AE 737: Mechanics of Damage Tolerance

Lecture 6 - Plastic Zone

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schedule

- 17 Feb - Plastic Zone, HW 2 Due, HW 1 Self-grade due
- 22 Feb - Fracture Toughness
- 24 Feb - Fracture Toughness, HW3 Due, HW 2 Self-grade due
- 1 Mar - Residual Strength

outline

- plastic zone
- plastic stress intensity ratio
- plastic zone shape
- group problems

plastic zone

plastic zone

- Previous developments assumed perfectly elastic materials
- Most common materials have some plasticity
- Any stress above the yield stress will undergo plastic deformation (no stress higher than σ_y will be present in the material)

plastic zone

- Plasticity helps retard crack propagation due to residual stresses
- After an overload, elastic regions will contract back to their undeformed shape
- The region which has undergone plastic deformation, however, holds its deformed shape
- This introduces a region of residual compressive stress near the crack tip
- Before the crack can propagate, a stress needs to overcome this residual stress

2D problems

- We often simplify the full 3D elasticity equations for planar problems
- For very thin panels, we assume that all out-of-plane stresses are 0
- This is called plane stress

plane stress

$$\sigma_z = \tau_{xz} = \tau_{zy} = 0$$
$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$
$$\epsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$
$$\epsilon_z = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{xz} = \gamma_{yz} = 0$$

2D problems

- When instead a panel is very thick, we assume that any strains through the thickness are small relative to other strains
- $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$
- This is known as plane strain

plane strain

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\
\epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\
0 &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \gamma_{yz} = 0
\end{aligned}$$

Irwin's first approximation

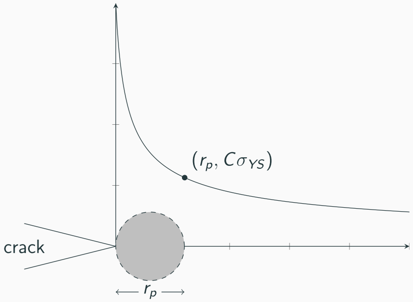
- If we recall the equation for opening stress (σ_y) near the crack tip

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{1.2}$$

- In the plane of the crack, when $\theta = 0$ we find

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}$$

Irwin's first approximation



Irwin's first approximation

- We use C, the *Plastic Constraint Factor* to convert between Plane Strain and Plane Stress solutions
- The plastic zone size can now be approximated

$$\sigma_{yy}(r = r_p) = C\sigma_{YS}$$

$$\frac{K_I}{\sqrt{2\pi r_p}} = C\sigma_{YS}$$

$$r_p = \frac{1}{2\pi}\left(\frac{K_I}{C\sigma_{YS}}\right)^2$$

Irwin's first approximation

- For plane stress (thin panels) we let $C = 1$ and find rp as

$$r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

- And for plane strain (thick panels) we let $C = \sqrt{3}$ and find

$$r_p = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

Intermediate panels

- For panels which lie between plane strain and plane stress states, we use the following expression to estimate the plastic zone size

$$r_p = \frac{1}{I\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

- Where I is defined as

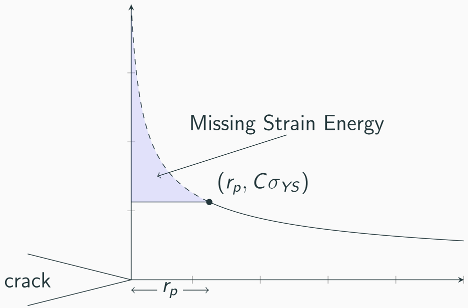
$$I = 6.7 - \frac{1.5}{t}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

- And $2 \leq I \leq 6$

Irwin's second approximation

- If our material is perfectly elastic-plastic, no stresses above $C\sigma_{ys}$ will exist in the material
- This ignores the strain energy (represented by the area under the curve) in the plastic zone

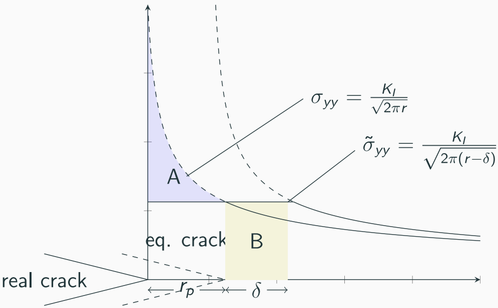
Irwin's second approximation



Irwin's second approximation

- To account for the additional strain energy, Irwin considered a plastic zone size increased by some δ
- He also needed to adjust the stress function, and considered an equivalent crack tip in these calculations

Irwin's second approximation



Irwin's second approximation

We need A=B, so we set them equivalent and solve for δ.

$$\begin{aligned} A &= \int_0^{r_p} \sigma_{yy} dr - r_p\sigma_{YS} \\ &= \int_0^{r_p} \frac{K_I}{\sqrt{2\pi r}} dr - r_p\sigma_{YS} \\ &= \frac{K_I}{\sqrt{2\pi}} \int_0^{r_p} r^{-1/2} dr - r_p\sigma_{YS} \\ &= \frac{2K_I\sqrt{r_p}}{\sqrt{2\pi}} - r_p\sigma_{YS} \end{aligned}$$

Irwin's second approximation

- We have already found *rp* as

$$r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

- If we solve this for *KI* we find

$$K_I = \sqrt{2\pi r_p}\sigma_{YS}$$

Irwin's second approximation

- We can now substitute back into the strain energy of A

$$\begin{aligned} A &= \frac{2\sqrt{2\pi r_p}\sigma_{YS}\sqrt{r_p}}{\sqrt{2\pi}} - r_p\sigma_{YS} \\ &= 2\sigma_{YS} r_p - r_p\sigma_{YS} \\ &= r_p\sigma_{YS} \end{aligned}$$

Irwin's second approximation

- B is given simply as $B = \delta\sigma_{ys}$ so we equate A and B to find δ

$$A = B$$
$$r_p\sigma_{YS} = \delta\sigma_{YS}$$
$$r_p = \delta$$

Irwin's second approximation

- This means the plastic zone size is simply $2rp$
- However, it also means that the effective crack length is $a+rp$
- Since rp depends on KI, we must iterate a bit to find the "real" rp and KI

Example

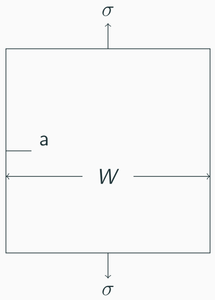


Figure 1: An edge crack of length a in a panel of width W is subjected to a remote load

equations

$$\beta = \left[1.122 - 0.231\frac{a}{W} + 10.55\left(\frac{a}{W}\right)^2 - 21.71\left(\frac{a}{W}\right)^3 + 30.82\left(\frac{a}{W}\right)^4\right]$$

$$I = 6.7 - \frac{1.5}{t}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

$$r_p = \frac{1}{I\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2$$

work[1]

[1] ../examples/Plastic%20Zone%20Example.html

plastic stress intensity ratio

plastic stress intensity ratio

- Engineers often use stress intensity to decide which material to use for a certain application
- The ratio of plastic stress intensity to elastic stress intensity, as a function of yield stress over applied stress, can help illustrate the effects of plasticity for different materials.

plastic stress intensity ratio

For an infinitely wide center-cracked panel, we can solve for KIe/KI symbolically, in plane stress

$$K_I = \sigma\sqrt{\pi a}$$
$$K_{Ie} = \sigma\sqrt{\pi(a + r_p)}$$
$$r_p = \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2$$
$$K_{Ie} = \sigma\sqrt{\pi\left(a + \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2\right)}$$

stress intensity ratio

$$K_{Ie}^2 = \sigma^2\pi\left(a + \frac{1}{2\pi}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2\right)$$
$$K_{Ie}^2 = \sigma^2\pi a + \frac{\sigma^2}{2}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2$$
$$K_{Ie}^2 - \frac{\sigma^2}{2}\left(\frac{K_{Ie}}{\sigma_{YS}}\right)^2 = \sigma^2\pi a$$
$$K_{Ie}^2\left(1 - \frac{\sigma^2}{2\sigma_{YS}^2}\right) = \sigma^2\pi a$$

Note: square both sides

plastic stress intensity ratio

$$K_{Ie}^2 = \frac{\sigma^2 \pi a}{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}$$

$$K_{Ie} = \frac{\sigma\sqrt{\pi a}}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}}$$

$$K_{Ie} = \frac{K_I}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}}$$

$$\frac{K_{Ie}}{K_I} = \frac{1}{\sqrt{1 - \frac{\sigma^2}{2\sigma_{YS}^2}}}$$

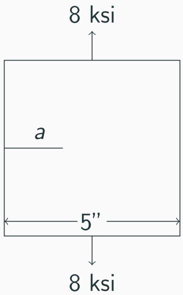
Note: We divide both sides by $\left(1 - \frac{\sigma^2}{2\sigma_{YS}^2}\right)$

plastic stress intensity ratio

- We can also find the plastic stress intensity ratio numerically for finite width panels
- Panel thickness, yield stress, panel width, crack length could all be variables in this case
- Different heat treatments of metal alloys can give a different yield stress, with most other properties remaining the same
- Typical crack lengths can vary based on inspection cycles

example

- You are asked to design an inspection cycle for a panel
- Consider the plastic stress intensity ratio, and the effect of varying crack lengths on it



example

online example here[2]

[2] ../examples/Plastic%20stress%20intensity%20ratio.html

plastic zone shape

plastic zone shape

- Although we drew a circle to give a rough idea of the plastic zone in Irwin's method, this solution was only 1D
- We considered $\theta = 0$.
- It is advantageous to model the plastic zone shape, we will do so using two different yield theories
- Von Mises and Tresca

principal stresses

- Principal stresses are often used in yield theories
- We can determine the principal stresses near the crack tip as

$$
\begin{aligned}
\sigma_1 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) \\
\sigma_2 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right) \\
\sigma_3 &= 0 && \text{(plane stress)} \\
\sigma_3 &= \frac{2\nu K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} && \text{(plane strain)}
\end{aligned}
$$

Von Mises yield theory

- The Von Mises yield theory is also known as the Distortion Energy Yield Theory
- In this yield theory, we assume that failure or yielding occurs when the strain energy exceeds some threshold
- It has been observed that hydrostatic pressure does not generally cause yielding
- We separate the strain energy into two parts, volumetric and distortional
- Only the distortional strain energy is used to determine the yield strength

Von Mises yield theory

- The distortional strain energy is given by

$$W_d = \frac{1}{12} G \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$

- Which for a uniaxially loaded point becomes

$$W_d = \frac{1}{6} G \sigma_{YS}^2$$

- We can equate the two cases and solve

$$\begin{aligned} \frac{1}{6} G \sigma_{YS}^2 &= \frac{1}{12} G \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \\ 2\sigma_{YS}^2 &= (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \end{aligned}$$

Von Mises yield theory

- We can find the plastic zone size, *rp* by substituting the principal stress relations into the distortional strain energy equation
- In plane stress we find

$$2\sigma_{YS}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - 0)^2 + (0 - \sigma_1)^2$$

Von Mises yield theory

$$2\sigma_{YS}^2 = \left(\frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) - \frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right)\right)^2 + \left(\frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right) - 0\right)^2 + \left(0 - \frac{K_I}{\sqrt{2\pi r_p}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right)\right)^2$$

Von Mises yield theory

- After solving we find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1+3\sin^2\frac{\theta}{2}\right)$$

- We can similarly solve for *rp* in plane strain to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1-4\nu+4\nu^2+3\sin^2\frac{\theta}{2}\right)$$

Tresca yield theory

- Tresca yield theory assumes that yielding begins when the maximum shear stress reaches a critical value
- In uniaxial tension this gives

$$\tau_0 = \tau_{max} = \frac{1}{2}\left(\sigma_{max} - \sigma_{min}\right) = \frac{1}{2}\left(\sigma_{YS} - 0\right) = \frac{\sigma_{YS}}{2}$$

Tresca yield theory

- Using the results for principal stress we found previously, we see that

$$\sigma_{max} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)$$
$$\sigma_{min} = 0$$

- We can substitute and solve as before to find

$$r_p = \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right)^2$$

Tresca yield theory

- In plane strain, it is not clear whether σ_2 or σ_3 will be σ_{min}
- We can solve for when σ_2 will be σ_{min}

$$\begin{aligned} \sigma_2 &\quad lt; \sigma_3 \\ \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right) &\quad lt; \frac{2\nu K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2} \\ 1-\sin\frac{\theta}{2} &\quad lt; 2\nu \\ \theta_t gt; 2\sin^{-1}(1-2\nu) \end{aligned}$$

Tresca yield theory

- When $2\pi - \theta_t < \theta < \theta_t$, σ_2 is the minimum, otherwise σ_3 is the minimum
- Note: Error(s) in text p. 101
- Once we have chosen the appropriate minimum stress (σ_2 or σ_3), we can solve for *rp* as before

Tresca yield theory

$$
\begin{aligned}
r_p &= \frac{2K_I^2}{\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} && \theta_t lt; \theta \qquad lt; 2\pi - \theta_t \\
r_p &= \frac{K_I^2}{2\pi\sigma_{YS}^2}\cos^2\frac{\theta}{2}\left(1 - 2\nu + \sin\frac{\theta}{2}\right)^2 && \theta lt; \theta_t, \theta \quad gt; 2\pi - \theta_t
\end{aligned}
$$

3D plastic zone shape

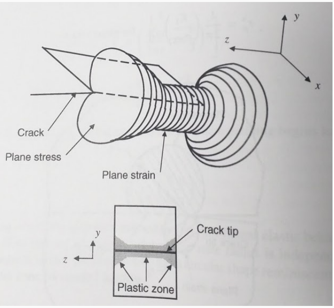


Figure 2: An image showing the 3D plastic zone shape, which looks a little bit like a dumbell. The plastic zone is much larger near the surface, where the material behaves as if in plane stress. In the

example

online example here[3]

[3] ../examples/Plastic%20Zone%20Shape.html

group problems

group one

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of $\sigma_{YS} = 55$ MPa, with an applied load of $\sigma = 20$ MPa
- Assume the panel is very thin

group two

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of $\sigma_{YS} = 55$ MPa, with an applied load of $\sigma = 20$ MPa
- Assume the panel is very thick

group three

- Calculate the plastic zone size for an infinitely wide, center-cracked panel
- Consider a crack-length of 4 cm, and a yield strength of $\sigma_{YS} = 55$ MPa, with an applied load of $\sigma = 20$ MPa
- The panel thickness is $t = 0.65$ cm

group four

- Find the plastic stress intensity ratio for an infinitely wide, center-cracked panel
- What factors will increase or decrease the plastic stress intensity ratio?