AE 737 - MECHANICS OF DAMAGE TOLERANCE

Lecture 5

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- Superposition: addition and subtraction are fine
- When comparing stress intensity factors, it is important for crack length to be the same
- ”Quarter circular crack” is a corner crack with $a = c$
- p. 51 My/I
- HW1 3 & 16 $a = c$
- Added ”last updated” to homework and title slide
- Homework can be turned in before class in my mail box or office

- 4 Feb - Plastic Zone, Homework 1 Due, Homework 2 Assigned
- 9 Feb - Fracture Toughness, Homework 2 Due, Homework 3 Assigned
- 11 Feb - Fracture Toughness
- 16 Feb - Residual Strength, Homework 3 Due, Homework 4 Assigned

OUTLINE

1. plastic zone
2. plastic stress intensity ratio
3. plastic zone shape

PLASTIC ZONE

- Previous developments assumed perfectly elastic materials
- Most common materials have some plasticity
- Any stress above the yield stress will undergo plastic deformation (no stress higher than σ_y will be present in the material)

- Plasticity helps retard crack propagation due to residual stresses
- After an overload, elastic regions will contract back to their undeformed shape
- The region which has undergone plastic deformation, however, holds its deformed shape
- This introduces a region of residual compressive stress near the crack tip
- Before the crack can propagate, a stress needs to overcome this residual stress

- We often simplify the full 3D elasticity equations for planar problems
- For very thin panels, we assume that all out-of-plane stresses are 0
- This is called plane stress

$$\sigma_z = \tau_{xz} = \tau_{zy} = 0 \tag{5.1a}$$
$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \tag{5.1b}$$
$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \tag{5.1c}$$
$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \tag{5.1d}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{5.1e}$$
$$\gamma_{xz} = \gamma_{yz} = 0 \tag{5.1f}$$

- When instead a panel is very thick, we assume that any strains through the thickness are small relative to other strains
- $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
- This is known as plane strain

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{5.2a}$$

$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \tag{5.2b}$$

$$0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \tag{5.2c}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{5.2d}$$

$$\gamma_{xz} = \gamma_{yz} = 0 \tag{5.2e}$$

IRWIN'S FIRST APPROXIMATION

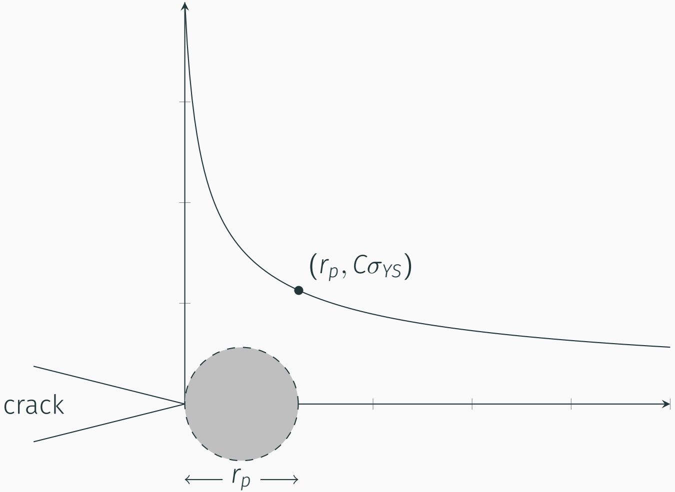
- If we recall the equation for opening stress (σ_y) near the crack tip

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{1.2}$$

- In the plane of the crack, when $\theta = 0$ we find

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}$$

IRWIN'S FIRST APPROXIMATION



- We use C "Plastic Constraint Factor" to convert between Plane Strain and Plane Stress solutions
- The plastic zone size can now be approximated by solving the equation $\sigma_{yy}(r = r_p) = C\sigma_{YS}$

$$\sigma_{yy}(r = r_p) = C\sigma_{YS} \tag{5.3a}$$

$$\frac{K_I}{\sqrt{2\pi r_p}} = C\sigma_{YS} \tag{5.3b}$$

$$r_p = \frac{1}{2\pi}\left(\frac{K_I}{C\sigma_{YS}}\right)^2 \tag{5.3c}$$

IRWIN'S FIRST APPROXIMATION

- For plane stress (thin panels) we let $C = 1$ and find r_p as

$$r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{5.4}$$

- And for plane strain (thick panels) we let $C = \sqrt{3}$ and find

$$r_p = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{5.5}$$

- For panels which lie between plane strain and plane stress states, we use the following expression to estimate the plastic zone size

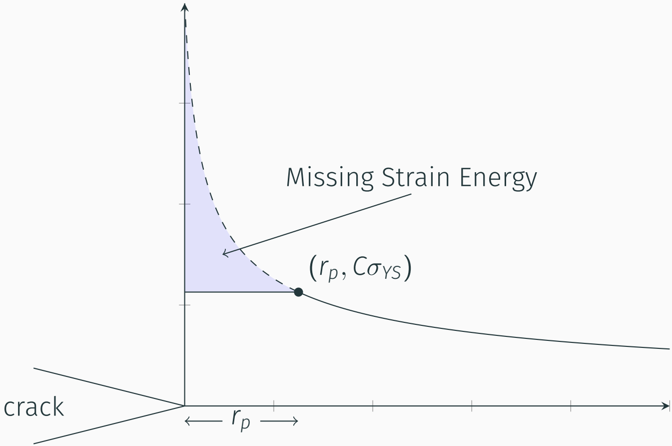
$$r_p = \frac{1}{I\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{5.6}$$

- Where I is defined as

$$I = 6.7 - \frac{1.5}{t}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{5.7}$$

- And $2 \leq I \leq 6$

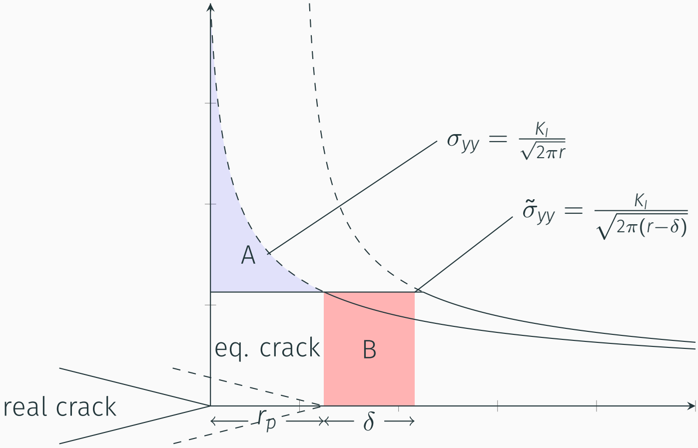
- If our material is perfectly elastic-plastic, no stresses above $C\sigma_{YS}$ will exist in the material
- This ignores the strain energy (represented by the area under the curve) in the plastic zone


Missing Strain Energy
$(r_p, C\sigma_{YS})$
crack
r_p

IRWIN'S SECOND APPROXIMATION

- To account for the additional strain energy, Irwin considered a plastic zone size increased by some δ
- He also needed to adjust the stress function, and considered an equivalent crack tip in these calculations

IRWIN'S SECOND APPROXIMATION



- We need $A = B$, so we set them equivalent and solve for δ.

$$A = \int_0^{r_p} \sigma_{yy} dr - r_p \sigma_{YS} \tag{5.8a}$$

$$= \int_0^{r_p} \frac{K_I}{\sqrt{2\pi r}} dr - r_p \sigma_{YS} \tag{5.8b}$$

$$= \frac{K_I}{\sqrt{2\pi}} \int_0^{r_p} r^{-1/2} dr - r_p \sigma_{YS} \tag{5.8c}$$

$$= \frac{2K_I \sqrt{r_p}}{\sqrt{2\pi}} - r_p \sigma_{YS} \tag{5.8d}$$

- We have already found r_p as

$$r_p = \frac{1}{2\pi} \left(\frac{K_I}{\sigma_{YS}} \right)^2 \tag{5.8e}$$

- If we solve this for K_I we find

$$K_I = \sqrt{2\pi r_p} \sigma_{YS} \tag{5.8f}$$

- We can now substitute back into the strain energy of A

$$A = \frac{2\sqrt{2\pi r_p}\sigma_{YS}\sqrt{r_p}}{\sqrt{2\pi}} - r_p\sigma_{YS} \tag{5.8g}$$

$$= 2\sigma_{YS}r_p - r_p\sigma_{YS} \tag{5.8h}$$

$$= r_p\sigma_{YS} \tag{5.8i}$$

- B is given simply as $B = \delta\sigma_{YS}$, so we equate A and B to find δ

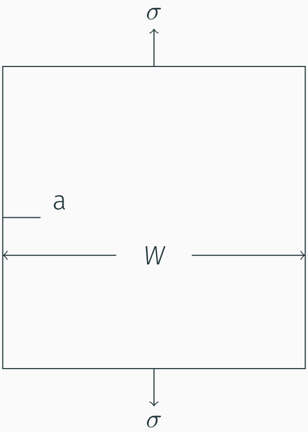
$$A = B \tag{5.8j}$$

$$r_p\sigma_{YS} = \delta\sigma_{YS} \tag{5.8k}$$

$$r_p = \delta \tag{5.8l}$$

- This means the plastic zone size is simply $2r_p$
- However, it also means that the effective crack length is $a + r_p$
- Since r_p depends on K_I, we must iterate a bit to find the "real" r_p and K_I

EXAMPLE



$$\beta = \left[1.122 - 0.231\frac{a}{W} + 10.55\left(\frac{a}{W}\right)^2 - 21.71\left(\frac{a}{W}\right)^3 + 30.82\left(\frac{a}{W}\right)^4\right] \tag{2.4a}$$

$$I = 6.7 - \frac{1.5}{t}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{4.13}$$

$$r_p = \frac{1}{I\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \tag{4.12}$$

PLASTIC STRESS INTENSITY RATIO

- Engineers often use stress intensity to decide which material to use for a certain application
- The ratio of plastic stress intensity to elastic stress intensity, as a function of yield stress over applied stress, can help illustrate the effects of plasticity for different materials.

- For an infinitely wide center-cracked panel, we can solve for K_{Ie}/K_I symbolically
- For plane stress we have:

$$K_I = \sigma\sqrt{\pi a} \tag{5.9a}$$

$$K_{Ie} = \sigma\sqrt{\pi(a + r_p)} \tag{5.9b}$$

$$r_p = \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2 \tag{5.9c}$$

$$K_{Ie} = \sigma\sqrt{\pi\left(a + \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2\right)} \tag{5.9d}$$

- We square both sides

$$K_{Ie}^2 = \sigma^2\pi\left(a + \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2\right) \tag{5.9e}$$

$$K_{Ie}^2 = \sigma^2\pi a + \frac{\sigma^2}{2}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2 \tag{5.9f}$$

$$K_{Ie}^2 - \frac{\sigma^2}{2}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2 = \sigma^2\pi a \tag{5.9g}$$

$$K_{Ie}^2\left(1 - \frac{\sigma^2}{2\sigma_{YS}^2}\right) = \sigma^2\pi a \tag{5.9h}$$

- We divide both sides by $\left(1 - \frac{\sigma^2}{2\sigma_{YS}^2}\right)$

$$K_{Ie}^2 = \frac{\sigma^2 \pi a}{1 - \frac{\sigma^2}{2\sigma_{YS}^2}} \tag{5.9i}$$

$$K_{Ie} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}} \tag{5.9j}$$

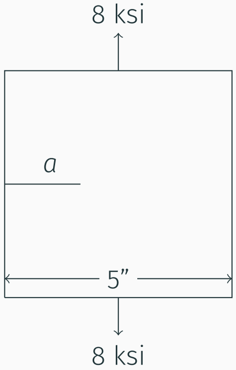
$$K_{Ie} = \frac{K_I}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}} \tag{5.9k}$$

$$\frac{K_{Ie}}{K_I} = \frac{1}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}} \tag{5.9l}$$

PLASTIC STRESS INTENSITY RATIO

- We can also find the plastic stress intensity ratio numerically for finite width panels
- Panel thickness, yield stress, panel width, crack length could all be variables in this case
- Different heat treatments of metal alloys can give a different yield stress, with most other properties remaining the same
- Typical crack lengths can vary based on inspection cycles

- You are trying to design an appropriate inspection cycle on a panel
- One item to consider is the plastic stress intensity ratio, consider the effect of varying crack lengths on the plastic stress intensity ratio.



PLASTIC ZONE SHAPE

- Although we drew a circle to give a rough idea of the plastic zone in Irwin's method, this solution was only 1D
- We considered $\theta = 0$.
- It is advantageous to model the plastic zone shape, we will do so using two different yield theories
- Von Mises and Tresca

PRINCIPAL STRESSES

- Principal stresses are often used in yield theories
- We can determine the principal stresses near the crack tip as

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right) \tag{5.10a}$$

$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right) \tag{5.10b}$$

$$\sigma_3 = 0 \qquad \text{(plane stress)} \tag{5.10c}$$

$$\sigma_3 = \frac{2\nu K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \qquad \text{(plane strain)} \tag{5.10d}$$

VON MISES YIELD THEORY

- The Von Mises yield theory is also known as the Distortion Energy Yield Theory
- In this yield theory, we assume that failure or yielding occurs when the strain energy exceeds some threshold
- It has been observed that hydrostatic pressure does not generally cause yielding
- We separate the strain energy into two parts, volumetric and distortional
- Only the distortional strain energy is used to determine the yield strength

- The distortional strain energy is given by

$$W_d = \frac{1}{12} G \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \tag{5.11}$$

- Which for a uniaxially loaded point becomes

$$W_d = \frac{1}{6} G \sigma_{YS}^2 \tag{5.12}$$

- We can equate the two cases and solve

$$\frac{1}{6} G \sigma_{YS}^2 = \frac{1}{12} G \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \tag{5.13a}$$

$$2\sigma_{YS}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \tag{5.13b}$$

- We can find the plastic zone size, r_p by substituting the principal stress relations (5.10a) into the distortional strain energy equation (5.11)
- In plane stress we find

$$2\sigma_{YS}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - 0)^2 + (0 - \sigma_1)^2 \tag{5.14a}$$

$$\begin{aligned} 2\sigma_{YS}^2 = &\left(\frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) - \frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)\right)^2 + \\ &\left(\frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right) - 0\right)^2 + \\ &\left(0 - \frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)\right)^2 \end{aligned} \tag{5.14b}$$

- After solving we find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1 + 3\sin^2\frac{\theta}{2}\right) \tag{5.15}$$

- We can similarly solve for r_p in plane strain to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1 - 4\nu + 4\nu^2 + 3\sin^2\frac{\theta}{2}\right) \tag{5.16}$$

- Tresca yield theory assumes that yielding begins when the maximum shear stress reaches a critical value
- In uniaxial tension this gives

$$\tau_0 = \tau_{max} = \frac{1}{2}\left(\sigma_{max} - \sigma_{min}\right) = \frac{1}{2}\left(\sigma_{YS} - 0\right) = \frac{\sigma_{YS}}{2} \tag{5.17}$$

- Using (5.10a), we see that

$$\sigma_{max} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right) \tag{5.18a}$$

$$\sigma_{min} = 0 \tag{5.18b}$$

- We can substitute and solve as before to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2} \cos^2\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)^2 \tag{5.19}$$

- In plane strain, it is not clear whether σ_2 or σ_3 will be σ_{min}
- We can solve for when σ_2 will be σ_{min}

$$\sigma_2 < \sigma_3 \tag{5.20a}$$

$$\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right) < \frac{2\nu K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \tag{5.20b}$$

$$1 - \sin\frac{\theta}{2} < 2\nu \tag{5.20c}$$

$$\theta_t > 2\sin^{-1}(1 - 2\nu) \tag{5.20d}$$

- When $2\pi - \theta_t < \theta < \theta_t$, σ_2 is the minimum, otherwise σ_3 is the minimum
- Note: Error(s) in text p. 101
- Once we have chosen the appropriate minimum stress ($\sigma_2 or \sigma_3$), we can solve for r_p as before

$$r_p = \frac{2K_I^2}{\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} \qquad \theta_t < \theta < 2\pi - \theta_t \tag{5.21a}$$

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1 - 2\nu + \sin\frac{\theta}{2}\right)^2 \qquad \theta < \theta_t, \theta > 2\pi - \theta_t \tag{5.21b}$$

3D plastic zone shape