties are:

- Anti-symmetric $\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$
- Trace $\epsilon_{iik} = \epsilon_{11k} + \epsilon_{22k} + \epsilon_{33k} = 0$
- Contraction $\epsilon_{ijs}\epsilon_{pqs} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$
- Determinant $det[a_{ij}] = \epsilon_{ijk}a_{1i}a_{2j}a_{3k}$
- Cross Product $\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k$ e.g., $\hat{e}_2 \times \hat{e}_3 = \epsilon_{23k}\hat{e}_k = \hat{e}_1$

Note that there is no simple representation for it using matrix notation since it is of size $[3 \times 3 \times 3]$.

Example 1.1: Use the tensor notation to perform both the dot product and cross product of two vectors $\hat{A}$ and $\hat{B}$.

This is accomplished by expanding each vector in terms of its components plus unit vectors and realizing that it is only the unit vectors that participate in the vector operations. That is,

$$\begin{aligned}
\hat{A} \cdot \hat{B} &= (A_i \hat{e}_i) \cdot (B_j \hat{e}_j) = A_i B_j (\hat{e}_i \cdot \hat{e}_j) = A_i B_j \delta_{ij} = A_i B_i \\
&= A_1 B_1 + A_2 B_2 + A_3 B_3 \\
\hat{A} \times \hat{B} &= (A_i \hat{e}_i) \times (B_j \hat{e}_j) = A_i B_j (\hat{e}_i \times \hat{e}_j) = A_i B_j \epsilon_{ijk} \hat{e}_k \\
&= (A_2 B_3 - A_3 B_2)\hat{e}_1 + (A_3 B_1 - A_1 B_3)\hat{e}_2 + (A_1 B_2 - A_2 B_1)\hat{e}_3
\end{aligned}$$

It is thus clear from this approach which vector operations result in vectors and which result in scalars. ■

1.2 Coordinate Transformation

The concept of component is fundamental to tensor analysis. A given tensor quantity will have different components in different coordinate systems, thus we wish to know how they change under a coordinate transformations.

Consider two Cartesian coordinates systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) as shown in Figure 1.1. Let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ be the base vectors for the (x_1, x_2, x_3) coordinate system, and $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ the base vectors for the (x'_1, x'_2, x'_3) system as also shown in the figure.

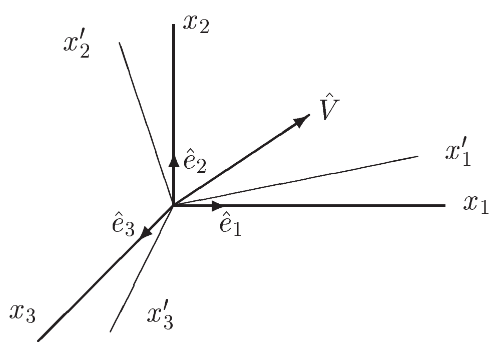


Figure 1.1: Base vectors and rotated coordinate system.

Since the coordinate axes are mutually orthogonal, we have

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}\,, \qquad \hat{e}'_i \cdot \hat{e}'_j = \delta_{ij}$$

A vector $\hat{V}$ can be projected onto the two coordinate systems with the result:

$$\hat{V} = V'_j \hat{e}'_j = V_j \hat{e}_j$$

Taking the scalar product of this equation with $\hat{e}'_i$, we obtain

$$V'_j \hat{e}'_j \cdot \hat{e}'_i = V_j \hat{e}_j \cdot \hat{e}'_i$$

which yields, on using the properties of the Kronecker delta,

$$V'_j \delta_{ij} = V_j \hat{e}_j \cdot \hat{e}'_i \qquad \text{or} \qquad V'_i = V_j \hat{e}_j \cdot \hat{e}'_i = V_j \hat{e}'_i \cdot \hat{e}_j$$

This can be further rewritten by introducing the matrix of *direction cosines*

$$\beta_{ij} \equiv \hat{e}'_i \cdot \hat{e}_j \tag{1.1}$$

and substituting to get

$$V'_i = \beta_{ij} V_j$$

This gives the relation for transformation of components in one coordinate system into components in another.

A similar procedure (but taking a scalar product with $\hat{e}_i$) leads to the inverse relation

$$V_i = \beta_{ji} V'_j$$

These two equations taken together yields the circular result

$$V'_i = \beta_{ij} \beta_{kj} V'_k$$

from which we conclude that

$$\beta_{ij} \beta_{kj} = \delta_{ik}$$

This can be written in matrix form as

$$[\ \beta\ ][\ \beta\ ]^T = \lceil\ I\ \rfloor \qquad \text{or} \qquad [\ \beta\ ]^T[\ \beta\ ] = \lceil\ I\ \rfloor$$

Thus, β_{ij} are orthogonal and the relation is known as the *orthogonality relation.* Also, it is easy to see that $\det[\ \beta\ ] = \pm 1$.

Example 1.2: Show that if two directions of an orthogonal triad are known, then the third direction can be found by using the orthogonality conditions.

Let the given vectors be

$$\begin{aligned} \hat{e}'_1 &= \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 + \tfrac{1}{\sqrt{2}}\hat{e}_3 \\ \hat{e}'_2 &= \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 - \tfrac{1}{\sqrt{2}}\hat{e}_3 \end{aligned}$$

and it is desired to obtain $\hat{e}'$. The given vectors are orthogonal since $\hat{e}'_1 \cdot \hat{e}'_2 = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} = 0$. We can obtain the third vector from knowledge that

$$\hat{e}'_3 = \hat{e}'_1 \times \hat{e}'_2 = \tfrac{1}{\sqrt{2}}\hat{e}_1 + \tfrac{1}{\sqrt{2}}\hat{e}_2 + 0\hat{e}_3$$

The first row of the direction cosines is $\beta_{1j} = \hat{e}'_1 \cdot \hat{e}_j = \hat{e}'_{1j}$ so that the direction cosines are easily established as

$$[\beta_{ij}] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

That is, each row consists of the components of the primed unit vectors. ■

1.3 Scalar, Vector and Tensor Fields

A quantity is a field quantity when it is a function of position. For example $\phi = \phi(x_1, x_2, x_3, t)$. Let $\{x_1, x_2, x_3\}$ and $\{x'_1, x'_2, x'_3\}$ be two fixed sets of rectangular Cartesian coordinates related by the transformation law

$$x'_i = \beta_{ij} x_j$$

where β_{ij} are the direction cosines. A system of quantities is called by different names depending on how the components of the system are defined in the variables x_1, x_2, x_3 and how they are transformed when the variables x_1, x_2, x_3 are changed to x'_1, x'_2, x'_3.

A system is called a *scalar* if it has only a single component ϕ in the variables x_i and a single component ϕ' in the variables x'_i and if ϕ and ϕ' are numerically equal at the corresponding points. That is,

$$\phi(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$$

A system is called a *vector field* or a tensor field of order one if it has three components v_i in the variables x'_i and the components are related by the transformation law

$$\hat{v} = v_i \hat{e}_i = v'_k \hat{e'}_k = v_i \beta_{ki} \hat{e}'_k$$

Dropping the unit vectors and rearranging the subscripts, these become

$$v'_i = \beta_{ik} v_k \qquad \text{or} \qquad \{v'\} = [\ \beta\ ]\{v\}$$

The tensor field of order two is a system which has nine components t_{ij} in the variables x_1, x_2, x_3, and nine components t'_{ij} in the variables x'_1, x'_2, x'_3, and the components are related by the transformation law

$$t'_{ij} = \beta_{im} \beta_{jn} t_{mn} \qquad \text{or} \qquad [\ t'\ ] = [\ \beta\ ][\ t\ ][\ \beta\ ]^T$$

The pattern is, each order of tensor requires n^3 components and n $[\ \beta\ ]$ matrices in the transformation.

We obtain from a generalization to a tensor of order n:

$$\begin{aligned} t'_{ijk\cdots} &= \beta_{im} \beta_{jn} \beta_{kp} \cdots t_{mnp\cdots} \\ t_{ijk\cdots} &= \beta_{mi} \beta_{nj} \beta_{pk} \cdots t'_{mnp\cdots} \end{aligned} \tag{1.2}$$

If all components of a tensor vanish in one coordinate system, then they vanish in all other coordinate systems.

A second order tensor t_{ij} is often represented as a square matrix $[t_{ij}]$. In this form matrix algebra applies. Higher order tensors, however, do not have a simple 2-D matrix representation.

Isotropic Tensors

An interesting type of tensor frequently occurs in the study of elasticity, namely, that of an isotropic tensor. An *isotropic tensor* is one whose components remain unchanged under a coordinate transformation. For a second order isotropic tensor, for example,

$$t'_{ij} = t_{ij} = \beta_{ik}\beta_{jl}t_{kl}$$

A collection of particular isotropic tensors of different order is:

order	isotropic tensor
0	1
1	
2	δ_{ij}
3	ϵ_{ijk}
4	$\delta_{ij}\delta_{pq}$, $\delta_{ip}\delta_{jq}$, $\delta_{pj}\delta_{iq}$
5	$\epsilon_{ijk}\delta_{pq}$, and permutations
6	$\delta_{ij}\delta_{pq}\delta_{rs}$, $\delta_{ip}\delta_{jq}\delta_{rs}$, $\delta_{pj}\delta_{iq}\delta_{rs}$, $\delta_{ip}\delta_{rs}\delta_{js}$, and permutations

It is noted that there cannot be a vector (tensor of order one) that is isotropic.

Calculus of Tensor Fields

A tensor field occurs when the tensor is a function of position. Examples of scalar, vector and second order tensor fields are, respectively,

$$\begin{aligned} \phi &= \phi(x_1, x_2, x_3, t) \\ v_i &= V_i(x_1, x_2, x_3, t) \\ t_{ij} &= T_{ij}(x_1, x_2, x_3, t) \end{aligned}$$

Since these are continuous functions of position then they are amenable to calculus operations such as differentiation and integration.

If $x'_i = \beta_{ij}x_j$, then for a vector v_j we have

$$v'_j(\hat{x}') = \beta_{jk}v_k(\hat{x})$$

Differentiating both sides of the equation, we obtain

$$\frac{\partial v'_j}{\partial x'_i} = \beta_{jk}\frac{\partial v_k}{\partial x'_i} = \beta_{jk}\frac{\partial v_k}{\partial x_m}\frac{\partial x_m}{\partial x'_i} = \beta_{jk}\beta_{im}\frac{\partial v_k}{\partial x_m}$$

This says that partial derivatives of any tensor field behave like the components of a Cartesian tensor. (It should be noted that this is not true in curvilinear coordinate systems.) From this it is apparent that the term

$$\frac{\partial t_{ij}}{\partial x_k}$$

is a third order tensor. In general differentiation with respect to x increases the order of a term by 1.

Consider a tensor field $T_{jkm\cdots}(x)$ in a volume V bounded by a surface S. Then, the *Integral theorem* states

$$\int_S n_i T_{jkm\ldots} dS = \int_V \frac{\partial}{\partial x_i} T_{jkm\ldots}\, dV \tag{1.3}$$

where n_i are components of the unit vector $\hat{n}$ along the exterior normal of S. Special cases of this are

$$\begin{aligned} \int_S n_i\phi\, dS &= \int_V \frac{\partial\phi}{\partial x_i}\, dV \\ \int_S n_i v_i\, dS &= \int_V \frac{\partial v_i}{\partial x_i}\, dV \\ \int_S \epsilon_{ijk} n_j v_k\, dS &= \int_V \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\, dV \end{aligned} \tag{1.4}$$

The second of these is usually referred to as *Gauss's theorem*.

Consider the case when the surface integral is of the product of a scalar and vector ϕv_i, then

$$\int_S n_i \phi v_i\, dS = \int_S \phi v_i n_i\, dS = \int_V [\phi \frac{\partial v_i}{\partial x_i} + \frac{\partial \phi}{\partial x_i} v_i]\, dV$$

Furthermore, let the vector be represented as the gradient of a scalar $v_i = \partial\psi/\partial x_i$ so that

$$\frac{\partial v_i}{\partial x_i} = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2} = \frac{\partial^2\psi}{\partial x_i \partial x_i} = \nabla^2\psi$$

then the integral theorem becomes

$$\int_S \phi \frac{\partial\psi}{\partial x_i} n_i\, dS = \int_V [\phi\nabla^2\psi + \frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_i}]\, dV$$

Now interchange the role of ϕ and ψ and subtract the two integral relations to get

$$\int_S \left[\phi\frac{\partial\psi}{\partial x_i} n_i - \psi\frac{\partial\phi}{\partial x_i} n_i\right] dS = \int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dV \tag{1.5}$$

This is usually referred to as *Green's theorem* and we will find a very useful application of it in Chapter 7.

1.4 Properties of Second Order Tensors

Since second order tensors are so prevalent in the theory of elasticity (they are used to represent stress and strain, for example), it is of value now to summarize some of their major properties. Because second order tensors can be represented conveniently by matrices, many of the following results can also be established simply from matrix theory.

Symmetry and Antisymmetry

A tensor S_{ij} is *symmetric* if

$$S_{ji} = S_{ij} \qquad \text{or} \qquad [\ S\ ]^T = [\ S\ ]$$

A tensor A_{ij} is *antisymmetric* if

$$A_{ji} = -A_{ij} \qquad \text{or} \qquad [\ A\ ]^T = -[\ A\ ]$$

It follows that an anti-symmetric tensor must have zeros on the diagonal.

Any second order tensor can be decomposed into the sum of a symmetric and an antisymmetric part by

$$\begin{aligned} B_{ij} &= \tfrac{1}{2}B_{ij} + \tfrac{1}{2}B_{ji} + \tfrac{1}{2}B_{ij} - \tfrac{1}{2}B_{ji} \\ &= \qquad S_{ij} \qquad + \qquad A_{ij} \end{aligned}$$

Note that the contraction of a symmetric and antisymmetric tensor is zero. That is, $S_{ij}A_{ij} = 0$.

Principal Values of Symmetric Tensors

Consider the relation

$$V_i = T_{ij}n_j\,, \qquad V_i = \text{vector} \qquad n_i = \text{unit vector}$$

and let T_{ij} be a symmetric tensor. That is, a vector V_i is produced by contracting the unit vector with the second order symmetric tensor T_{ij}. We will give an interesting geometrical interpretation of this relation that will aid in the understanding of the properties of stress and strain.

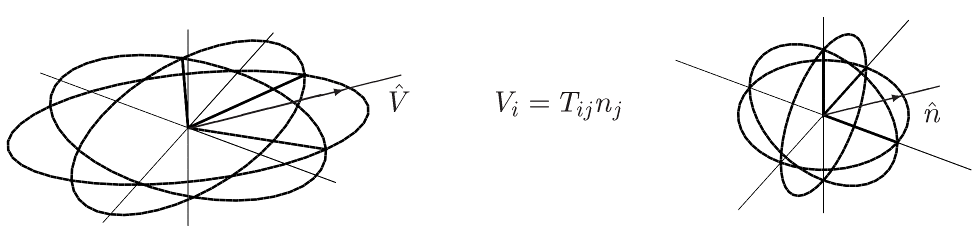


Figure 1.2: The ellipsoid associated with the transformation of second order tensors. As the vector $\hat{n}$ traces a sphere, the vector $\hat{V}$ traces an ellipsoid.

Consider different initial vectors $\hat{n}$ each of the same size but different orientations; they will correspond to different vectors $\hat{V}$. Figure 1.2 shows a collection of such vectors where $\hat{n}$ traces out the coordinate circles of a sphere. Note that $\hat{V}$ traces an ellipse but not necessarily in the coordinate planes; many such traces would form an ellipsoid. That is, the sphere traced by $\hat{n}$ is transformed into an ellipsoid traced by $\hat{V}$ and the principal axes of the ellipsoid do not necessarily coincide with the coordinate directions.

In general, the vectors $\hat{n}$ and $\hat{V}$ are not parallel. An interesting question to ask, however, is: Are there any initial vectors that have the same orientation before and after transformation? The answer will give an insight into the properties of all second order tensors, not just strain but also stress and moments of inertia to name two more.

Assume that there is an $\hat{n}$ such that it is parallel to $\hat{V}$, that is,

$$V_i = \lambda n_i = T_{ij} n_j \qquad \text{or} \qquad [T_{ij} - \lambda \delta_{ij}]\, n_j = 0$$

where λ is some scalar multiplier. This is given in expanded matrix form as

$$\begin{bmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{12} & T_{22}-\lambda & T_{23} \\ T_{13} & T_{23} & T_{33}-\lambda \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

This has a non-trivial solution for n_i only if the determinant of the coefficient matrix vanishes. Expanding the determinantal equation we obtain the characteristic equation

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \tag{1.6}$$

where the *invariants* I_1, I_2, I_3 are defined as

$$\begin{aligned} I_1 &= T_{ii} = T_{11} + T_{22} + T_{33} \\ I_2 &= \tfrac{1}{2}[T_{ii}T_{jj} - T_{ij}T_{ji}] = T_{11}T_{22} + T_{22}T_{33} + T_{33}T_{11} - T_{12}^2 - T_{23}^2 - T_{13}^2 \\ I_3 &= \det[T_{ij}] = T_{11}T_{22}T_{33} + 2T_{12}T_{23}T_{13} - T_{11}T_{23}^2 - T_{22}T_{13}^2 - T_{33}T_{12}^2 \end{aligned} \tag{1.7}$$

The characteristic equation yields three roots or possible values for λ

$$\lambda^{(1)},\ \lambda^{(2)},\ \lambda^{(3)}$$

These are called *eigenvalues* or *principal values*. For each principal value, there is a corresponding solution for vector $\hat{n}$. The three $\hat{n}$'s

$$\hat{n}^{(1)},\ \hat{n}^{(2)},\ \hat{n}^{(3)}$$

are called *eigenvectors* or *principal directions*. The principal values and directions are the same as those of the principal axes of the ellipsoid in Figure 1.2.

The principal values can be computed from the invariants [25] by first defining

$$Q = \tfrac{1}{9}[I_1^2 - 3I_2]\,, \qquad R = \tfrac{1}{54}[-2I_1^3 + 9I_1I_2 - 27I_3]\,, \qquad \theta = \cos^{-1}[R/\sqrt{Q^3}]$$

then

$$\lambda_1 = \tfrac{1}{3}I_1 - 2\sqrt{Q}\cos(\tfrac{1}{3}\theta)\,, \qquad \lambda_2, \lambda_3 = \tfrac{1}{3}I_1 - 2\sqrt{Q}\cos(\tfrac{1}{3}(\theta \pm 2\pi)) \tag{1.8}$$

For two-dimensional problems where $T_{13} = 0$ and $T_{23} = 0$, this simplifies to

$$\lambda_1, \lambda_2 = \tfrac{1}{2}[T_{11} + T_{22}] \pm \tfrac{1}{2}\sqrt{[T_{11} - T_{22}]^2 + 4T_{12}^2}\,, \qquad \lambda_3 = T_{33} \tag{1.9}$$

This coincides with the construction known as Mohr's circle.

Example 1.3: Prove that the principal directions are orthogonal.

Consider two particular directions (designated a and b), then they satisfy

$$T_{ij} n_j^a = \lambda^a n_i^a\,, \qquad T_{ij} n_j^b = \lambda^b n_i^b$$

Contract the first with n_i^b and the second with n_i^a to give the scalar relations

$$T_{ij} n_i^b n_j^a = \lambda^a n_i^b n_i^a\,, \qquad T_{ij} n_i^a n_j^b = \lambda^b n_i^a n_i^b$$

Subtract these two and noting that because of the symmetry of T_{ij}, the left hand terms are equal and cancel, to then give

$$0 = (\lambda^a - \lambda^b) n_i^a n_i^b$$

We conclude that as long as the two principal values are different then

$$n_i^a n_i^b = 0 \qquad \text{or} \qquad \hat{n}^a \cdot \hat{n}^b = 0$$

which proves the orthogonality. ■

Example 1.4: Transform the components of T_{ij} to the coordinate system defined by the principal directions.

Let the original coordinate system be defined by the base vectors

$$\hat{e}_1 = \{1, 0, 0\}\,, \qquad \hat{e}_2 = \{0, 1, 0\}\,, \qquad \hat{e}_3 = \{0, 0, 1\}$$

and the transformed system by

$$\hat{e}_1' = \hat{n}^{(1)}\,, \qquad \hat{e}_2' = \hat{n}^{(2)}\,, \qquad \hat{e}_3' = \hat{n}^{(3)}$$

Then the transformation matrix is given by

$$\beta_{ij} = \hat{e}_i' \cdot \hat{e}_j = \hat{n}^{(i)} \cdot \hat{e}_j = n_k^{(i)} \hat{e}_k \cdot \hat{e}_j = n_j^{(i)}$$

We note that since $n_i^{(k)}$ are the direction cosines of an orthogonal triad then

$$n_i^{(k)} = \beta_{ki} \qquad \text{e.g.,} \qquad n_i^{(2)} = \{\beta_{21}, \beta_{22}, \beta_{23}\}$$

From the eigenvalue problem, we have

$$T_{ij} n_j^{(k)} = \lambda^{(k)} n_i^{(k)}$$

(there is no sum on k). Use of the direction cosines leads to

$$T_{ij}\beta_{kj} = \lambda^{(k)}\beta_{ki}$$

Multiply both sides by β_{li} and recognizing the left hand side as the transform of T_{ij} and the right hand side as the Kronecker delta gives

$$T'_{lk} = \lambda^{(k)}\delta_{kl}$$

or in expanded matrix form

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ & T'_{22} & T'_{23} \\ Sym & & T'_{33} \end{bmatrix} = \begin{bmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{bmatrix}$$

That is, with respect to the new coordinate system, T_{ij} has a diagonal form.

We can show that $\lambda^{(i)}$'s are the maximums and minimums of the associated ellipsoid surface. ∎

Cayley-Hamilton Theorem

An interesting aspect of symmetric second order tensors is that higher order products of it can always be written in terms of the invariants. This is intimately related to the principal values.

For example, the principal values of $[T_{ij}]$ are obtained from the characteristic equation for the eigenvalue problem. The Cayley-Hamilton theorem states that the matrix $[T_{ij}]$ satisfies the same characteristic equation, that is,

$$[T_{ij}]^3 - I_1[T_{ij}]^2 + I_2[T_{ij}]^1 - I_3[\,\delta_{ij}\,] = 0$$

From this, we have

$$[T_{ij}]^3 = I_1[T_{ij}]^2 - I_2[T_{ij}]^1 + I_3[\,\delta_{ij}\,] \tag{1.10}$$

allowing the next higher order power to be written as

$$[T_{ij}]^4 = I_1[T_{ij}]^3 - I_2[T_{ij}]^2 + I_3[T_{ij}]^1$$

But the third power term is already known, hence this can be reduced to

$$[T_{ij}]^4 = (I_1^2 - I_2)[T_{ij}]^2 + (I_3 - I_1 I_2)[T_{ij}]^1 + I_1 I_2[\,\delta_{ij}\,]$$

In this way, it can be easily shown that terms with higher powers can be expressed in terms of $[T_{ij}]^2$, $[T_{ij}]^1$, and $[\,\delta_{ij}\,]$, and the invariants.

Exercises

1.1 Write out explicitly the components of

$$\sigma_{ij} = S_{ij} + H_{ijpq}E_{pq}$$

1.2 Show by expansion

$$\begin{array}{ll} (a) & [\hat{A} \times (\hat{B} \times \hat{C}] = (\hat{A} \cdot \hat{C})\hat{B} - (\hat{A} \cdot \hat{B})\hat{C} \\ (b) & \hat{\nabla} \cdot (\hat{\nabla} \times \hat{V}) = 0 \\ (c) & \epsilon_{ijk}\epsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \end{array}$$

1.3 Prove, by expansion into component form, that

$$\hat{\nabla} \times (\hat{A} \times \hat{B}) = \hat{A}(\hat{\nabla} \cdot \hat{B}) - \hat{B}(\hat{\nabla} \cdot \hat{A}) + (\hat{B} \cdot \hat{\nabla})\hat{A} - (\hat{A} \cdot \hat{\nabla})\hat{B}$$

1.4 Prove that the principal values of a symmetric second order tensor are extremal values.

1.5 Show that if $\hat{e}'_i = \beta_{ik}\hat{e}_k$ then $\hat{e}_i = \beta_{ki}\hat{e}'_k$, when $\hat{e}_i$ and $\hat{e}'_i$ are orthogonal triads.

1.6 Show that $\det[\beta_{ij}] = \pm 1$.

1.7 Find the components of the second order tensor

$$[T_{ij}] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

with respect to axes rotated about the 3-direction by an angle θ. For a number of angles, evaluate the invariants I_1, I_2, I_3.

1.8 Find the components of the second order tensor

$$[T_{ij}] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

with respect to a set of axes defined by

$$\hat{e}'_1 = \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 + \tfrac{1}{\sqrt{2}}\hat{e}_3\,, \qquad \hat{e}'_2 = \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 - \tfrac{1}{\sqrt{2}}\hat{e}_3\,, \qquad \hat{e}'_3 = \tfrac{1}{\sqrt{2}}\hat{e}_1 + \tfrac{1}{\sqrt{2}}\hat{e}_2$$

1.9 A second order tensor is given as

$$[T_{ij}] = \begin{bmatrix} 21 & 15 & 12 \\ 15 & -6 & 15 \\ 12 & 15 & -6 \end{bmatrix}$$

What are its principal values and their directions?

1.10 The principal directions for a tensor are

$$\hat{e}'_1 = \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 + \tfrac{1}{\sqrt{2}}\hat{e}_3\,, \qquad \hat{e}'_2 = \tfrac{1}{2}\hat{e}_1 - \tfrac{1}{2}\hat{e}_2 - \tfrac{1}{\sqrt{2}}\hat{e}_3\,, \qquad \hat{e}'_3 = \tfrac{1}{\sqrt{2}}\hat{e}_1 + \tfrac{1}{\sqrt{2}}\hat{e}_2$$

If the principal values are $\{1,\, 2,\, 3\}$, what are the component values of the tensor?

1.11 If $[S_{ij}]$ is symmetric and $[A_{ij}]$ is anti-symmetric, show that $S_{ij}A_{ij} = 0$.

Chapter 2

Deformation

This chapter considers how lengths, areas and volumes change during a deformation. These are important basic considerations because changes in lengths are related to strains, changes in areas are associated with stresses, and a change of volume is related to conservation of mass. The deformation gradient is shown to be the essential quantity necessary to describe the changes of these quantities. The duality of the Lagrangian and Eulerian formulations is maintained throughout.

2.1 General Description of Deformations

We set up a common global coordinate system and associate x_i^o with the undeformed configuration and x_i with the deformed configuration. That is,

$$\text{Initial position:}\quad \hat{x}^o = x_i^o \hat{e}_i \qquad\qquad \text{Final position:}\quad \hat{x} = x_i \hat{e}_i$$

where both vectors are referred to the common set of unit vectors $\hat{e}_i$.

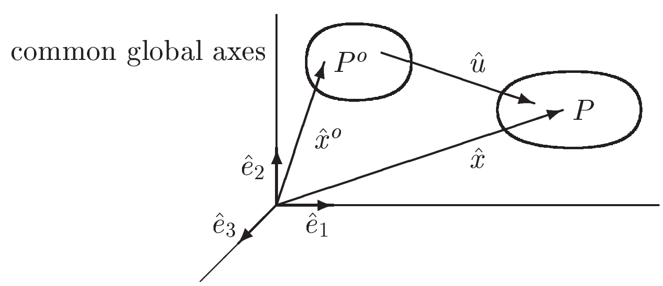


Figure 2.1: Undeformed and deformed configurations.

Motion

A *motion* is expressed in either of the following equivalent forms

$$x_i = x_i(x_1^o, x_2^o, x_3^o, t) \qquad \text{or} \qquad x_i^o = x_i^o(x_1, x_2, x_3, t)$$

The variables x_i^o and x_i are called the *Lagrangian* and *Eulerian* variables, respectively. In the Lagrangian system, all quantities are expressed in terms of the initial position coordinates and time; in the Eulerian system, the independent variables are x_i and t where x_i are the position coordinates at the time of interest. Realizing that the description of deformation is essentially geometric, then the difference between the Lagrangian and Eulerian descriptions can be stated as:

Lagrangian: Put a rectangular grid on the original (undeformed) body — determine what it will look like during the motion.

Eulerian: Put a rectangular grid on the current (deformed) body — determine what it looked like in the original state.

In other words, the Lagrangian grid is always superposed on the same material points and therefore deforms; the Eulerian grid is always rectangular (in the deformed state) and is superposed on a constantly changing set of material points. For this reason, the two descriptions are sometimes referred to as material and spatial descriptions, respectively. The significant difference in the two descriptions is in the designation of the neighboring points.

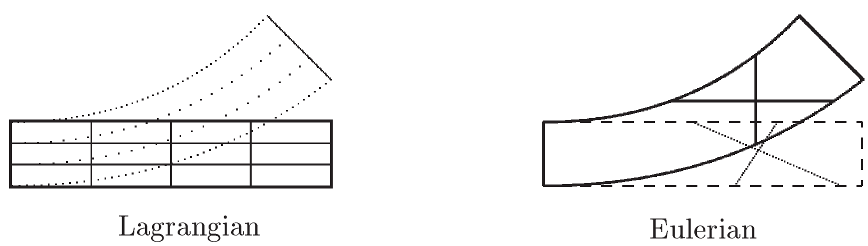


Figure 2.2: Grids illustrating Lagrangian and Eulerian descriptions.

The choice of description also has an effect on the time derivatives that must be used. That is, consider any quantity $\phi(x_1^o, x_2^o, x_3^o, t)$, the material derivative of ϕ is

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t}$$

If ϕ is expressed in Eulerian form as $\bar{\phi}(x_1, x_2, x_3, t) = \phi(x_1^o, x_2^o, x_3^o, t)$, then

$$\frac{d\bar{\phi}}{dt} = \frac{\partial \bar{\phi}}{\partial t} + \frac{\partial \bar{\phi}}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial \bar{\phi}}{\partial t} + v_j \frac{\partial \bar{\phi}}{\partial x_j}$$

where v_j is the convective velocity. Thus the velocity in the Eulerian description must take into account the fact that not only is the quantity itself changing ($\partial\bar{\phi}/\partial t$) but the particles being considered are also changing.

It must be reiterated that there is no essential difference between the two descriptions; the preference of one over the other is usually based on convenience for particular problems.

A *deformation* is a comparison of two states — the intermediate ones are not important and typically time does not appear explicitly. The deformation of a material point is expressed as

$$x_i = x_i(x_1^o, x_2^o, x_3^o) \qquad \text{or} \qquad x_i^o = x_i^o(x_1, x_2, x_3)$$

A displacement is the shortest distance traveled when a particle moves from one location to another, that is,

$$\hat{u} = \hat{r} - \hat{r}^o = x_i\hat{e}_i - x_i^o\hat{e}_i \qquad \text{or} \qquad u_i = (x_i - x_i^o)$$

Since displacement is a comparison of two states then it is the same in both the Lagrangian and Eulerian descriptions.

Example 2.1: Illustrate the duality between the Lagrangian and Eulerian descriptions of motion.

In the two-dimensional motion given by the Lagrangian description

$$\begin{aligned} x_1 &= \tfrac{1}{2}(x_1^o + x_2^o)e^t + \tfrac{1}{2}(x_1^o - x_2^o)e^{-t} \\ x_2 &= \tfrac{1}{2}(x_1^o + x_2^o)e^t - \tfrac{1}{2}(x_1^o - x_2^o)e^{-t} \end{aligned}$$

the displacement vector is

$$\begin{aligned} u_1 = x_1 - x_1^o &= [\tfrac{1}{2}(x_1^o + x_2^o)e^t + \tfrac{1}{2}(x_1^o - x_2^o)e^{-t}] - x_1^o \\ u_2 = x_2 - x_2^o &= [\tfrac{1}{2}(x_1^o + x_2^o)e^t - \tfrac{1}{2}(x_1^o - x_2^o)e^{-t}] - x_2^o \end{aligned}$$

Here, the displacements at time t are given in terms of the position (x_1^o, x_2^o) occupied by the particle at time $t = 0$. Alternatively, the same displacement could be specified in terms of the position (x_1, x_2) occupied by the particle at time t. Inverting the above relations gives the Eulerian descriptions

$$\begin{aligned} x_1^o &= \tfrac{1}{2}(x_1 + x_2)e^{-t} + \tfrac{1}{2}(x_1 - x_2)e^t \\ x_2^o &= \tfrac{1}{2}(x_1 + x_2)e^{-t} - \tfrac{1}{2}(x_1 - x_2)e^t \end{aligned}$$

and

$$\begin{aligned} u_1 = x_1 - x_1^o &= x_1 - [\tfrac{1}{2}(x_1 + x_2)e^{-t} + \tfrac{1}{2}(x_1 - x_2)e^t] \\ u_2 = x_2 - x_2^o &= x_2 - [\tfrac{1}{2}(x_1 + x_2)e^{-t} - \tfrac{1}{2}(x_1 - x_2)e^t] \end{aligned}$$

Thus, the displacement can be written either as $u = u(x_i^o)$ or $u = u(x_i)$ and we expect the relationship to be invertible. ■

Deformation Gradients

In the mechanics of deformable bodies, we are particularly interested in the deformation of neighboring points; that they are different is in the nature of deformable bodies.

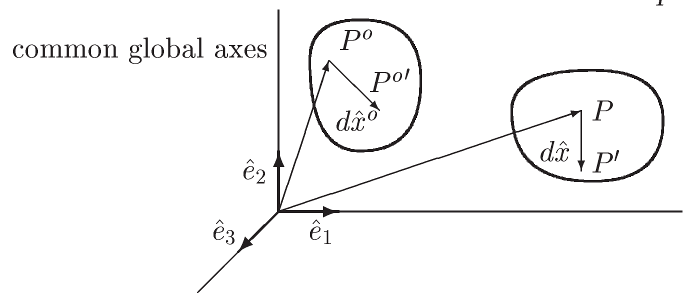


Figure 2.3: Deformation of neighboring points.

Consider a deformation in the vicinity of the point P; that is, consider two points P and P' separated by dx_i^o in the undeformed configuration and by dx_i in the deformed configuration. The positions of the two points are related through

$$\begin{aligned}
P \quad &: \quad x_i = x_i(x_i^o) \\
P' \quad &: \quad x_i + dx_i = x_i(x_i^o + dx_i^o) \\
&\qquad\qquad \approx x_i(x_i^o) + \frac{\partial x_i}{\partial x_j^o} dx_j^o + \tfrac{1}{2} \frac{\partial^2 x_i}{\partial x_j^o \partial x_k^o} dx_j^o dx_k^o + \ldots
\end{aligned}$$

Hence if dx_i^o is small, that is, the neighboring points are very close to each other, then

$$dx_i = \frac{\partial x_i}{\partial x_j^o} dx_j^o \tag{2.1}$$

This describes how the separation in the deformed configuration is related to the separation in the undeformed configuration. It is expressed in matrix form as

$$\left\{ \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \end{array} \right\} = \begin{bmatrix} \dfrac{\partial x_1}{\partial x_1^o} & \dfrac{\partial x_1}{\partial x_2^o} & \dfrac{\partial x_1}{\partial x_3^o} \\ \dfrac{\partial x_2}{\partial x_1^o} & \dfrac{\partial x_2}{\partial x_2^o} & \dfrac{\partial x_2}{\partial x_3^o} \\ \dfrac{\partial x_3}{\partial x_1^o} & \dfrac{\partial x_3}{\partial x_2^o} & \dfrac{\partial x_3}{\partial x_3^o} \end{bmatrix} \left\{ \begin{array}{c} dx_1^o \\ dx_2^o \\ dx_3^o \end{array} \right\}$$

The inverse (which should exist for continuous deformations) can also be written as

$$\left\{ \begin{array}{c} dx_1^o \\ dx_2^o \\ dx_3^o \end{array} \right\} = \begin{bmatrix} \dfrac{\partial x_1^o}{\partial x_1} & \dfrac{\partial x_1^o}{\partial x_2} & \dfrac{\partial x_1^o}{\partial x_3} \\ \dfrac{\partial x_2^o}{\partial x_1} & \dfrac{\partial x_2^o}{\partial x_2} & \dfrac{\partial x_2^o}{\partial x_3} \\ \dfrac{\partial x_3^o}{\partial x_1} & \dfrac{\partial x_3^o}{\partial x_2} & \dfrac{\partial x_3^o}{\partial x_3} \end{bmatrix} \left\{ \begin{array}{c} dx_1 \\ dx_2 \\ dx_3 \end{array} \right\} \qquad \text{or} \qquad dx_i^o = \frac{\partial x_i^o}{\partial x_j} dx_j$$

The quantities $\dfrac{\partial x_i}{\partial x_j^o}, \dfrac{\partial x_i^o}{\partial x_j}$ are called the *deformation gradients* and form the basis of the description of any deformation. Deformation gradients relate the behavior of neighboring particles and this is essential for forming derivatives. Derivatives in the Lagrangian and Eulerian coordinates are different because fundamentally different particle pairs are considered.

The relation

$$dx_i = \frac{\partial x_i}{\partial x_j^o} dx_j^o$$

uniquely specifies dx_1, dx_2, dx_3 in terms of dx_1^o, dx_2^o, dx_3^o. On the assumption that the deformation is continuous then we should be able to write

$$\{dx^o\} = [\frac{\partial x_i}{\partial x_j^o}]^{-1}\{dx\}$$

which is true only if

$$\det[\frac{\partial x_i}{\partial x_j^o}] \neq 0$$

Define the *Jacobians* referenced to the undeformed and deformed configurations as, respectively,

$$\begin{aligned} J^o \equiv \det[\frac{\partial x_i}{\partial x_j^o}] &= \epsilon_{ijk} \frac{\partial x_1}{\partial x_i^o}\frac{\partial x_2}{\partial x_j^o}\frac{\partial x_3}{\partial x_k^o} \\ J \equiv \det[\frac{\partial x_i^o}{\partial x_j}] &= \epsilon_{ijk} \frac{\partial x_1^o}{\partial x_i}\frac{\partial x_2^o}{\partial x_j}\frac{\partial x_3^o}{\partial x_k} \end{aligned} \tag{2.2}$$

Some other useful forms for the Jacobian are

$$\begin{aligned} J^o &= \epsilon_{ijk} \frac{\partial x_i}{\partial x_1^o}\frac{\partial x_j}{\partial x_2^o}\frac{\partial x_k}{\partial x_3^o}, \qquad J^o = \tfrac{1}{6}\epsilon_{pqr}\epsilon_{ijk} \frac{\partial x_i}{\partial x_p^o}\frac{\partial x_j}{\partial x_q^o}\frac{\partial x_k}{\partial x_r^o} \\ J &= \epsilon_{ijk} \frac{\partial x_i^o}{\partial x_1}\frac{\partial x_j^o}{\partial x_2}\frac{\partial x_k^o}{\partial x_3}, \qquad J = \tfrac{1}{6}\epsilon_{pqr}\epsilon_{ijk} \frac{\partial x_i^o}{\partial x_p}\frac{\partial x_j^o}{\partial x_q}\frac{\partial x_k^o}{\partial x_r} \end{aligned}$$

Note that the Jacobian is a scalar quantity.

We will impose the restriction on any deformation that no region of finite volume is deformed into a region of zero or infinite volume. That is, we restrict the values of the Jacobian as

$$0 < J^o < \infty\,, \qquad 0 < J < \infty$$

It is necessary to always check this to see if the deformation is physically possible.

Earlier, we said that a deformation is a comparison of two states — the intermediate ones do not matter. The Jacobian, therefore, plays a valuable role in restricting the path between the two states.

Example 2.2: A particular deformation is described by

$$x_1 = x_1^o + k\,x_2^o \qquad x_2 = 2x_1^o + 3x_2^o \qquad x_3 = x_3^o$$

where k is a parameter. What are the restrictions on the allowable deformation?

The deformation gradient is

$$[\frac{\partial x_i}{\partial x_j^o}] = \begin{bmatrix} 1 & k & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Jacobian is the determinant of this and is

$$J^o = 3 - 2k$$

Therefore, in order for this to be a real deformation, we must have the restriction that $k < 1.5$. Suppose, however, $k = 2$ so that

$$x_1 = x_1^o + 2\,x_2^o \qquad x_2 = 2x_1^o + 3x_2^o \qquad x_3 = x_3^o$$

The new points could be plotted (drawn) and the deformed shape would 'look reasonable'. However, if the deformation is followed through its history (as the parameter k is changed continuously) then at one stage ($k = 1.5$) the volume is zero and after that the body will have turned itself inside out. ■

2.2 Deformation of Lines, Areas, and Volumes

The deformation gradient forms the basis for the description of any deformation. This section demonstrates its use in the description of the deformation of lines, areas, and volumes.

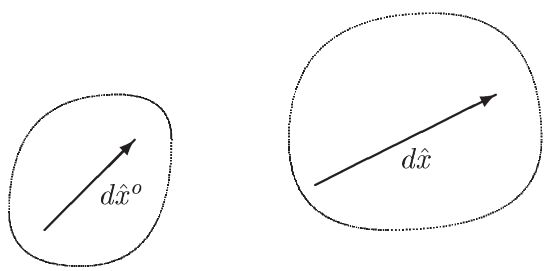


Figure 2.4: Deformation of lines.

Lines

The descriptions of a line segment before and after deformation are

$$d\hat{x}^o = dx_i^o \hat{e}_i\,, \qquad d\hat{x} = dx_i \hat{e}_i$$

A straight forward application of the deformation gradient gives

$$d\hat{x} = \frac{\partial x_i}{\partial x_j^o} dx_j^o \hat{e}_i$$

Consider the special case of a line $d\hat{x}^o$ originally oriented only along $\hat{e}_1$, that is,

$$d\hat{x}^o = dx_1^o\hat{e}_1 + dx_2^o\hat{e}_2 + dx_3^o\hat{e}_3 = dx_1^o\hat{e}_1$$

After deformation this line segment becomes

$$\begin{aligned} d\hat{x} &= dx_1\hat{e}_1 + dx_2\hat{e}_2 + dx_3\hat{e}_3 \\ &= \frac{\partial x_1}{\partial x_1^o}dx_1^o\hat{e}_1 + \frac{\partial x_2}{\partial x_1^o}dx_1^o\hat{e}_2 + \frac{\partial x_3}{\partial x_1^o}dx_1^o\hat{e}_3 \end{aligned}$$

Even though $d\hat{x}^o$ is only horizontal, $d\hat{x}$ has all three components.

Areas

The area of a parallelogram region can be calculated by considering the vector cross product of lines that bound it. That is, if the region is defined by two vectors $d\hat{x}^a$ and $d\hat{x}^b$, we have

$$\text{Area} = d\hat{x}^a \times d\hat{x}^b = \text{vector}$$

Note that we consider the area to be a vector; it has a direction as well as a magnitude and the direction is given by a normal to the surface.

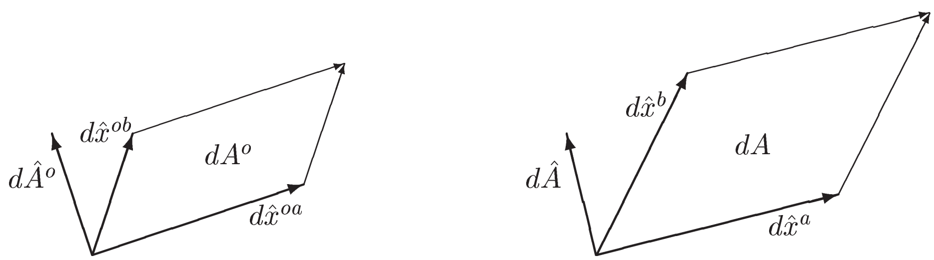


Figure 2.5: Deformation of areas.

The initial area is

$$d\hat{A}^o = d\hat{x}^{oa} \times d\hat{x}^{ob}$$

or, on substituting for the vectors,

$$dA_k^o\hat{e}_k = dx_i^{oa}\hat{e}_i \times dx_j^{ob}\hat{e}_j = \epsilon_{ijk}dx_i^{oa}dx_j^{ob}\hat{e}_k$$

In component form, this is

$$dA_k^o = \epsilon_{ijk}dx_i^{oa}dx_j^{ob}$$

Multiply both sides by ϵ_{pqk} and replacing the contraction of two permutation symbols by the delta functions leads to the following alternative form

$$\epsilon_{pqk}dA_k^o = dx_p^{oa}dx_q^{ob} - dx_q^{oa}dx_p^{ob}$$

Consider the situation of a rectangle so that the only non-zero components of length are dx_1^{oa} and dx_2^{ob}. That is, with $p = 1$ and $q = 2$ the area is $dA_3^o = dx_1^{oa}dx_2^{ob}$, as expected.

After deformation, the area becomes

$$d\hat{A} = d\hat{x}^a \times d\hat{x}^b \quad \text{or} \quad dA_k = \epsilon_{ijk} dx_i^a dx_j^b$$

and by use of the deformation gradient

$$dA_k = dA\, n_k = \epsilon_{ijk} \frac{\partial x_i}{\partial x_p^o} \frac{\partial x_j}{\partial x_q^o} dx_p^{oa} dx_q^{ob}$$

This can be rearranged (by noting that i, j, p, q are dummy indices) to also give

$$dA_k = -\epsilon_{ijk} \frac{\partial x_i}{\partial x_p^o} \frac{\partial x_j}{\partial x_q^o} dx_q^{oa} dx_p^{ob}$$

Adding this to the previous expression for dA_k results in

$$2\, dA_k = \epsilon_{ijk} \frac{\partial x_i}{\partial x_p^o} \frac{\partial x_j}{\partial x_q^o} (dx_p^{oa} dx_q^{ob} - dx_q^{oa} dx_p^{ob}) = \epsilon_{ijk} \frac{\partial x_i}{\partial x_p^o} \frac{\partial x_j}{\partial x_q^o} \epsilon_{pqr} dA_r^o$$

As a final step to simplify the relation between the deformed and undeformed areas, multiply both sides by the deformation gradient $\partial x_k / \partial x_1^o$ and expand on r. The only non-zero situation is for $r = 1$, hence we get

$$2dA_k \frac{\partial x_k}{\partial x_1^o} = \epsilon_{ijk} \left(\frac{\partial x_k}{\partial x_1^o} \frac{\partial x_i}{\partial x_2^o} \frac{\partial x_j}{\partial x_3^o} - \frac{\partial x_k}{\partial x_1^o} \frac{\partial x_j}{\partial x_2^o} \frac{\partial x_i}{\partial x_3^o} \right) dA_1 = 2\epsilon_{ijk} \frac{\partial x_i}{\partial x_1^o} \frac{\partial x_j}{\partial x_2^o} \frac{\partial x_k}{\partial x_3^o} dA_1$$

By introducing the Jacobian we get the simpler expression

$$dA_k \frac{\partial x_k}{\partial x_1^o} = J^o dA_1^o$$

Similar expressions are obtained by multiplying by the deformation gradients $\partial x_k / \partial x_2^o$ and $\partial x_k / \partial x_3^o$ to finally get

$$dA_k \frac{\partial x_k}{\partial x_i^o} = J^o dA_i^o \qquad \text{or} \qquad dA_i = J^o \frac{\partial x_k^o}{\partial x_i} dA_k^o \tag{2.3}$$

This elegant form for the deformation of areas is somewhat similar to the corresponding one for line segments; note, however, that it is the Eulerian form of the deformation gradient that is used.

Volume

Consider the parallelepiped of sides $d\hat{x}^{oa}, d\hat{x}^{ob}, d\hat{x}^{oc}$, which deforms into $d\hat{x}^a, d\hat{x}^c, d\hat{x}^c$. The volume before deformation is $dV^o = (d\hat{x}^{oa} \times d\hat{x}^{ob}) \cdot d\hat{x}^{oc}$ or in expanded form

$$\begin{aligned} dV^o &= (dx_p^{oa} \hat{e}_p^o \times dx_q^{ob} \hat{e}_q) \cdot dx_r^{oc} \hat{e}_r \\ &= dx_p^{oa} dx_q^{ob} \epsilon_{pqk} \hat{e}_k \cdot dx_r^{oc} \hat{e}_r \\ &= \epsilon_{pqr} dx_p^{oa} dx_q^{ob} dx_r^{oc} \end{aligned}$$

Similarly, the volume after deformation is

$$dV = \epsilon_{ijk} dx_i^a dx_j^b dx_k^c$$

Expand dV using the deformation gradient and recognizing the collection of gradients as J^o, rearrange to get

$$dV = \epsilon_{ijk} J^o dx_i^{oa} dx_j^{ob} dx_k^{oc} \qquad \text{or} \qquad dV = J^o dV^o \tag{2.4}$$

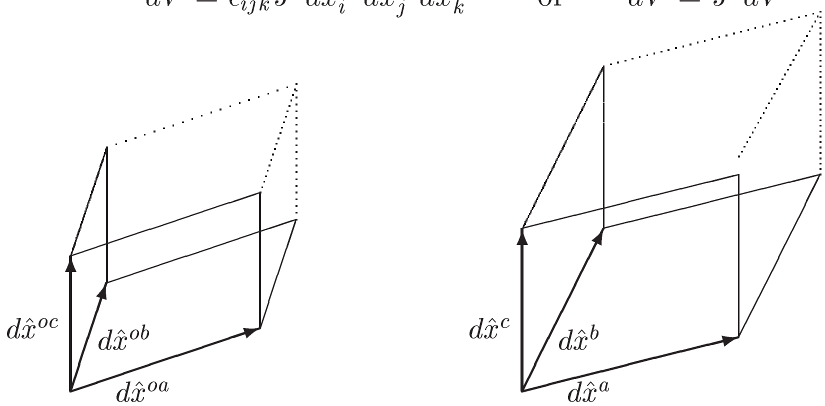


Figure 2.6: Three edges and three faces of a deforming volume.

An alternative statement of continuity is that mass is constant and since mass = density × volume = constant, then

$$\rho^o dV^o = \rho dV = \rho J^o dV^o$$

or simply

$$\rho^o = \rho J^o \,, \qquad J^o = \rho^o / \rho$$

In a similar manner, we have

$$\rho = \rho^o J \,, \qquad J = \rho / \rho^o$$

Note that J^o and J can change from point to point depending on the particular deformation gradient but an interesting result that can be obtained is

$$\begin{aligned} \frac{\partial}{\partial x_i}(J^o) &= \frac{\partial}{\partial x_i}(\rho^o/\rho) = 0 \\ \frac{\partial}{\partial x_i^o}(J) &= \frac{\partial}{\partial x_i^o}(\rho/\rho^o) = 0 \end{aligned} \tag{2.5}$$

These somewhat surprising relationships will be used in the next chapter when we consider stress.

Example 2.3: Reconsider the deformation described by

$$x_1 = x_1^o + k\,x_2^o \qquad x_2 = 2x_1^o + 3x_2^o \qquad x_3 = x_3^o$$

Determine how lines and areas are deformed.

The deformation gradients are

$$[\frac{\partial x_i}{\partial x_j^o}] = \begin{bmatrix} 1 & k & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\frac{\partial x_i^o}{\partial x_j}] = \begin{bmatrix} 3 & -k & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{(3-2k)}$$

A line initially in the 2-direction ($dx_i^o = \{0, 1, 0\} dS_o$) after deformation has the components

$$\begin{aligned} dx_1 &= 1 \times 0 + k \times dS_o + 0 \times 0 = k\,dS_o \\ dx_2 &= 2 \times 0 + 3 \times dS_o + 0 \times 0 = 3\,dS_o \\ dx_3 &= 0 \times 0 + 0 \times dS_o + 1 \times 0 = 0 \end{aligned}$$

Hence the new length and its direction are given by

$$dS = \sqrt{k^2 + 9}\,dS_o\,, \qquad \hat{n} = \frac{k\hat{e}_1 + 3\hat{e}_2}{\sqrt{9 + k^2}}$$

Now consider the area formed by this line projected in the 3-direction, that is, $dA_i^o = \{1, 0, 0\} dA_o$; after deformation it has the components

$$\begin{aligned} dA_1 &= (3-2k)[3 \times dA_o - 2 \times 0 + 0 \times 0]/(3-2k) = 3dA_o \\ dA_2 &= (3-2k)[-k \times dA_o + 1 \times 0 + 0 \times 0]/(3-2k) = -k\,dA_o \\ dA_3 &= (3-2k)[0 \times dA_o + 0 \times 0 + (3-2) \times 0]/(3-2k) = 0 \end{aligned}$$

From this, it is apparent that there is also a component of area generated in the 2-direction. The magnitude and direction of the deformed area are, respectively,

$$dA = \sqrt{9 + k^2}\,dA_o\,, \qquad \hat{n} = \frac{3\hat{e}_1 - k\hat{e}_2}{\sqrt{9 + k^2}}$$

The vector directions of the line and area are perpendicular to each other. ■

Example 2.4: Consider the following plane inhomogeneous deformation

$$x_1 = [R - x_2^o]\sin(x_1^o/R)\,, \qquad x_2 = R - [R - x_2^o]\cos(x_1^o/R)\,, \qquad x_3 = x_3^o \tag{2.6}$$

where R is a positive parameter. What (if any) are the restrictions for this to be a valid deformation? Draw the deformed shape of material initially lying between $-h < x_2^o < h$, and calculate the orientation and magnitude of the deformed areas.

This deformation is shown in Figure 2.7. Note that initially horizontal lines become arcs of concentric circles, while initially vertical lines become radial lines emanating from a common point. The location of the common point changes as the deformation changes. This deformation resembles that of bending.

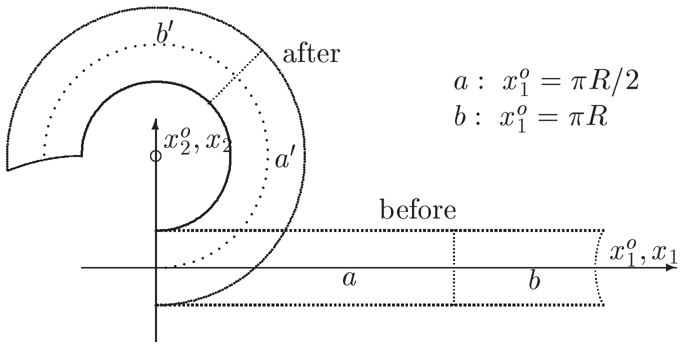


Figure 2.7: Shape before and after deformation.

The deformation gradient is given by

$$\left[\frac{\partial x_i}{\partial x_j^o}\right] = \begin{bmatrix} (R - x_2^o)\cos(x_1^o/R)/R & -\sin(x_1^o/R) & 0 \\ (R - x_2^o)\sin(x_1^o/R)/R & \cos(x_1^o/R) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Jacobian is the determinant of the deformation gradient matrix and on multiplication simplifies to

$$J^o = 1 - \frac{x_2^o}{R}$$

We note that as long as $x_2^o < R$ that the volume remains positive.

The areas are related through the inverse of the above deformation gradient. Since it is a 2-D problem, we get

$$\left\{ \begin{array}{c} dA_1 \\ dA_2 \end{array} \right\} = \begin{bmatrix} \cos(x_1^o/R) & \sin(x_1^o/R) \\ -J^o\sin(x_1^o/R) & J^o\cos(x_1^o/R) \end{bmatrix} \left\{ \begin{array}{c} dA_1^o \\ dA_2^o \end{array} \right\}$$

Areas that were initially vertical and facing the 1-direction are preserved; this can be checked by comparing dA_2 for $x_1^o = \pi R/2$ to dA_1 for $x_1^o = \pi R$. Areas that were initially horizontal either contract ($x_2^o > 0$) or expand ($x_2^o < 0$). This is the hallmark of a beam or plate in bending.

In the limit as $x_2^o/R << 1$, then $J^o \approx 1$ and areas are preserved. This is the situation that is be prevalent when the bending of thin-walled structures is considered. ■

2.3 Strain

Strain is a measure of the 'stretching' of the material particles within a body. That is, it is a measure of the relative displacement without rigid body motion and is an essential ingredient for the description of the constitutive behavior of materials.

Elementary Strain Measures

There are many measures of strain in existence, so it is worthwhile to review some of the more common ones so as to put into perspective the measures we wish to introduce. Assume that a line segment of original length of L_o is changed to L. Some of the common measures of strain are:

- Engineering strain:
$$e = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta L}{L_o}$$

- True strain:
$$e^T = \frac{\text{change in length}}{\text{final (current) length}} = \frac{\Delta L}{L} = \frac{\Delta L}{L_o + \Delta L}$$

- Logarithmic strain:
$$e^N = \int_{L_o}^{L} \text{true strain} = \int_{L_o}^{L} \frac{dl}{l} = \ln(\frac{L}{L_o})$$

An essential requirement of a strain measure is to allow the final length to be calculated. This is true of each of the above since

$$\begin{aligned} \text{Engineering:} \quad L &= L_o + \Delta L = L_o + L_o e = L_o(1+e) \\ \text{True:} \quad L &= L_o + \Delta L = L_o + \frac{e^T L_o}{(1-e^T)} = \frac{L_o}{(1-e^T)} \\ \text{Logarithmic:} \quad L &= L_o exp(e^N) \end{aligned}$$

All these measures are equivalent since they allow ΔL (or L) to be calculated knowing L_o. The relation among the measures are

$$e^T = \frac{e}{1+e} \ , \qquad e^N = \ln(1+e)$$

Because the measures are equivalent, then it is a matter of convenience as to which measure is to be chosen in an analysis.

The difficulty with these strain measures is that they do not have proper transformation properties. This would poses a problem in developing our three dimensional theory because the quantities involved should transform as tensors of the appropriate order.

Tensorial Strain Measures

As a medium deforms, various positions of the medium will translate and rotate. The easiest way to distinguish between deformation and the local rigid-body motion is to consider the change in distance between two neighboring material particles. We will use this to establish our strain measures.

Suppose that two material particles, before the motion, have coordinates (x_i^o) and $(x_i^o + dx_i^o)$; and after the motion, (x_i) and $(x_i + dx_i)$. The initial and final distances between these neighboring particles are given by

$$dS_o^2 = dx_i^o dx_i^o = (dx_1^o)^2 + (dx_2^o)^2 + (dx_3^o)^2$$

and

$$dS^2 = dx_i dx_i = \frac{\partial x_m}{\partial x_i^o}\frac{\partial x_m}{\partial x_j^o} dx_i^o dx_j^o$$

respectively. Only in the event of stretching or straining is dS^2 different from dS_o^2. That is,

$$dS^2 - dS_o^2 = dS^2 - dx_i^o dx_i^o = \left[\frac{\partial x_m}{\partial x_i^o}\frac{\partial x_m}{\partial x_j^o} - \delta_{ij}\right] dx_i^o dx_j^o$$

is a measure of the relative displacements. It is insensitive to rotation as can be easily demonstrated by considering a rigid body motion. On the other hand, if the Eulerian variables are employed, the same change of length can be expressed as

$$dS^2 - dS_o^2 = dx_i dx_i - dS_o^2 = \left[\delta_{ij} - \frac{\partial x_m^o}{\partial x_i}\frac{\partial x_m^o}{\partial x_j}\right] dx_i dx_j$$

Consequently, these equations can be written as

$$dS^2 - dS_o^2 \;\; = \;\; 2E_{ij} dx_i^o dx_j^o = 2e_{ij} dx_i dx_j \tag{2.7}$$

respectively, by introducing the strain measures

$$E_{ij} \equiv \tfrac{1}{2}\left[\frac{\partial x_m}{\partial x_i^o}\frac{\partial x_m}{\partial x_j^o} - \delta_{ij}\right], \qquad e_{ij} \equiv \tfrac{1}{2}\left[\delta_{ij} - \frac{\partial x_m^o}{\partial x_i}\frac{\partial x_m^o}{\partial x_j}\right] \tag{2.8}$$

It is easy to observe that both E_{ij} and e_{ij} are symmetric tensors of the second order and are called the Lagrangian and Eulerian strains, respectively. The matrix expressions for both strains are

$$2[E_{ij}] \equiv [\frac{\partial x_m}{\partial x_i^o}][\frac{\partial x_m}{\partial x_j^o}]^T - \lceil\ I\ \rfloor, \qquad 2[\,e_{ij}\,] \equiv \lceil\ I\ \rfloor - [\frac{\partial x_m^o}{\partial x_i}][\frac{\partial x_m^o}{\partial x_j}]^T$$

where $\lceil\ I\ \rfloor$ is the unit matrix.

The two strain measures are related by

$$E_{ij} = e_{pq}\frac{\partial x_p}{\partial x_i^o}\frac{\partial x_q}{\partial x_j^o}$$

Thus only knowing e_{ij}, for example, it is not possible to determine E_{ij} without knowledge of the full deformation gradient $\partial x_i/\partial x_j^o$. This is because both strain measures are missing the information about the rigid body rotation.

Sometimes the straining is described in terms of the *Cauchy-Green deformation tensor* defined as

$$C_{ij} \equiv \sum_k \frac{\partial x_k}{\partial x_i^o}\frac{\partial x_k}{\partial x_j^o} \qquad \text{or} \qquad [\,C\,] \equiv [\frac{\partial x}{\partial x^o}][\frac{\partial x}{\partial x^o}]^T$$

Then

$$E_{ij} = \tfrac{1}{2}C_{ij}$$

The principal values of $[C_{ij}]$ have the meaning of principal stretches and we will find a useful application in discussing rubber elasticity in Chapter 4.

Example 2.5: Show that the Lagrangian strain measure is a second order tensor.
Consider two coordinate systems; the change of lengths must be the same in the two coordinate systems

$$dS^2 - dS_o^2 = 2E_{ij}dx_i^o dx_j^o = 2E'_{ij}dx'^o_i dx'^o_j$$

Since dx_i^o is a first order tensor, its components in the primed system are

$$dx_i^{o'} = \beta_{ij}dx_j^o\,, \qquad dx_i^o = \beta_{ji}dx'^{o'}_j$$

Substituting the latter of these gives

$$dS^2 - dS_o^2 = 2E_{ij}\beta_{pi}\beta_{qj}dx'^{o'}_p dx'^{o'}_q = 2E'_{ij}dx'^o_i dx'^o_j$$

Interchanging subscripts we conclude that

$$E'_{ij} = \beta_{ip}\beta_{jq}E_{pq}$$

which is the transformation law for second order tensors. ■

Example 2.6: Determine the strain for the deforming body of Figure 2.7.
The displacement gradient is

$$\left[\frac{\partial u_i}{\partial x_j^o}\right] = \left[\frac{\partial x_i}{\partial x_j^o} - \delta_{ij}\right] = \begin{bmatrix} J^o\cos(x_1^o/R) - 1 & -\sin(x_1^o/R) & 0 \\ J^o\sin(x_1^o/R) & \cos(x_1^o/R) - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $J^o = 1 - x_2^o/R$. The strains are

$$\begin{aligned} E_{11} &= J^oC - 1 + \tfrac{1}{2}[(J^oC-1)^2 + (J^oS)^2 + (0)^2] = -\frac{x_2^o}{R} + \frac{1}{2}(\frac{x_2^o}{R})^2 \\ E_{22} &= C - 1 + \tfrac{1}{2}[(C-1)^2 + (-S)^2 + (0)^2] = 0 \\ E_{12} &= \tfrac{1}{2}[-S + J^oS] + \tfrac{1}{2}[(J^oC-1)(-S) + (J^oS)(C-1) + (0)^2] = 0 \end{aligned}$$

where we used $C \equiv \cos(x_1^o/R)$ and $S \equiv \sin(x_1^o/R)$. Only line segments initially in the x_1-direction are strained. There is a line, $x_2^o = 0$, which is not strained; other lines are strained in proportion to their distance from this line. In the limit of a very thin body ($x_2^o/R << 1$), the strain distribution is approximately linear

$$E_{11} \approx -\frac{x_2^o}{R} \approx -x_2^o \frac{dv}{dx_1^o}$$

where v is the u_2 displacement of the $x_2^o = 0$ line. These are the strain characteristics of a beam or plate in bending. ■

Principal Strains and Invariants

Because E_{ij} and e_{ij} are symmetric second order tensors, they enjoy all the properties as discussed in Section 1.4. In particular, they have principal values which have an interesting physical interpretation in terms of the ellipsoid of Figure 1.2.

Consider the line segment before deformation as a vector $d\hat{S}^o = dx_i^o \hat{e}_i$. We can then interpret Equation (2.7) as a vector product relation

$$dS^2 - dS_o^2 = 2V_j dx_j^o = 2\hat{V} \cdot d\hat{S}^o\,, \qquad V_j \equiv E_{ij} dx_i^o \tag{2.9}$$

Consider different initial vectors $d\hat{S}^o$ each of the same size but different orientations; they will correspond to different vectors $\hat{V}$. Figure 1.2 shows a collection of such vectors where $d\hat{S}^o$ traces out the coordinate circles of a sphere. Note that $\hat{V}$ traces an ellipse and many such traces would form an ellipsoid. That is, the sphere traced by $d\hat{S}^o$ is transformed into an ellipsoid traced by $\hat{V}$.

When $\hat{V}$ and $d\hat{S}^o$ are parallel, the dot product is extremal and consequently, so also is the strain. Thus, the axes of the ellipsoid will correspond to the principal strains. With this in mind, we have the interpretation of the above equation as

$$\begin{aligned} \tfrac{1}{2}[dS^2/dS_o^2 - 1] &= n_i^o E_{ij} n_j^o \\ &= E_{11} n_1^o n_1^o + E_{22} n_2^o n_2^o + E_{33} n_3^o n_3^o + 2E_{12} n_1^o n_2^o + 2E_{13} n_1^o n_3^o + 2E_{23} n_2^o n_3^o \end{aligned}$$

where $n_i^o = dx_i^o/dS_o$. This is the equation of the ellipsoid with respect to the coordinate directions. The extremum values can be established by differentiation with respect to n_p^o; when doing this, however, we must also take into account the constraint that $n_k^o n_k^o = 1$. We implement this by way of a Lagrange multiplier and extremize

$$\phi = E_{ij} n_j^o n_i^o - \lambda n_k^o n_k^o$$

where λ is the (as yet unknown) Lagrange multiplier. Differentiating this leads to

$$E_{ij} n_j^o - \lambda n_i^o = 0 \qquad \text{or} \qquad [E_{ij} - \lambda \delta_{ij}]^T n_j^o = 0$$

We now recognize this as the eigenvalue problem discussed in Section 1.4 and λ as the principal value of strain.

Physical Interpretation of Normal Strains

To relate these strain tensors to the strain quantities with which we are familiar, consider the line element

$$dx_1^o = dS_o\,, \qquad dx_2^o = dx_3^o = 0$$

at the initial state. After deformation, the line element is given by dx_i with magnitude dS.

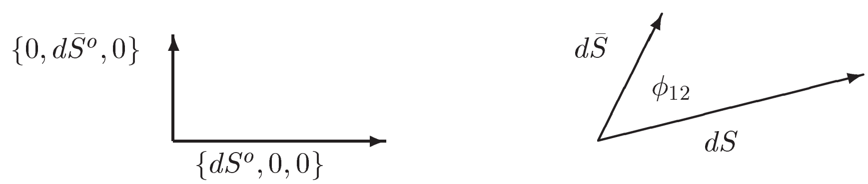


Figure 2.8: Deformation of two initially perpendicular line elements.

Let E_1 be the extension per unit original length of the element, that is,

$$E_1 = \frac{dS - dS_o}{dS_o} \qquad \text{or} \qquad dS = (1 + E_1)dS_o$$

For this line element we also have

$$dS^2 - dS_o^2 = 2E_{11}dS_o^2$$

and combining with the above yields

$$(1 + E_1)^2 dS_o^2 - dS_o^2 = 2E_{11}dS_o^2$$

Thus, the extension E_1 is related to the Lagrangian strain component E_{11} as

$$E_{11} = E_1 + \tfrac{1}{2}E_1^2 \qquad \text{or} \qquad E_1 = \sqrt{1 + 2E_{11}} - 1$$

Similar relations for line elements originally in the x_2^o and x_3^o directions can be obtained as

$$\begin{aligned} E_{22} &= E_2 + \tfrac{1}{2}E_2^2 \qquad \text{or} \qquad E_2 = \sqrt{1 + 2E_{22}} - 1 \\ E_{33} &= E_3 + \tfrac{1}{2}E_3^2 \qquad \text{or} \qquad E_3 = \sqrt{1 + 2E_{33}} - 1 \end{aligned}$$

The components E_{11}, E_{22} and E_{33} are called the normal components of strain.

Notice that there is a certain asymmetry (as regards the stretching direction) in the meaning of the normal components. For example, as the line is stretched, E_1 increases possibly without limit resulting in E_{11} doing the same. If, however, the line is shrunk so that E_1 is negative, then there is a definite limit given by $E_1 = -1$ which corresponds to $\Delta L = -L_o$ meaning that the line length has shrunk to zero. That is, we have the limits

$$-1 < E_1 < \infty\,, \qquad -0.5 < E_{11} < \infty$$

This asymmetry between the stretching and shrinking directions is important when considering the constitutive behavior.

Applying the binomial expansion to the expression for E_1 in terms of E_{11} gives

$$\begin{aligned} E_1 &= (1 + E_{11} - \tfrac{1}{2}E_{11}^2 + \cdots) - 1 \\ &= E_{11} - \tfrac{1}{2}E_{11}^2 + \cdots \end{aligned}$$

When E_{11} is small, that is, when $E_{11} \ll 1$, the above equation reduces to

$$E_1 \approx E_{11}$$

This simply says that E_{11} can be interpreted as an extension per unit length only when it is small.

Consider a line segment initially at an arbitrary angle θ in the $x_1^o - x_2^o$ plane in the undeformed configuration, then its length after deformation is obtained from

$$dS^2 - dS_o^2 = 2E_{ij}dx_i^o dx_j^o = 2\left[E_{11}dx_1^o dx_1^o + E_{21}dx_1^o dx_2^o + E_{12}dx_2^o dx_1^o + E_{22}dx_2^o dx_2^o\right]$$

Realizing that $dx_1^o = dS_o \cos\theta$ and so on, and that the strain of this arbitrary line is

$$dS^2 - dS_o^2 \equiv 2E_{\theta\theta}dS_o^2$$

leads to

$$E_{\theta\theta} = E_{11}\cos^2\theta + E_{21}\cos\theta\sin\theta + E_{12}\cos\theta\sin\theta + E_{22}\sin^2\theta$$

This gives us the transformation rule for the components of strain and so that they transform as second order tensors.

Physical Interpretation of Shear Strains

By rearrangement, the off-diagonal components of the strain tensor can be written in terms of only measurements of change of lengths. That is, the shear strain can be interpreted in terms of normal strains; this is the strain gage rosette interpretation of the components of the strain tensor. An alternative interpretation in terms of changes of angles follows.

A deformation can exhibit distortion in the configuration, that is, exhibit a change in the angle between two line elements. Consider, in the initial state, two line elements parallel to x_1^o and x_2^o, respectively. The two line elements are denoted by dx_i^o and $d\bar{x}_i^o$, respectively, with

$$\begin{aligned} dx_1^o &= dS_o\,, \qquad dx_2^o = dx_3^o = 0 \\ d\bar{x}_2^o &= d\bar{S}_o\,, \qquad dx_1^o = dx_3^o = 0 \end{aligned}$$

These two elements are perpendicular to each other initially. After deformation, dx_i^o is deformed into dx_i, and $d\bar{x}_i^o$ into $d\bar{x}_i$.

Denoting the angle between dx_i and $d\bar{x}_i$ by ϕ_{12} and taking the dot product of these two vectors, we obtain

$$d\hat{x}\cdot d\hat{\bar{x}} = dS\,d\bar{S}\cos\phi_{12} = dx_i\,d\bar{x}_i = \frac{\partial x_i}{\partial x_k^o}\frac{\partial x_i}{\partial x_m^o}dx_k^o\,d\bar{x}_m^o = \frac{\partial x_i}{\partial x_1^o}\frac{\partial x_i}{\partial x_2^o}dx_1^o dx_2^o = 2E_{12}dx_1^o dx_2^o$$

which can readily be rewritten as

$$dS\,d\bar{S}\cos\phi_{12} = 2E_{12}dS_o d\bar{S}_o$$

By substituting for dS and $d\bar{S}$ in terms of the extensions, we get

$$dS = (1+E_1)dS_o\,, \qquad d\bar{S} = (1+E_2)d\bar{S}_o$$

thus leading to

$$\cos\phi_{12} = \frac{2E_{12}}{(1+E_1)(1+E_2)}$$

Denoting the change in angle by

$$\alpha_{12} \equiv \tfrac{1}{2}\pi - \phi_{12}$$

and using the expressions for the extensions in terms of the strain components, we finally obtain

$$\sin\alpha_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

Thus, all the Lagrangian strain components E_{11}, E_{22}, and E_{12} contribute to the change of angle. However, it is only when $E_{12}=0$, that the angle between the two elements would be preserved. The component E_{12} therefore seems a good measure of the 'shearing' of perpendicular line segments.

Since the term $\sin\alpha_{12}$ must lie in the range ± 1, we then have the limits on E_{12} of

$$-\tfrac{1}{2}\pi < \alpha_{12} < \tfrac{1}{2}\pi \quad \text{or} \quad -\sqrt{1+2E_{11}}\sqrt{1+2E_{22}} < 2E_{12} < +\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}$$

The limits on E_{12} are a combination of those on α_{12} and on the stretches.

Consider two perpendicular lines segments initially oriented at an arbitrary angle θ from the 1- and 2-axis, respectively, in the undeformed configuration. The change of angle α'_{12} after deformation is obtained from

$$dS\,d\bar{S}\sin\alpha'_{12} = dx_i\,d\bar{x}_i = \frac{\partial x_i}{\partial x_k^o}\frac{\partial x_i}{\partial x_m^o}dx_k^o\,d\bar{x}_m^o = [2E_{km}+\delta_{km}]dx_k^o\,d\bar{x}_m^o$$

Expanding this gives

$$dS\,d\bar{S}\sin\alpha'_{12} = (2E_{11}+1)dx_1^o d\bar{x}_1^o + 2E_{21}dx_1^o d\bar{x}_2^o + 2E_{12}dx_2^o d\bar{x}_1^o + (2E_{22}+1)dx_2^o d\bar{x}_2^o$$

Realizing that the undeformed segment lengths are given by

$$dx_i^o = dS_o\{\cos\theta,\, \sin\theta,\, 0\}\,, \qquad d\bar{x}_i^o = d\bar{S}_o\{-\sin\theta,\, \cos\theta,\, 0\}$$

we obtain

$$dS\, d\bar{S} \sin\alpha_{12}' = 2E_{12}' dS_o d\bar{S}_o$$

where

$$E_{12}' = -(E_{11} - E_{22})\cos\theta\sin\theta + E_{12}(\cos^2\theta - \sin^2\theta)$$

The quantity E_{12}' can be shown to be the Lagrangian strain component E_{12} in the new coordinate system obtained by a rotation θ in the $x_1 - x_2$ plane (rotation about the x_3 axis).

Physical Interpretation of Eulerian Strains

A similar procedure leads to the geometrical interpretation of the Eulerian strain components. Consider a line element dx_i in the deformed state which is parallel to the x_1-axis

$$dx_1 = dS, \quad dx_2 = dx_3 = 0$$

Let the extension per unit deformed length for this element after deformation be

$$e_1 = \frac{dS - dS_o}{dS}$$

For this particular line element we have

$$dS^2 - dS_o^2 = 2e_{11} dS^2$$

Comparing the above two equations, we obtain

$$e_1 = 1 - \sqrt{1 - 2e_{11}}$$

Again, if e_{11} is small, then $e_1 = e_{11}$.

The Eulerian strains also exhibit limits. For example, as the line is stretched e_1 increases but only up to the definite limit given by $e_1 = 1$ which corresponds to $\Delta L = \infty$. This results in the limit $e_{11} = 0.5$. The full limits are

$$-\infty < e_1 < 1\,, \qquad -\infty < e_{11} < 0.5$$

Notice the complementarity between these limits and the corresponding ones for the Lagrangian strains.

Similarly, if we consider two line elements dx_i and $d\bar{x}_i$ which are parallel to x_1 and x_2 axes in the deformed state, then the angle change of these two elements from the initial state to the deformed state can be easily derived as

$$\sin\beta_{12} = \frac{2e_{12}}{\sqrt{1 - 2e_{11}}\sqrt{1 - 2e_{22}}}$$

where $\beta_{12} = \phi_{12} - \frac{1}{2}\pi$ in which ϕ_{12} is the angle between these two elements before deformation. Again, it is apparent that $-\frac{1}{2}\pi < \beta_{12} < \frac{1}{2}\pi$.

2.4 Rotation

A general deformation can be conceived as a straining action plus a rotation. While rotations of rigid bodies are fairly easy to comprehend, the description of the rotations of deformable bodies requires more development. The conceptual difficulty arises primarily because different lines through a given point on the body can have a different rotation. We will find it necessary to introduce the idea of an 'average' or mean rotation.

Rotation of Single lines

We wish to consider the rotation of a general line element OP which deforms to $O'P'$. For convenience, let the two lines occupy the same position at the origin so that O and O' coincide.

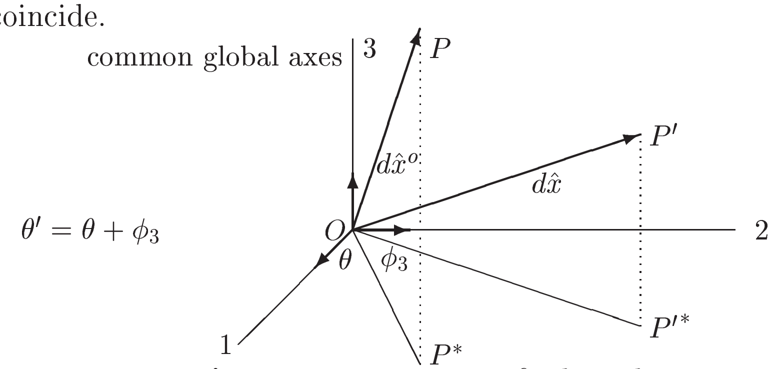


Figure 2.9: Rotation of a line element.

Consider a line segment that is initially lying in the 1-2 plane; that is, consider the lines OP^* and OP'^* (the latter is the projection of the deformed line onto the 1-2 plane) rotating about the x_3-axis. The rotation is obtained as a change in orientation as follows:

$$\text{orientation of } OP^*: \quad \tan\theta = \frac{dx_2^o}{dx_1^o}$$

$$\text{orientation of } OP'^*: \quad \tan\theta' = \frac{dx_2}{dx_1} = \frac{\dfrac{\partial x_2}{\partial x_1^o}dx_1^o + \dfrac{\partial x_2}{\partial x_2^o}dx_2^o}{\dfrac{\partial x_1}{\partial x_1^o}dx_1^o + \dfrac{\partial x_1}{\partial x_2^o}dx_2^o}$$

The rotation of the line projection is $\phi_3 = \theta' - \theta$ or

$$\begin{aligned}
\tan\phi_3 &= \tan(\theta' - \theta) = \frac{\tan\theta' - \tan\theta}{1 + \tan\theta'\tan\theta} \\
&= \frac{(\dfrac{\partial x_2}{\partial x_1^o} - \dfrac{\partial x_1}{\partial x_2^o}) + (\dfrac{\partial x_2}{\partial x_1^o} + \dfrac{\partial x_1}{\partial x_2^o})\cos 2\theta + (\dfrac{\partial x_2}{\partial x_2^o} - \dfrac{\partial x_1}{\partial x_1^o})\sin 2\theta}{(\dfrac{\partial x_1}{\partial x_1^o} + \dfrac{\partial x_2}{\partial x_2^o}) + (\dfrac{\partial x_1}{\partial x_1^o} - \dfrac{\partial x_2}{\partial x_2^o})\cos 2\theta + (\dfrac{\partial x_1}{\partial x_2^o} + \dfrac{\partial x_2}{\partial x_1^o})\sin 2\theta}
\end{aligned}$$

where

$$\cos\theta = dx_1^o/\sqrt{(dx_1^o)^2+(dx_2^o)^2}\,, \qquad \sin\theta = dx_2^o/\sqrt{(dx_1^o)^2+(dx_2^o)^2}$$

was used. From this, it is clear that different line elements will have different amounts of rotation (since θ will be different). In fact, some will be positive, some negative. To characterize the rotation at a point, it is necessary to remove the angle dependence. This will be done by averaging.

Averaged Measure of Rotation

A measure of average rotation is given as

$$\tan\bar{\phi}_3 \equiv \frac{1}{2\pi}\int_o^{2\pi}\tan\phi_3\,d\theta = \frac{1}{2\pi}\int\frac{A+B\cos 2\theta - C\sin 2\theta}{D+C\cos 2\theta + B\sin 2\theta}d\theta$$

where the coefficients

$$A = \frac{\partial x_2}{\partial x_1^o} - \frac{\partial x_1}{\partial x_2^o}\,,\quad B = \frac{\partial x_2}{\partial x_1^o} + \frac{\partial x_1}{\partial x_2^o}\,,\quad C = \frac{\partial x_1}{\partial x_1^o} - \frac{\partial x_2}{\partial x_2^o}\,,\quad D = \frac{\partial x_1}{\partial x_1^o} + \frac{\partial x_2}{\partial x_2^o}$$

are independent of θ. Let the denominator be written as

$$\begin{aligned} f &= D + C\cos 2\theta + B\sin 2\theta \\ df &= (0 - 2C\sin 2\theta + 2B\cos 2\theta)d\theta \end{aligned}$$

Therefore, the integral can be rewritten as

$$\tan\bar{\phi}_3 = \frac{1}{2\pi}\int_o^{2\pi}\frac{\frac{1}{2}df}{f} + \frac{A}{2\pi}\int_o^{2\pi}\frac{d\theta}{D+C\cos 2\theta + B\sin 2\theta} = I_1 + I_2$$

The first integral is simply

$$I_1 = \frac{1}{4\pi}\ln[\ f\ ]_o^{2\pi} = 0$$

The denominator of the second integral can be rearranged as (using the sine of the sum of two angles)

$$\begin{aligned} D + C\cos 2\theta + B\sin 2\theta &= D + \sqrt{C^2+B^2}(\sin\beta\cos 2\theta + \cos\beta\sin 2\theta) \\ &= D + \sqrt{C^2+B^2}\sin(\beta + 2\theta) \end{aligned}$$

with $\tan\beta \equiv C/B$. Hence the integral becomes using $\sigma \equiv \beta + 2\theta$

$$I_2 = \frac{A}{2\pi}\int_o^{2\pi}\frac{d\theta}{D+\sqrt{C^2+B^2}\sin(\beta+2\theta)} = \frac{A}{4\pi}\int_\beta^{\beta+4\pi}\frac{d\sigma}{D+\sqrt{C^2+B^2}\sin\sigma}$$

Providing that $D^2 > C^2 + B^2$, this gives

$$I_2 = \frac{2A}{4\pi}\frac{1}{\sqrt{D^2-(C^2+B^2)}}\tan^{-1}\left(\frac{D\tan(\sigma/2)+\sqrt{C^2+B^2}}{\sqrt{D^2-(C^2+B^2)}}\right)\Big|_\beta^{\beta+4\pi}$$

The $\tan^{-1}(\,)$ term is zero or multiples of 2π. Since for small deformations it is required that

$$A = \frac{\partial x_2}{\partial x_1^o} - \frac{\partial x_1}{\partial x_2^o} = \frac{\partial u_2}{\partial x_1^o} - \frac{\partial u_1}{\partial x_2^o}$$

be a measure of rotation, then let $\tan^{-1}(\,)$ be 2π. Hence

$$I_2 = \frac{\frac{1}{2}A}{\sqrt{D^2 - (C^2 + B^2)}}$$

Consequently, the average rotation becomes (after substituting for the coefficients)

$$\tan\bar{\phi}_3 = \frac{\frac{1}{2}(\frac{\partial x_2}{\partial x_1^o} - \frac{\partial x_1}{\partial x_2^o})}{\sqrt{(\frac{\partial x_1}{\partial x_1^o})(\frac{\partial x_2}{\partial x_2^o}) - \frac{1}{4}(\frac{\partial x_1}{\partial x_2^o} + \frac{\partial x_2}{\partial x_1^o})^2}}$$

Similarly, for lines initially in the other planes, we have

$$\begin{aligned} \tan\bar{\phi}_2 &= \frac{\frac{1}{2}(\frac{\partial x_1}{\partial x_3^o} - \frac{\partial x_3}{\partial x_1^o})}{\sqrt{(\frac{\partial x_1}{\partial x_1^o})(\frac{\partial x_3}{\partial x_3^o}) - \frac{1}{4}(\frac{\partial x_1}{\partial x_3^o} + \frac{\partial x_3}{\partial x_1^o})^2}} \\ \tan\bar{\phi}_1 &= \frac{\frac{1}{2}(\frac{\partial x_3}{\partial x_2^o} - \frac{\partial x_2}{\partial x_3^o})}{\sqrt{(\frac{\partial x_2}{\partial x_2^o})(\frac{\partial x_3}{\partial x_3^o}) - \frac{1}{4}(\frac{\partial x_2}{\partial x_3^o} + \frac{\partial x_3}{\partial x_2^o})^2}} \end{aligned}$$

In the limit of small deformations, these three angles are simply related to the anti-symmetric component of the deformation gradient.

Example 2.7: Consider a simple shear deformation parallel to the $x_1^o - x_2^o$ plane and given mathematically by

$$x_1 = x_1^o + k\,x_2^o\,, \qquad x_2 = x_2^o\,, \qquad x_3 = x_3^o$$

Determine the average rotation of a point.

Substituting for $x_i(x_i^o)$ into the formula for the average rotation, we get

$$\tan\bar{\phi}_1 = 0\,, \qquad \tan\bar{\phi}_2 = 0\,, \qquad \tan\bar{\phi}_3 = \frac{\frac{1}{2}(0-k)}{\sqrt{(1)(1) - \frac{1}{4}(k+0)^2}} = \frac{-\frac{1}{2}k}{\sqrt{1 - \frac{1}{4}k^2}}$$

If the deformation is small, this gives approximately

$$\bar{\phi}_3 \approx -\tfrac{1}{2}k$$

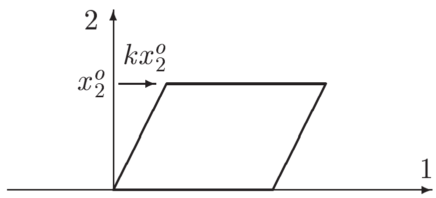


Figure 2.10: Simple shear deformation.

In looking at Figure 2.10, we see that the vertical and horizontal lines rotate angles of

$$\tan\phi_{90} = -k\,, \qquad \tan\phi_0 = 0$$

respectively. For small deformations, the average rotation is the average of the rotations of these two mutually perpendicular lines.

The constraint $D^2 > C^2 + B^2$ for this rotation about the 3-axis becomes

$$4(\frac{\partial x_1}{\partial x_1^o})(\frac{\partial x_2}{\partial x_2^o}) > (\frac{\partial x_1}{\partial x_2^o} + \frac{\partial x_2}{\partial x_1^o})^2$$

Specifically, for the simple shear problem, this is equivalent to

$$4 > k^2 \qquad \text{or} \qquad k < 2$$

When $D^2 < C^2 + B^2$, that is when the deformation is larger, then the integration must be performed differently. ■

2.5 Deformation in Terms of Displacement

Sometimes it is convenient to deal with displacements and displacement gradients instead of the deformation gradient. These are obtained by using the relations

$$x_m = x_m^o + u_m\,, \qquad x_m^o = x_m - u_m$$

giving the derivatives as

$$\frac{\partial x_m}{\partial x_i^o} = \frac{\partial u_m}{\partial x_i^o} + \delta_{im} \qquad \frac{\partial x_m^o}{\partial x_i} = \delta_{im} - \frac{\partial u_m}{\partial x_i}$$

The previous results will now be summarized in terms of these displacement gradients.

Strains and Rotations in Terms of Displacement

The Lagrangian strain tensor E_{ij} can be written in terms of the displacement by

$$\begin{aligned} E_{ij} &= \tfrac{1}{2}\Big[\Big(\frac{\partial u_m}{\partial x_i^o} + \delta_{im}\Big)\Big(\frac{\partial u_m}{\partial x_j^o} + \delta_{jm}\Big) - \delta_{ij}\Big] \\ &= \tfrac{1}{2}\Big[\frac{\partial u_m}{\partial x_i^o}\frac{\partial u_m}{\partial x_j^o} + \delta_{im}\frac{\partial u_m}{\partial x_j^o} + \delta_{jm}\frac{\partial u_m}{\partial x_i^o} + \delta_{im}\,\delta_{jm} - \delta_{ij}\Big] \\ &= \tfrac{1}{2}\Big[\frac{\partial u_i}{\partial x_j^o} + \frac{\partial u_j}{\partial x_i^o} + \frac{\partial u_m}{\partial x_i^o}\frac{\partial u_m}{\partial x_j^o}\Big] \end{aligned}$$

Similarly, the Eulerian strain tensor e_{ij} becomes

$$e_{ij} = \tfrac{1}{2}\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\Big]$$

Typical expressions for E_{ij} and e_{ij} in unabridged notations are given in the following:

$$\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial x_1^o} + \tfrac{1}{2}\Big[\Big(\frac{\partial u_1}{\partial x_1^o}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_1^o}\Big)^2 + \Big(\frac{\partial u_3}{\partial x_1^o}\Big)^2\Big] \\
E_{12} &= \tfrac{1}{2}\Big(\frac{\partial u_1}{\partial x_2^o} + \frac{\partial u_2}{\partial x_1^o}\Big) + \tfrac{1}{2}\Big[\frac{\partial u_1}{\partial x_1^o}\frac{\partial u_1}{\partial x_2^o} + \frac{\partial u_2}{\partial x_1^o}\frac{\partial u_2}{\partial x_2^o} + \frac{\partial u_3}{\partial x_1^o}\frac{\partial u_3}{\partial x_2^o}\Big] \\
e_{11} &= \frac{\partial u_1}{\partial x_1} - \tfrac{1}{2}\Big[\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_2}{\partial x_1}\Big)^2 + \Big(\frac{\partial u_3}{\partial x_1}\Big)^2\Big] \\
e_{12} &= \tfrac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) - \tfrac{1}{2}\Big[\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\Big]
\end{aligned}$$

These strain components cannot be interpreted as the normal strains and shear strains used in introductory courses on solid mechanics. The nonlinear terms in these strain components make their geometrical meaning less obvious.

A typical rotation term can be written in terms of displacement as

$$\tan\bar{\phi}_3 = \frac{\tfrac{1}{2}(\frac{\partial u_2}{\partial x_1^o} - \frac{\partial u_1}{\partial x_2^o})}{\sqrt{\Big(1 + \frac{\partial u_1}{\partial x_1^o}\Big)\Big(1 + \frac{\partial u_2}{\partial x_2^o}\Big) - \tfrac{1}{4}\Big(\frac{\partial u_1}{\partial x_2^o} + \frac{\partial u_2}{\partial x_1^o}\Big)^2}}$$

Thus the antisymmetric tensor defined as

$$\omega_{ij} = \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i^o} - \frac{\partial u_i}{\partial x_j^o}\right)$$

characterizes the rotation since $\bar{\phi}$ is always zero when ω_{ij} is zero.

Alternative Expressions for Strains

From

$$x_i = x_i^o + u_i \qquad \text{or} \qquad dx_i = dx_i^o + du_i$$

we have for the deformed length

$$dS^2 = dx_i dx_i = (dx_i^o + du_i)(dx_i^o + du_i)$$

Expanding this and moving the first term to the left-hand-side, we obtain

$$dS^2 - dS_o^2 = 2du_i dx_i^o + du_i du_i$$

By using the chain rule,

$$du_i = \frac{\partial u_i}{\partial x_j^o} dx_j^o$$

and decomposing the second order tensor $\partial u_i / \partial x_j^o$ into its symmetric and anti-symmetric parts, we have

$$du_i \;=\; (\bar{\epsilon}_{ji} + \bar{\omega}_{ji}) dx_j^o = (\bar{\epsilon}_{ij} - \bar{\omega}_{ij}) dx_j^o$$

where

$$\bar{\epsilon}_{ij} \;\equiv\; \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i^o} + \frac{\partial u_i}{\partial x_j^o}\right), \qquad \bar{\omega}_{ij} \equiv \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i^o} - \frac{\partial u_i}{\partial x_j^o}\right)$$

The symmetric tensor $\bar{\epsilon}_{ij}$ is recognized as the linear part of E_{ij} and is often referred to as the small or infinitesimal strain tensor. The tensor $\bar{\omega}_{ij}$ is called the (Lagrangian) rotation tensor. The change in length now becomes

$$dS^2 - dS_o^2 \;=\; 2(\bar{\epsilon}_{ij} - \bar{\omega}_{ij}) dx_i^o dx_j^o + (\bar{\epsilon}_{ij} - \bar{\omega}_{ij})(\bar{\epsilon}_{ik} - \bar{\omega}_{ik}) dx_j^o dx_k^o$$

Noting that

$$\bar{\omega}_{ij}\, dx_i^o dx_j^o = 0$$

and interchanging the dummy indices i and k, we have

$$dS^2 - dS_o^2 = [2\bar{\epsilon}_{ij} + (\bar{\epsilon}_{kj} - \bar{\omega}_{kj})(\bar{\epsilon}_{ki} - \bar{\omega}_{ki})] dx_i^o dx_j^o$$

Comparing the above equation with Equation (2.7), we arrive at

$$E_{ij} = \bar{\epsilon}_{ij} + \tfrac{1}{2}(\bar{\epsilon}_{kj} - \bar{\omega}_{kj})(\bar{\epsilon}_{ki} - \bar{\omega}_{ki})$$

Similarly, we can show that

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{2}(\epsilon_{kj} - \omega_{kj})(\epsilon_{ki} - \omega_{ki})$$

where

$$\epsilon_{ij} \;\equiv\; \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right), \qquad \omega_{ij} \equiv \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$$

These expressions make it clear that the nonlinear part of the strain measure involves the rotation.

From Equation (2.7), it is easy to show that the vanishing of the strain tensors is a necessary and sufficient condition for all small material elements at a point to displace without change in length. In other words, $E_{ij} = 0$ (or equivalently $e_{ij} = 0$) indicates, at most, a rigid body displacement. However, the condition $\bar{\epsilon}_{ij} = 0$ does not lead to $E_{ij} = 0$ if $\bar{\omega}_{ij} \neq 0$. On the other hand, it is possible that $E_{ij} = 0$ while $\bar{\epsilon}_{ij} \neq 0$ and $\bar{\omega}_{ij} \neq 0$.

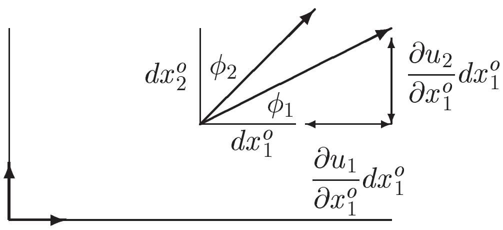


Figure 2.11: Rotation of line elements

Interpretation of $\bar{\omega}_{ij}$

In the case of finite deformation, the geometrical interpretation of $\bar{\omega}_{ij}$ is not easy to come by. An attempt will be made using the two-dimensional case as follows.

Consider two line elements dx_i^o and $d\bar{x}_i^o$ with

$$\begin{aligned} dx_1^o &= dS_o\,, \qquad dx_2^o = dx_3^o = 0 \\ d\bar{x_2^o} &= d\bar{S}_o\,, \qquad d\bar{x^o}_1 = d\bar{x^o}_3 = 0 \end{aligned}$$

which are parallel to the x_1- and x_2-axes, respectively, in the initial state. These line elements are deformed into dx_i and $d\bar{x}_i$, respectively. The rotation angles that dx_i^o and $d\bar{x}_i^o$ experience during the deformation are given by

$$\tan\phi_1 = \frac{\frac{\partial u_2}{\partial x_1^o}}{1 + \frac{\partial u_1}{\partial x_1^o}}\,, \qquad \tan\phi_2 = \frac{\frac{\partial u_1}{\partial x_2^o}}{1 + \frac{\partial u_2}{\partial x_2^o}} \tag{2.10}$$

Thus, the rotation involves the gradients $\frac{\partial u_2}{\partial x_1^o}$ and $\frac{\partial u_1}{\partial x_2^o}$, and the "projected extensions" $\bar{\epsilon}_{11}$ and $\bar{\epsilon}_{22}$. In the two-dimensional case, there is only one non-vanishing component in $\bar{\omega}_{ij}$, which is

$$\bar{\omega}_{12} = \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1^o} - \frac{\partial u_1}{\partial x_2^o}\right)$$

Using Equations (2.10), we rewrite the above as

$$\bar{\omega}_{12} = \tfrac{1}{2}\Big[\Big(1 + \frac{\partial u_1}{\partial x_1^o}\Big)\tan\phi_1 - \Big(1 + \frac{\partial u_2}{\partial x_2^o}\Big)\tan\phi_2\Big]$$

Thus, $\bar{\omega}_{12}$ is a weighted average of $\tan\phi_1$ and $\tan(-\phi_2)$. This should not be confused with $\bar{\phi}_3$. For small $\bar{\epsilon}_{11}, \bar{\epsilon}_{22}, \phi_1, \phi_2$, we obtain

$$\bar{\omega}_{12} \approx \tfrac{1}{2}(\phi_1 - \phi_2)$$

meaning that $\bar{\omega}_{12}$ is the average of these two rotations.

Example 2.8: A cubic element is rotated about the x_3-axis by $\pi/2$ as shown

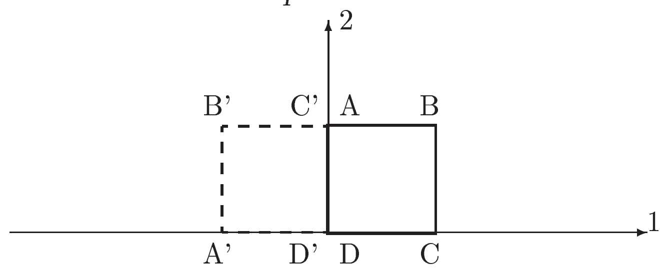


Figure 2.12: Rigid body rotation of a cube.

in Figure 2.12 where points A, B, C, D are displaced to A', B', C', D', respectively. Obtain the symmetric and anti-symmetric parts of the displacement gradient.

The deformed coordinates are given as

$$x_1 = -x_2^o\,, \qquad x_2 = x_1^o\,, \qquad x_3 = x_3^o$$

The displacement components are readily obtained as

$$u_1 = -x_2^o - x_1^o\,, \qquad u_2 = x_1^o - x_2^o\,, \qquad u_3 = 0$$

Thus the symmetric and anti-symmetric parts of the displacement gradient are

$$[\,\bar{\epsilon}_{ij}\,] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\bar{\omega}_{ij}] = \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It can be easily verified that in this case $E_{ij} = 0$ and $\tan\bar{\phi}_3 = \infty$ (thus $\bar{\phi}_3 = \pi/2$.) Consequently, in large displacements, existence of non-zero components of $\bar{\epsilon}_{ij}$ cannot be interpreted as the presence of strain. ■

Infinitesimal Strain and Rotation

The full nonlinear deformation analysis of problems is quite difficult and so simplifications are often sought. Three situations for the straining of a block are shown in Figure 2.13. The general case is that of large displacements, large rotations, and large strains. In the chapters dealing with the linear theory, the displacements, rotations, and strains are all small, and Case (c) prevails. Nonlinear analysis of thin-walled structures such as shells is usually restricted to Case (b), where the deflections and rotations can be large but the strains are small. This is a reasonable approximation because structural materials do not exhibit large strains without yielding or fracture and structures are designed to operate without this occurring.

If the displacement gradients are small in the solid, that is,

$$\left|\frac{\partial u_i}{\partial x_j^o}\right| \ll 1 \qquad \text{and} \qquad \left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$$

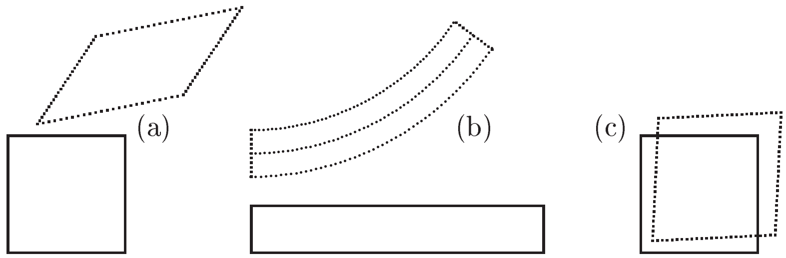


Figure 2.13: Combinations of displacements and strains. (a) Large displacements, rotations, and strains. (b) Large displacements and rotations but small strains. (c) Small displacements, rotations, and strains.

then the product terms in the Lagrangian strain tensor E_{ij} and the Eulerian strain tensor e_{ij} can be neglected. The results are

$$\begin{aligned} E_{ij} \approx \bar{\epsilon}_{ij} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j^o} + \frac{\partial u_j}{\partial x_i^o}\right) \\ e_{ij} \approx \epsilon_{ij} &= \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \end{aligned}$$

where $\bar{\epsilon}_{ij}$ and ϵ_{ij} are the infinitesimal strain tensors. This assumption also leads to the conclusion that the components E_{ij} and e_{ij} are small. Thus, the infinitesimal strain components have direct interpretations as extensions or change of angles. Further, the magnitudes of the strain components are small as compared with unity, indicating that the deformation (extension of a material element and the change of angle between two material elements) is small. Consequently, we have

$$E_1 \approx e_1 \quad , \quad E_2 \approx e_2 \quad , \quad E_3 \approx e_3$$

Similarly,

$$\alpha_{23} \approx \beta_{23} \quad , \quad \alpha_{13} \approx \beta_{13} \quad , \quad \alpha_{12} \approx \beta_{12} \quad ,$$

which leads to the conclusion

$$\bar{\epsilon}_{ij} \approx \epsilon_{ij}$$

meaning that the infinitesimal Lagrangian strain components $\bar{\epsilon}_{ij}$ and the infinitesimal Eulerian strain components ϵ_{ij} are equal in value.

If, in addition to the above, the following condition exists

$$\left|\frac{u_i}{L}\right| \ll 1$$

where L is the smallest dimension of the body, then

$$x_i \approx x_i^o$$

and the distinction between the Lagrangian and Eulerian variables vanishes. As a result, the functional forms of the displacement components u_i in these two variables become identical, and the strain components $\bar{\epsilon}_{ij}$ and ϵ_{ij} have identical functional forms with no distinction between them necessary. Henceforth, when we use the small strain approximations, ϵ_{ij} will be used to denote both the infinitesimal Lagrangian and Eulerian strain tensors, and x_i to denote both Lagrangian and Eulerian variables.

Example 2.9: Establish the relation between the incremental components of the Lagrangian strain tensor and the incremental components of the displacements.

Let the change of the new positions be rewritten as

$$x_i \longrightarrow x_i + \Delta x_i\,, \qquad dx_i \longrightarrow dx_i + d\Delta x_i$$

The change in length of a line element can be written in terms of the Lagrangian strain as

$$2\sum_{i,j} E_{ij} dx_i^o dx_j^o = dS^2 - dS_o^2 = \sum_i dx_i dx_i - \sum_i dx_i^o dx_i^o$$

The increments in strains due to the changes $d\Delta x_i$ are obtained from this as

$$2\sum_{i,j} \Delta E_{ij} dx_i^o dx_j^o = 2\sum_i d\Delta x_i dx_i - 0 \qquad \text{or} \qquad \sum_{i,j} \Delta E_{ij} dx_i^o dx_j^o = \sum_p d\Delta x_p dx_p$$

We will consider two representations for $d\Delta x_i$.

First, noting that dx_i^o is not changed,

$$d\Delta x_p = d\Delta(x_p^o + u_p) = d\Delta u_p = \sum_i \frac{\partial \Delta u_p}{\partial x_i^o} dx_i^o$$

Then, with the usual representation for dx_p,

$$\sum_{i,j} \Delta E_{ij} dx_i^o dx_j^o = \sum_{ij} \frac{\partial \Delta u_p}{\partial x_i^o} dx_i^o \frac{\partial x_p}{\partial x_j^o} dx_j^o \qquad \text{or} \qquad \Delta E_{ij} = \sum_p \frac{\partial x_p}{\partial x_i^o} \frac{\partial \Delta u_p}{\partial x_j^o} = \sum_p \frac{\partial x_p}{\partial x_i^o} \Delta[\frac{\partial u_p}{\partial x_j^o}]$$

The increments in strain are not solely related to the increments in displacement gradients. Further, if we use the decomposition of the displacement gradient,

$$\frac{\partial u_i}{\partial x_j^o} = \bar{\epsilon}_{ij} + \bar{\omega}_{ij}\,, \qquad 2\bar{\epsilon}_{ij} = \frac{\partial u_i}{\partial x_j^o} + \frac{\partial u_j}{\partial x_i^o}, \quad 2\bar{\omega}_{ij} = \frac{\partial u_i}{\partial x_j^o} - \frac{\partial u_j}{\partial x_i^o}$$

then

$$\Delta E_{ij} = \sum_p \frac{\partial x_p}{\partial x_i^o} \Delta \bar{\epsilon}_{pj}$$

The increments in strain are not solely related to the increments in the small strain tensor.

As a second choice, let

$$d\Delta x_i = d\Delta(x_i^o + u_i) = d\Delta u_i = \sum_k \frac{\partial \Delta u_i}{\partial x_k} dx_k = \sum_k \Delta[\frac{\partial u_i}{\partial x_k}] dx_k = \sum_k \Delta[\epsilon_{ik} + \omega_{ik}] dx_k$$

where the symmetric and antisymmetric decompositions are given by, respectively,

$$2\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\,, \qquad 2\omega_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\,,$$

Therefore,

$$\sum\nolimits_{i,j} \Delta E_{ij} dx_i^o dx_j^o = \sum\nolimits_{i,j} [\Delta\epsilon_{ij} + \omega_{ij}] dx_i dx_j = \sum\nolimits_{i,j} \Delta\epsilon_{ij} dx_i dx_j$$

where the anti-symmetry of ω_{ij} and symmetry of $dx_i dx_j$ was used to set the product to zero. Substituting for dx_i in terms of dx_i^o now gives

$$\Delta E_{mn} = \sum\nolimits_{i,j} \frac{\partial x_i}{\partial x_m^o} \frac{\partial x_j}{\partial x_n^o} \Delta\epsilon_{ij} \qquad \text{and} \qquad \Delta\epsilon_{mn} = \sum\nolimits_{i,j} \frac{\partial x_i^o}{\partial x_m} \frac{\partial x_j^o}{\partial x_n} \Delta E_{ij} \tag{2.11}$$

These surprising results shows that although ΔE_{ij}, $\Delta\bar{\epsilon}_{ij}$ and $\Delta\epsilon_{ij}$ are small, they are not equal. The main reason for this is because they are referred to different configurations. We will utilize this relation when we consider small variations of the strain field in Chapter 6. ■

2.6 Special Deformations

Some special deformations are considered here that will be useful to our discussions in the later chapters.

I: Homogeneous Deformation

If the final position of each particle is a linear function of its initial position, that is,

$$x_i = C_{ik} x_k^o + B_i \qquad \text{or} \qquad dx_i = C_{ik} dx_k^o$$

then the deformation gradient does not depend on x_i^o,

$$\frac{\partial x_i}{\partial x_j^o} = C_{ij} = \text{constants}$$

This deformation is said to be *homogeneous*. Some special characteristics of it are:

- The strain is the same irrespective of the point (x_i^o) considered.
- Straight lines deform into straight lines (but angles may change).
- All deformations can be treated as locally homogeneous.

II: Rigid Body Rotation

In a rigid body rotation all the points are given a displacement but the relative distance between points is unchanged. Consider the two dimensional case described by

$$\begin{aligned} x_1 &= x_1^o \cos\phi - x_2^o \sin\phi \\ x_2 &= x_1^o \sin\phi + x_2^o \cos\phi \\ x_3 &= x_3^o \end{aligned}$$

where ϕ is the angle of rotation. The corresponding displacements are

$$\begin{aligned} u_1 &= x_1^o(\cos\phi - 1) - x_2^o \sin\phi \\ u_2 &= x_1^o \sin\phi + x_2^o(\cos\phi - 1) \\ u_3 &= 0 \end{aligned}$$

from which the displacement gradients are determined to be

$$\left[\frac{\partial u_i}{\partial x_j^o}\right] = \begin{bmatrix} (\cos\phi - 1) & -\sin\phi & 0 \\ \sin\phi & (\cos\phi - 1) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{ij} = 0$$

It is now easy to show that all strain components are zero. However, the infinitesimal strain tensor given by

$$\bar{\epsilon}_{ij} = \tfrac{1}{2}\left(\frac{\partial u_j}{\partial x_i^o} + \frac{\partial u_i}{\partial x_j^o}\right)$$

has the components

$$[\,\bar{\epsilon}_{ij}\,] = \begin{bmatrix} (\cos\phi - 1) & 0 & 0 \\ 0 & (\cos\phi - 1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \approx \begin{bmatrix} -\frac{1}{2}\phi^2 & 0 & 0 \\ 0 & -\frac{1}{2}\phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is only when ϕ is very small is this strain tensor nearly zero.

III: Uniform Extension

A special case of homogeneous deformation is the uniform extension as shown in Figure 2.14. The motion is given explicitly by

$$\begin{aligned} x_1 &= \lambda_1 x_1^o + b_1 \\ x_2 &= \lambda_2 x_2^o + b_2 \\ x_3 &= \lambda_3 x_3^o + b_3 \end{aligned}$$

where the λ's are stretches and the b's are constants. The special case of simple extension is described by $\lambda_2 = \lambda_3 = \lambda_o$ and is analogous to uniaxial stress.

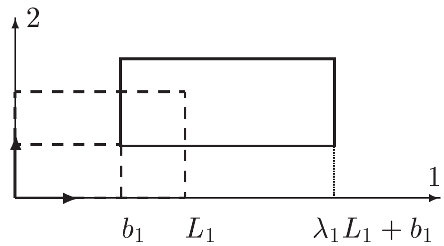


Figure 2.14: Simple extension in two dimensions.

The corresponding displacements are

$$\begin{aligned} u_1 &= (\lambda_1 - 1)x_1^o + b_1 \\ u_2 &= (\lambda_2 - 1)x_2^o + b_2 \\ u_3 &= (\lambda_3 - 1)x_3^o + b_3 \end{aligned}$$

from which the displacement gradients are determined to be

$$\left[\frac{\partial u_i}{\partial x_j^o}\right] = \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}$$

The Lagrangian strain tensor is

$$[E_{ij}] = \begin{bmatrix} \frac{1}{2}(\lambda_1^2 - 1) & 0 & 0 \\ 0 & \frac{1}{2}(\lambda_2^2 - 1) & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_3^2 - 1) \end{bmatrix}$$

showing it to be diagonal. That is, the coordinate directions are the principal strain directions. The extensions of line elements arranged in the x_1^o, x_2^o, and x_3^o directions are

$$E_1 = \sqrt{1 + 2E_{11}} - 1 = \lambda_1 - 1\,, \qquad E_2 = \lambda_2 - 1\,, \qquad E_3 = \lambda_3 - 1$$

and the rotations are given simply as

$$\bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0$$

The λ_i have meaning of "new length over original length".

IV: Simple Shear

A simple shear deformation parallel to the $x_1^o - x_2^o$ plane is shown in Figure 2.15 and given mathematically by

$$\begin{aligned} x_1 &= x_1^o + k\, x_2^o \\ x_2 &= x_2^o \\ x_3 &= x_3^o \end{aligned}$$

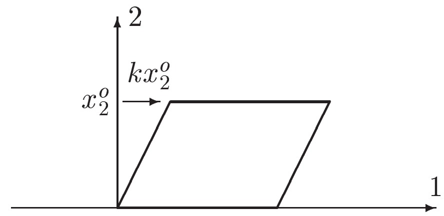


Figure 2.15: Simple shear.

The displacement components are readily obtained as

$$u_1 = k\,x_2^o\,, \qquad u_2 = 0\,, \qquad u_3 = 0$$

from which the Lagrangian strain tensor is calculated with the result

$$[E_{ij}] = \begin{bmatrix} 0 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & \frac{1}{2}k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note the presence of a non-zero E_{22} normal component. It can easily be verified that there is no extension in the x_1^o-direction ($E_1 = 0$), and the extension in the x_2^o-direction is

$$E_2 = \sqrt{1+k^2} - 1$$

The rotations are

$$\bar{\phi}_1 = 0, \quad \bar{\phi}_2 = 0, \quad \tan\bar{\phi}_3 = -\tfrac{1}{2}k/\sqrt{1-\tfrac{1}{4}k^2}$$

Also, as k approaches 2, $\bar{\phi}_3$ approaches $-\pi/2$.

V: Generalized Plane Homogeneous Deformation

Consider the deformation of Figure 2.16 where E_1 and E_2 are extensions and ϕ_1 and ϕ_2 are rotations and described by

$$\begin{aligned} x_1 &= (1+E_1)\cos\phi_1\,x_1^o + (1+E_2)\sin\phi_2\,x_2^o \\ x_2 &= (1+E_1)\sin\phi_1\,x_1^o + (1+E_2)\cos\phi_2\,x_2^o \\ x_3 &= x_3^o \end{aligned}$$

The displacement gradient is

$$\left[\frac{\partial u_i}{\partial x_j^o}\right] = \begin{bmatrix} (1+E_1)\cos\phi_1 - 1 & (1+E_2)\sin\phi_2 & 0 \\ (1+E_1)\sin\phi_1 & (1+E_2)\cos\phi_2 - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

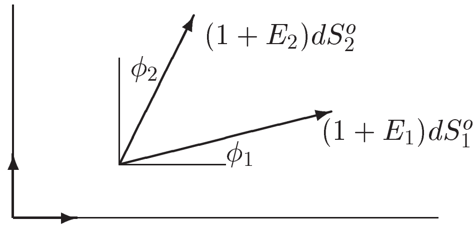


Figure 2.16: Generalized plane homogeneous deformation.

The strain components are obtained as

$$[E_{ij}] = \begin{bmatrix} E_1 + \frac{1}{2}E_1^2 & \frac{1}{2}(1+E_1)(1+E_2)\sin(\phi_1+\phi_2) & 0 \\ \frac{1}{2}(1+E_1)(1+E_2)\sin(\phi_1+\phi_2) & E_2 + \frac{1}{2}E_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The small strain approximation gives

$$[\,\bar{\epsilon}_{ij}\,] = \begin{bmatrix} (1+E_1)\cos\phi_1 - 1 & \frac{1}{2}(1+E_1)\sin\phi_1 + \frac{1}{2}(1+E_2)\sin\phi_2 & 0 \\ \frac{1}{2}(1+E_1)\sin\phi_1 + \frac{1}{2}(1+E_2)\sin\phi_2 & (1+E_2)\cos\phi_2 - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is only when E_1, E_2, ϕ_1, and ϕ_2 are separately very small that we get the approximation

$$[\,\bar{\epsilon}_{ij}\,] \approx \begin{bmatrix} E_1 & \frac{1}{2}(\phi_1+\phi_2) & 0 \\ \frac{1}{2}(\phi_1+\phi_2) & E_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.10: A block experiences a rigid body motion. Estimate the allowable rotation angle such that the error in strain is not to exceed $\pm 10\,\mu\epsilon$ when the small strain formulas are used.

The small strain estimate is

$$\bar{\epsilon}_{11} = \cos\phi - 1 \approx 1 - \tfrac{1}{2}\phi^2 + \cdots - 1 \approx -\tfrac{1}{2}\phi^2$$

Therefore we want

$$\tfrac{1}{2}\phi^2 < 10 \times 10^{-6} \qquad \text{or} \qquad \phi < 4.5 \times 10^{-3}\,[\text{radians}] < 0.25^o$$

This is a very small angle. In other words, it does not take much of a rigid body rotation in order to invalidate the use of the small strain approximations. ■

Exercises

2.1 A block rotates an angle θ about the x_3 axis. Write down its deformation and obtain the deformation gradient. Show that the volume change is zero.

2.2 Consider the following deformation with a large value of k

$$x_1 = x_1^o + k\,x_2^o \qquad x_2 = x_2^o \qquad x_3 = x_3^o$$

Draw the deformed shape of a volume that was initially a cube. Show that the formulas describing the deformation of areas are in agreement with the geometric construction.

2.3 Consider the following deformation

$$x_1 = 3x_1^o + k\,x_2^o \qquad x_2 = 2x_1^o + 4x_2^o \qquad x_3 = x_3^o$$

What (if any) are the restrictions on k for this to be a valid deformation? Draw the deformed shape. Show by measurement the consistency of the physical interpretation of the Lagrangian strains with their connection to the deformation gradient.

2.4 For the previous deformation, determine the principal strains. Draw the before and after positions of the principal element. Draw the deformed shape of a volume that was initially a cube. Show that the formulas describing the deformation of areas are in agreement with the geometric construction.

2.5 Consider the following inhomogeneous deformation

$$x_1 = [R - x_2^o]\sin(x_1^o/R) \qquad x_2 = R - [R - x_2^o]\cos(x_1^o/R) \qquad x_3 = x_3^o$$

What (if any) are the restrictions for this to be a valid deformation? Draw the deformed shape of material initially lying between $-h < x_2^o < h$. Calculate the orientation and magnitude of an area that was initially vertical and facing the 1-direction.

2.6 For the previous deformation, determine the Lagrangian and Eulerian strain tensors. Under what circumstance(s) are they the same?

2.7 Show that the Lagrangian and Eulerian strains are related by

$$E_{ij} = e_{pq}\frac{\partial x_p}{\partial x_i^o}\frac{\partial x_q}{\partial x_j^o}$$

2.8 Give the physical meaning to the components of the Eulerian strain e_{ij}.

2.9 Show that if

$$J^o = \epsilon_{ijk}\frac{\partial x_1}{\partial x_i^o}\frac{\partial x_2}{\partial x_j^o}\frac{\partial x_3}{\partial x_k^o} \qquad \text{then} \qquad J^o = \epsilon_{ijk}\frac{\partial x_i}{\partial x_1^o}\frac{\partial x_j}{\partial x_2^o}\frac{\partial x_k}{\partial x_3^o}$$

2.10 Show that

$$\frac{\partial}{\partial x_i^o}(J) = 0 \qquad \frac{\partial}{\partial x_i}(J^o) = 0$$

Physically, what are these saying?

2.11 The Lagrangian strain tensor at a point is

$$[E_{ij}] = \begin{bmatrix} 2 & -1 & \sqrt{2} \\ -1 & 3 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{bmatrix}$$

What is the engineering strain of a line element initially oriented as $\hat{n} = \frac{1}{2}\hat{e}_1 - \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$. What is the angle change between two line elements initially oriented as $\hat{n}^a = \frac{1}{2}\hat{e}_1 - \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$ and $\hat{n}^b = -\frac{1}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_3$.

2.12 Consider the homogeneous deformation of a square in a two-dimensional body such that the corners move as

$$(0,0) \Rightarrow (0,0) \qquad (1,0) \Rightarrow (1,1.5) \qquad (0,1) \Rightarrow (-1,2)$$

Describe the deformation mathematically. Determine the Lagrangian strain tensor. What can you say about the deformation given by

$$(0,0) \Rightarrow (0,0) \qquad (0,1) \Rightarrow (1,1.5) \qquad (1,0) \Rightarrow (-1,2)$$

2.13 For the previous deformation, find the principal strain values and directions. Determine and draw the final position of lines initially oriented along the principal directions.

2.14 Show that the two expressions for a deforming area

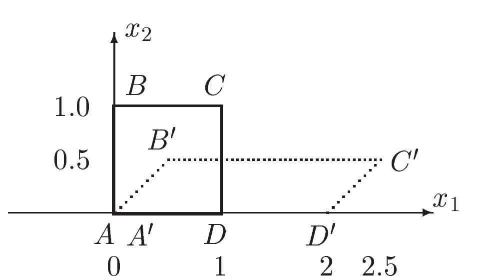
$$dA_k^o = \epsilon_{ijk} dx_i^{oa} dx_j^{ob}$$

and

$$\epsilon_{pqk} dA_k^o = dx_p^{oa} dx_q^{ob} - dx_q^{oa} dx_p^{ob}$$

are equivalent forms.

2.15 Consider a 2-D square body $ABCD$ which deforms homogeneously into $A'B'C'D'$.
(a) Write the displacement components in terms of the Lagrangian and Eulerian variables.
(b) Compute E_{ij} and e_{ij}.
(c) Find the extension of element AC after deformation by use of E_{ij}.
(d) Find the extension of element AB after deformation by use of E_{ij}.
(e) Find the engineering strain of a vertical element in the deformed body.



Chapter 3

Stress

The kinetics of rigid bodies are described in terms of forces; the equivalent concept for continuous media is stress (loosely defined as force over unit area). The deformed state is the natural configuration in which to consider stress and this gives rise to the Cauchy stress tensor. However, to maintain the duality of the Lagrangian and Eulerian descriptions, we also introduce stresses defined with respect to the undeformed configuration.

3.1 Cauchy Stress Principle

Actions can be exerted on a continuum through either contact forces or forces contained in the mass. There are three types of forces and moments worth distinguishing:

- **Extrinsic:** arise (in part) from outside the body, e.g., gravity, magnetic, electrostatic.
- **Mutual:** arise within the body and act upon pairs of particles, e.g., Newtonian gravitation.
- **Contact:** act upon bounding surfaces and are equipollent (same mechanical effect) to the loading of one portion of the material, e.g., stress, pressure.

The contact force is often referred to as a surface force or traction as its action occurs on a surface. We are primarily concerned with contact forces.

More general actions can take place in the form of surface moments and body moments. Theories that allow existence of these moments are called couple stress theories, or microstructural theories, and such solids are sometimes called Cosserat continua. These will not be considered in the following presentation.

Traction Vector

Consider a small surface element of area ΔA on our imagined exposed surface A in the deformed configuration as shown in Figure 3.1. There must be forces and moments acting on ΔA to make it equipollent to the effect of the rest of the material. That is, when the pieces are put back together these forces cancel each other. Let these forces be thought of as contact forces and so give rise to contact stresses (even though they are inside the body). Cauchy formalized this by introducing his concept of traction vector.

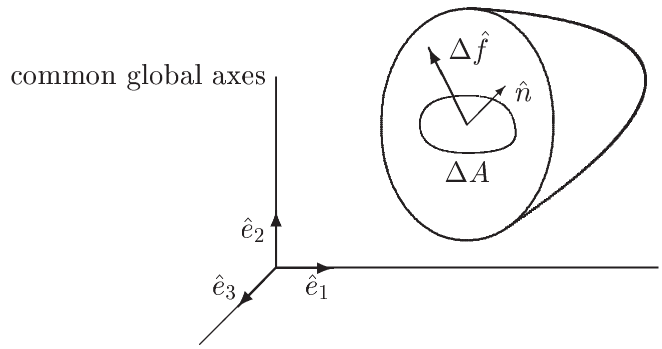


Figure 3.1: Exposed forces on an arbitrary section cut.

Let $\hat{n}$ be the unit vector which is perpendicular to the surface element ΔA and let $\Delta\hat{f}$ be the resultant force exerted from the other part of the surface element with the negative normal vector. We assume that as ΔA becomes vanishingly small, the ratio $\Delta\hat{f}/\Delta A$ approaches a definite limit $d\hat{f}/dA$. The vector obtained in the limiting process is

$$\lim_{\Delta A\to 0}\frac{\Delta\hat{f}}{\Delta A}=\frac{d\hat{f}}{dA}\equiv\hat{t}^{(\hat{n})}$$

which is called the *traction vector*. This vector represents the force per unit area acting on the surface and its limit exists because the material is assumed continuous. The superscript $\hat{n}$ is a reminder that the traction is dependent on the orientation of the area.

Also note that, in general, there are moments or torques acting on ΔA and in the limit

$$\hat{m}^{(\hat{n})}=\lim_{\Delta A\to 0}\frac{\Delta\hat{m}}{\Delta A}$$

We make the assumption that $\hat{m}$ is zero for all $\hat{n}$. Retaining $\hat{m}$ gives couple stress theory which is needed, for example, when doing notch problems where the notch geometry is on the order of the grain size. We will neglect couple stresses.

To give explicit representation of the traction vector, consider its components on the three faces of a cube as shown in Figure 3.2. The traction on the 1-face is

$$\hat{n}=\hat{e}_1: \qquad \hat{t}^{(\hat{n})}=t_i^{(\hat{e}_1)}\hat{e}_i=t_1^{(\hat{e}_1)}\hat{e}_1+t_2^{(\hat{e}_1)}\hat{e}_2+t_3^{(\hat{e}_1)}\hat{e}_3$$

while on the 2-face

$$\hat{n}=\hat{e}_2: \qquad \hat{t}^{(\hat{n})}=t_i^{(\hat{e}_2)}\hat{e}_i=t_1^{(\hat{e}_2)}\hat{e}_1+t_2^{(\hat{e}_2)}\hat{e}_2+t_3^{(\hat{e}_2)}\hat{e}_3$$

Note that, in general, $t_1^{(\hat{e}_2)}\neq t_1^{(\hat{e}_1)}$. Since this description is somewhat cumbersome, we simplify the notation by introducing

$$\sigma_{ij}\equiv t_j^{(\hat{e}_i)}$$

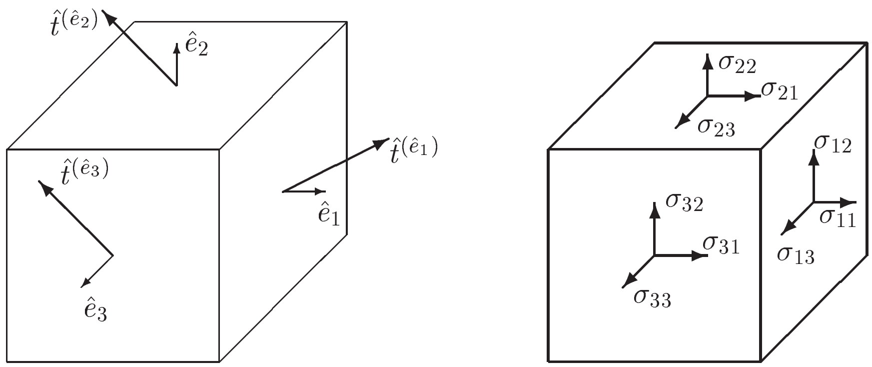


Figure 3.2: Stressed cube.

where 'i' refers to the face and 'j' to the component. More specifically,

$$\sigma_{11} \equiv t_1^{(\hat{e}_1)}, \qquad \sigma_{13} \equiv t_3^{(\hat{e}_1)}, \qquad \sigma_{31} \equiv t_1^{(\hat{e}_3)}, \qquad \ldots$$

The normal projections of $\hat{t}^{(\hat{n})}$ on these special faces are the normal stress components $\sigma_{11}, \sigma_{22}, \sigma_{33}$, while projections perpendicular to $\hat{n}$ are shear stress components σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31},σ_{32}.

It is important to realize that while $\hat{t}$ resembles the elementary idea of stress (force over area) it is not stress; $\hat{t}$ transforms as a vector and has only three components. The tensor σ_{ij} is our definition of stress; it has nine components with units of force over area, but at this stage we do not know how these components transform.

Relation between t_i and n_j

We know that the traction vector $\hat{t}^{(\hat{n})}$ acting on an area $dA\hat{n}$ depends on the normal $\hat{n}$ of the area. The particular relation can be obtained by considering a traction on an arbitrary surface of the tetrahedron shown in Figure 3.3. On the three faces perpendicular to the coordinate directions the components of the three traction vectors are denoted by σ_{ij}. The vector acting on the inclined surface ABC is $\hat{t}$ and the unit normal vector $\hat{n}$. The equilibrium of the tetrahedron requires that the resultant force acting on it must vanish.

The equation for the balance of forces in the x_1-direction for the tetrahedron is given by

$$t_1 dA - \sigma_{11} dA_1 - \sigma_{21} dA_2 - \sigma_{31} dA_3 + b_1 \rho dV = 0$$

where b_1 is the x_1-component of the body force $\hat{b}$ (which may also contain inertia terms), t_1 is the x_1-component of the traction vector, dA_i is the area of the face perpendicular to the x_i axis, dA is the area of the inclined surface, and

$$dV = \tfrac{1}{3} h dA$$

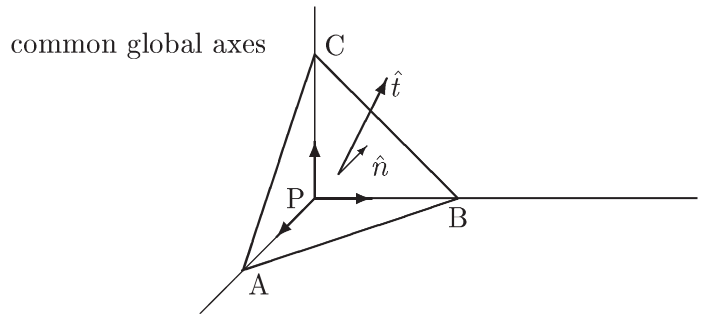


Figure 3.3: Tetrahedron.

is the volume of the tetrahedron. In this, h is the smallest distance from point P to the inclined surface ABC. Noting that the normal to the area has the components

$$\hat{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3$$

we conclude that the components of area are

$$\begin{aligned} \text{area of face 1:} \quad dA_1 &= n_1 dA \\ \text{area of face 2:} \quad dA_2 &= n_2 dA \\ \text{area of face 3:} \quad dA_3 &= n_3 dA \end{aligned}$$

Now divide through by dA in the equilibrium relation, and letting $h \to 0$, we obtain

$$t_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = \sigma_{j1} n_j$$

Similar equations can be derived by considering the balance of forces in the x_2- and x_3-directions. The results are, respectively,

$$\begin{aligned} t_2 &= \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 = \sigma_{j2} n_j \\ t_3 &= \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 = \sigma_{j3} n_j \end{aligned}$$

These three equations can be written in the indicial notation as

$$t_i = \sigma_{ji} n_j \tag{3.1}$$

This compact relation says that we need only know nine numbers $[\sigma_{ij}]$ to be able to determine the traction vector on any area passing through a point. These elements are called the Cauchy stress components and form the Cauchy stress tensor. It is a second order tensor (since t_i and n_j transform as first order tensors) but not necessarily symmetric (at this stage). The tensorial property indicates that the above relation is true in any Cartesian coordinate system.

Example 3.1: Consider a block of material with a uniformly distributed force acting on the 1-face. Determine the tractions on an interior plane.

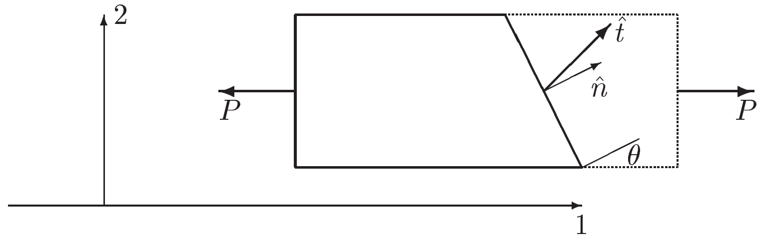


Figure 3.4: Block with end forces.

First consider a vertical cut to expose an interior 1-face, i.e., $\hat{n} = \{1, 0, 0\}$. Represent the components of the applied force as $P_i = \{P, 0, 0\}$, then equilibrium gives

$$t_i A - P_i = 0 \qquad \text{or} \qquad t_1 = \frac{P}{A} = \sigma_{11}\,, \quad t_2 = 0 = \sigma_{12}\,, \quad t_3 = 0 = \sigma_{13}$$

Similarly, for the 2-face and 3-face we obtain

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0\,; \qquad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$$

Thus the state of stress in the entire body is given by the stress tensor

$$[\,\sigma_{ij}\,] = \begin{bmatrix} P/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now make a cut at an angle of θ, that is, the unit normal vector $\hat{n}$ to the inclined face has the components $n_i = \{\cos\theta, \sin\theta, 0\}$. The traction t_i acting on this surface is given by $t_i = \sigma_{ji} n_j$ from which we have

$$\begin{aligned} t_1 &= \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = \sigma_{11}\cos\theta = \frac{P}{A}\cos\theta \\ t_2 &= 0\,, \quad t_3 = 0 \end{aligned}$$

Finally, consider the normal and tangential components of the traction acting on this area. Let $\hat{s}$ be the tangential unit vector, we have

$$t_n = \hat{t}\cdot\hat{n} = t_k n_k = t_1 n_1 = \frac{P}{A}\cos^2\theta\,, \qquad t_s = \hat{t}\cdot\hat{s} = t_k s_k = t_1 s_1 = -\frac{P}{A}\cos\theta\sin\theta$$

These two components are in fact the transformed components of the stress tensor σ'_{ij}, obtained by rotating the $x_1 - x_2$ axes about x_3 for an angle θ. That is

$$t_n = \sigma'_{11}\,, \quad t_s = \sigma'_{12}$$

Thus the transformation for t_1 and t_2 contain $\sin\theta$ at most whereas the transformation for t_n and t_t contain $\cos\theta^2$ and $\cos\theta\sin\theta$. ■

Equations of Motion in Terms of Stress

Recall that Newton's laws for the equation of motion of a rigid body can be written as

$$\begin{aligned} \sum \hat{F} &= m\hat{a} \\ \sum \hat{M} &= m\,\hat{x} \times \hat{a} \end{aligned}$$

where $\hat{a}$ is the acceleration and m the mass. These equations will now be used to establish the equations of motion of a deformable body.

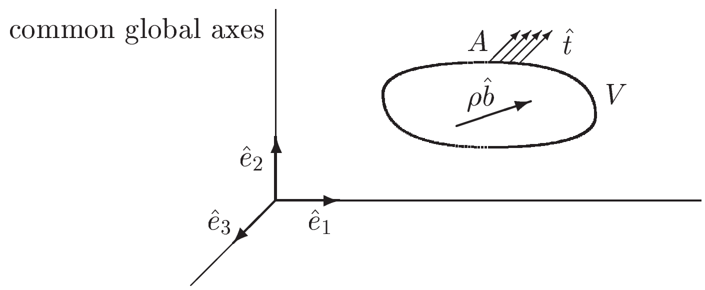


Figure 3.5: Arbitrary small volume.

Consider an arbitrary volume V taken from the deformed body, then the Newton's laws of motion become, respectively,

$$\int_A \hat{t}\, dA + \int_V \rho\hat{b}\, dV = \int_V \rho\hat{\ddot{u}}\, dV$$

and

$$\int_A \hat{x} \times \hat{t}\, dA + \int_V \hat{x} \times \hat{b}\rho\, dV = \int_V \hat{x} \times \hat{\ddot{u}}\rho\, dV$$

where $\hat{t}$ is the traction on the boundary surface A, and $\hat{b}$ is the body force per unit mass. In indicial notation, these are rewritten as

$$\begin{aligned} \int_A t_i dA + \int_V \rho b_i dV &= \int_V \rho\ddot{u}_i dV \\ \int_A \epsilon_{ijk} x_j t_k dA + \int_V \epsilon_{ijk} x_j b_k \rho dV &= \int_V \epsilon_{ijk} x_j \ddot{u}_k \rho dV \end{aligned}$$

These are the equations of motion in terms of t_i. We can now obtain the equations of motion in terms of the stress σ_{ij} by using $t_i = \sigma_{pi} n_p$ and noting that, by the integral theorem of Chapter 1,

$$\int_A t_i\, dA = \int_A \sigma_{pi} n_p\, dA = \int_V \frac{\partial \sigma_{pi}}{\partial x_p} dV$$

The equations of motion become

$$\int_V \left[\frac{\partial \sigma_{pi}}{\partial x_p} + \rho b_i - \rho\ddot{u}_i\right] dV = 0$$

$$\int_V \left[\frac{\partial}{\partial x_p}(\epsilon_{ijk} x_j \sigma_{pk}) + \rho\epsilon_{ijk} x_j b_k - \rho\epsilon_{ijk} x_j \ddot{u}_k\right] dV = 0$$

Noting that

$$\frac{\partial}{\partial x_p}(x_j \sigma_{pk}) = \sigma_{jk} + x_j \frac{\partial \sigma_{pk}}{\partial x_p}$$

we can expand the second equation and simplify it using the first equation. Furthermore, since the volume V is arbitrary, we conclude that the integrands must vanish and therefore

$$\begin{aligned} \frac{\partial \sigma_{pi}}{\partial x_p} + \rho b_i &= \rho \ddot{u}_i \\ \epsilon_{ijk} \sigma_{jk} &= 0 \end{aligned}$$

The second equation shows that σ_{ij} is a symmetric tensor since the contraction of a symmetric tensor with an anti-symmetric tensor is zero. Hence the two equations of motion become

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i &= \rho \ddot{u}_i \\ \sigma_{ij} &= \sigma_{ji} \end{aligned} \tag{3.2}$$

These equations of motion are written in expanded notation as

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho b_1 &= \rho \ddot{u}_1 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho b_2 &= \rho \ddot{u}_2 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 &= \rho \ddot{u}_3 \end{aligned}$$

It is worth repeating that due to the symmetry property of the stress tensor, only six components are independent. As a result, the number of independent stress components in the above are reduced since $\sigma_{12} = \sigma_{21}, \sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$.

It is also worth noting that the equations of motion in terms of the Cauchy stress measure is applicable to large deformations.

Example 3.2: Under what circumstances (if any) is the following symmetric stress field in static equilibrium?

$$\sigma_{11} = 3x_1 + k_1 x_2^2 \qquad \sigma_{22} = 2x_1 + 4x_2 \qquad \sigma_{12} = \sigma_{21} = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1x_2$$

all others being zero.

We will consider the case when the body forces (including inertia) are zero. The first two static equilibrium equations become

$$3 + d + 2ex_2 + fx_1 = 0\,, \qquad 4 + b + 2cx_1 + fx_2 = 0$$

These must be true for any x_1, x_2 which leads to

$$\begin{aligned} 3 + d = 0\,, \qquad e = 0\,, \qquad f = 0 \\ 4 + b = 0\,, \qquad c = 0\,, \qquad f = 0 \end{aligned}$$

It is interesting that the k_1 term does not affect the equilibrium. ■

3.2 Properties of the Cauchy Stress Tensor

Since the Cauchy stress tensor is symmetric and second order, then it inherits many properties as already outlined in Chapter 1 for second order tensors. It is worthwhile, however, to review these here in the physical context of stress.

The Cauchy Stress Tensor

The concept of stress introduced in the last section is different from the elementary idea of force over area. In preparation for the more subtle concepts of Lagrange and Kirchhoff stress to be introduced later, it is worth our while to recapitulate the developments of the Cauchy stress tensor.

On any surface of a deformed body, there exists a traction vector $\hat{t}$. The magnitude and the direction of this vector at a point changes on different surfaces with different unit normal vectors $\hat{n}$.

The three components of the traction vector on the x_1-face (the one that is perpendicular to x_1-axis) are denoted by

$$\sigma_{11},\ \sigma_{12},\ \sigma_{13}$$

Similarly, on the x_2-face and x_3-face we have the components of the traction vectors as

$$\sigma_{21},\ \sigma_{22},\ \sigma_{23} \qquad \text{and} \qquad \sigma_{31},\ \sigma_{32},\ \sigma_{33}$$

respectively. Although these nine scalars $[\sigma_{ij}]$ have the size of a second order tensor, they cannot be assumed automatically to have formed a tensor, as they have not yet been shown to have satisfied the coordinate transformation law.

By considering the equilibrium of a small arbitrary volume, the relation $t_i = \sigma_{ji} n_j$ was established for tractions acting on arbitrary areas. The quotient rule then showed that σ_{ij} is a second order tensor. From the above, we conclude that the state of stress in a body is completely given by the stress tensor σ_{ij}. Therefore, given any surface with the unit normal vector $\hat{n}$, one is able to determine the stress vector (force intensity) acting on that surface if the stress tensor is known.

The final step considered the equations of motion of an arbitrary volume from which we concluded the symmetry of the Cauchy stress tensor.

In this development, the physical entity is the traction vector; the stress tensor is viewed as a derived (or defined) quantity that connects the components of the traction vector acting on any plane.

Principal Stresses

It is noted the traction vector $\hat{t}$ acting on a surface depends on the direction $\hat{n}$ and is usually not parallel to $\hat{n}$. We now attempt to find an $\hat{n}$ such that the stress vector is acting in the direction of $\hat{n}$, that is,

$$t_i = \sigma n_i$$

where σ is a scalar representing the magnitude of the stress vector. A direction which satisfies this is called a *principal direction* of stress, and σ is called the corresponding *principal stress*.

By using the relation $t_i = \sigma_{ji} n_j$, the above yields

$$\sigma_{ji} n_j = \sigma n_i$$

which can be rewritten as

$$[\sigma_{ji} - \sigma\delta_{ij}]^T \{n_j\} = 0$$

Thus, three equations result for the three unknowns n_1, n_2, and n_3. It is well known that for the three homogeneous equations to yield a non-trivial solution, the determinant of the coefficient matrix must vanish, that is,

$$\det[\sigma_{ji} - \sigma\delta_{ij}] = 0$$

This is recognized as a standard eigenvalue problem. Viewing σ_{ij} as a real, symmetric matrix the eigenvalues σ_i are guaranteed to be real, and the corresponding eigenvectors n_i are mutually orthogonal. The proof of this theorem can be found in many books on matrix theory and is also discussed in Chapter 1.

Example 3.3: Consider a state of stress at a point given by the stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress invariants can be computed as $I_1 = 5, I_2 = 4$, and $I_3 = 0$. The characteristic equation is, therefore,

$$\sigma^3 - 5\sigma^2 + 4\sigma = 0$$

The principal values are readily obtained as follows

$$\sigma_1 = 4\,, \quad \sigma_2 = 1\,, \quad \sigma_3 = 0$$

The unit normal $\hat{n}^{(1)}$ corresponding to $\sigma_1 = 4$ can be obtained by substituting this value back into the system of equations to obtain

$$\begin{aligned} -2n_1^{(1)} + 2n_2^{(1)} &= 0 \\ 2n_1^{(1)} - 2n_2^{(1)} &= 0 \\ -3n_3^{(1)} &= 0 \end{aligned}$$

This indicates that the three equations are not independent. Thus, only two equations are available to determine the solution. Since there are three unknowns, two equations can determine only the ratios among the three quantities $n_1^{(1)}, n_2^{(1)}$, and $n_3^{(1)}$. However, with the specification that $\hat{n}$ is a unit vector then

$$n_1^2 + n_2^2 + n_3^2 = 1$$

and the solution is uniquely obtained as

$$n_1^{(1)} = \frac{1}{\sqrt{2}}\,, \qquad n_2^{(1)} = \frac{1}{\sqrt{2}}\,, \qquad n_3^{(1)} = 0$$

Following similar manipulations, the unit vectors $\hat{n}^{(2)}$ and $\hat{n}^{(3)}$ corresponding to σ_2 and σ_3, respectively, can be determined. We have

$$n_1^{(2)} = 0\,, \qquad n_2^{(2)} = 0\,, \qquad n_3^{(2)} = 1$$

and

$$n_1^{(3)} = \frac{1}{\sqrt{2}}\,, \qquad n_2^{(3)} = \frac{-1}{\sqrt{2}}\,, \qquad n_3^{(3)} = 0$$

It is straightforward to show that the $\hat{n}^{(i)}$ are orthogonal. ■

Normal Stress

The normal stress σ_n is the component of the traction vector in the direction of $\hat{n}$, the unit normal to the surface of interest. This component of stress can be obtained by taking the scalar product of $\hat{t}$ and $\hat{n}$ as

$$\sigma_N = t_i n_i = \sigma_{ij} n_i n_j$$

If the coordinate axes are chosen to coincide with the principal directions of stress, then σ_{ij} has the form of a diagonal matrix, that is,

$$\sigma_{ij} = 0 \quad \text{if} \quad i \neq j$$

and

$$\sigma_{11} = \sigma_1\,, \qquad \sigma_{22} = \sigma_2\,, \qquad \sigma_{33} = \sigma_3$$

In that case, the normal stress σ_N can be expressed as

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

If the principal stresses are ordered such that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

then we obtain

$$\sigma_1 > \sigma_N > \sigma_3$$

by using $n_1^2 + n_2^2 + n_3^2 = 1$. This indicates that σ_1 and σ_3 are the maximum and minimum normal stresses, respectively, at the point.

The extremum values of the normal stress can be established more formally by considering the behavior of the normal component of traction. We know that $\sigma_N =$

$\sigma_{ji}n_jn_i$ but when finding the extremum we must also take into account the constraint that $n_kn_k = 1$. We implement this by way of a Lagrange multiplier and extremize

$$\phi = \sigma_{ji}n_jn_i - \lambda n_kn_k$$

where λ is the (as yet unknown) Lagrange multiplier. Differentiating this with respect to n_p leads to

$$\sigma_{ji}n_j - \lambda n_i = 0 \qquad \text{or} \qquad [\sigma_{ji} - \lambda\delta_{ij}]^T\{n_j\} = 0$$

We now recognize λ as the principal value of stress.

Shear Stress

The shearing stress τ is the projection of the stress vector on the surface of interest. That is,

$$\hat{t} = \sigma_N\hat{n} + \tau\hat{n}' \quad \text{or} \quad \tau\hat{n}' = \hat{t} - \sigma_N\hat{n}$$

where $\hat{n}'$ is perpendicular to $\hat{n}$. The magnitude of the shearing stress, τ, is given by

$$\tau^2 = (\tau\hat{n}')\cdot(\tau\hat{n}') = (\hat{t} - \sigma_N\hat{n})\cdot(\hat{t} - \sigma_N\hat{n}) = |\hat{t}|^2 - \sigma_N^2 = t_it_i - \sigma_N^2$$

where $|\hat{t}|$ is the magnitude of the traction vector. If the principal directions are chosen as the coordinate axes, then

$$\begin{aligned}
t_1 = \sigma_{1j}n_j &= \sigma_1 n_1 \\
t_2 = \sigma_{2j}n_j &= \sigma_2 n_2 \\
t_3 = \sigma_{3j}n_j &= \sigma_3 n_3
\end{aligned}$$

Thus,

$$|\hat{t}|^2 = t_it_i = t_i^2 + t_2^2 + t_3^2 = (\sigma_1n_1)^2 + (\sigma_2n_2)^2 + (\sigma_3n_3)^2$$

These allow the magnitude of the shear stress to be obtained as

$$\begin{aligned}
\tau^2 &= (\sigma_1n_1)^2 + (\sigma_2n_2)^2 + (\sigma_3n_3)^2 - (\sigma_1n_1^2 + \sigma_2n_2^2 + \sigma_3n_3^2)^2 \\
&= n_1^2(1-n_1^2)\sigma_1^2 + n_2^2(1-n_2^2)\sigma_2^2 + n_3^2(1-n_3^2)\sigma_3^2 - 2\sigma_1\sigma_2n_1^2n_2^2 \\
&\quad -2\sigma_2\sigma_3n_2^2n_3^2 - 2\sigma_1\sigma_3n_1^2n_3^2
\end{aligned}$$

Using $n_in_i = 1$ to replace the terms in parenthesis, as for example, $1 - n_1^2 = n_2^2 + n_3^2$, we obtain finally

$$\tau^2 = n_1^2n_2^2(\sigma_1 - \sigma_2)^2 + n_2^2n_3^2(\sigma_2 - \sigma_3)^2 + n_3^2n_1^2(\sigma_3 - \sigma_1)^2$$

It is obvious from this that $\tau = 0$ on the surfaces with

$$\begin{aligned}
n_1 &= 1\,, \qquad & n_2 = n_3 = 0 \\
n_2 &= 1\,, \qquad & n_1 = n_3 = 0 \\
n_3 &= 1\,, \qquad & n_1 = n_2 = 0
\end{aligned}$$

while the normal stress σ_N either reaches the maximum values or the minimum value.

Consider all the surfaces that contain the x_2-axis with

$$n_1 \neq 0\,, \qquad n_3 \neq 0\,, \qquad n_2 = 0$$

The shear stress then is

$$\tau^2 = n_3^2 n_1^2 (\sigma_3 - \sigma_1)^2 = (1 - n_1^2) n_1^2 (\sigma_3 - \sigma_1)^2$$

since $n_1^2 + n_3^2 = 1$. The extremum value of τ occurs at

$$\frac{\partial(\tau^2)}{\partial n_1} = 0 = (2n_1 - 4n_1^3)(\sigma_3 - \sigma_1)^2$$

Solving for n_1 gives us the directions

$$n_1 = \pm\frac{1}{\sqrt{2}}\,, \qquad n_2 = 0\,, \qquad n_3 = \pm\frac{1}{\sqrt{2}}$$

That is, the maximum value of the shearing stress occurs on the surfaces bisecting the angle between the x_1- and x_3-axes. The corresponding value of the shearing stress is

$$\tau_{\max}^2 = \tfrac{1}{4}(\sigma_3 - \sigma_1)^2 \quad \text{or} \quad |\tau_{\max}| = \tfrac{1}{2}|\sigma_3 - \sigma_1|$$

This conclusion can be reached for the general case if the principal stresses are ordered as $\sigma_1 > \sigma_N > \sigma_3$.

3.3 Stresses Referred to the Undeformed State

To complete our duality of treatments of stress and deformation, we need to consider the equations of motion with respect to the undeformed configuration. Specifically, the Cauchy equations of motion are in terms of the spatial partial derivatives and these must be changed to derivatives with respect to the undeformed state. In doing so, we introduce two new measures of stress, the Lagrangian and Kirchhoff stresses, respectively.

Equations of Motion in the Undeformed State

The equations of motion were derived in terms of the Cauchy stress tensor as

$$\begin{aligned} \frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i &= \rho \ddot{u}_i \\ \sigma_{ij} &= \sigma_{ji} \end{aligned}$$

which are in reference to the deformed state.

There are advantages to expressing the equations of motion in terms of variables in the undeformed state. We begin with the body force $\hat{b}$, which is the body force per unit mass in the deformed configuration. Define the body force per unit mass in the undeformed state as $\hat{b}^o$ such that the resultant force is the same. That is,

$$b_i^o \rho^o \, dV^o = b_i \, \rho \, dV$$

In view of the mass conservation law, $\rho^o dV^o = \rho dV$, we obtain

$$b_i^o = b_i$$

This body force relation is also valid for inertial forces.

We change the spatial derivatives to material derivatives as follows:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \frac{\partial \sigma_{ji}}{\partial x_p^o}\frac{\partial x_p^o}{\partial x_j} = \frac{\partial}{\partial x_p^o}\left(\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}\right) - \sigma_{ji}\frac{\partial^2 x_p^o}{\partial x_p^o \partial x_j} = \frac{\partial}{\partial x_p^o}\left(\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}\right)$$

Thus, the equations of motion become

$$\frac{\partial}{\partial x_p^o}\left(\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}\right) + \rho b_i = \rho \ddot{u}_i$$

Noting that $\rho/\rho^o = J$ and $\partial J/\partial x_p^o = 0$ (see Equation(2.5)), we obtain

$$\frac{\partial}{\partial x_p^o}\left(\frac{\rho^o}{\rho}\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}\right) + \rho^o b_i = \rho^o \ddot{u}_i$$

It remains now to replace the term in parenthesis with a quantity that has meaning in the undeformed configuration.

I: Lagrangian Stress

The simplest approach is to define a new stress as

$$\sigma_{pi}^L \equiv \frac{\rho^o}{\rho}\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}$$

and substitution of it into the equation of motion yields

$$\frac{\partial \sigma_{pi}^L}{\partial x_p^o} + \rho^o b_i^o = \rho^o \ddot{u}_i$$

The symmetry condition $\sigma_{ij} = \sigma_{ji}$ is now given by

$$\sigma_{pi}^L \frac{\partial x_j}{\partial x_p^o} = \sigma_{pj}^L \frac{\partial x_i}{\partial x_p^o}$$

indicating that σ_{pi}^L is non-symmetric. Because of the role played by σ_{pi}^L we can interpret it as a stress tensor, and it is called the *Lagrange stress tensor*. Using this new stress, the equations of motion are relatively simple (indeed they resemble the Cauchy equations) but the non-symmetric components will complicate the constitutive relation.

II: Kirchhoff Stress

We would prefer to have a symmetric stress tensor, to that end let us replace the term in parenthesis with

$$\frac{\partial x_i}{\partial x_j^o}\sigma_{kj}^K \equiv \frac{\rho^o}{\rho}\sigma_{ji}\frac{\partial x_p^o}{\partial x_j}$$

from which we have

$$\sigma_{pq}^K = \frac{\rho^o}{\rho}\sigma_{ji}\frac{\partial x_p^o}{\partial x_i}\frac{\partial x_q^o}{\partial x_j}$$

It is evident that $\sigma_{ij}^K = \sigma_{ji}^K$ making σ_{ij}^K a symmetric tensor. Substitute this into the equations of motion to get

$$\frac{\partial}{\partial x_k^o}\left[\frac{\partial x_i}{\partial x_j^o}\sigma_{kj}^K\right] + \rho^o b_i^o = \rho^o \ddot{u}_i$$

Again, because of the role played by σ_{pi}^K we can interpret it as a stress tensor and it is called the *Kirchhoff stress tensor* or second Piola stress tensor. The Kirchhoff stress tensor is symmetric which simplifies the constitutive relation, but the equations of motion are slightly more complicated.

Interpretation of Lagrangian and Kirchhoff Stresses

In the analysis of stresses in a body with finite deformation, it may be convenient to use the Lagrangian variables. Since true loading exists only in the deformed state, the corresponding loading and stress in the body at the initial (undeformed) state could be considered as fictitious; however, we showed that even the Cauchy stress is a derived (or defined) quantity and therefore we are free to define other stresses. To appreciate the motivation in introducing the new definitions of stress, it is worthwhile to keep the following in mind:

- The traction vector is first defined in terms of a force divided by area.
- The stress tensor is defined according to a transformation relation for the traction and area normal.

To refer to the surface before deformation, a traction vector $\hat{t}^o$ acting on an area dA^o must be defined. The introduction of such a vector is arbitrary and two usual definitions will be discussed. But first the Cauchy stress will be reconsidered so as to motivate the developments.

I: Cauchy Stress

In the deformed state, on every plane surface passing through a point, there is a traction vector t_i defined in terms of the deformed surface area. That is, letting the

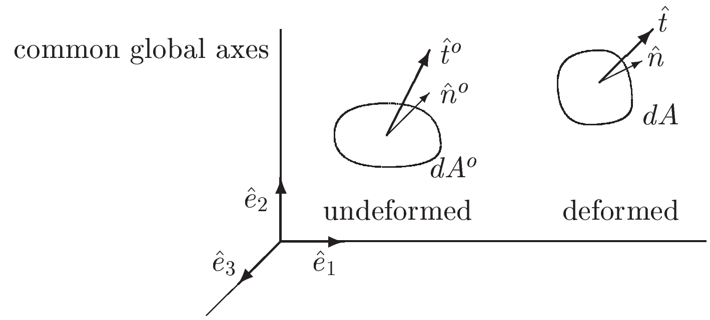


Figure 3.6: Traction vectors in the undeformed and deformed configurations.

traction vector be $\hat{t}$ and the total resultant force acting on dA be $d\hat{f}$ then

$$t_i \equiv \frac{df_i}{dA}$$

Let all the traction vectors and unit normals in the deformed body form two respective vector spaces. Then the Cauchy stress tensor σ_{ij} was shown to be the transformation between these two vector spaces, that is,

$$t_i = \sigma_{ji} n_j$$

The tensorial property of the Cauchy stress tensor can be established from the quotient rule. Defined in this manner, the Cauchy stress tensor is an abstract quantity; however, on special plane surfaces such as the ones with unit normals parallel to $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$, respectively, the nine components of σ_{ij} can be related to the stress vector and thus have physical meaning.

Thus the meaning of σ_{ij} are the components of stress derived from the force vector df_i divided by the deformed area. This, in elementary terms, is called *true stress*.

II: Lagrangian Stress

Let the resultant force $d\hat{f}^o$ referred to the undeformed configuration be identical to the force $d\hat{f}$ acting on the deformed area. That is,

$$df_i^o = df_i$$

The Lagrangian traction vector t_i^o is defined as

$$t_i^o \equiv \frac{df_i^o}{dA^o} = \frac{df_i}{dA^o}$$

The Lagrangian stress tensor σ_{ij}^L is defined as the second order tensor which relates the two vector spaces t_i^o and n_i^o in the following manner:

$$t_i^o = \sigma_{ji}^L n_j^o$$

The meaning of σ_{pi}^L are the components of stress derived from the force df_i divided by the original area. In elementary terms, this is called *engineering stress*.

III: Kirchhoff Stress

Let the resultant force $d\hat{f}^o$ referred to the undeformed configuration be given by a transformation of the force $d\hat{f}$ acting on the deformed area. In particular, let

$$df_i^o = \frac{\partial x_i^o}{\partial x_j} df_j$$

which follows the analogous rule as for the deformation of line segments

$$dx_i^o = \frac{\partial x_i^o}{\partial x_j} dx_j$$

It is important to realize that this is not a rotation transformation but that the force components are being 'deformed'.

The Kirchhoff traction vector is defined as

$$t_i^o \equiv \frac{df_i^o}{dA^o} = \frac{\partial x_i^o}{\partial x_j} \frac{df_j}{dA^o}$$

This leads to the definition of the Kirchhoff stress tensor σ_{ij}^K:

$$t_i^o = \sigma_{ji}^K n_j^o$$

Formally, this is the same as for the Lagrangian stress, but recall that the components of force are different.

The meaning of σ_{pq}^K are components of stress derived from the transformed components of the force vector, divided by the original area. There is no elementary equivalent to this stress.

Relations Among the Stress Tensors

From the definition of the Lagrangian traction vector, we have

$$df_i = t_i^o dA^o = t_i dA$$

Using the Lagrangian and Cauchy stress tensors, the above equation becomes

$$\sigma_{ji}^L n_j^o dA^o = \sigma_{pi} n_p dA$$

Since, from Chapter 2, we have the relation between the areas

$$n_p \, dA = J^o \frac{\partial x_j^o}{\partial x_p} n_j^o \, dA^o$$

we obtain

$$\sigma_{ji}^L n_j^o dA^o = J^o \sigma_{pi} \frac{\partial x_j^o}{\partial x_p} n_j^o dA^o$$

Thus

$$\sigma^L_{ji} = J^o \frac{\partial x^o_j}{\partial x_p} \sigma_{pi}$$

We have $J^o = \rho^o/\rho$, thus the relation between Lagrangian and Cauchy stress tensors can also be written as

$$\sigma^L_{ji} = \frac{\rho^o}{\rho} \frac{\partial x^o_j}{\partial x_p} \sigma_{pi} \,, \qquad \sigma_{ji} = \frac{\rho}{\rho^o} \frac{\partial x_j}{\partial x^o_p} \sigma^L_{pi} \tag{3.3}$$

Note that, in general, σ^L_{ij} is not a symmetric tensor.

The relation between the Kirchhoff stress tensor and the Cauchy stress tensor is derived in a similar manner. We get

$$\sigma^K_{ji} = \frac{\rho^o}{\rho} \frac{\partial x^o_i}{\partial x_m} \frac{\partial x^o_j}{\partial x_n} \sigma_{mn} \,, \qquad \sigma_{ji} = \frac{\rho}{\rho^o} \frac{\partial x_i}{\partial x^o_m} \frac{\partial x_j}{\partial x^o_n} \sigma^K_{mn} \tag{3.4}$$

The matrix form of the relation between the Cauchy and Kirchhoff stress is

$$[\sigma^K_{ij}] = J[\frac{\partial x_i}{\partial x^o_m}]^{-1} [\sigma_{mn}] [\frac{\partial x^o_j}{\partial x^o_n}]^{-1T} \,, \qquad [\sigma_{ij}] = \frac{1}{J} [\frac{\partial x_i}{\partial x^o_m}] [\sigma^K_{mn}] [\frac{\partial x_j}{\partial x^o_n}]^T \tag{3.5}$$

where the meaning of $[\partial x_i/\partial x^o_j]^{-1}$ is the ij^{th} component of the inverse of the matrix $[\partial x_p/\partial x^o_j]$.

Note that the Kirchhoff stress tensor is a symmetric tensor. Since the strain tensors we introduced are symmetric, it is more convenient for us to use the Kirchhoff stress tensor in the formulation of the stress-strain laws.

3.4 Stress in Special Deformations

The description of some special deformations was given in the previous chapter. It is the purpose of this section to investigate the corresponding stresses. The three fundamental cases considered are that of simple extension, simple shear, and a state similar to bending. In the first, the magnitude of the areas change but not their orientation. In the second and third, both the magnitude and orientation of the areas change.

Simple Extension

The deformation corresponding to a uniform extension is given by

$$x_1 = \lambda_1 x^o_1 \,, \qquad x_2 = \lambda_2 x^o_2 \,, \qquad x_3 = \lambda_3 x^o_3$$

The unit cubic solid in Figure 3.7 is subjected to simple extension where the applied load is acting in only one direction. In this particular instance, the stretches are

$$\lambda_1 = 2 \,, \qquad \lambda_2 = \lambda_3 = 0.5$$

For this problem, the basic information is given in terms of the forces and so the stresses will be established by using the connection between them, the tractions, and the stresses.

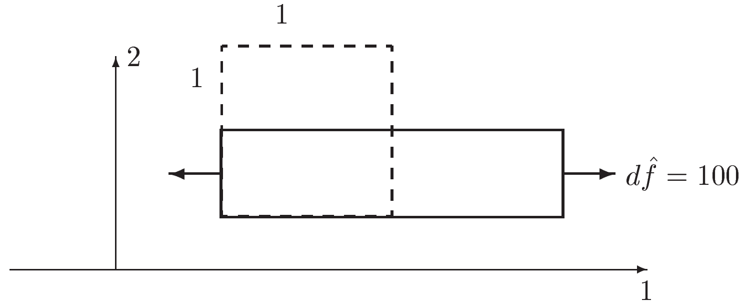


Figure 3.7: Cube with uniaxial load.

The deformation gradients are given by

$$[\frac{\partial x_i}{\partial x_p^o}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}, \qquad [\frac{\partial x_p^o}{\partial x_i}] = [\frac{\partial x_i}{\partial x_p^o}]^{-1} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The Jacobian is therefore

$$J^o = 2 \times .5 \times .5 = .5$$

which shows that there is a volume change.

I: Cauchy Stress

The Cauchy stress tensor is obtained from information about the traction vectors

$$t_i = \sigma_{ji} n_j \equiv \frac{df_i}{dA} = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3$$

On the x_1-face $\hat{n} = \hat{e}_1$, giving $t_i = \sigma_{1i}$, and the components of force and area are therefore

$$df_i = \{100,\ 0,\ 0\} : \qquad dA_1 = \lambda_2 \lambda_3 = 0.5 \times 0.5 = 0.25\,, \quad dA_2 = 0\,, \quad dA_3 = 0$$

On the x_2-face and x_3-face, we have $t_i = \sigma_{2i}$, and $t_i = \sigma_{3i}$, respectively, and in both cases

$$df_i = \{0,\ 0,\ 0\}$$

Thus, for the respective faces, we have

$$\begin{aligned} \hat{n}^{(1)} &= \{1,0,0\} : \qquad \sigma_{11} = \frac{df_1}{dA_1} = \frac{100}{.25} = 400, \quad \sigma_{12} = \sigma_{13} = 0 \\ \hat{n}^{(2)} &= \{0,1,0\} : \qquad \sigma_{21} = \sigma_{23} = \sigma_{22} = 0 \\ \hat{n}^{(3)} &= \{0,0,1\} : \qquad \sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \end{aligned}$$

In summary, the components of the stress tensor are

$$[\sigma_{ij}] = \begin{bmatrix} 400 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The components of the other stress definitions will be obtained by using this and the deformation gradients.

II: Lagrange Stress

Convert the Cauchy stress to Lagrangian stress by

$$\sigma^L_{pi} = \frac{\rho^o}{\rho}\sigma_{ij}\frac{\partial x^o_p}{\partial x_j} = \frac{\rho^o}{\rho}\left[\sigma_{i1}\frac{\partial x^o_p}{\partial x_1} + \sigma_{i2}\frac{\partial x^o_p}{\partial x_2} + \sigma_{i3}\frac{\partial x^o_p}{\partial x_3}\right]$$

Since $J^o = \rho^o/\rho = \frac{1}{2}$ then the components of Lagrange stress are now obtained by substitution in the above expression. For example,

$$\begin{aligned} \sigma^L_{11} &= \tfrac{1}{2}[400\,\tfrac{1}{2} + 0 \times 0 + 0 \times 0] = 100 \\ \sigma^L_{12} &= \tfrac{1}{2}[0\,\tfrac{1}{2} + 0 \times 0 + 0 \times 0] = 0 \end{aligned}$$

In summary,

$$[\sigma^L_{pi}] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that its magnitude is significantly different from that of the Cauchy stress.

III: Kirchhoff Stress

Convert the Cauchy stress to Kirchhoff stress by

$$\begin{aligned} \sigma^K_{pq} &= \frac{\rho^o}{\rho}\sigma_{ij}\frac{\partial x^o_p}{\partial x_i}\frac{\partial x^o_q}{\partial x_j} \\ &= J^o\Big[\sigma_{11}\frac{\partial x^o_p}{\partial x_1}\frac{\partial x^o_q}{\partial x_1} + \sigma_{12}\frac{\partial x^o_p}{\partial x_1}\frac{\partial x^o_q}{\partial x_2} + \sigma_{13}\frac{\partial x^o_p}{\partial x_1}\frac{\partial x^o_q}{\partial x_3} \\ &\quad + \sigma_{21}\frac{\partial x^o_p}{\partial x_2}\frac{\partial x^o_q}{\partial x_1} + \sigma_{22}\frac{\partial x^o_p}{\partial x_2}\frac{\partial x^o_q}{\partial x_2} + \sigma_{23}\frac{\partial x^o_p}{\partial x_2}\frac{\partial x^o_q}{\partial x_3} \\ &\quad + \sigma_{31}\frac{\partial x^o_p}{\partial x_3}\frac{\partial x^o_q}{\partial x_1} + \sigma_{32}\frac{\partial x^o_p}{\partial x_3}\frac{\partial x^o_q}{\partial x_2} + \sigma_{33}\frac{\partial x^o_p}{\partial x_3}\frac{\partial x^o_q}{\partial x_3}\Big] \end{aligned}$$

Since only $\sigma_{11} \neq 0$, we have simply

$$\sigma^K_{pq} = J^o\sigma_{11}\frac{\partial x^o_p}{\partial x_1}\frac{\partial x^o_q}{\partial x_1}$$

and this gives, for instance,

$$
\begin{aligned}
\sigma_{11}^K &= \tfrac{1}{2}(400)(\tfrac{1}{2})^2 = 50 \\
\sigma_{22}^K &= \tfrac{1}{2}(400)(0)^2 = 0 \\
\sigma_{33}^K &= \tfrac{1}{2}(400)(0)^2 = 0
\end{aligned}
$$

In summary,

$$
[\sigma_{pq}^K] = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

Since the original area is $dA^o = 1$ and the deformed area is $dA = 0.25$ then the Cauchy stress has the interpretation of force divided by deformed area while the Lagrange stress has the interpretation of force divided by original area. The Kirchhoff stress is the distorted force df_i^o divided by the original area $dA^o = 1$.

Simple Shear

Consider the simple shear deformation given by

$$
x_1 = x_1^o + kx_2^o\,, \qquad x_2 = x_2^o\,, \qquad x_3 = x_3^o
$$

and let the material have the following simple constitutive behavior

$$
\sigma_{ij} = 2\mu e_{ij}
$$

where μ is a modulus. We now wish to obtain the Lagrange and Kirchhoff stresses.

I: Cauchy Stress

The Cauchy stress will be obtained by use of the constitutive relation, and the others will then be obtained by transformations of it. The deformation gradients are

$$
[\frac{\partial x_p}{\partial x_i^o}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\frac{\partial x_p^o}{\partial x_i}] = [\frac{\partial x_p}{\partial x_i^o}]^{-1} = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Note that there is no volume change since $J = J_o = 1$. The Eulerian and Lagrangian strain tensors are, respectively,

$$
2e_{ij} = \delta_{ij} - \frac{\partial x_p^o}{\partial x_i}\frac{\partial x_p^o}{\partial x_j} = \begin{bmatrix} 0 & k & 0 \\ k & -k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad 2E_{ij} = \frac{\partial x_p}{\partial x_i^o}\frac{\partial x_p}{\partial x_j^o} - \delta_{ij} = \begin{bmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

The Cauchy stress tensor, therefore, is

$$
\sigma_{ij} = \begin{bmatrix} 0 & \mu k & 0 \\ \mu k & -\mu k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mu k \begin{bmatrix} 0 & 1 & 0 \\ 1 & -k & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

The negative σ_{22} component may seem counter-intuitive but note that it is inherited directly from the e_{22} component of strain.

II: Lagrange Stress

The Lagrangian stresses are obtained from

$$\sigma^L_{pi} = \frac{\rho^o}{\rho}\sigma_{ij}\frac{\partial x^o_p}{\partial x_j} = \sigma_{i1}\frac{\partial x^o_p}{\partial x_1} + \sigma_{i2}\frac{\partial x^o_p}{\partial x_2}$$

giving, for example,

$$\begin{aligned}
\sigma^L_{11} &= \sigma_{11}\frac{\partial x^o_1}{\partial x_1} + \sigma_{12}\frac{\partial x^o_1}{\partial x_2} = 0 + (\mu k)(-k) = -\mu k^2 \\
\sigma^L_{12} &= \sigma_{21}\frac{\partial x^o_1}{\partial x_1} + \sigma_{22}\frac{\partial x^o_1}{\partial x_2} = (\mu k)(1) + (-\mu k^2)(-k) = \mu k + \mu k^3 \\
\sigma^L_{21} &= \sigma_{11}\frac{\partial x^0_2}{\partial x_1} + \sigma_{12}\frac{\partial x^o_2}{\partial x_2} = 0 + (\mu k)(1) = \mu k
\end{aligned}$$

The complete stress tensor is

$$\sigma^L_{pi} = \mu k \begin{bmatrix} -k & 1+k^2 & 0 \\ 1 & -k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that this stress tensor is not symmetric. It is also worth noting that in contrast to the Cauchy stress, this tensor has a non-zero component σ^L_{11}.

III: Kirchhoff Stress

The Kirchhoff stresses are obtained from

$$\begin{aligned}
\sigma^K_{pq} &= \frac{\rho^o}{\rho}\sigma_{ij}\frac{\partial x^o_p}{\partial x_i}\frac{\partial x^o_q}{\partial x_j} \\
&= \sigma_{12}\left[\frac{\partial x^o_p}{\partial x_1}\frac{\partial x^o_q}{\partial x_2} + \frac{\partial x^o_p}{\partial x_2}\frac{\partial x^o_q}{\partial x_1}\right] + \sigma_{22}\left[\frac{\partial x^o_p}{\partial x_2}\frac{\partial x^o_q}{\partial x_2}\right]
\end{aligned}$$

Some particular evaluations are

$$\begin{aligned}
\sigma^K_{11} &= \mu k\,[(1)(-k) + (-k)(1)] + (-\mu k^2)(k^2) = -2\mu k^2 - \mu k^4 \\
\sigma^K_{12} &= \mu k\,[(1)(1) + (-k)(0)] + (-\mu k^2)(1)(-k) = \mu k + \mu k^3
\end{aligned}$$

The complete stress tensor is

$$\sigma^K_{pq} = \mu k \begin{bmatrix} -2k-k^3 & 1+k^2 & 0 \\ 1+k^2 & -k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the Kirchhoff stress tensor is symmetric in contrast to the Lagrangian stress tensor.

The magnitude of the shear deformation is governed by the parameter k. It is worth noting that when it is small, all three stress tensors approach the same values. Another point worth noting is that the simple constitutive relation $\sigma_{ij} = 2\mu e_{ij}$ in the Eulerian variables does not lead to an analogous simple relation between σ^K_{ij} and E_{ij}.

Meaning of Forces in Simple Shear

Continuing the discussion of the simple shear deformation problem, it is of interest to investigate the nature of the forces present and the areas they act on during a simple shear. The approach taken is to use the definition of the traction vector in terms of forces and areas, and the relation between it and the various stress components.

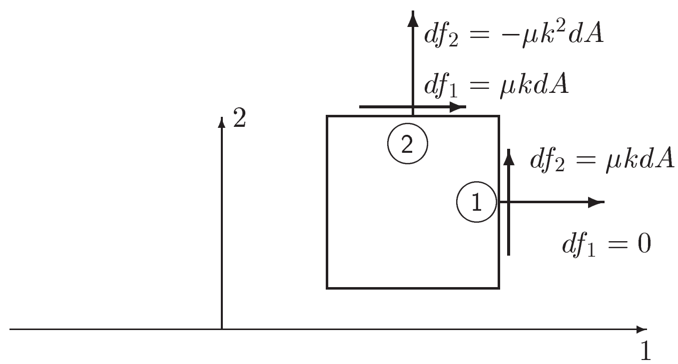


Figure 3.8: Cauchy forces acting on the deformed areas.

For the Cauchy stress, in general,

$$t_i = \sigma_{ji} n_j = \frac{df_i}{dA}$$

On face 1, $\hat{n} = \{1, 0, 0\}$ so that

$$\frac{df_i}{dA} = \sigma_{1i} n_1 + \sigma_{2i} n_2 + \sigma_{3i} n_3 = \sigma_{1i}$$

giving the various tractions as

$$\frac{df_1}{dA} = \sigma_{11} = 0\,, \qquad \frac{df_2}{dA} = \sigma_{12} = \mu k\,, \qquad \frac{df_3}{dA} = \sigma_{13} = 0$$

The forces themselves are

$$df_1 = 0\,, \qquad df_2 = \mu k\, dA\,, \qquad df_3 = 0$$

On the other face, $\hat{n} = \{0, 1, 0\}$, giving the tractions as

$$\frac{df_1}{dA} = \mu k\,, \qquad \frac{df_2}{dA} = -\mu k^2\,, \qquad \frac{df_3}{dA} = 0$$

and the forces as

$$df_1 = \mu k\, dA\,, \qquad df_2 = -\mu k^2\, dA\,, \qquad df_3 = 0$$

We interpret df_2 as a compressive force needed to keep the 2-face from deflecting in the 2-direction. The forces acting on the faces of the deformed body is shown in Figure 3.8.

We have for the Lagrange traction vector,

$$t_i^L = \sigma_{pi}^L n_p^o = \frac{df_i}{dA^o}$$

To use this, it is first necessary to determine the orientation of the faces and this is done easily by considering the deformation of the areas. Recall that the deformed and undeformed areas are related by

$$n_p^o dA^o = \frac{\rho}{\rho^o}\frac{\partial x_i}{\partial x_p^o} n_i \, dA$$

On face 1, $\hat{n} = \{1, 0, 0\}$ so that

$$n_p^o dA^o = \frac{\partial x_1}{\partial x_p^o} dA = \{1, k, 0\} dA$$

Consequently, the direction and magnitude of the undeformed area are, respectively,

$$n_p^o = \left\{ \frac{1}{\sqrt{1+k^2}} \, , \, \frac{k}{\sqrt{1+k^2}} \, , \, 0 \right\} \quad \text{and} \quad dA^o = \sqrt{1+k^2}\, dA$$

On face 2, $\hat{n} = \{0, 1, 0\}$ so that, as above,

$$n_p^o dA^o = \frac{\partial x_2}{\partial x_p^o} dA = \{0, 1, 0\} dA$$

giving the direction and magnitude as

$$n_p^o = \{0, 1, 0\} \qquad\qquad dA^o = dA$$

Now that we know the areas, the forces can be obtained. For example, on face 1

$$\frac{df_i}{dA^o} = \sigma_{1i}^L n_1^o + \sigma_{2i}^L n_2^o + \sigma_{3i}^L n_3^o$$

giving

$$\begin{aligned} \frac{df_1}{dA^o} &= -\mu k^2 \frac{1}{\sqrt{1+k^2}} - \mu k \frac{k}{\sqrt{1+k^2}} + 0 = 0 \\ \frac{df_2}{dA^o} &= \mu(1+k^2)\frac{k}{\sqrt{1+k^2}} - \mu k^2 \frac{k}{\sqrt{1+k^2}} + 0 = \frac{\mu k}{\sqrt{1+k^2}} \end{aligned}$$

Therefore, the forces themselves are

$$\begin{aligned} df_1 &= 0 \\ df_2 &= \frac{\mu k}{\sqrt{1+k^2}} dA^o = \frac{\mu k}{\sqrt{1+k^2}}\sqrt{1+k^2}\, dA = \mu k \, dA \end{aligned}$$

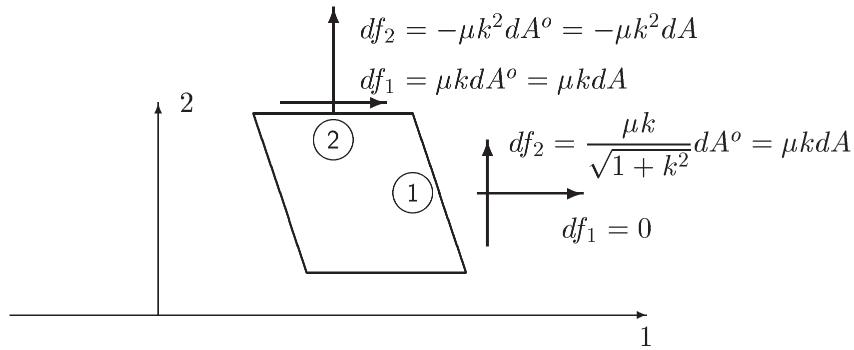


Figure 3.9: Lagrange forces acting on the undeformed areas.

Similarly, on face 2

$$\frac{df_i}{dA^o} = \sigma^L_{2i}$$

giving for the tractions and forces

$$\begin{aligned} \frac{df_1}{dA^o} &= \mu k \qquad \text{or} \qquad df_1 = \mu k dA^o = \mu k dA \\ \frac{df_2}{dA^o} &= -\mu k^2 \qquad \text{or} \qquad df_2 = -\mu k^2 dA^o = -\mu k^2 dA \end{aligned}$$

The forces on both faces are the same as for the Cauchy stress; this is expected from the definition of Lagrangian stress. Note that the components df_i on the 1-face are with respect to the coordinate directions and not with respect to the rotated face.

The traction vector associated with the Kirchhoff stress is

$$t^K_i = \sigma^K_{pi} n^o_p = \frac{df^o_i}{dA^o}$$

The orientation of the faces in the undeformed configuration is the same as obtained above in the Lagrange analysis. Hence on face 1

$$\frac{df^o_i}{dA^o} = \sigma^K_{1i} n^o_1 + \sigma^K_{2i} n^o_2 + \sigma^K_{3i} n^o_3$$

giving

$$\begin{aligned} \frac{df^o_1}{dA^o} &= -\mu k^2(2+k^2)\frac{1}{\sqrt{1+k^2}} + \mu k(1+k^2)\frac{k}{\sqrt{1+k^2}} + 0 = -\frac{\mu k^2}{\sqrt{1+k^2}} \\ \frac{df^o_2}{dA^o} &= \mu k(1+k^2)\frac{1}{\sqrt{1+k^2}} - \mu k^2 \frac{k}{\sqrt{1+k^2}} + 0 = \frac{\mu k}{\sqrt{1+k^2}} \end{aligned}$$

Therefore, the forces can be written as

$$\begin{aligned} df^o_1 &= \frac{-\mu k^2}{\sqrt{1+k^2}} dA^o = -\mu k^2 dA \\ df^o_2 &= \frac{\mu k}{\sqrt{1+k^2}} dA^o = \mu k dA \end{aligned}$$

On face 2,

$$\frac{df_i^o}{dA^o} = \sigma_{2i}^K$$

giving

$$\begin{aligned}
\frac{df_1^o}{dA^o} &= \mu k(1+k^2) \qquad \text{or} \qquad df_1^o = \mu k(1+k^2)dA^o = \mu k(1+k^2)dA \\
\frac{df_2^o}{dA^o} &= -\mu k^2 \qquad \text{or} \qquad df_2^o = -\mu k^2 dA^o = -\mu k^2 dA
\end{aligned}$$

It is seen that the 1 and 2 face forces (as a system) are different from both the Cauchy and Lagrange systems.

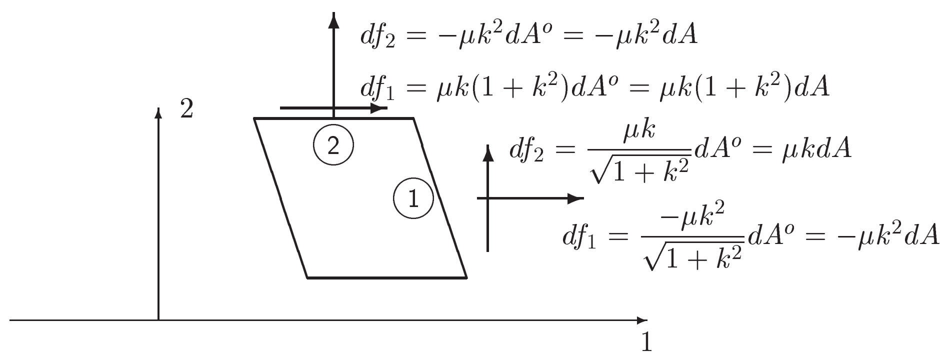


Figure 3.10: Kirchhoff forces acting on the undeformed areas.

Note that the components df_i on the 1-face are with respect to the coordinate directions and not with respect to the rotated face. The normal and tangential forces on the rotated face are

$$\begin{aligned}
df_n^o &= df_1^o n_1^o + df_2^o n_2^o = 0 \\
df_t^o &= -df_1^o n_2^o + df_2^o n_1^o = \mu k dA^o = \mu k\sqrt{1+k^2}dA
\end{aligned}$$

This is analogous to the Cauchy system in that there is a zero normal component and a tangential component that is $\mu k\times$ area.

Stresses in Bending

Consider the following plane inhomogeneous deformation

$$x_1 = [R - x_2^o]\sin(x_1^o/R)\,, \qquad x_2 = R - [R - x_2^o]\cos(x_1^o/R)\,, \qquad x_3 = x_3^o$$

where R is a positive parameter. Let the constitutive behavior be given by the linear relation

$$\sigma_{ij}^K = 2G\,E_{ij} + \lambda\delta_{ij}E_{kk}$$

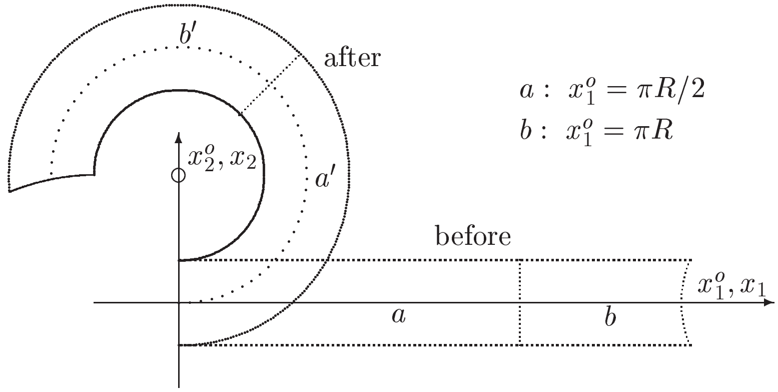


Figure 3.11: Shape before and after deformation.

where G and λ are moduli. We are interested in determining the Cauchy stress.

This deformation is shown in Figure 3.11. Note that initially horizontal lines become arcs of concentric circles, while initially vertical lines become radial lines emanating from a common point whose location changes with the deformation. The deformation gradient is given by

$$\left[\frac{\partial x_i}{\partial x_j^o}\right] = \begin{bmatrix} (R - x_2^o)\cos(x_1^o/R)/R & -\sin(x_1^o/R) & 0 \\ (R - x_2^o)\sin(x_1^o/R)/R & \cos(x_1^o/R) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Jacobian is the determinant of the deformation gradient matrix and on multiplication simplifies to

$$J^o = 1 - \frac{x_2^o}{R}$$

We note that as long as $x_2^o < R$ that the volume remains positive.

The Lagrangian strains are

$$E_{11} = -\frac{x_2^o}{R} + \frac{1}{2}(\frac{x_2^o}{R})^2 , \qquad \text{others:} \quad E_{ij} = 0$$

The Kirchhoff stresses are therefore

$$\sigma_{11}^K = (2G + \lambda)E_{11} , \qquad \sigma_{22}^K = \lambda E_{11} , \qquad \text{others:} \quad \sigma_{ij}^K = 0$$

We can think of these stresses as acting on the undeformed configuration. The tractions on a surface $x_1^o =$ constant are parabolic and independent of the position x_1^o; in the limit of small x_2^o, however, they have the familiar linear distribution of a beam in bending. This situation therefore resembles a beam in pure bending. The stress σ_{22}^K would give rise to a normal traction on the lateral surface; what this implies is that the given deformation could be achieved only with the aid of additional tractions on the lateral surfaces.

Since σ_{11}^K and σ_{22}^K are the only non-zero stresses, we have for the Cauchy stresses

$$\sigma_{ji} = \frac{\rho}{\rho_o}\Big[\frac{\partial x_i}{\partial x_1^o}\frac{\partial x_j}{\partial x_1^o}\sigma_{11}^K + \frac{\partial x_i}{\partial x_2^o}\frac{\partial x_j}{\partial x_2^o}\sigma_{22}^K\Big]$$

This leads to three non-zero components of the Cauchy stress

$$\begin{aligned}
\sigma_{11} &= \frac{\rho}{\rho_o}\left[\frac{1}{R^2}(R-x_2^o)^2C^2\sigma_{11}^K + S^2\sigma_{22}^K\right] = J^oC^2\sigma_{11}^K + \frac{1}{J^o}S^2\sigma_{22}^K \\
\sigma_{22} &= \frac{\rho}{\rho_o}\left[\frac{1}{R^2}(R-x_2^o)^2S^2\sigma_{11}^K + C^2\sigma_{22}^K\right] = J^oS^2\sigma_{11}^K + \frac{1}{J^o}C^2\sigma_{22}^K \\
\sigma_{12} &= \frac{\rho}{\rho_o}\left[\frac{1}{R^2}(R-x_2^o)^2SC\sigma_{11}^K - CS\sigma_{22}^K\right] = J^oCS\sigma_{11}^K - \frac{1}{J^o}CS\sigma_{22}^K
\end{aligned}$$

where $C \equiv \cos(x_1^o/R)$ and $S \equiv \sin(x_1^o/R)$. These stresses exhibit a rather complex dependence on both x_1^o and x_2^o.

The presence of the non-zero shear stress is, perhaps, a bit surprising. Keep in mind, however, as x_1^o is changed, that the components σ_{ij} are not necessarily oriented with respect to the deformed cross-section. It is instructive, therefore, to consider the components of the Cauchy stress with respect to the deformed cross-section. Consider a n area whose normal is initially horizontal, then after deformation the normal has the orientation

$$n_1 = \cos(x_1^o/R) = C\,, \qquad n_2 = \sin(x_1^o/R) = S$$

We now transform the stress components to get

$$\begin{aligned}
\sigma_{nn} &= \sigma_{11}C^2 + \sigma_{22}S^2 + 2\sigma_{12}CS = J^o\sigma_{11}^K \\
\sigma_{tt} &= \sigma_{11}S^2 + \sigma_{22}C^2 - 2\sigma_{12}CS = \frac{1}{J^o}\sigma_{22}^K \\
\sigma_{tn} &= -(\sigma_{11} - \sigma_{22})CS + \sigma_{12}(C^2 - S^2) = 0
\end{aligned}$$

Thus, the Cauchy stress components with respect to line preserving orientations show a close connection to the Kirchhoff stress. Indeed, if we consider the case when $x_2^o \ll R$ (that is, it is like a very slender beam) but we still allow the large deflections, then we get

$$J^o = \frac{\rho}{\rho_o} = \frac{1}{R}(R - x_2^o) \approx 1$$

leading to

$$\sigma_{nn} \approx \sigma_{11}^K\,, \qquad \sigma_{tt} \approx \sigma_{22}^K$$

There are many practical problems where the deflections are large but the strains small, under those circumstances it is found useful to invoke the above approximate relation between the Kirchhoff and Cauchy stresses.

Exercises

3.1 Given that $t_i = \sigma_{ji} n_j$ where t_i and n_j are first order tensors, prove that σ_{ij} is a second order tensor.

3.2 The concept of 'complementary shear' says that the shear stresses on two perpendicular faces are equal. Given that $t_i = \sigma_{ji} n_j$ with σ_{ij} being symmetric, show that the tractions on two arbitrary faces passing through a common point are related by

$$t_i^{(1)} n_i^{(2)} = t_i^{(2)} n_i^{(1)}$$

3.3 The normal component of the traction vector is given by

$$\sigma_N = t_i n_i = \sigma_{ji} n_j n_i$$

Find the directions that make this component an extremum. [Note that the normal vector must satisfy $n_1^2 + n_2^2 + n_3^2 = 1$.]

3.4 Determine the principal values and their orientations for the stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

3.5 Consider the following components of a stress tensor

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine the components of the traction vector with respect to an area rotated θ about the x_3 axis. Determine the components of stress transformed an angle θ about the same axis. How do the above compare or are they related ?

3.6 Under what circumstances (if any) is the following symmetric stress field in static equilibrium ?

$$\sigma_{11} = 3x_1 + g_2(x_2)\,, \qquad \sigma_{22} = 4x_2 + g_1(x_1)\,, \qquad \sigma_{12} = \sigma_{21} = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1x_2$$

3.7 Under what circumstances (if any) is the following symmetric stress field in static equilibrium with no body forces ?

$$\sigma_{11} = 3x_1\,, \qquad \sigma_{22} = 4x_2\,, \qquad \sigma_{12} = Ax_1 + cx_2$$

Establish the tractions on the three sides of the triangular body $(0,0)$, $(1,0)$, $(0,1)$. Is it in overall equilibrium ?

3.8 A circular bar in torsion has a symmetric stress field given by

$$\sigma_{13} = -Gx_2\,, \qquad \sigma_{23} = Gx_1$$

with all others being zero. Are the stresses in equilibrium? Determine the tractions on a boundary surface whose normal is

$$\hat{n} = \{\cos\theta,\, \sin\theta,\, 0\}$$

3.9 A stress field is given by

$$\sigma_{11} = \frac{-2F}{\pi h}\frac{x_1^2 x_2}{r^4}\,, \qquad \sigma_{22} = \frac{-2F}{\pi h}\frac{x_2^3}{r^4}\,, \qquad \sigma_{12} = \frac{-2F}{\pi h}\frac{x_1 x_2^2}{r^4} = \sigma_{21}$$

with all others being zero and $r^2 = x_1^2 + x_2^2$. Are the stresses in equilibrium? Determine the tractions along a line $x_2 = \alpha x_1$, where α is a parameter. What are the tractions along a curve $x_1^2 + x_2^2 = R^2 =$ constant.

3.10 The stress in a continuum is given as

$$[\sigma_{ij}] = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{bmatrix}$$

What form must the body forces take if equilibrium is to be satisfied everywhere?

3.11 Consider the simple shear deformation

$$x_1 = x_1^o + kx_2^o\,, \qquad x_2 = x_2^o\,, \qquad x_3 = x_3^o$$

and the constitutive behavior

$$\sigma_{ij}^K = 2G\,E_{ij} + \lambda\delta_{ij}E_{kk}$$

Determine the Lagrangian stress and Cauchy stress. Investigate the forces and the areas they act on.

3.12 Consider the deformation

$$x_1 = [R - x_2^o]\sin(x_1^o/R)\,, \qquad x_2 = R - [R - x_2^o]\cos(x_1^o/R)\,, \qquad x_3 = x_3^o$$

and the constitutive behavior

$$\sigma_{ij}^K = 2G\,E_{ij} + \lambda\delta_{ij}E_{kk}$$

Determine the Lagrangian stress and Cauchy stress. Investigate the forces and the areas they act on.

3.13 A rigid block has a Cauchy stress σ_{11} only acting on it. If the block is given a rigid body rotation about the x_3 axis (such that σ_{11} moves with it), what are the new Cauchy stresses? Determine the components of the Kirchhoff stress before and after the rotation. Show that the Lagrangian strain tensor is also invariant to the same rigid body rotation.

Chapter 4

Elastic Materials

Elasticity deals with a particular set of assumptions about the material behavior. Some of the considerations are the following:

- The stress at a point depends on geometric changes that take place only in its immediate vicinity. This is really a question of scale.
- There are no history effects — the present state of strain gives the present stress. This rules out, for example, plasticity effects.
- There is an instant recovery of the original shape when the forces are removed. This rules out creep, viscoelasticity, and other rate dependent effects.
- Temperature changes only cause a change in volume but otherwise do not directly affect the material parameters.

Two other material properties of relevance are *Inhomogeneity* and *Anisotropy*. A material is inhomogeneous when the properties vary with location, whereas a material is anisotropic when the properties vary with the coordinate reference frame. A special case of the latter of considerable interest is that of *isotropy* (no directional dependence) because many engineering materials of interest are approximated quite adequately by it.

4.1 Work and Strain Energy Concepts

The concepts of Cauchy stress and Lagrangian strain can be developed independently of each other. A constitutive relation involves both of them simultaneously. The concepts of work and energy also involve stress and strain simultaneously so we find it useful to begin our discussion of constitutive theories with a discussion of some work and energy concepts; these will provide a relevant setting for the rational construction of constitutive theories.

Work

Consider a typical force-deflection or stress-strain curve as shown in Figure 6.9. The elasticity requirement is that both the loading and unloading paths coincide.

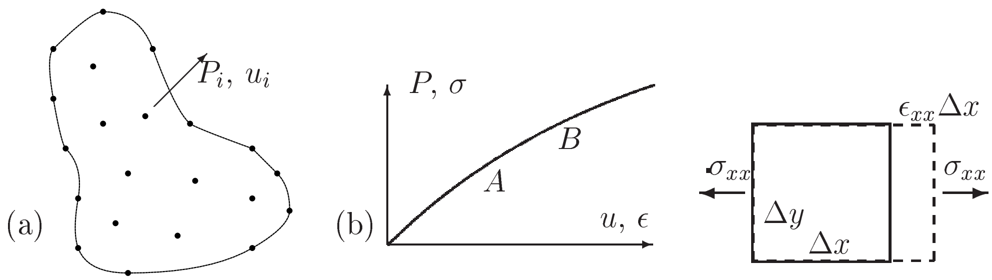


Figure 4.1: Typical elastic behavior. (a) Discretized arbitrary body. (b) Force-displacement behavior. (c) Stressed infinitesimal element showing strain.

The work done by a force at a point is the vector dot product of the force and the displacement at the point. For example, in terms of our global coordinate system,

$$d\mathcal{W} = \hat{P} \cdot d\hat{u} = P_x du + P_y dv + P_z dw$$

where the 'hat' indicates a vector. The force is understood to be constant during the infinitesimal displacement $d\hat{u}$. When the force moves along a path from State A to State B, the work done is

$$\Delta\mathcal{W} = \int_A^B d\mathcal{W} = \int_A^B \hat{P} \cdot d\hat{u} = \int_A^B (P_x du + P_y dv + P_z dw)$$

The loading curves of Figure 4.1 can be interpreted as a sequence of possible equilibrium states as the load (or deflection) is changed. Thus, A and B are two possible equilibrium states with different load conditions. We use the symbol Δ to signify that an *increment* of work is performed in moving from one state to the other — the change in configuration may be small or large. In the special case when the initial configuration is the unstressed, unstrained, virgin state we will simply use $\mathcal{W}$ without the Δ for the work done in reaching a certain state.

The systems we are interested in have multiple forces and moments, so we will generalize the above expression for work to

$$\Delta\mathcal{W} = \sum_i \int_A^B P_i du_i = \int_A^B \{P\}^T \{du\} \tag{4.1}$$

It is understood that P_i and u_i are common (pointing in the same direction) components of generalized forces and displacements, respectively. Thus the individual contribution to the work could refer to forces and displacements $P_3\, du_3$ or torques and twists $T_9\, d\phi_9$, and so on.

Strain Energy

Consider an arbitrary sub-volume V of a body, and let it have displacement increments du_i, then the external work increment due to the tractions t_i and body force b_i is

$$d\mathcal{W}_e = \int_A t_i du_i dA + \int_V \rho b_i du_i dV$$

But $t_i = \sigma_{ij} n_j$, so substitute and use the integral theorem and equilibrium conditions to get

$$d\mathcal{W}_e = \int_V \sigma_{ij} d\Big(\frac{\partial u_i}{\partial x_j}\Big) dV = \int_V \sigma_{ij} d\,(\epsilon_{ij} + \omega_{ij})\, dV$$

Recalling that the contraction of a symmetric and antisymmetric tensor is zero, then the term with ω_{ij} disappears giving

$$d\mathcal{W}_e = \int_V \sigma_{ij} d\epsilon_{ij} dV = d\mathcal{U}$$

It is interesting that the work is the product of the Cauchy stress and the small strain increment and not the Eulerian strain increment as perhaps anticipated. The term $\mathcal{U}$ is the strain energy so that the above relation is the formal equivalence between the external work done and the internal stored energy.

We would also like to write the work expression in terms of the undeformed configuration. Following developments similar to the example in Section 2.5 leading to Equation (2.11), and using the relation between Cauchy and Kirchhoff stress,

$$d\epsilon_{mn} = \frac{\partial x_i^o}{\partial x_m}\frac{\partial x_j^o}{\partial x_n} dE_{ij}\,, \qquad \sigma_{mn} = \frac{\rho}{\rho^o}\frac{\partial x_m}{\partial x_i^o}\frac{\partial x_n}{\partial x_j^o}\sigma_{ij}^K$$

and recalling that the deformed and undeformed volumes are related by $dV = dV^o \rho^o/\rho$, then the internal work term becomes

$$d\mathcal{U} = \int_{V^o} \sigma_{pq}^K \, dE_{pq} \, dV^o$$

Hence the Cauchy stress / Eulerian (small) strain combination is energetically equivalent to the Kirchhoff stress / Lagrangian strain combination.

Example 4.1: Show the explicit relation between the strain energy and the components of stress and strain.

Using the Lagrangian variables, we have

$$d\mathcal{U} = \int_{V^o} \Big[\sigma_{xx}^K \, dE_{xx} + \sigma_{yy}^K \, dE_{yy} + \sigma_{xy}^K 2\, dE_{xy} + \cdots\Big] \, dV^o$$

Note that there are no products such as $\sigma_{xx}^K \, dE_{yy}$ or $\sigma_{xx}^K \, dE_{xy}$ and that the shear stress multiplies twice the shear strain. ■

4.2 Elastic Constitutive Relations

The mathematical description of material behavior and its reactions to applied loads is called a *constitutive relation.* There are a number of restrictions on the allowable form of constitutive relations and Reference [22] gives a clear explanation. A primary concept is that of material objectivity and we begin with a discussion of this concept.

The deformation situations that arise in practice are usually of the type of cases (b) and (c) of Figure 2.13 (large rotations but small strains) and we utilize this to make some useful approximations. A second class of elastic materials is represented by the rubber-like materials; these generally exhibit very large strains but are essentially isotropic. We consider these in the next section.

Material Objectivity

It is generally accepted that material properties should be independent of the coordinate frame of the observer. Hence we would like to use entities in the constitutive relation so that the frame independence is guaranteed.

As shown in Chapter 1, first and second order tensors transform according to

$$\{\hat{v}^*\} = [\hat{Q}(t)] \cdot \{\hat{v}\}\,, \qquad [\hat{T}^*] = [\hat{Q}(t)] \cdot [\,\hat{T}\,] \cdot [\hat{Q}(t)]^T \tag{4.2}$$

where $[\,Q\,]$ is an orthogonal tensor that rotates the frame of reference. Requiring this to be true for time dependent rigid motions make the tensors *objective*. Not all tensors are objective; for example, consider the velocity and acceleration obtained from the displacement

$$\begin{aligned}
\{\hat{u}^*\} &= [\,\hat{Q}\,] \cdot \{\hat{u}\} \\
\{\dot{\hat{v}}^*\} &= [\,\hat{Q}\,] \cdot \{\dot{\hat{v}}\} + [\,\dot{\hat{Q}}\,] \cdot \{\hat{u}\} \\
\{\ddot{\hat{v}}^*\} &= [\,\hat{Q}\,] \cdot \{\ddot{\hat{v}}\} + 2[\,\dot{\hat{Q}}\,] \cdot \{\dot{\hat{u}}\} + [\,\ddot{\hat{Q}}\,] \cdot \{\hat{u}\}
\end{aligned}$$

Accordingly, velocity and accelerations should not be used in constitutive relations. Parenthetically, the acceleration is not objective is well known and is the reason for the Coriolis force in mechanics.

To summarize, in essence, material objectivity says that functions and fields whose values are scalars, vectors, or tensors are called *frame indifferent* (or *objective*) if both the dependent and independent vector and tensor variables transform according to Equation (4.2) [22]. Since our concept of elasticity does not include rate effects, then practically it means that the use of Lagrangian strain and Kirchhoff stress in the constitutive relations will automatically satisfy material objectivity.

Hyperelastic Materials

Consider a small volume of material under the action of applied loads on its surface. Then a straight forward assumption about elastic behavior is that:

> *For an elastic body, the stress depends only on deformation and not on the history of the deformation.*

This is expressed as

$$\sigma_{ij}^K = f_{ij}(E_{kl}) \tag{4.3}$$

which means that the nine components of stress are given by nine separate functions of all the strains. Using a Taylor series expansion in terms of strains will then give a rather complicated collection of functions and associated material parameters; such relations are not practical to use. This approach has the further disadvantage that it does not indicate, in a rational way, the terms that can be removed in formulating relations for special materials.

An alternative assumption about elastic behavior is that:

> *The work done by the applied forces is transformed completely into strain (potential) energy, and this strain energy is completely recoverable.*

That the work is transformed into potential energy that is completely recoverable means the material system is conservative. Using material variables (Lagrangian strain and Kirchhoff stress), the increment of work done on the small volume is

$$d\mathcal{W}_e = \int_{V^o} \Big[\sum\nolimits_{i,j} \sigma_{ij}^K dE_{ij}\Big] dV^o$$

The potential is comprised entirely of the strain energy $\mathcal{U}$; the increment of strain energy is

$$d\mathcal{U} = d\mathcal{U}(E_{ij}) = \int_{V^o} \Big[\sum\nolimits_{i,j} \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}} dE_{ij}\Big] dV^o$$

where $\overline{\mathcal{U}}$ is the strain energy density. From the hypothesis, we can equate $d\mathcal{W}_e$ and $d\mathcal{U}$, and because the volume is arbitrary, the integrands must be equal, hence we have

$$\sigma_{ij}^K = \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}} \tag{4.4}$$

A material described by this relation is called *hyperelastic*. Note that it is valid for large deformations and for anisotropic materials; however, rather than develop this general case, we will look at each of these cases separately.

We can recast the above in terms of the deformation tensor instead of the strain tensor — this is the standard formulation for large strains. That is, the constitutive relation is written as

$$\sigma_{ij}^K = \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}} = 2\frac{\partial \overline{\mathcal{U}}}{\partial C_{ij}}, \qquad C_{ij} = \sum\nolimits_m \frac{\partial x_m}{\partial x_i^o}\frac{\partial x_m}{\partial x_j^o} = 2E_{ij} + \delta_{ij} \tag{4.5}$$

where C_{ij} is called the *right Cauchy-Green deformation tensor*. This form is completely equivalent to the one written in terms of the Lagrangian strain but is slightly more convenient because analytical treatments of rubber elasticity, for example, tend to use principal stretches rather than strains.

While Relations (4.4) and (4.5) are simple and elegant, they too would lead to impractical expressions when expanded. As will be demonstrated, however, they afford a rational way to construct special elastic relations.

Example 4.2: The following quantities were recorded during the large deformation testing of a uniaxial specimen:

P, the applied force;

$\epsilon \equiv \Delta L/L_o$, the unit change of axial length;

$\epsilon_t \equiv \Delta W/W_o$, the unit change of transverse width.

Establish the relationships necessary to convert this information to stress and strain.

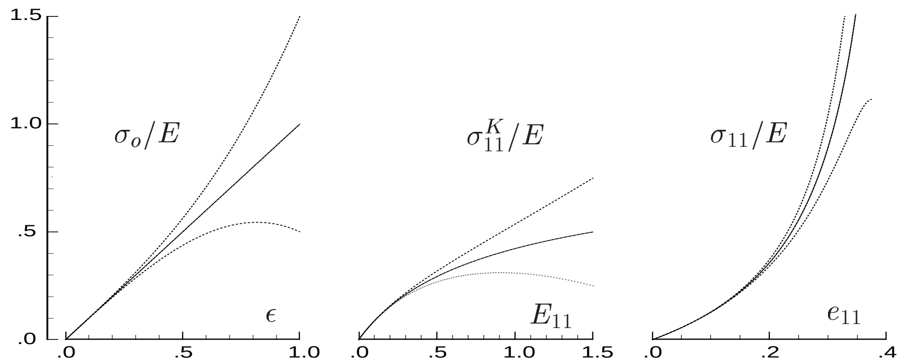


Figure 4.2: The forms of constitutive behavior for the experimental observed behavior. (a) Basic recorded data. (b) Kirchhoff stress against Lagrangian strain. (c) Cauchy stress against Eulerian strain.

Following from the examples of Section 2.3, we have that the stretches are

$$\lambda_1 = 1 + \epsilon\,, \qquad \lambda_2 = 1 + \epsilon_t = 1 - \nu\epsilon\,, \qquad \lambda_3 = 1 + \epsilon_t = 1 - \nu\epsilon = \lambda_2$$

where we have introduced $\nu \equiv -\epsilon_t/\epsilon$ as the ratio of the axial straining to the transverse straining. The Lagrangian and Eulerian strains in the axial direction are

$$E_{11} = \epsilon + \tfrac{1}{2}\epsilon^2\,, \qquad e_{11} = e - \tfrac{1}{2}e^2 = \frac{\epsilon}{(1+\epsilon)^2}[1 + \tfrac{1}{2}\epsilon]$$

The stresses are

$$\sigma_{11}^K = \frac{1}{\lambda_1}\frac{dP}{dA_1^o} = \frac{\sigma_o}{(1+\epsilon)}\,, \qquad \sigma_{11} = \frac{1}{\lambda_2\lambda_3}\frac{dP}{dA_1^o} = \frac{\sigma_o}{(1-\nu\epsilon)^2}\,, \qquad \sigma_o \equiv \frac{dP}{dA_1^o}$$

The stress σ_o can be thought of as the "force over original area," although here it is introduced solely as a normalizing factor.

As shown in Figure 4.2(a), there are three possibilities for the behavior of $\sigma_o = P/A_o$ against $\epsilon \equiv \Delta L/L_o$: it can be concave up indicating hardening, be concave down indicating softening, or be linear. The corresponding stress-strain curves are also shown in Figure 4.2. Note that for the range of nonlinear behaviors shown, all the Kirchhoff stress/Lagrangian strain relations show softening, whereas the Cauchy stress/Eulerian strain show hardening. Therefore, whether a material is physically linear or nonlinear, softening or hardening, is not a definite concept but depends on the measures used for the stress and strain. Of course, the mechanical problem can be objectively nonlinear even though the constitutive relation is linear because the description of the geometry can be nonlinear. ■

Example 4.3: Contrast the physical response of materials described by linear constitutive relations.

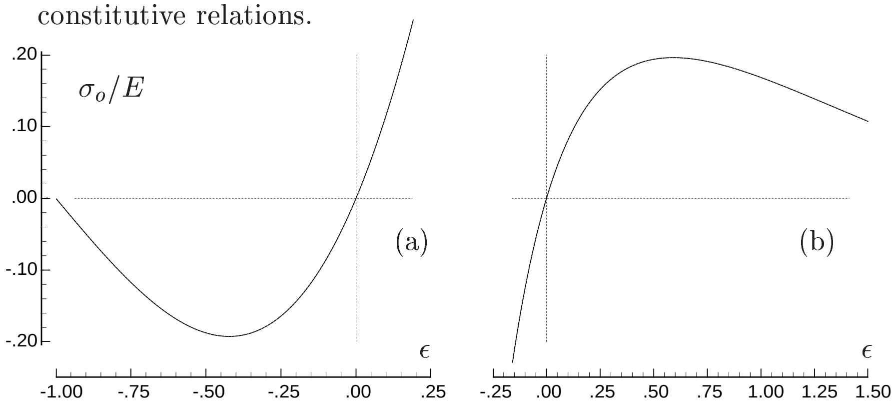


Figure 4.3: Physical responses for linear constitutive relations. (a) Material. (b) Spatial.

Consider the uniaxial constitutive relations

$$\text{material:} \quad \sigma^K_{11} = E E_{11}\,, \qquad \text{spatial:} \quad \sigma_{11} = E e_{11}$$

where, for simplicity, we let the modulus of both materials be the same. Substituting the respective expressions for stress and strain leads to

$$\text{material:} \quad \sigma_o = E\epsilon(1+\epsilon)(1+\tfrac{1}{2}\epsilon)\,, \qquad \text{spatial:} \quad \sigma_o = E\epsilon(\frac{1-\nu\epsilon}{1+\epsilon})^2[1+\tfrac{1}{2}\epsilon]$$

These are shown plotted in Figure 4.3.

There are two obvious implications from Figure 4.3: linear constitutive relations imply highly nonlinear physical behaviors, and the two descriptions are completely different. In addition, both descriptions exhibit instabilities. Consider the spatial description, for example: as the load is increased, a point is reached where further load increments cannot be sustained and large deformations ensue. This is an example of a *limit point* instability. It is worth noting that the instability occurs while the cross-sectional area is still of significant size. The material description exhibits an instability in compression.

In subsequent chapters, we will deal with large displacements and rotations, but relatively small strains. In those cases, we will use a linear constitutive relation and restrict ourselves to strain levels such that $\epsilon < 0.20$; this avoids both instabilities, and all relations can be reasonably approximated as linear. For structural materials such as aluminum and steel, these strain levels would have been associated with gross plastic yielding. The case of rubber elasticity will be given special treatment. ■

Small Strain Elastic Materials

Reinforced materials are likely to have directional properties and are therefore anisotropic. They are also more likely to have small operational strains. Thin-walled structures such as frames and shells are likely to have large displacements and rotations but rather small strains so as to operate without plasticity occurring. We take advantage of this small strain situation to effect a set of material approximations.

Because the strains are assumed small, we can take the Taylor series expansion of the strain energy density function

$$\overline{\mathcal{U}}(E_{ij}) = \overline{\mathcal{U}}(0) + \sum_{i,j} \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}}\Big|_0 E_{ij} + \tfrac{1}{2} \sum_{i,j,p,q} \frac{\partial^2 \overline{\mathcal{U}}}{\partial E_{ij} \partial E_{pq}}\Big|_0 E_{ij} E_{pq} + \cdots$$

By using the Lagrangian strain tensor, the expansion is valid for large deflections and rotations but for small strains. Noting that $\overline{\mathcal{U}}(0) = 0$ and

$$\frac{\partial E_{ij}}{\partial E_{pq}} = \delta_{ip}\delta_{jq}$$

then get

$$\sigma_{ij}^K = \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}} \approx \frac{\partial \overline{\mathcal{U}}}{\partial E_{ij}}\Big|_0 + \sum\nolimits_{r,s} \frac{\partial^2 \overline{\mathcal{U}}}{\partial E_{ij} \partial E_{rs}}\Big|_0 E_{rs} + \cdots = \sigma_{ij}^o + \sum\nolimits_{r,s} D_{ijrs} E_{rs} + \cdots$$

where σ_{ij}^o corresponds to an initial stress. Because of symmetry in σ_{ij}^K and E_{rs}, D_{ijrs} reduces to 36 coefficients. But because of the explicit form of D_{ijrs} in terms of derivatives, we have the further restrictions

$$D_{ijrs} = \frac{\partial^2 \overline{\mathcal{U}}}{\partial E_{ij} \partial E_{rs}}\Big|_0 = \frac{\partial^2 \overline{\mathcal{U}}}{\partial E_{rs} \partial E_{ij}}\Big|_0 = D_{rsij}$$

This additional symmetry reduces the elastic tensor to 21 constants. This is usually considered to be the most general small strain elastic material. Note that the corresponding result starting with Equation (4.3) gives 36 constants.

We can write this relation in the matrix form

$$\{\sigma^K\} = [\, D\,]\{E\}, \quad \{\sigma^K\} \equiv \{\sigma_{11}^K, \sigma_{22}^K, \sigma_{33}^K, \sigma_{12}^K, \cdots\}^T, \quad \{E\} \equiv \{E_{11}, E_{22}, E_{33}, 2E_{12}, \cdots\}^T$$

where $[\, D\,]$ is of size $[6 \times 6]$. Because of the symmetry of both the stress and strain, we have $[\, D\,]^T = [\, D\,]$. Special materials are reduced forms of this relation.

A later section considers material symmetries in detail, here we discuss just a couple of examples. An *orthotropic* material has three planes of symmetry and this reduces the number of material coefficients to nine and the elastic matrix is given by

$$[\, D\,] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(D_{11} - D_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(D_{11} - D_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(D_{11} - D_{12}) \end{bmatrix}$$

For a transversely isotropic material this reduces to five coefficients because $D_{55} = D_{44}$ and $D_{66} = (D_{11} - D_{12})/2$. A thin fiber-reinforced composite sheet is usually considered to be transversely isotropic [17]. A point worth noting is that the material coefficients are given with respect to a particular coordinate system. Hence, we must transform the coefficients into the new coordinate system when the axes are changed.

For the isotropic case, every plane is a plane of symmetry and every axis is an axis of symmetry. It turns out that there are only two independent elastic constants, and the elastic matrix is given as above but with

$$D_{11} = D_{22} = D_{33} = \lambda + 2\mu\,, \qquad D_{12} = D_{23} = D_{13} = \lambda$$

The constants λ and μ are called the Lamé constants. The stress-strain relation for isotropic materials (with no initial stress) are usually expressed in the form

$$\sigma^K_{ij} = 2\mu E_{ij} + \lambda\delta_{ij}\textstyle\sum_k E_{kk}\,, \qquad 2\mu E_{ij} = \sigma^K_{ij} - \dfrac{\lambda}{3\lambda+2\mu}\delta_{ij}\textstyle\sum_k \sigma^K_{kk} \tag{4.6}$$

The small deformation version of this relation is called *Hooke's law* and this version using Lagrangian strain and Kirchhoff stress is sometimes referred to as the *St. Venant-Kirchhoff law* [8]; we will refer to both of them as simply Hooke's law.

The expanded form of the Hooke's law for strains in terms of stresses is

$$\begin{aligned}
E_{xx} &= \frac{1}{E}[\sigma^K_{xx} - \nu(\sigma^K_{yy} + \sigma^K_{zz})] \\
E_{yy} &= \frac{1}{E}[\sigma^K_{yy} - \nu(\sigma^K_{zz} + \sigma^K_{xx})] \\
E_{zz} &= \frac{1}{E}[\sigma^K_{zz} - \nu(\sigma^K_{xx} + \sigma^K_{yy})] \\
2E_{xy} &= \frac{2(1+\nu)}{E}\sigma^K_{xy}\,, \quad 2E_{yz} = \frac{2(1+\nu)}{E}\sigma^K_{yz}\,, \quad 2E_{xz} = \frac{2(1+\nu)}{E}\sigma^K_{xz}
\end{aligned} \tag{4.7}$$

and for stresses in terms of strains

$$\begin{aligned}
\sigma^K_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)E_{xx} + \nu(E_{yy} + E_{zz})] \\
\sigma^K_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)E_{yy} + \nu(E_{zz} + E_{xx})] \\
\sigma^K_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)E_{zz} + \nu(E_{xx} + E_{yy})] \\
\sigma^K_{xy} &= \frac{E}{2(1+\nu)}2E_{xy}\,, \quad \sigma^K_{yz} = \frac{E}{2(1+\nu)}2E_{yz}\,, \quad \sigma^K_{xz} = \frac{E}{2(1+\nu)}2E_{xz}
\end{aligned} \tag{4.8}$$

where E is the Young's modulus and ν is the Poisson's ratio related to the Lamé coefficients by

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\,, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}\,, \quad \lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}\,, \quad \mu = G = \frac{E}{2(1+\nu)}$$

The coefficient $\mu = G$ is called the shear modulus.

Viewing the relation between the normal components of stress and strain as forming a $[3 \times 3]$ matrix of the material parameters, then it can be inverted only if the determinant remains positive. The determinant is

$$\det = (1 - 2\nu)(1 + \nu)^2 > 0$$

hence we conclude that

$$-1 < \nu < 0.5$$

The same conclusion can be drawn by considering the strain energy function [22]. A negative Poisson's ratio would indicate a material that, under uniaxial tension, would expand in the transverse direction; this is possible for some of the structured materials. A Poisson's ratio of 0.5 would indicate an infinite bulk modulus or very little volume change for a given stress level. This is sometimes referred to as *incompressibility*; note, however, that $\nu = 0.5$ is not the incompressibility condition under large deformations as we discuss shortly for rubber-like materials.

Example 4.4: Particularize Hooke's law to the case of plane stress.

A special case that arises in the analysis of thin-walled structures is that of *plane stress*. Here, the stress through the thickness of the plate or shell is approximately zero such that $\sigma^K_{zz} \approx 0$, $\sigma^K_{xz} \approx 0$, and $\sigma^K_{yz} \approx 0$. This leads to

$$E_{zz} = \frac{-\nu}{E}[\sigma^K_{xx} + \sigma^K_{yy}] = \frac{-\nu}{1-\nu}[E_{xx} + E_{yy}]$$

Substituting this into the 3-D Hooke's law then gives

$$\begin{aligned} E_{xx} &= \frac{1}{E}[\sigma^K_{xx} - \nu\sigma^K_{yy}]\,, & \sigma^K_{xx} &= \frac{E}{(1-\nu^2)}[E_{xx} + \nu E_{yy}] \\ E_{yy} &= \frac{1}{E}[\sigma^K_{yy} - \nu\sigma^K_{xx}]\,, & \sigma^K_{yy} &= \frac{E}{(1-\nu^2)}[E_{yy} + \nu E_{xx}] \end{aligned} \tag{4.9}$$

The shear relation is unaffected. ■

Example 4.5: Show how temperature can affect the constitutive relation.

A temperature change can affect the constitutive behavior in two ways: first, it can change the values of the material coefficients; and second, it causes a volumetric expansion. We are only concerned here with the latter effect — this is called *thermoelasticity*. Because the temperature change only causes a volume change, then only the normal strain components are affected and the Hooke's law of Equation (4.7) is modified to

$$\begin{aligned} E_{xx} &= \frac{1}{E}[\sigma^K_{xx} - \nu(\sigma^K_{yy} + \sigma^K_{zz})] + \alpha\Delta T \\ E_{yy} &= \frac{1}{E}[\sigma^K_{yy} - \nu(\sigma^K_{zz} + \sigma^K_{xx})] + \alpha\Delta T \\ E_{zz} &= \frac{1}{E}[\sigma^K_{zz} - \nu(\sigma^K_{xx} + \sigma^K_{yy})] + \alpha\Delta T \end{aligned} \tag{4.10}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change. ■

Example 4.6: Recover the bulk behavior from Hooke's law.

The sum of the normal components of stress and strain are related by

$$\sum\nolimits_k \sigma^K_{kk} = \frac{E}{(1-2\nu)} \sum\nolimits_k E_{kk}$$

Under small deformation situations, the first sum is three times the hydrostatic pressure while the second sum is the volume change, hence we can write this in the form of a bulk pressure-volume relation

$$-p = \frac{E}{3(1-2\nu)} \frac{\Delta V}{V} = K \frac{\Delta V}{V}\,, \qquad K = \frac{E}{3(1-2\nu)} = \frac{3\lambda + 2\mu}{3}$$

where K is called the *bulk modulus*. ■

Example 4.7: Obtain the isotropic Hooke's law on the assumption that D_{ijpq} is an isotropic tensor.

Section 1.3 gives a collection of isotropic tensors. Since D_{ijpq} is fourth order, it must have the form

$$D_{ijpq} = \alpha \delta_{ij} \delta_{pq} + \beta \delta_{ip} \delta_{jq} + \gamma \delta_{iq} \delta_{jp}$$

where the coefficients are constants. Because of the symmetry of the stress and strain tensors, D_{ijpq} must be symmetric in ij and pq which leads to $\gamma = \beta$.

The stress-strain relation then becomes

$$\sigma^K_{ij} = D_{ijpq} E_{pq} = [\alpha \delta_{ij} \delta_{pq} + \beta[\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}]] E_{pq}$$

Perform the contractions using the delta functions and compare with the usual form of Hooke's law using the Lamé parameters

$$\sigma^K_{ij} = \alpha E_{pp} \delta_{ij} + 2\beta E_{ij} = 2\mu E_{ij} + \lambda E_{kk} \delta_{ij}$$

We therefore conclude that the representation of D_{ijpq} is given by

$$D_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu[\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}] \tag{4.11}$$

This is the form that originally motivated the use of the Lamé parameters. ■

Example 4.8: Show how an initial stress state affects the current relation between an increment of stress and an increment of strain for an isotropic material.

Let the initial stress state σ^o_{ij} be associated with the displacement field u^o_i. Furthermore, let the current displacement u_i be represented as

$$u_i = u^o_i + \xi_i$$

where ξ_i is the (small) increment of displacement from the current value of u_i^o. Using this in the strain-displacement relation allows the total strain to be decomposed as

$$\begin{aligned} 2E_{ij} &= \frac{\partial u_i}{\partial x_j^o} + \frac{\partial u_j}{\partial x_i^o} + \sum\nolimits_k \frac{\partial u_k}{\partial x_i^o}\frac{\partial u_k}{\partial x_j^o} \\ &= \Big(\frac{\partial u_i^o}{\partial x_j^o} + \frac{\partial u_j^o}{\partial x_i^o} + \sum\nolimits_k \frac{\partial u_k^o}{\partial x_i^o}\frac{\partial u_k^o}{\partial x_j^o}\Big) \\ &\quad + \Big(\frac{\partial \xi_i}{\partial x_j^o} + \frac{\partial \xi_j}{\partial x_i^o} + \sum\nolimits_k \frac{\partial \xi_k}{\partial x_i^o}\frac{\partial \xi_k}{\partial x_j^o}\Big) + \Big(\sum\nolimits_k \frac{\partial u_k^o}{\partial x_i^o}\frac{\partial \xi_k}{\partial x_j^o} + \sum_k \frac{\partial \xi_k}{\partial x_i^o}\frac{\partial u_k^o}{\partial x_j^o}\Big) \end{aligned}$$

The various collections of terms in parentheses are labeled as follows

$$E_{ij} = E_{ij}^o + \epsilon_{ij} + \eta_{ij}$$

Note that ϵ_{ij} is an increment of strain from the current configuration but referenced to the zero configuration. The interaction term η_{ij} contains components of both u_i^o and ξ_i; this is the term we are especially interested in.

Let the constitutive relation be

$$\sigma_{ij}^K = 2\mu E_{ij} + \lambda\delta_{ij}\sum\nolimits_k E_{kk}\,, \qquad 2\mu E_{ij} = \sigma_{ij}^K - \frac{\lambda}{2\mu+3\lambda}\delta_{ij}\sum\nolimits_k \sigma_{kk}^K$$

Then, after substituting for the strains, the stresses are

$$\sigma_{ij}^K = \sigma_{ij}^o + 2\mu\epsilon_{ij} + \lambda\delta_{ij}\sum\nolimits_k \epsilon_{kk} + 2\mu\eta_{ij} + \lambda\delta_{ij}\sum\nolimits_k \eta_{kk}$$

We are interested in taking derivatives of this stress with respect to ϵ_{pq}. Since

$$\frac{\partial \xi_k}{\partial x_j^o} \approx \epsilon_{ij} + \omega_{ij}$$

then, for the purpose of differentiation, we can replace the gradient of ξ_i with ϵ_{ij}. We now get for the σ_{11}^K stress, for example,

$$\sigma_{11}^K = \sigma_{11}^o + 2\mu\epsilon_{11} + \lambda[\epsilon_{11} + \epsilon_{22} + \epsilon_{33}] + 2\mu\sum\nolimits_k \frac{\partial u_k^o}{\partial x_1^o}\epsilon_{k1} + \lambda\sum\nolimits_{k,p}\frac{\partial u_k^o}{\partial x_p^o}\epsilon_{kp}$$

with similar expressions for the other components. Let us define the current tangent moduli as

$$\begin{aligned} E_{T11} \equiv \frac{\partial \sigma_{11}^K}{\partial \epsilon_{11}} &= (2\mu+\lambda) + (2\mu+\lambda)\frac{\partial u_1^o}{\partial x_1^o} \approx (2\mu+\lambda)[1 + E_{11}^o] \\ &= (2\mu+\lambda)\Big[1 + \frac{1}{2\mu}\{\sigma_{11}^o - \frac{\lambda}{2\mu+3\lambda}(\sigma_{11}^o + \sigma_{22}^o + \sigma_{33}^o)\}\Big] \end{aligned}$$

The derivatives are taken such that the other strains are kept constant. The effect of the initial stress is to change the tangent modulus — an increase in stress causes an increase in modulus. This is the same phenomenon as observed when tuning a violin string, say. Suppose the initial stress is uniaxial such that only $\sigma^o_{11} \neq 0$, then two of the moduli are

$$
\begin{aligned}
E_{T11} &= (2\mu+\lambda)\Big[1+\frac{1}{2\mu}\{1-\frac{\lambda}{2\mu+3\lambda}\}\sigma^o_{11}\Big] \\
E_{T22} &= (2\mu+\lambda)\Big[1+\frac{1}{2\mu}\{\quad-\frac{\lambda}{2\mu+3\lambda}\}\sigma^o_{11}\Big]
\end{aligned}
$$

We see that the material becomes anisotropic due to the stress σ^o_{11}.

A way to visualize the above results is to consider a block of material that is under a quasi-static load state. Now superpose a stress wave disturbance of small amplitude. The current tangent modulus relates the small increments (due to the stress wave) of strain to the small increments of stress. Although the material is isotropic, the stress wave experiences the material as being anisotropic. As an aside, residual stresses can be detected by monitoring the small changes in wave speed caused by the small changes in tangent moduli [13]. ■

4.3 Finite Strain Isotropic Elastic Materials

Many structural materials (steel and aluminum, for example) and rubber-like materials are essentially isotropic in that the stiffness of a sheet is about the same in all directions. We will develop constitutive relations for this special case. We will emphasize, however, the large strain case of the rubber materials.

Mooney-Rivlin Materials

A way to specify isotropy is to specify that the strain energy is a function of the strain invariants only; that is, $\overline{\mathcal{U}} = \overline{\mathcal{U}}(I_1, I_2, I_3)$, where the invariants are computed by

$$
I_1 = \sum\nolimits_k E_{kk}\,, \qquad I_2 = \tfrac{1}{2}I_1^2 - \tfrac{1}{2}\sum\nolimits_{i,k} E_{ik}E_{ik}\,, \qquad I_3 = \det[E_{ij}]
$$

The stress-strain relation becomes, using the chain rule for differentiation,

$$
\sigma^K_{ij} = \frac{\partial\overline{\mathcal{U}}}{\partial I_1}\frac{\partial I_1}{\partial E_{ij}} + \frac{\partial\overline{\mathcal{U}}}{\partial I_2}\frac{\partial I_2}{\partial E_{ij}} + \frac{\partial\overline{\mathcal{U}}}{\partial I_3}\frac{\partial I_3}{\partial E_{ij}}
$$

The derivatives of the invariants with respect to strain are

$$
\frac{\partial I_1}{\partial E_{ij}} = \delta_{ij}\,, \qquad \frac{\partial I_2}{\partial E_{ij}} = I_1\delta_{ij} - E_{ij}\,, \qquad \frac{\partial I_3}{\partial E_{ij}} = I_2\delta_{ij} - I_1E_{ij} + \sum\nolimits_k E_{ik}E_{kj}
$$

On substituting these into the above constitutive relation, and rearranging, we get

$$
\sigma^K_{ij} = \beta_o\delta_{ij} + \beta_1E_{ij} + \beta_2\sum\nolimits_p E_{ip}E_{pj} \qquad (4.12)
$$

which is a nice compact relation. The coefficients β_i are functions of only the invariants [11] and have the explicit representation

$$\beta_o = \frac{\partial \overline{\mathcal{U}}}{\partial I_1} + \frac{\partial \overline{\mathcal{U}}}{\partial I_2} I_1 + \frac{\partial \overline{\mathcal{U}}}{\partial I_3} I_2 \,, \qquad \beta_1 = -\frac{\partial \overline{\mathcal{U}}}{\partial I_2} - \frac{\partial \overline{\mathcal{U}}}{\partial I_3} I_1 \,, \qquad \beta_2 = \frac{\partial \overline{\mathcal{U}}}{\partial I_3}$$

Although the tensor form of the relation in Equation (4.12) is quadratic, this does not imply that the stress-strain relation can only be quadratic in the components since the coefficients are arbitrary function of the invariants. Indeed, if $\overline{\mathcal{U}}(I_1, I_2, I_3)$ is considered to be expanded as a polynomial in the invariants, it is seen how this form gives an elasticity description with many material coefficients.

Before proceeding, we recast the above in terms of the deformation tensor instead of the strain tensor — this is the standard formulation for large strains. The invariants of the deformation tensor are computed by

$$\begin{aligned}
I_1 &= \textstyle\sum_j C_{ii} = C_{11} + C_{22} + C_{33} \\
&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2 &= \textstyle\sum_{i,j} \frac{1}{2}[C_{ii}C_{jj} - C_{ij}C_{ji}] = C_{11}C_{22} + C_{22}C_{33} + C_{33}C_{11} - C_{12}^2 - C_{23}^2 - C_{13}^2 \\
&= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\
I_3 &= \det[C_{ij}] = C_{11}C_{22}C_{33} + 2C_{12}C_{23}C_{13} - C_{11}C_{23}^2 - C_{22}C_{13}^2 - C_{33}C_{12}^2 \\
&= \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned} \tag{4.13}$$

When the body is unloaded, the principal stretches λ_i are each unity giving the invariants as 3, 3, and 1, respectively. Also, in the following we will use the designation $J \equiv \sqrt{I_3}$ for the jacobian.

Following as above, the stress-deformation relation becomes

$$\sigma_{ij}^K = 2\frac{\partial \overline{\mathcal{U}}(I_1, I_2, I_3)}{\partial C_{ij}} = 2\frac{\partial \overline{\mathcal{U}}}{\partial I_1}\frac{\partial I_1}{\partial C_{ij}} + 2\frac{\partial \overline{\mathcal{U}}}{\partial I_2}\frac{\partial I_2}{\partial C_{ij}} + 2\frac{\partial \overline{\mathcal{U}}}{\partial I_3}\frac{\partial I_3}{\partial C_{ij}}$$

The derivatives of the invariants with respect to C_{ij} are

$$\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij} \,, \qquad \frac{\partial I_2}{\partial C_{ij}} = I_1\delta_{ij} - C_{ij} \,, \qquad \frac{\partial I_3}{\partial C_{ij}} = I_3[C_{ij}]^{-1} \tag{4.14}$$

where the meaning of $[C_{ij}]^{-1}$ is the ij^{th} component of the inverse of $[C_{ij}]$. The stress-deformation relation can be written as

$$\sigma_{ij}^K = \beta_0\delta_{ij} + \beta_1 C_{ij} + \beta_2 [C_{ij}]^{-1} \tag{4.15}$$

where the coefficients are now given by

$$\beta_0 = 2\frac{\partial \overline{\mathcal{U}}}{\partial I_1} + 2\frac{\partial \overline{\mathcal{U}}}{\partial I_2} I_1 \,, \qquad \beta_1 = -2\frac{\partial \overline{\mathcal{U}}}{\partial I_2} \,, \qquad \beta_2 = 2\frac{\partial \overline{\mathcal{U}}}{\partial I_3} I_3$$

There is no essential difference between this form and the one using strains.

Let $\bar{\mathcal{U}}(I_1, I_2, I_3)$ be expanded as a polynomial in the invariants as

$$\overline{\mathcal{U}} = \sum\nolimits_{p,q,r} \alpha_{pqr}(I_1 - 3)^p(I_2 - 3)^q(I_3 - 1)^r$$

where the α_{pqr} are constants. A reduced form for elastomers can be taken to be

$$\overline{\mathcal{U}} = \alpha_{100}(I_1 - 3) + \alpha_{010}(I_2 - 3) + \alpha_{200}(I_1 - 3)^2 + \alpha_{300}(I_1 - 3)^3 + \alpha_{001}(I_3 - 1)^2$$

The number of terms retained will depend on the quality and type of experimental data available for the characterization. Reference [8] suggests not including any I_2 dependent terms because the resulting strain energy density function is not polyconvex.

Many materials that are capable of sustaining large deformations (such as elastomers and biological materials) usually exhibit incompressibility; this can be imposed by setting $I_3 = 1$ in the energy density expansion. Under these conditions, the first term alone in the expansion gives what is called a *neo-Hookean* material, while the first two terms alone is often referred to as a *Mooney-Rivlin* material. However, as shown in an example to follow, the energy balance is unaffected by the addition of a pressure, and hence the constitutive relation must be amended to give

$$\sigma^K_{ij} = 2\frac{\partial \overline{\mathcal{U}}(I_1, I_2)}{\partial C_{ij}} - pJ[C_{ij}]^{-1}$$

where the pressure p is treated as an unknown to be specified by the conditions of the problem. An example problem to follow shows how it is determined in a plane stress case.

Example 4.9: Consider a state of hydrostatic stress where the pressure is related to the Cauchy stress by $\sigma_{ij} = -p\delta_{ij}$. Determine the corresponding Kirchhoff stresses.

The Kirchhoff stresses are related to the Cauchy stresses by

$$[\sigma^K_{ij}] = J\sum\nolimits_{m,n}[\frac{\partial x^o_i}{\partial x_m}][-p\delta_{mn}][\frac{\partial x^o_j}{\partial x_n}]^T = -pJ\sum\nolimits_{m}[\frac{\partial x^o_i}{\partial x_m}][\frac{\partial x^o_j}{\partial x_m}]^T = -pJ[C_{ij}]^{-1}$$

The deformation tensors reduce to

$$[\frac{\partial x_i}{\partial x^o_j}] = \lambda[\,\delta_{ij}\,]\,, \qquad [C_{ij}] = \lambda^2[\,\delta_{ij}\,]\,, \qquad [C_{ij}]^{-1} = [\,\delta_{ij}\,]/\lambda^2$$

Consequently, the constitutive relation reduces to

$$\sigma^K_{ij} = [\beta_0 + \beta_1 + \beta_2]\delta_{ij} = 2[\frac{\partial \overline{\mathcal{U}}}{\partial I_1} + 2\frac{\partial \overline{\mathcal{U}}}{\partial I_2}I_1 - \frac{\partial \overline{\mathcal{U}}}{\partial I_2} + \frac{\partial \overline{\mathcal{U}}}{\partial I_3}I_3]\delta_{ij} = -pJ[\,\delta_{ij}\,]$$

This shows that all invariants contribute to the pressure; as a result, even if $I_3 = 1$ is imposed for incompressible materials (so that I_3 terms do not appear in the energy expansion) the constitutive relation still gives a hydrostatic pressure. To be explicit, when the condition $I_3 = 1$ is imposed, the energy balance is unaffected by the addition of a pressure, and hence the constitutive relation must be amended to give

$$\sigma_{ij}^K = 2\frac{\partial \overline{\mathcal{U}}(I_1, I_2)}{\partial C_{ij}} - pJ[C_{ij}]^{-1}$$

where p is treated as an unknown to be specified by the conditions of the problem. ■

Example 4.10: Particularize the Mooney-Rivlin material constitutive relations for plane stress under nearly incompressible conditions.

We will work with the principal stretches, hence begin with the deformation written as

$$x_1 = \lambda_1 x_1^o\,, \qquad x_2 = \lambda_2 x_2^o\,, \qquad x_3 = \lambda_3 x_3^o$$

where the λ's are stretches. The special case of simple extension is described by $\lambda_2 = \lambda_3 = \lambda_o$ and is analogous to uniaxial stress.

The deformation gradients are

$$\left[\frac{\partial x_i}{\partial x_j^o}\right] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad \left[\frac{\partial x_i^o}{\partial x_j}\right] = \begin{bmatrix} 1/\lambda_1 & 0 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix}$$

The Cauchy-Green deformation tensor is

$$[C_{ij}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

Because of near incompressibility, we can set

$$J = \lambda_1\lambda_2\lambda_3 \approx 1\,, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \approx 1$$

The deviatoric Mooney-Rivlin material energy function is

$$\overline{\mathcal{U}}_d = \alpha_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \alpha_{01}(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3)$$

The Kirchhoff stresses are given by (all stress and strain values are principal values)

$$\sigma_{ii}^K = \frac{\partial \overline{\mathcal{U}}}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial E_{ii}} - \frac{1}{\lambda_i^2}p = \frac{1}{\lambda_i}\Big[2\alpha_{10}\lambda_i - 2\alpha_{01}\frac{1}{\lambda_i^3}\Big] - \frac{1}{\lambda_i^2}p$$

since $E_{ii} = \frac{1}{2}[\lambda_i^2 - 1]$. The Cauchy stresses are related to the Kirchhoff stresses by

$$[\ \sigma\ ] = [\frac{\partial x}{\partial x^o}][\sigma^K][\frac{\partial x}{\partial x^o}]^T \qquad \Rightarrow \qquad \sigma_{ii} = \lambda_i^2\sigma_{ii}^K = \lambda_i[2\alpha_{10}\lambda_i - 2\alpha_{01}\frac{1}{\lambda_i^3}] - p$$

Both constitutive relations show a strong nonlinear dependence on the stretches λ_i.

Let the membrane stretching be in the $1-2$ plane so that the plane stress condition specifies that $\sigma_{33}=0$; from this we conclude that the pressure is given by

$$p=\lambda_3[2\alpha_{10}\lambda_3-2\alpha_{01}\frac{1}{\lambda_3^3}]=2\alpha_{10}\frac{1}{\lambda_1^2\lambda_2^2}-2\alpha_{01}\lambda_1^2\lambda_2^2$$

where again the incompressibility condition is used. The two Kirchhoff stresses can now be written as

$$\sigma_{11}^K=2\alpha_{10}[1-\frac{1}{\lambda_1^4\lambda_2^2}]-2\alpha_{01}[\frac{1}{\lambda_1^4}-\lambda_2^2]\,,\qquad \sigma_{22}^K=2\alpha_{10}[1-\frac{1}{\lambda_2^4\lambda_1^2}]-2\alpha_{01}[\frac{1}{\lambda_2^4}-\lambda_1^2]$$

The corresponding Cauchy stresses are

$$\sigma_{11}=2\alpha_{10}[\lambda_1^2-\frac{1}{\lambda_1^2\lambda_2^2}]-2\alpha_{01}[\frac{1}{\lambda_1^2}-\lambda_1^2\lambda_2^2]\,,\qquad \sigma_{22}=2\alpha_{10}[\lambda_2^2-\frac{1}{\lambda_2^2\lambda_1^2}]-2\alpha_{01}[\frac{1}{\lambda_2^2}-\lambda_1^2\lambda_2^2]$$

since $\sigma_{ii}=\lambda_i^2\sigma_{ii}^K$ in principal coordinates. An interesting feature of these relations is that, although the expansion of the strain energy density function is relatively simple, it nonetheless gives rise to highly nonlinear stress-deformation relations. ■

Example 4.11: Use the neo-Hookean material model to discuss the inflation of a spherical balloon.

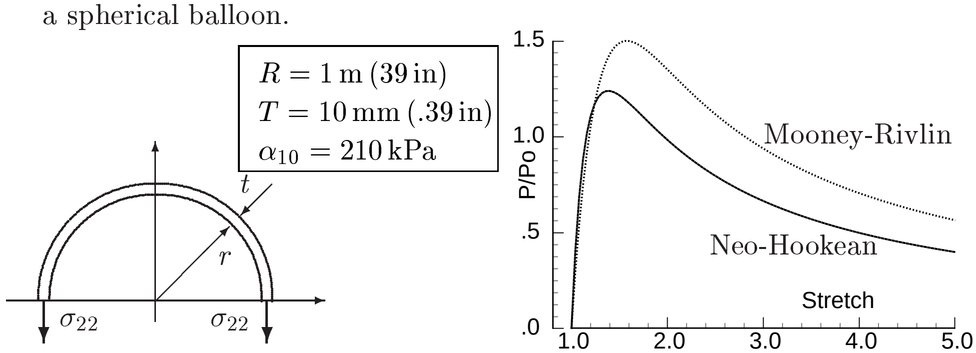


Figure 4.4: Inflation of a spherical balloon. (a) Geometry and free body diagram. (b) Pressure-stretch response

Let the initial radius and thickness be R and T, respectively; then during inflation their current values are

$$r=\lambda_1 R\,,\qquad t=\lambda_3 T$$

It is assumed that the balloon is stretching in the local $1-2$ plane so that the thickness direction is 3. The spherical symmetry gives $\lambda_1=\lambda_2=\lambda$ and $\lambda_3=1/(\lambda_1\lambda_2)=1/\lambda^2$ because of near incompressibility. Under these conditions, the plane stress neo-Hookean relation becomes

$$\sigma_{11}^K=\sigma_{22}^K=2\alpha_{10}[1-\frac{1}{\lambda^6}]\,,\qquad \sigma_{11}=\sigma_{22}=\lambda^2\sigma_{11}^K=2\alpha_{10}[\lambda^2-\frac{1}{\lambda^4}]$$

Consider a free body cut so that the sphere is split in two; then equilibrium between the membrane stress and the pressure gives

$$\pi r^2 p_o = 2\pi r t \sigma_{11} \qquad \text{or} \qquad p_o = \sigma_{11} 2t/r$$

This is the same relation as given in elementary treatments of thin-walled spherical pressure vessels. Here, however, we allow for the change of thickness and radius under pressure and the relation becomes

$$p_o = 2\alpha_{10}[\lambda^2 - \frac{1}{\lambda^4}]\frac{2\lambda_3 T}{\lambda R} = 2\alpha_{10}[\frac{1}{\lambda} - \frac{1}{\lambda^7}]\frac{2T}{R}$$

This relation is shown plotted in Figure 4.4 and exhibits the interesting behavior that the pressure peaks at $\lambda = 7^{1/6} \approx 1.38$ and thereafter decreases. What this means is that once the pressure reaches the peak value there is a sudden expansion of the balloon with a consequent decrease of the internal pressure. This is behavior regularly observed for toy balloons. ∎

Example 4.12: Discuss the inflation of a spherical balloon governed by Hooke's law.

From Equation (4.9), under the conditions $\sigma_{11}^K = \sigma_{22}^K$, the three principal strains reduce to

$$E_{11} = E_{22} = \frac{1-\nu}{E}\sigma_{11}^K \,, \qquad E_{33} = \frac{-2\nu}{E}\sigma_{11}^K$$

The principal stretches are

$$\lambda_1 = \lambda_2 = \sqrt{2E_{11}+1} = \sqrt{1 + \frac{2(1-\nu)}{E}\sigma_{11}^K} \,, \qquad \lambda_3 = \sqrt{2E_{33}+1} = \sqrt{1 - \frac{4\nu}{E}\sigma_{11}^K}$$

Solve these to give

$$\sigma_{11}^K = \frac{E}{2(1-\nu)}[\lambda_1^2 - 1] \,, \qquad \lambda_3 = \sqrt{1 - \frac{2\nu}{(1-\nu)}[\lambda_1^2 - 1]}$$

and the Jacobian

$$J = \lambda_1\lambda_2\lambda_3 = \lambda_1^2\sqrt{1 - \frac{2\nu}{(1-\nu)}[\lambda_1^2 - 1]} \,, \qquad J_{\nu=0.5} = \lambda_1^2\sqrt{3 - 2\lambda_1^2}$$

The Jacobian relation shows that the local material volume can become zero at

$$\lambda_1 = \sqrt{(1+\nu)/(2\nu)} \approx 1.22$$

where the numerical value is for $\nu = 0.5$. This value of maximum stretch is significantly smaller than the stretch achieved using the neo-Hookean material of the previous example. Furthermore, the material is not nearly incompressible even if $\nu \approx 0.5$.

We conclude that Equation (4.8) is not a good material description to use when large strains ($> 20\%$) are expected. Indeed, we will reserve that relation for the case of large displacement and rotations but small strains, the situations that prevail for thin-walled structures. ∎

Nearly Incompressible Materials: rubber-like behavior

The imposition of pure incompressibility can cause numerical difficulties. Additionally, it seems that for rubber-like materials it is a better approximation, anyway, to assume that they are only nearly incompressible materials (the bulk modulus is several orders of magnitude larger than the shear modulus). For these materials, it is usual then to split the stored energy function into an *isochoric* or *deviatoric* part (no volume/density change) and a volumetric part. We demonstrate this modification here, the material discussed is given deeper coverage in References [4, 7, 8].

In the previous developments, the Jacobian of the deformation is given by

$$J = I_3^{1/2}$$

The incompressibility condition under large deformations is that $J = 1$ and not $\nu = 0.5$ as shown earlier. As shown in a previous example, all invariants contribute to the pressure; as a result, even if $I_3 = 1$ is imposed then the constitutive relation gives a hydrostatic pressure.

Since the invariants I_1 and I_2 contribute to the pressure, we first introduce modified stretches given by

$$\lambda_i^* = \lambda_i / I_3^{1/6}\,, \qquad J^* = \lambda_1^*\lambda_2^*\lambda_3^* = 1$$

leading to reduced invariants defined as

$$J_1 \equiv I_1/I_3^{1/3} = I_1/J^{2/3}\,, \qquad J_2 \equiv I_2/I_3^{2/3} = I_2/J^{4/3}\,, \qquad J_3 \equiv I_3^{1/2} = J$$

The strain energy function then takes the form

$$\overline{\mathcal{U}} = \overline{\mathcal{U}}_d + \tfrac{1}{2}K(J_3-1)^2 = \sum\nolimits_{m,n}\alpha_{mn}(J_1-3)^m(J_2-3)^n + \tfrac{1}{2}K(J_3-1)^2$$

where K plays the role of a bulk modulus and the associated term accounts for the energy due to bulk compression.

The required derivatives are

$$\frac{\partial J_1}{\partial C_{pq}} = \frac{\partial I_1}{\partial C_{pq}}\frac{1}{I_3^{1/3}} - \tfrac{1}{3}\frac{\partial I_3}{\partial C_{pq}}\frac{I_1}{I_3^{4/3}}, \qquad \frac{\partial J_2}{\partial C_{pq}} = \frac{\partial I_2}{\partial C_{pq}}\frac{1}{I_3^{2/3}} - \tfrac{2}{3}\frac{\partial I_3}{\partial C_{pq}}\frac{I_1}{I_3^{5/3}}$$

$$\frac{\partial J_3}{\partial C_{pq}} = \tfrac{1}{2}\frac{\partial I_3}{\partial C_{pq}}\frac{J_3}{J_3 I_3^{1/2}}$$

and the derivatives of the invariants with respect to C_{ij} are as given in Equation (4.14). The stress-deformation relation for the two term Mooney-Rivlin material, for example, can now be written as

$$\sigma_{ij}^K = \beta_0\delta_{ij} + \beta_1 C_{ij} + \beta_3[C_{ij}]^{-1} + K[J_3-1]J_3[C_{ij}]^{-1} \tag{4.16}$$

where the coefficients are given by

$$\beta_0 = 2\alpha_{10}\frac{1}{I_3^{1/3}} + 2\alpha_{01}\frac{I_1}{I_3^{2/3}}\,, \qquad \beta_1 = -2\alpha_{01}\frac{1}{I_3^{2/3}}\,, \qquad \beta_3 = -\tfrac{2}{3}\alpha_{10}\frac{I_1}{I_3^{1/3}} - \tfrac{4}{3}\alpha_{01}\frac{I_2}{I_3^{2/3}}$$

Equation (4.16) resembles the relations developed earlier, the difference is that the compressibility condition $J \neq 1$ is retained. As a consequence, Mooney-Rivlin material, for example, has three material parameters.

As shown in an example to follow, the small deformation modulus is

$$D_{ijkl} = [-\tfrac{4}{3}(\alpha_{10}+\alpha_{01}) + K]\delta_{ij}\delta_{kl} + 2(\alpha_{10}+\alpha_{01})[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

The equivalent Lamé parameters are

$$\lambda_{eq} = -\tfrac{4}{3}(\alpha_{10}+\alpha_{01}) + K\,, \qquad \mu_{eq} = 2(\alpha_{10}+\alpha_{01})$$

Consequently, this material has three parameters that can be utilized to model actual material behavior. The pure incompressibility condition is achieved by setting K very large which effectively reduces the modeling to two parameters. Reference [8] recommends choosing K so that the effective Poisson's ratio (for small deformations) is $\nu \approx 0.475$. That is,

$$K = \frac{3\lambda + 2\mu}{3} = \frac{2\mu[1+\nu]}{3[1-2\nu]} \approx 20\,\mu\,, \qquad \mu = 2(\alpha_{10}+\alpha_{01})$$

This is the form implemented in many FEM codes.

The neo-Hookean material has the relatively simple form

$$\sigma_{ij}^K = \mu\Big[\delta_{ij} - \tfrac{1}{3}I_1[C_{ij}]^{-1}\Big]/I_3^{1/3} + K[J_3 - 1]J_3[C_{ij}]^{-1}$$

where $\mu = 2\alpha_{10}$ is like a shear modulus. If K is chosen as above, then this has a single parameter. Keep in mind, however, that many authors [4, 8] choose the Mooney-Rivlin parameters such that

$$2\alpha_{10} = \mu[1-\gamma]\,, \qquad 2\alpha_{01} = \mu[\,\gamma\,]$$

where γ is a parameter varying in the range $0 < \gamma < 0.10$.

Example 4.13: Determine the tangent modulus for the Mooney-Rivlin material.

The tangent modulus, in general, is given by

$$D_{ijkl} = \frac{\partial\sigma_{ij}^K}{\partial E_{kl}} = 2\frac{\partial\sigma_{ij}^K}{\partial C_{kl}}$$

Performing the indicated derivatives then leads to [7]

$$\begin{aligned} D_{ijkl} &= \gamma_1 C_{ij}^{-1}C_{kl}^{-1} + \gamma_2[\delta_{ij}C_{kl}^{-1} + C_{ij}^{-1}\delta_{kl}] + \gamma_3[C_{ik}^{-1}C_{jl}^{-1} + C_{il}^{-1}C_{jk}^{-1}] \\ &\quad + \gamma_4\delta_{ij}\delta_{kl} + \gamma_5[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + \gamma_6[C_{ij}C_{kl}^{-1} + C_{ij}^{-1}C_{kl}] + KJ^2C_{ij}^{-1}C_{kl}^{-1} \end{aligned}$$

where

$$\gamma_1 = \tfrac{4}{9}\alpha_{10}\frac{I_1}{I_3^{1/3}} + \tfrac{16}{9}\alpha_{01}\frac{I_2}{I_3^{2/3}} + K[J-1]J, \quad \gamma_3 = \tfrac{2}{3}\alpha_{10}\frac{I_1}{I_3^{1/3}} + \tfrac{4}{3}\alpha_{01}\frac{I_2}{I_3^{2/3}} - K[J-1]J$$

$$\gamma_2 = -\tfrac{4}{3}\alpha_{10}\frac{1}{I_3^{1/3}} - \tfrac{8}{3}\alpha_{01}\frac{I_1}{I_3^{2/3}}, \quad \gamma_4 = 4\alpha_{01}\frac{1}{I_3^{2/3}}, \quad \gamma_5 = -2\alpha_{01}\frac{1}{I_3^{2/3}}, \quad \gamma_6 = \tfrac{8}{3}\alpha_{01}\frac{1}{I_3^{1/3}}$$

We can write this relation in the matrix form

$$\{\sigma^K\} = [\ D\ ]\{E\}, \quad \{\sigma\} \equiv \{\sigma^K_{11}, \sigma^K_{22}, \sigma^K_{33}, \sigma^K_{12}, \cdots\}^T, \quad \{E\} \equiv \{E_{11}, E_{22}, E_{33}, 2E_{12}, \cdots\}^T$$

where $[\ D\ ]$ is of size $[6 \times 6]$ and populated as

$$[\ D\ ] = \begin{bmatrix} D_{1111} & D_{1122} & \cdots & D_{1123} \\ D_{2211} & D_{2222} & \cdots & D_{2223} \\ \vdots & \vdots & \ddots & \vdots \\ D_{2311} & D_{2322} & \cdots & D_{2323} \end{bmatrix}$$

Because of the symmetry of both the stress and strain, we have $[\ D\ ]^T = [\ D\ ]$. However, although the material is isotropic in the sense that it is a function of only the invariants, the moduli are anisotropic under large deformation. This is a severe example of the initial load interaction case considered earlier.

The small deformation modulus is recovered by setting

$$I_1 = 3, \quad I_2 = 3, \quad I_3 = 1, \quad C_{ij} = \delta_{ij}, \quad C^{-1}_{ij} = \delta_{ij}$$

to get

$$D_{ijkl} = [-\tfrac{4}{3}(\alpha_{10} + \alpha_{01}) + K]\delta_{ij}\delta_{kl} + 2(\alpha_{10} + \alpha_{01})[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

The equivalent Lamé parameters are obtained by utilizing the result of Equation (4.11) to get

$$\lambda_{eq} = -\tfrac{4}{3}(\alpha_{10} + \alpha_{01}) + K\,, \qquad \mu_{eq} = 2(\alpha_{10} + \alpha_{01})$$

The linear behavior is valid only for very small strains. ■

4.4 Elastic Symmetries

If the internal composition of a material possesses symmetry of any kind, then symmetry can be observed in its elastic properties. Structured materials such as composites and crystals exhibit these special symmetries. We discuss them within the context of small strain theories.

For the purpose of this discussion, we take the generalized Hooke's law in the form

$$\{\sigma\} = [\ c\ ]\{\epsilon\}$$

where $[\ c\ ]$ is of size $[6 \times 6]$ and is symmetric. The presence of material symmetries reduces the number of independent constants still further. Such simplifications in the generalized Hooke's law can be obtained as follows. Let x, y, z be a coordinate system and x', y', z' be the second system which is symmetric to the first in accordance with the form of its elastic symmetry. Since the directions of similar axes of both systems are equivalent with respect to elastic properties, the equations of the generalized Hooke's law will have the same form in both coordinate systems, and the corresponding elastic constants should also be identical.

Monoclinic System: one elastic symmetry plane

Let the symmetric coordinate system be x', y', z' with the base vectors

$$\hat{e}'_1 = \{1, 0, 0\}, \qquad \hat{e}'_2 = \{0, 1, 0\}, \qquad \hat{e}'_3 = \{0, 0, -1\}$$

The transformation matrix is given by

$$[\beta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The new stress components referred to the primed system are

$$[\sigma'_{ij}] = [\ \beta\ ][\sigma_{ij}][\ \beta\ ]^T = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & -\sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & -\sigma_{yz} \\ -\sigma_{zx} & -\sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

The strain components in the new coordinate system can similarly be obtained as

$$[\,\epsilon'_{ij}\,] = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & -\epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & -\epsilon_{yz} \\ -\epsilon_{zx} & -\epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

The elastic symmetry requires that

$$\begin{Bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{yz} \\ \sigma'_{xz} \\ \sigma'_{xy} \end{Bmatrix} = [c_{ij}] \begin{Bmatrix} \epsilon'_{xx} \\ \epsilon'_{yy} \\ \epsilon'_{zz} \\ 2\epsilon'_{yz} \\ 2\epsilon'_{xz} \\ 2\epsilon'_{xy} \end{Bmatrix}$$

By using the above relations this can be rewritten as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & -c_{14} & -c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & -c_{24} & -c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & -c_{34} & -c_{35} & c_{36} \\ -c_{14} & -c_{24} & -c_{34} & c_{44} & c_{45} & -c_{46} \\ -c_{15} & -c_{25} & -c_{35} & c_{45} & c_{55} & -c_{56} \\ c_{16} & c_{26} & c_{36} & -c_{46} & -c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{yz} \\ 2\epsilon_{xz} \\ 2\epsilon_{xy} \end{Bmatrix}$$

Comparison of this with the general matrix leads to the conclusion

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0$$

The matrix of elastic constants simplifies to the form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix}$$

Note that the number of independent elastic constants reduces to 13.

Orthotropic System: three orthogonal planes of symmetry

Let x, y, z be perpendicular to the three symmetry planes, respectively. The orthotropy assures that no change in mechanical behavior will be incurred when the x, y, z directions are reversed. Following the procedure described previously, the matrix of elastic constants for orthotropic materials assumes the form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

The number of independent elastic constants reduces to 9.

Hexagonal System: transversely isotropic

This system has a plane of symmetry in addition to an axis of symmetry perpendicular to the plane. Assume that the plane of symmetry coincides with the $x - y$ plane, and the axis of symmetry is the z-axis. Thus, any pair of orthogonal axes (x', y') lying on the $x - y$ plane are similar to (x, y). Hence, the stress-strain relations with respect to (x', y', z') where $z' = -z$ should remain identical to those with respect to the (x, y, z) system. In order to satisfy this invariant property, the c_{ij} matrix assumes the following form

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12}) \end{bmatrix}$$

It is noted that for transversely isotropic solids, there are five independent elastic constants.

Isotropic System

For the isotropic case every plane is a plane of symmetry and every axis is an axis of symmetry. It turns out, there are only two independent elastic constants, and the elastic constant matrix is given by

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix}$$

in which

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu\,, \qquad c_{12} = c_{23} = c_{13} = \lambda$$

The constants λ and μ are called the Lamé constants. The stress-strain relations for isotropic materials are usually expressed in the form

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}\,, \qquad 2\mu\epsilon_{ij} = \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij} \tag{4.17}$$

Note that except for the isotropic material, the material coefficients are given with respect to a particular coordinate system. Generally, we must transform the constants into the new coordinate system when the system is changed.

Linear Elastic Isotropic Materials

Most structural materials are adequately represented by their linear elastic behavior. We therefore summarize this behavior and in the process introduce some of the common material constants used.

In practice, the elastic constants commonly used for an isotropic material are K, E, G and ν. These four constants are called the *bulk modulus*, the *Young's modulus*, the *shear modulus*, and the *Poisson's ratio*, respectively. We shall study some special states of stress in order to reveal the physical significance of these engineering constants and their relation to the Lamé constants.

I. Hydrostatic Pressure

A state of hydrostatic pressure is given by

$$\sigma_{ij} = -p\delta_{ij}$$

where p is the pressure. In view of the traction relation

$$t_i = \sigma_{ij} n_j = -p\delta_{ij} n_j = -p n_i$$

we note that on any plane passing through a point, the traction vector is always perpendicular to the plane. Taking the contraction of the stress, we have

$$\sigma_{ii} = -3p$$

From the stress-strain relations, we also have

$$\sigma_{ii} = 2\mu\epsilon_{ii} + \lambda\delta_{ii}\epsilon_{kk} = (3\lambda + 2\mu)\epsilon_{kk}$$

Comparison of these leads to

$$p = -(3\lambda + 2\mu)\tfrac{1}{3}\epsilon_{kk} = -K\epsilon_{kk} = -K\frac{\Delta V}{V}$$

where

$$K \equiv \tfrac{1}{3}(3\lambda + 2\mu) = \tfrac{1}{3}\frac{\sigma_{kk}}{\epsilon_{kk}}$$

is called the bulk modulus. Since ϵ_{kk} denotes the volume change, it is obvious that K measures the rigidity in the dilatational deformation.

II. Simple Tension

A state of simple tension applied in the x_1-direction is characterized by the state of stress $\sigma_{11} = \sigma_o$, and all other σ_{ij} vanish. The quantity σ_o is the uniaxial tensile stress. The ratio $\sigma_{11}/\epsilon_{11}$ is defined to be the Young's modulus E.

The condition that $\sigma_{ij} = 0$ except $\sigma_{11} = \sigma_o$ leads to the strain conditions $\epsilon_{ij} = 0$ if $i \neq j$ by use of the stress-strain relations. The normal strain ϵ_{11} is

$$2\mu\epsilon_{11} = \sigma_{11} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{11} = \frac{2(\lambda + \mu)}{3\lambda + 2\mu}\sigma_{11}$$

The expression for the Young's modulus is then obtained as

$$E \equiv \frac{\sigma_{11}}{\epsilon_{11}} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

It is noted that in simple tension $\epsilon_{22} \neq 0$ and $\epsilon_{33} \neq 0$. In fact,

$$\epsilon_{22} = \epsilon_{33} = \frac{1}{2\mu}\Big[0 - \frac{\lambda}{3\lambda + 2\mu}\sigma_{11}\Big] = -\frac{\lambda}{2(\lambda + \mu)}\epsilon_{11}$$

The ratio

$$\nu \equiv -\frac{\epsilon_{22}}{\epsilon_{11}} = \frac{\lambda}{2(\lambda + \mu)}$$

is called the Poisson's ratio.

III. Simple Shear

Consider a state of simple shear in the $x_1 - x_2$ plane where the only non-vanishing stress component is $\sigma_{12} = \sigma_{21}$, and the corresponding non-vanishing strain component is $\epsilon_{12} = \epsilon_{21}$. From the stress-strain relations, we have

$$\sigma_{12} = 2\mu\epsilon_{12}$$

The shear modulus is defined as the ratio of the shear stress to the total angle change $2\epsilon_{12}$, that is,

$$\mu \equiv \frac{\sigma_{12}}{2\epsilon_{12}} = G$$

In this context, the quantity $2\epsilon_{12}$ is often referred to as the *engineering shear strain.*

Summary

The expanded form of the Hooke's law for strains in terms of stresses is

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \\
\epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \\
\epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \\
2\epsilon_{xy} &= \frac{2(1+\nu)}{E}\sigma_{xy}\,, \quad 2\epsilon_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}\,, \quad 2\epsilon_{xz} = \frac{2(1+\nu)}{E}\sigma_{xz}
\end{aligned}$$

and for stress in terms of strain

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{xx} + \nu(\epsilon_{yy} + \epsilon_{zz})] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{yy} + \nu(\epsilon_{zz} + \epsilon_{xx})] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})] \\
\sigma_{xy} &= \frac{E}{2(1+\nu)}2\epsilon_{xy}\,, \quad \sigma_{yz} = \frac{E}{2(1+\nu)}2\epsilon_{yz}\,, \quad \sigma_{xz} = \frac{E}{2(1+\nu)}2\epsilon_{xz}
\end{aligned}$$

Strain Energy for Linear Elastic Materials

When the strains are small, we need not distinguish between the undeformed and deformed configurations. Under this circumstance, let the material obey Hooke's law and be summarized as

$$\mathcal{U} = \tfrac{1}{2}\int_V [\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\gamma_{xy} + \cdots]\, dV = \tfrac{1}{2}\int_V \{\sigma\}^T\{\,\epsilon\,\}\, dV$$

For a thin plate under plane stress conditions where $\sigma_{zz} = 0$, $\sigma_{xz} = 0$, $\sigma_{yz} = 0$, this reduces to

$$\mathcal{U} = \tfrac{1}{2}\int_V [\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \sigma_{xy}\gamma_{xy}]\, dV$$

Substituting Hooke's law gives the alternative forms

$$\begin{aligned}\mathcal{U} &= \tfrac{1}{2}\int_V \frac{1}{E}[\sigma_{xx}^2 + \sigma_{yy}^2 - 2\nu\sigma_{xx}\sigma_{yy} + 2(1+\nu)\sigma_{xy}^2]\, dV \\ &= \tfrac{1}{2}\int_V \frac{E}{1-\nu^2}[\epsilon_{xx}^2 + \epsilon_{yy}^2 + 2\nu\epsilon_{xx}\epsilon_{yy} + \tfrac{1}{2}(1-\nu)\gamma_{xy}^2]\, dV\end{aligned}$$

Energy considerations can put a limit on the allowable values for the material properties as we now discuss.

The strain energy density can be written in terms of the principle strains as

$$\overline{\mathcal{U}} = \frac{E}{(1+\nu)(1-2\nu)}\Big[\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + 2\nu(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)\Big]$$

If either $\nu < -1$ or $\nu > 0.5$ then the coefficient is negative and it would be easy to identify normal strain states that have negative strain energies. Hence we conclude that

$$(1-2\nu)(1+\nu) > 0 \qquad \text{or} \qquad -1 < \nu < 0.5$$

for normal materials. A negative Poisson ratio would indicate a material that, under uniaxial tension, would expand in the transverse direction. This is possible for some of the so-called structured materials.

The relation among the elastic constants are presented as follows in tabular form:

in terms of	$\lambda =$	$\mu = G =$	$E =$	$\nu =$	$K =$
λ, G			$\frac{G(3\lambda + 2G)}{\lambda + G}$	$\frac{\lambda}{2(\lambda + G)}$	$\frac{(3\lambda + 2G)}{3}$
λ, E		$\frac{C + (E - 3\lambda)}{4}$		$\frac{C - (E + \lambda)}{4\lambda}$	$\frac{C + (3\lambda + E)}{6}$
λ, ν		$\frac{\lambda(1 - 2\nu)}{2\nu}$	$\frac{\lambda(1 + \nu)(1 - 2\nu)}{\nu}$		$\frac{\lambda(1 + \nu)}{3\nu}$
λ, K		$\frac{3(K - \lambda)}{2}$	$\frac{9K(K - \lambda)}{3K - \lambda}$	$\frac{\lambda}{3K - \lambda}$	
G, E	$\frac{G(2G - E)}{E - 3G}$			$\frac{E - 2G}{2G}$	$\frac{GE}{3(3G - E)}$
G, ν	$\frac{2G\nu}{(1 - 2\nu)}$		$2G(1 + G)$		$\frac{2G(1 + G)}{3(1 - 2G)}$
G, K	$\frac{3K - 2G}{3}$		$\frac{9KG}{3K + G}$	$\frac{3K - 2G}{2(3K + G)}$	
E, ν	$\frac{\nu E}{(1 + \nu)(1 - 2\nu)}$	$\frac{E}{2(1 + \nu)}$			$\frac{E}{3(1 - 2\nu)}$
K, E	$\frac{3K(3K - E)}{9K - E}$	$\frac{3EK}{9K - E}$		$\frac{3K - E}{6K}$	
ν, K	$\frac{3K\nu}{1 + \nu}$	$\frac{3K(1 - 2\nu)}{2(1 + \nu)}$	$3K(1 - 2\nu)$		

where $C = \sqrt{E^2 + 2\lambda E + 9\lambda^2}$

Exercises

4.1 Consider the 2-D stress-strain relation

$$\{\sigma_{11},\, \sigma_{22},\, \sigma_{12}\}^T = [\ C\ ]\{\epsilon_{11},\, \epsilon_{22},\, 2\epsilon_{12}\}^T$$

Reduce $[\ C\ ]$ by imposing isotropy in the $x_1 - x_2$ plane.

4.2 Find the components c_{ijpq} in terms of the Lame constants for 3-D isotropic solids.

4.3 A cube of steel of side 250 mm is loaded with a uniformly distributed pressure of 200 MPa on the four faces having normals in the x and y directions. Rigid constraints limit the total deformation of the cube in the z direction to 0.05 mm. Determine the normal stress, if any, which develops in the z direction.

4.4 A block of aluminum $4 \times 1 \times 0.25\,\mathrm{in}^3$ is loaded in the long direction by 1000 lb. What loads must be added to the narrow side faces in order to prevent their motion? What is the resultant motion of the other faces?

4.5 Show that the principal directions of stress and strain coincide for an isotropic material.

4.6 Show, using the Cayley-Hamilton theorem, that the third invariant of strain is given by

$$I_3 = \tfrac{1}{3}E_{ip}E_{pq}E_{qi} - \tfrac{1}{3}I_1^3 + I_1 I_2$$

4.7 Use the previous result to show that

$$\frac{\partial I_3}{\partial E_{ij}} = I_2\delta_{ij} - I_1 E_{ij} + E_{ik}E_{kj}$$

4.8 During the large deformation testing of a uniaxial specimen, the following quantities are recorded:

P: Applied Force

$\dfrac{\Delta L}{L_o}$: Unit change of axial Length

$\dfrac{\Delta W}{W_o}$: Unit change of transverse Length

Establish the relationships necessary to convert these measured quantities to Kirchhoff stress and Lagrangian strain.

Chapter 5

Linear Elasticity Problems

The developments of the last four chapters form the basis of the field equations of the theory of elasticity. This chapter shows how these are put in a form for solving boundary value problems. In the general case, the equations are non-linear and therefore can only be solved approximately. Therefore, this chapter emphasizes the formulation of the linear theory.

The stress function approach is a very powerful method for solving plane elasticity problems. As will be seen, satisfying all the field equations reduce to finding a single bi-harmonic function. As a consequence, the main difficulty in solving boundary value problems is in satisfying the boundary conditions and not so much the field equations. This problem is further exacerbated if the functional form of the tractions are not 'similar' to the functional form of the boundary geometry. We therefore have a need to be able to solve our elasticity equations in general coordinate systems dictated by the geometry of the boundaries. We will illustrate the approach using the transformation of coordinates to cylindrical coordinates as well as the use of complex variables approach.

5.1 Reduction to the Linear Theory of Elasticity

In the previous developments of the theory of elasticity, two main sources of non-linearity arose. They were associated with large deformations and nonlinear material behavior. We will now show the process of reduction to the linear theory by simplifying both of these.

Non-linear Elasticity Problems

All the essential equations are collected here and put in a form that constitutes the complete set necessary for a solution. It is emphasized that, at this stage, they are applicable to large deformations and to general non-linear elastic material behavior.
Equilibrium:

$$\frac{\partial}{\partial x_k^o}\left[\frac{\partial x_i}{\partial x_j^o}\sigma_{kj}^K\right]+\rho^o b_i^o=\rho^o\ddot{u}_i\,,\qquad \sigma_{ij}^K=\sigma_{ji}^K$$

Strain-displacement:

$$2E_{ij} = \frac{\partial u_i}{\partial x_j^o} + \frac{\partial u_j}{\partial x_i^o} + \frac{\partial u_k}{\partial x_i^o}\frac{\partial u_k}{\partial x_j^o}$$

Constitutive relation:

$$\sigma_{ij}^K = \frac{\partial \mathcal{U}}{\partial E_{ij}}$$

Boundary conditions:

$$\begin{array}{rrcl} \text{specify} & u_i & \text{on} & A_u^o \\ & t_i^o = \sigma_{ij}^K n_j^o & \text{on} & A_t^o \end{array}$$

There are fifteen unknowns (three displacements, six strains, and six stresses) and fifteen field equations (three equilibrium, six strain-displacement, and six constitutive). The boundary conditions are necessary since the field equations, being differential equations, will give rise to additional unknown functions after integration. In general, closed-form analytical solutions cannot be obtained and approximate numerical solutions must be resorted to.

Classical (Linear) Theory of Elasticity

We will simplify the equations in an effort to obtain analytical solutions. The primary assumptions of the linear theory of elasticity adopted here are:

- All deformations are small.
- The constitutive relation is linear and isotropic.
- There are no inertial effects.

The basic equations then reduce to

$$\begin{array}{rrcl} & \dfrac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i & = & 0 \\ & \sigma_{ij} & = & \sigma_{ji} \\ & \epsilon_{ij} & = & \dfrac{1}{2}(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}) \\ & \sigma_{ij} & = & 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \\ \text{specify:} & u_i & \text{or} & t_i = \sigma_{ij}n_j \quad \text{for } BC \end{array}$$

Note that linearizing the equations does not reduce the number of unknowns.

While it is possible to attempt to solve all the field equations simultaneously, it is more common to first reduce the total number of unknowns. Historically two major formulations have emerged: Displacement and Stress, respectively.

I: Displacement Formulation — Navier's Equation

The displacements are taken as the basic unknowns, that is, at each point there are three unknown functions u_1, u_2, u_3. These must be determined subjected to the constraint that the stresses derived from them are equilibrated.

Using Hooke's law, we write the stress in terms of displacement

$$\sigma_{ij} = \mu\Big[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big] + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

and substitute into the equilibrium equations to get

$$\mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + (\lambda+\mu)\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \rho b_i = 0$$

These are the Navier's equations and are three equations with three unknowns. (While we have reduced the number of unknowns, we have increased the order of the derivatives.) The boundary conditions must be specified in terms of u_i and are

$$\begin{array}{lll} \text{on } A_u: & u_i & \text{are specified} \\ \text{on } A_t: & \lambda\dfrac{\partial u_k}{\partial x_k}n_i + \mu\Big[\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\Big]n_j = t_i & \text{are specified} \end{array}$$

Notice that the second boundary condition is actually a set of inhomogeneous differential equations.

These equations are difficult to solve directly and so inverse methods are usually used. A formalization of this process uses the idea of displacement potentials. That is, using Helmholz's theorem, the displacement vector field can be decomposed into a scalar potential ϕ and a vector potential ψ in the form

$$u_i = \frac{\partial \phi}{\partial x_i} + \epsilon_{ijk}\frac{\partial \psi_k}{\partial x_j}\,, \qquad \frac{\partial \psi_k}{\partial x_k} = 0$$

If the body force is absent, then the Navier's equations can be expressed as

$$(\lambda+2\mu)\frac{\partial}{\partial x_i}\nabla^2\phi + \mu\epsilon_{ijk}\frac{\partial}{\partial x_j}\nabla^2\psi_k = 0$$

This equation is satisfied if

$$\nabla^2\phi = \text{constant}\,, \qquad \nabla^2\psi_k = \text{constant}$$

Thus, the problem reduces to solving a set of Poisson's equations in the region to find particular solutions for the displacement potentials; these particular solutions automatically satisfy the governing field equations. The complete solution to the boundary value problem is synthesized from a collection of particular solutions written as

$$\phi = \sum_n a_n\phi_n\,, \qquad \psi_k = \sum_n b_{kn}\psi_n$$

where a_n and b_{kn} are undetermined coefficients. From these, the displacements themselves are obtained by differentiation and the coefficients obtained by satisfying the boundary conditions.

II: Stress Formulation — Beltrami-Mitchell Equations

As an alternative formulation, the stresses are assumed as the basic unknowns. That is, at each point in the body there are six unknown functions $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}$. These obviously must satisfy equilibrium. However, there are only three equilibrium equations, hence, further restrictions must be imposed. These restrictions come from the requirement that the strains associated with the stresses must be *compatible*.

Suppose a stress field is proposed and it is equilibrated. The use of Hooke's law converts it to a strain field. Suppose now it is desired to obtain the displacements. This can be done by integrating the strain-displacement relation

$$\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} = 2\epsilon_{ij}$$

This can be viewed as a system of six independent partial differential equations for three unknown u_i. Theoretically, only three equations are needed to determine the displacement fields. If the six strain components ϵ_{ij} are arbitrarily assigned, then multiple values of the displacements would result. For a unique solution in u_i, some restrictions must be placed on the strains ϵ_{ij}. By differentiating the above, we obtain, for instance

$$2\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} = \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} + \frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l}$$

Interchanging subscripts in this relation leads to

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} = 0$$

Among the 81 equations given here, some of them are identically satisfied, and some of them are repetitions. Only six equations are nontrivial and independent, and in unabridged notation, these equations are

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\Big[-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\Big] \\
\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y}\Big[-\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\Big] \\
\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z}\Big[-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\Big] \\
2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} &= \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} \\
2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} &= \frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} \\
2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x} &= \frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2}
\end{aligned}$$

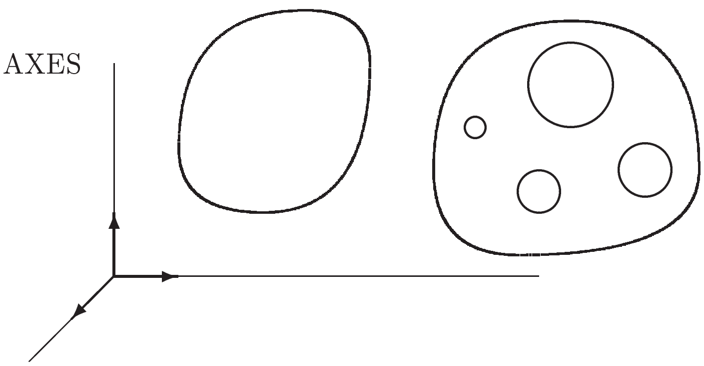


Figure 5.1: Simply and multiply connected bodies.

These six equations are known, collectively, as the equations of compatibility, first obtained by St. Venant in 1860.

A body is said to be simply connected if every closed curve drawn in the body can be shrunk to a point without passing out of the body. For example, a hollow sphere is simply-connected while an open-ended hollow cylinder is a multiply-connected body. The equations of compatibility are necessary and sufficient for a simply-connected body. For a multiply-connected body, they are necessary but no longer sufficient; additional conditions must be imposed to ensure the single-valuedness of displacement.

To obtain compatibility in terms of stress, use Hooke's law in the form

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

to replace the strains in the compatibility equations with stresses and simplify this by utilizing the equilibrium equations to get

$$\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \Big(\frac{1}{1+\nu}\Big)\frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} + \Big(\frac{\nu}{1-\nu}\Big)\rho\frac{\partial b_k}{\partial x_k}\delta_{ij} + \rho\Big(\frac{\partial b_i}{\partial x_j} + \frac{\partial b_j}{\partial x_i}\Big) = 0$$

The stress field must satisfy this equation and the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$$

in order to be admissible. The boundary conditions to be satisfied are

$$\begin{array}{lrcl} \text{on } A_t: & \sigma_{ij}n_j & = & t_i = \text{ given} \\ \text{on } A_u: & u_i & = & \text{given} \end{array}$$

Note that the second set of boundary conditions are obtained by integrating the strain-displacement relations in conjunction with the stress-strain relations.

Example 5.1: Analyze Levy's Problem (1898).

As an example of solving an elasticity problem, we will find the stresses in a semi-infinite wedge of mass density ρ, due to fluid pressure of specific weight γ. We will use the inverse approach; that is, we assume the stresses to be of a particular form and then determine the coefficients from the field equations and boundary conditions.

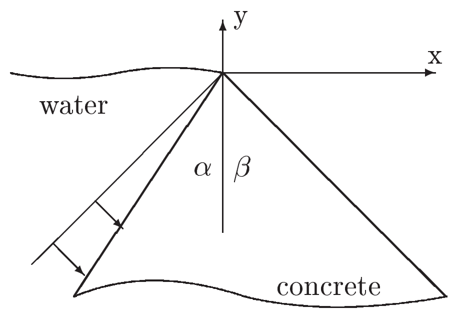


Figure 5.2: Dam with water and own weight loading.

I. Stress Fields

Since the pressure exerted by the water varies linearly with depth, assume that a linear expansion for the stresses is sufficient, that is,

$$\begin{aligned}
\sigma_{xx} &= a_1 x + b_1 y + c_1 \\
\sigma_{yy} &= a_2 x + b_2 y + c_2 \\
\sigma_{xy} &= a_{12} x + b_{12} y + c_{12}
\end{aligned}$$

There are a total of 9 coefficients to be determined from

- Equilibrium
- Compatibility
- Boundary Conditions

Let the origin be at the apex of the dam, then

$$c_1 = c_2 = c_{12} = 0$$

Equilibrium gives

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho b_x = 0 \qquad \text{or} \qquad a_1 + b_{12} + 0 = 0 \\
&\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho b_y = 0 \qquad \text{or} \qquad a_{12} + b_2 - \rho g = 0
\end{aligned}$$

Hence the stresses reduce to

$$\begin{aligned}
\sigma_{xx} &= a_1 x + b_1 y \\
\sigma_{yy} &= a_2 x + b_2 y \\
\sigma_{xy} &= -b_2 x - a_1 y + \rho g x
\end{aligned}$$

That is, any values substituted for a_1, a_2, b_1, b_2 will give a system of stresses that satisfy the equilibrium equations.

Compatibility is automatically satisfied since the stresses are linear functions of position (hence, so are the strains) and the compatibility equation has only double derivatives.

II. Boundary Conditions

We now must choose values for the coefficients that satisfy our particular boundary value problem. In general, the tractions on any boundary are related to the stresses by

$$\begin{aligned} t_x &= \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \\ t_y &= \sigma_{xy} n_x + \sigma_{yy} n_y + \sigma_{zy} n_z \end{aligned}$$

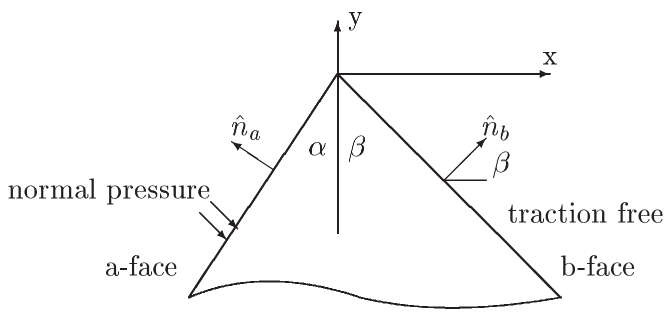


Figure 5.3: Tractions on the dam faces.

For the b-face, we have $\{n_x, n_y, n_z\} = \{\cos\beta, \sin\beta, 0\}$ and substituting into the traction equations gives

$$\begin{aligned} t_x &= 0 = (a_1 x + b_1 y)\cos\beta + (-b_2 x - a_1 y + \rho g x)\sin\beta + 0 \\ t_y &= 0 = (-b_2 x - a_1 y + \rho g x)\cos\beta + (a_2 x + b_2 y)\sin\beta + 0 \end{aligned}$$

Along the b-face, x and y are related by

$$\tan\beta = \frac{x}{-y} \qquad \text{or} \qquad x = -y\tan\beta$$

Substituting for x and canceling the y's gives

$$\begin{aligned} -2a_1\tan\beta + 0 + b_1 + b_2\tan^2\beta &= 0 + \rho g\tan^2\beta \\ a_1 + a_2\tan^2\beta + 0 + 2b_2\tan\beta &= 0 - \rho g\tan\beta \end{aligned}$$

Notice that there is no x or y dependence.

For the a-face, we have $\{n_x, n_y, n_z\} = \{-\cos\alpha, \sin\alpha, 0\}$ and $x = y\tan\alpha$, giving the tractions as (noting that the pressure is $p = -\gamma y$)

$$t_x = p\cos\alpha = -\gamma y\cos\alpha\,, \qquad t_y = -p\sin\alpha = \gamma y\sin\alpha$$

These two equations become

$$\begin{aligned}
2a_1 \tan\alpha + 0 + b_1 + b_2 \tan^2\alpha &= -\gamma + \rho g \tan^2\alpha \\
a_1 + a_2 \tan^2\alpha + 0 + 2b_2 \tan\alpha &= -\gamma \tan\alpha + \rho g \tan\alpha
\end{aligned}$$

Again, notice that we do not have x or y appearing in these equations.

Solving all four of these equations simultaneously gives

$$\begin{aligned}
a_2 &= \gamma\left[\frac{\tan^2\alpha + 3\tan\alpha\tan\beta - 2}{(\tan\alpha + \tan\beta)^3}\right] - \rho g\left[\frac{\tan\alpha - \tan\beta}{(\tan\alpha + \tan\beta)^2}\right] \\
a_1 &= \tan\alpha\tan\beta\left[\frac{\gamma}{\tan\alpha + \tan\beta} - a_2\right] \\
b_2 &= \tfrac{1}{2}\left[-\frac{\gamma\tan\alpha}{\tan\alpha + \tan\beta} + a_2(\tan\alpha - \tan\beta) - \rho g\right] \\
b_1 &= \tfrac{1}{2}\tan^2\beta\left[-\frac{3\gamma\tan\alpha}{\tan\alpha + \tan\beta} + a_2(3\tan\alpha + \tan\beta) - \rho g\right]
\end{aligned}$$

These coefficients can be substituted back to give the stress distributions.

The critical stage of this problem was in satisfying the boundary conditions; we were successful because the functional form of the stresses at the boundaries coincided with the functional form of the tractions. For example, on the a-face the relevant functions are linear in x and y. Note, for example, that if the dam surface had a relation of the form $x = 3y^2$, say, then we would not have been able to satisfy the boundary conditions. ■

Example 5.2: A block rests on a horizontal plane under the action of gravity. Investigate the stresses and displacements.

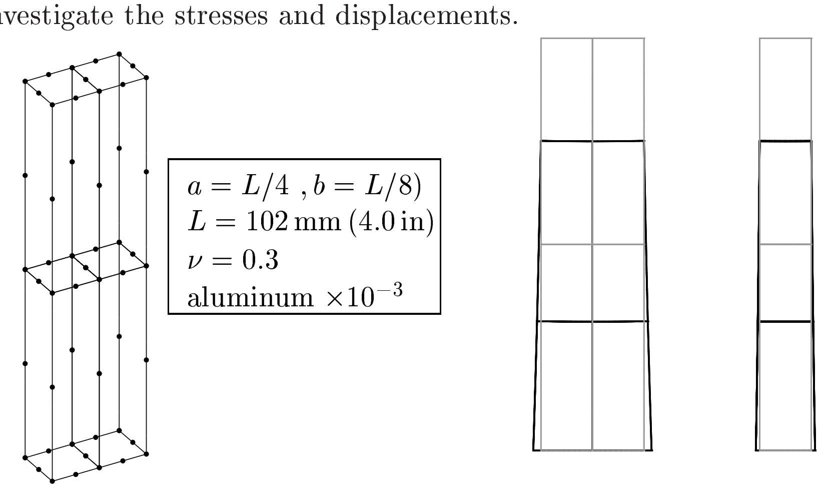


Figure 5.4: Deformation of a block under gravity loading. (a) Geometry and node positions. (b) Front and side view of deformed shape. (deformations are exaggerated $\times 5000$).

Let the coordinate system be as shown in Figure 5.4 with gravity acting vertically. The body force per unit volume is $f_z^b = -\rho g$. Assume all the stresses are zero except the vertical stress, then from the equilibrium equations we get

$$\sigma_{zz} = \rho g[z-L]\,, \qquad \sigma_{ij} = 0$$

We will discuss the consequences of this assumed stress state on the displacement field.

Using Hooke's law and the strain-displacement relations gives

$$\begin{array}{lcl}
\epsilon_{xx} = \dfrac{\partial u}{\partial x} = -\dfrac{\nu\rho g}{E}[z-L] & \longrightarrow & u = -\dfrac{\nu\rho g}{E}x[z-L] + u_o(y,z) \\
\epsilon_{yy} = \dfrac{\partial v}{\partial y} = -\dfrac{\nu\rho g}{E}[z-L] & \longrightarrow & v = -\dfrac{\nu\rho g}{E}y[z-L] + v_o(x,z) \\
\epsilon_{zz} = \dfrac{\partial w}{\partial z} = +\dfrac{\rho g}{E}[z-L] & \longrightarrow & w = \dfrac{\rho g}{2E}[z^2-2zL] + w_o(x,y)
\end{array}$$

Substitute these into the shear strain relations to get

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \quad \longrightarrow \quad u_o = \bar{u}(z) + \alpha y + \beta_1, \quad v_o = \bar{v}(z) - \alpha x + \beta_2$$

The other shear strain relations give

$$\begin{array}{lcl}
\gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = 0 = -\dfrac{\nu\rho g x}{E} + \dfrac{\partial \bar{u}}{\partial z} + \dfrac{\partial w_o}{\partial x} & \longrightarrow & w_o = \dfrac{\nu\rho g x^2}{2E} - \dfrac{\partial \bar{u}}{\partial z}x + f(y) \\
\gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} = 0 = -\dfrac{\nu\rho g y}{E} + \dfrac{\partial \bar{v}}{\partial z} + \dfrac{\partial w_o}{\partial y} & \longrightarrow & w_o = \dfrac{\nu\rho g y^2}{2E} - \dfrac{\partial \bar{v}}{\partial z}y + g(y)
\end{array}$$

Since both expressions for w_o must be the same and since they are independent of z, we conclude that

$$\bar{u} = c_1 z + c_2, \quad \bar{v} = c_3 z + c_4\,, \qquad f(y) = \frac{\nu\rho g y^2}{2E} + c_5 y + c_6, \quad g(x) = \frac{\nu\rho g x^2}{2E} + c_7 x + c_8$$

Putting all these together, we get the displacement fields as

$$\begin{array}{lcl}
u & = & -\dfrac{\nu\rho g}{E}x[z-L] + \alpha y + c_1 z + \beta_1 \\
v & = & -\dfrac{\nu\rho g}{E}y[z-L] - \alpha x + c_3 z + \beta_2 \\
w & = & +\dfrac{\rho g}{2E}[z^2-2zL] + \dfrac{\nu\rho g}{2E}[x^2+y^2] - c_1 x - c_3 y + \beta_3
\end{array}$$

The constants are obtained from the boundary conditions.

Let the bottom center of the block be at the origin and let it have zero displacements then $\beta_i = 0$. Furthermore, because of symmetry, the vertical line $x = 0$, $y = 0$ remains straight and vertical hence the slopes $\partial u/\partial z$ and $\partial v/\partial z$ at the origin are zero. Finally, to suppress rotation about the z-axis set $\partial u/\partial y - \partial v/\partial x$ to zero. The resulting displacement fields are

$$\begin{array}{lcl}
u & = & -\dfrac{\nu\rho g}{E}x[z-L] \\
v & = & -\dfrac{\nu\rho g}{E}y[z-L] \\
w & = & \dfrac{\rho g}{2E}[z^2-zL] + \dfrac{\nu\rho g}{2E}[x^2+y^2]
\end{array}$$

The most interesting aspect of this solution is that the plane $z = 0$ does not remain horizontal; in fact, it has the vertical displacement

$$w = \frac{\nu\rho g}{2E}[x^2 + y^2]$$

To suppress this would require a complicated set of tractions localized to the plane, in other words, stress concentrations would be induced at the edges.

Figure 5.4(a) shows the finite element mesh using Hex20 elements. The gravity loads for each node are computed by

$$P_{zi} = \int_{v^o} \rho g h_i \, dV^o = \rho g \int_v h_i |J| \, dV$$

This is computed numerically using full integration. The boundary conditions imposed are that there is no vertical displacement at the bottom, the center is fixed and the nodes along $x = 0$, $z = 0$ are restrained to move only along the x-direction.

The exaggerated deformed shapes are shown in Figure 5.4(b). Note that there is no lateral contraction at the top. The normalized displacements are

$$\frac{w_{max}}{-\rho g L^2/2E} = 1.00200\,, \qquad \frac{v_{max}}{\nu\rho g b L/E} = 0.98667$$

These are quite close to the analytical solution. When the base is fully constrained, the corresponding results are

$$\frac{w_{max}}{-\rho g L^2/2E} = 0.9662\,, \qquad \frac{v_{max}}{\nu\rho g b L/E} = 0.00$$

The constraining effect also affects the stresses with the unconstrained being

$$\frac{\sigma_{xx}}{\rho g L} = -0.0011\,, \qquad \frac{\sigma_{yy}}{\rho g L} = -0.0028\,, \qquad \frac{\sigma_{zz}}{\rho g L} = -0.9483 \times \frac{4}{3.77} = -1.0061$$

compared to

$$\frac{\sigma_{xx}}{\rho g L} = +0.2940\,, \qquad \frac{\sigma_{yy}}{\rho g L} = +0.2989\,, \qquad \frac{\sigma_{zz}}{\rho g L} = -0.9901 \times \frac{4}{3.77} = -1.0505$$

The stresses are at the integration point closest to the bottom center and the σ_{zz} is given an approximate correction to estimate the stress at the base. It is clear that the constraint generates a stress that is on the order of Poisson's ratio times the axial stress. ■

Uniqueness of Solutions

For an elasticity problem, the body force b_i and the boundary conditions over the boundary surface A, are usually prescribed. There are two types of boundary conditions: displacement-prescribed and traction-prescribed. In general, the total boundary surface can be divided into two parts, A_u and A_t over which the displacements and tractions are prescribed, respectively. A solution to an elasticity problem is one for which the stresses satisfy the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0$$

the strains are compatible, and the boundary conditions in terms of the tractions and displacements are satisfied. This solution is unique in the sense that the state of stress (and strain) is determinate without ambiguity. By contrast, the nonlinear field equations do not lead to a unique solution.

To prove the uniqueness of solution, we start with the assumption that there are two possible solutions u'_i and u''_i to the same problem. Denote the difference of the solution by

$$\begin{aligned} u_i &= u'_i - u''_i \\ \epsilon_{ij} &= \epsilon'_{ij} - \epsilon''_{ij} \\ \sigma_{ij} &= \sigma'_{ij} - \sigma''_{ij} \end{aligned}$$

Since σ'_{ij} and σ''_{ij} satisfy the same equilibrium equations and boundary conditions, it is easy to see that the difference solution satisfies

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{in } V$$

and that the boundary conditions are

$$\begin{aligned} u_i &= 0 \qquad \text{on } A_u \\ t_i &= 0 \qquad \text{on } A_t \end{aligned}$$

Since the strain energy density function is given as

$$\overline{\mathcal{U}} = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij} \qquad (\neq \mathcal{U}' - \mathcal{U}'')$$

we have

$$\int_V \overline{\mathcal{U}}\, dV = \tfrac{1}{2}\int_V \sigma_{ij}\epsilon_{ij} dV = \tfrac{1}{2}\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j} dV$$

Integrating by parts, we obtain

$$\begin{aligned} \tfrac{1}{2}\int_V \sigma_{ij}\frac{\partial u_i}{\partial x_j} dV &= \tfrac{1}{2}\int_V \frac{\partial}{\partial x_j}(\sigma_{ij}u_i) dV - \tfrac{1}{2}\int_V \frac{\partial \sigma_{ij}}{\partial x_j} u_i dV \\ &= \tfrac{1}{2}\int_A \sigma_{ij}u_i n_j dA \\ &= \tfrac{1}{2}\int_A t_i u_i dA \\ &= \tfrac{1}{2}\int_{A_u} t_i u_i dA + \tfrac{1}{2}\int_{A_t} t_i u_i dA \end{aligned}$$

Since u_i vanishes on A_u, and t_i vanish on A_t, we have

$$\int \overline{\mathcal{U}}\, dV = 0$$

This is possible only if $\overline{\mathcal{U}} = 0$, which in turn requires ϵ_{ij} and σ_{ij} to be zero. Thus, we conclude that the only difference between the two solutions is a rigid body motion (since this does not contribute to the stresses or strains). Moreover, if the displacements are known on any part of the boundary, then they are determined uniquely in V and on A.

5.2 Plane Problems

It is very difficult to solve general 3-D problems, and therefore various schemes of simplification have arisen. One such scheme, which has a lot of practical use, is to reduce the dimensionality of the problem. In this section, we consider those problems where the essential behaviors occur in a plane. Chapter 7 considers other types of reductions of the field equations in forming approximate structural theories.

Reduction to 2-D Equations

The reduction of a 3-D problem to an equivalent 2-D one involves approximation and therefore is not valid for general cases. To make explicit the situations of interest, consider a 3-D body such that:

- the body is bounded by two flat surfaces lying in the $1-2$ plane,
- the cross-section of the body is uniform in the x_3 direction,
- the load is uniformly distributed in the x_3 direction,
- there are no shears on the flat faces.

Note that this body does not have to be thin. The second of these restrictions rules out bending while the third rules out torsion.

Plane Strain Formulation

The *plane strain* assumption is that $u_3 = 0$. This leads to the zero strains

$$\epsilon_{33} = 0\,, \qquad \epsilon_{13} = 0\,, \qquad \epsilon_{23} = 0$$

In summary, the strains and strain-displacement relation become

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \epsilon_{\alpha\beta} = \tfrac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right] \qquad \alpha,\beta = 1,2$$

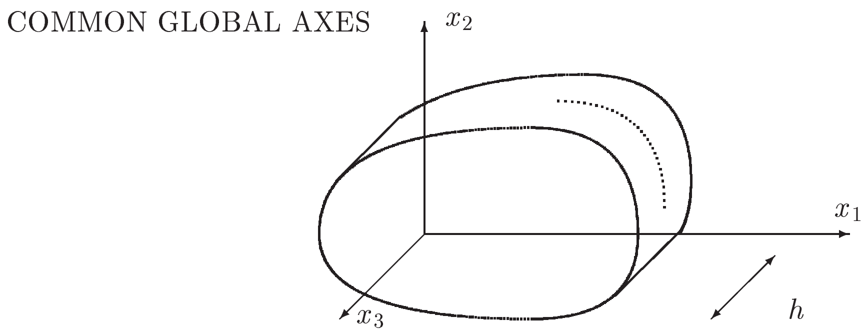


Figure 5.5: Solid body bounded by two planes.

The only non-trivial compatibility condition is

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

Based on the linear, isotropic Hooke's law, the stresses under plane strain conditions must be

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$

In expanded form, this is

$$\begin{aligned}
\sigma_{11} &= 2\mu\epsilon_{11} + \lambda(\epsilon_{11} + \epsilon_{22} + 0) \\
\sigma_{22} &= 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22} + 0) \\
\sigma_{33} &= 2\mu 0 \quad + \lambda(\epsilon_{11} + \epsilon_{22} + 0) \\
\sigma_{12} &= 2\mu\epsilon_{12} \\
\sigma_{13} &= 0 \\
\sigma_{23} &= 0
\end{aligned}$$

This can be rewritten so as to group the $1, 2$ terms together as

$$\sigma_{\alpha\beta} = 2\mu\epsilon_{\alpha\beta} + \lambda\delta_{\alpha\beta}\epsilon_{\delta\delta}\,, \qquad \sigma_{33} = \lambda\epsilon_{\delta\delta}$$

with α, β ranging from 1 to 2. Thus, in plane strain problems, the ϵ_{33} normal strain is zero, but the corresponding σ_{33} normal stress is not.

The reduced stress tensor is given by

$$\sigma_{ij} = \begin{bmatrix} \sigma_{\alpha\beta} & 0 \\ 0 & \sigma_{33} \end{bmatrix} \qquad \alpha,\, \beta = 1, 2$$

Provided that the body force ρb_3 is zero, the equilibrium equations reduce to

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho b_\alpha = 0 \qquad \alpha,\, \beta = 1, 2$$

Thus we are left with two equilibrium equations.

Approximate Nature of Plane Stress

The *plane stress* assumption is that if the plate is very thin then the stresses σ_{13}, σ_{23}, σ_{33}, are zero. If we can impose these conditions then we can simplify our equations as done for the plane strain case. However, unlike the plane strain case were the imposition of $u_3 = 0$ leads to an exact two-dimensional formulation, it is not clear *a priori* that the above stress state can actually be imposed. This we will first investigate.

Let us seek an exact solution under the restriction

$$\sigma_{13} = 0\,, \qquad \sigma_{23} = 0\,, \qquad \sigma_{33} = 0$$

This solution must satisfy the equilibrium and compatibility equations, which under the assumption of no body forces reduce to, respectively,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0\,, \qquad \frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \left(\frac{1}{1+\nu}\right) \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = 0$$

In what follows, roman subscripts range from 1 to 3, while greek subscripts range from 1 to 2.

The compatibility equation with $ij = 13$, 23, 33 become

$$\frac{\partial^2 \Psi}{\partial x_1 \partial x_3} = 0\,, \qquad \frac{\partial^2 \Psi}{\partial x_2 \partial x_3} = 0\,, \qquad \frac{\partial^2 \Psi}{\partial x_3 \partial x_3} = 0\,, \qquad \Psi \equiv \sigma_{11} + \sigma_{22}$$

We therefore conclude that the distribution of the sum of the normal stresses is of the form

$$\Psi = c x_3 + f(x_1,\, x_2)$$

Let us now restrict the stress distributions to be symmetric about the middle plane, then $c = 0$ and we have

$$\Psi = \Psi(x_1,\, x_2)$$

The normal stresses are only a function of the in-plane coordinates. By setting $ij =$ 11, 22 in the compatibility equation and adding, we get

$$\nabla_2^2 \Psi = 0\,, \qquad \nabla_2^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

showing that Ψ is a harmonic function.

The explicit form for the two equilibrium equations is

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0\,, \qquad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

We can satisfy these equations by introducing a function such that

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2}\,, \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2}\,, \qquad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

The function ϕ is called a *stress function* which we will develop in more detail later in this chapter. Adding the normal stresses, we get

$$\Psi = \nabla_2^2\phi\,, \qquad \nabla_2^2\nabla_2^2\phi = \nabla_2^2\Psi = 0$$

showing that ϕ is a bi-harmonic function.

The compatibility equation with $ij = 11$ is

$$\nabla_2^2\sigma_{11} + \frac{\partial^2\sigma_{11}}{\partial x_3^2} + \frac{1}{1+\nu}\frac{\partial^2\Psi}{\partial x_1^2} = 0$$

Noting that

$$\nabla_2^2\sigma_{11} = \frac{\partial^2}{\partial x_2^2}\nabla_2^2\phi = \frac{\partial^2}{\partial x_2^2}\Psi\,, \qquad \frac{\partial^2\Psi}{\partial x_1^2} = -\frac{\partial^2\Psi}{\partial x_2^2}$$

The compatibility equation can be arranged as

$$\frac{\partial^2}{\partial x_2^2}\left\{\frac{\nu}{1+\nu}\Psi + \frac{\partial^2\phi}{\partial x_3^2}\right\} = 0$$

The other two compatibility equations (with $ij = 22$ and 12, respectively) give

$$\frac{\partial^2}{\partial x_1^2}\left\{\frac{\nu}{1+\nu}\Psi + \frac{\partial^2\phi}{\partial x_3^2}\right\} = 0\,, \qquad \frac{\partial^2}{\partial x_1\partial 2}\left\{\frac{\nu}{1+\nu}\Psi + \frac{\partial^2\phi}{\partial x_3^2}\right\} = 0$$

By integrating these we conclude that

$$\frac{\nu}{1+\nu}\Psi + \frac{\partial^2\phi}{\partial x_3^2} = a + bx_1 + cx_2$$

where a, b and c are arbitrary constants. Integrating the stress function ϕ then gives

$$\phi(x_1,x_2,x_3) = -\frac{\nu}{1+\nu}\Psi(x_1,x_2)\tfrac{1}{2}x_3^2 + [a + bx_1 + cx_2]\tfrac{1}{2}x_3^2 + \phi_1(x_1,x_2)x_3 + \phi_0(x_1,x_2)$$

The coefficients a, b, c do not participate in the stress solution and hence may be taken as zero. We are imposing that the stresses are symmetric in x_3 hence we also take $\phi_1(x_1,x_2)$ as zero. We thus have the stress function representation

$$\phi(x_1,x_2,x_3) = \phi_0(x_1,x_2) - \frac{\nu}{1+\nu}\Psi(x_1,x_2)\tfrac{1}{2}x_3^2$$

where Ψ is a harmonic function and ϕ_0 is a bi-harmonic function.

From the above solution representation, we see that the stresses in our 3-D problem is comprised of two parts; the first depends on the bi-harmonic function ϕ_0 and the latter on the harmonic function Ψ. This latter solution, being proportional to x_3^2, may be made as small as we please compared to the first if we restrict the plate to be sufficiently thin.

In conclusion, if we impose the plane stress conditions, we can get a solution that satisfies equilibrium but the compatibility conditions will be approximated. The error can be made as minimal as necessary if the plate thickness is restricted to being very thin. Chapter 7 provides an alternative justification for plane stress theories.

Plane Stress Formulation

Based on the foregoing discussion, we will assume

$$\sigma_{13} = 0\,, \qquad \sigma_{23} = 0\,, \qquad \sigma_{33} = 0$$

That is, the stress tensor reduces to

$$\sigma_{ij} \quad \Rightarrow \quad \sigma_{ij} = \begin{bmatrix} \sigma_{\alpha\beta} & 0 \\ 0 & \sigma_{33} \end{bmatrix} \qquad \alpha,\, \beta = 1,2$$

The equilibrium equations become

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho b_\alpha = 0 \qquad \alpha\,,\, \beta = 1,2$$

Thus we are left with two equilibrium equations.

Assume the material obeys the linear, isotropic Hooke's law such that

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}$$

In expanded form, this is

$$\begin{aligned} \sigma_{11} &= 2\mu\epsilon_{11} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \\ \sigma_{22} &= 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) \\ \sigma_{33} &= 2\mu\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0 \\ \sigma_{12} &= 2\mu\epsilon_{12} \\ \sigma_{13} &= 2\mu\epsilon_{13} = 0 \\ \sigma_{23} &= 2\mu\epsilon_{23} = 0 \end{aligned}$$

In contrast to the plane strain case, we have a non-zero ϵ_{33} strain component. This is related to the other normal strains by

$$\epsilon_{33} = \Big(\frac{-\lambda}{2\mu + \lambda}\Big)[\epsilon_{11} + \epsilon_{22})$$

This can be rewritten so as to group the $1,2$ terms together as

$$\sigma_{\alpha\beta} = 2\mu\epsilon_{\alpha\beta} + \left(\frac{\lambda 2\mu}{2\mu + \lambda}\right)\delta_{\alpha\beta}\epsilon_{\gamma\gamma}$$

In summary, the strains and strain-displacement relation become

$$\epsilon_{ij} \quad \Rightarrow \quad \epsilon_{ij} = \begin{bmatrix} \epsilon_{\alpha\beta} & 0 \\ \epsilon_{33} & \epsilon_{33} \end{bmatrix}, \qquad \epsilon_{\alpha\beta} = \tfrac{1}{2}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right] \qquad \alpha, \beta = 1,2$$

We assume in our approximation that the only strains participating in the compatibility equations are the in-plane strains. That is,

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

When the body is only somewhat thin, the resulting case is referred to as *Generalized Plane Stress.*

Summary of 2-D Plane Elasticity Problems

The basic unknowns are

2	Displacements :	u_1 , u_2	or	u_α	$\alpha = 1, 2$
3	Strains :	$\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$	or	$\epsilon_{\alpha\beta}$	$\beta = 1, 2$
3	Stresses:	$\sigma_{11}, \sigma_{22}, \sigma_{12}$	or	$\sigma_{\alpha\beta}$	

Note that σ_{33} in plane strain and ϵ_{33} in plane stress, are not part of the basic unknowns but are obtained after the solution. The appropriate field and material relations are

Strain-Displacement: $$\epsilon_{\alpha\beta} = \tfrac{1}{2}\Big[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\Big]$$

Compatibility: $$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

Equilibrium: $$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho b_\alpha = 0$$

Stress-Strain: $$\sigma_{\alpha\beta} = 2\mu[\epsilon_{\alpha\beta} + \frac{3-\kappa}{2\kappa-2}\delta_{\alpha\beta}\epsilon_{\gamma\gamma}] = 2\mu[\epsilon_{\alpha\beta} + \nu_\sigma\delta_{\alpha\beta}\epsilon_{\gamma\gamma}]$$
$$\epsilon_{\alpha\beta} = \frac{1}{2\mu}[\sigma_{\alpha\beta} - \frac{3-\kappa}{4}\delta_{\alpha\beta}\sigma_{\gamma\gamma}] = \frac{1}{2\mu}[\sigma_{\alpha\beta} - \nu_\epsilon\delta_{\alpha\beta}\sigma_{\gamma\gamma}]$$

Plane Strain: $$\kappa = 3 - 4\nu\,, \qquad \nu_\sigma = \frac{\nu}{1-2\nu}\,, \qquad \nu_\epsilon = \nu$$
$$\epsilon_{33} = 0$$
$$\sigma_{33} = \lambda(\epsilon_{11} + \epsilon_{22}) = \nu(\sigma_{11} + \sigma_{22})$$

Plane Stress: $$\kappa = \frac{3-\nu}{1+\nu}\,, \qquad \nu_\sigma = \nu\,, \qquad \nu_\epsilon = \frac{\nu}{1+\nu}$$
$$\sigma_{33} = 0$$
$$\epsilon_{33} = \frac{-\lambda}{2\mu+\lambda}(\epsilon_{11} + \epsilon_{22}) = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

Substituting the strains in terms of stresses into the compatibility equations, using the equilibrium equations and rearranging gives

$$\nabla^2(\sigma_{11} + \sigma_{22}) = -\frac{4\rho}{(1+\kappa)}\left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2}\right)$$

This is called the Beltrami-Mitchell equation. Note that if no body forces are present then

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0$$

That is, the first invariant of stress is an harmonic function.

Cylindrical Coordinates

Some, but not all, entities follow the usual transformation law with a change of coordinate system, For example, in the cylindrical coordinates (r, θ, z), the strain components $\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}, \epsilon_{rz}, \epsilon_{r\theta}, \epsilon_{z\theta}$ are related to the rectangular components $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$, $\epsilon_{xy}, \epsilon_{yz}, \epsilon_{zx}$ by the usual tensor transformation law. That is, the stress and strain components can be referred to a local rectangular frame of reference oriented in the direction of the curvilinear coordinates.

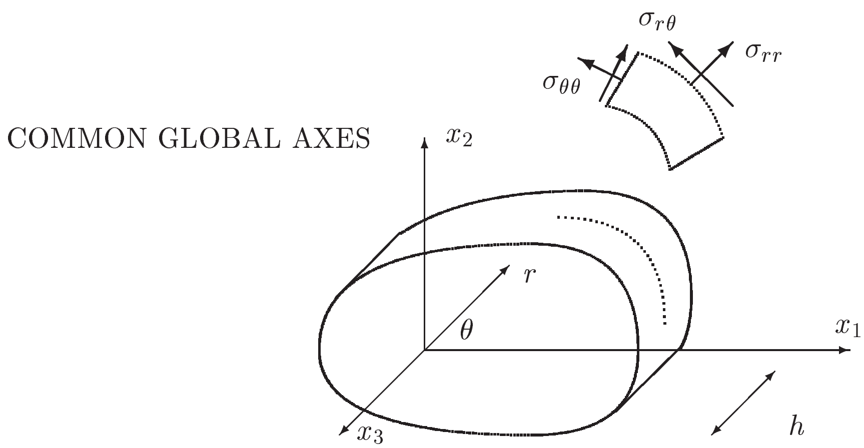


Figure 5.6: Cylindrical coordinates.

However, if displacement vectors are resolved into components in the directions of the curvilinear coordinates, the strain-displacement relationship involves derivatives of the displacement components and, therefore, is influenced by the curvature of the coordinate system. The strain-displacement relations may appear quite different from the corresponding formulas in rectangular coordinates.

I: Transformation of Derivatives

We start with the relations between the cylindrical coordinates (r, θ, z) and the rectangular coordinates (x, y, z) given by

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,, \qquad z = z$$

and

$$r^2 = x^2 + y^2\,, \qquad \theta = \tan^{-1}\frac{y}{x}\,, \qquad z = z$$

The derivatives are

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\
\frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos\theta}{r}
\end{aligned}
$$

By using the chain rule for differentiation, it follows that any derivatives with respect to x and y in the Cartesian equations may be transformed into derivatives with respect to r and θ as

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}
\end{aligned}
$$

II: Displacements

In the cylindrical coordinate system, the components of the displacement vector are denoted by u_r, u_θ, u_z. Components of the same vector resolved in the directions of the rectangular coordinates are u_x, u_u, u_z. These components of displacement are related according to

$$
\begin{aligned}
u_x &= u_r\cos\theta - u_\theta\sin\theta \\
u_y &= u_r\sin\theta + u_\theta\cos\theta \\
u_z &= u_z
\end{aligned}
$$

The transformation can be written alternatively in matrix form as

$$
\left\{\begin{array}{c} u_r \\ u_\theta \end{array}\right\} = \left[\begin{array}{cc} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{array}\right]\left\{\begin{array}{c} u_x \\ u_y \end{array}\right\}, \qquad \left\{\begin{array}{c} u_x \\ u_y \end{array}\right\} = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{array}\right]\left\{\begin{array}{c} u_r \\ u_\theta \end{array}\right\}
$$

III: Strain-Displacement

Set a local Cartesian system $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ at point (r, θ, z) in which $\hat{e}_r, \hat{e}_\theta$, and $\hat{e}_z$ are the unit base vectors in the r, θ, and z direction, respectively. Denoting the strain components with respect to this coordinate system by ϵ'_{ij}, that is,

$$
[\,\epsilon'_{ij}\,] = \left[\begin{array}{ccc} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{array}\right]
$$

and using the transformation law

$$
\epsilon'_{ij} = \beta_{im}\beta_{jn}\epsilon_{mn}\,, \qquad [\beta_{ij}] = \left[\begin{array}{ccc} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]
$$

we obtain the relation between the two sets of strain as

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_{rr}\cos^2\theta + \epsilon_{\theta\theta}\sin^2\theta - \epsilon_{r\theta}\sin 2\theta \\
\epsilon_{yy} &= \epsilon_{rr}\sin^2\theta + \epsilon_{\theta\theta}\cos^2\theta + \epsilon_{r\theta}\sin 2\theta \\
\epsilon_{xy} &= (\epsilon_{\theta\theta} - \epsilon_{rr})\cos\theta\sin\theta + \epsilon_{r\theta}(\cos^2\theta - \sin^2\theta) \\
\epsilon_{zx} &= \epsilon_{zr}\cos\theta + \epsilon_{z\theta}\sin\theta \\
\epsilon_{zy} &= -\epsilon_{zr}\sin\theta + \epsilon_{z\theta}\cos\theta \\
\epsilon_{zz} &= \epsilon_{zz}
\end{aligned}$$

Substituting the strain-displacement relation in Cartesian coordinates into the above, and recognizing such terms, for example, as

$$\begin{aligned}
\frac{\partial u_x}{\partial x} &= (\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(u_r\cos\theta - u_\theta\sin\theta) \\
&= \cos^2\theta\frac{\partial u_r}{\partial r} + \frac{\sin^2\theta}{r}u_r - \frac{\cos\theta\sin\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\cos\theta\sin\theta}{r}u_\theta - \frac{\cos\theta\sin\theta}{r}\frac{\partial u_r}{\partial\theta}
\end{aligned}$$

we obtain

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\epsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
2\epsilon_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\
2\epsilon_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
2\epsilon_{\theta z} &= \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}
\end{aligned}$$

IV: Stress and Equilibrium

Using the local Cartesian system $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$ the components of the stress tensor at a point (r, θ, z) are denoted by

$$[\sigma'_{ij}] = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

From the coordinate transformation law we obtain

$$\begin{aligned}
\sigma_{xx} &= \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta \\
\sigma_{yy} &= \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta
\end{aligned}$$

$$
\begin{aligned}
\sigma_{zz} &= \sigma_{zz} \\
\sigma_{xy} &= (\sigma_{\theta\theta} - \sigma_{rr}) \sin\theta \cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta) \\
\sigma_{xz} &= \sigma_{rz} \cos\theta - \sigma_{\theta z} \sin\theta \\
\sigma_{yz} &= \sigma_{rz} \sin\theta + \sigma_{\theta z} \cos\theta
\end{aligned}
$$

The derivation of the equilibrium equations in the cylindrical coordinate system is a straightforward exercise following closely to that of the strain-displacement. We get

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} + \rho b_\theta &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + \rho b_z &= 0
\end{aligned}
$$

where b_r, b_θ, and b_z are the components of the body force vector $\hat{b}$ in the r, θ, and z directions, respectively.

V: Summary of Plane Elasticity in Cylindrical Coordinates

The foregoing equations apply to 3-D bodies, we now restrict the equations to the case of plane elasticity. The summary is similar to that for rectangular coordinates for plane problems in that we assume we have removed the z dependence.

The basic unknowns are

2 Displacements:	$u_r\ ,\ u_\theta$
3 Strains:	$\epsilon_{rr}\ ,\ \epsilon_{\theta\theta}\ ,\ \epsilon_{r\theta}$
3 Stresses:	$\sigma_{rr}\ ,\ \sigma_{\theta\theta}\ ,\ \sigma_{r\theta}$

and the corresponding field equations in terms of these are

Strain-Displacement:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}$$

$$2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)$$

Compatibility:

$$r\frac{\partial \epsilon_{rr}}{\partial r} - \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} - \frac{\partial}{\partial r}\left(r^2\frac{\partial \epsilon_{\theta\theta}}{\partial r}\right) + 2\frac{\partial}{\partial r}\left(r\frac{\partial \epsilon_{r\theta}}{\partial \theta}\right) = 0$$

Equilibrium:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \rho b_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \rho b_\theta = 0$$

$$
\begin{aligned}
\text{Stress-Strain:} \quad & \sigma_{rr} = 2G[\epsilon_{rr} + \frac{3-\kappa}{2\kappa-2}(\epsilon_{rr}+\epsilon_{\theta\theta})] \\
& \sigma_{\theta\theta} = 2G[\epsilon_{\theta\theta} + \frac{3-\kappa}{2\kappa-2}(\epsilon_{rr}+\epsilon_{\theta\theta})] \\
& \sigma_{r\theta} = 2G\epsilon_{r\theta} \\
& \epsilon_{rr} = \frac{1}{2G}[\sigma_{rr} - \frac{3-\kappa}{4}(\sigma_{rr}+\sigma_{\theta\theta})] \\
& \epsilon_{\theta\theta} = \frac{1}{2G}[\sigma_{\theta\theta} - \frac{3-\kappa}{4}(\sigma_{rr}+\sigma_{\theta\theta})] \\
& \epsilon_{r\theta} = \frac{1}{2G}\sigma_{r\theta} \\
\text{Plane Strain:} \quad & \kappa = 3 - 4\nu \\
& \epsilon_{zz} = 0 \\
& \sigma_{zz} = \lambda(\epsilon_{rr}+\epsilon_{\theta\theta}) = \nu(\sigma_{rr}+\sigma_{\theta\theta}) \\
\text{Plane Stress:} \quad & \kappa = \frac{3-\nu}{1+\nu} \\
& \epsilon_{zz} = -\frac{\lambda}{(2G+\lambda)}(\epsilon_{rr}+\epsilon_{\theta\theta}) = -\frac{\nu}{E}(\sigma_{rr}+\sigma_{\theta\theta}) \\
& \sigma_{zz} = 0
\end{aligned}
$$

Substituting the strains in terms of stresses into the compatibility equations, using the equilibrium equations and rearranging gives

$$
\nabla^2(\sigma_{rr}+\sigma_{\theta\theta}) = -\frac{4\rho}{(1+\kappa)}\left(\frac{\partial b_r}{\partial r} + \frac{1}{r}\frac{\partial b_\theta}{\partial \theta}\right)
$$

This is the Beltrami-Mitchell equation in cylindrical coordinates.

5.3 Airy Stress Function Formulation

The plane theory of elasticity has eight unknown functions described by a set of eight coupled partial differential equations. A traditional approach to solving a system of equations is to reduce the number of unknowns at the expense of increasing the order of the governing differential equations; in the theory of elasticity we reduce the system to a single unknown function called the Airy stress function.

Cartesian and Cylindrical Coordinates

The stress formulation reduces to finding a set of stresses that satisfy equilibrium

$$
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho b_x = 0\,, \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho b_y = 0
$$

and compatibility

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \frac{-4\rho}{(1+\kappa)}\left[\frac{\partial b_x}{\partial x} + \frac{\partial b_x}{\partial y}\right] = \frac{4}{(1+\kappa)}\nabla^2 V$$

We now further reduce this formulation to the determination of a single function.

Suppose the body forces can be derived from a potential $\mathcal{V}(x, y)$ as

$$\rho b_x = -\frac{\partial \mathcal{V}}{\partial x}, \qquad \rho b_y = -\frac{\partial \mathcal{V}}{\partial y}$$

For example, gravity loading in the y- direction is described by $\mathcal{V} = \rho g y$, then

$$\rho b_x = 0, \qquad \rho b_y = -\rho g$$

(Note that ρb_y is a force per volume since $\rho g = \rho W/M = \rho W/\rho V = W/V$.) Further, let the stresses be obtained from a stress function $\phi(x, y)$ as

$$\begin{aligned} \sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} + \mathcal{V} \\ \sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} + \mathcal{V} \\ \sigma_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \tag{5.1}$$

It can be easily verified by substitution that stresses obtained in this manner will automatically satisfy equilibrium.

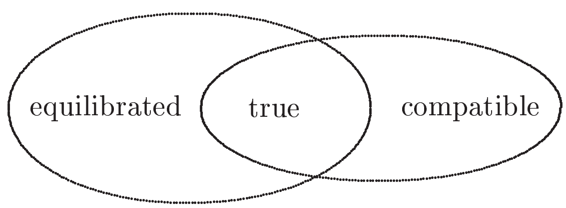


Figure 5.7: A true stress field satisfies equilibrium and compatibility.

But the stresses must also satisfy compatibility; that is, on substituting for the stresses in terms of the stress function, compatibility becomes

$$\nabla^2\nabla^2\phi = -\frac{2(\kappa - 1)}{(1+\kappa)}\nabla^2\mathcal{V}$$

The function ϕ is called the Airy Stress Function. Note that

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\phi$$

The general solution to the above equation can be put in the form

$$\phi = \phi_c + \phi_p$$

where the functions ϕ_c, ϕ_p are the complementary and particular solutions, respectively. They satisfy

$$\begin{aligned} \nabla^2\nabla^2\phi_c &= 0 \\ \nabla^2\nabla^2\phi_p &= -\frac{2(\kappa - 1)}{(1 + \kappa)}\nabla^2\mathcal{V} \end{aligned}$$

Thus, ϕ_c is a bi-harmonic function, while ϕ_p depends on the body force field and is not necessarily bi-harmonic.

The Airy stress function is a scalar function, hence most of the results for cylindrical coordinates follow directly from the corresponding Cartesian results.

The stresses are related to the stress function by

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \mathcal{V} \\ \sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} + \mathcal{V} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \end{aligned}$$

The radial and hoop components of the body force are given by

$$\rho b_r = -\frac{\partial\mathcal{V}}{\partial r}, \qquad \rho b_\theta = -\frac{1}{r}\frac{\partial\mathcal{V}}{\partial\theta}$$

The Airy stress function still satisfies the bi-harmonic equation but written as

$$\nabla^4\phi = \nabla^2\nabla^2\phi = -2\frac{\kappa - 1}{\kappa + 1}\nabla^2 V, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

That is, the only difference is that the Laplace operator is written in cylindrical coordinates.

Axisymmetric Problems

We begin the use of the Airy stress function by looking at a couple of problems that are axisymmetric in the stresses. Note that this does not necessarily mean that the displacements are also axisymmetric. To make the discussion explicit, we will consider a curved beam problem that is referred to as *Golovin's Curved Beam Problem (1881)*.

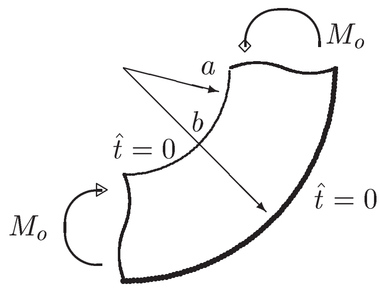


Figure 5.8: Curved beam with resultant moments.

I: Stresses

A curved beam is subjected to end moments M_o as shown in Figure 5.8. From the moment balance condition, it is evident that the moment on any radial cross-section along the beam is constant. In addition, the surface tractions are independent of θ. Hence, this is an axisymmetric problem in stress (although not necessarily in displacements).

For axisymmetric stress problems, the potential functions ϕ and $\mathcal{V}$, are independent of θ. We get for the bi-harmonic equation

$$\begin{aligned} \nabla^2 &= \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} = \frac{1}{r}\left(r\frac{d}{dr}\right) \\ \nabla^4 &= \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]\right\} \end{aligned}$$

By direct integration of $\nabla^2\phi_c = 0$, we obtain the general solution for ϕ_c as

$$\phi_c = A\log_n r + Br^2\log_n r + Cr^2 + D$$

where A, B, C, D are constants of integration. Since D will not contribute to the stress field it can be dropped.

The body force is absent in our problem, therefore the Airy stress function is given by entirely by ϕ_c

$$\phi(r,\theta) = \phi(r) = A\log_n r + Br^2\log_n r + Cr^2$$

This gives the stresses as

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\log_n r) + 2C \\ \sigma_{\theta\theta} &= \frac{\partial\phi^2}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\log_n r) + 2C \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \end{aligned}$$

There are three coefficients to be solved for by satisfying the boundary conditions. The boundary conditions to be imposed are

$$
\begin{aligned}
\text{at } r = a: \quad & t_r &=& \ \sigma_{rr} n_r + \sigma_{r\theta} n_\theta = -\sigma_{rr} = 0 \\
& t_\theta &=& \ \sigma_{r\theta} n_r + \sigma_{\theta\theta} n_\theta = -\sigma_{r\theta} = 0 \\
\text{at } r = b: \quad & t_r &=& \ \sigma_{rr} = 0 \\
& t_\theta &=& \ \sigma_{r\theta} = 0
\end{aligned}
$$

These become

$$
\begin{aligned}
\sigma_{rr}|_{r=a} = 0 &= \frac{A}{a^2} + B(1 + 2\log_n a) + 2C \\
\sigma_{rr}|_{r=b} = 0 &= \frac{A}{b^2} + B(1 + 2\log_n b) + 2C
\end{aligned}
$$

since the shear traction condition is automatically satisfied. One more equation, in addition to the above, is needed to determine the constants A, B and C. We cannot impose tractions on the ends as the boundary conditions simply because we do not know them. So we impose conditions on the resultants instead. That is,

$$
\begin{aligned}
F_\theta &= \int_a^b \sigma_{\theta\theta}\, dr = 0 \\
F_r &= \int_a^b \sigma_{r\theta}\, dr = 0 \\
M &= \int_a^b \sigma_{\theta\theta}\, r dr = M_o
\end{aligned}
$$

These become, on substituting for the stresses,

$$
\begin{aligned}
F_\theta &= \int \frac{\partial^2\phi}{\partial r^2} dr = \left.\frac{\partial\phi}{\partial r}\right|_b - \left.\frac{\partial\phi}{\partial r}\right|_a = b\,\sigma_{rr}(b) - a\,\sigma_{rr}(a) = 0 \\
F_r &= 0 \\
M_o &= \int \frac{\partial^2\phi}{\partial r^2} r dr = r\left.\frac{\partial\phi}{\partial r}\right|_a^b - \int \frac{\partial\phi}{\partial r} dr = b^2\sigma_{rr}(b) - a^2\sigma_{rr}(a) - \phi(b) + \phi(a) \\
&= -A\log_n(\frac{b}{a}) - B[b^2\log_n b - a^2\log_n a] - C[b^2 - a^2]
\end{aligned}
$$

Solving these equations for the coefficients in terms of M_o gives

$$
\begin{aligned}
A &= \frac{4M_o}{N} b^2 a^2 \log_n\left(\frac{b}{a}\right) \\
B &= \frac{2M_o}{N}[b^2 - a^2] \\
C &= -\frac{2M_o}{N}\left[b^2 - a^2 + 2(b^2\log_n b - a^2\log_n a)\right] \\
N &= (b^2 - a^2)^2 - 4b^2a^2\left[\log_n \frac{b}{a}\right]^2
\end{aligned}
$$

and we finally have for the stresses

$$\begin{aligned}
\sigma_{rr} &= \frac{4M_o b^2}{N}\Big[\ \frac{a^2}{r^2}\log_n\left(\frac{b}{a}\right) + \log_n\left(\frac{r}{b}\right) + \frac{a^2}{b^2}\log_n\left(\frac{a}{r}\right)\Big] \\
\sigma_{\theta\theta} &= \frac{4M_o b^2}{N}\Big[-\frac{a^2}{r^2}\log_n\left(\frac{b}{a}\right) + \log_n\left(\frac{r}{b}\right) + \frac{a^2}{b^2}\log_n\left(\frac{a}{r}\right) + 1 - \frac{a^2}{b^2}\Big] \\
\sigma_{r\theta} &= 0
\end{aligned}$$

The stress distribution is sketched in Figure 5.9. Notice the very large increase in hoop stress at the inner radius.

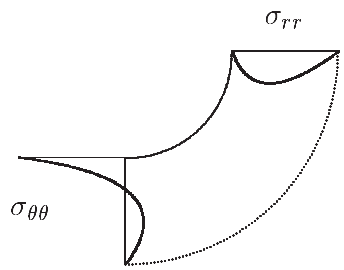


Figure 5.9: Stress distributions in curved beam.

A significant aspect of this solution is that the hoop stress $\sigma_{\theta\theta}$ is not linearly distributed on the cross-section (as is found in elementary beam theory). However, with $b = a+h$ and we consider the limit of small h then indeed we do recover a linear distribution of stress.

II: Displacements from Stresses

We will now obtain the displacements for the curved beam problem. This will be done by first obtaining the strains from the stresses and then integrating the strain-displacement relations.

The derivatives of displacement can be obtained from Hooke's law and the strain-displacement relation as

$$\begin{aligned}
\epsilon_{rr} = \frac{\partial u_r}{\partial r} &= \frac{1}{2G}\left[(1-\bar{\nu})\sigma_{rr} - \bar{\nu}\sigma_{\theta\theta}\right] \qquad \bar{\nu} \equiv \frac{3-\kappa}{4} \\
&= \frac{1}{2G}\left[+\frac{A}{r^2} + B\left\{1 - 4\bar{\nu} + 2(1-2\bar{\nu})\log_n r\right\} + 2(1-2\bar{\nu})C\right]
\end{aligned}$$

Note that the displacements u_θ , u_r could be functions of θ, hence the full expression for the hoop and shear strains must be used. That is,

$$\begin{aligned}
\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} &= \frac{1}{2G}\left[(1-\bar{\nu})\sigma_{\theta\theta} - \bar{\nu}\sigma_{rr}\right] \\
&= \frac{1}{2G}\left[-\frac{A}{r^2} + B\left\{3 - 4\bar{\nu} + 2(1-2\bar{\nu})\log_n r\right\} + (1-2\bar{\nu})2C\right] \\
2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} &= \frac{1}{G}\sigma_{r\theta} = 0
\end{aligned}$$

Integrate the radial strain equation to get

$$2Gu_r = -\frac{A}{r} + Br\left[-1 + 2\log_n r - 4\bar{\nu}\log_n r\right] + 2(1-2\bar{\nu})Cr + g_1'(\theta)$$

where $g_1(\theta)$ is a function of integration. Combine this with the hoop strain relation to get

$$2G\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 4B(1-\bar{\nu}) - \frac{1}{r}g_1'(\theta)$$

Now integrate this with respect to θ

$$2Gu_\theta = 4B(1-\bar{\nu})r\theta - g_1(\theta) + g_2(r)$$

where $g_2(r)$ is another function of integration.

Both displacements are known to within two arbitrary functions $g_1(\theta)$ and $g_2(r)$. At this stage we have not used the shear strain-displacement relation; imposition of it will remove the arbitrariness in the integration functions. Substitute for the displacements into the shear strain relation to get

$$\begin{aligned} 0 = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \quad &= \quad \frac{1}{r}\left[g_1''(\theta)\right] + \left[4B(1-\bar{\nu})\theta + g_2'(r)\right] \\ &\quad -\frac{1}{r}\left[4B(1-\bar{\nu})r\theta - g_1(\theta) + g_2(r)\right] \end{aligned}$$

which simplifies to

$$g_1''(\theta) + g_1(\theta) + rg_2'(r) - g_2(r) = 0$$

Collect like functions of θ and r and since they separately must, at most, be constant, then

$$g_1''(\theta) + g_1(\theta) = \lambda\,, \qquad rg_2'(r) - g_2(r) = -\lambda$$

where λ is the constant. Solving these differential equations gives

$$\begin{aligned} g_1(\theta) &= C_1\cos\theta + C_2\sin\theta + \lambda \\ g_2(r) &= C_3 r + \lambda \end{aligned}$$

The functions $g_1(\theta)$ and $g_2(r)$ are seen to be rigid body motions. In other words, the stress function can be used to obtain the displacements within a rigid body motion. Note also that u_θ is linear in r and therefore “plane sections remain plane” even though the strains (and stresses) are not linearly distributed on the cross-section.

Example 5.3: Derive the Lamé solution for a pressurized cylinder.

Consider a hollow cylinder subjected to uniform external and internal pressures p_o and p_i, respectively. This is an axisymmetric problem and therefore we can take the stress function as

$$\phi = A\log_n r + Br^2\log_n r + Cr^2$$

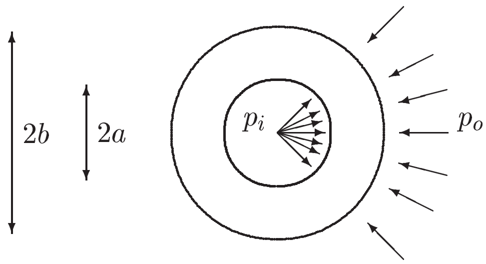


Figure 5.10: Pressurized cylinder.

which gives the stresses (same as for Golovin's problem)

$$\begin{aligned}
\sigma_{rr} &= \frac{A}{r^2} + B\left[1 + 2\log_n r\right] + 2C \\
\sigma_{\theta\theta} &= -\frac{A}{r^2} + B\left[3 + 2\log_n r\right] + 2C \\
\sigma_{r\theta} &= 0
\end{aligned}$$

The boundary conditions to be satisfied are

$$\begin{aligned}
\text{at } r = a: \qquad t_r &= -\sigma_{rr} = p_i \\
t_\theta &= -\sigma_{r\theta} = 0 \\
\text{at } r = b: \qquad t_r &= \sigma_{rr} = -p_o \\
t_\theta &= \sigma_{r\theta} = 0
\end{aligned}$$

Again, the two shear conditions are satisfied automatically. This gives two equations, but there are three unknowns A, B and C.

Our difficulty here is that we have a solution that appears to satisfy all the requirements: equilibrium and compatibility are satisfied since the stresses are derived from a bi-harmonic stress function, and any pair of the three coefficients are capable of satisfying the traction boundary conditions. Lamé's problem is an example of a multiply connected region; that is, a body with two or more independent boundary contours. In these cases, the compatibility equations are an incomplete statement of compatibility for the body. To see this, look at the displacements; in particular, note that

$$2Gu_\theta = B(1+\kappa)r\theta$$

This cannot be allowed since, for continuity, we have

$$\begin{aligned}
\text{at} \quad \theta = 0: \qquad u_\theta &= 0 \\
\text{at} \quad \theta = 2\pi: \qquad u_\theta &= \frac{B}{2G}(1+\kappa)r2\pi
\end{aligned}$$

which gives both a zero and a non-zero value at the same point. That is, the presence of B gives rise to multi-valued displacements, and only $B = 0$ can satisfy the continuity condition. Bi-harmonic functions are guaranteed to satisfy compatibility only for simply connected regions; for multiply connected regions, we impose the additional constraint that the displacement be single valued.

The boundary conditions now become

$$\text{at} \quad r = a: \qquad -p_i = \frac{A}{a^2} + 2C$$
$$r = b: \qquad -p_o = \frac{A}{b^2} + 2C$$

which gives

$$A = (p_o - p_i)\frac{a^2 b^2}{(b^2 - a^2)}, \qquad 2C = \frac{a^2 p_i - b^2 p_o}{(b^2 - a^2)}$$

and the stresses as

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{(1 - a^2/b^2)}\left[\frac{a^2}{b^2}p_i - p_o + \frac{a^2}{r^2}(p_o - p_i)\right] \\
\sigma_{\theta\theta} &= \frac{1}{(1 - a^2/b^2)}\left[\frac{a^2}{b^2}p_i - p_o - \frac{a^2}{r^2}(p_o - p_i)\right]
\end{aligned}$$

This is the Lamé solution.

Consider the special case when $a = 0$, that is, it is a solid cylinder. Then the stresses reduce to

$$\begin{aligned}
\sigma_{rr} &= -p_o \\
\sigma_{\theta\theta} &= -p_o
\end{aligned}$$

There is no dependence on r and the stress is a state of hydrostatic stress. Consider the companion problem of an infinite sheet with a circular hole and let the remote stress in the radial direction be $-p_o$. The stresses are

$$\begin{aligned}
\sigma_{rr} &= [-1 + \frac{a^2}{r^2}]p_o \\
\sigma_{\theta\theta} &= [-1 - \frac{a^2}{r^2}]p_o
\end{aligned}$$

Note that at $r = a$ the radial stress goes to zero but the hoop stress increases to twice the applied pressure. That is, if there is no hole in the sheet then the maximum stress is p_o; the presence of the hole causes a redistribution of stress and in so doing the maximum stress rises to $2p_o$. This is an example of a *stress riser* or *stress concentration*.

Now consider the case $b \to \infty$ and $p_o = 0$; that is, it is a pressurized hole in an infinite sheet. The stresses are

$$\begin{aligned}
\sigma_{rr} &= -\frac{a^2}{r^2}p_i \\
\sigma_{\theta\theta} &= +\frac{a^2}{r^2}p_i
\end{aligned}$$

These stresses are the same at $r = a$ and decay as $1/r^2$. As we impose a finite boundary at $r = b$, σ_{rr} still has the same boundary conditions but is forced to zero more rapidly. Hence, to compensate, $\sigma_{\theta\theta}$ increases giving a maximum stress of

$$\sigma_{\theta\theta}\text{max} = \frac{b^2 + a^2}{b^2 - a^2}p_i$$

on the inner boundary.

A final special case of interest is that of a thin walled pressure vessel. Consider the approximation $a \to b$, that is, $b = a + t$, then

$$\frac{a^2}{b^2} = \frac{(b-t)^2}{b^2} = \frac{b^2 - 2bt + t^2}{b^2} = 1 - \frac{2t}{b} + \frac{t^2}{b^2} \simeq 1 - \frac{2t}{b}$$

Substituting into the Lamé solution with $p_o = 0$ gives

$$\begin{aligned} \sigma_{rr} &\approx 0 \\ \sigma_{\theta\theta} &\approx p_i \frac{b}{t} \end{aligned}$$

The hoop stress is the dominant stress. If the cylinder is thin-walled with the radius being twenty times that of the thickness, then the generated hoop stress is twenty times that of the applied pressure. ■

Example 5.4: A disk of radius b rotates at a constant angular velocity Ω. Determine the resulting stresses.

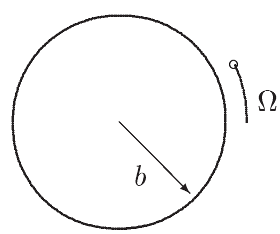


Figure 5.11: Rotating disk.

This is an example where the body force terms are important — the centrifugal force constitutes the body force field. These body forces are given by

$$\begin{aligned} \rho b_r &= \rho r \Omega^2 \\ \rho b_\theta &= 0 \end{aligned}$$

It can be easily verified that a suitable body force potential is

$$\mathcal{V} = -\tfrac{1}{2}\rho r^2 \Omega^2$$

Thus, the stress function satisfies the nonhomogeneous equation

$$\nabla^4 \phi = -\frac{2(\kappa - 1)}{\kappa + 1} \nabla^2 \mathcal{V} = \frac{4(\kappa - 1)}{(\kappa + 1)} \rho \Omega^2$$

Since the body force field and the boundary conditions are axisymmetrical, the resulting stress field must be also axisymmetrical. Thus, the solution is given by

$$\phi = A \log_n r + B r^2 \log_n r + C r^2 + \phi_p$$

where ϕ_p is a particular solution. Note, however, that both $\log_n r$ and $r^2 \log_n r$ produce singular stresses at $r = 0$ (see the tables at the end of this chapter) and therefore are not admissible. Accordingly, we set $A = B = 0$. It is easy to show that

$$\phi_p = \frac{\kappa - 1}{16(\kappa + 1)} r^4 \rho \Omega^2$$

is a particular solution and thus ϕ reduces to

$$\phi = Cr^2 + \frac{\kappa - 1}{16(\kappa + 1)} \rho \Omega^2 r^4$$

The stresses corresponding to this function are

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \mathcal{V} = 2C - \frac{\kappa + 3}{4(\kappa + 1)} \rho \Omega^2 r^2 \\
\sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} + \mathcal{V} = 2C + \frac{\kappa - 5}{4(\kappa + 1)} \rho \Omega^2 r^2 \\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = 0
\end{aligned}$$

The boundary conditions at $r = b$ are

$$\begin{aligned}
t_r &= \sigma_{rr} = 0 \\
t_\theta &= \sigma_{r\theta} = 0
\end{aligned}$$

the second equation is satisfied automatically. From the first equation, we obtain

$$C = \frac{\kappa + 3}{8(\kappa + 1)} \rho \Omega^2 b^2$$

Thus, the complete solution for the rotating disk is given by

$$\phi = \frac{\kappa + 3}{8(\kappa + 1)} \rho \Omega^2 b^2 r^2 + \frac{\kappa - 1}{16(\kappa + 1)} \rho \Omega^2 r^4$$

and the stresses are

$$\begin{aligned}
\sigma_{rr} &= \frac{\kappa + 3}{4(\kappa + 1)} \rho \Omega^2 [b^2 - r^2] \\
\sigma_{\theta\theta} &= \frac{\kappa + 3}{4(\kappa + 1)} \rho \Omega^2 [b^2 + \frac{(\kappa - 5)}{(\kappa + 3)} r^2]
\end{aligned}$$

The maximum stress 1s

$$\sigma_{rr}|_{max} = \sigma_{\theta\theta}|_{max} = \frac{\kappa + 3}{4(\kappa + 1)} \rho \Omega^2 b^2$$

and occurs at the center of the disk ($r = 0$).

The problem of a hollow disk is solved in the analogous manner, it is just a matter of retaining the $A \log_n r$ and $Br^2 \log_n r$ terms in the stress function. ■

5.4 A Guide to Selecting Stress Functions

The key in using the Airy stress function to solve plane elasticity problems lies in the selection of candidate stress functions. Since stress functions satisfy the equilibrium and compatibility equations, they are required only to satisfy the boundary conditions. Thus, the boundary conditions give us the clue about the nature of the functions to be selected. We establish a very general collection of stress functions and then show how they are synthesized to solve particular boundary value problems.

Collection of Particular Solutions of $\nabla^4\phi = 0$

The general solution for the bi-harmonic equation in cylindrical coordinates was obtained by J.H. Michell (1899) by direct substitution of $\phi = f(r)e^{\alpha\theta}$. The solutions are summarized as

$$
\begin{aligned}
\phi(r,\theta) &= A_o + B_o\theta + A\log_n r + Br^2\log_n r + Cr^2 + Dr^2\theta \\
&+ \Big[A_1 r + B_1 r^3 + C_1\frac{1}{r} + D_1 r\log_n r + Er\theta\Big]\left\{\begin{array}{l}\sin\theta\\ \cos\theta\end{array}\right. \\
&+ \sum_{m=2}^{\infty}\Big[A_m r^m + B_m r^{m+2} + C_m\frac{1}{r^m} + D_m\frac{r^2}{r^m}\Big]\left\{\begin{array}{l}\sin m\theta\\ \cos m\theta\end{array}\right. \\
&+ \sum_{m=2}^{\infty} r^m\Big[A_m\cos m\theta + B_m\sin m\theta + C_m\cos(m-2)\theta + D_m\sin(m-2)\theta\Big]
\end{aligned}
$$

The brace indicates that either term can be used. The constant term A_o does not yield any non-trivial stresses and is therefore usually omitted. Stresses and displacements obtained from these can be found in the charts of Tables 5.1 & 5.2 at the end of this chapter.

A quick way to obtain harmonic functions in Cartesian coordinates is to extract separately the real and imaginary parts of an analytic function. For example, if

$$
\phi = \phi_R + i\phi_I = (x+iy)^n\,, \qquad i \equiv \sqrt{-1}
$$

then

n	ϕ_R	ϕ_I
1	x	y
2	$x^2 - y^2$	$2xy$
3	$x^3 - 3xy^2$	$3x^2y - y^3$
4	$x^4 - 6x^2y^2 + y^4$	$4x^3y - 4xy^3$
5	$x^5 - 10x^3y^2 + 5xy^4$	$5x^4y - 10x^2y^3 + y^5$

Each of these is a harmonic function. If $\phi(x,y)$ is harmonic, then the product functions $x\phi$ and $y\phi$ are bi-harmonic because

$$
\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)[x\phi] = 2\frac{\partial\phi}{\partial x} + x\frac{\partial^2\phi}{\partial x\partial x} + x\frac{\partial^2\phi}{\partial y\partial y} = 2\frac{\partial\phi}{\partial x} + x\nabla^2\phi = 2\frac{\partial\phi}{\partial x}
$$

Therefore,

$$\nabla^2\nabla^2[x\phi] = 2\frac{\partial}{\partial x}[\nabla^2\phi] = 0$$

Similarly for the y product. This gives a quick scheme for obtaining bi-harmonic functions. For example,

$x\phi$			$y\phi$		
x^2	:	xy	xy	:	y^2
$x^3 - xy^2$	:	$2x^2y$	$x^2y - y^3$	:	$2xy^2$
$x^4 - 3x^2y^2$	:	$3x^3y - xy^3$	$xy - 3xy^3$	:	$3x^2y^2 - y^4$

is a collection of bi-harmonic functions obtained from the table of harmonic functions above. This can be generalized to the statement: Let ϕ_o, ϕ_1, and ϕ_2 be any harmonic functions, then a representation of a bi-harmonic function can be formed by the linear combination

$$\phi(x, y) = \phi_o(x, y) + x\phi_1(x, y) + y\phi_2(x, y)$$

Generating bi-harmonic functions is thus a straightforward procedure.

A bi-harmonic stress function is always the exact solution to some problem — the art of solving practical problems is finding the right combination of these functions to satisfy the given boundary conditions.

Example 5.5: A hollow cylinder is subjected to a non-uniform outer pressure given by $p = p_o \sin^2\theta$. Determine the stress distribution.

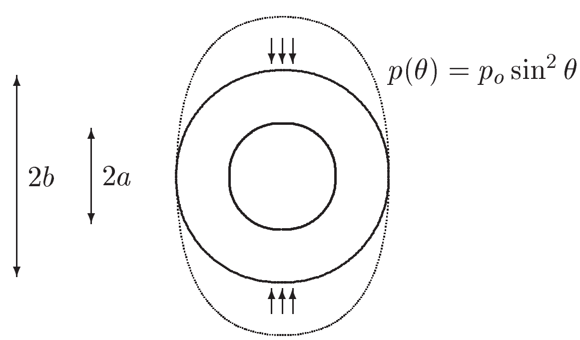


Figure 5.12: Hollow cylinder with non-uniform pressure.

This is a variation on the Lamé problem and we will use it to illustrate how the stress functions can be selected. We begin by rewriting the pressure distribution in the equivalent form

$$p = \tfrac{1}{2}p_o(1 - \cos 2\theta)$$

This is done because the charts of Tables 5.1 & 5.2 use trigonometric functions with multiple arguments and not multiple powers. The boundary conditions are

$$\begin{aligned} \text{at } r = a: &\qquad \sigma_{rr} = \sigma_{r\theta} = 0 \\ \text{at } r = b: &\qquad \sigma_{rr} = -\tfrac{1}{2}p_o + \tfrac{1}{2}p_o\cos 2\theta \\ &\qquad \sigma_{r\theta} = 0 \end{aligned}$$

From the θ dependence nature of σ_{rr} at the boundaries, we select the stress functions that produce σ_{rr} either independent of θ or dependent on $\cos 2\theta$. From the stress function table, we find the following stress functions which satisfy these requirements

$$r^2\,, \quad \log_n r\,, \quad \theta\,, \quad r^2 \log_n r\,, \quad r^2 \cos 2\theta\,, \quad r^4 \cos 2\theta\,, \quad \frac{1}{r^2} \cos 2\theta\,, \quad \cos 2\theta$$

Further examination of these functions reveal that $r^2 \log_n r$ produces multiple–valued displacement and is therefore not suitable for the present problem. Thus, a proper Airy stress function is given by

$$\phi = Ar^2 + B \log_n r + C\theta + [Dr^2 + Er^4 + F\frac{1}{r^2} + G] \cos 2\theta$$

The constants $A, .., G$ are to be determined by the boundary conditions. ■

Example 5.6: Determine the state of stress in a large plate, with a small hole, uniformly loaded in the y direction remote from the hole.

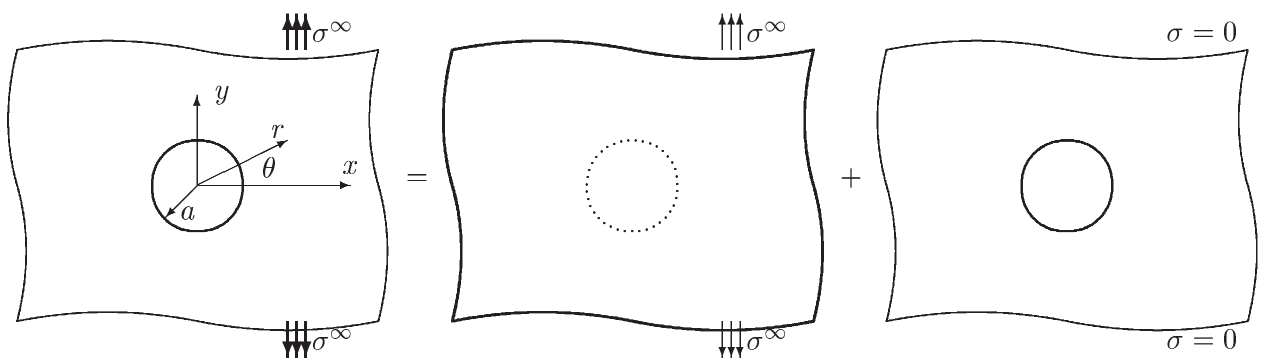


Figure 5.13: Hole in an infinite sheet.

This problem is usually referred to as the *Kirsch's Hole in an Infinite Sheet Problem*. The basic strategy we will apply is to add two stress systems together: the first gives the correct applied tractions at infinity while the second enforces the zero tractions around the edge of the hole without affecting the stresses at infinity.

I: Remote Stress Function

Initially, neglect the hole and obtain a stress function for the remote stress. That is, knowing

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \phi_o}{\partial y^2} = 0 \\
\sigma_{yy} &= \frac{\partial^2 \phi_o}{\partial x^2} = \sigma^\infty \\
\sigma_{xy} &= -\frac{\partial^2 \phi_o}{\partial x \partial y} = 0
\end{aligned}$$

leads us to choose the stress function as

$$\phi_o = \frac{\sigma^\infty}{2} x^2$$

In the vicinity of the hole, we will need to use cylindrical coordinates when satisfying the boundary conditions, hence rewrite ϕ_o as

$$\phi_o = \frac{\sigma^\infty}{2} r^2 \cos^2\theta = \frac{\sigma^\infty}{2} r^2 (\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) = \tfrac{1}{4}\sigma^\infty r^2 + \tfrac{1}{4}\sigma^\infty r^2 \cos 2\theta$$

Our plan is to add to this a stress function that will satisfy the boundary conditions at $r = a$. Whatever form it takes, the stresses must be consistent with this at $r \to \infty$ and therefore they must go to zero at $r \to \infty$.

Although the ϕ_o obtained above satisfies the stress condition at $r \to \infty$, it does not satisfy the boundary condition at $r = a$ of

$$\begin{aligned} 0 &= t_r = \sigma_{rr} n_r + \sigma_{r\theta} n_\theta = -\sigma_{rr} \\ 0 &= t_\theta = -\sigma_{r\theta} \end{aligned}$$

The stress function ϕ_o yields the following stresses at $r = a$

$$\begin{aligned} \sigma_{rr} &= \tfrac{1}{2}\sigma^\infty - \tfrac{1}{2}\sigma^\infty \cos 2\theta \\ \sigma_{r\theta} &= \frac{\sigma^\infty}{2} \sin 2\theta \end{aligned}$$

Additional bi-harmonic functions must be added to ϕ_o in order to clear these tractions without disturbing the stress condition at $r \to \infty$ which are already satisfied by ϕ_o

II: Complete Stress Function

Using the above mentioned boundary conditions as a guide, the added bi-harmonic functions must produce stresses which are either independent of θ or dependent on $\cos 2\theta$ (for σ_{rr}) and $\sin 2\theta$ (for $\sigma_{r\theta}$). Meanwhile, the additional stresses must vanish as $r \to \infty$. From the bi-harmonic function table, the suitable candidate stress functions are

$$\log_n r\,, \quad \frac{1}{r^2}\cos 2\theta\,, \quad \cos 2\theta$$

The general stress function that satisfies the remote conditions is therefore

$$\phi = A \log_n r + \frac{\sigma^\infty}{4} r^2 + [\frac{\sigma^\infty}{4} r^2 + C_2 \frac{1}{r^2} + D_2] \cos 2\theta$$

giving the stresses

$$\begin{aligned} \sigma_{rr} &= \frac{A}{r^2} + \frac{\sigma^\infty}{2} - \left[\frac{\sigma^\infty}{2} + \frac{6C_2}{r^4} + \frac{4D_2}{r^2}\right] \cos 2\theta \\ \sigma_{\theta\theta} &= -\frac{A}{r^2} + \frac{\sigma^\infty}{2} + \left[\frac{\sigma^\infty}{2} + \frac{6C_2}{r^4}\right] \cos 2\theta \\ \sigma_{r\theta} &= 0 - \left[\frac{-\sigma^\infty}{2} + \frac{6C_2}{r^4} + \frac{2D_2}{r^2}\right] \sin 2\theta \end{aligned}$$

There are three constants A, C_2, D_2 to be determined by the boundary conditions at $r = a$. Note that as r becomes very large that the additional terms do indeed vanish.

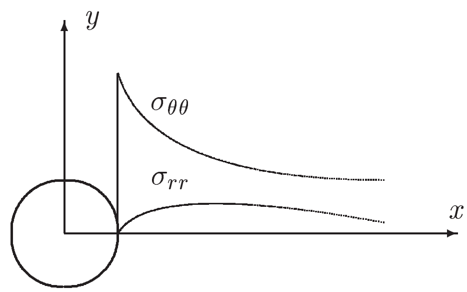


Figure 5.14: Stress distributions near the hole.

III: Hole Boundary Conditions

The boundary conditions at the edge of the hole are that the tractions are zero, that is,

$$\begin{aligned} t_r = -\sigma_{rr} &= 0 = \frac{A}{a^2} + \frac{\sigma^\infty}{2} - \left[\frac{\sigma^\infty}{2} + \frac{6C_2}{a^4} + \frac{4D_2}{a^2}\right] \sin 2\theta \\ t_{r\theta} = -\sigma_{r\theta} &= 0 = -\left[-\frac{\sigma^\infty}{2} + \frac{6C_2}{a^4} + \frac{2D_2}{a^2}\right] \cos 2\theta \end{aligned}$$

Since this must be true for any θ then

$$\begin{aligned} \frac{A}{a^2} + \frac{\sigma^\infty}{2} &= 0 \\ \frac{\sigma^\infty}{2} + \frac{6C_2}{a^4} + \frac{4D_2}{a^2} &= 0 \\ -\frac{\sigma^\infty}{2} + \frac{6C_2}{a^4} + \frac{2D_2}{a^2} &= 0 \end{aligned}$$

Solving these simultaneously gives the coefficients as

$$A = -\frac{\sigma^\infty}{2} a^2 , \qquad C_2 = \frac{\sigma^\infty}{4} a^4 , \qquad D_2 = -\frac{\sigma^\infty}{2} a^2$$

The stresses are, finally,

$$\begin{aligned} \sigma_{rr} &= \frac{\sigma^\infty}{2}\left\{1 - \frac{a^2}{r^2} - \left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right) \cos 2\theta\right\} \\ \sigma_{\theta\theta} &= \frac{\sigma^\infty}{2}\left\{1 + \frac{a^2}{r^2} + \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta\right\} \\ \sigma_{r\theta} &= \frac{\sigma^\infty}{2}\left\{1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right\} \sin 2\theta \end{aligned}$$

Figure 5.14 shows the distribution of the stress along the $\theta = 0$ axis.

The hoop stress around the edge of the hole is

$$\sigma_{\theta\theta} = \sigma^\infty \left\{1 + 2\cos 2\theta\right\}$$

showing that at $\theta = 0$, the maximum stress is three times the remote stress. Also note that at $\theta = \pi/2$, $\sigma_{\theta\theta} = -\sigma^\infty$. ■

Problems with Concentrated Forces

These examples demonstrates a common approach to solving problems using stress functions. After the function is chosen, it is first necessary to verify that it is a solution of the bi-harmonic equation. This will guarantee that the stresses are equilibrated and the strains compatible, everywhere in the body. Then the function is investigated for the types of tractions it gives rise to. The behavior of these tractions is what specifies the type of boundary value problem being solved. For example, zero tractions are associated with a free surface, a linear traction distribution with an applied load that has a simple distribution.

I: Flamant's Problem (1892)

The stress function we will investigate is

$$\phi = Ax \tan^{-1}\left(\frac{x}{y}\right)$$

This has the representation in cylindrical coordinates as

$$\phi(r,\theta) = Ar[\theta + \tfrac{1}{2}\pi]\sin(\theta + \tfrac{1}{2}\pi)$$

Since this can be expanded into forms that are in Tables 5.1 & 5.2 we conclude that it is biharmonic.

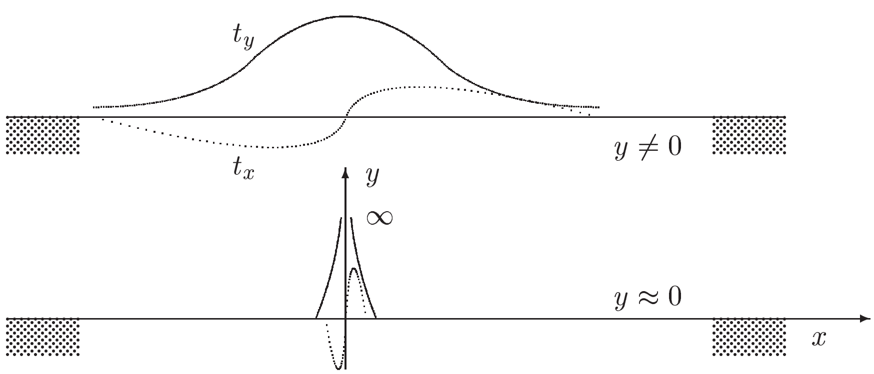


Figure 5.15: Tractions on two surfaces at constant y.

We will work primarily in Cartesian coordinates. The stresses are

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2A\frac{x^2 y}{r^4}\,, \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 2A\frac{y^3}{r^4}\,, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 2A\frac{xy^2}{r^4}$$

Keep in mind that the tractions can be evaluated for any (x,y) but recognizing a boundary is really a question of identifying some obvious feature of the tractions; for example, zero traction for a free surface. Note that for the present case, the stresses are singular at $r = 0$, that is

$$\sigma \to \infty \qquad \text{as} \qquad r \to 0$$

Also note that the stresses are zero at infinity

$$\sigma \to 0 \qquad \text{as} \qquad x \text{ or } y \to \infty$$

To consider a particular boundary value problem we must look at the tractions on particular lines. Consider a horizontal plane $y \neq 0$ such that $n_x = 0$, $n_y = 1$, then the tractions are

$$\begin{aligned} t_x &= \sigma_{xx} n_x + \sigma_{xy} n_y = \sigma_{xy} = 2A\frac{xy^2}{r^4} \\ t_y &= \sigma_{xy} n_x + \sigma_{yy} n_y = \sigma_{yy} = 2A\frac{y^3}{r^4} \end{aligned}$$

Note that along $y = 0$, the traction vanishes except for $r = 0$.

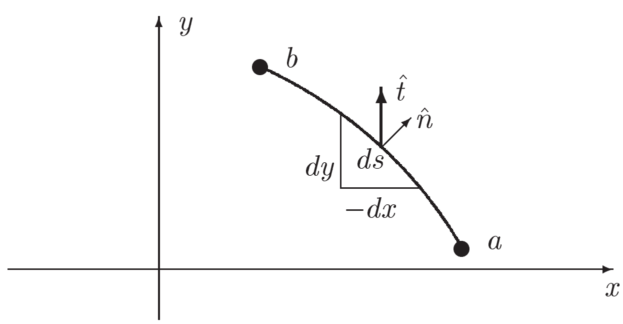


Figure 5.16: Traction resultants on an arbitrary curved surface.

We will look the resultants. Consider a general boundary S with positive direction as shown in the Figure 5.16. The components of the unit normal vector to S are given by

$$n_x = \frac{dy}{dS}, \qquad n_y = -\frac{dx}{dS}$$

The traction has two components given by

$$\begin{aligned} t_x &= \sigma_{xx} n_x + \sigma_{xy} n_y = \left(\frac{\partial^2 \phi}{\partial y^2}\right)\left(\frac{dy}{dS}\right) + \left(-\frac{\partial^2 \phi}{\partial x \partial y}\right)\left(-\frac{dx}{dS}\right) = \frac{d}{dS}\left(\frac{\partial \phi}{\partial y}\right) \\ t_y &= \sigma_{xy} n_x + \sigma_{yy} n_y = \left(-\frac{\partial^2 \phi}{\partial x \partial y}\right)\left(\frac{dy}{dS}\right) + \left(\frac{\partial^2 \phi}{\partial x^2}\right)\left(-\frac{dx}{dS}\right) = -\frac{d}{dS}\left(\frac{\partial \phi}{\partial x}\right) \end{aligned}$$

The resultant forces on the contour length between points a and b are

$$F_x = \int_a^b t_x dS = +\left[\frac{\partial \phi}{\partial y}\right]_a^b, \qquad F_y = \int_a^b t_y dS = -\left[\frac{\partial \phi}{\partial x}\right]_a^b$$

which are dependent only on the values of the stress function at the end points. The resultant forces on the entire horizontal section $y \neq 0$ are

$$\begin{aligned} F_x &= +\left[\frac{\partial \phi}{\partial y}\right]_a^b = \left[-A\frac{x^2}{r^2}\right]_{x=+\infty}^{x=-\infty} = 0 \\ F_y &= -\left[\frac{\partial \phi}{\partial x}\right]_a^b = -\left[+A\tan^{-1}\left(\frac{x}{y}\right) + A\frac{xy}{r^2}\right]_{x=+\infty}^{x=-\infty} = -A\pi \end{aligned}$$

Let the resultant force be equal to the applied force P (force per unit length), that is,

$$-A\pi = -P$$

from which the constant is determined.

The stress field is thus obtained as

$$\sigma_{xx} = \frac{2P}{\pi}\frac{x^2 y}{r^4}\,, \qquad \sigma_{yy} = \frac{2P}{\pi}\frac{y^3}{r^4}\,, \qquad \sigma_{xy} = \frac{2P}{\pi}\frac{xy^2}{r^4}$$

These are the stresses in a half-plane subjected to a downward vertical force applied at the origin of the coordinate system. This is referred to as Flamant's solution for the half-plane.

There is a subtlety in the solution that is worth discussing. Recall that a bi-harmonic stress function always gives rise to a stress field that is the solution of a well posed elasticity problem. This problem solved, however, may not be the problem of immediate interest. For example, if the half-plane problem is posed as "concentrated normal traction and zero shear traction" then the Flamant result is not the solution. The reason is that the Flamant solution also has a singular behavior in the shear traction. To effect a practical solution, it is often necessary to relax the statement of the problem. For example, if the half-plane problem is posed as "concentrated normal traction and concentrated shear traction with a zero resultant" then the Flamant result is the solution. Note that the shear distribution in not known in advance.

II: Partial Load on a Half Plane

Consider the stress function

$$\phi(r,\theta) = Ar^2[\theta - \sin\theta\cos\theta] = Ar^2[\theta - \tfrac{1}{2}\sin 2\theta]$$

This is in Tables 5.1 & 5.2 hence it is bi-harmonic. We will now discuss what boundary problem it is associated with.

Either by differentiation, or from the tables, we get that the stresses are

$$\sigma_{rr} = A[2\theta + \sin 2\theta]\,, \qquad \sigma_{\theta\theta} = A[2\theta - \sin 2\theta]\,, \qquad \sigma_{r\theta} = A[-1 + \cos 2\theta]$$

Note that these stresses do not depend on r. Thus on lines $r = a =$ constant the tractions are

$$t_r = A[2\theta + \sin 2\theta]\,, \qquad t_\theta = A[-1 + \cos 2\theta]$$

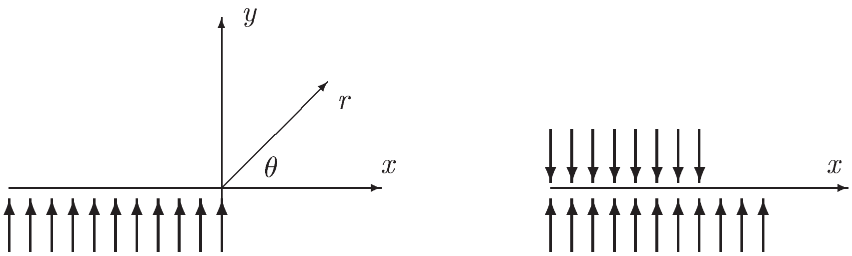


Figure 5.17: Loading on half of a half plane.

which also do not depend on r. These tractions, however, do not have an easily recognizable distribution.

Along the lines $\theta = 0$ and $\theta = \pi$, the tractions are

$$\theta = 0: \quad t_r = 0 \quad t_\theta = 0; \qquad \theta = \pi: \quad t_r = 0 \quad t_\theta = A2\pi$$

This is a constant normal traction on half of the surface. We therefore recognize this as the loading shown in Figure 5.17. The coefficient A would be chosen to give the appropriate intensity of the surface traction.

An interesting solution is obtained by combining two of the partial load solutions but shifted a distance apart. With opposite magnitudes, the result in a normal applied load over a finite segment of the surface. Thus,

$$\begin{aligned}\phi \;=\; & A\Big[[(x-a)^2+y^2]\tan^{-1}(\frac{y}{x-a}) - (x-a)y \\ & -[(x+a)^2+y^2]\tan^{-1}(\frac{y}{x+a}) + (x+a)y\Big]\end{aligned}$$

Let the contact area $2a$ shrink but also let the intensity correspondingly increase so that $A2a =$ constant, then we approach, in the limit of small a, the solution to a concentrated point load.

Note that the solution has a more complicated expression than the Flamant solution; this is because there are no surface shear stresses.

Example 5.7: Show by differentiation that the function $\phi = Ax\tan^{-1}(x/y)$ is bi-harmonic.

Using the relations

$$\frac{\partial}{\partial x}\tan^{-1}(\frac{x}{y}) = \frac{y}{x^2+y^2}, \qquad \frac{\partial}{\partial y}\tan^{-1}(\frac{x}{y}) = \frac{-x}{x^2+y^2}$$

we have for some of the derivatives

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= A\tan^{-1}\left(\frac{x}{y}\right) + A\frac{xy}{r^2} \qquad r^2 \equiv x^2+y^2 \\ \frac{\partial^2\phi}{\partial x^2} &= \frac{2A}{r^4}y^3, \qquad \frac{\partial^2\phi}{\partial y^2} = \frac{2A}{r^4}x^2y\end{aligned}$$

Therefore,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{2A}{r^2}y$$

This shows that ϕ is not harmonic. Further differentiation gives

$$\frac{\partial^2}{\partial x^2}\nabla^2\phi = \frac{16A}{r^6}x^2y - \frac{4A}{r^4}y\,, \qquad \frac{\partial^2}{\partial y^2}\nabla^2\phi = -\frac{12A}{r^4}y + \frac{16A}{r^6}y^3$$

Therefore,

$$\nabla^2\nabla^2\phi = -\frac{16A}{r^4}y + \frac{16A}{r^6}y(x^2+y^2) = 0$$

Hence, ϕ is indeed bi-harmonic. ■

Example 5.8: Determine the stresses in a disk under diametral compression.

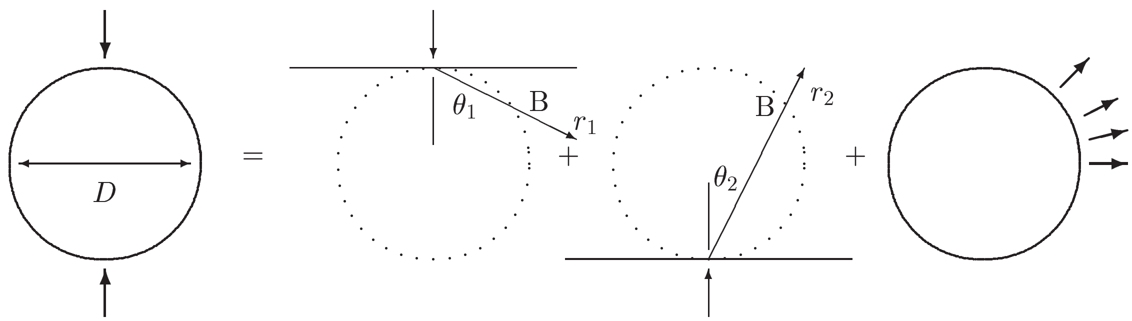


Figure 5.18: A disk under diametral compression is solved as the superposition of two Flamant solutions plus a uniform traction.

Let the center of the disk be the origin of the coordinate system, then the two Flamant solutions have the stress functions

$$\phi = x\,\tan^{-1}\left[\frac{x}{\frac{1}{2}D \pm y}\right]$$

At a point B on the circumference there is a compression in the radial directions of r_1 and r_2 of amount

$$\frac{2P}{\pi}\frac{\cos\theta_1}{r_1}\,, \qquad \frac{2P}{\pi}\frac{\cos\theta_2}{r_2}$$

respectively. But these radial lines are perpendicular to each other and therefore

$$\frac{\cos\theta_1}{r_1} = \frac{\cos\theta_2}{r_2} = \frac{1}{D}$$

We conclude that the two principal stresses at B are $2P/\pi D$. Since this conclusion is true for any point B on the circumference then a uniform traction of amount $2P/\pi D$ applied around the circumference will then give a zero traction boundary. The stress function to be added is

$$\phi = \frac{P}{\pi D}[x^2+y^2]$$

The stresses on the horizontal line of symmetry are given by

$$\sigma_{xx} = \frac{2P}{\pi h D}\left[\frac{D^2 - 4x^2}{D^2 + 4x^2}\right]^2, \qquad \sigma_{yy} = \frac{-2P}{\pi h D}\left[\frac{4D^4}{[D^2 + 4x^2]^2} - 1\right], \qquad \sigma_{xy} = 0$$

where h is the thickness of the disk. ■

Approximate Boundary Conditions

A well posed elasticity problem has boundary conditions in the form of imposed tractions or imposed displacements. However, in the solution approach where we synthesize the solution as a collection of particular solutions, we do not always have the ability to exactly match the boundary conditions. We illustrate this with the problem of a deep cantilever beam.

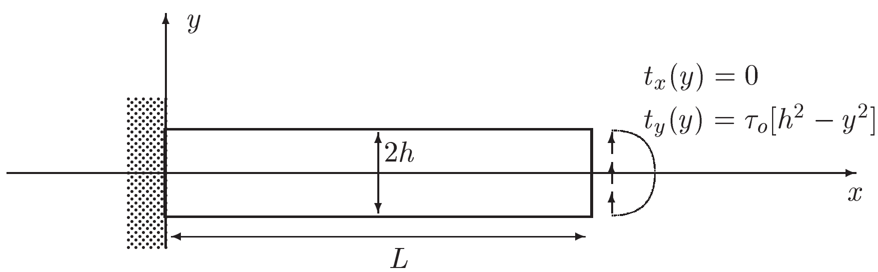


Figure 5.19: Cantilever beam with end shear traction.

Consider a deep cantilever beam with a parabolic shear traction distribution on the end. Based on the traction distribution, it seems that the stress function

$$\phi(x, y) = Axy + Bxy^3 + Cy^3$$

is sufficient to solve the problem because it leads to a parabolic shear distribution. This is what we want to investigate.

First, it is clear that ϕ is bi-harmonic because the highest power in the polynomial is three. The stresses are obtained as

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 6Bxy + 6Cy\,, \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0\,, \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -A - 3By^2$$

Consider the tractions on the horizontal planes $y = \pm h$ such that $n_x = 0$, $n_y = 1$. That is,

$$t_x = 0 = \sigma_{xy} = -A - 3Bh^2\,, \qquad t_y = 0 = \sigma_{yy} = 0$$

Note that the normal traction condition is automatically satisfied. In fact, we only get one equation from the four traction conditions and this leads to $A = -3Bh^2$.

Now look at the face at $x = L$ where $n_x = 1$, $n_y = 0$. The tractions are

$$t_x = 0 = \sigma_{xx} = 6BLy + 6Cy\,, \qquad t_y = \tau_o[h^2 - y^2] = \sigma_{xy} = -A - 3By^2$$

These two conditions leads to three equations

$$6BL + 6C = 0\,, \qquad \tau_o h^2 = -A\,, \qquad \tau_o = -3B$$

Solving gives the coefficients

$$2A = -\tau_o h^2\,, \qquad B = \tau_o/3\,, \qquad C = -BL$$

Thus the stress solution is

$$\sigma_{xx} = 2\tau_o[x - L]y\,, \qquad \sigma_{yy} = 0\,, \qquad \sigma_{xy} = \tau_o[h^2 - y^2]$$

At this stage, we have a stress field that satisfies the tractions on three sides of the body. In order to guarantee that this is indeed the solution we must also satisfy the boundary conditions along the face at $x = 0$. But what are the traction conditions? These were not specified as part of the problem.

This is an example of a 'mixed boundary value problem', that is, some of the boundary conditions are traction specified while others are displacement specified. To obtain the displacements we must integrate the strain-displacement relations. Thus, from the normal strains (after using Hooke's law with $\nu = 0$ for simplicity) get

$$Eu_x(x, y) = 2\tau_o[x^2y/2 - xyL] + f_1(y)\,, \qquad u_y(x, y) = f_2(x)$$

where f_1 and f_2 are functions of integrations. The displacements must also satisfy the shear strain-displacement relation, hence substitute and regroup in terms of only x and y. The separate groups must be equal to a constant (λ, say), therefore integration gives the separate functions $f_1(y)$ and $f_2(x)$. We finally get for the displacements

$$Eu_x(x, y) = 2\tau_o[x^2/2 - xL + h^2 - y^2/3]y - \lambda y + c_1\,, \qquad Eu_y(x, y) = \tau_o[L - x/3]x^2 + \lambda x + c_2$$

where λ, c_1, c_2 are unknowns. These contribute a rigid body motion.

Look at the displacements at $x = 0$

$$Eu_x(0, y) = 2\tau_o[h^2 - y^2/3]y - \lambda y + c_1\,, \qquad Eu_y(0, y) = c_2$$

The horizontal displacement is non-zero, not what we wanted for the fixed end condition. The above solution is not the exact solution for the fixed cantilever beam problem; the simple polynomial stress function is not capable of representing the singular stress behavior at the fixed end where $y = \pm h$. The solution, however, is the exact solution if the tractions at $x = 0$ were specified as

$$t_x = +2\tau_o Ly\,, \qquad t_y = -\tau_o[h^2 - y^2]$$

Note that if these tractions were arbitrarily specified then global equilibrium is probably violated.

The above solution gives a good approximation to the cantilever beam problem because it satisfies the exact traction conditions top and bottom, and as can be verified, satisfies an approximate version of the tractions in the form of resultants on the ends. In fact, this is a very useful approach to obtaining practical solutions: satisfy some of the traction conditions exactly, and the others approximately in the form of resultants. If the region of interest is remote from these latter boundaries, then the solution will be quite insensitive to the specific distributions of the applied tractions. This is known as *St. Venant's principle*.

5.5 Applications of Complex Variables

The previous sections demonstrated the fundamental role played by harmonic and bi-harmonic functions in the plane theory of elasticity. These functions have a very simple representation in terms of complex variables; as a consequence, many of the results of the plane theory can be summarized very elegantly and conveniently using these harmonic functions. In addition, the use of complex variables allows the introduction of conformal mapping techniques which facilitate the solution of problems with a larger range of boundary geometries.

Representation of Harmonic Functions

The essence of the complex variable approach is the introduction of two new variables which can be viewed simply as a linear coordinate transformation. By suitably choosing the transformation we can simplify many of the relations.

Consider two new variables defined as

$$z \equiv x + iy\,, \qquad \bar{z} \equiv x - iy$$

where $i = \sqrt{-1}$. These can be viewed simply as a linear coordinate transformation in which i is just a constant. A quantity with a bar is referred to as the *complex conjugate*; that is, in an expression replace i with $-i$. The notation $\overline{g(z)}$ means that wherever z is seen in the function $g(z)$, replace it with $\bar{z}$. Also $\overline{\overline{g(z)}} = g(z)$ as seen from the following

$$g(z) = e^z\,, \qquad \overline{g(z)} = e^{\bar{z}}\,, \qquad \overline{\overline{g(z)}} = e^{\bar{\bar{z}}} = e^z = g(z)$$

We also have that

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$

This indicates that it will be easy to alternate between Cartesian and cylindrical coordinates.

We will now transform all functions and their derivatives according to

$$\phi(x, y) \quad \longrightarrow \quad \phi(z, \bar{z})$$

The derivatives can be transformed as

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial}{\partial z}(1) + \frac{\partial}{\partial \bar{z}}(1) = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \\
\frac{\partial}{\partial y} &= i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}} \qquad \text{or} \qquad i\frac{\partial}{\partial y} = -\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}
\end{aligned}$$

Consequently, we have

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\Big[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big], \qquad \frac{\partial}{\partial \bar{z}} = \tfrac{1}{2}\Big[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big]$$

As a simple illustration of the use of complex variables, consider the expression for the volumetric strain

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial z} - i\frac{\partial v}{\partial \bar{z}} = \frac{\partial(u+iv)}{\partial z} + \frac{\partial(u-iv)}{\partial \bar{z}} = 2\text{Real}\Big[\frac{\partial(u+iv)}{\partial z}\Big]$$

Notice how natural the complex variable $u + iv$ arose.

Laplace's equation can now be rewritten as

$$\nabla^2\phi \;=\; \Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big]\phi = \Big[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big]\Big[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big]\phi = 4\Big[\frac{\partial}{\partial z}\Big]\Big[\frac{\partial}{\partial \bar{z}}\Big]\phi$$

Hence, harmonic functions must satisfy the equation

$$\frac{\partial^2\phi}{\partial z\partial \bar{z}} = 0$$

This is easily integrated to give

$$\frac{\partial\phi}{\partial z} = f(z) \qquad \text{or} \qquad \phi = \int f(z)dz + g(\bar{z}) = F(z) + \overline{g(z)}$$

Since the harmonic function ϕ is real, then

$$2\phi \;=\; F(z) + \overline{g(z)} + \overline{F(z)} + \overline{\overline{g(z)}} = [F + g] + [\overline{F + g}]$$

That is, a solution of $\nabla^2\phi = 0$ can be written as

$$\phi = \tfrac{1}{2}[\phi_1(z) + \overline{\phi_1(z)}] = \text{Real}[\phi_1(z)]$$

which is simply the real part of an analytic function.

The bi-harmonic equation can similarly be transformed to

$$\nabla^2\nabla^2\Phi = 0 \qquad \Rightarrow \qquad \frac{\partial^4\Phi}{\partial z^2\partial\bar{z}^2} = 0$$

On integration get

$$\begin{aligned}
\frac{\partial^3\Phi}{\partial z\partial z\partial\bar{z}} &= f(z) \\
\frac{\partial^2\Phi}{\partial z\partial z} &= \bar{z}f(z) + g(z) \\
\frac{\partial\Phi}{\partial z} &= \bar{z}\int f(z)dz + \int g(z)dx + \overline{h_1(z)} \\
\Phi &= \bar{z}\int\int f(z)dz\int\int g(z)dz + z\overline{h_1} + \overline{h_2}
\end{aligned}$$

Redefining some terms, we have

$$\Phi = \bar{z}F(z) + G(z) + z\overline{h_1(z)} + \overline{h_2(z)}$$

Since Φ is real, then

$$2\Phi = \bar{z}F + G + z\overline{h_1} + \overline{h_2} + z\bar{F} + \bar{G} + \bar{z}h_1 + h_2$$

or, after regrouping,

$$\begin{aligned}
2\Phi &= \bar{z}[F + h_1] + [G + h_2] + z[\overline{F + h_1}] + [\overline{G + h_2}] \\
&= \bar{z}\phi_1 + \phi_2 + z\bar{\phi}_1 + \bar{\phi}_2
\end{aligned}$$

Hence, a general representation of a bi-harmonic function is

$$\Phi = \mathrm{Real}[\bar{z}\phi_1 + \phi_2]$$

where $\phi_1 = \phi_1(z)$, $\phi_2 = \phi_2(z)$. Compare this representation to $\Phi = x\phi_1 + \phi_2$ used earlier in the chapter.

Application to Navier's Equations

Recall that the field equations of the linear theory of elasticity can be reduced to a single set called the Navier's equations, written for plane problems as

$$\begin{aligned}
\mu\nabla^2 u + (\lambda + \mu)\frac{\partial e}{\partial x} + \rho b_x &= 0 \\
\mu\nabla^2 v + (\lambda + \mu)\frac{\partial e}{\partial y} + \rho b_y &= 0
\end{aligned}$$

A solution of these equations is a solution to the field equations. We will transform this set using complex variables.

Multiply the second equation by i and add to the first to get

$$\mu\nabla^2[u+iv]+(\lambda+\mu)[\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}]e=0$$

(We neglect body forces for the present.) This becomes, on replacing (x,y) with $(z,\bar{z})$

$$4\mu\frac{\partial^2}{\partial z\partial\bar{z}}[u+iv]+2(\lambda+\mu)\frac{\partial}{\partial\bar{z}}[\ e\ ]=0$$

Integrate with respect to $\bar{z}$ and get

$$4\mu\frac{\partial}{\partial z}(u+iv)+2(\lambda+\mu)e=g(z)$$

Add this equation to the conjugate of itself and obtain

$$4\mu\left[\frac{\partial}{\partial z}(u+iv)+\frac{\partial}{\partial\bar{z}}(u-iv)\right]+4(\lambda+\mu)e=g+\bar{g}$$

Since the bracketed term is e as shown earlier, then

$$4(\lambda+2\mu)e=g+\bar{g}$$

That is, e is the real part of an analytic function and is therefore harmonic. Hence the displacements can be written as

$$2\mu[u+iv]=\frac{-(\lambda+\mu)}{4(\lambda+2\mu)}\Big[\int g\,dz+z\bar{g}\Big]+\tfrac{1}{2}\int g dz+\bar{h}$$

Define the new functions

$$\phi\equiv\frac{(\lambda+\mu)}{4(\lambda+2\mu)}\int g\,dz\,,\qquad \overline{h(z)}\equiv-\psi(z)$$

giving the final expression for the displacements as

$$2\mu[u+iv]=\left(\frac{\lambda+3\mu}{\lambda+\mu}\right)\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}$$

This says that for any two analytic functions ϕ and ψ, the displacements, strains, stresses and so on, obtained from them will automatically satisfy all the necessary field equations.

These results are easily rewritten in cylindrical coordinates. Utilizing the transformation equations

$$\begin{aligned}u_r &= u_x\cos\theta+u_y\sin\theta\\ u_\theta &= -u_x\sin\theta+u_y\cos\theta\end{aligned}$$

we can introduce a complex displacement function as

$$u_r + iu_\theta = u_x[\cos\theta - i\sin\theta] + u_y[\sin\theta + i\cos\theta] = [u_x + iu_y]e^{-i\theta}$$

These become, in terms of the stress function,

$$2\mu[u_r + iu_\theta] = \left[\left(\frac{\lambda + 3\mu}{\lambda + \mu}\right)\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right]e^{-i\theta}$$

The only difference in comparison to the Cartesian form is the presence of the exponential term.

Stresses and Boundary Tractions

By using the strain-displacement relation and Hooke's law, the stresses can be obtained as

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2[\phi' + \bar{\phi}'] = 4R_e[\phi'] \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2[\bar{z}\phi'' + \psi'], \qquad ()' \equiv \frac{d()}{dz} \end{aligned}$$

In Cartesian coordinates, the boundary tractions are usually given in the combinations $[\sigma_{xx}, \sigma_{xy}]$ or $[\sigma_{yy}, \sigma_{xy}]$ hence a convenient form for the above is

$$\begin{aligned} t_x - it_y &= \sigma_{xx} - i\sigma_{xy} = \phi' + \bar{\phi}' - \bar{z}\phi'' - \psi\text{''} \\ t_x + it_y &= \sigma_{yy} + i\sigma_{xy} = \phi'' + \bar{\phi}'' + \bar{z}\phi'' + \psi\text{''} \end{aligned}$$

These are special cases of the traction on an arbitrary plane given by

$$\begin{aligned} t_n - it_t &= [t_x - it_y]e^{i\alpha} \\ &= \tfrac{1}{2}[\sigma_{xx} + \sigma_{yy}] - \tfrac{1}{2}[\sigma_{yy} - \sigma_{xx} - i2\sigma_{xy}]e^{i2\alpha} \\ &= \phi' + \bar{\phi}'' - [\bar{z}\phi'' + \psi']e^{i2\alpha} \end{aligned}$$

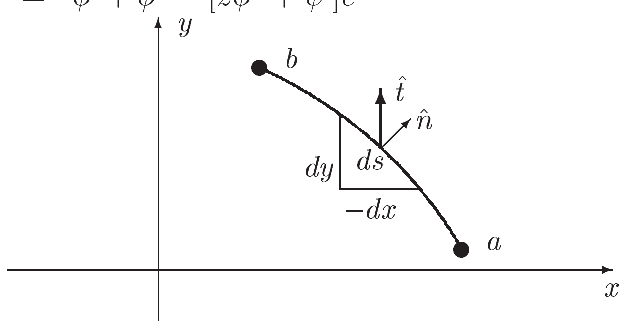


Figure 5.20: Tractions on an arbitrary boundary.

Similarly for cylindrical coordinates, the stresses are

$$\begin{aligned} \sigma_{rr} + \sigma_{\theta\theta} &= \sigma_{xx} + \sigma_{yy} \\ \sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} &= [\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}]e^{2i\theta} \end{aligned}$$

In terms of the stress function, these become

$$\begin{aligned}\sigma_{rr}+\sigma_{\theta\theta} &= 4R_e[\phi'(z)]\\ \sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta} &= 2[\bar{z}\phi''(z)+\psi'(z)]e^{2i\theta}\end{aligned}$$

We also have the relation

$$\sigma_{rr}-i\sigma_{r\theta}=[\phi'(z)+\overline{\phi'(z)}]-[\bar{z}\phi''(z)+\psi'(z)]e^{2i\theta}$$

When performing evaluations of the functions, it may be necessary to replace z with $z=re^{i\theta}$.

Recall that the resultant forces on any arc-length is given (in terms of the Airy stress function) by

$$F_x=\left.\frac{\partial\Phi}{\partial y}\right|_A^B, \qquad F_y=\left.\frac{\partial\Phi}{\partial x}\right|_A^B$$

This is written in terms of the complex potential functions as

$$F_y+iF_x \;=\; -\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}=-\left[\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right]\Phi\Big|_A^B=-2\frac{\partial\Phi}{\partial z}\Big|_A^B$$

Further, since the stress function has the general representation

$$\Phi=\bar{z}\phi_1+\phi_2+z\bar{\phi}_1+\bar{\phi}_2$$

then the resultant forces are represented by

$$-\tfrac{1}{2}\Big[F_y+iF_x\Big]=\bar{z}\phi_1'+\phi_2'+\bar{\phi}_1$$

Hence, on any boundary where the traction is specified

$$\bar{z}\phi_1'(z)+\overline{\phi_1(z)}+\phi_2'(z)=f(z) \qquad z=z_{boundary}$$

Associate the following $\phi\rightarrow\phi_1$, $\phi_2'\rightarrow\psi$, then the traction boundary condition becomes

$$\bar{z}\phi'(z)+\overline{\phi(z)}+\psi(z)=f(z) \qquad z=z_{boundary}$$

A special case of importance is that of a traction free boundary, there we have

$$\bar{z}\phi'(z)+\overline{\phi(z)}+\psi(z)=0 \qquad z=z_{boundary}$$

The application of this useful formula will be demonstrated on the elliptical hole problem.

Structure of $\phi(z)$ and $\psi(z)$

In some cases it may be possible to obtain $\phi(z)$ and $\psi(z)$ by integration, but typically they are determined by synthesizing particular solutions.

In a finite, simply connected region, the functions $\phi(z)$ and $\psi(z)$ are single valued analytic functions of z and therefore can be represented by

$$\phi(z) = \sum_o^\infty a_n z^n \,, \qquad \psi(z) = \sum_o^\infty b_n z^n$$

Note that each term in the series is a harmonic function.

If the region is not simply connected then ϕ and ψ need not be single valued and other solutions are possible. For the displacements and stresses to be single valued in a finite multiply connected region, the functions are represented by

$$\begin{aligned}
\phi(z) &= -\frac{1}{2\pi(1+\kappa)} \sum_{k=1}^m [(F_1^k + iF_2^k)\log(z - z_k)] + \phi_o(z) \\
\psi(z) &= -\frac{\kappa}{2\pi(1+\kappa)} \sum_{k=1}^m [(F_1^k - iF_2^k)\log(z - z_k)] + \psi_o(z) \\
\phi_o(z) &= \sum_o^\infty a_n z^n \,, \qquad \psi_o(z) = \sum_o^\infty b_n z^n
\end{aligned}$$

where F_1^k, F_2^k are force resultants on each contour, z^k is an arbitrary point in each 'hole', and m is the number of contours.

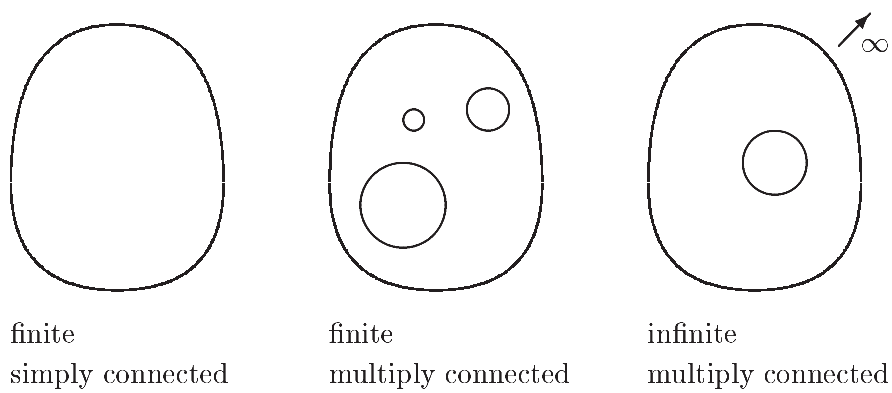


Figure 5.21: Types of connected regions.

In an infinite region with several internal contours

$$\begin{aligned}
\phi(z) &= -\frac{(F_1 + iF_2)}{2\pi(1+\kappa)} \log(z) + [\tfrac{1}{4}(\sigma_{11}^\infty + \sigma_{22}^\infty)]z + \phi_o \\
\psi(z) &= -\frac{\kappa(F_1 + iF_2)}{2\pi(1+\kappa)} \log(z) + [\tfrac{1}{2}(\sigma_{11}^\infty - \sigma_{22}^\infty) + i\sigma_{12}^\infty]z + \psi_o \\
\phi_o(z) &= \sum_o^\infty a_n z^{-n} \,, \qquad \psi_o(z) = \sum_o^\infty b_n z^{-n}
\end{aligned}$$

Note that the added series give stresses that decay at infinity.

Example 5.9: Express the case of a uniform stress distribution using complex variables.

The appropriate stress functions and their derivatives are

$$\begin{aligned} \phi(z) &= A_1 z\,, \qquad \psi(z) = B_1 z \\ \phi'(z) &= A_1\,, \qquad \psi' = B_1 \\ \phi'' &= 0 \end{aligned}$$

The stresses are

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4R_e[\,A_1\,] = 4R_e[A_{11} + iA_{12}] = 4A_{11} \\ \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= 2[\bar{z}0 + B_1] = 2[B_{11} + iB_{12}] \end{aligned}$$

These can be separated out to give

$$\sigma_{xx} = 2A_{11} - B_{11}\,, \qquad \sigma_{yy} = 2A_{11} + B_{11}\,, \qquad \sigma_{xy} = B_{12}$$

This corresponds to a uniform stress state. These equations may also be rearranged as

$$B_{12} = \sigma_{xy}^{\infty}\,, \qquad B_{11} = \tfrac{1}{2}[\sigma_{yy}^{\infty} - \sigma_{xx}^{\infty}]\,, \qquad A_{11} = \tfrac{1}{4}[\sigma_{yy}^{\infty} + \sigma_{xx}^{\infty}]$$

This form allows us to determine the coefficients from knowledge of the remote stresses. Note that we can always set $A_{12} = 0$. ■

Example 5.10: An infinite sheet has a circular hole with arbitrary tractions applied around the edge of the hole and a constant remote stress. Determine the stress distribution.

This problem is a generalization of the Kirsch solution obtained earlier in the chapter. This example also serves as an application of Fourier series.

Choose the stress functions as

$$\phi'(z) = \sum_o^\infty A_n \frac{1}{z^n}\,, \qquad \psi'(z) = \sum_o^\infty B_n \frac{1}{z^n}\,, \qquad \phi''(z) = -\sum A_n \frac{n}{z^{n+1}}$$

since this gives constant stresses at infinity. The tractions around the hole are related to σ_{rr} and $\sigma_{r\theta}$, hence obtain

$$\begin{aligned} \sigma_{rr} - i\sigma_{r\theta} &= \phi'(z) + \overline{\phi'(z)} - [\bar{z}\phi''(z) + \psi'(z)]e^{2i\theta} \\ &= \sum A_n \frac{1}{z^n} + \sum \bar{A}_n \frac{1}{z^n} - \Big[-\sum A_n \frac{n\bar{z}}{z^{n+1}} + \sum B_n \frac{1}{z^n}\Big]e^{2i\theta} \end{aligned}$$

Let $z = re^{i\theta}$ and regroup to get

$$\sigma_{rr} - i\sigma_{r\theta} = \sum_o^\infty \Big[A_n(n+1) - \frac{1}{r^2}B_{(n+2)}\Big]\frac{e^{-in\theta}}{r^n} + \sum_o^\infty \bar{A}_n \frac{e^{in\theta}}{r^r} - B_o e^{i2\theta} - B_1 \frac{1}{r} e^{i\theta}$$

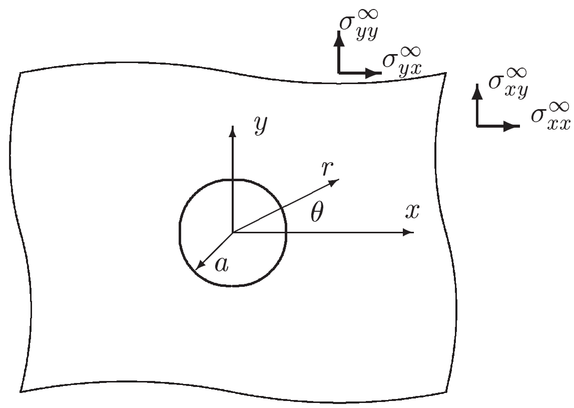


Figure 5.22: Generalized hole problem.

These stresses are recognized as a Fourier series in θ for fixed r. Hence, let the applied tractions at $r = a$ be represented by

$$[t_r - it_{r\theta}] = [\sigma_{rr} - i\sigma_{r\theta}]\Big|_{r=a} \approx \sum_{n=-\infty}^{\infty} C_n e^{in\theta}$$

That is, the arbitrary traction distribution is specified through the Fourier coefficients C_n. The stress coefficients are related to these by

$$\sum_{o}^{\infty}\left[A_n(n+1) - \frac{1}{a^2}B_{n+2}\right]\frac{e^{-in\theta}}{a^n} + \sum_{o}^{\infty}\bar{A}_n\frac{e^{in\theta}}{a^n} - B_o e^{i2\theta} - B_1\frac{e^{i\theta}}{a} = \sum_{n=-\infty}^{\infty} C_n e^{in\theta}$$

Equate the corresponding powers of θ to give the simultaneous equations necessary to solve for the coefficients:

$$\begin{aligned}
\bar{A}_n\frac{1}{a^n} &= C_n \qquad n \geq 3 \\
\bar{A}_2\frac{1}{a^2} - B_o &= C_2 \\
\bar{A}_1\frac{1}{a} - B_1\frac{1}{a} &= C_1 \\
A_o - \frac{1}{a^2}B_2 + \bar{A}_o &= C_o \\
A_o(n+1)\frac{1}{a^2} - B_{n+2}\frac{1}{a^{n+2}} &= C_{-n} \qquad n \geq 1
\end{aligned}$$

Solving these gives the recurrence relations for the coefficients as

$$\begin{aligned}
A_n &= a^n\bar{C}_n \qquad n \geq 3 \\
B_n &= -a^n C_{-n+2} + A_{n-2}(n-1)a^2 \qquad n \geq 3
\end{aligned}$$

Since A_o and B_o are associated with the stresses at infinity, let these be known also, hence the starter values for the recurrence relations are

$$\begin{aligned}
A_2 &= a^2B_o + a^2C_2 \\
B_2 &= (A_o + \bar{A}_o)a^2 - a^2C_o - B_1 + \bar{A}_1 = aC_1
\end{aligned}$$

We need an additional relation to obtain B_1 and A_1, separately.

Look at the contributions B_1 and A_1 to the displacement. In particular, consider

$$\begin{aligned} \phi' &= A_1 \frac{1}{z}\,, & \psi' &= B_1 \frac{1}{z} \\ \phi &= A_1 \ln z\,, & \psi &= B_1 \ln z \end{aligned}$$

The displacements are

$$\begin{aligned} 2\mu[u_r + iu_\theta] &= \left[\left(\frac{\lambda+3\mu}{\lambda+\mu}\right) A_1 \ln z - z\bar{A}_1 \frac{1}{\bar{z}z} - \bar{B}_1 \ln \bar{z}\right] e^{-i\theta} \\ &= \left[\left(\frac{\lambda+3\mu}{\lambda+\mu}\right) A_1 [\ln r + i\theta] - \bar{A}_1 e^{2i\theta} - \bar{B}_1 [\ln r - i\theta]\right] e^{-i\theta} \end{aligned}$$

Looking at these displacements at $\theta = 0$ and $\theta = 2\pi$ gives, respectively,

$$\begin{aligned} 2\mu[u_r + iu_\theta] &= [(\cdot)A_1[\ln r + 0] - \bar{A}_1 - \bar{B}_1[\ln r - 0]] \\ 2\mu[u_r + iu_\theta] &= [(\cdot)A_1[\ln r + i2\pi] - \bar{A}_1 - \bar{B}_1[\ln r - i2\pi]] \end{aligned}$$

The difference is

$$\left(\frac{\lambda+3\mu}{\lambda+\mu}\right) A_1 i2\pi + \bar{B}_1 i2\pi$$

Therefore, to have single valued displacements requires that this difference be zero. That is,

$$\bar{B}_1 = -\left(\frac{\lambda+3\mu}{\lambda+\mu}\right) A_1$$

This completes the set of equations. Some of the interesting special cases are discussed next.

If the applied tractions around the edge of the hole are zero, then $C_n = 0$ giving

$$\begin{aligned} A_2 &= a^2 B_o \\ A_n &= 0 \qquad n \geq 3 \\ B_1 &= A_1 = 0 \\ B_2 &= (A_o + \bar{A}_o)a^2 \\ B_3 &= A_1 2a^2 = 0 \\ B_4 &= A_2 3a^2 = 3a^4 B_o \\ B_n &= 0 \qquad n \geq 5 \end{aligned}$$

The stress functions for this case are therefore

$$\phi' = A_o + B_o \frac{a^2}{z^2}\,, \qquad \psi' = B_o + (A_o + \bar{A}_o)\frac{a^2}{z^2} + B_o 3\frac{a^4}{z^4}$$

Let the stresses at infinity be

$$\sigma_{xx} = 0\,, \quad \sigma_{xy} = 0\,, \quad \sigma_{yy} = \sigma^\infty$$

Hence the coefficients are

$$A_o = \tfrac{1}{4}\sigma^\infty\,, \qquad B_o = \tfrac{1}{2}\sigma^\infty$$

and the stress functions become

$$\begin{aligned} \phi' &= \tfrac{1}{4}\sigma^\infty[1 + 2\frac{a^2}{z^2}], \qquad \psi' = \tfrac{1}{2}\sigma^\infty[1 + \frac{a^2}{z^2} + 3\frac{a^4}{z^4}] \\ \phi'' &= \tfrac{1}{4}\sigma^\infty 4[-4\frac{a^2}{z^3}] \end{aligned} \tag{5.2}$$

These recover the solution to the Kirsch problem as obtained earlier in the chapter.

The case of a hole with a uniform pressure is easily obtained by setting

$$\begin{aligned} C_n &= 0 \quad \text{except} \quad C_o \neq 0 \\ A_o &= 0 \\ B_o &= 0 \end{aligned}$$

The elegance of the Fourier series approach is that any traction distribution can be represented simply by changing the coefficients C_n. In particular, for the point loaded hole set $C_n = 1$.

The above solutions are represented in terms of infinite series; however, they can be truncated depending on the accuracy required. With this in mind, the fast Fourier transform (FFT) algorithm can be used to do the summation very efficiently on a computer. ■

Example 5.11: Consider an elliptical hole in an infinite sheet under remote tension. Determine the stress distribution.

This, of course, is analogous to the Kirsch problem; what makes it different is that the hole is not circular and therefore it is necessary to introduce a scheme for treating such boundaries. One approach is to introduce elliptical coordinates and transform all the governing equations similar to what was done earlier for cylindrical coordinates. Instead, we introduce a much more powerful technique that uses conformal mappings.

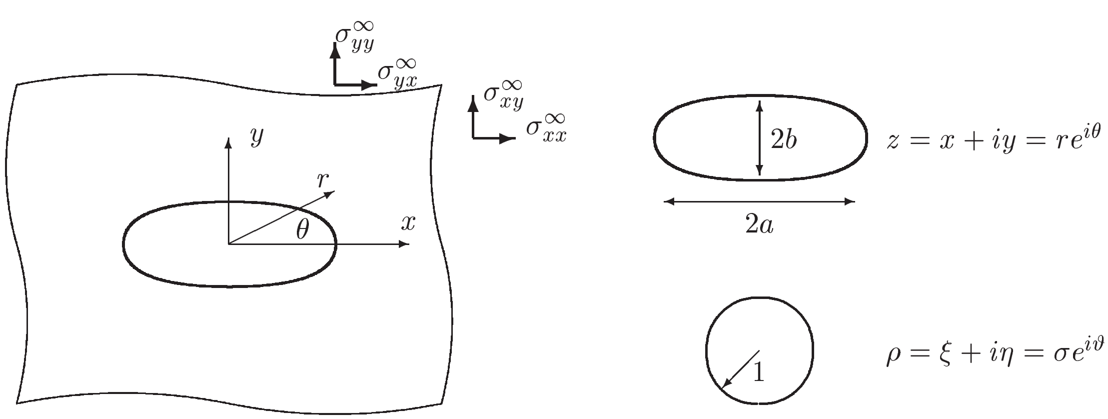


Figure 5.23: Geometry of an elliptical hole.

The conformal mapping idea is quite general but to concretize the method we will consider only the case of an ellipse. Let $z = x + iy$ be the physical plane and $\rho = \xi + i\eta$ be the transform plane. The mapping that relates the two is then

$$z = f(\rho) = c[\rho + m\frac{1}{\rho}] \qquad c = \frac{a+b}{2}\,, \quad m = \left(\frac{a-b}{a+b}\right)$$

In cylindrical coordinate form with $z = re^{i\theta}$ and $\rho = \sigma e^{i\vartheta}$, we have

$$re^{i\theta} = c\Big[\sigma e^{+i\vartheta} + \frac{m}{\sigma}e^{-i\vartheta}\Big]$$

A special cases of this is, for example, at $\vartheta = 0$

$$re^{i\theta} = c\Big[\sigma + \frac{m}{\sigma}\Big]$$

Hence if $\sigma = 1$ then $x = c[1+m]$, $y = 0$ which corresponds to the end of the ellipse. Another special case is when $\vartheta = 90^o$, then

$$re^{i\theta} = c[i\sigma - im\sigma]$$

Now if $\sigma = 1$ then $x = 0$, $y = +c[1-m]$ which corresponds to the top of the ellipse.

The traction free boundary conditions, expressed in terms of the conformal mapping, becomes

$$\overline{f(\rho)}\frac{d\phi}{d\rho}\frac{d\rho}{dz} + \bar{\phi} + \frac{d\psi}{d\rho}\frac{d\rho}{dz} = 0 \qquad \rho = \rho_{boundary}$$

We wish to find a set of stress functions that satisfies this condition.

Based on the behavior of a circular hole, let the stress functions be

$$\phi(z) = \phi(\rho) = A_0\rho - A_2\frac{1}{\rho}\,, \qquad \psi(z) = \psi(\rho) = B_o\rho - B_2\frac{1}{\rho} - B_4\frac{1}{3\rho^3}$$

giving for their derivatives

$$\phi'(z) = \frac{d}{dz}\phi(z) = \{A_o + A_2\frac{1}{\rho^2}\}\frac{d\rho}{dz}\,, \qquad \psi'(z) = \{B_o + B_2\frac{1}{\rho^2} + B_4\frac{1}{\rho^4}\}\frac{d\rho}{dz}$$

Substituting these into the traction free boundary condition, requires that

$$\overline{C[\rho + \frac{m}{\rho}]}\{A_o + A_2\frac{1}{\rho^2}\}\frac{d\rho}{dz} + \overline{\{A_o\rho - A_2\frac{1}{\rho}\}} + \{B_o\rho - B_2\frac{1}{\rho} - B_4\frac{1}{3\rho^3}\}\frac{d\rho}{dz} = 0$$

with $z = z_{boundary}$. On the boundary $\rho = \sigma e^{i\vartheta} = e^{i\vartheta}$; therefore, the mapping is given as

$$\frac{dz}{d\rho} = c[1 - \frac{m}{\rho^2}] \qquad \text{or} \qquad \frac{d\rho}{dz} = \frac{\rho^2}{c(\rho^2 - m)}$$

Hence the loading condition now becomes

$$\begin{aligned} c[e^{-i\vartheta} + me^{+i\vartheta}]\{A_o + A_2e^{-i\vartheta 2}\}\frac{e^{i\vartheta 2}}{c} \quad &+ \quad (e^{i2\vartheta} - m)\{\bar{A}_oe^{-i\vartheta} - \bar{A}_2e^{i\vartheta}\} \\ &+ \quad \frac{e^{i2\vartheta}}{c}\{B_oe^{i\vartheta} - B_2e^{-i\vartheta} - B_4e^{-i3\vartheta}\} = 0 \end{aligned}$$

Equating equal powers of ϑ results in

$$\begin{array}{rr} e^{i3\vartheta}: & mA_o - \bar{A}_2 + \frac{1}{c}B_o = 0 \\ e^{i\vartheta}: & A_o + mA_2 + \bar{A}_o + m\bar{A}_2 - \frac{1}{c}B_2 = 0 \\ e^{-i\vartheta}: & A_2 - mA_o - \frac{1}{c}B_4\frac{1}{3} = 0 \end{array}$$

giving the solution

$$\begin{aligned} A_2 &= mA_o + \frac{1}{c}B_o \\ \frac{1}{c}B_2 &= A_o + \bar{A}_o + m(A_2 + \bar{A}_2) = A_o + \bar{A}_o + m^2(A_o + \bar{A}_o) \\ \frac{1}{3c}B_4 &= -mA_o + mA_0 + \frac{1}{c}B_o = \frac{1}{c}B_o \end{aligned}$$

The stress functions can now be written as

$$\begin{aligned} \phi'(z) &= \{A_o + [mA_o + \frac{1}{c}B_o]\frac{1}{\rho^2}\}\frac{\rho^2}{c(\rho^2 - m)} \\ \psi'(z) &= \{B_o + [(1+m^2)(A_o + \bar{A}_0) + \frac{m}{C}(B_o + \bar{B}_o)]\frac{c}{\rho^2} + 3B_o\frac{1}{\rho^4}\}\frac{\rho^2}{c(\rho^2 - m)} \end{aligned}$$

Imposing the remote stress limits of $\sigma_{xx} = 0$, $\sigma_{xy} = 0$, and $\sigma_{yy} = \sigma^\infty$, gives

$$A_o = \tfrac{1}{4}c\sigma^\infty\,, \qquad B_o = \tfrac{1}{2}c\sigma^\infty$$

The stress functions are then

$$\begin{aligned} \phi'(z) &= \tfrac{1}{2}\sigma^\infty\{1 + (m+2)\frac{1}{\rho^2}\}\frac{\rho^2}{(\rho^2 - m)} \\ \psi'(z) &= \tfrac{1}{2}\sigma^\infty\{1 + ((1+m^2) + 2m)\frac{1}{\rho^2} + \frac{1}{3\rho^4}\}\frac{\rho^2}{(\rho^2 - m)} \end{aligned} \tag{5.3}$$

These can now be used to obtain the stresses at any point.

Compare this solution with the stress function for the circular hole. Obviously if $a = b$ then $m = 0$ and we recover the Kirsch solution. But more importantly, we note that both have the same structure with the associations $\rho \leftrightarrow z$ and that the geometry appears in the $\rho^2/(\rho^2 - m)$ term.

We now consider some special cases. The stress invariant is given by

$$\sigma_x + \sigma_y = \sigma^\infty R_e\left[\{1 + (m+2)\frac{1}{\rho^2}\}\frac{\rho^2}{(\rho^2 - m)}\right], \qquad z = c[\rho + \frac{m}{\rho}]$$

A special case of considerable interest is the stress along the inner boundary. Here one of the normal stresses is zero allowing the other to be determined directly from the invariant as

$$\sigma_t = \sigma^\infty R_e\left[\{1 + (m+2)e^{-2i\vartheta}\}\frac{e^{2i\vartheta}}{(e^{2i\vartheta} - m)}\right]$$

The maximum stress occurs at $\vartheta = 0$ and is

$$\sigma_t = \sigma^\infty R_e \left[\{1 + (m+2)\} \frac{1}{1-m} \right] = \sigma^\infty \frac{(3+m)}{(1-m)}$$

Note that if $m = 0$ the boundary is circular and then $\sigma_t = 3\sigma^\infty$. At the other extreme, if $m = 1$ the boundary corresponds to that of a crack and then $\sigma_t = \infty$. Consider this second case in more detail. In particular, consider when the radius is sharp but not zero, then

$$R = \frac{b^2}{a}, \qquad m = \left(\frac{a-b}{a+b} \right) = \left(\frac{a - \sqrt{aR}}{a + \sqrt{aR}} \right)$$

This gives the maximum stress as

$$\sigma_t = \sigma^\infty \left[\frac{3a + 3\sqrt{aR} + a - \sqrt{aR}}{a + \sqrt{aR} - a + \sqrt{aR}} \right] = \sigma^\infty \left[\frac{4a + 2\sqrt{aR}}{2\sqrt{aR}} \right] = \sigma^\infty \left[1 + 2\sqrt{\frac{a}{R}} \right]$$

Consequently, the sharper the radius the higher the stress. ■

Exercises

5.1 Use the Fourier series representation to solve the problem of a circular hole with diametrically opposite point loads.

5.2 Solve the problem of a circular hole with a traction distribution given by $p_o \sin^2\theta$.

5.3 Solve the problem of a crack loaded by a uniform pressure.

5.4 Use the Fourier series representation to solve the problem of a crack with opposite point loads at its center.

5.5 Determine if the following strain field is compatible.

$$\epsilon_{xx} = 2x^2 + 3y^2 + z + 1 \qquad \epsilon_{yy} = x^2 + 2y^2 + 3z + 2 \qquad \epsilon_{zz} = 3x + 2y + z^2 + 1$$
$$\epsilon_{xy} = 4xy \qquad \epsilon_{yz} = \epsilon_{xz} = 0$$

5.6 The stresses in a 3-D stressed body are

$$\sigma_{xx} = -A(L-x)y\,, \qquad \sigma_{xy} = \tfrac{1}{8}A(h^2 - 4y^2) = \sigma_{yx}$$

with $\sigma_{ij} = 0$ otherwise, and A, L, h being constants. Is this stress system in equilibrium? If the problem is one of plane stress, are the strains compatible? What are the displacements? Show that if u, v, dv/dx are zero at $x = 0$, $y = 0$ then the vertical deflection of the line $y = 0$ is

$$v(x) = \frac{A}{6E}[3L - x]x^2$$

5.7 Let it be required to have a displacement field of the form

$$v(x, y) = Ax^2 \qquad u(x, y) = ?$$

What should the functional form of u be so that all the field equations are satisfied?

5.8 Show that while the non-trivial stresses

$$\sigma_{xx} = Py[1 - \frac{y^2}{3b^2}]$$

(other stresses being zero) satisfies equilibrium, the corresponding strains are not compatible.

5.9 What must be the relationship between the coefficients A and B in order for the following strains to be compatible?

$$\epsilon_{xx} = A[x^2 + y^2] \qquad \epsilon_{yy} = A[x^2 + y^2] \qquad 2\epsilon_{xy} = Bxy = \epsilon_{yx} \qquad \text{others} = 0$$

5.10 The tractions on the upper and lower surfaces of a rectangular block are

$$t_y = x^2 , \qquad t_x = 0$$

while on the ends they are zero. Prove that the stress system

$$\sigma_{xx} = 0 , \qquad \sigma_{yy} = x^2 , \qquad \sigma_{xy} = 0 = \sigma_{yx}$$

is not a solution to the problem.

5.11 Show that while the stress function

$$\phi(x, y) = \tfrac{1}{2} P y^2 [1 - y^2/(6b^2)]$$

gives stress that are in equilibrium, the corresponding strains are not compatible.

5.12 Consider the Airy stress function

$$\phi(x, y) = A \log_n(\sqrt{x^2 + y^2})$$

Sketch the stress distributions along a few coordinate lines. What are the tractions along the surface $x^2 + y^2 = a^2$.

5.13 What class of problems is solved by the following Airy stress function?

$$\phi(x, y) = Ax^2 + Bxy + Cy^2$$

5.14 Consider the following polynomial stress function

$$\phi(x, y) = Ax^2 + Bx^2 y + Cxy^2 + Dy^3$$

Under what circumstance(s) is it bi-harmonic? Use it to solve the problem of pure bending of a prismatic bar.

5.15 Show that the following polynomial stress function

$$\phi(x, y) = Axy + Bx^3 + Cx^3 y + Dxy^3 + Ex^3 y^3 + Fxy^5$$

can be used to solve the rectangular dam problem. Note that since this polynomial is not symmetric in x and y that the orientation of the axes must be chosen appropriately.

5.16 Motivated by the desire to use Fourier series to represent the applied tractions, it is proposed to use the following stress function

$$\phi(x, y) = \cos(n\pi x/L) f(y)$$

where $n = 0, 1, \ldots$ and L is a constant. Determine the allowable form for $f(y)$ for this to be an acceptable Airy stress function. If the applied tractions are represented as

$$P(x) \approx \sum_n a_n \cos(n\pi x/L)$$

determine the stress function in terms of a_n.

5.17 From the previous problem, show that the stress σ_{xx} at a point on the surface of a half-plane is a compression equal to the applied pressure at that point.

5.18 If the body forces are absent, show that the displacements can be given by the Airy stress function as

$$2\mu u_x = -\frac{\partial \phi}{\partial x} + \frac{1+x}{4}\frac{\partial \psi}{\partial y}, \qquad 2\mu u_y = -\frac{\partial \phi}{\partial y} + \frac{1+x}{4}\frac{\partial \psi}{\partial x}$$

where $\nabla^2\psi = 0$ and $\nabla^2\phi = \dfrac{\partial^2\psi}{\partial x \partial y}$.

5.19 An infinite plate containing a circular hole is subjected to a pure shear stress at infinity. Given the Airy stress function

$$\phi(r,\theta) = [Ar^2 + Br^{-2} + C]\sin\theta$$

Find A, B, and C. Plot $\sigma_{\theta\theta}$ around the hole.

5.20 Show that the problem of a curved cantilever beam with an end shear force can be solved with the following stress function.

$$\phi(r,\theta) = [Ar^3 + Br^{-1} + Cr\log_n r]\sin\theta$$

5.21 Show that Flamant's problem of a point load on a half-plane can be solved with the following stress function.

$$\phi(r,\theta) = Ar\theta\sin\theta$$

5.22 A ring, inner radius b and outer radius a, is split and the two ends moved radially apart an amount Δ. Show that the following stress function solves the problem.

$$\phi(r,\theta) = -\frac{2\mu\Delta}{\pi(\kappa+1)}\left[r\log_n r\sin\theta + \frac{1}{2(1+\beta^2)}\left(\beta^2 a^2\frac{\sin\theta}{r} - \frac{1}{a^2}r^3\sin\theta\right)\right], \qquad \beta = \frac{b}{a}$$

5.23 A rigid disk has a resultant moment applied to it. What is the simplest distribution of traction on its edge that will keep it in equilibrium?

5.24 A rigid disk is solidly embedded in an infinite sheet. Determine the stress distribution in the sheet due to an applied moment acting on the disk.

5.25 An elastic disk is bonded to a rigid ring. The composite disk rotates with angular constant speed Ω. Find the stress and displacement fields.

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2\log r + 1$	0	$2\log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$r^4\sin 2\theta$	0	$-6r^2\cos 2\theta$	$12r^2\sin 2\theta$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^2\sin 2\theta$	$-2\sin 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$\sin 2\theta$	$-4\sin 2\theta/r^2$	$2\cos 2\theta/r^2$	0
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\sin 2\theta/r^2$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$	$6\sin 2\theta/r^4$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$(n+1)nr^n\sin n\theta$	$(n+2)(n+1)r^n\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-(n+1)nr^n\cos n\theta$	$(n+2)(n+1)r^n\sin n\theta$
$\cos n\theta/r^n$	$-(n+1)n\cos n\theta/r^{n+2}$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$
$\sin n\theta/r^n$	$-(n+1)n\sin n\theta/r^{n+2}$	$(n+1)n\cos n\theta/r^{n+2}$	$(n+1)n\sin n\theta/r^{n+2}$
$\cos n\theta/r^{n-2}$	$-(n+2)(n-1)\cos n\theta/r^n$	$-n(n-1)\sin n\theta/r^n$	$(n-1)(n-2)\cos n\theta/r^n$
$\sin n\theta/r^{n-2}$	$-(n+2)(n-1)\sin n\theta/r^n$	$n(n-1)\cos n\theta/r^n$	$(n-1)(n-2)\sin n\theta/r^n$

Table 5.1: Stresses

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$\log r$	$-1/r$	0
θ	0	$-1/r$
$r^2\log r$	$(\kappa-1)r\log r - r$	$(\kappa+1)r\theta$
$r^2\theta$	$(\kappa-1)r\theta$	$-(\kappa+1)r\log r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$2r\theta\sin\theta$	$(\kappa-1)\theta\sin\theta+(\kappa+1)\log r\cos\theta-\cos\theta$	$(\kappa-1)\theta\cos\theta-(\kappa-1)\log r\sin\theta-\sin\theta$
$2r\theta\cos\theta$	$(\kappa-1)\theta\cos\theta+(\kappa-1)\log r\sin\theta-\sin\theta$	$-(\kappa-1)\theta\sin\theta-(\kappa+1)\log r\cos\theta-\cos\theta$
$2r\log r\cos\theta$	$(\kappa+1)\theta\sin\theta+(\kappa-1)\log r\cos\theta-\cos\theta$	$(\kappa+1)\theta\cos\theta-(\kappa-1)\log r\sin\theta-\sin\theta$
$2r\log r\sin\theta$	$-(\kappa+1)\theta\cos\theta+(\kappa-1)\log r\sin\theta-\sin\theta$	$(\kappa+1)\theta\sin\theta+(\kappa-1)\log r\cos\theta+\cos\theta$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^4\cos 2\theta$	$-(3-\kappa)r^3\cos 2\theta$	$(3+\kappa)r^3\sin 2\theta$
$r^4\sin 2\theta$	$-(3-\kappa)r^3\sin 2\theta$	$-(3+\kappa)r^3\cos 2\theta$
$r^2\cos 2\theta$	$-2r\cos 2\theta$	$2r\sin 2\theta$
$r^2\sin 2\theta$	$-2r\sin 2\theta$	$-2r\cos 2\theta$
$\cos 2\theta$	$(\kappa+1)\cos 2\theta/r$	$-(\kappa-1)\sin 2\theta/r$
$\sin 2\theta$	$(\kappa+1)\sin 2\theta/r$	$(\kappa-1)\cos 2\theta/r$
$\cos 2\theta/r^2$	$2\cos 2\theta/r^3$	$2\sin 2\theta/r^3$
$\sin 2\theta/r^2$	$2\sin 2\theta/r^3$	$-2\cos 2\theta/r^3$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{n+2}\cos n\theta$	$-(n+1-\kappa)r^{n+1}\cos n\theta$	$(n+1+\kappa)r^{n+1}\sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1-\kappa)r^{n+1}\sin n\theta$	$-(n+1+\kappa)r^{n+1}\cos n\theta$
$\cos n\theta/r^n$	$n\cos n\theta/r^{n+1}$	$n\sin n\theta/r^{n+1}$
$\sin n\theta/r^n$	$n\sin n\theta/r^{n+1}$	$-n\cos n\theta/r^{n+1}$
$\cos n\theta/r^{n-2}$	$(n-1+\kappa)\cos n\theta/r^{n-1}$	$-(n-1-\kappa)\sin n\theta/r^{n-1}$
$\sin n\theta/r^{n-2}$	$(n-1+\kappa)\sin n\theta/r^{n-1}$	$-(n-1-\kappa)\cos n\theta/r^{n-1}$

Table 5.2: Displacements.

Chapter 6

Nonlinear Elasticity Problems

In the general case of deformable bodies, we can have large displacements, large rotations, and large strains. This renders the governing equations highly nonlinear and therefore they can only be solved using computational methods. Furthermore, because of the complicated load history dependence, this suggests a time or load incremental type solution. We combine these two requirements into an incremental/iterative solution algorithm. The total Lagrangian scheme is introduced as a specific example of a solution scheme and this is combined with Newton-Raphson equilibrium iterations to give accurate solutions of the nonlinear equations.

An essential step is formulating the problem as a set of discrete unknowns; this is demonstrated using a finite element formulation. Furthermore, the field equations must be recast in terms of discrete systems and this is done via the principle of virtual work.

6.1 Discretizing Continuous Media

An effective strategy for solving general nonlinear problems for continuous media is to divide the body into many subregions each of which have a relatively simple deformation distribution. We illustrate this process here; while the process is approximate, we can establish conditions under which the exact result can be obtained.

1-D Spatial Interpolations

Consider the 1-D space shown in Figure 6.1 and discretized as indicated by the black dots. The values of the functions, u_i, at the points are the discretized representation. Consider a typical line segment between two dots of length L divided into two segments where the common point is at (x) as shown in Figure 6.2. Define

$$h_1 = L_1/L\,, \qquad h_2 = L_2/L\,, \qquad h_i = h_i(x)$$

We have the obvious constraint that $h_1 + h_2 = 1$. The lengths of these segments uniquely define the position of the common point.

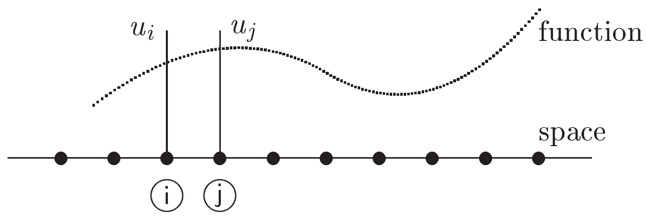


Figure 6.1: A discretized 1-D medium.

The position of a point (x) along the line can be written as

$$\left\{ \begin{array}{c} 1 \\ x \end{array} \right\} = \left[\begin{array}{cc} 1 & 1 \\ x_1 & x_2 \end{array} \right] \left\{ \begin{array}{c} h_1 \\ h_2 \end{array} \right\} \tag{6.1}$$

where the subscripts $1, 2$ refer to the node. We can invert this to get the expressions for the coordinates

$$\left\{ \begin{array}{c} h_1 \\ h_2 \end{array} \right\} = \frac{1}{L} \left[\begin{array}{cc} x_2 & -1 \\ -x_1 & 1 \end{array} \right] \left\{ \begin{array}{c} 1 \\ x \end{array} \right\} = \frac{1}{L} \left[\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right] \left\{ \begin{array}{c} 1 \\ x \end{array} \right\}$$

with $L \equiv x_2 - x_1$. From this, it is apparent that functions of (x) can equally well be written as functions of (h_1, h_2). That is, any function of interest can be written as

$$u(x) = \sum_i^2 h_i(x) u_i = \frac{1}{L}[x_2 - x]u_1 + \frac{1}{L}[-x_1 + x]u_2 = h_1(x)\, u_1 + h_2(x)\, u_2$$

where u_i are the nodal values. The functions $h_i(x)$ are called *interpolation* functions.

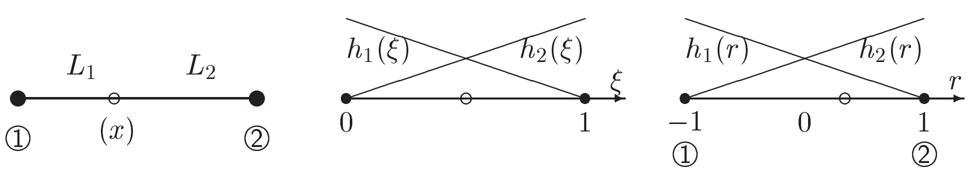


Figure 6.2: Coordinates for 1-D spaces. (a) Physical. (b) Natural. (c) Isoparametric.

In order for the two coordinates $\{h_1, h_2\}$ to describe the single coordinate x, it must be supplemented by the constraint $h_1 + h_2 = 1$. We can invoke this constraint explicitly by introducing *natural coordinates* given as

$$h_1 = 1 - \xi\,, \qquad h_2 = \xi \tag{6.2}$$

These are shown in Figure 6.2(b) where ξ ranges from 0 to 1. We have for a typical function

$$u(x) = \sum_i^2 h_i(\xi) u_i = [1 - \xi]u_1 + [\xi]u_2$$

We can introduce yet a third set of coordinates which we will call *isoparametric* coordinates given as

$$h_1 = \tfrac{1}{2}[1-r]\,, \qquad h_2 = \tfrac{1}{2}[1+r] \tag{6.3}$$

These are shown in Figure 6.2(c) where r ranges from -1 to $+1$. We have for a typical function

$$u(x) = \textstyle\sum_i^2 h_i(r)u_i = \tfrac{1}{2}[1-r]u_1 + \tfrac{1}{2}[1+r]u_2$$

This form better generalizes for higher dimensions and is especially useful when numerical methods are used for integration purposes.

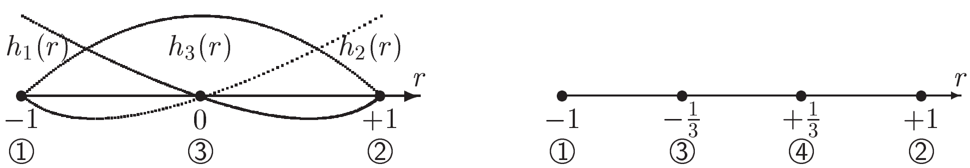


Figure 6.3: Higher order interpolations. (a) 3-Node. (b) 4-Node.

For any segment length, we can increase the order of function by increasing the number of nodes. Thus Figure 6.3(a) has 3 nodes and therefore allows for the interpolations

$$h_1 = \tfrac{1}{2}[1-r] - \tfrac{1}{2}[1-r^2]\,, \qquad h_2 = \tfrac{1}{2}[1+r] - \tfrac{1}{2}[1-r^2]\,, \qquad h_3 = [1-r^2]$$

Note that each function has the value 1 at its corresponding node. These expressions could be simplified but the current form best expresses the hierarchical form of the interpolations. In a similar way, the addition of a fourth node allows the interpolations

$$\begin{aligned}
h_1 &= \tfrac{1}{2}[1-r] - \tfrac{1}{2}[1-r^2] - \tfrac{1}{16}[1+9r-r^2-9r^3] \\
h_2 &= \tfrac{1}{2}[1+r] - \tfrac{1}{2}[1-r^2] - \tfrac{1}{16}[1+9r-r^2-9r^3] \\
h_3 &= \qquad\qquad\quad + [1-r^2] - \tfrac{1}{16}[7+27r-7r^2-27r^3] \\
h_4 &= \qquad\qquad\qquad\qquad\qquad - \tfrac{1}{16}[9+27r-9r^2-27r^3]
\end{aligned}$$

where the hierarchical form is clear. What the higher order interpolations afford, quite obviously, is the use of higher order functions.

In the form presented, the interpolations also apply to the general curved line in space that deforms into another general line in space. That is, we can think of the coordinates themselves as being interpolated. To clarify this point, let the original coordinates be designated with a superscript 'o', then any point along the line is given by

$$x^o = \sum_{i=1}^{N} h_i(r) x_i^o$$

where (x_i^o) are the original coordinates of the nodes. Thus each (r) coordinate has a corresponding (x^o) location.

A typical variable is therefore represented by

$$x^o = \sum_i^N h_i(r)\, x_i^o\,, \qquad u = \sum_i^N h_i(r)\, u_i$$

where u_i and x_i are the nodal values and N is the total number of nodes. The element strains, for example, are obtained in terms of derivatives of element displacements. Using the isoparametric coordinate system, we get, for example,

$$\frac{\partial}{\partial x^o} = \frac{\partial r}{\partial x^o}\frac{\partial}{\partial r}$$

The above interpolation has x^o as a function of r, but to evaluate the derivatives of r with respect to x^o we need to have the inverse relation between the two sets of variables. This is obtained as

$$\frac{\partial}{\partial r} = \frac{\partial x^o}{\partial r}\frac{\partial}{\partial x^o} \qquad \text{or} \qquad \{\frac{\partial}{\partial r}\} = [\, J_e \,]\{\frac{\partial}{\partial x^o}\}$$

The notation we use anticipates the generalizations needed when we consider multiple dimensions. The function $[\, J_e \,]$ is called the *Jacobian operator* relating the isoparametric coordinates to the physical coordinates. The relation for the derivatives requires

$$\{\frac{\partial}{\partial x^o}\} = [J_e^{-1}]\{\frac{\partial}{\partial r}\}$$

which requires that $[J_e^{-1}]$ exists. In most cases, the existence is clear; however, in cases where the segment is much distorted or folds back on itself the Jacobian transformation can become singular.

Example 6.1: Determine the relation between the strain and the degrees of freedom of a three-noded line segment.

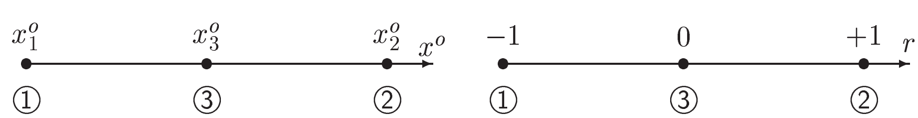


Figure 6.4: Three noded 1-D element.

Let the ends of the segment have positions x_1^o and x_2^o and let the middle node be at x_3^o as shown in Figure 6.4. The position of an arbitrary point is given by

$$\begin{aligned} x^o &= \sum_i^3 h_i(r)x_i^o = h_1(r)x_1^o + h_2(r)x_2^o + h_3(r)x_3^o \\ &= [\tfrac{1}{2}[1-r] - \tfrac{1}{2}[1-r^2]]x_1^o + [\tfrac{1}{2}[1+r] - \tfrac{1}{2}[1-r^2]]x_2^o + [1-r^2]x_3^o \\ &= \tfrac{1}{2}r[-1+r]x_1^o + \tfrac{1}{2}r[1+r]x_2^o + [1-r^2]x_3^o \end{aligned}$$

The derivative of the coordinate is

$$\frac{\partial x^o}{\partial r} = \tfrac{1}{2}[-1+2r]x_1^o + \tfrac{1}{2}[1+2r]x_2^o + [0-2r]x_3^o = \tfrac{1}{2}[x_2^o - x_1^o] + [x_1^o - 2x_3^o + x_2^o]r = J_e$$

Typically, J_e is a function of the coordinate r but in this case where

$$x_2^o = x_1^o + L\,, \qquad x_3^o = x_1^o + L/2$$

then

$$J_e = L/2\,, \qquad J_e^{-1} = 2/L$$

and is constant.

The displacement at an arbitrary interpolated point is given by

$$u(x^o) = u(r) = \sum\nolimits_i^3 h_i(r) u_i$$

and the strain is therefore

$$\begin{aligned}\epsilon_{xx} = \frac{\partial u}{\partial x^o} = \frac{\partial r}{\partial x^o}\frac{\partial u}{\partial r} = [J_e^{-1}]\frac{\partial u}{\partial r} &= [J_e^{-1}]\sum\nolimits_i \frac{\partial h_i}{\partial r} u_i \\ &= \frac{2}{L}\Big[\tfrac{1}{2}[-1+2r]u_1 + \tfrac{1}{2}[1+2r]u_2 + [0-2r]u_3\Big] \\ &= \frac{1}{L}[u_2 - u_1] + \frac{r}{L}[u_1 - 2u_3 + u_2]\end{aligned}$$

This gives a linear distribution of strain along the segment. Furthermore, in the limit of small segment size, we would get $u_3 \approx \frac{1}{2}(u_1 + u_2)$ causing the second term to be zero leaving a constant strain state given by $\epsilon_{xx} = [u_2 - u_1]/L$. As will be seen, this is an essential requirement for convergence. ■

Hexahedral Discretization for 3-D Bodies

Consider the general blocks of material in Figures 6.5(a) & 6.6(a) undergoing deformation. We will describe the deformation just in terms of the edge behaviors; in turn, these edges are discretized using nodal values as just described. Both 8-noded and 20-noded hexahedrals are considered.

Consider the 8-noded hexahedral solid shown in Figure 6.5(a), in the isoparametric coordinates of Figure 6.5(b) it is a cube with sides of length 2. The coordinates are given by

$$x^o = \sum\nolimits_i^8 h_i(r,s,t)x_i^o\,, \qquad y^o = \sum\nolimits_i^8 h_i(r,s,t)y_i^o\,, \qquad z^o = \sum\nolimits_i^8 h_i(r,s,t)z_i^o$$

where the interpolation functions are

$$\begin{aligned} h_1 &= \tfrac{1}{8}(1-r)(1-s)(1-t)\,, & h_5 &= \tfrac{1}{8}(1-r)(1-s)(1+t) \\ h_2 &= \tfrac{1}{8}(1+r)(1-s)(1-t)\,, & h_6 &= \tfrac{1}{8}(1+r)(1-s)(1+t) \\ h_3 &= \tfrac{1}{8}(1+r)(1+s)(1-t)\,, & h_7 &= \tfrac{1}{8}(1+r)(1+s)(1+t) \\ h_4 &= \tfrac{1}{8}(1-r)(1+s)(1-t)\,, & h_8 &= \tfrac{1}{8}(1-r)(1+s)(1+t) \end{aligned}$$

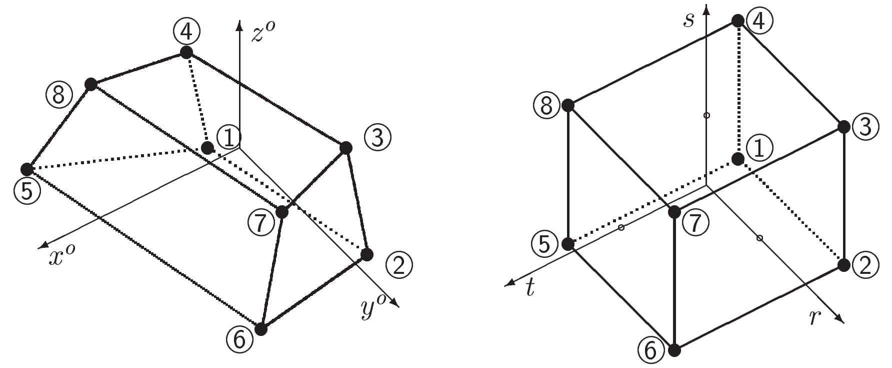


Figure 6.5: Deformation of a straight edged 3-D block using the Hex8 discretization. (a) Physical coordinates. (b) Isoparametric coordinates.

These functions have the value unity at each of the indicated nodes which additionally coincides with the subscripted number. For future reference, we can write these in the shorthand notation

$$h_i = \tfrac{1}{8}(1 + r_i r)(1 + s_i s)(1 + t_i t)$$

where (r_i, s_i, t_i) are the nodal coordinates and are specified in Table 6.1.

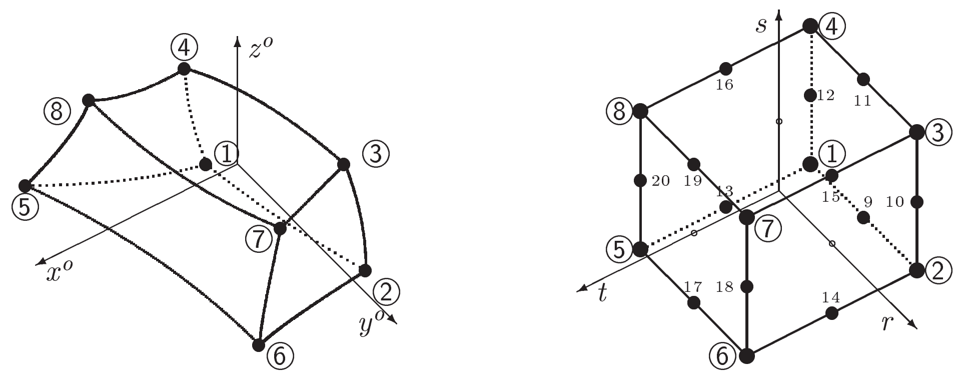


Figure 6.6: Deformation of a curved 3-D block using the Hex20 discretization. (a) Physical coordinates. (b) Isoparametric coordinates.

The power of the isoparametric approach to discretization is that the procedure is easily generalized to higher order interpolation functions. We illustrate the idea with the curved block shown in Figure 6.6(a). The block is discretized in terms of 20 points, the first eight being the same as for the linear hexahedron and the additional twelve corresponding to mid-edge nodes — the numberings for these nodes are shown in Figure 6.6(b). The coordinates of an arbitrary point are then given by

$$x^o = \sum\nolimits_i^{20} h_i(r,s,t) x_i^o \,, \qquad y^o = \sum\nolimits_i^{20} h_i(r,s,t) y_i^o \,, \qquad z^o = \sum\nolimits_i^{20} h_i(r,s,t) z_i^o$$

i	1	2	3	4	5	6	7	8				
r_i	-1	$+1$	$+1$	-1	-1	$+1$	$+1$	-1				
s_i	-1	-1	$+1$	$+1$	-1	-1	$+1$	$+1$				
t_i	-1	-1	-1	-1	$+1$	$+1$	$+1$	$+1$				
i	9	10	11	12	13	14	15	16	17	18	19	20
r_i	0	$+1$	0	-1	-1	$+1$	$+1$	-1	0	$+1$	0	-1
s_i	-1	0	$+1$	0	-1	-1	$+1$	$+1$	-1	0	$+1$	0
t_i	-1	-1	-1	-1	0	0	0	0	$+1$	$+1$	$+1$	$+1$

Table 6.1: Coefficients for interpolation functions.

where the interpolation functions are

$$
\begin{aligned}
i = 1, 8: \qquad & h_i = \tfrac{1}{8}(1 + r_i r)(1 + s_i s)(1 + t_i t)(r_i r + s_i s + t_i t - 2) \\
i = 9, 11, 17, 19: \qquad & h_i = \tfrac{1}{4}(1 - r^2)(1 + s_i s)(1 + t_i t) \\
i = 10, 12, 18, 20: \qquad & h_i = \tfrac{1}{4}(1 + r_i r)(1 - s^2)(1 + t_i t) \\
i = 13, 14, 15, 16: \qquad & h_i = \tfrac{1}{4}(1 + r_i r)(1 + s_i s)(1 - t^2)
\end{aligned}
$$

As before, (r_i, s_i, t_i) are the nodal coordinates and are specified in Table 6.1.

The element strains are obtained in terms of derivatives of element displacements. Using the isoparametric coordinate system, we get, for example,

$$\frac{\partial}{\partial x^o} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x^o} + \frac{\partial}{\partial s}\frac{\partial s}{\partial x^o} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x^o}$$

But to evaluate the derivatives of $(r,\, s,\, t)$ with respect to $(x^o,\, y^o,\, z^o)$, we need to have the explicit relation between the two sets of variables. As illustrated earlier, we first form the inverse relation according to

$$\left\{\begin{matrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial t} \end{matrix}\right\} = \begin{bmatrix} \dfrac{\partial x^o}{\partial r} & \dfrac{\partial y^o}{\partial r} & \dfrac{\partial z^o}{\partial r} \\ \dfrac{\partial x^o}{\partial s} & \dfrac{\partial y^o}{\partial s} & \dfrac{\partial z^o}{\partial s} \\ \dfrac{\partial x^o}{\partial t} & \dfrac{\partial y^o}{\partial t} & \dfrac{\partial z^o}{\partial t} \end{bmatrix} \left\{\begin{matrix} \dfrac{\partial}{\partial x^o} \\ \dfrac{\partial}{\partial y^o} \\ \dfrac{\partial}{\partial z^o} \end{matrix}\right\} \qquad \text{or} \qquad \{\frac{\partial}{\partial r}\} = [\, J_e\,]\{\frac{\partial}{\partial x^o}\}$$

where $[\, J_e\,]$ is called the *Jacobian operator* relating the isoparametric coordinates to the global coordinates. The relation for the derivatives requires

$$\{\frac{\partial}{\partial x^o}\} = [J_e^{-1}]\{\frac{\partial}{\partial r}\}$$

which requires that $[J_e^{-1}]$ exists. The structure of $[\, J_e\,]$ is

$$[\, J_e\,] = \sum_i^N \begin{bmatrix} h_{i,r} x_i^o & h_{i,r} y_i^o & h_{i,r} z_i^o \\ h_{i,s} x_i^o & h_{i,s} y_i^o & h_{i,s} z_i^o \\ h_{i,t} x_i^o & h_{i,t} y_i^o & h_{i,t} z_i^o \end{bmatrix}$$

where the subscript 'comma' indicates partial differentiation. This can be expressed in matrix form as

$$[\,J_e\,] = \begin{bmatrix} h_{1,r} & h_{2,r} & h_{3,r} & h_{4,r} & \cdots & h_{N,r} \\ h_{1,s} & h_{2,s} & h_{3,s} & h_{4,s} & \cdots & h_{N,s} \\ h_{1,t} & h_{2,t} & h_{3,t} & h_{4,t} & \cdots & h_{N,t} \end{bmatrix} \begin{bmatrix} x_1^o & y_1^o & z_1^o \\ x_2^o & y_2^o & z_2^o \\ \vdots & \vdots & \vdots \\ x_N^o & y_N^o & z_N^o \end{bmatrix} = [\partial_1 h][\,X\,]$$

The derivatives for the linear hexahedron Hex8, for example, are given by

$$\begin{aligned} \frac{\partial x^o}{\partial r} &= -\tfrac{1}{8}(1-s)(1-t)x_1^o + \tfrac{1}{8}(1-s)(1-t)x_2^o + \tfrac{1}{8}(1+s)(1-t)x_3^o \\ &\qquad - \tfrac{1}{8}(1+s)(1-t)x_4^o - \tfrac{1}{8}(1-s)(1+t)x_5^o + \cdots \\ \frac{\partial x^o}{\partial s} &= -\tfrac{1}{8}(1-r)(1-t)x_1^o - \tfrac{1}{8}(1+r)(1-t)x_2^o + \tfrac{1}{8}(1+r)(1-t)x_3^o \\ &\qquad + \tfrac{1}{8}(1-r)(1-t)x_4^o - \tfrac{1}{8}(1-r)(1+t)x_5^o + \cdots \end{aligned}$$

with analogous expressions for the other derivatives. It is therefore clear that both $[\,J_e\,]$ and $[\,J_e\,]^{-1}$ depend on (r, s, t) as well as the nodal coordinates; however, it is independent of the absolute global position of the element. If the block is rectangular, the Jacobian matrix is constant with diagonal only terms. Additionally, the product of the diagonal terms (which is the determinant) is related to the volume as $V^o_{cube} = det|J_e|\,8$.

In general, integrals involving the deformation will need to be performed numerically; this is demonstrated later in the chapter after the form of the system matrices are established.

A final point of crucial importance concerns the compatibility of many elemental blocks after assemblage. Each block uses the same interpolation functions, and shared faces use the same nodes; consequently, the complete shared face has the same displacement shape and no gaps are formed. (This is true even for very large deformations.) The blocks are therefore compatible. While the displacements are continuous across blocks, derivatives of displacement are not necessarily so; that is, the strain distribution may be discontinuous. This raises the issue of convergence to the exact (continuous) solution — we leave that to later when the field equations have been reformulated.

Example 6.2: Establish the explicit form for determining the displacement gradients and the strains for the Hex20 element.

We have for a typical displacement function

$$u(x^o, y^o, z^o) = \sum_i^{20} h_i(r, s, t) u_i$$

where u_i are the nodal values of the function. The displacement gradients can therefore be computed as

$$\frac{\partial u}{\partial x^o} = \sum_i \frac{\partial h_i}{\partial x^o} u_i\,, \qquad \frac{\partial v}{\partial x^o} = \sum_i \frac{\partial h_i}{\partial x^o} v_i\,, \qquad \frac{\partial v}{\partial y^o} = \sum_i \frac{\partial h_i}{\partial y^o} v_i$$

and so on. By use of the Jacobian, we have

$$\{\frac{\partial}{\partial x^o}\} = [J_e^{-1}]\{\frac{\partial}{\partial r}\}$$

so that we can express the displacement gradients in the matrix form

$$\left\{\begin{array}{c} u_{,x} \\ u_{,y} \\ u_{,z} \end{array}\right\} = [J_e^{-1}]\left[\begin{array}{ccccc} h_{1,r} & h_{2,r} & h_{3,r} & \cdots & h_{20,r} \\ h_{1,s} & h_{2,s} & h_{3,s} & \cdots & h_{20,s} \\ h_{1,t} & h_{2,t} & h_{3,t} & \cdots & h_{20,t} \end{array}\right]\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{20} \end{array}\right\}$$

where as before the subscript 'comma' indicates partial differentiation. Introduce the matrix of derivatives as

$$\left\{\begin{array}{c} \cdot_{,x^o} \\ \cdot_{,y^o} \\ \cdot_{,z^o} \end{array}\right\} \longrightarrow [J_e^{-1}]\left[\begin{array}{c} h_{i,r} \\ h_{i,s} \\ h_{i,t} \end{array}\right]_{i=1,N} \equiv \left[\begin{array}{c} A_x \\ A_y \\ A_z \end{array}\right]_{i=1,N} \tag{6.4}$$

The same matrix $[\ A\ ]_i$ gets applied to each of the displacements. The subscript i is associated with the shape function (and hence also the node). Arrange the degrees of freedom as

$$\{u\} = \{u_1, v_1, w_1;\ u_2, v_2, \cdots, w_{20}\}^T$$

and define the expanded nodal matrix of derivatives $[B_D]_i$

$$\{u_{,x}\} = \left\{\begin{array}{c} u_{,x} \\ u_{,y} \\ u_{,z} \\ v_{,x} \\ v_{,y} \\ v_{,z} \\ w_{,x} \\ w_{,y} \\ w_{,z} \end{array}\right\}, \qquad [B_D]_i = \left[\begin{array}{ccc} A_x & 0 & 0 \\ A_y & 0 & 0 \\ A_z & 0 & 0 \\ 0 & A_x & 0 \\ 0 & A_y & 0 \\ 0 & A_z & 0 \\ 0 & 0 & A_x \\ 0 & 0 & A_y \\ 0 & 0 & A_z \end{array}\right]_{i=1,N}$$

The vector of displacement gradients can then be evaluated from

$$\{u_{,x}\} = [B_D]\{u\}, \qquad [B_D] = [9 \times 60] = [B_{D1},\ B_{D2},\ \cdots,\ B_{D20}]$$

The interpretation is that $[B_D]$ is expanded left to right for each node in the element. As formulated, neither $[\ J_e\ ]$ nor the $[\ A\ ]_i$ matrices changes during the deformation and hence neither does the displacement gradient operator $[B_D]$; however, since this is different for each element and for each integration point in the element, then it is usually computed 'on-the-fly' during a deformation.

Typical terms for the strains are obtained from $\{u_{,x}\}$ according to

$$\begin{aligned} E_{xx} &= u_{,x} \quad + \tfrac{1}{2}[u_{,x}^2 + v_{,x}^2 + w_{,x}^2] \\ E_{yy} &= v_{,y} \quad + \tfrac{1}{2}[u_{,y}^2 + v_{,y}^2 + w_{,y}^2] \\ 2E_{xy} &= u_{,y} + v_{,x} + [u_{,x}\, u_{,y} + v_{,x}\, v_{,y} + w_{,x}\, w_{,y}] \end{aligned}$$

The vector of displacement gradients is thus considered the prime quantity to be evaluated. ■

Example 6.3: Evaluate the ability of the Hex20 interpolation functions to determine strains.

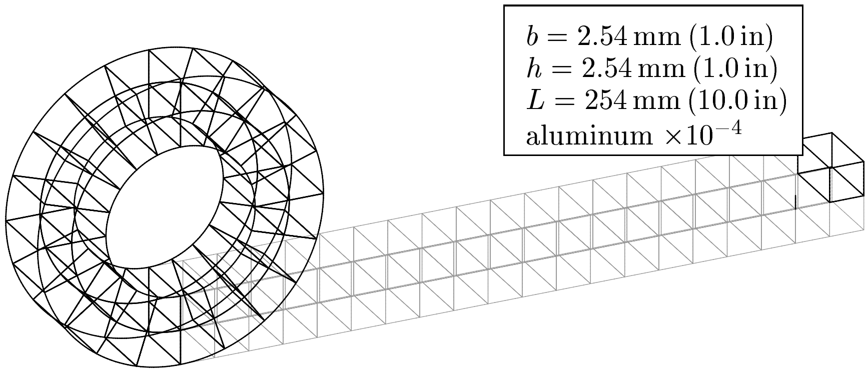


Figure 6.7: Bent block of square cross-section. [20×2] mesh.

The block in Figure 6.7 is given the deformation corresponding to Figure 2.7. That is, each node is given the exact displacements corresponding to Equation (2.6) and the interpolation functions are used to compute the strains. The middle plane of the block shown should not experience any stretching.

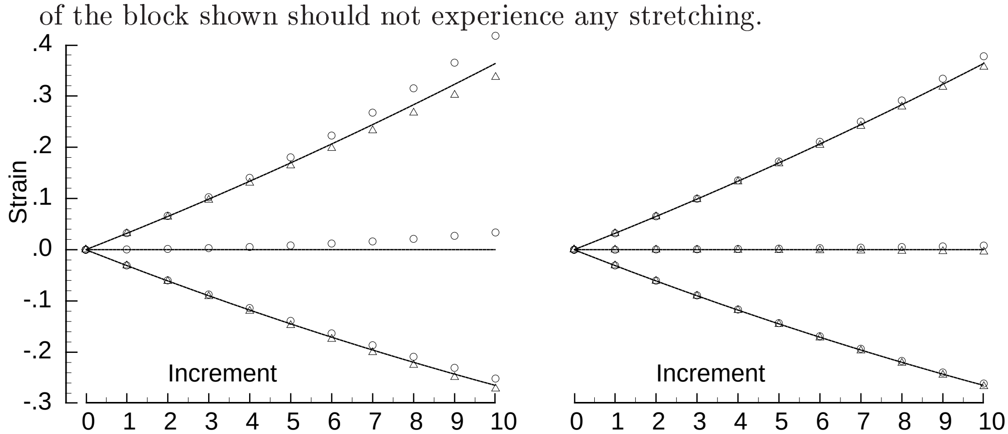


Figure 6.8: Strain comparisons. (a) [10×1] mesh. (b) [20×2] mesh.

Figure 6.8 shows the nodal strain for two levels of discretization; circles correspond to corner nodes while triangles correspond to mid-edge nodes. It is clear that two elements through the thickness enables an accurate evaluation of the strains from the given displacements. ■

6.2 Equilibrium of Discretized Systems

The formulation of a problem in terms of a set of governing differential equations plus a set of boundary conditions is known as the *strong formulation* of the problem. This is not always the most convenient form for solving actual problems and is especially true for nonlinear problems where the displacement fields are approximated as just discussed. We now recast the equilibrium developments of Chapter 3 in the form suitable for discretized systems. The underlying principle invoked is that of virtual work.

Virtual Work Formulation of Equilibrium

Consider a typical force-deflection or stress-strain curve as shown in Figure 6.9. The elasticity requirement, as discussed in depth in Chapter 4, is that both the loading and unloading paths coincide. That chapter also introduced some work and energy concepts which we utilize here.

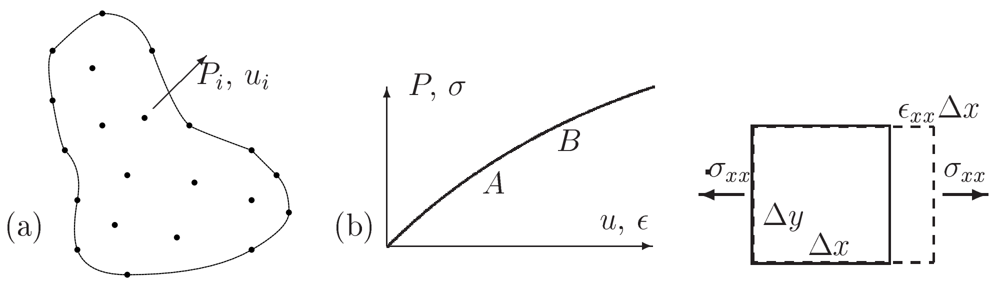


Figure 6.9: Typical elastic behavior. (a) Discretized arbitrary body. (b) Force-displacement behavior. (c) Stressed infinitesimal element showing strain.

Let $u_i(x_i^o)$ be the displacement field which satisfies the equilibrium equations in V. On the surface A, the surface traction t_i is prescribed on A_t and the displacement on A_u. Consider a variation of displacement δu_i (we will call this the *virtual displacement*), then

$$\bar{u}_i = u_i + \delta u_i$$

where u_i satisfy the equilibrium equations and the given boundary conditions. Thus, δu_i must vanish over A_u but be arbitrary over A_t; that is, δu_i must satisfy the geometric constraints of the problem.

Let δW_e be the external virtual work done by the body force b_i (which could include inertia effects) and the traction t_i; that is,

$$\delta \mathcal{W}_e = \int_V \rho b_i \delta u_i \delta V + \int_{A_t} t_i \delta u_i \, dA + \int_{A_u} t_i \delta u_i \, dA \tag{6.5}$$

The last term is zero hence we can replace A_t in the second integral with A on the understanding that δu_i can not be varied over the portion A_u. We can then express this virtual work as

$$\delta \mathcal{W}_e \;\; = \;\; \int_V \rho b_i \delta u_i dV + \int_A \sigma_{ji} \delta u_i n_j \, dA$$

$$
\begin{aligned}
&= \int_V \rho b_i \delta u_i dV + \int_V \frac{\partial}{\partial x_j}(\sigma_{ji}\delta u_i) dV \\
&= \int_V \Big[\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j}\Big]\delta u_i dV + \int_V \sigma_{ji}\,\delta(\frac{\partial u_i}{\partial x_j}) dV \\
&= \int_V \Big[\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j}\Big]\delta u_i dV + \int_V \sigma_{ji}\delta\epsilon_{ji} dV
\end{aligned}
$$

where the last term was reduced using the decomposition of the deformation gradient into $\epsilon_{ij} + \omega_{ij}$ and noting that the contraction of the antisymmetric rotation with the symmetric stress is zero. We also used the integral theorem of Equation (1.3).

Define the total virtual work as

$$
\delta\mathcal{W} = \delta\mathcal{W}_e - \int_V \sigma_{ji}\delta\epsilon_{ji} dV = \int_V \Big[\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j}\Big]\delta u_i dV
$$

These developments actually paralleled what was done in deriving the Cauchy stress equations of motion in Chapter 3, therefore, we can look at it in one of two ways. First, because the term in brackets is zero due to equilibrium, then we conclude that the total virtual work is zero. That is,

$$
\delta\mathcal{W} = \delta\mathcal{W}_e - \int_V \sigma_{ji}\delta\epsilon_{ji} dV = 0
$$

On the other hand, if we say that the total virtual work is zero for every independent virtual displacement δu_i, then we conclude that the term in brackets is zero. That is, we obtain the equilibrium equations in terms of the Cauchy stress. Phrased more formally, the principle of virtual work states that *a deformable body is in equilibrium if the total virtual work is zero for every independent kinematically admissible virtual displacement*. We will interpret the symbol δ as meaning a *variation* and the above equation as a variational principle. A final point worth mentioning is that the product in the integral involves the Cauchy stress with the *small* Eulerian strain components and not the Eulerian strain itself.

We would also like to write the virtual work expression in terms of the undeformed configuration. Following developments similar to the example in Section 2.5 leading to Equation (2.11), we get the relation

$$
\delta\epsilon_{mn} = \frac{\partial x_i^o}{\partial x_m}\frac{\partial x_j^o}{\partial x_n}\delta E_{ij}
$$

The relation between Cauchy and Kirchhoff stress is

$$
\sigma_{mn} = \frac{\rho}{\rho^o}\frac{\partial x_m}{\partial x_i^o}\frac{\partial x_n}{\partial x_j^o}\sigma_{ij}^K
$$

and recalling that the deformed and undeformed volumes are related by $dV = dV^o\rho^o/\rho$, the internal virtual work term becomes

$$
\sigma_{mn}\delta\epsilon_{mn}\, dV = \sigma_{pq}^K\delta E_{pq}\, dV^o
$$

Hence the Cauchy stress / Eulerian (small) strain combination is virtual work equivalent to the Kirchhoff stress / Lagrangian strain combination. The equivalent result was established in Chapter 4 for the strain energy.

We are now in a position to write the virtual work form of equilibrium as

$$\delta\mathcal{W} = \delta\mathcal{W}_e - \int_V \sigma_{mn}\delta\epsilon_{mn}dV = \delta\mathcal{W}_e - \int_{V^o} \sigma^K_{pq}\delta E_{pq}dV^o = 0 \tag{6.6}$$

In contrast to the differential equations of motion, there are no added complications using the undeformed state as the reference state. It is useful to realize that, during a deformation, the reference state $t = 0$ could be any one of the previous equilibrium positions and not necessarily the original stress-free state. We will make use of this in our incremental formulation for the computer.

Some Particular Stationary Principles

The virtual work form is completely general, but there are further developments that are more convenient to use in some circumstances. We now look at some of these developments.

I: Stationary Potential Energy

A system is *conservative* if the work done in moving the system around a closed path is zero. We say that the external force system is conservative if it can be obtained from a potential function. For example, for a set of discrete forces, we have

$$P_i = -\frac{\partial\mathcal{V}}{\partial u_i} \qquad \text{or} \qquad \mathcal{V} = -\textstyle\sum_i P_i u_i$$

where u_i is the displacement associated with the load P_i. The negative sign in the definition of $\mathcal{V}$ is arbitrary, but choosing it so gives the interpretation of $\mathcal{V}$ as the capacity (or potential) to do work. The external virtual work term now becomes

$$\delta\mathcal{W}_e = \textstyle\sum_i P_i\delta u_i = -\sum_i \frac{\partial\mathcal{V}}{\partial u_i}\delta u_i = -\delta\mathcal{V}$$

We get almost identical representations for conservative body forces and conservative traction distributions. Follower forces, as well as dissipative forces such as friction, are examples of *nonconservative* forces.

The internal virtual work is associated with the straining of the body and therefore we will use the representation

$$\delta\mathcal{U} = \int_V \sigma_{ij}\delta\epsilon_{ij}dV = \int_{V^o} \sigma^K_{ij}\delta E_{ij}dV^o$$

and call $\delta\mathcal{U}$ the virtual strain energy of the body. When straining is conservative, we can view $\mathcal{U}$ as a potential function: examples of such materials were developed in Chapter 4 and generally are referred to as hyperelastic materials.

The principle of virtual work for conservative systems can now be rewritten as

$$\delta\mathcal{U} + \delta\mathcal{V} = 0 \qquad \text{or} \qquad \delta\Pi \equiv \delta[\mathcal{U} + \mathcal{V}] = 0 \tag{6.7}$$

The term inside the brackets is called the *total potential energy* and this relation is called the *principle of stationary potential energy.* We may now restate the principle of virtual work as: *For a conservative system to be in equilibrium, the first-order variation in the total potential energy must vanish for every independent admissible virtual displacement.* Another way of stating this is that among all the displacement states of a conservative system that satisfy compatibility and the boundary constraints, those that also satisfy equilibrium make the total potential energy stationary. In comparison to the conservation of energy theorem, this is much richer, because instead of one equation it leads to as many equations are there are degrees of freedom (independent displacements).

If there are nonconservative forces (those for which we cannot write a potential) the principle is extended to

$$\delta[\mathcal{U} + \mathcal{V}] - \textstyle\sum_j Q_j \delta u_j = 0 \tag{6.8}$$

where Q_j are the individual nonconservative forces. This is sometimes referred to as the *extended* principle of stationary potential energy. Thus the main difference between the principle of virtual work and the (unextended) principle of stationary potential energy is that the former is more general because it also applies to nonconservative systems which the latter does not.

Example 6.4: Determine the equilibrium conditions for the nonlinear system shown in Figure 6.10.

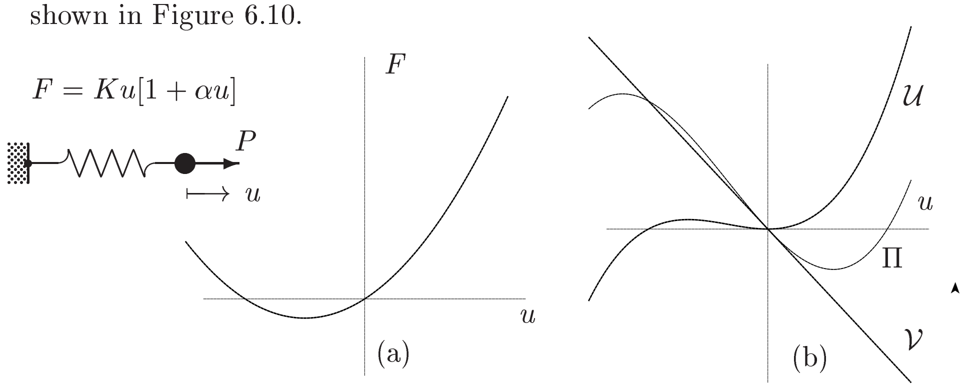


Figure 6.10: Equilibrium of a nonlinear spring. (a) Force-deflection relation. (b) Potential energy.

Identify u, the resulting displacement at the point of application of the load, as the independent admissible displacement. The response of the nonlinear spring is shown in Figure 6.10(a): under tension it stiffens, under compression it shows softening. The virtual work for this spring is

$$\delta\mathcal{W} = F\delta u = K[1 + \alpha u]u\delta u$$

This is also the virtual strain energy δU. Integrating then gives

$$\mathcal{U} = \tfrac{1}{2}Ku^2[1 + \tfrac{2}{3}\alpha u]$$

The potential of the applied force is

$$\mathcal{V} = -Pu$$

The total potential energy of the system is, therefore,

$$\Pi = \tfrac{1}{2}Ku^2[1 + \tfrac{2}{3}\alpha u] - Pu$$

These terms are shown plotted in Figure 6.10(b) for different values of displacement u. It is apparent that Π can achieve two stationary values — a valley and a peak. The principle indicates that both occur at equilibrium positions.

A stationary potential energy requires that

$$\frac{\partial \Pi}{\partial u} = 0 \qquad \Rightarrow \qquad Ku[1 + \alpha u] - P = 0$$

We recognize this as the equilibrium balance between the external applied load P and the internal force F of the spring. If the spring were linear ($\alpha = 0$), it would reduce to the single equilibrium equation

$$Ku = P \qquad \text{or} \qquad u = P/K$$

In the nonlinear case, however, we have two possible equilibrium positions

$$u = \frac{-1 \pm \sqrt{1 + 4\alpha P/K}}{2\alpha} \approx \frac{P}{K}, \ \frac{-1}{\alpha}$$

(The approximation is for slight nonlinearity when α is small.) The first is close to the linear equilibrium position, but what is the meaning of the second position? Furthermore, this second position corresponds to a negative displacement, which surely cannot happen because the load is positive. A hallmark of nonlinear systems is the possibility of multiple equilibrium positions for a given load state. Indeed, looking at Figure 6.10(a), we see that even at zero load ($F = 0$) there is the equal possibility of two deflections. A crucial discussion is the distinction between the two equilibrium points in terms of the stability of their equilibrium — we leave that discussion to later.

It is hard to imagine an "ordinary" material or spring behaving in this way. However, engineering structures and many of the structured materials (an example is corrugated cardboard) do behave this way. ■

II: Discrete Systems

For computer solution of nonlinear problems, the governing equations must be reduced somehow to equations using discrete unknowns. That is, we introduce some generalized coordinates (or degrees of freedom with the constrained degrees removed). At present, we need not be explicit about which coordinates we are considering but the situations discussed in Section 6.1 where the nodal displacements associated with the discretization of a continuum are what we have in mind.

Accept that we can write a function as

$$u = u(u_1, u_2, \ldots, u_N)$$

where u_i are the generalized coordinates. We get these generalized coordinates by the imposition of *holonomic* constraints — the constraints are geometric of the form $f_i(u_1, u_2, \ldots, u_N; t) = 0$ and do not depend on the velocities.

The total potential of the conservative system can be written in the form

$$\mathcal{U} + \mathcal{V} = \Pi = \Pi(u_1, u_2, \ldots, u_N)$$

and its variation is given by

$$\delta\Pi = \sum_{j=1}^{N} \frac{\partial \Pi}{\partial u_j} \delta u_j$$

Couple this with the virtual work of the nonconservative forces to get

$$\sum_{j=1}^{N} \frac{\partial \Pi}{\partial u_j} \delta u_j - \sum_j Q_j \delta u_j = 0 \qquad \text{or} \qquad \sum_{j=1}^{N} \Big[\frac{\partial \Pi}{\partial u_j} - \sum_j Q_j \Big] \delta u_j = 0$$

Since the δu_i can be varied arbitrarily, then the principle of stationary potential energy in terms of generalized coordinates leads to

$$\mathcal{F}_i = \frac{\partial}{\partial u_i} \big[\mathcal{U} + \mathcal{V} \big] - Q_i = 0 \qquad \text{for} \quad i = 1,\, 2,\, \ldots,\, N \tag{6.9}$$

The expression, $\mathcal{F}_i = 0$, is our statement of static equilibrium. It is important to realize that the single equation $\delta\Pi = 0$ (for a conservative system) is actually a system of N equations if there are N degrees of freedom in the system. Thus, in comparison to the conservation of energy theorem, this is much richer, because instead of just one equation, it leads to as many equations are there are degrees of freedom.

Example 6.5: Consider a long slender rod, fixed at one end, with an applied load at the other. Determine the equilibrium conditions for the long rod assuming a linear elastic constitutive relation.

Let the unknown displacement fields be discretized as

$$u(x^o) = c_1 x^o\,, \qquad v(x^o) = 0\,, \qquad w(x^o) = 0$$

where c_1 is the single discrete unknown. This gives the strains

$$E_{11} = \frac{\partial u}{\partial x^o} + \tfrac{1}{2} \Big[(\frac{\partial u}{\partial x^o})^2 + (\frac{\partial v}{\partial x^o})^2 + (\frac{\partial w}{\partial x^o})^2 \Big] = c_1 + \tfrac{1}{2} c_1^2$$

with all other strains being zero. Assume the simple constitutive relation $\sigma_{ij}^K = E E_{ij}$ so that

$$\sigma_{11}^K = E E_{11} = [c_1 + \tfrac{1}{2} c_1^2]$$

with all other stresses being zero.

The virtual work of the applied load is

$$\delta \mathcal{W}_e = P\,\delta u_L = \delta c_1 L$$

The virtual strain energy is

$$\begin{aligned}\delta \mathcal{U} &= \int_{V^o} \sigma_{11}^K \delta E_{11} dV^o = \int_{V^o} E[c_1 + \tfrac{1}{2}c_1^2][\delta c_1 + c_1 \delta c_1^2] dV^o \\ &= EV^o[c_1 + \tfrac{1}{2}c_1^2][1 + c_1]\delta c_1 = EV^o[c_1 + \tfrac{3}{2}c_1^2 + \tfrac{1}{2}c_1^3]\delta c_1\end{aligned}$$

The principal of virtual work becomes

$$\delta \mathcal{W}_e - \delta \mathcal{U} = 0 \qquad \Rightarrow \qquad \Big[PL - EV^o[c_1 + \tfrac{3}{2}c_1^2 + \tfrac{1}{2}c_1^3]\Big]\delta c_1 = 0$$

Since c_1 is arbitrary then so also is δc_1 and we conclude that the bracketed term must be zero. That is,

$$\Big[1 + \tfrac{3}{2}c_1 + \tfrac{1}{2}c_1^2\Big]c_1 = \frac{PL}{EV^o}$$

This simple problem has lead to a nonlinear relation between the unknown deformation (as represented by c_1) and the applied loads. The origin of the nonlinearity is in the strain-displacement relation.

The typical problem in solid mechanics is where the applied loads are specified and the displacements are unknown, therefore we can anticipate that these problems usually require some nonlinear solver routine. Most of them are iterative and a popular one is discussed in Section 6.4; here we mention a scheme called *Picard iteration*. The basic idea is to arrange the equation so that the nonlinearities are on the left hand side and are estimated from a previous guess. A possibility is

$$c_1^{i+1} = \frac{PL/EV^o}{[1 + \frac{3}{2}c_1 + \frac{1}{2}c_1^2]^i}$$

where i is the iteration counter. The first few iterations are

$$\begin{aligned}c_1^0 &= 0 \\ c_1^1 &= PL/EV^o \\ c_1^2 &= \frac{PL/EV^o}{[1 + \frac{3}{2}(PL/EV^o) + \frac{1}{2}(PL/EV^o)^2]}\end{aligned}$$

This solution illustrates a characteristic of many nonlinear systems; namely, that they can exhibit instabilities. For the approximation shown, when $PL/EV^o = -1$ the deflection is indicated as being infinite. ■

III: Finite Element Formulation

Rather than use the result of Equation (6.9), we find it more convenient to begin with the equations of motion themselves, Equation (6.6), written at the current time t

$$\int_{V^o} \sigma_{ij}^K \delta E_{ij} dV^o = \delta W_e$$

where inertia effects are neglected. In anticipation of using matrix notation, introduce

$$\begin{aligned} \delta E_{kl} \quad &\to \quad \{6\times 1\} = \{\delta E\} = \{\delta E_{11},\, \delta E_{22},\, \delta E_{33},\, 2\delta E_{12},\, 2\delta E_{23},\, 2\delta E_{13}\}^T \\ \sigma^K_{kl} \quad &\to \quad \{6\times 1\} = \{\sigma^K\} = \{\sigma^K_{11},\, \sigma^K_{22},\, \sigma^K_{33},\, \sigma^K_{12},\, \sigma^K_{23},\, \sigma^K_{13}\}^T \end{aligned}$$

so that the integrand of the internal virtual work can be written as

$$\int_{V^o} \{\sigma^K\}^T \{\delta E\}\, dV^o$$

There is nothing special about the sequence of components in the vectors, the above sequence is one of the standard forms [4].

The variation of the Lagrangian strain is given by

$$2\delta E_{ij} = \frac{\partial \delta u_i}{\partial x^o_j} + \frac{\partial \delta u_j}{\partial x^o_i} + \sum\nolimits_k \Big[\frac{\partial u_k}{\partial x^o_i}\frac{\partial \delta u_k}{\partial x^o_j} + \frac{\partial \delta u_k}{\partial x^o_i}\frac{\partial u_k}{\partial x^o_j}\Big]$$

The basic idea of the discretization is that, since the actual distribution of displacements is quite complicated, it was approximated as a collection of piecewise simple distributions over many regions and these distributions were characterized by the discrete nodal values. That is, let the displacements in a small region of volume V^o_m be represented by

$$u_i(x^o_1, x^o_2, x^o_3) = \sum\nolimits_k h_k(x^o_1, x^o_2, x^o_3) u_{ik} = [\ h\ ]\{u_i\} \quad \text{or} \quad \{u\} = [H]\{u\}$$

where $h_k(x^o_1, x^o_2, x^o_3)$ are known shape functions and u_{ik} are the unknown nodal values. All relevant quantities can now be written in terms of both of these. For example, the derivatives and variations are given by

$$\frac{\partial u_i}{\partial x^o_j} = \sum\nolimits_k \frac{\partial h_k}{\partial x^o_j} u_{ik} = [\partial h]\{u\}\,, \qquad \frac{\partial \delta u_i}{\partial x^o_j} = \sum\nolimits_k \frac{\partial h_k}{\partial x^o_j} \delta u_{ik} = [\partial h]\{\delta u\}$$

Hence the variation of strain can be written symbolically as

$$\{\delta E\} = [B_E]\{\delta u\}$$

where $[B_E]$ contains various spatial derivatives of h_k as well as the nodal displacement values $\{u\}$; its content will be made explicit later.

The internal virtual work becomes

$$\begin{aligned} &\int_{V^o} \{\delta E\}^T \{\sigma^K\} dV^o \\ &\quad \Rightarrow \quad \sum\nolimits_m \{\delta u\}^T_m \{ \int_{V^o_m} [B_E]^T \{\sigma^K\}\, dV^o_m\} = \sum\nolimits_m \{\delta u\}^T_m \{F\}_m = \{\delta u\}^T \{F\} \end{aligned}$$

where $\{F\}$ is the assemblage of all the element forces $\{F\}_m$ given by

$$\{F\} = \sum\nolimits_m \int_{V^o_m} [B_E]^T \{\sigma^K\}\, dV^o_m \tag{6.10}$$

The force vector $\{F\}$ is interpreted as the set of nodal forces that are (virtual) work equivalent to the actual distributed stresses and strains. The integral is done for each element and the assemblage process forms the vector $\{F\}$ of size $\{N \times 1\}$.

The virtual work of the applied body forces (without inertia) and surface tractions leads to

$$\sum_i \int_{V^o} \rho f_i \delta u_i \, dV^o + \sum_i \int_{A^o} t_i \delta u_i \, dA^o$$
$$\Rightarrow \quad \{P\} = \{\int_{V^o} [H]^T \{f\} dV^o\} + \{\int_{A^o} [H]^T \{\,t\,\} dA^o\}$$

Assemblage is done as in the linear case, and these give the equations of motion as

$$\{\mathcal{F}\} = \{P\} - \{F\} = 0 \tag{6.11}$$

This is our governing equation for a general nonlinear system discretized in the form of finite elements. It is completely equivalent to Equation (6.9), but because we began with the principle of virtual work we do not need to discuss the difference between conservative and nonconservative systems. That is, the present form of the equations is applicable to situations involving plasticity as well as those with follower forces.

Note that the the load distribution is determined initially (although its history will change); only the body stress term $\{F\}$ needs to be updated.

Example 6.6: Establish the explicit form for the strain operator matrix $[B_E]$ for the Hex20 discretization.

Typical terms for the variation of strains are obtained from $\{u_{,x}\}$ according to

$$\begin{aligned}
\delta E_{xx} &= \delta u_{,x} \quad + [u_{,x}\, \delta u_{,x} + v_{,x}\, \delta v_{,x} + w_{,x}\, \delta w_{,x}\,] \\
\delta E_{yy} &= \delta v_{,y} \quad + [u_{,y}\, \delta u_{,y} + v_{,y}\, \delta v_{,y} + w_{,y}\, \delta w_{,y}\,] \\
2\delta E_{xy} &= \delta u_{,y} + \delta v_{,x} + [\delta u_{,x}\, u_{,y} + u_{,x}\, \delta u_{,y} + \delta v_{,x}\, v_{,y} + v_{,x}\, \delta v_{,y} + \delta w_{,x}\, w_{,y} + w_{,x}\, \delta w_{,y}\,]
\end{aligned}$$

We can replace all functions using the interpolation functions and then express these in matrix form as

$$\left\{ \begin{array}{c} \delta E_{xx} \\ \delta E_{yy} \\ \vdots \\ 2\delta E_{yz} \\ 2\delta E_{xz} \end{array} \right\} = \Big[[B_L] + [B_N] \Big] \left\{ \begin{array}{c} \delta u_1 \\ \delta v_1 \\ \vdots \\ \delta v_{20} \\ \delta w_{20} \end{array} \right\} \qquad \text{or} \qquad \{\delta E\} = [B_E]\{\delta u\}$$

The matrices are defined as

$$[B_L]_i = \begin{bmatrix} A_x & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & A_z \\ A_y & A_x & 0 \\ 0 & A_z & A_y \\ A_z & 0 & A_x \end{bmatrix}_{i=1,N}$$

$$[B_N]_i = \begin{bmatrix} u_{,x}A_x & v_{,x}A_x & w_{,x}A_x \\ u_{,y}A_y & v_{,y}A_y & w_{,y}A_y \\ u_{,z}A_z & v_{,z}A_z & w_{,z}A_z \\ u_{,x}A_y + u_{,y}A_x & v_{,x}A_y + v_{,y}A_x & w_{,x}A_y + w_{,y}A_x \\ u_{,y}A_z + u_{,z}A_y & v_{,y}A_z + v_{,z}A_y & w_{,y}A_z + w_{,z}A_y \\ u_{,x}A_z + u_{,z}A_x & v_{,x}A_z + v_{,z}A_x & w_{,x}A_z + w_{,z}A_x \end{bmatrix}_{i=1,N} \tag{6.12}$$

where the matrices $[\ A\ ]_i$ have already been established in Equation (6.4). The strain operator matrix is

$$[B_E] = [6 \times 60] = [B_{L1} + B_{N1},\ B_{L2} + B_{N2},\ \cdots,\ B_{L120} + B_{N120}]$$

The nonlinear matrix $[B_N]$ contains the displacement gradients and therefore changes during the deformation. ■

Example 6.7: Determine the equivalent nodal loads for a distributed loading.

The basic idea is to say that the virtual work of the equivalent nodal loads is equal to the virtual work of the actual loading. Consider, for example, a body force acting in the x-direction; the virtual work is

$$\sum_i^N P_{xi}\delta u_i = \int_{V^o} f_x^b(x^o, y^o, z^o)\delta u\, dV^o = \sum_i^N \Big[\int_V f_x^b(r,s,t)h_i(r,s,t)\,|J_e|dV\Big]\delta u_i$$

We conclude that

$$P_{xi} = \int_V f_x^b(r,s,t)h_i(r,s,t)\,|J_e|dV$$

In a similar manner, the nodal forces due to surface tractions and line loads are, respectively,

$$P_{xi} = \int_A t_x(r,s,t)h_i^A(r,s,t)\,|J_A|dA\,, \qquad P_{xi} = \int_S q_x(r,s,t)h_i^S(r,s,t)\,|J_S|dS$$

The notations $h_i^A(r,s,t)$ and $h_i^S(r,s,t)$ mean that the functions are evaluated on the appropriate surface and line, respectively. Also note that the $|J_e|$ to be used is the one for areas or lines as appropriate.

In general, each of these integrals need to be evaluated numerically. It is useful to keep in mind, however, that lower order interpolations can be used for loads than is used for the element nodal forces and in this way the applied loads can be determined separately from the continuum modeling. For example, if the linear hexahedral interpolations are used then the nodal loads due to a line load are

$$\text{uniform:}\quad P_1 = \tfrac{1}{2}q_oL,\ P_2 = \tfrac{1}{2}q_oL\,, \qquad \text{linear:}\quad P_1 = \tfrac{1}{3}q_mL,\ P_2 = \tfrac{2}{3}q_mL$$

with similar expressions for the tractions and body forces. (In both cases $P_3 = 0$). When the expectation is that small elements are needed then these representations of the load are sufficiently accurate. ■

6.3 Stiffness Properties of Discretized Systems

Earlier in the chapter, we introduced the elastic modulus as a measure of the stiffness of a linear material; this is a material property. When we deal with linear structures or structured materials we introduce the concept of structural stiffness which depends on both geometry and material properties. When we deal with nonlinear problems, we must further introduce the very important concept of the tangent stiffness. Unlike the elastic stiffness of linear structures, the tangent stiffness changes as the load changes giving rise to some surprising consequences such as its effect on the stability of the equilibrium.

General Stiffness Relations

The assembled element nodal forces are computed from Equation (6.10) as

$$\{F\} = \sum_m \int_{V_m^o} [B_E]^T \{\sigma^K\}\, dV_m^o$$

The variation of the nodal forces leads to

$$\{\delta F\} = [\frac{\partial F}{\partial u}]\{\delta u\} = [K_T]\{\delta u\}$$

where $[K_T]$ is the tangent (or total) stiffness of the system. This is the matrix we wish to establish in explicit form.

Substituting for $\{F\}$ in terms of the stress leads to

$$\{\delta F\} = \sum_m \int_{V_m^o} \Big[[B_E]^T\{\delta\sigma^K\} + [\delta B_E]^T\{\sigma^K\}\Big]\, dV_m^o$$

The stresses are a function of the strains so that the first term becomes

$$[B_E]^T\{\delta\sigma^K\} = [B_E]^T[\frac{\partial \sigma_i^K}{\partial E_j}]\{\delta E\} = [B_E]^T[\ D\ ][B_E]\{\delta u\}$$

The matrix $[\ D\ ]$ is the tangent modulus for the material.

The strain operator is considered a function of the displacement gradients so that the second term becomes

$$[\delta B_E]^T\{\sigma^K\} = \{\delta u_{,x}\}^T[\frac{\partial B_E}{\partial u_{,x}}]^T\{\sigma^K\} = \{\delta u\}^T[B_D]^T[\sigma^K]\{B_D\}$$

The final form will be demonstrated for the Hex20 element in the example problem to follow.

We therefore have for the total stiffness

$$[K_T] = \sum_m \int_{V_m^o} \Big[[B_E]^T[\ D\ ][B_E] + [B_D]^T[\sigma^K]\{B_D\}\Big]\, dV_m^o = [K_E] + [K_G]$$

The integral is over the element volume V_m^o and the summation is associated with the assemblage of the collection of elements. We see that the total stiffness relation is comprised of two parts: one is related to the tangent modulus properties of the material, the other is related to the current value of stress. The first matrix is often called the elastic stiffness because in the linear case it is primarily a function of the elastic material properties. The second matrix is called the *initial stress matrix* because in the linear case it depends on the stress and not on the material properties. The combination of both matrices is sometimes referred as the *tangent stiffness*. For nonlinear problems, both matrices depend on the stress and/or current deformation with the first distinctly related to the material tangent modulus $\partial\sigma_{ij}^K/\partial E_{pq}$ and the second distinctly related to the changing geometry of the element (through δB_E). It would seem appropriate to call the first matrix the tangent stiffness and the second the *geometric stiffness*; we will refer to the combination as the *total stiffness*. All matrices are symmetric.

The stiffness relation is quite general in that it is not restricted to any particular constitutive relation. Thus, for example, for elastic-plastic materials it is necessary only to replace

$$[D_{ep}] \quad \longrightarrow \quad [\,D\,]$$

The geometric stiffness is unaffected by the constitutive relation.

Both stiffnesses change during a general deformation. Indeed, both are complicated functions of space and this requires an efficient scheme for the computation of the volume integrals; we discuss this next.

Example 6.8: Establish the geometric stiffness matrix for the Hex20 element.

From Equation (6.12), only the $[B_N]$ part of $[B_E]$ is a function of the displacement gradients. Consequently,

$$[\delta B_E]_i = \begin{bmatrix} \delta u_{,x}A_x & \delta v_{,x}A_x & \delta w_{,x}A_x \\ \delta u_{,y}A_y & \delta v_{,y}A_y & \delta w_{,y}A_y \\ \delta u_{,z}A_z & \delta v_{,z}A_z & \delta w_{,z}A_z \\ \delta u_{,x}A_y + \delta u_{,y}A_x & \delta v_{,x}A_y + \delta v_{,y}A_x & \delta w_{,x}A_y + \delta w_{,y}A_x \\ \delta u_{,y}A_z + \delta u_{,z}A_y & \delta v_{,y}A_z + \delta v_{,z}A_y & \delta w_{,y}A_z + \delta w_{,z}A_y \\ \delta u_{,x}A_z + \delta u_{,z}A_x & \delta v_{,x}A_z + \delta v_{,z}A_x & \delta w_{,x}A_z + \delta w_{,z}A_x \end{bmatrix}_{i=1,N}$$

Each column can be expanded to the form

$$\begin{bmatrix} A_x & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & A_z \\ A_y & A_x & 0 \\ 0 & A_z & A_y \\ A_z & 0 & A_x \end{bmatrix} \begin{Bmatrix} \delta\cdot_{,x} \\ \delta\cdot_{,y} \\ \delta\cdot_{,z} \end{Bmatrix}$$

Note that the matrix is actually $[B_L]_i$. The product $\{\sigma^K\}^T\{\delta B_E\}$ has the component products

$$\{\sigma^K_{xx}, \sigma^K_{yy}, \sigma^K_{zz}, \sigma^K_{xy}, \sigma^K_{yz}, \sigma^K_{xz}\}\begin{bmatrix} A_x & 0 & 0 \\ 0 & A_y & 0 \\ 0 & 0 & A_z \\ A_y & A_x & 0 \\ 0 & A_z & A_y \\ A_z & 0 & A_x \end{bmatrix}$$

This can be re-arranged to

$$\left[\begin{array}{ccc|ccc} \sigma^K_{xx} & \sigma^K_{xy} & \sigma^K_{xz} & 0 & 0 & \cdots \\ \sigma^K_{yx} & \sigma^K_{yy} & \sigma^K_{yz} & 0 & 0 & \cdots \\ \sigma^K_{zx} & \sigma^K_{zy} & \sigma^K_{zz} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sigma^K_{xx} & \sigma^K_{xy} & \cdots \\ 0 & 0 & 0 & \sigma^K_{xy} & \sigma^K_{yy} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]\begin{bmatrix} A_x & 0 & 0 \\ A_y & 0 & 0 \\ A_z & 0 & 0 \\ 0 & A_x & 0 \\ 0 & A_y & 0 \\ 0 & A_z & 0 \\ 0 & 0 & A_x \\ 0 & 0 & A_y \\ 0 & 0 & A_z \end{bmatrix}_{i=1,N} = [\sigma^K][B_D]_i$$

Since $\{\delta u_{,x}\} = [B_D]\{\delta u\}$ then

$$[\delta B_E]^T\{\sigma^K\} = \{\delta u_{,x}\}^T[\frac{\partial B_E}{\partial u_{,x}}]^T\{\sigma^K\} = \{\delta u\}^T[B_D]^T[\sigma^K]\{B_D\}$$

from which it follows that

$$[K_G] = \sum\nolimits_m \int_{V^o_m} \Big[[B_D]^T[\sigma^K]\{B_D\}\Big]\, dV^o_m$$

This matrix plays a significant role in the buckling instability of structures. ∎

Numerical Quadrature

The foregoing stiffness relations (and some of the earlier results) show that we have a need to evaluate integrals over lengths, areas and volumes. This type of integration (where the integrand is specified) is called *quadrature*. Since the integration occurs repeatedly in a nonlinear problem, it is essential that it be performed efficiently and accurately. Clearly, it must be done numerically for general elements.

An advantage of the isoparametric formulation is that general volume integrals, for example, get reduced to integrals on a $[2 \times 2 \times 2]$ cube irrespective of the physical size of the block. That is,

$$\int_{V^o} Q(x^o, y^o, z^o)\, dV^o = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} Q(r,s,t)\ \det|J_e(r,s,t)|\, dr\, ds\, dt$$

where the physical dimensions enter through the Jacobian. This significantly facilitates standardizing the numerical process.

I: Lagrange Interpolation

The basic idea is to replace the integral with a summation. Thus, for example, the line and volume integrations become, respectively,

$$\int F(r)\,dr = \textstyle\sum_i \alpha_i F(r_i)\,, \qquad \int F(r,s,t)\,drdsdt = \textstyle\sum_{i,j,k} \alpha_{ijk} F(r_i,s_j,t_k)$$

The key is the proper choice of α_{ijk} at the appropriate locations (r_i, s_j, t_k) so as to maximize the accuracy.

To motivate the procedure, consider the 1-D function $F(r)$ evaluated at the $(n+1)$ distinct points r_0, $r_1, \cdots$. Represent this with the polynomial

$$F(r) \approx a_a + a_1 r + a_2 r^2 + \cdots$$

and set up the $(n+1)$ simultaneous equations to evaluate the a_i coefficients. The integral is then given by

$$I = \int_{-1}^{+1} F(r)\,dr \approx a_0 r + a_1 r^2 + a_2 r^3 + \cdots \Big|_{-1}^{+1}$$

Needless to say that if $F(r)$ is a polynomial of degree equal to or less than the representation, then the integral is exact. This approach, however, is a relatively costly procedure.

An efficient way to produce a polynomial representation is through Lagrangian interpolation. That is, we use the representation

$$F(r) \approx F_0 L_0(r) + F_1 L_1(r) + F_2 L_2(r) + \cdots$$

where

$$L_j(r) = \frac{(r-r_0)(r-r_1)\cdots[r-r_j]\cdots(r-r_{n-1})(r-r_n)}{(r_j-r_0)(r_j-r_1)\cdots[r_j-r_j]\cdots(r_j-r_{n-1})(r_j-r_n)}\,, \qquad L_j(r_i) = \delta_{ij}$$

In the functions $L_j(r)$ shown, the center terms in square brackets are missing and are only shown to help in establishing the permutations for evaluating $L_j(r)$. For example, if we have a quadratic polynomial, $n = 2$, then

$$L_0(r) = \frac{(r-r_1)(r-r_2)}{(r_0-r_1)(r_0-r_2)}, \quad L_1(r) = \frac{(r-r_0)(r-r_2)}{(r_1-r_0)(r_1-r_2)}, \quad L_2(r) = \frac{(r-r_0)(r-r_1)}{(r_2-r_0)(r_2-r_1)}$$

It is clear that when $r = r_0$, r_1, r_2 we get $F(r) = F_0$, F_1, F_2, respectively. The accomplishment of Lagrangian interpolation is that it avoids the necessity of having to solve the simultaneous equations indicated above.

n	r_i	W_i
1	$+0.0$	2.0
2	$-0.57735\,02691\,89626 = -1/\sqrt{3}$	1.0
	$+0.57735\,02691\,89626 = +1/\sqrt{3}$	1.0
3	$-0.77459\,66692\,41483 = -\sqrt{0.6}$	$0.55555\,55555\,55555 = -5/9$
	$+0.0$	$0.88888\,88888\,88888 = +8/9$
	$+0.77459\,66692\,41483 = +\sqrt{0.6}$	$0.55555\,55555\,55555 = +5/9$
4	$-0.86113\,63115\,94053$	$0.34785\,48451\,37454$
	$-0.33998\,10435\,84856$	$0.65214\,51548\,62546$
	$+0.33998\,10435\,84856$	$0.65214\,51548\,62546$
	$+0.86113\,63115\,94053$	$0.34785\,48451\,37454$

Table 6.2: Sampling points and weights for Gauss-Lagrange Quadrature

II: Gaussian Integration Points

There is one further refinement that can be implemented. That is, rather than choose equi-spaced points, we preselect the discretized points so as to minimize the error for a given level of polynomial. These preselected locations are shown in Table 6.2.

That is, the integration variable r is chosen so that the integration limits are from $r = -1$ to $r = +1$ and the integral is

$$I = \int_{-1}^{+1} F(r)\,dr \approx W_0 F(r_0) + W_1 F(r_1) + W_2 F(r_2) + \cdots$$

where the weights W_i are also shown in Table 6.2. The equispaced approximation integrates a polynomial of order n exactly, but the Gauss-Lagrange unequal spaced approximation integrates exactly a polynomial of degree $(2n-1)$ [4]. A further point is that the approximation for n and $n+1$ have the same accuracy and so the even formulas ($n = 0$, 2, 4, ...) are typically used.

These formulas are directly extended to multiple dimensions by

$$\begin{aligned}
\int_{-1}^{+1} F(r)\,dr &= \textstyle\sum_i W_i F(r_i) \\
\int_{-1}^{+1}\int_{-1}^{+1} F(r,s)\,drds &= \textstyle\sum_i\sum_j W_i W_j F(r_i, s_j) \\
\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} F(r,s,t)\,drdsdt &= \textstyle\sum_i\sum_j\sum_k W_i W_j W_k F(r_i, s_j, t_k)
\end{aligned}$$

That the same weighting factors and coordinates are used irrespective of the dimension of the problem makes this integration scheme very easy to implement in a computer code.

Example 6.9: Use numerical quadrature to evaluate the integral

$$I = \int_1^3 \frac{dx}{x}$$

We will use the order $n = 2$ (that is, three-point) rule. First introduce $r = x - 2$ so that the integral becomes

$$I = \int_{-1}^{+1} \frac{dr}{r+2}$$

The approximation is

$$\begin{aligned} I &\approx W_0 F(r_0) + W_1 F(r_1) + W_2 F(r_2) \\ &\approx \frac{5}{9}\frac{1}{1-\sqrt{0.6}} + \frac{8}{9}\frac{1}{1-0} + \frac{5}{9}\frac{1}{1+\sqrt{0.6}} \approx 1.09803 \end{aligned}$$

The exact result is $I = \ln(3) = 1.09861$. This is surprisingly accurate for a three term polynomial approximation. ■

6.4 Total Lagrangian Incremental Formulation

All nonlinear problems are solved in an incremental/iterative manner with some sort of linearization done at each time or load step. In this section, we develop the total Lagrangian incremental formulation. The incremental idea is shown in Figure 6.11 where everything is assumed known at time step t and it is desired to obtain the solution at time $t + \Delta t$. The fundamental equation we deal with is the static version of the equation of motion, Equation (6.11), written at the next time

$$\{\mathcal{F}\} = 0 = \{P\}^{t+\Delta t} - \{F\}^{t+\Delta t}, \qquad \{F\} = \textstyle\sum_m \displaystyle\int_{V_m^o} [B_E]^T \{\sigma^K\}$$

Since $\{F\}^{t+\Delta t}$ is unknown the developed scheme is implicit.

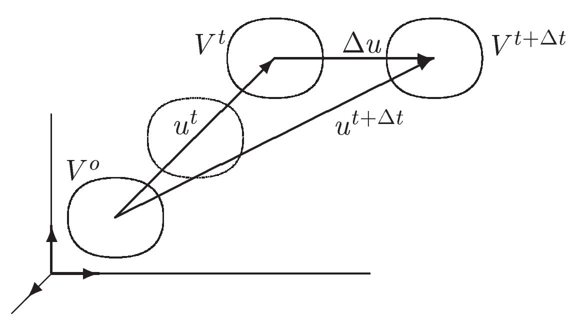


Figure 6.11: Decomposition of displacement.

This scheme seems quite suitable for the 3-D continuum elements where the degrees of freedom are the nodal displacements. There are many other formulations, indeed for 3-D thin-walled structures, a corotational scheme is quite popular.

Simple Truss Example

A truss is composed of slender members that support only axial load; consequently, these members must be triangulated for equilibrium under normal loads. We use the truss as an introductory example to illustrate the effect of axial loads on the stiffness properties of a structure. This will also serve as a test case for our computer formulation. Some geometric approximations are made but these will not be part of our final procedure for general continuous solids.

Consider the simple truss whose geometry is shown in Figure 6.12. The members are of original length L_o and the unloaded condition has the apex at a height of h. The two ends are on pinned rollers. For simplicity, assume the truss is elastic with the axial force related to the axial strain by

$$\bar{F}_o = EA_o\epsilon_{xx}$$

More general constitutive relations is left to later.

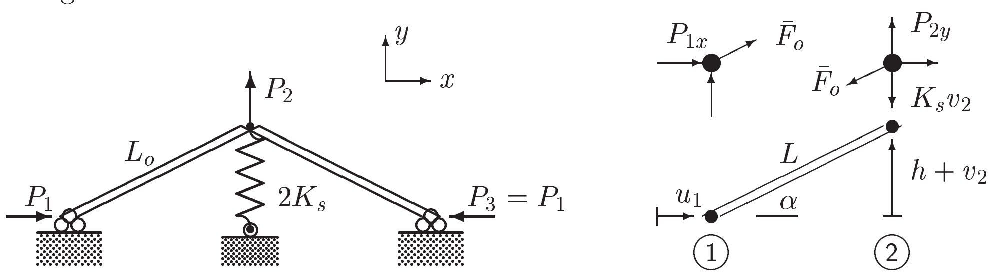


Figure 6.12: Simple pinned truss with a grounded spring.

Let the height h be small relative to the member length, and let the deflections be somewhat small; then we have the geometric approximations

$$L_x = L\cos\alpha \approx L_o - u_1\,, \qquad L_y = L\sin\alpha \approx h + v_2\,, \qquad u_1 \ll v_2$$

The deformed length of the member is

$$L = \sqrt{(L_o - u_1)^2 + (h + v_2)^2} \approx L_o - u_1 + \frac{h}{L_o}v_2 + \tfrac{1}{2}L_o(\frac{v_2}{L_o})^2$$

The axial force is computed from the strain as

$$\bar{F}_o = EA_o\epsilon = EA_o\frac{L - L_o}{L_o} = EA_o\Big[-\frac{u_1}{L_o} + \frac{h}{L_o}\frac{v_2}{L_o} + \tfrac{1}{2}(\frac{v_2}{L_o})^2\Big]$$

Note that we consider the parameters of the constitutive relation to be unchanged during the deformation.

Look at equilibrium in the deformed configuration; specifically, consider the resultant horizontal force at Node 1 and vertical force at Node 2 giving

$$\begin{aligned} 0 &= P_{1x} + \bar{F}_o\cos\alpha \approx P_{1x} + \bar{F}_o \\ 0 &= P_{2y} - \bar{F}_o\sin\alpha - K_s v_2 \approx P_{2y} - \beta\bar{F}_o - K_s v_2\,, \qquad \beta \equiv \frac{h + v_2}{L_o} \end{aligned}$$

Rewrite these in vector form as

$$\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} P_{1x} \\ P_{2y} \end{matrix}\right\} - \left\{\begin{matrix} -\bar{F}_o \\ \beta\bar{F}_o \end{matrix}\right\} - \left\{\begin{matrix} 0 \\ K_s v_2 \end{matrix}\right\} \qquad \text{or} \qquad \{\mathcal{F}\} = \{P\} - \{F\} = \{0\}$$

We refer to the last form of the equation as the *loading equation*; $\{P\}$ is the vector of applied loads, $\{F\}$ is the vector of element nodal forces, and $\{\mathcal{F}\}$ is the vector of out-of-balance forces. For equilibrium, we must have that $\{\mathcal{F}\} = \{0\}$, but, as we will see, this is not necessarily (numerically) true during an incremental approximation of the solution.

Example 6.10: Determine the deflections when the loads are $P_{1x} = P$, $P_{2y} = 0$.

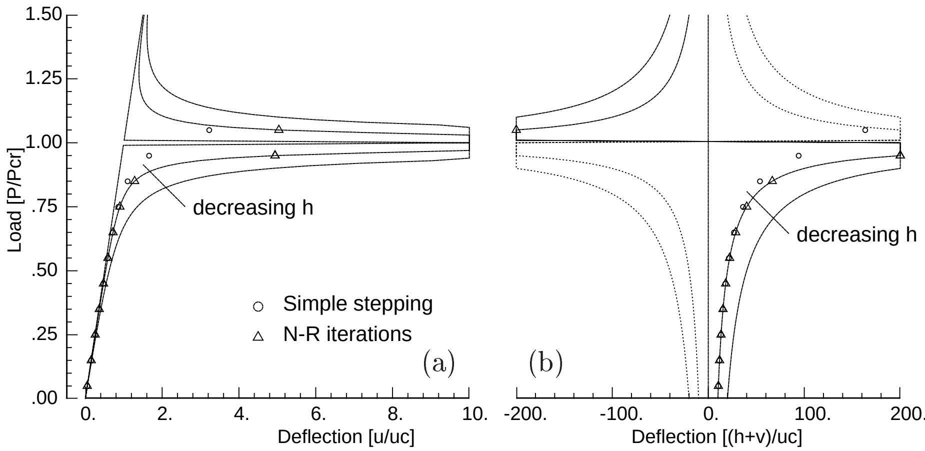


Figure 6.13: Load-deflection behavior for the simple truss. (a) Horizontal displacement u_1. (b) Vertical displacement v_2.

For this special case, we get $\bar{F}_o = -P$ and the two deflections are

$$\begin{aligned} u_1 &= \left[\frac{P}{EA} + (\frac{h}{L_o})^2 \frac{P}{K_s L_o - P} + \frac{1}{2}(\frac{h}{L_o})^2 (\frac{P}{K_s L_o - P})^2\right] L_o \\ v_2 &= \left[\frac{h}{L_o}\frac{P}{K_s L_o - P}\right] L_o \end{aligned}$$

The load-deflection relations are nonlinear even though the deflections are assumed to be somewhat small. Furthermore, when the applied load is close to $K_s L_o$, we get very large deflections. (This is inconsistent with our above stipulation that the deflections are “somewhat small,” let us ignore that issue for now and accept the results as indicated.) That is, at $P = P_{cr} = K_s L_o$, we get very large deflections meaning that the structure has become unstable. We say P has reached a critical value.

The full solutions are shown plotted in Figure 6.13 for different values of h. The effect of a decreasing h is to cause the transition to be more abrupt. Also shown are the behaviors for $P > K_s L_o$. These solutions could not be reached using monotonic loading, but they do in fact represent equilibrium states that can cause difficulties for a numerical scheme that seeks the equilibrium path approximately. That is, it is possible to accidentally (begin to) converge on these spurious equilibrium states.

■

Incremental Solution Scheme

We formulate the solution in an incremental fashion. That is, we view the deformation as occurring in a sequence of steps associated with time increments Δt, and at each step it is the increment of displacements that are considered to be the unknowns.

To help fix ideas, look again at the truss in Figure 6.12. We have already shown that the equilibrium equation is

$$\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} = \left\{\begin{matrix} P_{1x} \\ P_{2y} \end{matrix}\right\} - \left\{\begin{matrix} -\bar{F}_o \\ \beta\bar{F}_o \end{matrix}\right\} - \left\{\begin{matrix} 0 \\ K_s v_2 \end{matrix}\right\} \qquad \text{or} \qquad \{\mathcal{F}\} = \{P\} - \{F\} = \{0\} \tag{6.13}$$

with $\beta = (h+v_2)/L_o$ and the axial force is the nonlinear function of the displacements

$$\bar{F}_o = EA\Big[-\frac{u_1}{L_o} + \frac{h}{L_o}\frac{v_2}{L_o} + \tfrac{1}{2}(\frac{v_2}{L_o})^2\Big]$$

Consider the equilibrium equation at time step t_{n+1}

$$\{\mathcal{F}\}_{n+1} = \{P\}_{n+1} - \{F(u)\}_{n+1} = \{0\}$$

We do not know the displacements $\{u\}_{n+1}$, hence we cannot compute the axial force $\bar{F}_o$ nor the nodal forces $\{F\}_{n+1}$. As is usual in such nonlinear problems, we linearize about a known state. That is, assume we know everything at time step t_n, then write the Taylor series approximation (for small displacement increment) for the element nodal forces

$$\{F(u)\}_{n+1} \approx \{F(u)\}_n + [\frac{\partial F}{\partial u}]_n\{\Delta u\} + \cdots = \{F(u)\}_n + [K_T]_n\{\Delta u\} + \cdots \tag{6.14}$$

The square matrix $[K_T]$ is sometimes called the *tangent stiffness matrix* but as discussed in the previous section we will call it the total stiffness. The explicit form it takes for our truss problem is

$$[K_T]_n = [\frac{\partial F}{\partial u}]_n = \begin{bmatrix} \dfrac{\partial F_{1x}}{\partial u_1} & \dfrac{\partial F_{1x}}{\partial v_2} \\ \dfrac{\partial F_{2y}}{\partial u_1} & \dfrac{\partial F_{2y}}{\partial v_2} \end{bmatrix}_n = \begin{bmatrix} -\dfrac{\partial \bar{F}_o}{\partial u_1} & -\dfrac{\partial \bar{F}_o}{\partial v_2} \\ \beta\dfrac{\partial \bar{F}_o}{\partial u_1} & \dfrac{1}{L_o}\bar{F}_o + \beta\dfrac{\partial \bar{F}_o}{\partial v_2} + K_s \end{bmatrix}_n$$

Performing the differentiations

$$\frac{\partial \bar{F}_o}{\partial u_1} = EA\Big[-\frac{1}{L_o}\Big], \qquad \frac{\partial \bar{F}_o}{\partial v_2} = EA\Big[\frac{h}{L_o^2} + \frac{v_2}{L_o^2}\Big] = \frac{EA}{L_o}\beta$$

then leads to the stiffness

$$[K_T] = [\frac{\partial F}{\partial u}] = \frac{EA}{L_o}\begin{bmatrix} 1 & -\beta \\ -\beta & \beta^2 + \dfrac{K_s L_o}{EA} \end{bmatrix} + \frac{\bar{F}_o}{L_o}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = [K_E] + [K_G]$$

Note that both matrices are symmetric. The total stiffness has the same decomposition as discussed earlier. The first matrix is the elastic stiffness of a truss member oriented slightly off the horizontal by the angle $\beta = (h + v_2)/L_o$. The second matrix shows the effect of the axial loading.

We are now in a position to solve for the increments of displacement; re-arrange the approximate equilibrium equation into a loading equation as

$$\{P\}_{n+1} - \{F\}_n - [K_T]_n\{\Delta u\} \approx 0 \qquad \Longrightarrow \qquad [K_T]_n\{\Delta u\} = \{P\}_{n+1} - \{F\}_n$$

Again, consider the special case when $P_{1x} = P$, $P_{2y} = 0$; then the system of equations to be solved is

$$\left[\frac{EA}{L_o}\begin{bmatrix} 1 & -\beta \\ -\beta & \beta^2 + \gamma \end{bmatrix} + \frac{\bar{F}_o}{L_o}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]_n \begin{Bmatrix} \Delta u_1 \\ \Delta v_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}_{n+1} - \begin{Bmatrix} -\bar{F}_o \\ -\beta\bar{F}_o + K_s v_2 \end{Bmatrix}_n$$

with $\gamma = K_s L_o/EA$. Note that the right-hand-side has a load term associate with the current known state. A simple solution scheme involves computing the increments at each step and updating the displacements as

$$u_{1(n+1)} = u_{1(n)} + \Delta u_1\,, \qquad v_{2(n+1)} = v_{2(n)} + \Delta v_2$$

The axial force and orientation β also need to be updated as

$$\bar{F}_o|_{n+1} = EA\Big[-\frac{u_1}{L_o} + \frac{h}{L_o}\frac{v_2}{L_o} + \tfrac{1}{2}(\frac{v_2}{L_o})^2\Big]_{n+1}\,, \qquad \beta_{n+1} = \frac{h + v_2}{L_o}|_{n+1}$$

Table 6.3 and Figure 6.13 show the results using this simple stepping scheme, where $P_{cr} = K_s L_o$ and $u_{cr} = P_{cr}/EA$.

Table 6.3 also shows the out-of-balance force

$$\{\mathcal{F}\}_{n+1} = \{P\}_{n+1} - \{F\}_n$$

computed at the end of each step. Clearly, nodal equilibrium is not being satisfied and it deteriorates as the load increases. In order for this simple scheme to give reasonable results, it is necessary that the increments be small. This can be computationally prohibitive for large systems because, at each step, the total stiffness must be formed and decomposed. A more-refined incremental version that uses an iterative scheme to enforce nodal equilibrium will now be developed.

P/P_{cr}	u_1/u_{cr}	$(h+v_2)/u_{cr}$	P	$\mathcal{F}_{1x}$	$\mathcal{F}_{2y}$
.0500	.051000	10.5000	100.00	.050003	.989497E-02
.1500	.153401	11.6315	300.00	.256073	.446776E-01
.2500	.257022	13.1329	500.00	.450836	.589465E-01
.3500	.362372	15.0838	700.00	.761169	.759158E-01
.4500	.470656	17.7037	900.00	1.37274	.100333
.5500	.584416	21.3961	1100.0	2.72668	.137041
.6500	.709589	26.9600	1300.0	6.19165	.192208
.7500	.862467	36.1926	1500.0	17.0481	.257334
.8500	1.09898	53.8742	1700.0	62.5275	.938249E-01
.9500	1.66455	94.1817	1900.0	324.939	-4.00430
1.050	3.23104	163.411	2100.0	958.535	-24.0588

Table 6.3: Incremental results using simple stepping.

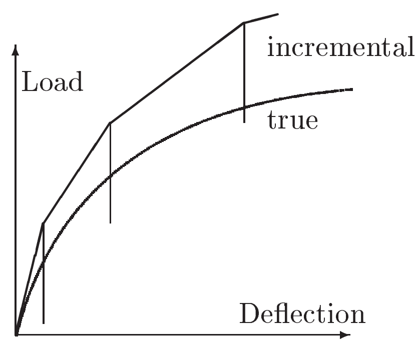


Figure 6.14: Illustration of the simple incremental scheme and its deviation from the true equilibrium path. The increments are load controlled.

Newton-Raphson Equilibrium Iterations

As was just pointed out, if the estimates for the new displacements during an incremental solution are substituted into Equation (6.13), then this equation will not be satisfied because the displacements were obtained using only an approximation of the nodal forces given by Equation (6.14). What we can do, however, is repeat the above process at the same applied load level until we get convergence. We illustrate this here.

We begin by writing the loading equation at time step t_{n+1} as

$$\{\mathcal{F}\}_{n+1} = \{P\}_{n+1} - \{F\}^i_{n+1}$$

where i is an iteration counter. The current displacement is estimated according to

$$\{u\}^i_{n+1} \approx \{u\}^{i-1}_{n+1} + \{\Delta u\}^i$$

Note that the increment is from the previous estimate of the current displacement $\{u\}_{n+1}$ and not $\{u\}_n$ as done previously. The Taylor series approximation for the

i	u_1	v_2	$u_1 - u_{1ex}$	$v_2 - v_{2ex}$	$\mathcal{F}_{1x}$	$\mathcal{F}_{2y}$
1	-3.02304	76.0009	-3.12184	72.2009	.3207E+7	-.244E+7
2	29.0500	75.9986	28.9512	72.1986	.21923	-72.3657
3	-25.8441	3.95793	-25.9429	.157933	.2594E+7	-107896.
4	.105242	3.95793	.64416E-2	.157926	.6835E-2	-.158213
5	.09867	3.80001	-.12436E-3	.8344E-5	12.4694	-.498780
6	.09880	3.80000	.15646E-6	.4053E-5	.000000	-.3948E-5
7	.09880	3.80000	.000000	.000000	-.4882E-3	.1878E-4
exact	.09880	3.80000				

Table 6.4: Newton-Raphson iterations for a load step $0.95\,P_{cr}$.

element nodal forces is

$$\{F(u)\}^i_{n+1} \approx \{F(u)\}^{i-1}_n + [\frac{\partial F}{\partial u}]^{i-1}_n\{\Delta u\}^i + \cdots = \{F(u)\}^{i-1}_n + [K_T]^{i-1}_n\{\Delta u\}^i + \cdots \tag{6.15}$$

The new estimates for the displacements are obtained by solving

$$\{K_T\}^{i-1}_n\{\Delta u\}^i = \{P\}_{n+1} - \{F\}^{i-1}_n\,, \qquad \{u\}^i_{n+1} \approx \{u\}^{i-1}_n + \{\Delta u\}^i$$

This is repeated until $\{\Delta u\}^i$ are sufficiently small.

The algorithm during each time step increment may thus be stated as

$$\begin{aligned}
\textbf{form:}\quad & \{K_T\}^{i-1}_{n+1}\,, \qquad \{F\}^{i-1}_{n+1} \\
\textbf{solve:}\quad & \{K_T\}^{i-1}_{n+1}\{\Delta u\}^i = \{P\}_{n+1} - \{F\}^{i-1}_{n+1} \\
\textbf{update:}\quad & \{u\}^i_{n+1} = \{u\}^{i-1}_{n+1} + \{\Delta u\}^i \\
\textbf{update:}\quad & \{K_T\}^i_{n+1}\,, \qquad \{F\}^i_{n+1}
\end{aligned}$$

and repeat until $\{\Delta u\}^i$ becomes less than some tolerance value. The iteration process is started (at each increment) using the starter values

$$\{u\}^0_{n+1} = \{u\}_n\,, \qquad \{K_T\}^0_{n+1} = \{K_T\}_n\,, \qquad \{F\}^0_{n+1} = \{F\}_n$$

This basic algorithm is known as the *full Newton-Raphson method*.

Combined incremental and iterative results are given in Figure 6.13. We see that it essentially gives the exact solution. Iteration results for a load level equal $0.95\,P_{cr}$ are given in Table 6.4 where the initial guesses correspond to the linear elastic solution. We see that convergence is quite rapid and the out-of-balance forces go to zero.

It is worth pointing out the converged value above P_{cr} in Figure 6.13; this corresponds to a vertical deflection where the truss has “flipped” over to the negative side. Such a situation would not occur physically, but does occur here due to a combination of linearizing the problem (i.e., the small angle approximation) and the nature of the iteration process (i.e., no restriction is placed on the size of the iterative increments).

General Nonlinear Algorithm

The previous development are applicable to general continua, it is necessary only to utilize the appropriate discretizations. As shown in the previous section, the total stiffness has the representation

$$[K_T] = \sum_m \int_{V_m^o} \Big[[B_E]^T[\ D\ ][B_E] + [B_D]^T[\sigma^K]\{B_D\}\Big]\, dV_m^o = [K_E] + [K_G]$$

The corresponding element nodal forces are compted as

$$\{F\} = \sum_m \int_{V_m^o} [B_E]^T\{\sigma^K\}$$

Otherwise, the scheme is identical to that just described for the simple truss.

In the following, we concentrate on the basic algorithm for the full Newton-Raphson method because it best illustrates the essential ingredients. This algorithm, for monotonically increasing loads, can be stated as:

Step 1: Specify parameters of the algorithm such as tolerances, and maximum iterations.

Step 2: Read the initial geometry and material properties.

Step 3: Specify load increments, number of steps.

Step 4: *Begin loop over time (load) increments:*

Step T.1: Increment the load vector $\{P\}_{t+\Delta t}$.

Step T.2: Initialize for equilibrium iterations

$$\{u\}^0_{t+\Delta t} = \{u\}_t\,, \qquad [K_T]^0_{t+\Delta t} = [K_T]_t\,, \qquad \{F\}^0_{t+\Delta t} = \{F\}_t$$

Step T.3: *Begin loop over equilibrium iterations:*

Step I.1: ITERATE:

Step I.2: Assemble nodal force vector $\{F\}^i$.

Step I.3: Form the effective load vector

$$\{\Delta P_{eff}\}^i_{t+\Delta t} \equiv \{P\}_{t+\Delta t} - \{F\}^i_{t+\Delta t}$$

Step I.4: Assemble the elastic stiffness matrix $[K_E]$.

Step I.5: Assemble the geometric stiffness matrix $[K_G]$.

Step I.6: Form the total stiffness matrix as

$$[K_T] = [K_E] + \gamma[K_G]$$

Step I.7: Decompose the total stiffness to

$$[K_T] = [\ U\ ]^T \lceil\ D\ \rfloor[\ U\ ]$$

Step I.8: Solve for the new displacement increments from

$$[\ U\ ]^T \lceil\ D\ \rfloor [\ U\ ] \{\Delta u\}^i = \{\Delta P_{eff}\}^i_{t+\Delta t}$$

Step I.9: Update the displacements

$$\{u\}^i_{t+\Delta t} \quad = \quad \{u\}^{i-1}_{t+\Delta t} + \beta \{\Delta u\}^i$$

Step I.10: Test for convergence.

if: $|\{\Delta u\}^i| / |\{u\}^i| < tol$ converged, goto UPDATE
if: $|\{\Delta u\}^i| / |\{u\}^i| > tol$ not converged, goto ITERATE

Step T.4: *End loop over iterations.*
Step T.5: UPDATE:

$$u_{t+\Delta t} = u^i_{t+\Delta t}\,, \qquad xyz_{t+\Delta t} = xyz^i_{t+\Delta t}$$

Step T.6: Store results for this time step.
Step T.7: If maximum load not exceeded continue looping over loads.

Step 5: *End loop over time (load) increments.*

Step 6: END

It is possible to enhance this algorithm by including automatic step changes, automatic testing for appropriate time step size, and monitoring the spectral properties of the total stiffness. The parameters β and γ can also be adjusted automatically.

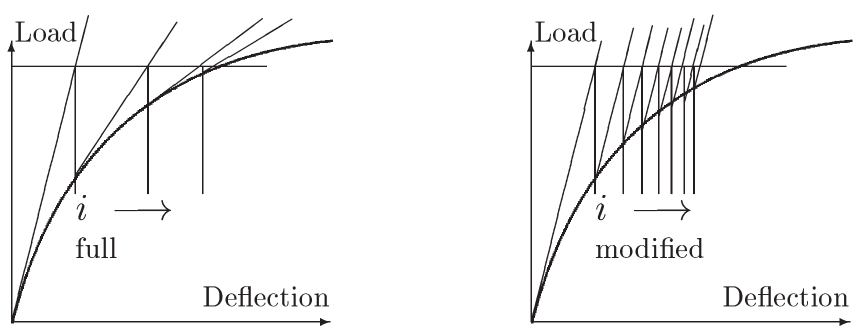


Figure 6.15: Full and modified Newton-Raphson methods.

The full Newton-Raphson method has the disadvantage that, during each iteration, the tangent stiffness matrix must be formed and decomposed. The cost of this can be quite prohibitive for large systems. Thus, effectively, the computational cost is like that of the incremental solution with many steps. It must be realized, however, that because of the quadratic convergence, six Newton-Raphson iterations, say, are much more effective than six load sub-increments.

The *modified Newton-Raphson method* is basically as above except that the total stiffness is not updated during the iterations but only after each load increment. This generally requires more iterations and sometimes is less stable but it is less computationally costly when it succeeds.

Both schemes are illustrated in Figure 6.15 where the starting point is from the zero load state. It is clear why the modified method will take more iterations. The plot for the modified method has the surprising implication that we do not need to know the actual total stiffness in order to compute correct results — what must be realized in the incremental/iterative scheme is that we are imposing equilibrium (iteratively) in terms of the applied loads and resultant nodal forces; we need good quality *element* stiffness matrices in order to get the good quality element nodal forces, but the assembled total stiffness matrix is used only to suggest a direction for the iterative increments. To get the correct converged results we need to have good element stiffness relations, but not necessarily a good assembled total stiffness matrix. Clearly, however, a good quality tangent stiffness will give more rapid convergence as well as increase the radius of convergence.

Example 6.11: Use Hex20 elements to model the large deflection, large strain of the block shown in Figure 6.16. Compare the results to those obtained using the Tet4 tetrahedron elements.

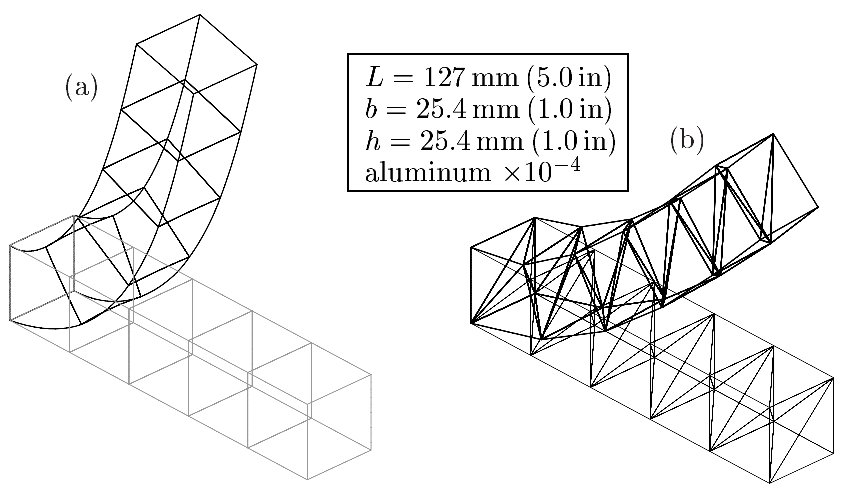


Figure 6.16: Large deformation of a thick beam. (a) Initial and deformed shapes using Hex20 elements. (b) Initial and deformed shapes using Tet4 elements.

One end of the block is fixed while the end has vertical loads which do not change their direction during the loading history. The z axis is along the length.

In this type of problem, the initial linear displacement is vertical only and consequently the axial straining is exaggerated. This in turn leads to very large unbalanced forces which require the Newton-Raphson iterations to bring them back to zero. It is possible, however, that the initial forces are so large (because they are associated with axial stretching) that convergence is impossible to achieve without using a small time step.

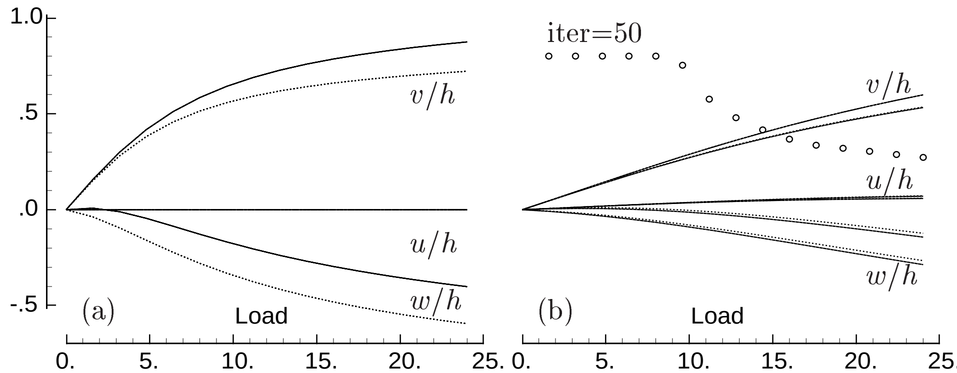


Figure 6.17: Large deformation of a thick beam. (a) Displacements against load using Hex20 elements. (b) Displacements against load showing the anisotropic behavior of the tetrahedron arrangement.

While it is possible to devise sophisticated step sizing schemes to help cope with these situations, a simple but effective scheme is described here. First, the modified Newton-Raphson iteration scheme (where the stiffnesses are updated only at the beginning of a load step) is used because these assure the displacement increments will be estimated based on a converged total stiffness. Second, the full increment of load is applied but the displacements are updated only partially by

$$\{u\}^i = \{u\}^i + \beta^i \{\Delta u\}^i$$

where

$$\beta^i = \frac{2^i}{2^{i_{max}}}\beta_o, \quad i < i_{max}\,, \qquad \beta^i = \beta_o, \quad i > i_{max}$$

Typically, $i_{max} = 5$ which gives increments of $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{1}$, Clearly, this increases the number of iterations, but the computational cost is not exorbitant since only the element nodal forces need be updated at each iteration. The significant advantage is that it gives the out-of-balance forces time to adjust before the full increment of load is effectively applied.

Figure 6.17(b) shows the displacements at all four end nodes. It is expected that the u displacements be nonzero because of the Poisson's ratio effect but that all four be centered about zero. It is clear from the figure that there is a definite sideways drift for the Tet4 results indicating that the tetrahedron arrangement exhibits some anisotropy.

Interestingly for the Tet4 element, the initial iterations reach the maximum limit set at 50 but after about six or so increments of load decrease significantly. Actually, the out-of-balance forces during the initial increments are reasonably well converged but seem to oscillate about small deviations. ■

Example 6.12: Treat the simple shear deformation of a block as a load control problem and show the relation between the Cauchy and Kirchhoff stresses.

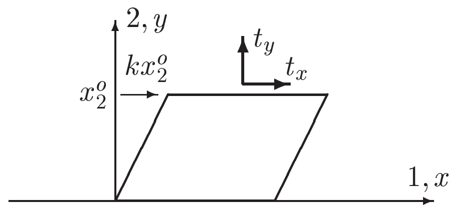


Figure 6.18: A block in simple shear.

A simple shear deformation parallel to the $x_1^o - x_2^o$ plane is shown in Figure 6.18 and given mathematically by

$$x_1 = x_1^o + k\, x_2^o\,, \qquad x_2 = x_2^o\,, \qquad x_3 = x_3^o$$

The displacement components are readily obtained as

$$u_1 = k\, x_2^o\,, \qquad u_2 = 0\,, \qquad u_3 = 0$$

indicating that horizontal lines move horizontally only. The deformation gradients are

$$[\frac{\partial x_p}{\partial x_i^o}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [\frac{\partial x_p^o}{\partial x_i}] = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that there is no volume change because $J = J_o = 1$. The Lagrangian strain tensor is

$$2E_{ij} = \sum\nolimits_p \frac{\partial x_p}{\partial x_i^o}\frac{\partial x_p}{\partial x_j^o} - \delta_{ij} = \begin{bmatrix} 0 & k & 0 \\ k & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is no straining of horizontal lines. There is also no straining in the third direction indicating that this is a state of plane strain.

Let the material have the following linear constitutive behavior:

$$\sigma_{ij}^K = 2\mu E_{ij} + \lambda\delta_{ij}\sum\nolimits_k E_{kk}$$

where μ and λ are the Lamé constants. The Kirchhoff stress tensor, therefore, is

$$\sigma_{ij}^K = \begin{bmatrix} \frac{1}{2}\lambda k^2 & \mu k & 0 \\ \mu k & \mu k^2 + \frac{1}{2}\lambda k^2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda k^2 \end{bmatrix} = \mu k \begin{bmatrix} \gamma k & 1 & 0 \\ 1 & (1+\gamma)k & 0 \\ 0 & 0 & \gamma k \end{bmatrix}$$

where $\gamma = \lambda/2\mu$. The tensile σ_{22}^K component arises from the fact that lines originally in the 2-direction are being stretched. The Cauchy stresses are obtained from

$$\begin{aligned} \sigma_{pq} &= \sum\nolimits_{i,j} \frac{\rho}{\rho^o}\sigma_{ij}^K \frac{\partial x_p}{\partial x_i^o}\frac{\partial x_q}{\partial x_j^o} \\ &= \sigma_{11}^K \left[\frac{\partial x_p}{\partial x_1^o}\frac{\partial x_q}{\partial x_1^o}\right] + \sigma_{12}^K \left[\frac{\partial x_p}{\partial x_1^o}\frac{\partial x_q}{\partial x_2^o} + \frac{\partial x_p}{\partial x_2^o}\frac{\partial x_q}{\partial x_1^o}\right] + \sigma_{22}^K \left[\frac{\partial x_p}{\partial x_2^o}\frac{\partial x_q}{\partial x_2^o}\right] + \sigma_{33}^K \left[\frac{\partial x_p}{\partial x_3^o}\frac{\partial x_q}{\partial x_3^o}\right] \end{aligned}$$

Substituting for the deformation gradients leads to the complete stress tensor as

$$\sigma_{pq} = \mu k \begin{bmatrix} (2+\gamma)k + (1+\gamma)k^3 & 1+(1+\gamma)k^2 & 0 \\ 1+(1+\gamma)k^2 & (1+\gamma)k & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

The Cauchy stress tensor, as expected, is symmetric.

The magnitude of the shear deformation is governed by the parameter k. It is worth noting that when it is small, both stress tensors approach the same values.

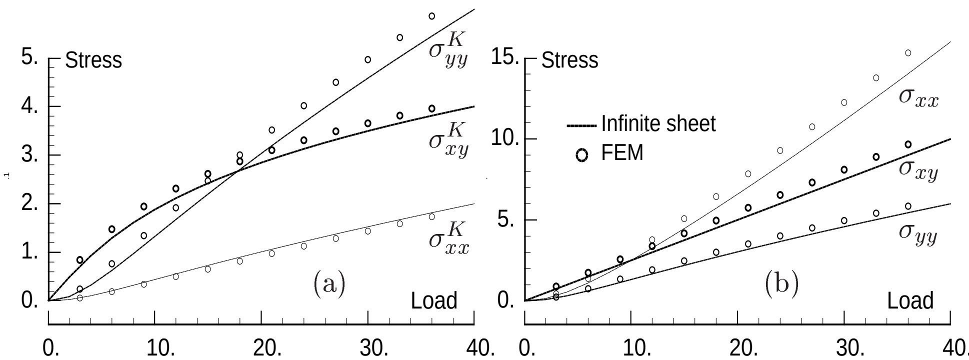


Figure 6.19: Stresses for a block under shear load control. (a) Kirchhoff stress. (b) Cauchy stress.

Imagine a free body cut parallel to the x-axis; this will expose two tractions related to the Cauchy stress by

$$t_x = \sigma_{xy}\,, \qquad t_y = \sigma_{yy}$$

The t_x traction, when multiplied by the area, gives a resultant horizontal force that we will consider to be the applied load. The resulting deformation is then related to the traction (and hence load) as

$$\sigma_{xy} = \mu k[1 + (1+\gamma)k^2] = t_x = P/hL$$

where hL is the area over which the resulting force P acts. Let us consider the problem to be load driven: then the deformation parameter k is a nonlinear function of the load. However, we can easily solve for it using a Newton-Raphson iterative scheme as

$$k^{i+1} = k^i + \frac{P/hL - f_o}{\mu[1 + (1+\gamma)3(k^i)^2]}\,, \qquad f_o = \mu k^i[1 + (1+\gamma)(k^i)^2]$$

where i is the iteration counter. This converges very rapidly.

Once we know k we can then determine the Kirchhoff stresses. The results are shown plotted in Figure 6.19 as the continuous lines. It is interesting to note that the Cauchy σ_{xx} is the largest of the stresses. ■

Example 6.13: Use the CST element to obtain a numerical solution of the shear problem.

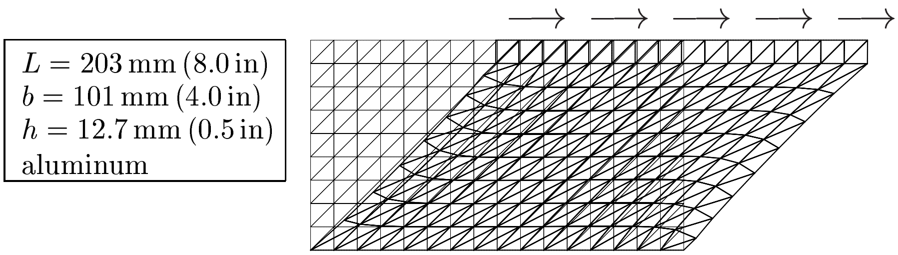


Figure 6.20: Undeformed and deformed shape of a block under shear.

The analytical solution just developed was for a very large sheet under homogeneous deformation. This is impractical to achieve here so we will model the block as shown in Figure 6.20.

The top row of elements have a stiffness 1000 times that of the other elements, it is also constrained to move only horizontally. In the infinite sheet case the lateral sides have shear components but clearly in the present case the normal tractions are zero.

Figure 6.21 shows the contours of Cauchy stress at the maximum load drawn on the deformed block. What they all have in common is that they show a nearly uniform region of stress in the middle portion. We therefore expect to have a reasonable comparison with the infinite sheet solution in this region. Figure 6.19 shows a comparison of the stress histories with that for the infinite sheet — all the trends are in agreement.

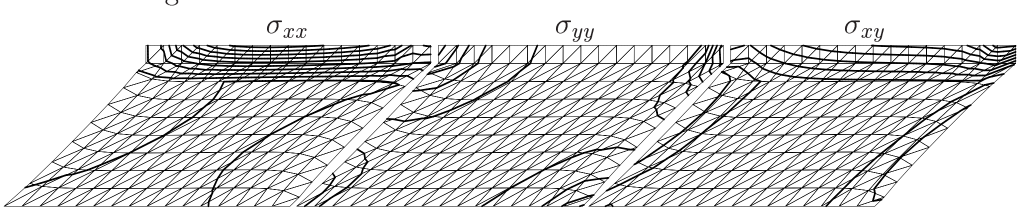


Figure 6.21: Contours of Cauchy stresses on the deformed block.

We do not expect the Cauchy σ_{xx} stress to go to zero at the boundaries because these boundaries are inclined in the deformed configuration ■

6.5 Stability of Discrete Systems

Up to now we have concentrated on establishing the equilibrium of deformable bodies. This section discusses a very important aspect of equilibrium; namely, its stability. This is a quintessentially nonlinear phenomenon and therefore could not be part of the equilibrium analyses of Chapter 5. Many of the ideas to be presented are developed in more depth in the paper by Allman [2] and the book by Crisfield [7].

Some Notions of Stability

There are a number of definitions of stability and the one chosen should be appropriate for the phenomena being investigated. Our basic notion is that if a system is slightly disturbed from its equilibrium position and eventually returns to the original position after removal of the disturbance, then it is stable. Conversely, if it gives a disproportionate response, then it is unstable. This basic notion is shown in Figure 6.22. There are other concepts of stability such as the original as proposed by Euler, which was phrased in terms of alternate equilibrium positions not "too far" from the configuration under discussion. We look at both of these concepts.

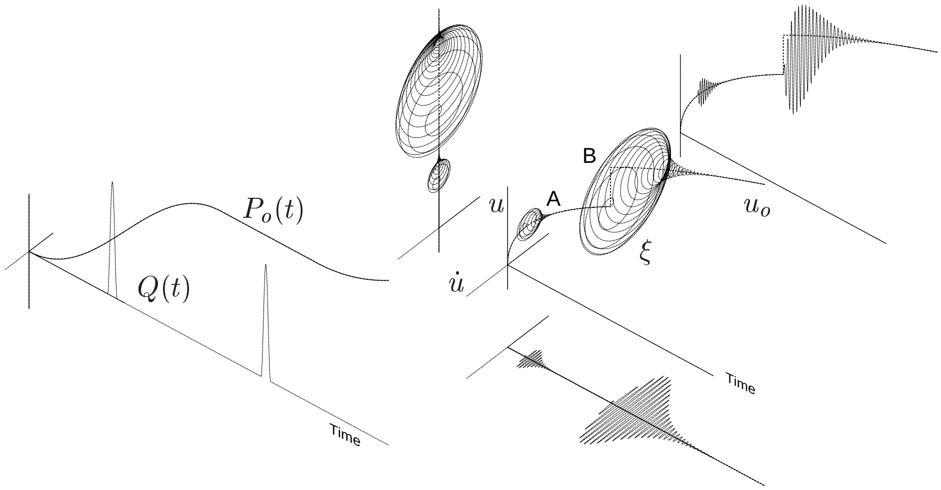


Figure 6.22: Dynamic view of a static instability. (a) Small ping loads applied in addition to the slowly increasing primary load. (b) Response to the two loads, the second ping causes a change in primary loading path indicating in instability.

I: Potential Energy View of Stability

Consider the equilibrium situations shown in Figure 6.23 where the black dot represents a ball and the line represents a surface it is resting on. In each case, the black ball is in equilibrium as can be easily verified by drawing a free body diagram. Now consider displacing the ball a very small amount away from the equilibrium position — this is indicated by the white ball. What will happen?

In Figure 6.23(a), the ball will roll back toward the original position; if there is dissipation in the system, it will oscillate and eventually settle down to the original position. We therefore say that the original (and not the disturbed) equilibrium is stable. In Figure 6.23(c), the ball will roll away from the original position; we therefore say that the original equilibrium is unstable. The ball in Figure 6.23(b) will stay put and therefore we say that the original equilibrium is neutral.

In simple terms, we can think of the position of the ball as its total potential energy. The change in this potential energy as the ball is displaced can be expressed as

$$\Delta\Pi \approx \delta\Pi + \tfrac{1}{2}\delta^2\Pi + \cdots = \sum\nolimits_i \frac{\partial\Pi}{\partial u_i}\delta u_i + \tfrac{1}{2}\sum\nolimits_i\sum\nolimits_j \frac{\partial^2\Pi}{\partial u_i \partial u_j}\delta u_i \delta u_j + \cdots$$

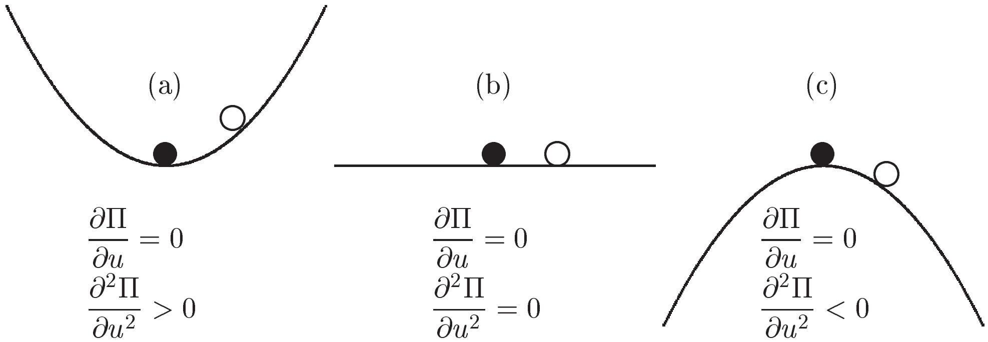


Figure 6.23: Disturbed equilibrium state of a ball. (a) Stable equilibrium. (b) Neutral equilibrium. (c) Unstable equilibrium.

We have already shown in Section 6.2 that the condition $\delta\Pi = 0$ governs equilibrium; a study of the second order term $\delta^2\Pi$ therefore governs the nature of the equilibrium, that is,

$$\begin{array}{ll} \delta^2\Pi > 0 & \text{:stable equilibrium} \\ \delta^2\Pi = 0 & \text{:neutral equilibrium} \\ \delta^2\Pi < 0 & \text{:unstable equilibrium} \end{array}$$

Actually, if $\delta^2\Pi = 0$ then we must check the next higher order terms also. We conclude that for stable equilibrium, the potential energy is a relative minimum as depicted in Figure 6.23(a).

To clarify the meaning of the second variation, let the equation of the surface be quadratic near the equilibrium point; that is,

$$y = \alpha x^2 \qquad \text{or} \qquad v = \alpha u^2$$

where α is a parameter. The three cases shown in Figure 6.23 correspond to $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$, respectively. The total potential when a small horizontal force Q is applied is

$$\Pi = \mathcal{U} + Mgv - Qu = \mathcal{U} + \alpha Mgu^2 - Qu$$

(In this case, the strain energy $\mathcal{U}$ is zero.) The different orders of variation of the potential are

$$\begin{aligned} \delta\Pi &= \frac{\partial \Pi}{\partial u}\delta u \quad = [\frac{\partial \mathcal{U}}{\partial u} + 2\alpha Mgu - Q]\,\delta u = [2\alpha Mgu - Q]\,\delta u \\ \delta^2\Pi &= \frac{\partial^2 \Pi}{\partial u^2}\delta u^2 = [\frac{\partial^2 \mathcal{U}}{\partial u^2} + 2\alpha Mg]\,\delta u^2 = [\alpha Mg]\,\delta u^2 \\ \delta^3\Pi &= \frac{\partial^3 \Pi}{\partial u^3}\delta u^3 = [\frac{\partial^3 \mathcal{U}}{\partial u^3} + 0]\,\delta u^3 = [\ 0\ ]\,\delta u^3 \end{aligned}$$

The first equation gives the equilibrium condition; when the system is in equilibrium this term is zero, therefore it is the second equation that determines the sign of $\Delta\Pi$.

Depending on the value of α, this second term can be either positive or negative. One of the important points to note is that the relation for the second variation does not contain the applied load Q but does contain the gravity load Mg.

II: Instability as Loss of Structural Stiffness

To clarify our stability idea in an actual situation, consider the simple pinned structure shown in Figure 6.24(a). The initially vertical bar is assisted in remaining vertical by the action of the horizontal spring; the spring is unstretched when the bar is vertical. An equilibrium analysis in the undeformed state shows that the displacements are related to the forces through the stiffness relations

$$Ku = Q\,, \qquad K_b v = -P \tag{6.16}$$

where K_b is the axial stiffness of the bar. For the purpose of the later discussion, let the bar be very stiff, from which we conclude that the displacement is only horizontal, that is, $v = 0$.

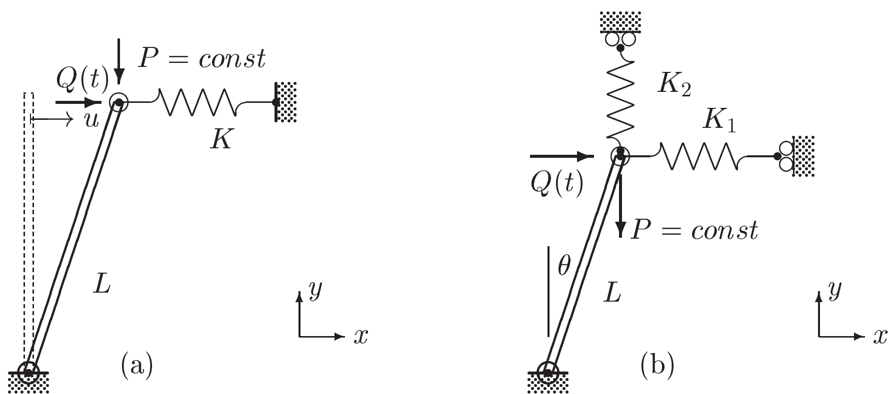


Figure 6.24: Disturbed equilibrium state of a pinned bar. (a) Small deflection model. (b) Large deflection model.

A key point in stability analysis is to look at equilibrium of the structure in its (slightly) deformed state. In this first linear analysis, we will consider all displacements to be small. Consider the situation when the bar has already displaced by an amount u as shown in Figure 6.24(a). Since it is in equilibrium, summing the moments about the base gives

$$-QL - Pu + KuL = 0 \qquad \text{or} \qquad [K - P/L]u = Q$$

It is worth noting at this stage the different roles played by the two forces: Q appears as a regular applied load on the right-hand side, but P appears on the left-hand side almost like a stiffness term. We refer to this as an initial stress term and is the same as the geometric stiffness as discussed earlier.

Clearly, we can solve for the displacements for any combinations of loads, thus

$$u = \frac{Q}{K - P/L}$$

These are shown plotted in Figure 6.25(a) for different values of Q. The obvious point of note is that in the vicinity of $P = KL$ the displacements are very large even for negligible Q. (Note that it is a consequence of the linear approximations that the infinity occurs — deflections in a real nonlinear situation would be finite as we see later.) Keep in mind that every point shown in the figure is an equilibrium position, and based on our discussion above we would declare that the structure is unstable near $P = KL$. One could imagine, therefore, a loading sequence that need only "skip over" this special value of load to land on one of the other equilibrium paths with finite displacement; we would then not have to worry about stability. The situation is much subtler than this, and we need to refine our concepts of equilibrium and stability in order to address it.

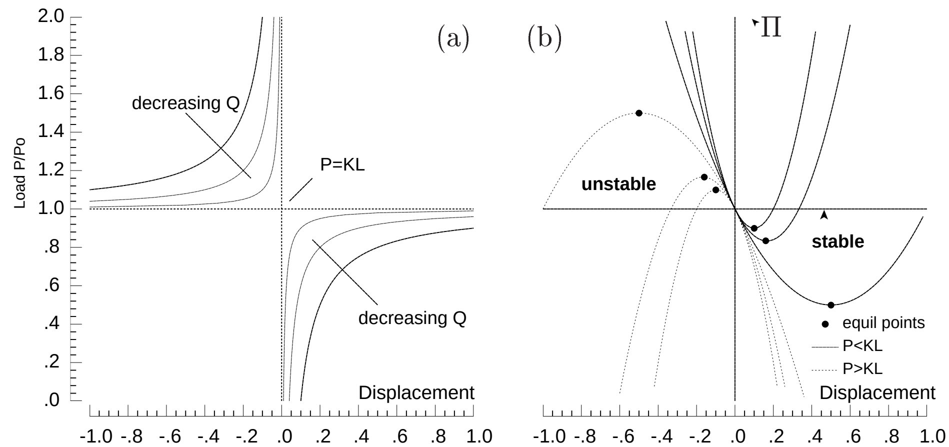


Figure 6.25: Behavior of the linear bar model. (a) Displacements for various values of loads. (b) Potential energy for different P values and constant Q.

The strain energy and potential of the loads for our simple system are, respectively,

$$\mathcal{U} = \tfrac{1}{2}Ku^2\,, \qquad \mathcal{V} = -Pv - Qu \approx -\tfrac{1}{2}[P/L]u^2 - Qu$$

The approximation is based on small deflections. The total potential is

$$\Pi = \mathcal{U} + \mathcal{V} = \tfrac{1}{2}Ku^2 - \tfrac{1}{2}[P/L]u^2 - Qu$$

This is shown plotted in Figure 6.25(b) for different values of P but the same value of Q. Note that most points on these plots are not equilibrium points — equilibrium is where

$$\mathcal{F} = \frac{\partial \Pi}{\partial u} = Ku - [P/L]u - Q = 0$$

and are shown as dots in the figure.

For $P < KL$, the plots show a series of valleys the bottoms of which correspond to the equilibrium positions. For $P > KL$, the plots show a series of peaks the top of which correspond to the equilibrium positions. We can draw a direct comparison between these plots and the hill and valley of Figure 6.23. We thus go beyond declaring just the point $P = KL$ unstable and now see that all points beyond $P = KL$ are unstable. The special point $P = KL$ is called the critical value or sometimes a bifurcation point. In those cases when Q is small, the deflections in the region close by are small; we will take advantage of this to do a linearized eigenvalue analysis to determine these bifurcation points. In the literature (see Reference [30] for a very large collection of solutions) this is generally known as a *buckling analysis*. It is emphasized, however, that such an analysis determines only the critical points and says nothing of the post-buckling behavior.

The second variation in strain energy is related to the elastic stiffness as

$$K_{ij} = \sum_i \sum_j \frac{\partial^2 \mathcal{U}}{\partial u_i \partial u_j} \qquad \text{or} \qquad [K_T] = [K_E] + [K_G]$$

Hence the second variation of the total potential energy must be related to a stiffness like term; actually, it is called the total stiffness as discussed in Section 6.3. Hence the conditions for stability of equilibrium can be applied to the total stiffness matrix; that is, we inspect whether this matrix is positive definite, positive semi-definite, or negative definite. A necessary and sufficient condition that $\delta^2\Pi$ be positive definite is that the determinant of the matrix $[K_T]$ and all its principal minors be positive. Usually, it is unnecessary to examine a sequence of minors since the determinant itself vanishes before any of its minors. Thus,

$$\det[K_T] = 0$$

is the criterion for the onset of instability.

III: Euler's Bifurcation View

Consider the loaded column shown in Figure 6.24(b); this is similar to that of Figure 6.24(a) except for the additional spring and that the deflection can be large. We use the angle off the vertical, θ, as the independent variable and allow it to vary in the range from zero to π.

Since the frame member is rigid, we easily establish the relationship between the vertical and horizontal displacements as

$$u = L\sin\theta\,, \qquad v = L\cos\theta - L$$

The total potential energy is given by

$$\begin{aligned}\Pi &= \tfrac{1}{2}K_1 u^2 + \tfrac{1}{2}K_2 v^2 - [-P]v - Qu \\ &= \tfrac{1}{2}KL^2\sin^2\theta + \tfrac{1}{2}K_2L^2(1-\cos\theta)^2 - PL(1-\cos\theta) - QL\sin\theta\end{aligned}$$

The equilibrium configurations are found by setting

$$\mathcal{F} = \frac{\partial \Pi}{\partial \theta} = \Big[K_1 \cos\theta + K_2(1-\cos\theta) - \frac{P}{L}\sin\theta\Big]\sin\theta - \frac{Q}{L}\cos\theta = 0$$

Rearranging gives

$$P = K_1\cos\theta + K_2(1-\cos\theta) - \frac{Q\cos\theta}{L\sin\theta}$$

where we view Q simply as a parameter.

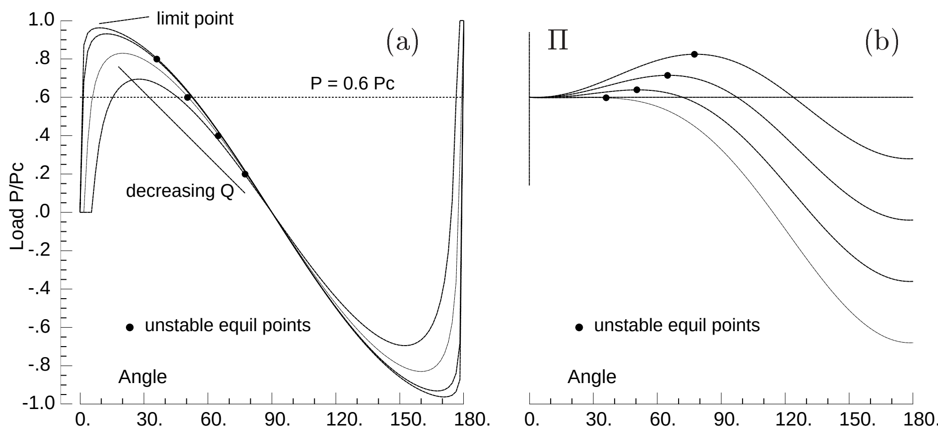


Figure 6.26: Behavior of the nonlinear bar model with $K_2 = 0$. (a) Rotation for various values of loads. (b) Potential energy for different P values and small constant Q.

Figure 6.26(a) shows a plot of the force P against angle for different values of Q when $K_2 = 0$. In each case, the force cannot exceed a particular maximum value and in the limit of small Q this corresponds to the critical value $P_c = K_1 L$. One interpretation of this figure is to view it as imposing the displacement and the force is the resulting reaction. Suppose, however, we wish to impose the force, then what is the interpretation of the decreasing force? More specifically, suppose we are at the peak value and we increment the load a small amount ΔP, what happens?

To answer this question, we need to look at the potential energy plotted in Figure 6.26(b). The total potential energy (when $K_2 = 0$) is given by

$$\Pi = \tfrac{1}{2}K_1 L^2 \sin^2\theta - PL[1-\cos\theta] - QL\sin\theta$$

Again, it is emphasized that most points on the potential plots are nonequilibrium points. Consider the line corresponding to the load values $P = 0.6P_c$ in Figure 6.26(a). It intersects the equilibrium curve at three points, but only two of the points (near 0 and 180) are stable. In other words, all points immediately past the peak are unstable and therefore a load/angle combination in this range would cause a large displacement. The member would rotate until it found the second stable equilibrium point at a large angle.

The maximum load point is called a *limit point*, because the load cannot exceed this limiting value. The phenomenon of quickly jumping from one equilibrium configuration to another distant one is called *snap-through*.

In the previous developments we saw that $Q \longrightarrow 0$ is a special case. Let us now deal directly with this situation. The equilibrium relation is

$$\mathcal{F} = \frac{\partial \Pi}{\partial \theta} = \Big[K_1 \cos\theta + K_2(1 - \cos\theta) - \frac{P}{L}\sin\theta\Big]\sin\theta = 0$$

This has two solutions. The first is

$$\sin\theta = 0 \qquad \text{or} \qquad \theta = 0,\ \pm\pi,\ \pm 2\pi,\ \ldots$$

These correspond to when the bar is in a vertical alignment. The other solution gives

$$P = K_1 \cos\theta + K_2[1 - \cos\theta]$$

Both solutions are shown plotted in Figure 6.27 for different values of K_2/K_1. We see that the solutions intersect at $P/K_1L = 1$, which is the critical value we previously identified. The presence of K_2 does not affect this critical value but it does change the shape of the equilibrium curve. We now investigate its stability.

The first variation of the potential energy is related to the equilibrium solution, while the second variation determines its stability. This is given by

$$K_T = \frac{\partial \mathcal{F}}{\partial \theta} = \frac{\partial^2 \Pi}{\partial \theta^2} = K_1 L^2(2\cos^2\theta - 1) + K_2 L^2(1 - 2\cos^2\theta + \cos\theta) - PL\cos\theta$$

For the solution $\theta = 0$, this becomes

$$K_T = \frac{\partial^2 \Pi}{\partial \theta^2} = K_1 L^2 - PL$$

Hence, when $P > K_1 L$ this solution path is unstable. That is, the intersection with the second solution has caused this solution path to become unstable. On the other hand, for $\theta = \pm\pi$ this becomes

$$K_T = \frac{\partial^2 \Pi}{\partial \theta^2} = K_1 L^2 - K_2 L^2 + PL$$

When $K_2 = 0$, say, then the solution is stable as long as P is not reversed. If $K_2 > K_1$, then a minimum value of P is required to maintain stability otherwise the bar will snap back to the original upright position.

For the second solution with $P = K_1 \cos\theta + K_2[1 - \cos\theta]$, we get

$$K_T = \frac{\partial^2 \Pi}{\partial \theta^2} = [K_2 - K_1]L^2 \sin^2\theta$$

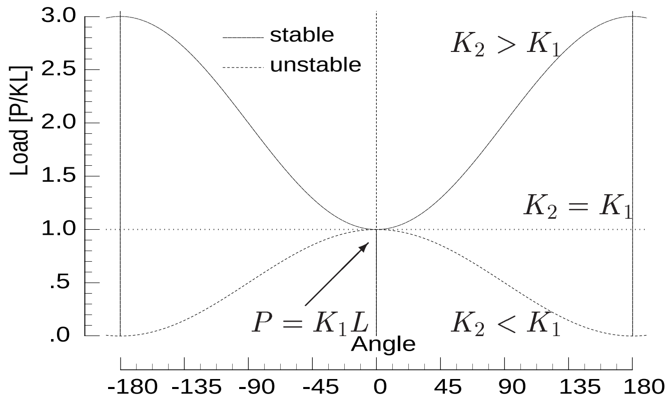


Figure 6.27: Bifurcation view, two equilibrium paths intersect.

Thus, so long as $K_2 > K_1$ the solution paths are stable. In particular, if $K_2 = 0$, the solution is always unstable, confirming what we saw in Figure 6.26(a). The complete picture is shown in Figure 6.27.

We now have the following bifurcation view of the loading process: as P is increased, the solution ($\theta = 0$) is stable until the critical point $P = K_1 L$ is reached; a further increment of load along this path makes the solution unstable — any disturbance will cause it to snap. Where it goes depends on the relative values of K_1 and K_2: for $K_2 > K_1$ there is a small increase in θ onto the stable path and the load can then continue to be increased. For $K_2 < K_1$, there is a very large increase in θ where the bar snaps to the inverted position. This snap, in reality, would involve a dynamic process.

IV: Dynamic View of Static Instabilities

It was difficult to discuss the previous notions of stability without simultaneously mentioning dynamics. After all, even our simple notion asks the question of what happens to the ensuing dynamics once the disturbance is applied.

A schematic of our dynamic view is illustrated in Figure 6.22. A stable loading state (resulting from the slowly applied load $P(t)$) is illustrated by the segment A; a disturbance (in the form of a short duration ping load $Q(t)$) causes oscillations about the equilibrium path; these are temporary and the structure eventually comes back to the equilibrium path. At, or beyond, a critical point, however, a small disturbance will cause a significant dynamic process to ensue. Depending on the particular problem, a nearby equilibrium path may or may not be found. It is worth noting that the new equilibrium path may not be statically connected to the original one; that is, we could not devise a proportional loading sequence (where the ratio of all the loads is kept constant) to connect the two equilibrium states.

The nonlinear governing system of Equation (6.11) is easily modified to account for inertia effects and leads to

$$[\,M\,]\{\ddot{u}\} + [\,C\,]\{\dot{u}\} = \{P\} - \{F(u)\}$$

where $[\,M\,]$ and $[\,C\,]$ are the mass and damping matrix respectively. Let us conceive of the total applied load and response as made up of two parts

$$\{P(t)\} = \{P_o(t)\} + \epsilon\{Q(t)\}\,, \qquad \{u(t)\} = \{u_o(t)\} + \epsilon\{\xi(t)\}$$

That is, there is the primary response $\{u\}_o$, which is due to $\{P_o\}$, and the smaller perturbation response $\{\xi\}$, which is due to the ping load $\{Q\}$. Expanding $F(u_o+\epsilon\xi)$ in a Taylor series for small ϵ

$$\{F(u_o+\epsilon\xi)\} \approx \{F(u_o)\} + [\frac{\partial F}{\partial u}]_o\{\epsilon\xi\} + \cdots = \{F(u_o)\} + \epsilon[K_T]_o\{\xi\} + \cdots$$

and setting the corresponding powers of ϵ to zero leads to the two equations

$$\begin{aligned} \epsilon^0 : \qquad & [\,M\,]\{\ddot{u}_o\} + [\,C\,]\{\dot{u}_o\} = \{P_o\} - \{F(u_o)\} \approx 0 \\ \epsilon^1 : \qquad & [\,M\,]\{\ddot{\xi}\} + [\,C\,]\{\dot{\xi}\} + [K_T]_o\{\xi\} = \{Q\} \end{aligned}$$

The first equation is for quasi-static primary loading. The second equation, which is often referred to as the variational equation [20], shows that the response due to the ping is that of a linear system with a constant stiffness $[K_T]_o$; how the stiffness changes during quasi-static loading is governed by the first equation.

As we showed for the total Lagrangian scheme, the incremental form of the quasi-static loading relation essentially becomes

$$[K_T]\{\Delta u\}_{t+\Delta t} = \{P\}_{t+\Delta t} - \{F\}_t$$

and so long as $[K_T]$ is nonsingular we can uniquely determine the deformation. As the deformation unfolds, however, the total stiffness can change, and in particular it can become singular — this is precisely the situation of interest here. After the ping is applied, the system is in free vibration governed by

$$[\,M\,]\{\ddot{\xi}\} + [\,C\,]\{\dot{\xi}\} + [K_T]_o\{\xi\} = \{0\}$$

We look for solutions of the form $\{\xi(t)\} = \{\phi\}e^{i\mu t}$. Substitute into the equation of motion to get

$$\Big[[K_T] + i\mu[\,C\,] - \mu^2[\,M\,]\Big]\{\phi\}e^{i\mu t} = 0$$

There can be nontrivial solutions only if the determinant is zero, which leads to a characteristic equation to determine the eigenvalues μ_i and eigenvectors $\{\phi\}_i$. The general solution is written as a combination of

$$\{\phi\}_1 e^{i\mu_1 t}, \quad \{\phi\}_2 e^{i\mu_2 t}, \quad \cdots, \quad \{\phi\}_N e^{i\mu_N t}$$

We state our stability criterion in terms of the properties of the eigenvalues μ_i. For the system to be asymptotically stable we want

$$\mathrm{Im}\,[\mu] > 0 \qquad \text{since} \qquad \{\xi\}e^{i\mu t} = \{\phi\}e^{i(\mu_R + i\mu_I)t} = \{\phi\}e^{-\mu_I t}e^{i\mu_R t}$$

Thus, a negative imaginary component of μ_i would give an exponentially increasing function of time. If the criterion is not true for any one of the roots, then the system is unstable.

The static instability criterion of Euler is essentially the case of $\mu_1 = 0$; that is, both the real and imaginary parts are zero simultaneously. There are structural problems, however, where this criterion is insufficient. For example, for follower-force type problems (and related problems such as aeroelastic flutter) instability occurs when the real part of μ_1 is still positive. Such situations are usually referred to as dynamic (or kinetic) instabilities [35].

Thus, a key ingredient of the dynamic approach is to monitor the spectral behavior of $[K_T]$. Since the total stiffness is available (when using the total Lagrangian scheme) an expedient method is to do an undamped vibration eigenanalysis — we will then refer to the eigenvalues as $\mu \to \lambda = \omega^2$, which are real only.

Agents and Imperfections

To summarize, motivated by dynamic heuristics, our view of static instability is that the structure is in a state of unstable equilibrium when the second derivative of the total potential energy is negative. The *agent* for the change in the above analysis was an applied load Q quite separate from the primary loading P, but note that we get a critical value even in the limit of $Q = 0$.

The agent Q is also referred to as an *imperfection* since it can be thought of as a slightly mis-applied P, and since slight geometric imperfections have the same result. The structure (and the solution) is called perfect when $Q = 0$. Imperfect structures exhibit limit type instabilities while the perfect structure can also exhibit bifurcations.

In the remainder of this chapter, we will concentrate on static instabilities and generally adopt the Euler approach of looking at equilibrium in the deformed configuration. We focus on monitoring the eigenvalues of the total stiffness matrix, our stability criterion being that if one of them goes to zero then the load is at a singular point. This rules out static problems that have a non-symmetric total stiffness — we leave some of these cases until the next chapter where a fully dynamic view is developed. The discussion of stability involves an analysis of the post-buckling behavior — this clearly needs our full nonlinear analysis techniques developed in the earlier chapters. In many situations it is sufficient just to know the first critical load and in the next few sections we develop methods for obtaining this information without doing a complete nonlinear post-buckling analysis. In the final sections, however, we navigate through a complete load/unload cycle for a structure undergoing large deformations and buckling.

Proportional Loading Along the Primary Path

With reference to Figure 6.22, we concentrate on the quasi-static primary loading path, and assume all inertia effects are negligible. The total potential energy of our general nonlinear system is

$$\Pi(u,\chi) = \mathcal{U}(u) - \chi\{u\}^T\{P\}$$

where $\mathcal{U}(u)$ is the strain energy that is a function only of the discrete displacement vector $\{u\}$, $\{P\}$ is a fixed load vector, and χ is a scalar load multiplier. Since the load vector never changes, we say that this loading is *proportional*, being controlled by the single parameter χ. This is shown schematically in Figure 6.28

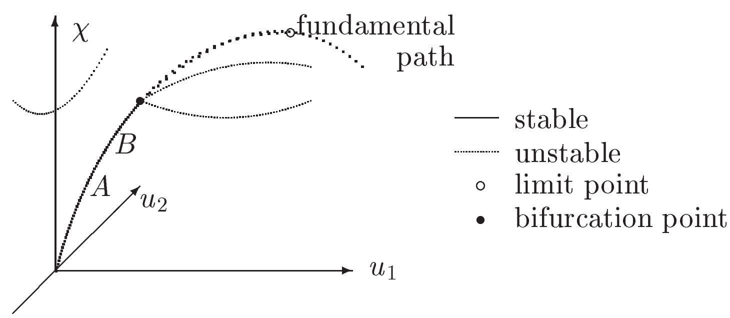


Figure 6.28: Examples of limit and bifurcation singular points occurring along a fundamental loading path.

First consider small changes in the total potential due to small changes in displacement with χ fixed

$$\delta\Pi(u,\chi) = \{\frac{\partial\Pi}{\partial u}\}\{\delta u\} + \tfrac{1}{2}\{\delta u\}^T[\frac{\partial^2\Pi}{\partial u\partial u}]\{\delta u\} + \cdots$$

We have that

$$\{\frac{\partial\Pi}{\partial u}\} = \{\frac{\partial\mathcal{U}}{\partial u}\} - \chi\{P\} \equiv \{\mathcal{F}\}(u,\chi)\,, \qquad [\frac{\partial^2\Pi}{\partial u\partial u}] = [\frac{\partial^2\mathcal{U}}{\partial u\partial u}] = [\frac{\partial\mathcal{F}}{\partial u}] \equiv [K_T]$$

where $[K_T]$ is the total stiffness. The variation of the potential is therefore given by

$$\delta\Pi(u,\chi) = \{\mathcal{F}\}^T\{\delta u\} + \tfrac{1}{2}\{\delta u\}^T[K_T]\{\delta u\} + \cdots$$

The energy changes should be stationary for an equilibrium loading path; that is, the first variation should be zero irrespective of $\{\delta u\}$. Hence, we have that

$$\{\frac{\partial\Pi}{\partial u}\} = \{\mathcal{F}\}(u,\chi) = \{\frac{\partial\mathcal{U}}{\partial u}\} - \chi\{P\} = \{F\} - \chi\{P\} = 0$$

This is the equation that defines the equilibrium path; this path can be viewed as a continuous curve in $(\{u\}^T,\chi)$ space. In this, only the nodal force vector $\{F\}$ is a function of the displacements.

Now consider a loading history along a sequence of equilibrium states. In particular, consider two equilibrium states, A and B, a small $\{\Delta u\}$ and $\Delta\chi$ apart, as shown in Figure 6.28. We have

$$\begin{aligned}\{\mathcal{F}\}(u,\chi)|_B &= \{\mathcal{F}\}(u_A+\Delta u,\chi_A+\Delta\chi)\\ &= \{\mathcal{F}\}(u,\chi)|_A+[\frac{\partial\mathcal{F}}{\partial u}]|_A\{\Delta u\}+\{\frac{\partial\mathcal{F}}{\partial\chi}\}|_A\Delta\chi+\cdots\end{aligned}$$

Since A and B are equilibrium states, then $\{\mathcal{F}\}|_A$ and $\{\mathcal{F}\}|_B$ are both zero giving

$$\{\frac{\partial\mathcal{F}}{\partial u}\}|_A\{\Delta u\}+[\frac{\partial\mathcal{F}}{\partial\chi}]|_A\Delta\chi=[\frac{\partial F}{\partial u}]|_A\{\Delta u\}-\Delta\chi\{P\}=[K_T]\{\Delta u\}-\Delta\chi\{P\}\approx 0$$

The approximation is because we are neglecting the higher-order terms. We will refer to this as the loading equation. Provided that $\det|[K_T]|\neq 0$, we get

$$\{\Delta u\}=\Delta\chi[K_T]^{-1}\{P\}$$

This is the standard tangential solution used in a nonlinear analysis and described earlier as part of the total Lagrangian scheme.

For stable equilibrium, the small changes of energy should be positive for any small perturbation $\{\delta u\}$ about the equilibrium point, hence we require that

$$\Pi(u+\delta u,\chi)-\Pi(u,\chi)>0 \qquad \text{or} \qquad \{\delta u\}^T[K_T]\{\delta u\}>0 \qquad \text{for all } \{\delta u\}$$

since $\{\mathcal{F}\}^T\{\delta u\}=0$ through equilibrium. For this to be true, we require that $[K_T]$ be positive definite. There are two situations of interest to us here. First, when

$$\{\delta u\}^T[K_T]\{\delta u\}<0 \qquad \text{for some } \{\delta u\}$$

then $[K_T]$ is not positive definite and it will have at least one negative eigenvalue. It is therefore unstable. The other case is when

$$\{\delta u\}^T[K_T]\{\delta u\}=0 \qquad \text{for some } \{\delta u\}$$

which is a neutral equilibrium state and $[K_T]$ has a zero eigenvalue. Consequently, $\det|[K_T]|=0$ and a full investigation of the nature of its equilibrium requires the use of higher-order terms in the expansion of the potential.

For the neutral equilibrium case, we cannot find a unique $\{\Delta u\}$ using the loading equation and we have a singular point. This singular point can be either a *limit point* or a *bifurcation point*. To see the distinction between these two, we must look at the spectral properties of the total stiffness and this is done in the examples to follow. Figure 6.28 shows examples of the two singular points. As shown in the figure, a 2-D plot of u_2 against load shows little or no motion until the singular point is reached, then it has two possible paths to take. In the simple example illustrated, the paths

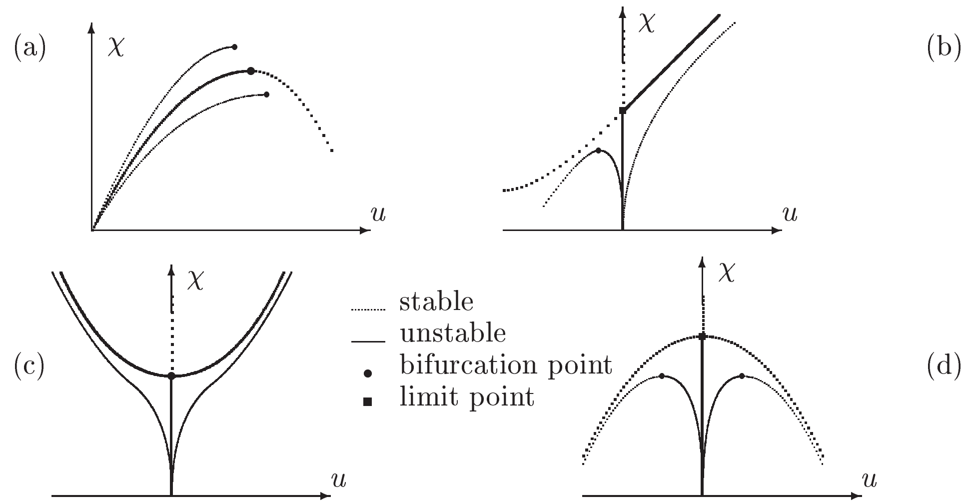


Figure 6.29: Classification of singular points and the effect of initial imperfections. (a) Limit point. (b) Asymmetric bifurcation. (c) Stable symmetric bifurcation. (d) Unstable symmetric bifurcation.

can be concave up (stable), concave down (unstable), or asymmetric (one stable, one unstable).

We are now in a position to classify each of the singular points. For a limit point, $\Delta\chi = 0$, giving two solutions symmetrically placed about the limit point. In either case, there is a direction in which negative energy results and hence a limit point is unstable.

For a bifurcation point, there are two cases that can arise. For an asymmetric bifurcation there are two solutions, one corresponding to the fundamental path and is unstable. For the other solution, the energy change is the same as for the limit point and hence an asymmetric bifurcation point is unstable.

For a symmetric bifurcation, again there are two solutions, The primary path is unstable. The bifurcated path may be stable or unstable.

Path Following Methods

Path-following schemes [7] are the computational implementation of the static methods; Reference [12] gives an excellent application and discussion (with many references) of current generalized path-following procedures. Severe difficulties can be encountered with limit points where the load-deflection curve becomes horizontal. The *arc-length* methods were introduced to overcome these difficulties. The essence of the arc-length method is that the load parameter becomes a variable just like the displacement variables; these $N + 1$ unknowns are solved by using N equilibrium equations and a constraint equation. Various forms of constraint equations can be used, a good discussion of some of the simpler ones is given in Reference [6] and a comprehensive survey is given in Reference [33].

Implementing the arc-length method requires a level of programming sophistication beyond the level directed by this book. As an alternative, the dynamic approach accepts that the structural behavior is dynamic in the vicinity of a critical point and follows the ensuing motion. The next section discusses this approach.

Example 6.14: Show the connection between a buckling eigenanalysis and a vibration eigenanalysis at the elevated load.

The vibration eigenvalue problem is

$$[K_T]\{\phi\}_i - \omega_i^2[\,M\,]\{\phi\}_i = 0$$

At the static singular point, $\omega_1 = 0$, we have

$$[K_T]\{\phi\}_1 = 0$$

Now consider the case when the structural response is only slightly nonlinear, then we can represent the total stiffness in the form

$$[K_T] \approx [K_E] + \chi[K_G]$$

where $[K_E]$ is the elastic stiffness, $[K_G]$ is the geometric stiffness, and χ is a loading factor. At the singular point, we therefore have

$$[K_T]\{\phi\}_1 = [K_E + \chi K_G]\{\phi\}_1 = 0$$

This is the eigenvalue problem for the buckling of the structure, and we conclude that the vibration mode shape is the same as the first buckling mode shape. This result will be demonstrated later for plates. ■

Example 6.15: Use a vibration analysis to distinguish between limit and bifurcation singular points.

Consider the free undamped vibration of the system when loaded near a critical point; that is, let $\{\Delta u\} = \{\,\xi\,\}e^{i\mu t}$ leading to

$$[K_T]\{\,\xi\,\} - \mu^2[\,M\,]\{\,\xi\,\} = 0$$

It is therefore sufficient for us to introduce the eigenvalues λ_m and eigenvectors $\{\phi\}_m$ of the total stiffness such that

$$[K_T]\{\phi\}_m = \lambda_m[\,M\,]\{\phi\}_m\,, \qquad \lambda_m = \mu_m^2$$

and let the eigenvectors be normalized such that

$$\{\phi\}_i^T[\,M\,]\{\phi\}_j = \delta_{ij}\,, \qquad \{\phi\}_m^T[K_T]\{\phi\}_m = \lambda_m$$

When the solution follows the path from a stable state, the lowest eigenvalue, λ_1, is zero at the singular point. Hence, we have that

$$[K_T]\{\phi\}_1 = 0$$

Now multiply the transpose of the loading equation by $\{\phi\}_1$ to get

$$\Big\{\{\Delta u\}^T[K_T] - \Delta\chi\{P\}^T\Big\}\{\phi\}_1 = 0 \qquad \text{giving} \qquad \Delta\chi\{P\}^T\{\phi\}_1 = 0$$

This relation is used to distinguish between limit and bifurcation points as follows:

$$\begin{aligned} \text{limit point:} \qquad & \Delta\chi = 0\,, \qquad \{P\}^T\{\phi\}_1 \neq 0 \\ \text{bifurcation point:} \qquad & \Delta\chi \neq 0\,, \qquad \{P\}^T\{\phi\}_1 = 0 \end{aligned} \tag{6.17}$$

Figure 6.28 shows examples of the two singular points. As shown in the figure, a 2-D plot of u_2 against load shows little or no motion until the singular point is reached, then it has two possible paths to take. In the simple example illustrated, the paths can be concave up (stable), concave down (unstable), or asymmetric (one stable, one unstable).

■

Example 6.16: Use a modal analysis to discuss the displacement increment shape at singular points.

We can get further insight into these singular points by using a modal representation of the displacement increment in terms of the eigenvectors

$$\{\Delta u\} = \eta_1\{\phi\}_1 + \eta_2\{\phi\}_2 + \cdots = \sum \eta_m\{\phi\}_m$$

Now substitute this into the loading equation, multiply the resulting equation by $\{\phi\}_m$, then making use of the orthogonality properties leads to

$$\eta_m\lambda_m - \Delta\chi\{P\}^T\{\phi\}_m = 0 \qquad \text{or} \qquad \eta_m = \frac{1}{\lambda_m}\Delta\chi\{P\}^T\{\phi\}_m$$

We therefore have the general modal representation for the displacement increment

$$\{\Delta u\} = \eta_1\{\phi\}_1 + \Delta\chi\sum_{m=2}\frac{1}{\lambda_m}\Big[\{P\}^T\{\phi\}_m\Big]\{\phi\}_m = \eta_1\{\phi\}_1 + \Delta\chi\{v\}$$

Note that $\{\phi\}_1^T\{v\} = 0$. For the limit point where $\Delta\chi = 0$ and assuming $\lambda_m \neq 0$, we get

$$\{\Delta u\} = \eta_1\{\phi\}_1$$

The displacement increment has the shape of the first eigenmode. The displacement increment for the bifurcation point, on the other hand, depends on all the modes.

■

Exercises

6.1 The total potential of a certain system is

$$\Pi = \tfrac{1}{5}x^5 - \tfrac{1}{4}ax^4 - \tfrac{2}{3}ax^3 + a^2$$

where a is a parameter and x is the generalized coordinate. Determine all of the equilibrium configurations and indicate which ones are stable and unstable.

6.2 If in the derivation of the beam geometric stiffness matrix, we use the rod shape functions instead of the beam shape functions, i.e.,

$$v(x) = (1 - \frac{x}{L})v_1 + (\frac{x}{L})v_2 = f_1(x)v_1 + f_2(x)v_2$$

Show that the derived inconsistent geometric stiffness matrix is given by

$$[\,k_G\,] = \frac{F_o}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Where might such an element be more useful than its consistent counterpart?

6.3 Determine a constraint element by adding

$$\tfrac{1}{2}k[\alpha_i u_i + \alpha_j u_j - \alpha_o]^2$$

to the potential energy term. Generalize the results for more than two constraints.

6.4 A particle in the (x, y) plane moves in the force field $F_x = -ky$, $F_y = kx$ (k is a constant). Prove that when the particle describes any closed path in the counterclockwise sense, the work performed on the particle is $2kA$ where A is the area enclosed by the path.

6.5 Suppose the rotation distribution in a circular shaft can be written in terms of the nodal rotations as

$$\phi(x) = (1 - \frac{x}{L})\phi_1 + (\frac{x}{L})\phi_2 \equiv f_1(x)\phi_1 + f_2(x)\phi_2$$

Use minimum potential energy to derive the torsion element.

6.6 For a uniform column with clamped ends, assume $v = ax^2(L - x)^2$ and determine the critical load. $[P_{cr} = 42EI/L^2]$

6.7 For a uniform column with clamped at one end and free at the other, assume $v = ax^2$ and determine the critical load. $[P_{cr} = 2.5EI/L^2]$

6.8 For a uniform column with clamped at one end and free at the other, assume the mode shape is the static deflection shape due to a point load at the tip and determine the critical load. $[P_{cr} = 42EI/17L^2]$

Chapter 7

Variational Methods

Variational methods can be used as an alternative statement of the basic equations of mechanics. Consequently, they can be used as a means of deriving the basic governing equations of the theory of elasticity and this will be demonstrated by way of the Calculus of Variations. Perhaps more importantly, however, they can also be used to obtain approximate solutions; in particular they form the basis for constructing rational approximate structural theories.

The concepts to be discussed have an intimate relationship to the principle of virtual derived in Chapter 6. Here the emphasis is on the calculus of variations as a mathematical tool to formalize the calculus for the variations associated with virtual work and the stationary principles. While the principles are not necessarily mechanics based, it must be kept in mind, however, that their application to solids and structures can only be justified by the same mechanics reasoning behind virtual work.

7.1 Calculus of Variations

We are familiar with finding the extreme values of a function $u(x)$. We first set

$$\frac{du}{dx} = 0$$

and then solve the equation to obtain $x = x_1, x_2, \cdots$ at which the function $u(x)$ assumes extrema. We shall be concerned with the calculation of the extreme values of functions defined by certain integrals whose integrands contain one or several functions assuming the roles of arguments. We shall use the term *functional* to refer to functions defined by integrals whose arguments themselves are functions.

Single Integrals

Consider the integral

$$J[\,u\,] = \int_{x_o}^{x_1} F(x, u, u')dx \qquad \text{e.g.,} \quad F = \tfrac{1}{2}\alpha[u']^2 + \beta u - q(x)$$

where F is a known, real function of the real arguments $x, u, u' \equiv du/dx$, and α, β are parameters of the problem. The value of the integral depends on the choice of $u = u(x)$, hence, the notation $J[\,u\,]$. To make the symbol $J[\,u\,]$ meaningful, it is clearly necessary to impose some restrictions on the choice of the argument $u(x)$ and on the prescribed function F appearing in the integrand. We shall suppose that the admissible arguments belong to a class C^2 (smooth in the second derivatives) and assume at the end of the interval (x_o, x_1) the specified values are u_o and u_1. Thus,

$$u(x_o) = u_o\,, \qquad u(x_1) = u_1$$

where u_o and u_1 are prescribed in advance. The entire set of admissible arguments $u(x)$ can thus be viewed as a family of smooth curves passing through (x_o, u_o) and (x_1, u_1).

For a given curve $u = u(x)$ of the set, the integral yields a definite numerical value $J[\,u\,]$, and we pose a problem of determining that particular curve $u(x)$ in the competing set which makes the integral a minimum. If $u(x)$ minimizes this integral, then every function $\bar{u}(x)$ in the neighborhood of $u(x)$ can be represented in the form

$$\bar{u} = u(x) + \epsilon\eta(x) \qquad\qquad \eta = \frac{d\bar{u}}{d\epsilon}$$

where ϵ is a small real parameter. We note that the function $u(x)$ is determined with $\epsilon = 0$. We shall call the difference $\bar{u}(x) - u(x) = \epsilon\eta(x)$ the *variation* of $u(x)$ and write

$$\delta u \equiv \epsilon\eta(x) = \epsilon\frac{d\bar{u}}{d\epsilon}$$

Moreover, every function in the set $\{u(x)\}$ satisfies the end conditions and we must have $\eta(x_o) = 0$ and $\eta(x_1) = 0$ at the ends.

Since $u(x)$ minimizes the integral, then

$$J[\,\bar{u}\,] = J[u + \epsilon\eta] \geq J[\,u\,]$$

The left-hand member in this inequality is a continuously differentiable function of ϵ, and therefore, a necessary condition that $u(x)$ minimize the integral is

$$\frac{dJ[u + \epsilon\eta]}{d\epsilon}|_{\epsilon=0} = 0$$

That is,

$$\frac{dJ[u + \epsilon\eta]}{d\epsilon}|_{\epsilon=0} = \frac{d}{d\epsilon}\int_{x_o}^{x_1} F(x,\, u + \epsilon\eta,\, u' + \epsilon\eta')dx = \int_{x_o}^{x_1}(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u'}\eta')dx = 0$$

Denote the variation of J as

$$\delta J \equiv \frac{dJ[u + \epsilon\eta]}{d\epsilon}|_{\epsilon=0}$$

then the above condition can be denoted as $\delta J = 0$. In anticipation of later developments, we now introduce a subscript notation to indicate the partial derivatives. We have, for example,

$$F_{,u} \equiv \frac{\partial F}{\partial u}, \qquad F_{,u'} \equiv \frac{\partial F}{\partial u'}$$

With this notation, we rewrite the derivative as

$$\frac{dJ[u+\epsilon\eta]}{d\epsilon}\Big|_{\epsilon=0} = \int_{x_o}^{x_1} (F_{,u}\,\eta + F_{,u'}\,\eta')dx = 0$$

Integrate the second term in this by parts to get

$$\int_{x_o}^{x_1} F_{,u'}\,\eta dx = F_{,u'}\,\eta\Big|_{x_o}^{x_1} - \int_{x_o}^{x_1} \frac{dF_{,u'}}{dx}\eta\,dx$$

and since $\eta(x_o) = \eta(x_1) = 0$, we can write it in the form

$$\int_{x_o}^{x_1} \left(F_{,u} - \frac{dF_{,u'}}{dx}\right)\eta(x)\,dx = 0$$

This integral must vanish for every $\eta(x)$, and we conclude therefore that

$$F_{,u} - \frac{dF_{,u'}}{dx} = 0$$

is a necessary condition that the integral be minimized by $u = u(x)$. On expanding it we get the second order differential equation

$$F_{,u'u'}\,\frac{d^2u}{dx^2} + F_{,u'u}\,\frac{du}{dx} + F_{,u'x} - F_{,u} = 0$$

This is usually called the Euler equation associated with the minimizing of the integral $J[\,u\,] = \int_{x_o}^{x_1} F(x,u,u')dx$.

Similar calculations performed on the functional

$$J[\,u\,] = \int_{x_o}^{x_1} F(x,u,u',u'',\cdots,u^{(n)})dx$$

yield the Euler equation

$$F_{,u} - \frac{d}{dx}F_{,u'} + \frac{d^2}{dx^2}F_{,u''} \quad \cdots \quad (-1)^n\frac{d^nF_{,u^{(n)}}}{dx^n} = 0$$

It is noted that a functional involving n derivatives of u results in an ordinary differential equation involving n derivatives of the function F.

Double Integrals

We now consider the problem of minimizing the double integral

$$J[\,u\,] = \int_R\!\!\int F(x,y,u,u_{,x}\,,u_{,y}\,)dxdy\,, \qquad u_{,x} \equiv \frac{\partial u}{\partial x}\,, \quad u_{,y} \equiv \frac{\partial u}{\partial y}$$

on the set $\{u(x,y)\}$ of functions of class C^2 where each $u(x,y)$ in the set takes specified continuous values, $u = \phi(s)$, on the boundary S of the region R.

Let us suppose that a certain function $u(x,y)$ in this set is included in the formula

$$\bar{u}(x,y) = u(x,y) + \epsilon\eta(x,y)\,, \qquad \delta u \equiv \epsilon\eta$$

where ϵ is a small parameter. Since $\bar{u} = \phi(s)$ on the boundary of R, $\eta(x,y) = 0$ on S. We form the integral $J[u+\epsilon\eta]$ and observe that

$$\frac{dJ[u+\epsilon\eta]}{d\epsilon}|_{\epsilon=0} = 0 \quad \text{or} \quad \delta J = 0$$

since $u(x,y)$ minimizes the functional. But,

$$J[u+\epsilon\eta] = \int_R\!\!\int F(x,y,u+\epsilon\eta,u_{,x}+\epsilon\eta_{,x}\,,u_{,y}+\epsilon\eta_{,y}\,)dxdy$$

so that

$$\delta J = \epsilon\int_R\!\!\int (F_{,u}\,\eta + F_{,u_{,x}}\,\eta_{,x} + F_{,u_{,y}}\,\eta_{,y}\,)dxdy = \epsilon\int_R\!\!\int (F_{,u}\,\delta u + F_{,u_{,x}}\,\delta u_{,x} + F_{,u_{,y}}\,\delta u_{,y}\,)dxdy$$

Notice that we do not take variations with respect to x or y. We rewrite this as

$$\delta J \;=\; \epsilon\int_R\!\!\int \left(F_{,u} - \frac{\partial F_{,u_{,x}}}{\partial x} - \frac{\partial F_{,u_{,y}}}{\partial y}\right)\eta\,dxdy + \epsilon\int_R\!\!\int \left[\frac{\partial}{\partial x}(F_{,u_{,x}}\,\eta) + \frac{\partial}{\partial y}(F_{,u_{,y}}\,\eta)\right]dxdy$$

Recall the Green's Theorem in a plane from Section 1.3,

$$\int_R\!\!\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_S (Pdx + Qdy)$$

Apply this to the second integral to obtain

$$\delta J = \int_R\!\!\int \left(F_{,u} - \frac{\partial F_{,u_{,x}}}{\partial x} - \frac{\partial F_{,u_{,y}}}{\partial y}\right)\delta u\,dxdy + \epsilon\int_S (F_{,u_{,x}}\,dy - F_{,u_{,y}}\,dx)\eta$$

But $\eta = 0$ on S, and since δJ vanishes for an arbitrary choice of η in R, we conclude that

$$F_{,u} - \frac{\partial F_{,u_{,x}}}{\partial x} - \frac{\partial F_{,u_{,y}}}{\partial y} = 0 \qquad \text{in R}$$

for the minimizing function $u(x,y)$.

A calculation similar in every respect to the foregoing for the functional

$$J[\,u\,] = \int\!\!\int_R F(x, y, u, u_{,x}\,, u_{,y}\,, u_{,xx}\,, u_{,yy}\,)\,dxdy$$

in which the admissible u assume specified continuous values on the boundary S of R, leads to the Euler Equation:

$$F_{,u} - \frac{\partial}{\partial x}(F_{,u_{,x}}\,) - \frac{\partial}{\partial y}(F_{,u_{,y}}\,) + \frac{\partial^2}{\partial x^2}(F_{,u_{,xx}}\,) + \frac{\partial^2}{\partial x \partial y}(F_{,u_{,xy}}\,) + \frac{\partial^2}{\partial y^2}(F_{,u_{,yy}}\,) = 0$$

These examples show that we can associate a differential formulation with a variational formulation and *vice versa*.

Example 7.1: Consider the following functional and boundary conditions

$$J[\,u\,] = \int\!\!\int_R \Big[(u_{,x}\,)^2 + (u_{,y}\,)^2 + 2f(x, y)u\Big]\,dxdy\,, \qquad u = \phi(s) \quad \text{on} \quad S$$

Determine the corresponding differential equation.

The Euler equation corresponding to $J[\ u\ ]u) = \min$ is

$$2f - 2u_{,xx} - 2u_{,yy} = 0$$

This can be written more familiarly as

$$\nabla^2 u = f(x, y) \qquad \text{in R}$$

This is Laplace's equation. ■

Example 7.2: Consider the inhomogeneous bi-harmonic equation given by

$$\nabla^4 \phi = c = \text{constant}$$

with the homogeneous boundary conditions on the rectangle

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{on } x = \pm a$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{on } y = \pm b$$

What is the corresponding variational statement of the problem?

The differential equation for ϕ can be identified as the Euler equation for the variational problem

$$J[\,\phi\,] = \int_{-b}^{b}\int_{-a}^{a} \Big[(\nabla^2\phi)^2 - 2c\phi\Big]\,dxdy = \ \min$$

These two examples show the connection between satisfying differential equations and minimizing a functional. ■

The Variational Operator

We now introduce the notation of 'variations' in order to show the analogy between the calculus of variations and differential calculus.

We drop the $\epsilon\eta$ notation and replace it with the formal operator δ where

$$\delta u \equiv \epsilon\eta$$

We say that δu is the variation of u. Corresponding to the variation in u, we have the first variation of the functional $F(x, u, u', \cdots)$

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \cdots$$

In the more general case of a functional $F(x, y, u, v, u_{,x}, v_{,y}, \cdots)$, we have

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial u_{,x}}\delta u_{,x} + \frac{\partial F}{\partial v_{,y}}\delta v_{,y} + \cdots$$

Note that we do not vary x or y, that is, $\delta x = 0$ and $\delta y = 0$. With this in mind, the analogy with the definition of the differential is complete. That is, the differential of a function is a first order approximation to the change in that function along a particular line, while the variation of a functional is a first order approximation to the change from curve to curve.

The laws of variations of sums, products, powers and so forth, follow those of the differential. Thus,

$$\delta(uv) = \delta u\, v + u\, \delta v\,, \qquad \delta(\frac{u}{v}) = \frac{\delta u\, v - u\, \delta v}{v^2}$$

The operators δ, ∂, and d are commutative, that is

$$\frac{d}{dx}\delta u = \delta\frac{du}{dx}\,, \quad \frac{\partial}{\partial u}\delta F = \delta\frac{\partial F}{\partial u}$$

We also have that

$$\delta J = \delta \int_1^2 F\, dx = \int_1^2 \delta F\, dx$$

since the limits of the integral do not vary.

Boundary Conditions

Up to now, we have considered the boundary conditions where u is specified and thus $\delta u = 0$. We now generalize this to encompass a wider range of problems.

Consider the 1-D functional equation

$$J[\,u\,] = \int_{x_1}^{x_2} F(x; u, ..)\, dx - \lambda(u_2 - u_1) = \min$$

where λ is a parameter, and u_1, u_2 are the end values of $u(x)$. To minimize this, we apply the variational operator to get

$$\delta J = \int_{x_1}^{x_2} \delta F \, dx - \lambda(\delta u_2 - \delta u_1) = 0$$

We get for terms inside the integral

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' + \cdots$$

Integrating the second term by parts gives

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\delta u' \, dx = \int_{x_1}^{x_2} \frac{\partial F}{\partial u'}\frac{d\delta u}{dx} dx = [\frac{\partial F}{\partial u'}\delta u]\Big|_{x1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}(\frac{\partial F}{\partial u'})\delta u \, dx$$

We must integrate the third term by parts twice. This results in

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u''}\delta u'' \, dx = [\frac{\partial F}{\partial u''}\delta u']_{x1}^{x_2} - [\frac{d}{dx}(\frac{\partial F}{\partial u''})\delta u]\Big|_{x1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2}(\frac{\partial F}{\partial u'})\delta u \, dx$$

The variational terms can be collected as

$$\delta J = \int_{x_1}^{x_2} \Big[\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial u'}\Big]\delta u \, dx + \Big[\frac{\partial F}{\partial u'} - \frac{d}{dx}\frac{\partial F}{\partial u''} - \lambda\Big]\delta u\Big|_{x1}^{x_2} + \Big[\frac{\partial F}{\partial u''}\Big]\delta u'\Big|_{x1}^{x_2} = 0$$

All three groups of terms must vanish. The first group gives rise to the governing differential equation, or the Euler equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial u'} = 0$$

The remaining terms give rise to the boundary conditions. Since they are associated with different variations of u, then we have separately

$$\Big[\frac{\partial F}{\partial u'} - \frac{d}{dx}\frac{\partial F}{\partial u''} - \lambda\Big]\delta u\Big|_{x1}^{x_2} = 0\,, \qquad \Big[\frac{\partial F}{\partial u''}\Big]\delta u'\Big|_{x1}^{x_2} = 0$$

Each of these represent two distinct types of boundary conditions. The first of these, for example, says at $x = x_1$ that

$$\delta u = 0,\ u = \text{specified} \qquad \text{or} \qquad \frac{\partial F}{\partial u'} - \frac{d}{dx}\frac{\partial F}{\partial u''} - \lambda = 0$$

The first is called a *geometric* boundary condition, while the second is called a *natural* boundary condition, respectively.

When we apply the stationary principles, we need to identify two classes of boundary conditions, called *essential* and *natural* boundary conditions. The essential boundary conditions are also called *geometric* boundary conditions because they correspond to prescribed displacements. The natural boundary conditions are sometimes called the *force* boundary conditions because they correspond to prescribed boundary forces in structural mechanics problems.

Example 7.3: Consider the function

$$F = \tfrac{1}{2}EA[u']^2 - q(x)$$

where EA is possibly a function of x but not u. Determine the governing differential equation and the corresponding boundary conditions.

The various derivatives are

$$\frac{\partial F}{\partial u} = -q\,, \qquad \frac{\partial F}{\partial u'} = EAu'\,, \qquad \frac{\partial F}{\partial u''} = 0$$

The Euler equation is

$$-\frac{d}{dx}[EA\frac{du}{dx}] - q = 0$$

with the associated boundary conditions

$$u = \text{specified} \qquad \text{or} \qquad EA\frac{du}{dx} - \lambda = 0$$

This corresponds to the axial loading of a rod of variable stiffness EA, and we recognize λ as the applied load.

The variational formulation can be written as

$$J[\,u\,] = \int_{x_1}^{x_2} \Big[\tfrac{1}{2}EA[u']^2 - q(x)\Big]\,dx - \lambda u\Big|_{x_1}^{x_2} = \min$$

As we will see next, the terms $\int_{x_1}^{x_2} q(x)\,dx + \lambda u|_{x_1}^{x_2}$ will have the interpretation of the potential of applied loads. ■

7.2 Direct Methods of Solution

Classical methods in the calculus of variations reduce the fundamental question of the existence of a solution for an arbitrary problem to the question of the existence of solutions of differential equations (the Euler equations). This approach is not always effective, and is greatly complicated by the fact that what is needed to solve a given variational problem is not a solution of the corresponding differential equation in a small neighborhood of some point, but rather a solution in some fixed region, which satisfies prescribed boundary conditions on the boundary of R. The difficulties inherent in this approach (especially when several independent variables are involved) have led to a search for variational methods of a different kind, known as *direct methods*, which do not entail the reduction of variational problems to problems involving differential equations.

Once direct variational methods have been developed, they can be used to solve differential equations, and this technique, the inverse of the one we have discussed, plays an important role in the modern theory of the subject. The basic idea is the following: Suppose it can be shown that a given differential equation is the Euler equation of some functional, and suppose it has been proven somehow that this functional has an extremum for sufficiently smooth admissible functions, then this very fact proves that the differential equation has a solution satisfying the boundary conditions corresponding to the given variational problem. Moreover, variational methods can be used not only to prove the existence of a solution of the original differential equation, but also to calculate a solution to any desired accuracy.

Ritz Method

The Ritz (or Rayleigh-Ritz) method provides a powerful way to obtain approximate solutions directly from the variational problem. That is, we will seek

$$J[\,u\,] = \text{ min} \qquad \text{or} \qquad \delta J = 0$$

without relying on solving the corresponding Euler equations. In elasticity problems, the energy principles conveniently set up the variational problems ready for application of the Ritz method. However, this method can be also used to solve differential equations if they can be identified as the Euler equations of a variational problem.

In general, a continuously distributed deformable body consists of an infinity of material points and therefore has infinitely many degrees of freedom. The Ritz method is an approximate procedure by which continuous systems are reduced to systems with finite degrees of freedom. The fundamental characteristic of the method is that we operate on the functional corresponding to the problem, either J or the total potential energy Π. Suppose we are looking for the solution for $\delta J = 0$ with prescribed boundary conditions on u. Let

$$u(x, y, z) = \textstyle\sum_{i=1}^{\infty} a_i \phi_i(x, y, z)$$

where ϕ_i are independent expansion or *trial functions*, and the a_i are multipliers to be determined in the solution. The trial functions satisfy the essential (geometric) boundary conditions but not necessarily the natural boundary conditions. The variational problem states that

$$J[\,u\,] = J[a_1\phi_1 + a_2\phi_2 + \cdots] = \text{ min}$$

Thus $J[a_1\phi_1 + a_2\phi_2 + \cdots]$ can be regarded as a function of the variables $a_1, a_2, \cdots$. To satisfy $J = \text{min.}$, we require that

$$\frac{\partial J}{\partial a_1} = 0\,, \qquad \frac{\partial J}{\partial a_2} = 0\,, \qquad \cdots$$

These equations are then used to determine the coefficients a_i. Normally, we only include a finite number of terms in the expansion.

An important consideration is the selection of the trial functions ϕ_i. Selecting efficient admissible functions may not be easy; fortunately, many problems closely resemble other problems that have been solved before, and the literature is full of examples that can serve as a guide. It must also be kept in mind that these functions need only satisfy the essential boundary conditions and not (necessarily) the natural boundary conditions. For practical analyses, this is a significant point and largely accounts for the effectiveness of the displacement-based finite element analysis procedure as will be shown later in this chapter.

For convenience in satisfying the boundary conditions on u, we usually set

$$u = u_o + \sum\nolimits_n a_n \phi_n$$

where u_o conforms to the non-homogeneous boundary conditions. For homogeneous displacement boundary conditions, we set $u_o = 0$.

Example 7.4: Consider a bar fixed at one end and subjected to an axial concentrated force at the other end, as shown in Figure 7.1. The variation of Young's modulus is $E(x) = E_o(1 + x/L)^2$. Obtain a Ritz approximate solution.

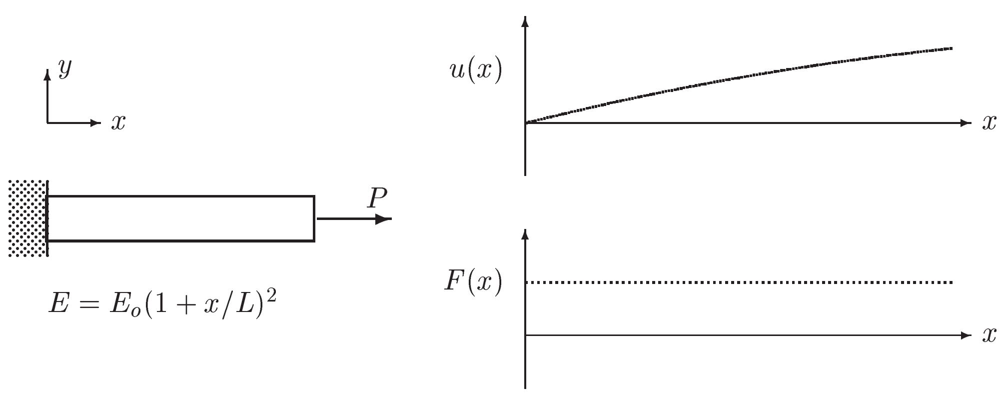


Figure 7.1: Bar with variable modulus.

The boundary conditions for this problem are:

$$\text{essential:} \quad u|_{x=0} = 0 \qquad \text{natural:} \quad EA\frac{du}{dx}|_{x=L} = P$$

The exact solution is easily calculated to give

$$u(x) = \int_0^x \epsilon(x)dx = \frac{PL}{E_oA}\frac{x/L}{(1 + x/L)}\,, \qquad F(x) = EA\frac{du}{dx} = P = \text{constant}$$

Both the displacement distribution and force distribution are shown plotted in Figure 7.1. We will use these results to evaluate the quality of the Ritz approximate solutions. Specifically, we wish to investigate the use of different trial functions.

Since the deformation is one-dimensional, then the strain is

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du}{dx}$$

and the total potential energy of the body is

$$\Pi = \tfrac{1}{2} \int_0^L EA \left(\frac{du}{dx} \right)^2 dx - P u_L$$

The integration over the cross-sectional area has already been performed. We will calculate the displacement and force distributions using the following assumed form for the displacement:

$$u(x) = a_0 + a_1 x + a_2 x^2$$

This must satisfy the essential boundary condition, hence $a_0 = 0$. Note that the remaining polynomial does not necessarily satisfy the natural boundary condition. Substituting the assumed displacements into the total potential energy expression, we obtain

$$\Pi = \tfrac{1}{2} \int_0^L E_o A (1 + x/L)^2 (a_1 + 2a_2 x)^2 dx - P(a_1 L + a_2 L^2)$$

Invoking the stationarity of Π with respect to the coefficients a_n, we obtain the following equations for a_1 and a_2

$$\begin{aligned} \frac{\partial \Pi}{\partial a_1} &= \int_0^L E_o A (1 + x/L)^2 (a_1 + 2a_2 x) dx - PL = 0 \\ \frac{\partial \Pi}{\partial a_2} &= \int_0^L E_o A (1 + x/L)^2 (a_1 + 2a_2 x) 2x \, dx - PL^2 = 0 \end{aligned}$$

Performing the required integrations gives

$$\frac{E_o A}{30} \begin{bmatrix} 70L & 85L^2 \\ 85L^2 & 124L^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} PL \\ PL^2 \end{Bmatrix}$$

Note that this is symmetric. Solving this system gives for the two coefficients

$$a_1 = \frac{78}{97} \frac{P}{E_o A}, \qquad a_2 = \frac{-30}{97} \frac{P}{E_o A L}$$

This Ritz analysis, therefore, yields the approximate solution

$$u(x) = \frac{78P}{97 E_o A} [x - \frac{10}{26L} x^2]$$

and the force distribution is

$$F(x) = EA \frac{du}{dx} = \frac{78P}{97} [1 - \frac{10}{13L} x](1 + x/L)^2$$

These results are shown in Table 7.1 as the 2-term columns. The most striking fact of these results is the accuracy of the displacements and yet the axial force is

	displ				force			
x/L	exact	1 term	2 term	bi-linear	exact	1 term	2 term	bi-linear
0.0	0.0	0.0	0.0	0.0	1.0	0.428	0.804	0.6316
0.5	0.3333	.2143	.3247	.3158	1.0	0.96	1.113	1.421
0.5	0.3333	.2143	.3247	.3158	1.0	0.96	1.113	0.729
1.0	0.5	.4285	.4948	.4779	1.0	1.714	0.740	1.297

Table 7.1: Displacement and force results for the non-uniform rod.

not constant and equal to P. This reiterates the fact that the Ritz approach only approximates equilibrium.

An interesting result is obtained if we use only a linear expansion for the displacements. In this circumstance, after imposing the essential boundary condition, we have the one term expansion

$$u(x) = a_1 x$$

Substituting this into the potential energy expression and minimizing, we obtain

$$\frac{\partial \Pi}{\partial a_1} = \int_0^L E_o A(1 + x/L)^2 (a_1) dx - PL = 0$$

Performing the required integration gives

$$\frac{E_o A}{30}[70L]a_1 = PL$$

We recognize this as precisely the first term in the above $[2 \times 2]$ matrix form. That is, as we increase the expansion of $u(x)$, then each additional term adds a row and column to the matrices but otherwise the existing matrices are unaffected.

Solving for a_1 gives

$$a_1 = \frac{3}{7}\frac{P}{E_o A}$$

The approximate solution for the displacement and force are, respectively,

$$u(x) = \frac{3P}{7E_o A}[x], \qquad F(x) = \frac{3P}{7}[1](1 + x/L)^2$$

These results are also shown in Table 7.1 as the 1-term columns. Note that this force distribution does not satisfy the differential equation of equilibrium. ■

Example 7.5: Use the Ritz method to find an approximate solution for a rectangular body fixed at one end and loaded with a concentrated force P at $x = L$ as shown in Figure 7.2.

The solution given by the elementary beam theory is

$$v(x) = \frac{PL^3}{6EI}\Big[3(\frac{x}{L})^2 - (\frac{x}{L})^3\Big], \qquad u(x) = -y\phi(x) = -\frac{PL^2}{2EI}\Big[2(\frac{x}{L}) - (\frac{x}{L})^2\Big]y$$

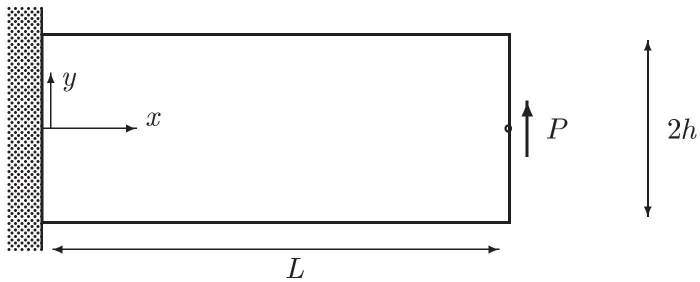


Figure 7.2: Cantilevered rectangular body.

The tip deflection is

$$v_{tip} = \frac{PL^3}{6Eb(2h)^3/12}[3-1] = \frac{PL^3}{2Ebh^3}$$

We will use this to motivate the Ritz functions as well as for comparison.

Consider the problem as a plane problem such that either $\epsilon_{zz} = 0$ (plane strain) or $\sigma_{zz} = 0$ (plane stress). In either case, the strain energy expression reduces to

$$\mathcal{U} = \tfrac{1}{2}\int [(2\mu(1+\bar{\nu})(\epsilon_{xx}^2 + \epsilon_{yy}^2) + 2\mu\bar{\nu}\epsilon_{xx}\epsilon_{yy} + 4\mu\epsilon_{xy}^2]\, dV$$

where $\bar{\nu}$ is chosen appropriately as given in Section 5.2. Motivated by the elementary solution, let the displacements be approximated as

$$u(x,y) \approx a_1[2Lx - x^2]y\,, \qquad v(x,y) \approx a_2[3Lx^2 - x^3]$$

This expansion satisfies the geometric boundary conditions of all displacements being zero at $x = 0$. There are of course many displacement fields that can satisfy these conditions.

The strains associated with this displacement field are

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = 2a_1[L-x]y\,, \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} = 0\,, \qquad 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (a_1 + a_2 3)[2Lx - x^2]$$

Note that these strains are automatically compatible, but the stresses associated with them are not necessarily in equilibrium. That is, we must determine a_n so as to obtain a 'good' solution. The strain energy is calculated as

$$\begin{aligned} \mathcal{U} &= \tfrac{1}{2}\int_{-h}^{+h}\int_{o}^{L} \Big[2\mu(1+\bar{\nu})4a_1^2[L^2 - 2Lx + x^2]y^2 + 0 + \mu(a_1+3a_2)^2[4L^2x^2 - 4Lx^3 + x^4]\Big]\, dx\, dy\, dz \\ &= \tfrac{1}{2}\left[2\mu(1+\bar{\nu})\frac{8}{9}a_1^2L^3h^3b + \mu(a_1+3a_2)^2\frac{16}{15}L^5hb\right] \end{aligned}$$

The potential of the external load is given by

$$\mathcal{V} = -Pv(x=L, y=0) = -2Pa_2L^3$$

The total potential energy is therefore

$$\Pi = [\mathcal{U} + \mathcal{V}] = \tfrac{1}{2}\Big[2\mu(1+\bar{\nu})\frac{8}{9}a_1^2L^3h^3b + \mu(a_1+3a_2)^2\frac{16}{15}L^5hb\Big] - 2Pa_2L^3$$

This is the function to be minimized.

We consider Π to be a function of a_1, a_2, and differentiate with respect to a_n to get the stationary values. That is,

$$\frac{\partial\Pi}{\partial a_1} = 0 = 2\mu(1+\bar{\nu})\frac{8}{9}a_1 L^3 h^3 b + \mu(a_1+3a_2)\frac{16}{15}L^5 hb$$
$$\frac{\partial\Pi}{\partial a_2} = 0 = \mu(a_1+a_2 3)\frac{16}{5}L^5 hb - 2PL^3$$

which gives the solution

$$a_1 = \frac{-3P}{2\mu(1+\bar{\nu})4bh^3}\,, \qquad a_2 = \frac{P}{2\mu(1+\bar{\nu})4bh^3}\Big[1+\frac{5(1+\bar{\nu})h^2}{9L^2}\Big]$$

The displacement field is given by

$$u(x,y) = \frac{-3P}{2\mu(1+\bar{\nu})4bh^3}[2Lx-x^2]y\,, \qquad v(x,y) = \frac{P}{2\mu(1+\bar{\nu})4bh^3}[1+\frac{5(1+\bar{\nu})h^2}{9L^2}][3Lx^2-x^3]$$

The tip deflection is

$$v_{tip} = \frac{PL^3}{2\mu(1+\bar{\nu})2bh^3}[1+\frac{5(1+\bar{\nu})h^2}{9L^2}]$$

The difference with the elementary theory occurs in two places. First, the Ritz solution has an extra term that depends on the ratio h/L; this disappears for slender beams — it is the shear deformation contribution. The leading terms are compared as

$$\frac{PL^3}{2\mu(1+\bar{\nu})2bh^3} \longleftrightarrow \frac{PL^3}{2Ebh^3} \qquad \text{or} \qquad \frac{1}{(1+\bar{\nu})} \longleftrightarrow \frac{1}{(1+\nu)}$$

where $E = 2\mu(1+\nu)$ was used. The elementary theory is based on uniaxial stress which is different than both plane problems. ■

Weighted Residual Methods

When the differential equation is available, we can work directly with it to form an approximate solution. Let it be required to solve a linear differential equation

$$L(u) = 0 \qquad \text{in } R$$

subjected to some linear homogeneous boundary conditions. Suppose we have an assumed solution $\tilde{u}$ then $L(\tilde{u})$ will give some error or residual. In the weighted residual method, this error is minimized over the domain by

$$\int\!\!\int_R W(x,y)L(\tilde{u}(x,y))\,dxdy = \min$$

where W is some weighting function. There are a number of methods that fall into this category — they differ only in their specification of the weighting function. In any event, we see that the differential formulation has been recast as a variational formulation.

I. Galerkin Method

In 1915, Galerkin proposed a method of approximate solution of the boundary-value problems in mathematical physics that is of much wider scope than the method of Ritz. The Galerkin method when applied to variational problems with quadratic functionals, reduces to the Ritz method. It is important to note that in Galerkin's formulation there is no reference to any connection of a potential. Indeed, the Galerkin method can be applied to a broad class of problems phrased in terms of integral and other types of functional equations.

Assume, for simplicity of exposition, that the domain R is two-dimensional, we seek an approximate solution of the problem in the form

$$U^N(x,y) = \sum_{i=1}^{N} \alpha_i \phi_i(x,y)$$

where U^N satisfy the same boundary conditions as $u(x,y)$. The finite sum ordinarily will not satisfy the differential equation, and the substitution of U^N will yield an error

$$L(U^N) = \epsilon_N(x,y)\,, \qquad \epsilon_N(x,y) \neq 0 \quad \text{in } R$$

If max $\epsilon_N(x,y)$ is small, $U^N(x,y)$ can be considered a satisfactory approximation of $U(x,y)$. Thus, $\epsilon_N(x,y)$ can be viewed as an error function, and the task is then to select the α_i to minimize $\epsilon_N(x,y)$.

A reasonable minimization technique is suggested by the following: if we represents $U(x,y)$ by the series $U(x,y) = \sum_{i=1}^{\infty} \alpha_i \phi_i$ and consider the N^{th} partial sum $U^N = \sum_{i=1}^{N} \alpha_i \phi_i$ then the orthogonality condition

$$\int\int_R L(U^N)\phi_j(x,y)dxdy = 0 \qquad \text{as } N \to \infty$$

is equivalent to the statement that $L(u) = 0$. This led Galerkin to impose on the error function $L(U^N)$ a set of orthogonality conditions

$$\int\int_R L(U^N)\phi_j(x,y)dxdy = 0 \qquad (j = 1, 2, \ldots N)$$

yielding the set of N equations

$$\int\int_R L(\sum_{i=1}^{N} \alpha_i \phi_i)\phi_j dxdy = 0 \qquad (j = 1, 2, \ldots N)$$

for the determination of the constants α_i in the approximate solution.

The resulting equations are always symmetric and positive definite if the operator L is symmetric and positive definite. Note that the addition of natural boundary conditions also destroys the symmetry. In comparison to the Ritz method, this approach can be applied to problems for which the governing equations are known but for which a potential does not exist. However, it has the disadvantage that higher order continuity conditions must be imposed when applied to a discretized domain. When the operator is symmetric and positive definite, integration by parts recover the Ritz formulation.

II. Least Squares

We can also form the squared error as

$$\int\int_R L(U^N)L(U^N)dxdy = \epsilon \qquad (j = 1, 2, \ldots N)$$

This becomes

$$\int\int_R L(\sum_{i=1}^{N} \alpha_i\phi_i)L(\phi_j)dxdy = 0 \qquad (j = 1, 2, \ldots N)$$

The resulting equations are always symmetric and positive definite. This also has the disadvantage that higher order continuity conditions must be imposed when applied to a discretized domain. Note that integration by parts will not reduce the order of derivatives occurring in the functional.

Example 7.6: Consider the rectangular panel shown in Figure 7.3 with end parabolic tractions. Regarding the panel being in a state of plane stress, determine the distribution of stresses.

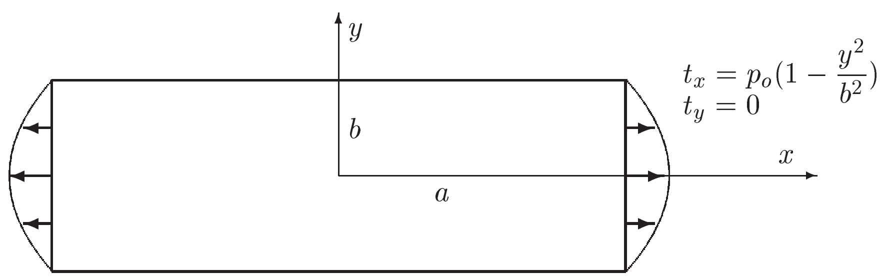


Figure 7.3: Parabolic traction distribution.

As shown in Chapter 5, for plane problems if we take the stresses derived from a stress function as

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y\partial y}, \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x\partial x}, \qquad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

then the stresses automatically satisfy the equilibrium equations. Furthermore, if the stress function is restricted to being bi-harmonic

$$\nabla^4\phi = 0$$

then the compatibility conditions are also satisfied. In the following, we will seek an approximate solution for $\phi(x, y)$.

The boundary conditions in terms of $\phi(x, y)$ are given by

$$\text{on } x = \pm a, \qquad t_y = \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0, \qquad t_x = \sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = p_o\Big[1 - \frac{y^2}{b^2}\Big]$$

$$\text{on } x = \pm b, \qquad t_x = \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0, \qquad t_y = \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 0$$

Let the stress function be composed of two parts

$$\phi = \phi_o + \phi^* \qquad \text{where} \qquad \phi_o = \tfrac{1}{2} p_o y^2 \Big[1 - \frac{y^2}{6b^2}\Big]$$

This will satisfy the traction boundary conditions. Then the bi-harmonic equation becomes

$$\nabla^4 \phi^* = \frac{2p_o}{b^2}$$

and the boundary conditions reduces to

$$\begin{aligned} \frac{\partial^2 \phi^*}{\partial x \partial y} &= \frac{\partial^2 \phi^*}{\partial y^2} = 0 \qquad && \text{on } x = \pm a \\ \frac{\partial^2 \phi^*}{\partial x \partial y} &= \frac{\partial^2 \phi^*}{\partial x^2} = 0 \qquad && \text{on } y = \pm b \end{aligned}$$

The differential equation for ϕ^* can be identified as the Euler equation for the variational problem

$$J[\,\phi^*\,] = \int_{-b}^{b} \int_{-a}^{a} \Big[(\nabla^2 \phi^*)^2 - \frac{4p_o}{b^2} \phi^* \Big] \, dx dy = \text{ min}$$

Thus, the problem can be solved by the direct method. Assume

$$\phi^* = (x^2 - a^2)^2 (y^2 - b^2)^2 (a_1 + a_2 x^2 + a_3 y^2 + \cdots)$$

Such an expansion satisfies the boundary conditions and the expansion functions are independent. If one term is taken, then

$$\frac{\partial J}{\partial a_1} = 0 \quad \Rightarrow \quad a_1 \left(\frac{64}{7} + \frac{256}{49} \frac{b^2}{a^2} + \frac{64}{7} \frac{b^4}{a^4} \right) = \frac{p_o}{a^4 b^2}$$

For a square plate ($a = b$), we find

$$a_1 = 0.04253 \frac{p_o}{a^6}$$

Thus the approximate stress function is given by

$$\phi = \tfrac{1}{2} p_o y^2 \Big[1 - \frac{y^2}{6a^2}\Big] + 0.04253 \frac{p_o}{a^6} (x^2 - a^2)^2 (y^2 - b^2)^2$$

The corresponding stress components are obtained from ϕ as

$$\begin{aligned} \sigma_{xx} &= p_o (1 - \frac{y^2}{a^2}) - 0.1702\, p_o (1 - \frac{3y^2}{a^2})(1 - \frac{x^2}{a^2})^2 \\ \sigma_{yy} &= - 0.1702\, p_o (1 - \frac{3x^2}{a^2})(1 - \frac{y^2}{a^2})^2 \\ \sigma_{xy} &= - 0.6805\, p_o (1 - \frac{x^2}{a^2})(1 - \frac{y^2}{a^2}) \frac{xy}{a^2} \end{aligned}$$

If three terms are taken, then (again for a square plate ($a = b$))

$$a_1 = 0.0404\frac{p_o}{a^6}\,, \qquad a_2 = a_3 = 0.01174\frac{p_o}{a^8}$$

and the stress along the axis $x = 0$ is

$$\sigma_{xx} = p_o(1 - \frac{y^2}{a^2}) - 0.1616\,p_o(1 - 3\frac{y^2}{a^2}) + 0.0235\,p_o(1 - 12\frac{y^2}{a^2} + 15\frac{y^4}{a^4})$$

These stresses satisfy equilibrium and boundary conditions, but not compatibility.

■

Relation between the Ritz and Finite Element Methods

At this stage, it is worthwhile to summarize some of the characteristics of the Ritz and Galerkin methods. The most important ones are:

- Usually, the accuracy of the assessed displacement is increased with an increase in the number of trial functions.
- While fairly accurate expressions for the displacements are obtained, the corresponding forces may differ significantly from the exact values.
- Equilibrium is satisfied in an average sense through minimization of the total potential energy. Therefore, forces (computed on the basis of the displacements) do not, in general, satisfy the equilibrium equations.
- The approximate system is stiffer than the actual system so that, for example, buckling loads and vibration resonances are overestimated while displacements are underestimated.

The question that arises is that of the appropriate additional terms to be used if more terms are to be included so as to achieved a converged accurate solution; the main attributes required is that it *complete* and *compatible*. For example, for a 1-D problem the simple polynomial

$$1 \quad x \quad x^2 \quad x^3 \quad \ldots$$

is complete. The trigonometric sequence

$$1 \quad \sin x \quad \cos x \quad \sin 2x \quad \cos 2x \quad \sin 3x \quad \cos 3x \quad \ldots$$

is also complete. Note, however, that the cosines on their own could not represent an asymmetric distribution.

As we go to higher dimensions, the question of completeness gets a little more involved. In the 2-D case, there is the *Pascal triangle*

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & x & & y & & \\
 & x^2 & & xy & & y^2 & \\
x^3 & & x^2y & & xy^2 & & y^3
\end{array}$$

An analogous sequence can be written in terms of the trigonometric functions.

A number of observations can now be made about the use of stationary principles. First, if the Ritz functional contains derivatives up to order m, then there must be continuity of displacement derivative up to $m-1$, and the order of the highest derivative that is present in the governing differential equation is then $2m$. For example, in a beam bending problem where the strain is d^2v/dx^2, $m = 2$ because the highest derivative in the functional is of order 2, and there must be continuity of v and dv/dx. The reason for obtaining a derivative of order $2m = 4$ in the governing differential equation is that integration by parts is employed $m = 2$ times. The Galerkin functional has higher order derivatives.

A second observation is that through the stationarity condition we obtain the governing differential equations *and* the proper boundary conditions. Hence, the effect of the natural boundary conditions are implicitly contained in the expression for the potential Π. (Note that the essential boundary conditions must be stated separately.)

The classical or conventional Ritz method is the discretized implementation of the stationary potential principle. Some of its most important characteristics are:

- The trial functions usually span the entire domain.
- Usually, the accuracy of the assessed displacement is increased with an increase in the number of trial functions.
- While fairly accurate expressions for the displacements are obtained, the corresponding forces may differ significantly from the exact values.
- Equilibrium is satisfied in an average sense through minimization of the total potential energy. Therefore, forces (computed on the basis of the displacements) do not, in general, satisfy the equilibrium equations.
- The approximate system is stiffer than the actual system.

One disadvantage of the classical Ritz analysis is that the trial functions are defined over the whole region. This causes a particular difficulty in the selection of appropriate functions; in order to solve accurately for large stress gradients, say, we may need many functions. However, these functions are defined over the regions in which the stresses vary rather slowly and where not many functions are required. Another difficulty arises when the total region is made up of subregions with different kinds of strain distributions. As an example, consider a building modeled by plates for the floors and beams for the vertical frame. In this situation, the trial functions used for one region (e.g., the floor) are not appropriate for the other region (e.g., the frame), and special displacement continuity conditions and boundary relations must be introduced. We conclude that the conventional Ritz analysis is, in general, not particularly computer-oriented.

We can view the finite element method as an application of the Ritz method, where instead of the trial functions spanning the complete domain, the individual functions span only subdomains (the finite elements) of the complete region. Figure 7.4 shows

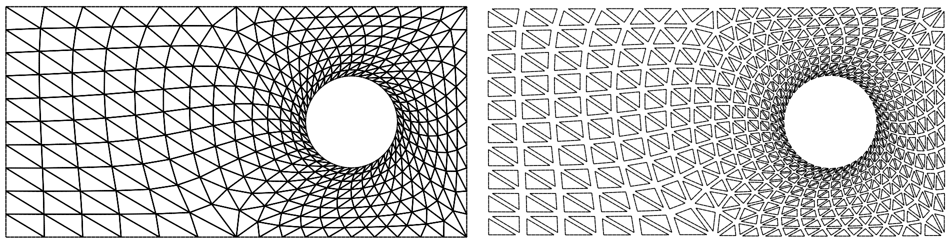


Figure 7.4: Continuous domain discretized as finite elements.

an example of a bar with a hole modeled as a collection of many triangular regions. The use of relatively many functions in regions of high strain gradients is made possible simply by using many elements as shown around the hole in the figure. The combination of domains with different kinds of strain distributions may be achieved by using different kinds of elements to idealize the domains.

In order that a finite element solution be a Ritz analysis, it must satisfy the essential boundary conditions. This refers to the actual boundaries of the problems and to inter-element compatibility.

However, in the selection of the displacement functions, no special attention need be given to the natural boundary conditions, because these conditions are imposed with the load vector and are satisfied approximately in the Ritz solution. The accuracy with which these natural boundary conditions are satisfied depends on the specific trial functions employed, and on the number of elements used to model the problem.

7.3 Semi-Direct Methods

Another application of the variational approach is the conversion of one set of governing differential equations into another, usually lower order, set. This is known as the *semi-direct method* because it still yields a strong formulation of the problem. The approach will be demonstrated with the construction of structural theories from the 3-D theory of elasticity.

Bending of Deep Beams

Consider a rectangular beam of length L, thickness h, and width b, as shown in Figure 7.5. If b is small, then the beam can be regarded as in a state of plane stress.

We begin by expanding the displacements in the beam by a Taylor series about the mid-plane displacements $\bar{u}(x, 0)$ and $\bar{v}(x, 0)$; however, since we are interested in flexural deformations, we set $\bar{u}(x, 0) = 0$ and retain only odd powers of y for $\bar{u}(x, y)$

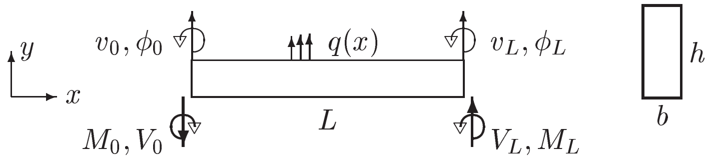


Figure 7.5: Timoshenko beam with distributed and end loads.

and even powers of y for $\bar{v}(x, y)$. Thus

$$\begin{aligned}\bar{u}(x,y) &\approx \bar{u}(x,0) + y\frac{\partial \bar{u}}{\partial y}|_{y=0} + \cdots = -y\phi(x) + O(y^3) \\ \bar{v}(x,y) &\approx \bar{v}(x,0) + y\frac{\partial \bar{v}}{\partial y}|_{y=0} + \cdots = v(x) + y^2\psi(x) + O(y^4)\end{aligned}$$

where we have used the notations

$$v(x) = \bar{v}(x,0)\,, \quad \phi(x) = -\frac{\partial \bar{u}}{\partial y}|_{y=0}\,, \quad \psi(x) = \tfrac{1}{2}\frac{\partial^2 \bar{v}}{\partial y^2}|_{y=0}$$

This approximation says that the deformation is governed by three independent functions, $v(x)$, $\phi(x)$, and $\psi(x)$, that depend only on the position along the centerline. We could, of course, retain more terms in the expansion and develop an even more refined theory. In fact, we will use the lateral traction boundary conditions to reduce the representation down to effectively only two functions.

The strains corresponding to the above deformation are

$$\epsilon_{xx} = \frac{\partial \bar{u}}{\partial x} = -y\frac{\partial \phi}{\partial x} + O(y^3)\,, \quad \epsilon_{yy} = \frac{\partial \bar{v}}{\partial y} = y2\psi + O(y^3)\,, \quad \gamma_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = -\phi + \frac{\partial v}{\partial x} + O(y^2)$$

The normal strains are predominantly linear on the cross-section, whereas the shear strain is predominantly constant. Substitute these strains into the Hooke's law for plane stress to get, for example,

$$\bar{\sigma}_{yy} = \frac{E}{1-\nu^2}\Big[\epsilon_{yy} + \nu\epsilon_{xx}\Big] = \frac{E}{(1-\nu^2)}\Big[2\psi - \nu\frac{\partial \phi}{\partial x} + O(y^2)\Big]y$$

We expect the normal tractions to be zero on the lateral surfaces when there is no distributed load, so choose

$$2\psi = \nu\frac{\partial \phi}{\partial x} \qquad \text{or} \qquad \epsilon_{yy} = -\nu\epsilon_{xx}$$

This gives the other normal stress as

$$\bar{\sigma}_{xx} = \frac{E}{1-\nu^2}\Big[\epsilon_{xx} + \nu\epsilon_{yy}\Big] = \frac{E}{(1-\nu^2)}\Big[-\frac{\partial \phi}{\partial x} + \nu^2\frac{\partial \phi}{\partial x} + O(y^2)\Big]y = -Ey\frac{\partial \phi}{\partial x} + O(y^3)$$

To the indicated order, the normal stresses form a uniaxial system of stresses. We assume the presence of a distributed lateral load does not greatly affect this conclusion.

The shear stress is

$$\sigma_{xy} = G\Big[-\phi + \frac{\partial v}{\partial x} + O(y^2)\Big]$$

This is predominantly constant on the cross-section and nonzero on the boundary. The lateral boundaries are also shear traction free; so if we take

$$\phi = \frac{\partial v}{\partial x}$$

this would impose that condition. It would also impose that there is no shear at all in the beam. Instead we will interpret the constant shear as actually representing the average shear as indicated in Figure 7.6.

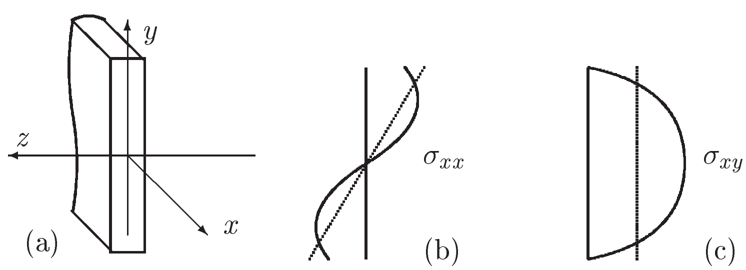


Figure 7.6: Distributions of stress an the cross-section. (a) Arbitrary cross-section. (b) Normal stress. (c) Shear stress.

We therefore conclude that the stresses are essentially

$$\sigma_{xx} = -yE\frac{\partial \phi}{\partial x}\,, \qquad \sigma_{xy} = G\Big[-\phi + \frac{\partial v}{\partial x}\Big]\,, \qquad \text{others} = 0$$

The strain energy now becomes

$$\mathcal{U} = \tfrac{1}{2}\int_V [\sigma_{xx}\epsilon_{xx} + \sigma_{xy}\gamma_{xy}]\,dV = \tfrac{1}{2}\int_V [E\epsilon_{xx}^2 + G\gamma_{xy}^2]\,dV$$

Substitute for the strains to get the total strain energy as

$$\mathcal{U} = \tfrac{1}{2}\int_o^L\int_{-h/2}^{h/2}\Big[Ey^2(\frac{\partial\phi}{\partial x})^2 + G(\phi - \frac{\partial v}{\partial x})^2\Big]b\,dydx = \tfrac{1}{2}\int_o^L\Big[EI(\frac{\partial\phi}{\partial x})^2 + GA(\phi - \frac{\partial v}{\partial x})^2\Big]dx$$

where the cross-sectional properties

$$A \equiv \int_{-h/2}^{h/2} bdy = bh\,, \qquad I \equiv \int_{-h/2}^{h/2} by^2dy = \tfrac{1}{12}bh^3$$

were introduced. If the applied surface tractions and end loads on the beam are as shown in Figure 7.6, then the potential of these loads is

$$\mathcal{V} = -\int_o^L q(x)v\,dx - M_L\phi_L + M_0\phi_0 - V_Lv_L + V_0v_0 = -\int_o^L q(x)v\,dx - M\phi\Big|_0^L - Vv\Big|_0^L$$

Our variational principle for the beam may now be stated as

$$\delta\left\{\int_0^L\left[\tfrac{1}{2}EI(\frac{\partial\phi}{\partial x})^2 + \tfrac{1}{2}GA(-\phi + \frac{\partial v}{\partial x})^2 + qv\right]dx + M\phi\Big|_0^L + Vv\Big|_0^L\right\} = 0$$

There are two entities, $v(x)$ and $\phi(x)$, which are subject to variation.

Taking the variation inside the integrals and using integration by parts, we get

$$\begin{aligned}&\int_0^L\left[GA(-\phi + \frac{\partial v}{\partial x}) + \frac{\partial}{\partial x}\Big[EI\frac{\partial\phi}{\partial x}\Big]\right]\delta\phi\,dx + \int_0^L\left[\frac{\partial}{\partial x}\Big[GA(-\phi + \frac{\partial v}{\partial x})\Big] + q\right]\delta v\,dx\\ &+\Big[EI\frac{\partial\phi}{\partial x} - M\Big]\delta\phi\Big|_0^L + \Big[GA[-\phi + \frac{\partial v}{\partial x}] - V\Big]\delta v\Big|_0^L = 0 \end{aligned} \tag{7.1}$$

The fact that the variations δv and $\delta\phi$ can be varied separately and arbitrarily, and that the limits on the integrals are also arbitrary, lead us to conclude from the terms in square brackets that

$$\begin{aligned}\frac{\partial}{\partial x}\Big[GA(\frac{\partial v}{\partial x} - \phi)\Big] &= -q\\ \frac{\partial}{\partial x}\Big[EI\frac{\partial\phi}{\partial x}\Big] + GA\Big[\frac{\partial v}{\partial x} - \phi\Big] &= 0\end{aligned} \tag{7.2}$$

The associated boundary conditions (at each end of the beam) are specified in terms of any pair of conditions selected from the following groups:

$$\left\{v \quad \text{or} \quad V = GA(\frac{\partial v}{\partial x} - \phi)\right\}, \qquad \left\{\phi \quad \text{or} \quad M = EI\frac{\partial\phi}{\partial x}\right\} \tag{7.3}$$

Thus a free boundary is specified as

$$V = 0\,, \qquad M = 0$$

An inadmissible set of boundary conditions are

$$v = 0\,, \quad V = 0 \qquad \text{or} \qquad \phi = 0\,, \quad M = 0$$

If either of these are imposed then there is no guarantee that the remaining term in Equation (7.1) is zero.

These are the *Timoshenko equations* for a deep beam. In comparison to the elementary Bernoulli-Euler beam theory, this theory accounts for the shear deformation associated with deep beams.

If the surface tractions on the body are as shown, then

$$\begin{aligned}\mathcal{V} &= \int_o^L bq_+\bar{v}dx - \int_o^L bq_-(x)\bar{v}dx \\ &\quad + \int_{-h/2}^{h/2} bf_1(y)\bar{u}(L,y)dy - \int_{-h/2}^{h/2} bf_o(y)\bar{u}(o,y)dy \\ &\quad + \int_{-h/2}^{h/2} bg_1(y)\bar{v}(L,y)dy - \int_{-h/2}^{h/2} bg_o(y)\bar{v}(o,y)dy\end{aligned}$$

Substituting for u and v and integrating gives

$$\mathcal{V} = \int_o^L p(x)vdx + M_e\phi(L) - M_o\phi(o) + Q_1v(L) - q_ov(o)$$

where the various resultants are given by

$$\begin{aligned}p(x) &= \int_o^L b[q_+(x) - q_-(x)]\,dx \\ M_o &= \int_{-h/2}^{h/2} bf_o(y)ydy\,, \qquad M_1 = \int_{-h/2}^{h/2} bf_1(y)ydy \\ Q_o &= \int_{-h/2}^{h/2} bg_ody\,, \qquad Q_1 = \int_{-h/2}^{h/2} bg_1dy\end{aligned}$$

In order to assure pure flexure we require that

$$\int_{-h/2}^{h/2} bf_1(y)ydy = 0 \qquad \text{and} \qquad \int_{-h/2}^{h/2} bf_o(y)ydy = 0$$

that is, the resultant forces in the x-direction must vanish.

In order to account for the truncation error of the expansion $\bar{u} \approx -y\phi$, a correction coefficient κ is often added to the expansion; that is, $u = -\kappa y\phi$. The coefficient κ can be evaluated many ways: for a rectangular cross-section, it is usually taken to be $2/3$ in static problems and $\pi^2/12$ for dynamic problems.

Example 7.7: Recover the Bernoulli-Euler beam equations from the previous formulation.

In the above developments, make the further assumption that

$$\phi = \frac{\partial v}{\partial x} \qquad \text{or} \qquad u = -y\frac{\partial v}{\partial x}$$

then the shear strain is zero and the potential becomes

$$\Pi \;=\; = \int_o^L \Big[\tfrac{1}{2}EI\Big(\frac{\partial^2 v}{\partial x^2}\Big)^2 - q(x)v(x)\Big]dx$$

The Euler equation becomes

$$\frac{\partial^2}{\partial x^2}\Big[EI\frac{\partial^2 v}{\partial x^2}\Big] - q(x) = 0$$

This is known as the Bernoulli-Euler beam equation. The expansion $\bar{u} \approx -y\phi$ assures that plane sections remain plane after deformation. The assumption $\phi = \partial v/\partial x$ further requires that the plane section remains normal to the neutral axis. It is clear that ϕ can be regarded as the rotation of the cross-section. ■

Twisting of Long Structural Members

Consider a long straight bar of constant cross-section in equilibrium under the action of end torques. The lateral boundaries are therefore traction free and the predominant response is an axial twisting. There are two cases of interest: one is when the cross-section is solid, the other is when it is in the form of a thin-walled tube.

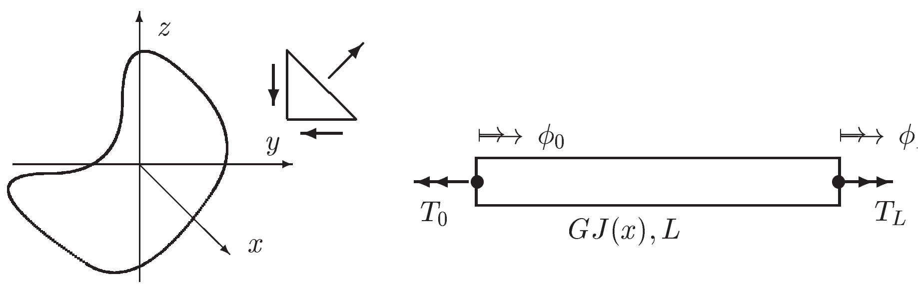


Figure 7.7: Torsion member with arbitrary cross-section. The x-axis is along the length.

I: Torsion of Solid Bars

Because the bar is slender, we begin by expanding the displacements in a Taylor series (in terms of y and z) about the mid-point values. While the dominant action is a rotation of the cross-section about the origin, there is also a warping of the cross-section. This leads to the approximate displacements

$$\bar{u}(x,y,z) \approx u(y,z) + \cdots\,, \quad \bar{v}(x,y,z) \approx -\phi_x xz + \cdots\,, \quad \bar{w}(x,y,z) \approx \phi_x xy + \cdots$$

where ϕ_x is the angle of twist per unit length and $u(y,z)$ is the warping.

The three normal strains are zero, the shear strains are

$$\gamma_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = \frac{\partial u}{\partial y} - \phi_x z\,, \qquad \gamma_{xz} = \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} = \frac{\partial u}{\partial z} + \phi_x y\,, \qquad \gamma_{yz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} = 0$$

We conclude that there are only two nonzero stresses given by

$$\tau_{xy} = \gamma_{xy} = G[\frac{\partial u}{\partial y} - \phi_x z]\,, \qquad \tau_{xz} = \gamma_{xz} = G[\frac{\partial u}{\partial z} + \phi_x y]$$

Since there are only two non-zero stresses, we can represent them in the form of a potential function. That is, let

$$\tau_{xy} = \frac{\partial \psi}{\partial z}\,, \qquad \tau_{xz} = -\frac{\partial \psi}{\partial y}$$

We will refer to $\psi(y,z)$ as the *torsion stress function.* The tractions are zero on the boundary. Consider a small boundary wedge as shown in Figure 7.7, the resultant force in the normal direction should be zero

$$F_n = 0 = \Delta A\tau_{xy}\sin\theta + \Delta A\tau_{xz}\cos\theta = \Delta A\frac{\partial\psi}{\partial z}\frac{dz}{ds} - \Delta A\frac{\partial\psi}{\partial y}(-\frac{dy}{ds}) = \Delta A\frac{d\psi}{ds}$$

where ds is a segment of the circumference and ΔA is the area of the triangle. We conclude from this that along the lateral boundary

$$\frac{d\psi}{ds} = 0 \qquad \text{or} \qquad \psi(s) = \text{constant}$$

For simply connected regions (no cut-outs) we can take the constant as zero.

The resultant torque on the cross-section is

$$T_x = \int_A dT_x = \int_A [-\tau_{xy}z + \tau_{xz}y]\,dydz = -\int_A [\frac{\partial\psi}{\partial z}z + \frac{\partial\psi}{\partial y}y]\,dydz$$

Noting that

$$\frac{\partial(z\psi)}{\partial z} = \psi + z\frac{\partial\psi}{\partial z}\,, \qquad \frac{\partial(y\psi)}{\partial y} = \psi + y\frac{\partial\psi}{\partial y}$$

we can divide the integral into two parts

$$T_x = \int_A 2\psi\,dA - \int_A [\frac{\partial(z\psi)}{\partial z} + \frac{\partial(y\psi)}{\partial y}]\,dydz = \int_A 2\psi\,dA - \int_A [d(z\psi)\,dy + d(y\psi)\,dz]$$

The $d(z\psi)$ and $d(y\psi)$ terms integrate to give boundary values, but as already noted, ψ can be taken as zero on the boundary, and hence we conclude that the second integral terms evaluates to zero. Thus the moment is given simply by

$$T_x = \int_A 2\psi\,dA$$

We have thus reduced the torsion problem to finding a function $\psi(y,z)$ that vanishes along the boundary. We will use our stationary principle to effect solutions for arbitrary cross-sections.

The strain energy expression reduces to

$$\mathcal{U} = \tfrac{1}{2}\int_V [\tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz}]\,dV = \frac{1}{2G}\int_V \Big[(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial y})^2\Big]\,dA\,dx = \frac{L}{2G}\int_A \Big[(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial y})^2\Big]\,dA$$

The only applied loads are the end torques; consider one end as fixed then the other end rotates an amount $\phi_x L$, consequently, the total potential of the problem is

$$\Pi = \frac{L}{2G}\int_A \Big[(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial y})^2\Big]\,dA - \phi_x L\int_A 2\psi\,dA$$

Our variational principle now becomes

$$\delta \int_A \Big[(\frac{\partial \psi}{\partial y})^2 + (\frac{\partial \psi}{\partial z})^2 - 4G\phi_x \psi \Big] \, dA = 0$$

There is just the single entity $\psi(y, z)$ subject to variation. The strong formulation reduces to the differential equation

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2G\phi_x$$

subject to ψ being zero on the boundary.

Suppose we find a function $\psi(x, y)$ that satisfies the zero boundary conditions, then $c_1\psi(x, y)$ also satisfies the boundary conditions. Substitute into the stationary principle and minimize with respect to c_1 to get

$$\Big[2c_1 \int_A [(\frac{\partial \psi}{\partial z})^2 + (\frac{\partial \psi}{\partial y})^2] \, dA - \int_A [4G\phi_x \psi] \, dA \Big] \, \delta c_1 = 0$$

Solving for c_1 and substituting into the strain energy gives

$$\mathcal{U} = \tfrac{1}{2} GJ\phi_x^2 \,, \qquad J \equiv \Big[4 \int_A \psi \, dA \Big]^2 \Big/ \int_A \Big[(\frac{\partial \psi}{\partial z})^2 + (\frac{\partial \psi}{\partial y})^2 \Big] \, dA$$

where GJ is called the *torsional stiffness* per unit length. We will now establish this quantity for some typical cross-sections.

Example 7.8: Evaluate the torsional stiffness of a bar with a rectangular cross-section.

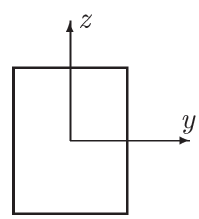


Figure 7.8: Rectangular cross-section.

We take a Ritz approach and expand ψ in terms of bases functions that satisfy the boundary condition. For example, we could take

$$\psi = [y^2 - b^2/4][z^2 - h^2/4] \sum_m \sum_n c_{mn} y^m z^n$$

Because of symmetry, m and n, must be even. For illustrative purposes, we will just do a single term expansion. Thus let

$$\psi = c_1 [y^2 - b^2/4][z^2 - h^2/4]$$

Substituting into the potential energy and evaluating the integrals leads to

$$\delta\Big[c_1^2\frac{b^3h^3}{90}[b^2+h^2] - 4G\phi_x c_1\frac{b^3h^3}{36}\Big] = 0$$

Performing the variation on c_1 leads to $c_1 = 5G\phi_x/[b^2+h^2]$ and the function as

$$\psi = \frac{5G\phi_x}{[b^2+h^2]}[y^2-b^2/4][z^2-h^2/4]$$

At this stage everything is determined about the solution.

For example, the stresses are given by

$$\tau_{xy} = \frac{\partial\psi}{\partial z} = \frac{10G\phi_x z}{[b^2+h^2]}[y^2-b^2/4]\,, \qquad \tau_{xz} = -\frac{\partial\psi}{\partial y} = -\frac{10G\phi_x y}{[b^2+h^2]}[z^2-h^2/4]$$

The maximum stresses occur at the middle of the sides and not at the extremities; indeed, the stresses are zero at the extremities. The torque rotation relation is

$$T_x = GJ\phi_x\,, \qquad J \equiv \frac{5}{18}\frac{b^3h^3}{b^2+h^2}$$

When cross-sections is square of side a, the torsional stiffness is

$$GJ = G\frac{5}{36}a^4 = 0.1389\,a^4\,, \qquad GJ_{\text{exact}} = 0.1406\,a^4$$

The comparison with the exact value [31] is remarkably good being different by only -1%. The accuracy depends on the aspect ratio being -3% for $h/b = 2$, -14% for $h/b = 10$, and -20% in the limit of $h/b \to \infty$. The next example problem will get a better estimate for large h/b. ■

Example 7.9: Estimate the stresses and torsional stiffness for a cross-section that is narrow.

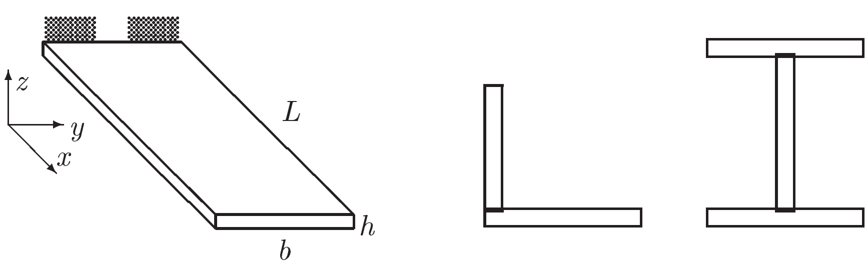


Figure 7.9: Bar with narrow rectangular cross-section.

Choose the torsion function as

$$\psi = c_1[z^2-h^2/4]$$

This gives zero values on the top and bottom but not the sides. However, in the limit of large b and/or small h, the effects of the sides will be negligible. Substituting into the potential energy and evaluating the integrals leads to

$$\delta\Big[\frac{4}{3}\frac{bh^3}{4}c_1^2 + 4G\phi_x\frac{2}{3}\frac{bh^3}{4}c_1\Big] = 0$$

Performing the variation on c_1 leads to $c_1 = -G\phi_x$ and the function as

$$\psi = -\phi_x[z^2 - h^2/4]$$

The stresses are given by

$$\tau_{xy} = -2G\phi_x z\,, \qquad \tau_{xz} = 0$$

and the torque rotation relation is

$$T_x = GJ\phi_x\,, \qquad J \equiv \tfrac{1}{3}bh^3$$

The torsional stiffness is exact in the limit of $b/h \to \infty$ and only off by $+6\%$ when $b/h = 10$.

An estimate of the torsional stiffness properties of rolled sections such as angles, channels and I-beams can be obtained simply by summing the stiffness of each segment. Thus,

$$GJ = \sum\nolimits_i GJ = \sum\nolimits_i G\tfrac{1}{3}bh^3$$

If this is not adequate, then the section can be treated as a folded plate as discussed in the next subsection. ■

II. Torsion of Tubes

The general problem of the torsion of a cross-section with multiple cut-out is very difficult to solve. Even approximate methods such as that of Ritz are also difficult to use because of the requirement to satisfy multiple boundary conditions simultaneously. There is one situation that is relatively easy to solve and that is when the interior boundaries coincide with a stress line. A special case of this is that of thin-walled tubes, this is tractable which we now illustrate.

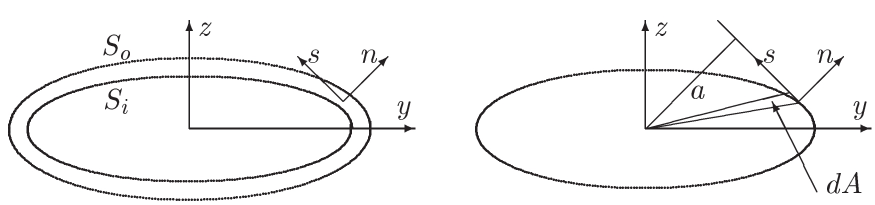


Figure 7.10: Torsion of thin-walled tubes.

Consider the general thin-walled tube shown in Figure 7.10. Introduce a set of local axes n and s also as shown. Previously it was established that on a free surface the torsion function must satisfy $\partial\psi/\partial s = 0$, from which we conclude that the function is constant on the inner and outer surfaces. For solid sections we set the constant as zero, for multiply connected sections we can set only one as zero, and the other is a constant to be determined as part of the solution. Here we choose

$$\psi = c_1 \quad \text{on} \quad S_i\,, \qquad\qquad \psi = 0 \quad \text{on} \quad S_o$$

Since the wall is thin, then expand the torsion function as

$$\psi(n,s) \approx \psi_0(s) + \psi_1(s)n + \psi_3(s)\tfrac{1}{2}n^2 + \cdots$$

Retaining just the linear terms and imposing the boundary values leads to

$$\psi(n,s) = \frac{c_1}{2} - \frac{c_1}{h}n\,, \qquad \psi_0 = \frac{c_1}{2}\,, \quad \psi_1 = -\frac{c_1}{h}$$

where h is the local thickness of the tube.

The local (n,s) axes can be thought of as a point transformation from the (y,z) axes from which we immediately get the stress relations

$$\tau_{xn} = \frac{\partial\psi}{\partial s} = 0\,, \qquad \tau_{xs} = -\frac{\partial\psi}{\partial n} = \frac{c_1}{h}$$

If the wall thickness is constant, we conclude that the shear stress is constant. The strain energy for a tube segment of length L reduces to

$$\mathcal{U} = \frac{L}{2G}\int_A (\frac{\partial\psi}{\partial n})^2 dA = \frac{L}{2G}\oint h(\frac{\partial\psi}{\partial n})^2 ds = \frac{L}{2G}c_1^2\oint\frac{ds}{h}$$

The torque caused by the stress τ_{xs} on an arc segment of length ds is

$$dT_x = a\tau_{xs}h\,ds = a\frac{c_1}{h}h\,ds = ac_1\,ds$$

where a is the perpendicular distance from the origin to the line of action of τ_{xs}. A simple geometric construction based on triangles with base ds shows that $a\,ds = 2\,dA$ from which we get that

$$T_x = \int_{\bar{A}} 2c_1\,dA = 2c_1\bar{A}$$

where $\bar{A}$ is the area enclosed the centerline of the tube.

The total potential of our problem is

$$\Pi = \mathcal{U} - T\phi_x L = \frac{L}{2G}c_1^2\oint\frac{ds}{h} - 2c_1\bar{A}\phi_x L$$

Minimizing with respect to c_1 gives

$$c_1 = 2G\bar{A}\phi_x/\oint\frac{ds}{h}\,, \qquad \psi(n,s) = G\bar{A}\phi_x L[1 - \frac{n}{h/2}]/\oint\frac{ds}{h}$$

We can now summarize the various relations

$$\phi_x = \frac{q}{2G\bar{A}} \oint \frac{ds}{h}\,, \qquad T_x = GJ\phi_x\,, \qquad J = 4\bar{A}^2 \Big/ \oint \frac{ds}{h}$$

where $q \equiv \tau_{xs} h = c_1 =$ constant, is called the *shear flow*.

Example 7.10: Compare the stiffness properties of a thin-walled circular tube with and without a longitudinal slit.

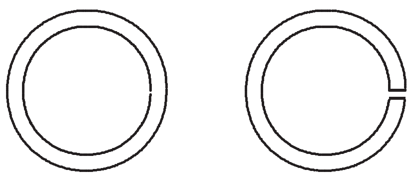


Figure 7.11: Thin-walled tubes.

Let the diameter be D. For the tube without a slit

$$J_1 = 4\bar{A}^2 \Big/ \oint \frac{ds}{h} = 4(\pi D^2/4)^2/(\pi D/h) = \tfrac{1}{4}\pi D^3 h$$

The stiffness of the split tube can be estimated by thinking of it as a narrow rectangle. Then,

$$J_2 = \tfrac{1}{3} b h^3 = \tfrac{1}{3}\pi D h^3$$

The ratio of the two is

$$\frac{J_1}{J_2} = \frac{3}{4}\frac{D^2}{h^2}$$

A tube with D/h of 20 gives a ratio of 300, that is a significant difference. ■

Flat Plate Theory

Fundamentally, as discussed in Section 5.2 for plane stress, thin plate theory is an approximate structural theory and therefore it is best to approach it by way of a variational principle. We will begin by developing a plate theory (called *Mindlin plate theory*) that takes the shear deformation into account — this is the plate equivalent of the Timoshenko beam. The transition to achieve the classical or (very) thin-plate theory is then more transparent.

I: General Governing Equations

Consider a rectangular plate of thickness h as shown in Figure 7.12. The plate lies in the x-y plane and is subjected to both in-plane and transverse loads. The mid-plane of the plate is taken at $z = 0$.

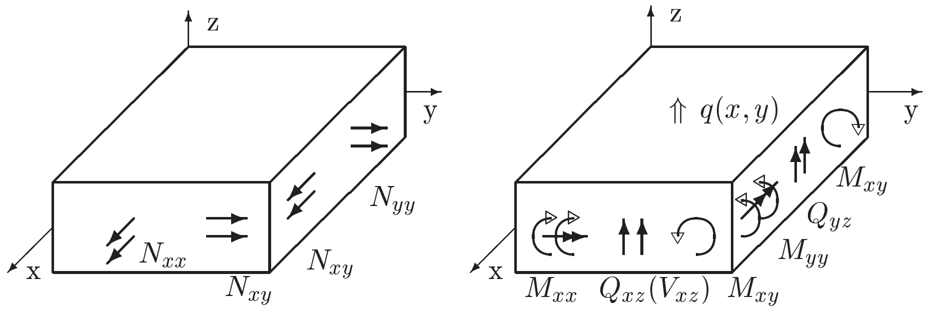


Figure 7.12: Element of stressed plate.

Because the plate is thin, we begin by expanding the displacements in a Taylor series (in terms of z) about the mid-plane values as

$$\begin{aligned}\bar{u}(x,y,z) &\approx u(x,y) - z\psi_x(x,y)\\ \bar{v}(x,y,z) &\approx v(x,y) - z\psi_y(x,y)\\ \bar{w}(x,y,z) &\approx w(x,y)\end{aligned} \tag{7.4}$$

where ψ_x and ψ_y are rotations of the subscripted faces in the directions of the curvatures. These say that the deformation is governed by five independent functions: $u(x,y)$, $v(x,y)$ are the in-plane displacements; $w(x,y)$ is the out-of-plane displacement; and $\psi_x(x,y)$, $\psi_y(x,y)$ are the rotations of the mid-plane. The normal and shear strains corresponding to these deformations are

$$\begin{aligned}\bar{\epsilon}_{xx} &= \frac{\partial\bar{u}}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial\psi_x}{\partial x}\\ \bar{\epsilon}_{yy} &= \frac{\partial\bar{v}}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial\psi_y}{\partial y}\\ \bar{\gamma}_{xy} &= \frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x} = \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) - z\Big(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\Big)\\ \bar{\gamma}_{xz} &= \frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x} = \Big(-\psi_x + \frac{\partial w}{\partial x}\Big)\\ \bar{\gamma}_{yz} &= \frac{\partial\bar{v}}{\partial z} + \frac{\partial\bar{w}}{\partial y} = \Big(-\psi_y + \frac{\partial w}{\partial y}\Big)\end{aligned}$$

Because the plate is thin, the stress in the z direction cannot be very large. A special case that arises in the analysis of thin-walled structures is that of *plane stress*. Here, the stress through the thickness of the plate is approximately zero such that $\sigma_{zz} \approx 0$, $\sigma_{xz} \approx 0$, and $\sigma_{yz} \approx 0$. This leads to

$$\epsilon_{zz} = \frac{-\nu}{E}[\sigma_{xx} + \sigma_{yy}] = \frac{-\nu}{1-\nu}[\epsilon_{xx} + \epsilon_{yy}]$$

Substituting this into the 3-D Hooke's law then gives

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}]\,, \qquad \sigma_{xx} = \frac{E}{(1-\nu^2)}[\epsilon_{xx} + \nu\epsilon_{yy}]$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}]\,, \qquad \sigma_{yy} = \frac{E}{(1-\nu^2)}[\epsilon_{yy} + \nu\epsilon_{xx}] \tag{7.5}$$

The shear relation is unaffected. Substituting for the strains in the Hooke's law then leads to

$$\begin{aligned}
\bar\sigma_{xx} &= \frac{E}{1-\nu^2}\,[\bar\epsilon_{xx} + \nu\bar\epsilon_{yy}] = \frac{E}{1-\nu^2}\Big[\Big(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\Big) - z\Big(\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\Big)\Big]\\
\bar\sigma_{yy} &= \frac{E}{1-\nu^2}\,[\bar\epsilon_{yy} + \nu\bar\epsilon_{xx}] = \frac{E}{1-\nu^2}\Big[\Big(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\Big) - z\Big(\frac{\partial \psi_y}{\partial y} + \nu\frac{\partial \psi_x}{\partial x}\Big)\Big]\\
\bar\sigma_{zz} &= 0 \qquad\qquad (7.6)\\
\bar\sigma_{xy} &= G\gamma_{xy} = G\Big[\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) - z\Big(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\Big)\Big]\\
\bar\sigma_{xz} &= G\gamma_{xz} = G\Big[-\psi_x + \frac{\partial w}{\partial x}\Big], \qquad \bar\sigma_{yz} = G\gamma_{yz} = G\Big[-\psi_y + \frac{\partial w}{\partial y}\Big]
\end{aligned}$$

Although the plate is treated as being in plane stress, we still retain the $\bar\sigma_{xz}$ and $\bar\sigma_{yz}$ shear stresses.

The strain energy for the plate is

$$\mathcal{U} = \tfrac{1}{2}\int_V [\bar\sigma_{xx}\bar\epsilon_{xx} + \bar\sigma_{yy}\bar\epsilon_{yy} + \bar\sigma_{xy}\bar\gamma_{xy} + \bar\sigma_{xz}\bar\gamma_{xz} + \bar\sigma_{yz}\bar\gamma_{yz}]\; dV$$

Substitute for the stresses and strains and integrate with respect to the thickness to get the total strain energy as

$$\begin{aligned}
\mathcal{U} = {}& \tfrac{1}{2}\int_A D\Big[\Big(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\Big)^2 - \tfrac{1}{2}(1-\nu)\Big[4\frac{\partial \psi_x}{\partial x}\frac{\partial \psi_y}{\partial y} - \Big(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y}\Big)^2\Big]\Big]dxdy\\
&+\tfrac{1}{2}\int_A Gh\Big[\Big(\psi_x - \frac{\partial w}{\partial x}\Big)^2 + \Big(\psi_y - \frac{\partial w}{\partial y}\Big)^2\Big]dx\,dy \qquad\qquad (7.7)\\
&+\tfrac{1}{2}\int_A \Big[E^*h\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\Big] + Gh\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big]dx\,dy
\end{aligned}$$

where $D \equiv Eh^3/12(1-\nu^2)$ is called the *plate bending stiffness* and $E^* \equiv E/(1-\nu^2)$. If the applied surface tractions and loads on the plate are as shown in Figure 7.12, then the potential of these loads is

$$\mathcal{V} = -\int q_w(x,y)w\,dx\,dy - N_{xx}u - N_{xy}v - M_{xx}\psi_x - M_{xy}\psi_y - V_{xz}w + \cdots$$

where the edge loads can be on each face. The energies de-couple into in-plane (u and v) and out-of-plane (w, ψ_x and ψ_y) sets; hence, we now find it convenient to treat them separately.

II: In-Plane Membrane Behavior

The energy and potential for the in-plane behavior are

$$\begin{aligned} U &= \tfrac{1}{2}\int_A \Big[E^* h\big[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\big] + Gh\big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\big)^2\Big]\,dx\,dy \\ \mathcal{V} &= -N_{xx}u - N_{xy}v + \cdots \end{aligned} \tag{7.8}$$

Application of our variational principle with variations in δu and δv leads to two differential equations

$$\begin{aligned} \frac{Eh}{1-\nu^2}\Big[\nabla^2 u - \tfrac{1}{2}(1+\nu)\big(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y}\big)\Big] &= 0 \qquad (7.9) \\ \frac{Eh}{1-\nu^2}\Big[\nabla^2 v - \tfrac{1}{2}(1+\nu)\big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\big)\Big] &= 0\,, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

These are the *Navier's equations.* as used in Section 5.5; the derivation here, however, shows that they are not restricted to very thin plates.

For the associated boundary conditions, we specify one condition from either set:

$$\Big\{u \quad \text{or} \quad N_{xx} = \frac{Eh}{1-\nu^2}\big[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\big]\Big\}\,, \qquad \Big\{v \quad \text{or} \quad N_{xy} = Gh\big[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\big]\Big\}$$

We can give interpretation of the boundary conditions in terms of resultants on the cross-section. For example, the resultants of the normal and shear stresses are defined on the cross-section as

$$N_{xx}(x,y) \equiv \int \bar{\sigma}_{xx}(x,y,z)\,dz\,, \qquad N_{xy}(x,y) \equiv \int \bar{\sigma}_{xy}(x,y,z)\,dz$$

and leads to

$$\begin{aligned} N_{xx} &= \frac{Eh}{(1-\nu^2)}\big[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\big] = \sigma_{xx}h \\ N_{yy} &= \frac{Eh}{(1-\nu^2)}\big[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\big] = \sigma_{yy}h \\ N_{xy} &= \frac{Eh}{2(1+\nu)}\big[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\big] = \sigma_{xy}h \end{aligned} \tag{7.10}$$

That is, N_{xx} and so on, are the resultant forces per unit length due to the stresses acting on the edge faces.

Example 7.11: Specialize the Navier's equation to the case where there is only an x dependence.

There are no derivatives with respect to y. The first of the two Navier's equations becomes

$$\frac{Eh}{1-\nu^2}\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{or} \qquad E^* h\frac{\partial^2 u}{\partial x^2} = 0$$

which is the one-dimensional rod equation. The second of the Navier's equations becomes

$$\frac{Eh}{1-\nu^2}\Big[\frac{\partial^2 v}{\partial x^2} - \tfrac{1}{2}(1+\nu)\Big(\frac{\partial^2 v}{\partial x^2}\Big)\Big] = 0 \qquad \text{or} \qquad Gh\frac{\partial^2 v}{\partial x^2} = 0$$

This is also a one-dimensional equation but it is for the transverse shear behavior. This is not the transverse flexural shear behavior.

For the associated boundary conditions, we specify one condition from either set:

$$\Big\{u \quad \text{or} \quad N_{xx} = \frac{Eh}{1-\nu^2}\frac{\partial u}{\partial x}\Big\}, \qquad \Big\{v \quad \text{or} \quad N_{xy} = Gh\frac{\partial v}{\partial x}\Big\}$$

These mimic the boundary conditions for the elementary theory. ■

III: Out-of-Plane Flexural Behavior

The energy and potential associated with the out-of-plane behavior are

$$\begin{aligned}
\mathcal{U} = {} & \tfrac{1}{2}\int_A D\Big[\Big(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\Big)^2 - \tfrac{1}{2}(1-\nu)\Big[4\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y} - \Big(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}\Big)^2\Big]\Big]\,dxdy \\
& + \tfrac{1}{2}\int_A Gh\Big[\Big(\psi_x - \frac{\partial w}{\partial x}\Big)^2 + \Big(\psi_y - \frac{\partial w}{\partial y}\Big)^2\Big]dx\,dy \\
\mathcal{V} = {} & -\int q_w(x,y)w\,dx\,dy - M_{xx}\psi_x - M_{xy}\psi_y - V_{xz}w + \cdots
\end{aligned} \tag{7.11}$$

An application of our variational principle with the variations of δw, $\delta\psi_x$ and $\delta\psi_y$, leads to, respectively,

$$\begin{aligned}
q_w + Gh\frac{\partial}{\partial x}\Big[\frac{\partial w}{\partial x} - \psi_x\Big] + Gh\frac{\partial}{\partial y}\Big[\frac{\partial w}{\partial y} - \psi_y\Big] &= 0 \\
\tfrac{1}{2}D\Big[(1-\nu)\nabla^2\psi_x + (1+\nu)\frac{\partial}{\partial x}\Big(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\Big)\Big] + Gh\Big[\frac{\partial w}{\partial x} - \psi_x\Big] &= 0 \\
\tfrac{1}{2}D\Big[(1-\nu)\nabla^2\psi_y + (1+\nu)\frac{\partial}{\partial y}\Big(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\Big)\Big] + Gh\Big[\frac{\partial w}{\partial y} - \psi_y\Big] &= 0
\end{aligned} \tag{7.12}$$

These are the governing equations for the *Mindlin plate*; this theory accounts for the shear deformation. The associated boundary conditions (on each edge face of the plate) are specified in terms of any three conditions selected from the following groups:

$$\begin{aligned}
&\Big\{w \quad \text{or} \quad V_{xz} = Gh\Big[\frac{\partial w}{\partial x} - \psi_x\Big]\Big\} \\
&\Big\{\psi_x \quad \text{or} \quad M_{xx} = D\Big[\frac{\partial\psi_x}{\partial x} + \nu\frac{\partial\psi_y}{\partial y}\Big]\Big\} \\
&\Big\{\psi_y \quad \text{or} \quad M_{xy} = \tfrac{1}{2}(1-\nu)D\Big[\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\Big]\Big\}
\end{aligned}$$

These are specified for an x-face, the other faces are similar.

We can give interpretation to the boundary conditions in terms of resultants of the stresses on the cross-section. For example, taking resultants for the shear stress defined as

$$Q_{xz}(x,y) \equiv \int \bar{\sigma}_{xz}(x,y,z)\,dz = \int G\Big[-\psi_x + \frac{\partial w}{\partial x}\Big]\,dz$$

leads to

$$Q_{xz} = Gh\Big[-\psi_x + \frac{\partial w}{\partial x}\Big] = V_{xz}\,, \quad Q_{yz} = Gh\Big[-\psi_y + \frac{\partial w}{\partial y}\Big] = V_{yz} \tag{7.13}$$

We can also take a moment due to the stresses acting on the edge faces. For example,

$$M_{xx} \equiv -\int \sigma_{xx} z\,dz = \frac{Eh^3}{12(1-\nu^2)}\Big[\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\Big]$$

and all resultants can be written as

$$\begin{aligned} M_{xx} &= D\Big[\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\Big] \\ M_{yy} &= D\Big[\frac{\partial \psi_y}{\partial y} + \nu\frac{\partial \psi_x}{\partial x}\Big] \\ 2M_{xy} &= D\Big[\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y}\Big](1-\nu) \end{aligned} \tag{7.14}$$

These moment resultants are related only to the rotations.

In order to account for the truncation error of the expansions $\bar{u}$ and $\bar{v}$, we could add correction coefficients to the energies as was done with the Timoshenko beam theory.

IV: Flexural Behavior of Very Thin Plates

The plate theory derived here (called classical plate theory) is the 2-D equivalent of the Bernoulli-Euler beam theory. Rather than go directly to the governing equations, we will retrace the developments of the Mindlin plate, but with the assumptions of the classical theory.

We modify the Mindlin equations to the thin-plate theory in two steps. First, we assume that the transverse shear deformation is negligible; this is equivalent to saying that the shear stiffness in the transverse direction is infinite. This leads to

$$\frac{\partial w}{\partial x} - \psi_x = 0\,, \qquad \frac{\partial w}{\partial y} - \psi_y = 0$$

It is important to realize that while these combinations are zero, their product with Gh is nonzero (because it is related to the transverse shear resultant). The displacements for the flexural motion are approximated as

$$\bar{u}(x,y,z) \approx -z\frac{\partial w}{\partial x}(x,y), \quad \bar{v}(x,y,z) \approx -z\frac{\partial w}{\partial y}(x,y), \quad \bar{w}(x,y,z) \approx w(x,y)$$

The normal and shear strains corresponding to these deformations are

$$
\begin{aligned}
\bar{\epsilon}_{xx} &= \frac{\partial \bar{u}}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}\,, \qquad \bar{\epsilon}_{yy} = \frac{\partial \bar{v}}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\
\bar{\gamma}_{xy} &= \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \\
\bar{\gamma}_{xz} &= \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} = 0\,, \qquad \bar{\gamma}_{yz} = \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} = 0
\end{aligned}
$$

We reiterate that, although the transverse shear strains are zero, the transverse shear forces are nonzero. Also note that there is an in-plane shear that depends on the distance from the midplane — there is no comparable quantity in beam theories. Substituting the above strains into the Hooke's law for plane stress gives

$$
\begin{aligned}
\bar{\sigma}_{xx} &= \frac{-Ez}{1-\nu^2}\Big[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\Big] \\
\bar{\sigma}_{yy} &= \frac{-Ez}{1-\nu^2}\Big[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\Big] \\
\bar{\sigma}_{xy} &= -2Gz\Big[\frac{\partial^2 w}{\partial x \partial y}\Big]
\end{aligned}
\tag{7.15}
$$

The strain energy for a plate in plane stress is

$$
\mathcal{U} = \tfrac{1}{2}\int_V [\bar{\sigma}_{xx}\bar{\epsilon}_{xx} + \bar{\sigma}_{yy}\bar{\epsilon}_{yy} + \bar{\sigma}_{xy}\bar{\gamma}_{xy}]\, dV
$$

Substitute for the stresses and strains and integrate with respect to the thickness to get the total strain energy as

$$
\mathcal{U} = \tfrac{1}{2}\int D\Big[(\nabla^2 w)^2 + 2(1-\nu)[(\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}]\Big]\, dx\, dy
$$

The potential of the applied loads is

$$
\mathcal{V} = -\int q_w(x,y) w\, dx\, dy - M_{xx}\frac{\partial w}{\partial x} - V_{xz} w + \cdots
$$

where the edge loads are on each face. Using variational principle with the variation of only δw then leads to the governing equation

$$
D\nabla^2\nabla^2 w = q \tag{7.16}
$$

Performing the integration by parts required to get the boundary conditions is rather involved for an arbitrary boundary — a detailed description is given in Reference [27]. The associated boundary conditions are found to be

$$
\begin{aligned}
&\Big\{ w \quad \text{or} \quad V_{xz} = -D\Big[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\Big]\Big\} \\
&\Big\{ \frac{\partial w}{\partial x} \quad \text{or} \quad M_{xx} = D\Big[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\Big]\Big\}
\end{aligned}
\tag{7.17}
$$

The shear to be specified is called the *Kirchhoff shear*. This shear is not the resultant Q_{xz} but is actually given by

$$\text{Kirchhoff shear:} \qquad V_{xz} = Q_{xz} - \frac{\partial M_{xy}}{\partial y}$$

This can be understood physically by realizing that the shear moment M_{xy} can be interpreted as a couple comprised of vertical forces a small distance apart. Then, because the moment is distributed, so too are the vertical forces, which consequently at any given location will have an imbalance in the vertical forces. Alternatively, the classical plate theory has restrictive degrees of freedom, where the shear strains γ_{xz} and γ_{yz} are zero. That is, the shear resultants Q_{xz} and Q_{yz} do not have a relationship to the corresponding deformation. While this can be rationalized in the constitutive relation by saying that the shear modulus in the transverse direction is very large, it means that the resultant force is associated with higher-order derivatives of the deformation.

The resultants can be written as

$$\begin{aligned} M_{xx} &= D\,[\kappa_{xx} + \nu\kappa_{yy}] = D\Big[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\Big] \\ M_{yy} &= D\,[\kappa_{yy} + \nu\kappa_{xx}] = D\Big[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\Big] \\ M_{xy} &= M_{yx} = D(1-\nu)\kappa_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{7.18}$$

These resultants are related only to the out-of-plane deflection. The stresses are obtained from equations such as

$$\sigma_{xx} = -\frac{M_{xx} z}{I_p}$$

with $I_p \equiv h^3/12$.

Example 7.12: Specialize the thin-plate flexural equations when there is no y dependence.

There are no derivatives with respect to y, and the summary of plate equations becomes

$$\begin{aligned} \text{Displacement}: &\quad w = w(x,t) \\ \text{Slope}: &\quad \psi_x = \frac{\partial w}{\partial x} \\ \text{Moment}: &\quad M_{xx} = +D\frac{\partial^2 w}{\partial x^2} \\ \text{Shear}: &\quad V_{xz} = -D\frac{\partial^3 w}{\partial x^3} \\ \text{Loading}: &\quad q = D\frac{\partial^4 w}{\partial x^4} \end{aligned} \tag{7.19}$$

These are the equations for a beam if we make the associations

$$D \iff EI \qquad \text{or} \qquad E/(1-\nu^2) \iff Eb$$

A plate deforming as assumed here is called *cylindrical bending*. ■

Exercises

7.1 Show that the variational problem

$$J[\,u\,] = \int_0^{10} [\tfrac{1}{2}(\frac{du}{dx})^2 - 100u]\,dx = \min$$

corresponds to the following differential equation

$$\frac{d^2u}{dx^2} + 100 = 0 \qquad 0 \le x \le 10$$

subject to the boundary conditions $u(0) = u(10) = 0$.

7.2 Obtain a Ritz solution to the previous exercise using the trial function $f(x) = x(10 - x)$. Compare this approximate solution with the exact solution.

7.3 In general, it is not possible to obtain a functional for problems whose governing differential equation contains odd-power derivatives. A special exception is the following case

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = 0$$

where a and b are constants. Show that the variational problem

$$J(u) = \int [\tfrac{1}{2}e^{ax}(\frac{du}{dx})^2 - bu]\,dx = \min$$

corresponds to the differential equation.

7.4 Suppose we wish to derive a "higher-order" rod element and to that end we take the deflection function in the cubic form

$$u(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

Introducing the nodal degree of freedoms $u_1 = u(0)$, $\phi_1 = du(0)/dx$, $u_2 = u(L)$, $\phi_2 = du(L)/dx$, show that the displacement can be represented in terms of the nodal values as

$$u(x) = g_1(x)u_1 + g_2(x)L\phi_1 + g_3(x)u_2 + g_4(x)L\phi_2$$

where the functions $g_n(x)$ are identical to those for the beam shape functions. Show that the higher-order rod element stiffness matrix is

$$[\ k\ ] = \frac{EA}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

Note that this is almost identical to the geometric stiffness matrix for the beam; Why?

7.5 Suppose that only the axial forces F_1, F_2 are taken as the nodal loads in the previous exercise, show that the elementary rod stiffness relation is recovered. Propose some nodal loads that do not give the trivial result.

7.6 With reference to Figure 7.1, use the Ritz method to find an approximate solution using

$$u(x) = a_0 + a_1 x$$

7.7 Consider a cantilever beam, fixed at the end $x = 0$ and subjected to a given displacement $v_L = c$ at the other. Show that the following is a set of admissible displacements and obtain the corresponding Ritz solution.

$$v(x) = \frac{cx^2}{L^2} + \sum a_n[1 - \cos(2n\pi x/L)]$$

7.8 Consider a cantilever beam, fixed at the end $x = 0$ and subjected to a concentrated lateral applied force at the other. Using the Ritz method, show that the following displacements

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

leads to the exact solution. Show that the addition of extra terms have zero contributions.

7.9 A uniformly loaded beam is simply supported at both ends. Find the deflection and bending moment at the center using the Ritz method. First use a quadratic function in x and then use a sine function in x. Compare the results with the exact solution and say why the second solution is better than the first.

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