

Lecture 6 - Plastic Zone

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering

3 February 2022

1

## schedule

- 3 Feb - Plastic Zone, HW 2 Due, HW 1 Self-grade due
- 8 Feb - Fracture Toughness
- 10 Feb - Fracture Toughness, HW3 Due, HW 2 Self-grade due
- 15 Feb - Residual Strength

2

- plastic stress intensity ratio
- plastic zone shape
- group problems

## plastic stress intensity ratio

---

## plastic stress intensity ratio

- Engineers often use stress intensity to decide which material to use for a certain application
- The ratio of plastic stress intensity to elastic stress intensity, as a function of yield stress over applied stress, can help illustrate the effects of plasticity for different materials.

4

## plastic stress intensity ratio

For an infinitely wide center-cracked panel, we can solve for  $K_{Ie}/K_I$  symbolically, in plane stress

$$K_I = \sigma \sqrt{\pi a}$$

$$K_{Ie} = \sigma \sqrt{\pi(a + r_p)}$$

$$r_p = \frac{1}{2\pi} \left( \frac{K_{Ie}}{\sigma_{YS}} \right)^2$$

$$K_{Ie} = \sigma \sqrt{\pi \left( a + \frac{1}{2\pi} \left( \frac{K_{Ie}}{\sigma_{YS}} \right)^2 \right)}$$

5

## stress intensity ratio

$$\begin{aligned}K_{Ie}^2 &= \sigma^2 \pi \left( a + \frac{1}{2\pi} \left( \frac{K_{Ie}}{\sigma_{YS}} \right)^2 \right) \\K_{Ie}^2 &= \sigma^2 \pi a + \frac{\sigma^2}{2} \left( \frac{K_{Ie}}{\sigma_{YS}} \right)^2 \\K_{Ie}^2 - \frac{\sigma^2}{2} \left( \frac{K_{Ie}}{\sigma_{YS}} \right)^2 &= \sigma^2 \pi a \\K_{Ie}^2 \left( 1 - \frac{\sigma^2}{2\sigma_{YS}^2} \right) &= \sigma^2 \pi a\end{aligned}$$

Note: square both sides

6

## plastic stress intensity ratio

$$\begin{aligned}K_{Ie}^2 &= \frac{\sigma^2 \pi a}{1 - \frac{\sigma^2}{2\sigma_{YS}^2}} \\K_{Ie} &= \frac{\sigma \sqrt{\pi a}}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}} \\K_{Ie} &= \frac{K_I}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}} \\\frac{K_{Ie}}{K_I} &= \frac{1}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}}\end{aligned}$$

Note: We divide both sides by  $\left( 1 - \frac{\sigma^2}{2\sigma_{YS}^2} \right)$

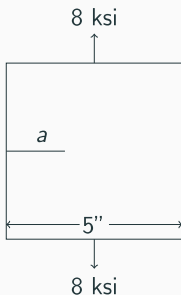
7

- We can also find the plastic stress intensity ratio numerically for finite width panels
- Panel thickness, yield stress, panel width, crack length could all be variables in this case
- Different heat treatments of metal alloys can give a different yield stress, with most other properties remaining the same
- Typical crack lengths can vary based on inspection cycles

8

### example

- You are asked to design an inspection cycle for a panel
- Consider the plastic stress intensity ratio, and the effect of varying crack lengths on it



9

online example here<sup>1</sup>

---

<sup>1</sup>[https://colab.research.google.com/drive/1Bb-eznneklW\\_BILR8po3\\_fQROB\\_t56\\_w?usp=sharing](https://colab.research.google.com/drive/1Bb-eznneklW_BILR8po3_fQROB_t56_w?usp=sharing)

10

## plastic zone shape

---

- Although we drew a circle to give a rough idea of the plastic zone in Irwin's method, this solution was only 1D
- We considered  $\theta = 0$ .
- It is advantageous to model the plastic zone shape, we will do so using two different yield theories
- Von Mises and Tresca

11

## principal stresses

- Principal stresses are often used in yield theories
- We can determine the principal stresses near the crack tip as

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \right)$$

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \right)$$

$$\sigma_3 = 0 \quad \text{(plane stress)}$$

$$\sigma_3 = \frac{2\nu K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \quad \text{(plane strain)}$$

12

## Von Mises yield theory

- The Von Mises yield theory is also known as the Distortion Energy Yield Theory
- In this yield theory, we assume that failure or yielding occurs when the strain energy exceeds some threshold
- It has been observed that hydrostatic pressure does not generally cause yielding
- We separate the strain energy into two parts, volumetric and distortional
- Only the distortional strain energy is used to determine the yield strength

13

## Von Mises yield theory

- The distortional strain energy is given by

$$W_d = \frac{1}{12} G \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$

- Which for a uniaxially loaded point becomes

$$W_d = \frac{1}{6} G \sigma_{YS}^2$$

- We can equate the two cases and solve

$$\frac{1}{6} G \sigma_{YS}^2 = \frac{1}{12} G \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$

$$2\sigma_{YS}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

14



## Von Mises yield theory

- We can find the plastic zone size,  $r_p$  by substituting the principal stress relations into the distortional strain energy equation
- In plane stress we find

$$2\sigma_{YS}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - 0)^2 + (0 - \sigma_1)^2$$

15

## Von Mises yield theory

$$\begin{aligned} 2\sigma_{YS}^2 = & \left( \frac{K_I}{\sqrt{2\pi}r_p} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \right) - \right. \\ & \left. \frac{K_I}{\sqrt{2\pi}r_p} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \right) \right)^2 + \\ & \left( \frac{K_I}{\sqrt{2\pi}r_p} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \right) - 0 \right)^2 + \\ & \left( 0 - \frac{K_I}{\sqrt{2\pi}r_p} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \right) \right)^2 \end{aligned}$$

16

## Von Mises yield theory

- After solving we find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2} \cos^2 \frac{\theta}{2} \left( 1 + 3 \sin^2 \frac{\theta}{2} \right)$$

- We can similarly solve for  $r_p$  in plane strain to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2} \cos^2 \frac{\theta}{2} \left( 1 - 4\nu + 4\nu^2 + 3 \sin^2 \frac{\theta}{2} \right)$$

17

## Tresca yield theory

- Tresca yield theory assumes that yielding begins when the maximum shear stress reaches a critical value
- In uniaxial tension this gives

$$\tau_0 = \tau_{max} = \frac{1}{2} (\sigma_{max} - \sigma_{min}) = \frac{1}{2} (\sigma_{YS} - 0) = \frac{\sigma_{YS}}{2}$$

18

## Tresca yield theory

- Using the results for principal stress we found previously, we see that

$$\sigma_{max} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \right)$$
$$\sigma_{min} = 0$$

- We can substitute and solve as before to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2} \cos^2 \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \right)^2$$

19

## Tresca yield theory

- In plane strain, it is not clear whether  $\sigma_2$  or  $\sigma_3$  will be  $\sigma_{min}$
- We can solve for when  $\sigma_2$  will be  $\sigma_{min}$

$$\sigma_2 < \sigma_3$$
$$\frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \right) < \frac{2\nu K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2}$$
$$1 - \sin \frac{\theta}{2} < 2\nu$$
$$\theta_t > 2 \sin^{-1}(1 - 2\nu)$$

20

- When  $2\pi - \theta_t < \theta < \theta_t$ ,  $\sigma_2$  is the minimum, otherwise  $\sigma_3$  is the minimum
- Note: Error(s) in text p. 101
- Once we have chosen the appropriate minimum stress ( $\sigma_2$  or  $\sigma_3$ ), we can solve for  $r_p$  as before

21

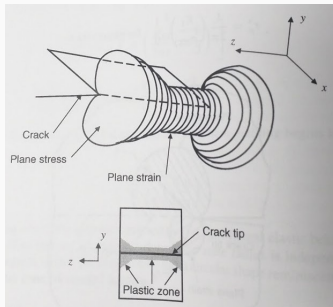
# Tresca yield theory

$$r_p = \frac{2K_I^2}{\pi\sigma_{YS}^2} \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \quad \theta_t \leq \theta < 2\pi - \theta_t$$

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2} \cos^2 \frac{\theta}{2} \left( 1 - 2\nu + \sin \frac{\theta}{2} \right)^2 \quad \theta_t < \theta < 2\pi - \theta_t$$

22

## 3D plastic zone shape



**Figure 1:** An image showing the 3D plastic zone shape, which looks a little bit like a dumbbell. The plastic zone is much larger near the surface, where the material behaves as if in plane stress. In the 23

### example

online example here<sup>2</sup>

<sup>2</sup><https://colab.research.google.com/drive/1ALdCMw3BzNDn-5clui2nrvfHfWMcN7-5?usp=sharing>

## group problems

---

### group one

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of  $\sigma_{YS} = 55$  MPa, with an applied load of  $\sigma = 20$  MPa
- Assume the panel is very thin

## group two

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of  $\sigma_{YS} = 55$  MPa, with an applied load of  $\sigma = 20$  MPa
- Assume the panel is very thick

26

## group three

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of  $\sigma_{YS} = 55$  MPa, with an applied load of  $\sigma = 20$  MPa
- The panel thickness is  $t = 0.65$  cm

27

- Find the plastic stress intensity ratio for an infinitely wide, center-cracked panel
- What factors will increase or decrease the plastic stress intensity ratio?