

Lecture 11 - Multiple Site Damage, Mixed-Mode Fracture

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schedule

- 22 Feb - Multiple Site Damage, Mixed-Mode
- 24 Feb - Exam 1 Review, HW5 Due, HW 4 Self-grade due
- 1 Mar - Exam 1
- 3 Mar - Stress-based Fatigue

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- multiple site damage
- mixed mode fracture

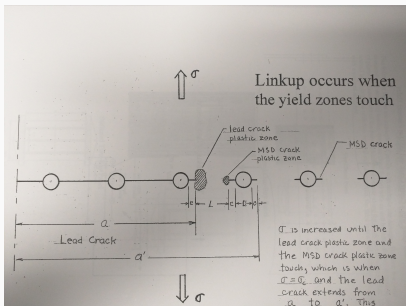
multiple site damage

multiple site damage

- Often damage can accumulate among multiple sources
- This is very common when there are a series of holes, each can develop cracks with a potential to link up
- “link up” occurs when the plastic zones between two adjacent cracks touch

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linkup



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linkup equation

- We know that

$$R_p = \frac{1}{2\pi} \left(\frac{K_{Ia}}{\sigma_{YS}} \right)^2$$

$$r_p = \frac{1}{2\pi} \left(\frac{K_{II}}{\sigma_{YS}} \right)^2$$

- Where we define the stress intensity factors at a and L as

$$K_{Ia} = \sigma \sqrt{\pi a} \beta_a$$

$$K_{II} = \sigma \sqrt{\pi l} \beta_l$$

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linkup equation

- Since fast cracking occurs when $R_p + r_p = L$ we solve for the condition where $R_p + r_p < L$

$$\frac{1}{2\pi} \left(\frac{K_{Ia}}{\sigma_{YS}} \right)^2 + \frac{1}{2\pi} \left(\frac{K_{II}}{\sigma_{YS}} \right)^2 < L$$
$$\frac{1}{2\pi\sigma_{YS}^2} [K_{Ia}^2 + K_{II}^2] < L$$

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$$\frac{1}{2\pi\sigma_{YS}^2} \left[\sigma^2 \pi a \beta_a^2 + \sigma^2 \pi l \beta_l^2 \right] < L$$

$$\frac{\sigma^2}{2\sigma_{YS}^2} \left[a \beta_a^2 + l \beta_l^2 \right] < L$$

$$\sigma_c = \sigma_{YS} \sqrt{\frac{2L}{a \beta_a^2 + l \beta_l^2}}$$

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example

worked link-up example here¹

¹<https://colab.research.google.com/drive/11Nt8oT53jTECs1JZO2FFhCpphy1gwXqi?usp=sharing>

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modified linkup equations

- We see that for a brittle material (with a small plastic zone) we predict no effect of “link-up”
- This does not agree with test data
- Even the 2024 predictions don't agree well with test data
- WSU used some empirical parameters to modify the linkup equations and better predict residual strength when multiple site damage is present

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modified 2024

- For 2024-T3 we use the following procedure
- First find σ_c from the unmodified equation

$$\sigma_{c,mod} = \frac{\sigma_c}{A_1 \ln(L) + A_2}$$

- Where $A_1 = 0.3065$ and $A_2 = 1.3123$ for A-basis yield strength and $A_1 = 0.3054$ and $A_2 = 1.3502$ for B-basis yield strength
- The same equation can also be used for 2524 with $A_1 = 0.1905$, $A_2 = 0.9683$ for A-basis yield and $A_1 = 0.2024$, $A_2 = 1.0719$ for B-basis yield

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- A similar modification was made for 7075

$$\sigma_{c,mod} = \frac{\sigma_c}{B_1 + B_2 L}$$

- Where $B_1 = 1.377$, $B_2 = 1.042$ for A-basis yield strength
and $B_1 = 1.417$, $B_2 = 1.073$ for B-basis yield strength

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- However, since general fracture had a closer prediction to real failure than the linkup equation, it may make more sense to modify the brittle fracture equation

$$\sigma_{c,mod} = \frac{K_c}{\sqrt{\pi a}(0.856 - 0.496 \ln(L))}$$

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mixed mode fracture

mixed-mode fracture

- Most cracks are primarily Mode I, but sometimes Mode II can also have an effect
- We can look at the combined stress field for Mode I and Mode II
- Recall the stress field near the crack tip

$$\begin{aligned}\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\end{aligned}$$

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mixed-mode fracture

- For Mode II we have

$$\begin{aligned}\sigma_x &= \frac{-K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ \sigma_y &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)\end{aligned}$$

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- In mixed-mode fracture problems, the crack will generally propagate in a different direction from the initial crack plane
- It is more convenient to handle this scenario in Polar Coordinates
- We can convert stress from Cartesian coordinates to Polar Coordinates using the stress transformation equations

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$$\sigma_r = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta$$

$$\sigma_\theta = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta$$

$$\tau_{r\theta} = -\sigma_x \sin \theta \cos \theta + \sigma_y \sin \theta \cos \theta + \tau_{xy}(\cos^2 \theta - \sin^2 \theta)$$

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combined stress field

- When we convert the stress fields from Mode I and Mode II into polar coordinates and combine them, we find

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4} \sin \frac{\theta}{2} + \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4} \sin \frac{\theta}{2} - \frac{3}{4} \sin \frac{3\theta}{2} \right)$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4} \cos \frac{\theta}{2} + \frac{3}{4} \cos \frac{3\theta}{2} \right)$$

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max circumferential stress

- The Maximum Circumferential Stress Criterion assumes that a crack will propagate in the principal direction
- In this direction, the shear stress is 0
- The fracture toughness is determined by the Mode I fracture toughness of the material

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max circumferential stress

- **Note:** In this discussion, we will use K_{IC} to differentiate Mode I fracture toughness from Mode II fracture toughness. This does NOT necessarily mean we are referring to plane strain fracture toughness
- Thus fracture begins when

$$\sigma_{\theta}(\theta_P) = \sigma_{\theta}(\theta = 0, K_{II} = 0, K_I = K_{Ic}) = \frac{K_{IC}}{\sqrt{2\pi r}}$$

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max circumferential stress

- Following the above assumptions, we can solve these equations to find θ_P
- Note: This assumes that we know both K_I and K_{II} , in this class we have not discussed any Mode II stress intensity factors, so they will be given.

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- In this case it simplifies to

$$K_I \sin \theta_p + K_{II} (3 \cos \theta_p - 1) = 0$$

- and

$$4K_{IC} = K_I \left(3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right) - 3K_{II} \left(\sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right)$$

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maximum circumferential stress criterion

- The general form for a Mode II stress intensity factor is

$$K_{II} = \tau \sqrt{\pi a} \beta'$$

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Assuming $\sigma = 4\tau$, $K_{IC} = 60 \text{ ksi}\sqrt{\text{in}}$, and $2a = 1.5 \text{ in}$.

Note: Assume $\beta = \beta' = 1$

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principal stress

- In the maximum circumferential stress criterion, we found the principal stress direction near the crack tip in polar coordinates
- We can also find the principal direction (if there were no crack) in Cartesian coordinates
- **Note:** This is not mathematically rigorous, but much easier to calculate and sometimes it's close enough

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principal stress

- If we make a free body cut along some angle θ we find, from equilibrium

$$0 = \sigma_{\theta} dA - \sigma_x dA \sin^2 \theta - \sigma_y dA \cos^2 \theta + 2\tau_{xy} dA \cos \theta \sin \theta$$

$$\sigma_{\theta} = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta$$

$$\frac{\partial \sigma_{\theta}}{\partial \theta} = (\sigma_x - \sigma_y) \sin 2\theta_p - 2\tau_{xy} \cos 2\theta_p$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

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principal stress

- As before, we consider crack propagation purely due to Mode I
- In the principal stress criterion, we find the maximum Mode I stress as a function of the remote applied stress

$$\sigma_{P1} = C\sigma$$

- We then find the remote failure stress by

$$\sigma_c = \frac{K_{IC}}{C\sqrt{\pi a}\beta}$$

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example

Assuming $\sigma = 4\tau$, $K_{IC} = 60 \text{ ksi}\sqrt{\text{in}}$, and $2a = 1.5 \text{ in}$.

Note: Assume $\beta = \beta' = 1$

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example

worked mixed-mode fracture example here²

²<https://colab.research.google.com/drive/15vHyABV7Q98UF9ZRZhQoDnn2858nNE0F?usp=sharing>

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