Name:

Homework 1

Due 5 September 2019

1. Find the stress field in an infinite body with a hole under remote shear, as shown.

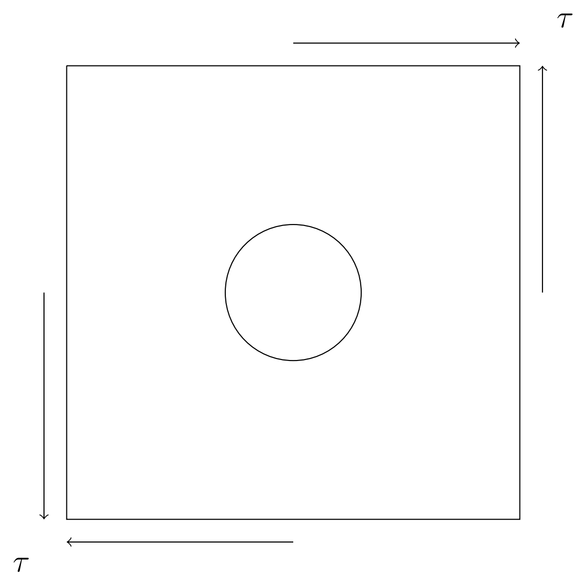

Figure 1: Figure for Problem 1

- First we solve the problem at the remote boundary conditions.
- Using Cauchy's stress theorem ($\sigma_{ij}n_j = t_i$), we find at $x = \pm\infty$

$$\sigma_{xx} = 0$$
$$\sigma_{xy} = \tau$$

- and at $y = \pm\infty$

$$\sigma_{xy} = \tau$$
$$\sigma_{yy} = 0$$

- Since σ_{xy} is the only non-zero stress term, we can solve this problem if $\sigma_{xy} = \tau$ and $\sigma_{xx} = \sigma_{yy} = 0$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \tau$$

- Which gives $\phi = -\tau xy$

- We can convert this solution to polar coordinates in preparation for adding the stress-free boundary conditions at $r = a$, where $x = r\cos\theta$ and $yr\sin\theta$, we find

$$\phi(r,\theta) = -\tau r^2 \sin\theta\cos\theta = -\frac{1}{2}\tau r^2 \sin 2\theta$$

- Which satisfies the BC's at $r \to \infty$
- The BC's at $r = a$ are

$$\sigma_{rr} = \sigma_{r\theta} = 0$$

- The current Airy stress function has a $-2\sin 2\theta$ term in σ_{rr} and a $-2\cos 2\theta$ term for $\sigma_{r\theta}$, we need to include terms that can cancel both of these near the hole, but will not affect the far-field stress as $r \to \infty$
- We observe that $\phi = \sin 2\theta$ and $\phi = \sin 2\theta / r^2$ both have similar terms and vanish as $r \to \infty$, so we consider the trial function

$$\phi(r,\theta) = -\frac{1}{2}\tau r^2 \sin 2\theta + A\sin 2\theta + B\frac{1}{r^2}\sin 2\theta$$

- We differentiate (or use the table) to find $\sigma_{rr}(a,\theta)$ and $\sigma_{r\theta}(a,\theta)$ and we find

$$\sigma_{rr}(a,\theta) = \tau\sin 2\theta - 4A\sin 2\theta/a^2 - 6B\sin 2\theta/a^4$$
$$\sigma_{r\theta}(a,\theta) = \tau\cos 2\theta + 2A\cos 2\theta/a^2 + 6B\cos 2\theta/a^4$$

- Solving these equations for A and B we find $A = \tau a^2$ and $B = -3\tau$ which gives for the stress field

$$\sigma_{rr} = \tau\sin 2\theta - \frac{4\tau a^2}{r^2}\sin 2\theta + \frac{18\tau}{r^4}\sin 2\theta$$
$$\sigma_{r\theta} = \tau\cos 2\theta + \frac{2\tau a^2}{r^2}\cos 2\theta - \frac{18\tau}{r^4}\cos 2\theta$$
$$\sigma_{\theta\theta} = -\tau\sin 2\theta - \frac{18\tau}{r^4}\sin 2\theta$$

2. Use simple beam theory to find the strain energy release rate of the following cracked beam

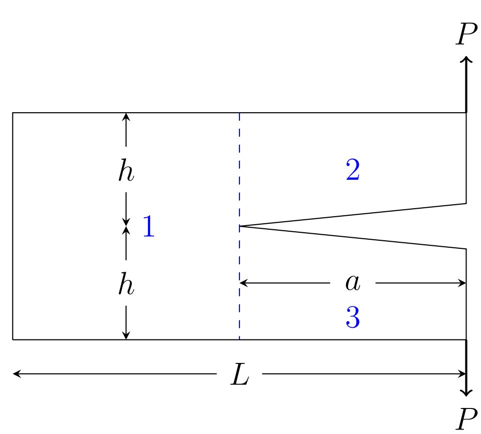

Figure 2: Figure for Problem 2

- We can divide the double cantilever beam into three sub-sections as shown and calculate the strain energy in each of the sub-sections
- Assuming a beam-like geometry and exactly opposing forces, we can neglect the strain energy in section 1 and consider sections 2 and 3 to be cantilever beams (hence the term 'double-cantilever beam'), we notice by symmetry that the strain energy in 2 and 3 should be equal, thus

$$U_T = 2U_2 = \int_0^a \frac{M^2}{2EI} dx$$

- Since the beam has a constant stiffness and cross-section, we need only integrate the bending moment, where $M = (a - x)P$, this gives

$$U_t = \frac{P^2}{EI} \int_0^a (a^2 - 2ax + x^2) dx = \frac{P^2 a^3}{3EI}$$

- We also need to find the external work done ($P\delta$), which for (two) cantilever beams is

$$W_e = 2P\delta = 2P \left(\frac{Pa^3}{3EI} \right) \tag{1}$$

- This gives a surface energy of

$$W_s = W_e - U_t = \frac{2P^2 a^3}{3EI} - \frac{P^2 a^3}{3EI} = \frac{P^2 a^3}{3EI} \tag{2}$$

- The strain energy release rate uses the energy per unit thickness, thus we divide the energy by the thickness, b

$$W_s = \frac{P^2 a^3}{3EIb}$$

- And we find G by taking the partial derivative with respect to a

$$G = \frac{\partial W_s}{\partial a} = \frac{\partial}{\partial a} \left(\frac{P^2 a^3}{3bEI} \right) = \frac{P^2 a^2}{bEI}$$

3. A cracked beam is subjected to a pair of forces at the center of the crack. Find the minimum P that can split the beam, where $E = 70$ GPa and $G_c = 200$ Nm/m^2. Note: the thickness is 0.5 cm, all dimensions are in cm.

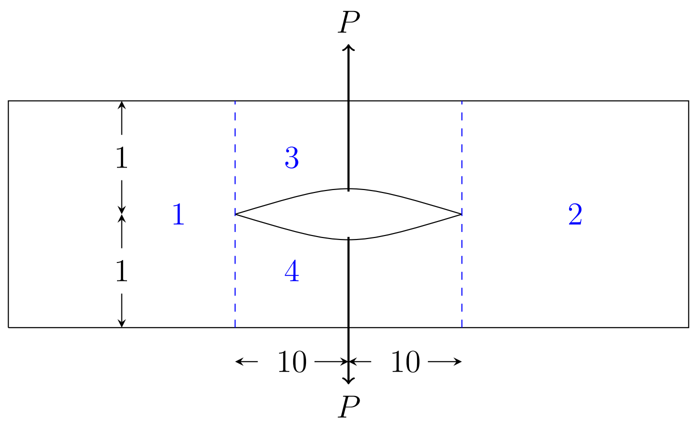

Figure 3: Figure for Problem 3

- As in Problem 2, we subdivide the beam and use symmetry with the assumption of beam geometry to neglect the strain energy in sub-sections 1 and 2
- We also note that the strain energy in 3 and 4 will be equivalent, so we need only calculate one of the two
- Finally, the expression for moment is piecewise, but since the fixed-fixed beam in 3 or 4 is symmetric about the center, we can calculate the strain energy in only one half of the beam
$$U_T = 2U_3 = 2(2)\int_0^a \frac{M^2}{2EI}dx$$
- The moment for one half of a fixed-fixed beam is given as $M = \frac{P}{8}(4x - L) = \frac{P}{8}(4x - 2a)$ which gives
$$U_T = \frac{P^2}{32EI}\int_0^a (16x^2 - 16xa + 4a^2)dx = \frac{P^2a^3}{24EI}$$
- To find the external work we find the beam displacement
$$\delta = \frac{Pa^3}{24EI} \tag{3}$$
- Which gives
$$W_s = W_e - U_t = \frac{2P^2a^3}{24EI} - \frac{P^2a^3}{24EI} = \frac{P^2a^3}{24EI} \tag{4}$$
- To find the strain energy release rate, we divide by the beam thickness and take the partial derivative with respect to a
$$G = \frac{\partial W_s}{\partial a} = \frac{\partial}{\partial a}\left(\frac{P^2a^3}{24EIb}\right) = \frac{P^2a^2}{8EIb}$$
- Before calculating the critical load, we should check units of G to ensure consistency. We have (using generic SI units)
$$\frac{\mathrm{N^2m^2}}{\mathrm{N/m^2m^4m}}$$

- This simplifies to N/m or, to match the units given for G, Nm/m^2
- We can now substitute values for G, a, E, I and b, careful to convert units to m to match, and we solve for P_c to find

$$P_c = \sqrt{\frac{8GEIb}{a^2}} = 152.8\text{N}$$