# AE837

## Advanced Mechanics of Damage Tolerance

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Wichita State University, Department of Aerospace Engineering August 20, 2019

# upcoming schedule

- Aug 20 - Syllabus, Elasticity Review
- Aug 22 - Elasticity Review
- Aug 27 - Griffith Fracture
- Aug 29 - Griffith Fracture

# outline

- introduction
- syllabus and schedule
- fracture introduction
- elasticity
- coordinate transformation
- examples
- principal values
- invariants
- principal directions
- examples

# introduction

# about me

# education

- B.S. in Mechanical Engineering from Brigham Young University
  - Worked with ATK to develop tab-less gripping system for tensile testing composite tow specimens
  - Needed to align the specimen, as well as grip it without causing a stress concentration

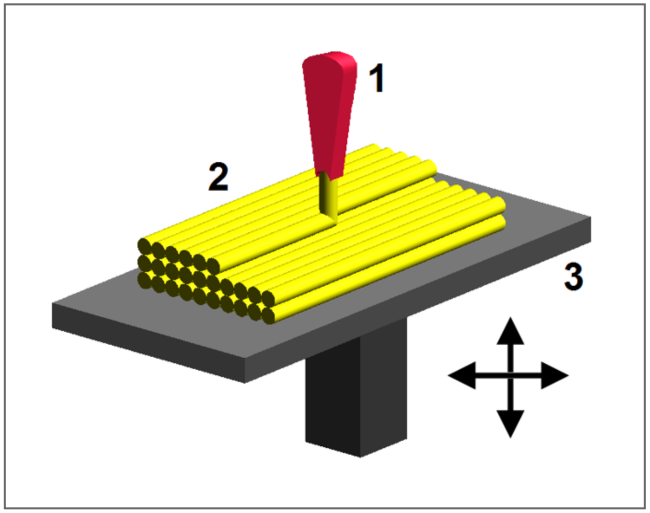
# education

- M.S. and Ph.D. from School of Aeronautics and Astronautics at Purdue University
  - Worked with Boeing to simulate mold flows
  - First ever mold simulation with anisotropic viscosity

# research

# research

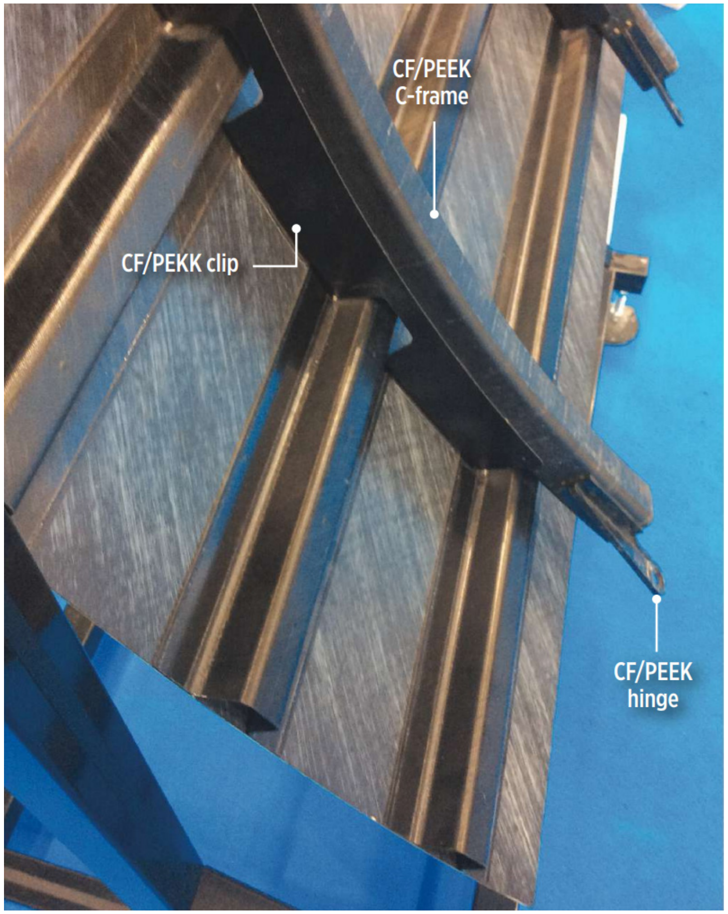
# research




- Composites are being used in 3D printing now
- Printing patterns are optimized for isotropic materials
- Sometimes composites hurt more than they help when not utilized properly

# research

- Thermoplastic composites offer many advantages over thermoset
- Production speed, recyclability
- Also have challenges, such as bonding/welding



# classes

- AE 731 Elasticity Theory (odd years in fall)
- AE 737 Mechanics of Damage Tolerance (every year in spring)
- AE 837 Advanced Mechanics of Damage Tolerance (odd years in fall)
- AE 760AA Micromechanics and Multiscale Modeling (odd years in spring)
- AE 831 Continuum Mechanics (even years in fall)

# introductions

- Name
- Student status (Undergrad, Masters, Ph.D)
- Full time or part time student?
- One interesting thing to remember you by

# syllabus and schedule

# course textbook

- Text is available as a pdf on Blackboard and the **class website**, you may purchase any version if you prefer a hard copy
- Homework will be given in handouts provided online
- Supplemental textbooks are listed in the syllabus for further study

# office hours

- I will e-mail everyone in the course a Doodle link we can use to find the optimal office hours
- Let me know if you do not receive the e-mail, you may need to update your information in Blackboard
- Take advantage of office hours, this is time that I have already set aside for you
- If the regular office hours do not work for your schedule, send me an e-mail and we can work out a time to meet

# tentative course outline

- Section 1 - linear elastic fracture mechanics
  - Elasticity Review (Aug 20-22)
  - Griffith Fracture (Aug 27-29)
  - Elastic Stress Field (Sept 3-12)
  - Exam 1 (Sept 19)

# tentative course outline

- Section 2 - advanced analytical fracture
  - Energy Approach (Sept 24-26)
  - Finite Elements (Oct 1-12)
  - Plasticity (Oct 17-24)
  - Mixed-Mode Fracture (Oct 29-31)
  - Exam 2 (Nov 7)

# tentative course outline

- Section 3 - computational fracture
  - XFEM (Nov 12-14)
  - Cohesive Zone Modeling (Nov 19-28)
  - Interfacial Cracks (Dec 3-5)
  - Final project (due Dec 6)

# grades

- Grade breakdown
  - Homework 15%
  - Exam 1 30%
  - Exam 2 30%
  - Final Project 25%
- Follow a traditional grading scale

# final project

- Perform computational fracture analysis on a real-life part (or test specimen) of your choosing
- Use the principles developed in this class to provide an analytical validation of your computational methods
- Examples: mixed-mode fracture study, progressive failure, adhesive bond failure, etc.
- Individual project
- More discussion after Exam 1

# class expectations

- Consider the cost (to you or others) of your being in class
- I ask that you refrain from distracting behaviors during class
- When you have something more important than class to take care of, please take care of it outside of class

# software

- You will be required to do finite element analysis in this class, we have a class kit license for COMSOL, but you are welcome to use any software package you want
- The student version of Abaqus is also free and has a good XFEM module (not all FEA tools have XFEM)
- Although we will not use FEA until later in the course, I advise you set up the software as soon as possible

# fracture introduction

# damage

- In linear elasticity, we generally consider materials in their pristine state
- Realities of manufacturing, cyclic loads, and unforeseen loads result in a material which is something other than pristine
- When stress is uniform, simple analysis can often predict failure ($\sigma > \sigma_f$)
- When damage or stress concentrations are present, however, failure is more accurately modeled as fracture

# fracture

- There are two primary approaches to modeling fracture
- In the first we examine the elastic stress and displacement fields near a crack tip
- A fracture criterion is then defined based on the so-called stress intensity factor
- The other approach considers the global energy of a body with a crack
- We consider the potential energy of a cracked solid and consider its variation with a virtual crack extension
- The energy release rate is then used to quantify failure

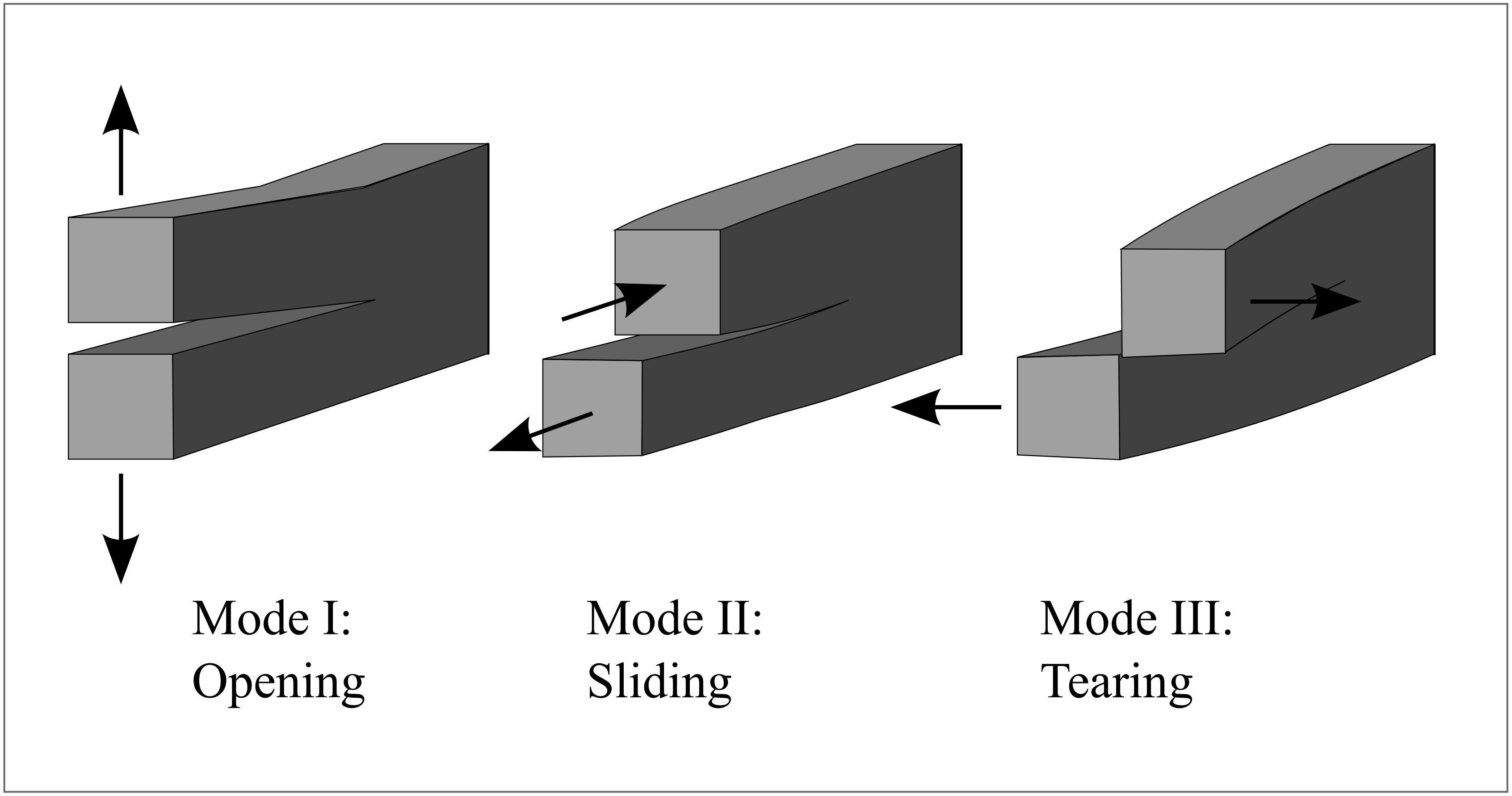
# fracture

- Several other parameters have been used to quantify failure in addition to the stress intensity factor and the energy release rate, with somewhat less success
- J-integral
- Crack Tip Opening Displacement
- Crack Tip Opening Angle

# fracture mechanics

- In fracture mechanics we consider three different modes
- Mode I is known as the “opening mode”
- Mode II is known as the “sliding mode”
- Mode III is known as the “tearing mode”

# fracture mechanics



# elasticity

# linear elasticity

- We cannot cover everything from elasticity, and you can get by in this course without it, but we will be using many principles from elasticity in this course
- Some of the things we will review are
  - Index notation (briefly)
  - Coordinate transformation
  - Principal values/directions
  - Spherical/cylindrical coordinates
  - Strain energy
  - General solution strategy

# big picture

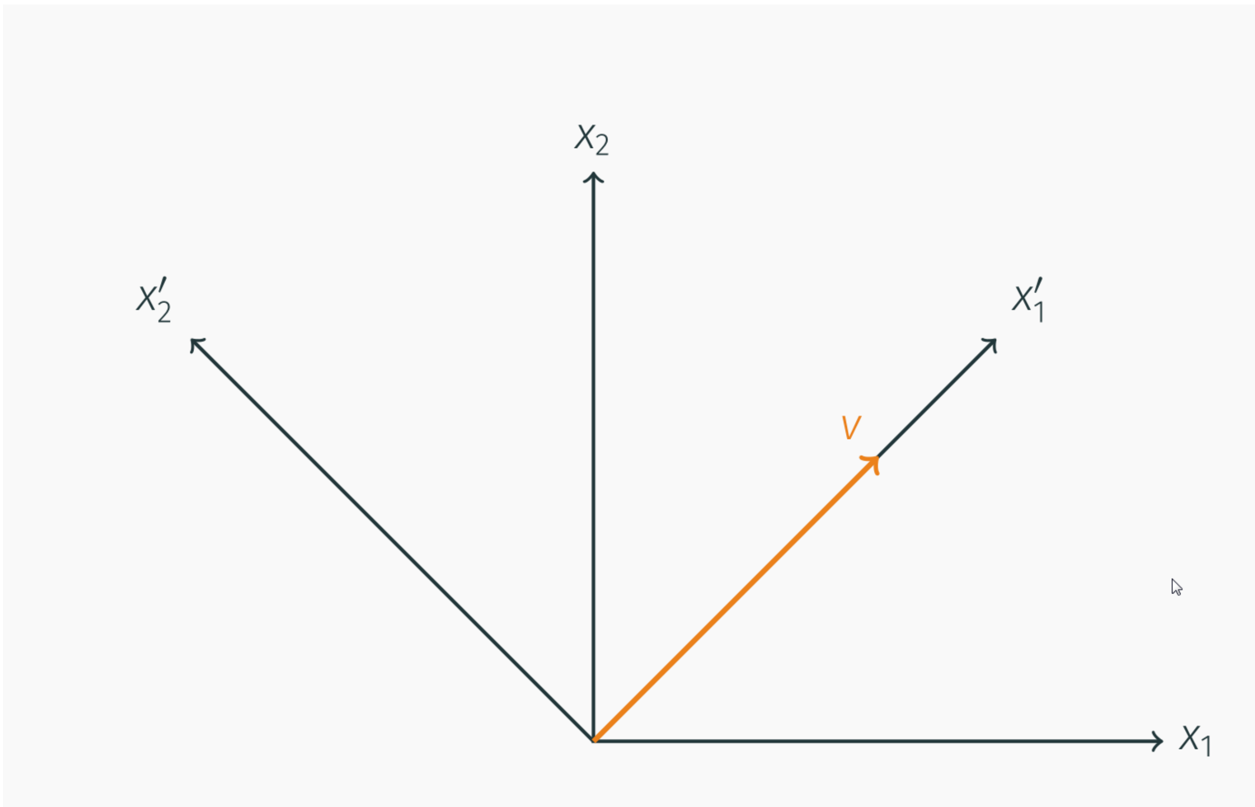
- Perhaps more important than the details we will review is to keep in mind the "big picture"
- To solve any problem in elasticity, we need to satisfy:
  1. The equilibrium equations (in the appropriate coordinate system)
  2. The boundary conditions
  3. Without violating strain compatibility

# big picture

- Most often, we assume a state of plane stress or plane strain and solve the problem in 2D
- Even “3D” problems (i.e. Mode III fracture) have reduced variables
- Stress functions are often formulated to automatically satisfy equilibrium, or displacement functions to automatically satisfy compatibility

# coordinate transformation

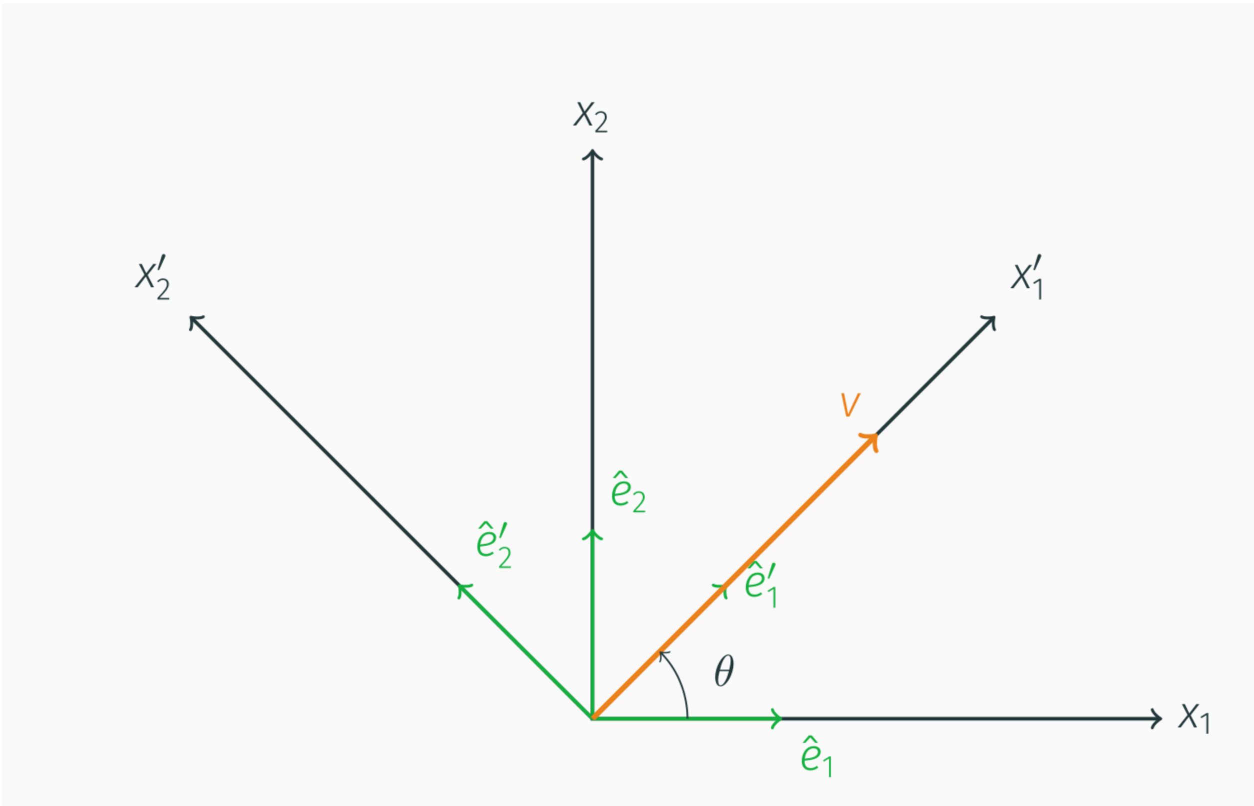
# two dimensions



## two dimensions

- The vector, $v$, remains fixed, but we transform our coordinate system
- In the new coordinate system, the ${x_2}'$ portion of $v$ is zero.
- To transform the coordinate system, we first define some unit vectors.
- $\hat{e}_1$ is a unit vector in the direction of $x_1$, while $\hat{e}'_1$ is a unit vector in the direction of ${x_1}'$

# two dimensions



# two dimensions

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45^\circ$
- We can write the transformed unit vectors, $\hat{e}_1'$ and $\hat{e}_2'$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, $\theta$.
  $\hat{e}_1' = \langle \hat{e}_1 \cos\theta, \hat{e}_2 \sin\theta \rangle$
  $\hat{e}_2' = \langle -\hat{e}_1 \sin\theta, \hat{e}_2 \cos\theta \rangle$

# two dimensions

- We can write the vector, $v$, in terms of the unit vectors describing our axis system
- $v = v_1\hat{e}_1 + v_2\hat{e}_2$
- (note: $\hat{e}_1 = \langle 1, 0\rangle$ and $\hat{e}_2 = \langle 0, 1\rangle$)
- $v = \langle 2, 2\rangle = 2\langle 1, 0\rangle + 2\langle 0, 1\rangle$

# two dimensions

- When expressed in the transformed coordinate system, we refer to $v^{'}$
- $v^{'} = \langle v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta \rangle$
- $v' = \langle 2\sqrt{2}, 0 \rangle$
- We can recover the original vector from the transformed coordinates:
- $v = v_1' \hat{e}_1' + v_2' \hat{e}_2'$
- (note: $\hat{e}_1' = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ and $\hat{e}_2' = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$)
- $v = 2\sqrt{2} \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle, 0 \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle 2, 2 \rangle$

# general

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- It is convenient to define a general form of the coordinate transformation in index notation
- We define $Q_{ij}$ as the cosine of the angle between the $x_i{}'$ axis and the $x_j$ axis.
- This is also referred to as the “direction cosine” $Q_{ij} = \cos(x_i{}', x_j)$

# general

- We can use this form on our 2D transformation example

$$
\begin{aligned}
Q_{ij} &= \cos(x'_i, x_j) \\
&= \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

# general

- We can transform any-order tensor using $Q_{ij}$
- Vectors (first-order tensors): $v_i' = Q_{ij} v_j$
- Matrices (second-order tensors): $\sigma_{mn}' = Q_{mi} Q_{nj} \sigma_{ij}$
- Fourth-order tensors: $C_{ijkl}' = Q_{im} Q_{jn} Q_{ko} Q_{lp} C_{mnop}$

# general

- We can similarly use $Q_{ij}$ to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_i = Q_{ji}v_j{}'$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im}Q_{jn}\sigma_{ij}{}'$
- Fourth-order tensors: $C_{ijkl} = Q_{mi}Q_{nj}Q_{ok}Q_{pl}C_{mnop}{}'$

# mental/emotional health warning

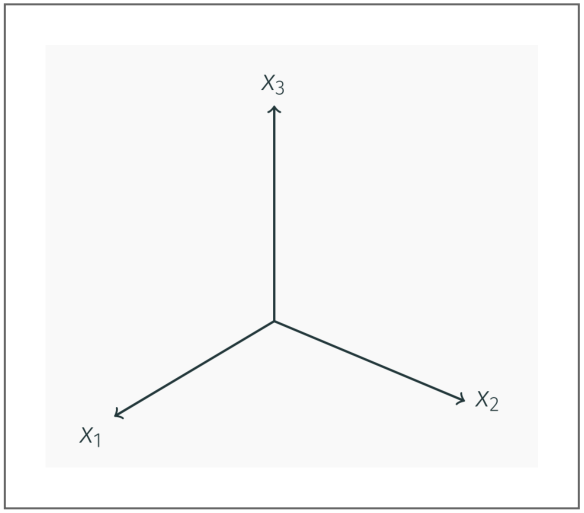
- Some texts flip the definition of $Q_{ij}$, and then flip their transformation law accordingly
- Any time you use tensor transformation, make sure you are following a consistent set of transformation laws

# general

- We can derive some interesting properties of the transformation tensor, $Q_{ij}$
- We know that $v_i = Q_{ji}v_j^{'}$ and that $v_i^{'} = Q_{ij}v_j$
- If we substitute (changing the appropriate indexes) we find:
- $v_i = Q_{ji}Q_{jk}v_k$
- We can now use the Kronecker Delta to substitute $v_i = \delta_{ik}v_k$ which gives
- $\delta_{ik}v_k = Q_{ji}Q_{jk}v_k$

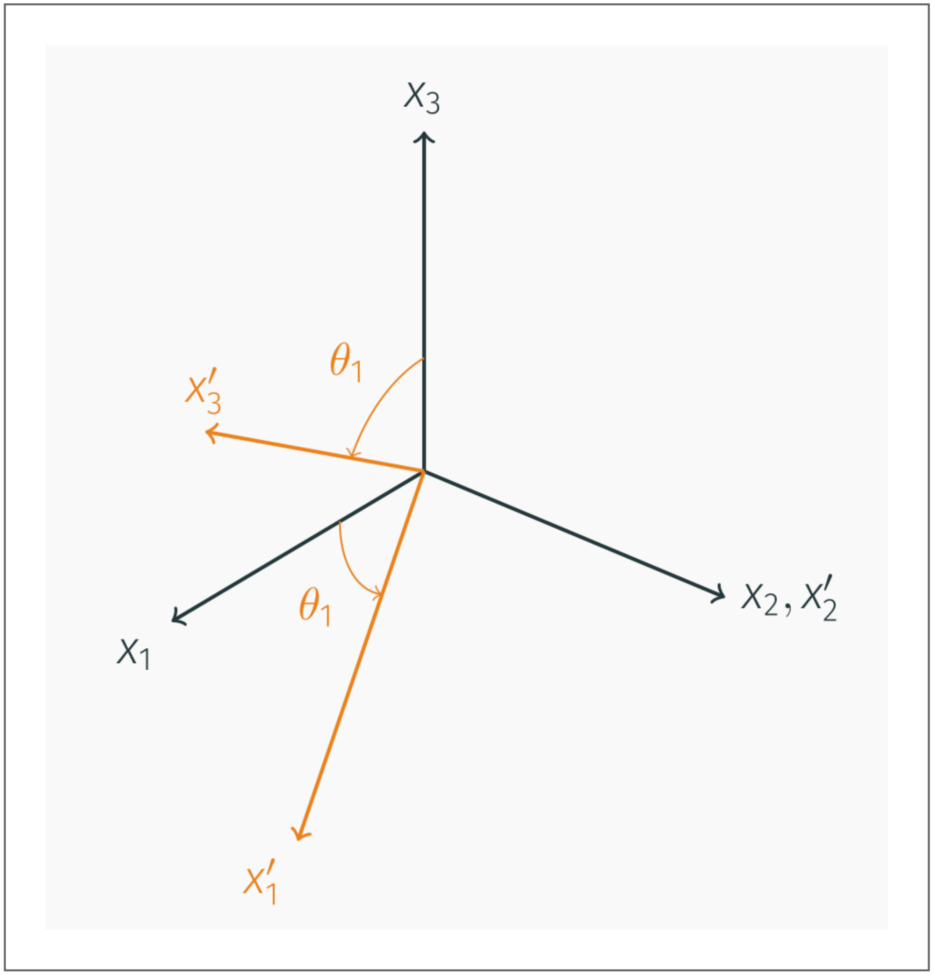
# examples

# example

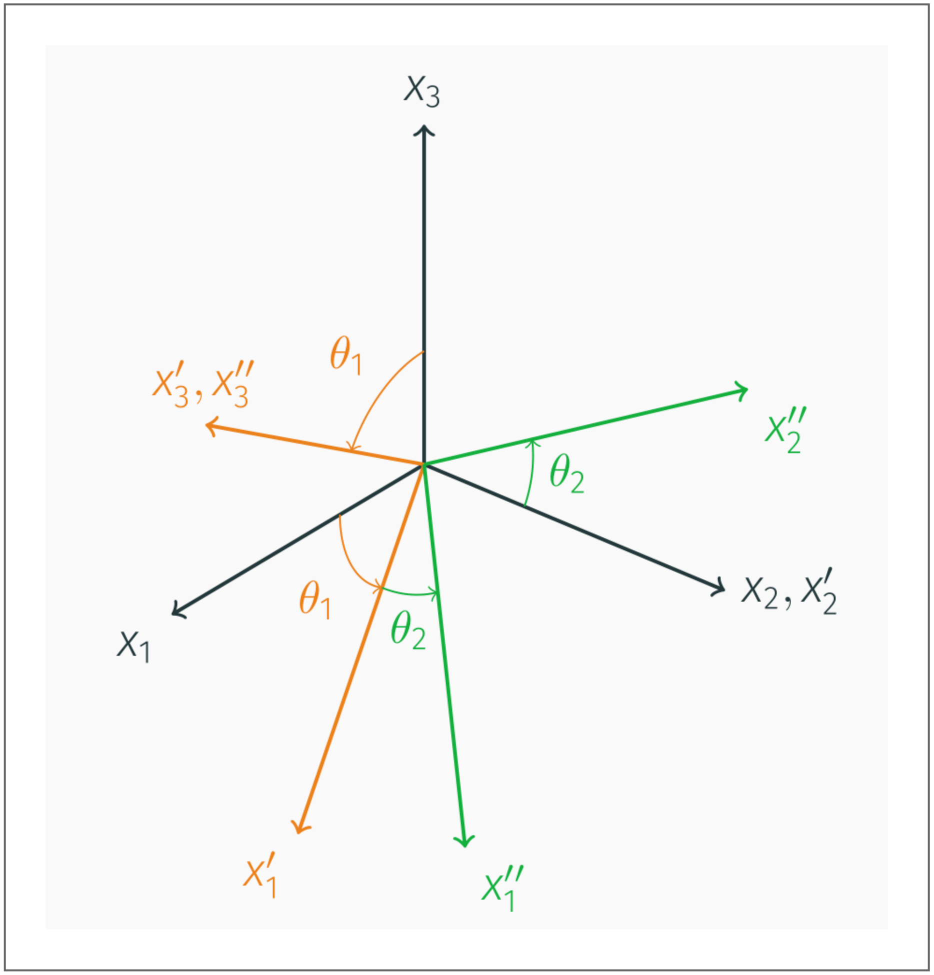



- Find $Q_{ij}^{1}$ for rotation of 60° about $x_2$
- Find $Q_{ij}^{2}$ for rotation of 30° about $x_3{}'$
- Find $e_i{}''$ after both rotations

# example



# example



# example

- ${Q_{ij}}^1 = \cos({x_i}', x_j)$
- ${Q_{ij}}^2 = \cos({x_i}'', {x_j}')$

$$Q^1_{ij} = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150 \\ \cos 90 & \cos 0 & \cos 90 \\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}$$

$$Q^2_{ij} = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90 \\ \cos 120 & \cos 30 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}$$

# example

- We now use $Q_{ij}$ to find $\hat{e}_i'$ and $\hat{e}_i''$
- First, we need to write $\hat{e}_i$ in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. $\hat{e}_1 = e_i^1$
- $e_i{}' = Q_{ij}{}^1 e_j$
- $e_i{}'' = Q_{ij}{}^2 e_j{}'$
- How do we find $e_i{}''$ in terms of $e_i$?
- $e_i{}'' = Q_{ij}{}^2 Q_{jk}{}^1 e_k$

# principal values

# principal values

- In the 2D coordinate transformation example, we were able to eliminate one value from a vector using coordinate transformation
- For second-order tensors, we desire to find the “principal values” where all non-diagonal terms are zero

# principal directions

- The direction determined by the unit vector, $n_j$, is said to be the *principal direction* or *eigenvector* of the symmetric second-order tensor, $a_{ij}$ if there exists a parameter, $\lambda$, such that $a_{ij}n_j = \lambda n_i$
- Where $\lambda$ is called the *principal value* or *eigenvalue* of the tensor

# principal values

- We can re-write the equation $(a_{ij} - \lambda\delta_{ij})n_j = 0$
- This system of equations has a non-trivial solution if and only if det$[a_{ij} - \lambda\delta_{ij}]$=0
- This equation is known as the characteristic equation, and we solve it to find the principal values of a tensor

# example

- Find the principal values of the tensor

$$A_{ij} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- From the characteristic equation, we know that det[$A_{ij} - \lambda\delta_{ij}$]=0, or

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

## example

- Calculating the determinant gives $(1-\lambda)(4-\lambda)-4 = 0$
- Multiplying out and simplifying, we find $\lambda^2 - 5\lambda = \lambda(\lambda - 5)=0$
- This has the solution $\lambda = 0, 5$

# invariants

# invariants

- Every tensor has some invariants which do not change with coordinate transformation
- These are known as *fundamental invariants*
- The characteristic equation for a tensor in 3D can be written in terms of the invariants $\det[a_{ij} - \lambda\delta_{ij}] = -\lambda^3 + I_\alpha\lambda^2 - II_\alpha\lambda + III_\alpha = 0$

# invariants

- The invariants can be found by the following equations

$$
\begin{aligned}
I_\alpha &= a_{ii} \\
II_\alpha &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) \\
III_\alpha &= \det[a_{ij}]
\end{aligned}
$$

# invariants

- In the principal direction, ${a_{ij}}'$ will be

$$a'_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

- Since invariants do not change with coordinate systems, we can also write the invariants as

$$\begin{aligned} I_\alpha &= \lambda_1 + \lambda_2 + \lambda_3 \\ II_\alpha &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ III_\alpha &= \lambda_1\lambda_2\lambda_3 \end{aligned}$$

# principal directions

# principal directions

- We defined principal directions earlier $(a_{ij} - \lambda\delta_{ij})n_j = 0$
- $\lambda$ are the principal values and $n_j$ are the principal directions
- For each eigenvalue there will be a principal direction
- We find the principal direction by substituting the solution for $\lambda$ back into this equation

# example

- Find the principal directions for the earlier principal values example
- Recall $\lambda = 0, 5$, let us say $\lambda_1 = 5$, we find $n_j^{(1)}$ by

$$\begin{bmatrix} 1-\lambda_1 & 2 \\ 2 & 4-\lambda_1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- This gives

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

# example

- We proceed to solve the equations using row-reduction, but we find

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- This means we cannot uniquely solve for $n_j$
- We are only concerned with the direction, magnitude is not important
- Choose $n_2 = 1$, solve for $n_1$
- $n^{(1)} = \langle \frac{1}{2}, 1 \rangle$

## example

- Similarly, for $\lambda_2 = 0$, we find

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- Which, after row-reduction, becomes

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = 0$$

- If we choose $n_2 = 1$, we find $n_1 = -2$
- This gives $n^{(2)} = \langle -2, 1 \rangle$

# example

- We can assemble a transformation matrix, $Q_{ij}$, from the principal directions
- First we need to normalize them to unit vectors
- $||n^{(1)}|| = \sqrt{\frac{5}{4}}$
- $\hat{n}^{(1)} = \frac{2}{\sqrt{5}}\langle\frac{1}{2}, 1\rangle = \langle\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$
- $||n^{(2)}|| = \sqrt{5}$
- $\hat{n}^{(2)} = \langle\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

# example

- This gives

$$Q_{ij} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- And we find $A_{mn}{}' = Q_{mi} Q_{nj} A_{ij}$

$$A'_{ij} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

# examples

# example

- Find principal values, principal directions, and invariants for the tensor

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

# example

- Characteristic equation simplifies to
- $-\lambda^3 + 14\lambda^2 - 56\lambda + 64 = 0$
- Which has the solutions $\lambda = 2, 4, 8$

# example

- To find the principal direction for $\lambda_1 = 8$

$$\begin{bmatrix} 8-8 & 0 & 0 \\ 0 & 3-8 & 1 \\ 0 & 1 & 3-8 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- After row-reduction, we find

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & 1 & -5 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_3 = 0$ and, as a result, $n_2 = 0$.
- $n_1$ can be any value, we choose $n_1 = 1$ to give a unit vector.
- $n^{(1)} = \langle 1, 0, 0 \rangle$

# example

- To find the principal direction for $\lambda_2 = 4$

$$\begin{bmatrix} 8-4 & 0 & 0 \\ 0 & 3-4 & 1 \\ 0 & 1 & 3-4 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- After row-reduction, we find

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = n_3$, so we choose $n_2 = n_1 = 1$
- This gives $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle$ after normalization

## example

- To find the principal direction for $\lambda_3 = 2$

$$\begin{bmatrix} 8-2 & 0 & 0 \\ 0 & 3-2 & 1 \\ 0 & 1 & 3-2 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

# example

- After row-reduction, we find

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0$$

- This means that $n_1 = 0$
- We also know that $n_2 = -n_3$, so we choose $n_2 = 1$ and $n_1 = -1$
- This gives $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$ after normalization

# example

- In summary, for $c_{ij}$ we have:
- $\lambda_1 = 8$ and $n^{(1)} = \langle 1, 0, 0\rangle$
- $\lambda_2 = 4$ and $n^{(2)} = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle$
- $\lambda_3 = 2$ and $n^{(3)} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$
- We can assemble $n^{(i)}$ into a transformation tensor

$$Q_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

# example

- What is ${c_{ij}}'$?
- ${c_{ij}}' = Q_{im}Q_{jn}c_{mn}$

$$c'_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

# example

- We can also find the invariants for

$$c_{ij} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

- Recall:

$$\begin{aligned} I_\alpha &= a_{ii} \\ II_\alpha &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) \\ III_\alpha &= \det[a_{ij}] \end{aligned}$$

# example

- First invariant $I_\alpha = a_{ii} = 8 + 3 + 3 = 14$
- Second invariant

$$II_\alpha = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij})$$

$a_{ii}a_{jj} = 14 \times 14$ $a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + ... + a_{33}a_{33}$

$$II_\alpha = \frac{1}{2}(196 - 84) = 56$$

# example

- Third invariant $III_\alpha = \det[a_{ij}]$ $III_\alpha = 8 \times (3 \times 3 - 1 \times 1)$=64