AE837

Advanced Mechanics of Damage Tolerance

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upcoming schedule

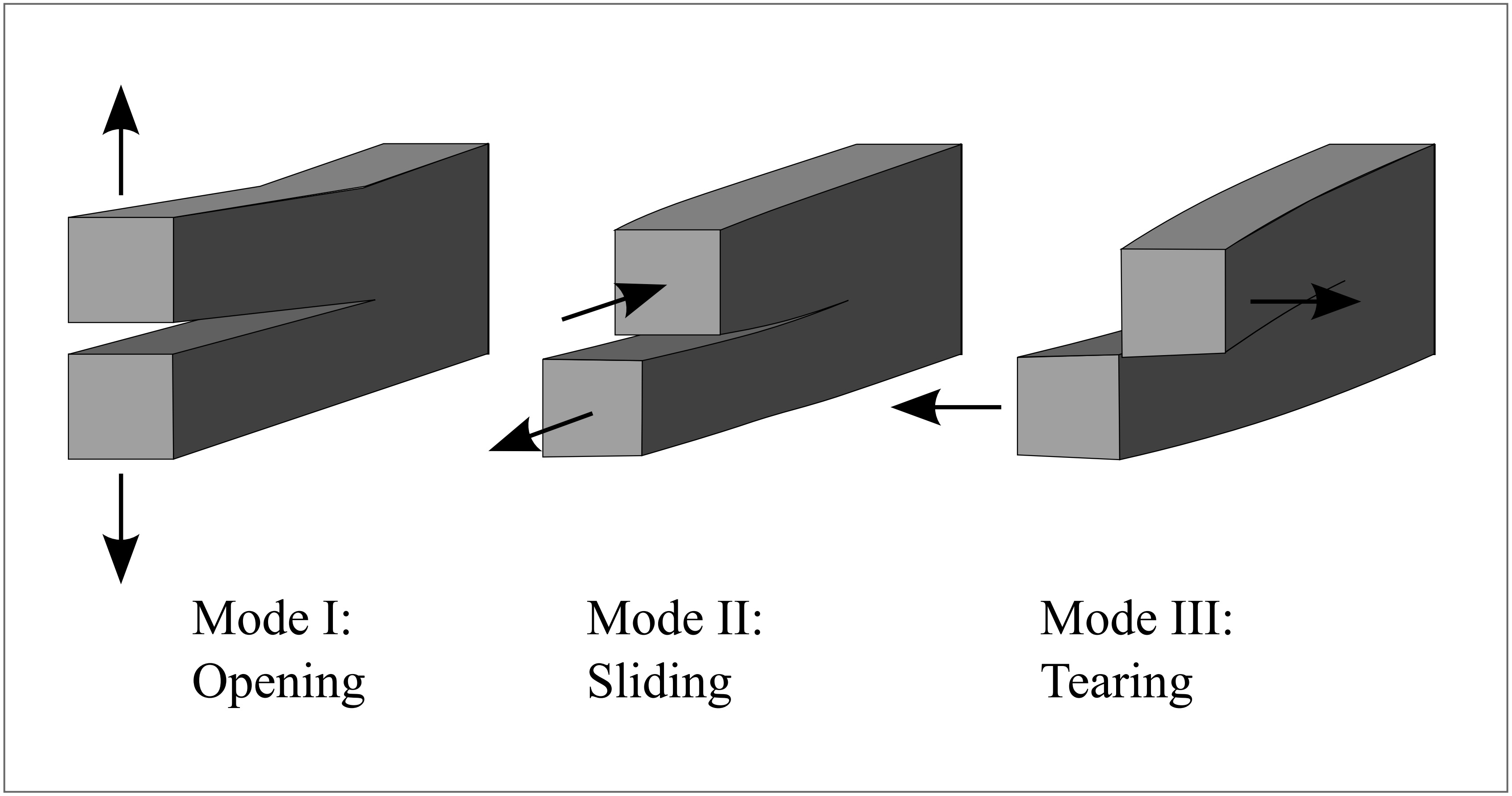
- Sep 3 - Elastic Stress Field, Homework 1 Due
- Sep 5 - Elastic Stress Field
- Sep 10 - Elastic Stress Field
- Sep 12 - Elastic Stress Field

outline

- review
- complex airy stress
- westergaard function method
- solutions using westergaard

review

fracture modes



stress field

- The approximate stress field for the various modes can be expressed in terms of the stress intensity factor

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}} + O(\sqrt{x}) \qquad \sigma_{xy} = \sigma_{yz} = 0$$

$$\sigma_{xy} = \frac{K_I I}{\sqrt{2\pi x}} + O(\sqrt{x}) \qquad \sigma_{yy} = \sigma_{yz} = 0$$

$$\sigma_{yz} = \frac{K_I II}{\sqrt{2\pi x}} + O(\sqrt{x}) \qquad \sigma_{yy} = \sigma_{xy} = 0$$

airy stress function

- A stress function technique that can be used to solve many planar problems is known as the *Airy stress function*
- This method reduces the governing equations for a planar problem to a single unknown function

airy stress function

- We assume first that body forces are derivable from a *potential function*, V

$$F_x = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y}$$

- How restrictive is this assumption?
- Most body forces are linear (gravity) and can easily be represented this way

airy stress function

- Consider the following

$$
\begin{aligned}
\sigma_{xx} &= \frac{\partial^2 \phi}{\partial y^2} + V \\
\sigma_{yy} &= \frac{\partial^2 \phi}{\partial x^2} + V \\
\tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y}
\end{aligned}
$$

- The function $\phi = \phi(x, y)$ is known as the Airy stress function
- Equilibrium is automatically satisfied

compatibility

- Substituting the Airy Stress function and potential function into the relationships, we find

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = -\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad \text{plane strain}$$

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \qquad \text{plane stress}$$

compatibility

- If there are no body forces, or the potential function satisfies Laplace's Equation

$$\nabla^2 V = 0$$

- Then both plane stress and plane strain reduce to

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

complex airy stress

complex conjugates

- In the cartesian system we can express complex conjugates as

$$
\begin{aligned}
z &= x + iy \\
\bar{z} &= x - iy
\end{aligned}
$$

- And in polar coordinates we have

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta) = re^{i\theta} \\
\bar{z} &= r(\cos\theta - i\sin\theta) = re^{-i\theta}
\end{aligned}
$$

analytic functions

- A complex function can be written as
$$f(z)=u(x,y)+iv(x,y)$$
- A complex function is said to be analytic if
$$\frac{\partial}{\partial x}f(z) = f'(z)\frac{\partial z}{\partial x} = f'(z)$$
- and
$$\frac{\partial}{\partial y}f(z) = f'(z)\frac{\partial z}{\partial y} = if'(z)$$

Cauchy-Riemann

- This means that

$$\frac{\partial}{\partial x} f(z) = -i \frac{\partial}{\partial y} f(z)$$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Cauchy-Riemann

- From this we obtain the Cauchy-Riemann equations
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y}$$
- From which we can easily derive the following
$$\nabla^2 u = \nabla^2 v = 0$$

airy stress functions

- The Airy stress function is biharmonic, we can write the following

$$\nabla^2\phi = P$$

- From which we have

$$\nabla^2 P = \nabla^2\nabla^2\phi = 0$$

- And thus P will satisfy the compatibility equations

airy stress functions

- We can say that P is the real part of a complex function
  $$P = Re\{f(z)\} \qquad \text{where} \qquad f(z)=P + iQ$$
- Now we let
  $$\psi(z) = \frac{1}{4}\int f(z)dz = p + iq$$
- ψ will also be analytic, thus
  $$\psi'(z) = \frac{1}{4}f(z)$$

airy stress functions

- According to the Cauchy-Riemann equations we now have
$$\psi'(z) = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{\partial q}{\partial y} - i\frac{\partial p}{\partial y}$$
- And we find that
$$P = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y}$$

airy stress functions

- If we now consider $\phi - (xp + yq)$, we can show that
$$\nabla^2[\phi - (xp + yq)] = 0$$
- This means that $\phi - (xp + yq)$ is harmonic, and can be taken as either the real or imaginary portion of some analytic function, $\chi(z)$
$$\phi - (xp + yq) = \mathrm{Re}\{\chi(z)\}$$
- We can now say that
$xp + yq = \mathrm{Re}\{\bar{z}\psi(z)\}$

complex airy stress

- The complex representation of the Airy stress function can now be written as

$$\phi = \mathrm{Re}\{\bar{z}\,\psi(z) + \chi(z)\}$$
$$2\phi(x,y) = \bar{z}\,\psi(z) + z\bar{\psi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z})$$

- And from the definition of the Airy stress function we obtain

$$\sigma_{xx} + i\sigma_{xy} = \frac{\partial^2\phi}{\partial y^2} - i\frac{\partial^2\phi}{\partial x\partial y} \qquad = -i\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$
$$\sigma_{yy} - i\sigma_{xy} = \frac{\partial^2\phi}{\partial x^2} + i\frac{\partial^2\phi}{\partial x\partial y} \qquad = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$

properties of analytic functions and conjugates

$$
\begin{aligned}
\frac{\partial f(z)}{\partial x} &= f' \frac{\partial z}{\partial x} & &= f'(z) \\
\frac{\partial f\bar{(}z)}{\partial x} &= \left(\frac{\partial \bar{f}(z)}{\partial x}\right) & &= f'\bar{(}z) \\
\frac{\partial f(z)}{\partial y} &= f'(z)\frac{\partial z}{\partial y} & &= if'(z) \\
\frac{\partial f\bar{(}z)}{\partial y} &= \left(\frac{\partial \bar{f}(z)}{\partial y}\right) & &= -if'\bar{(}z)
\end{aligned}
$$

complex airy stress functions

- The above properties allow us to write
  $$\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}$$
- Which gives

$$\sigma_{xx} + i\sigma_{xy} = \psi'(z) + \overline{\psi'(z)} - z\overline{\psi''(z)} - \overline{\chi''(z)}$$
$$\sigma_{yy} - i\sigma_{xy} = \psi'(z) + \overline{\psi'(z)} + z\overline{\psi''(z)} + \overline{\chi''(z)}$$

- We can add the two equations to find
  $$\sigma_{xx} + \sigma_{yy} = 2(\psi'(z) + \overline{\psi'(z)}) = 4\mathrm{Re}\{\psi'(z)\}$$
- Similarly we can subtract the equations to find
  $$\sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy} = 2(z\overline{\psi''(z)} + \overline{\chi''(z)})$$
- We can re-write using the complex conjugate
  $$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2(\bar{z}\,\psi''(z) + \chi''(z))$$

displacement

- We can use Hooke's law to find displacements in terms of the complex functions
- After some algebra, we find
  $2\mu(u_x + iu_y) = \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)}$
- Where
  $$\kappa = \frac{\lambda^* + 3\mu}{\lambda^* + \mu}$$

westergaard function method

mode I

- If we consider an infinite plate with cracks along the x-axis, and external loads are symmetric with respect to the x-axis, then $\sigma_{xy} = 0$ along $y = 0$
  $\text{Im}\{\bar{z}\psi''(z) + \chi''(z)\} = 0 \qquad \text{at } y = 0$
- At $y = 0$, $z = \bar{z}$, therefore we have (for all y)

$$\chi''(z) + z\psi''(z) + A = 0$$

- Where A is some real constant

mode I

- After substituting $\chi''(z) = -z\psi''(z) - A$ into the Airy stress relationships we find

$$\begin{aligned}
\sigma_{xx} &= 2\mathrm{Re}\{\psi'\} - 2y\mathrm{Im}\{\psi''\} + A \\
\sigma_{yy} &= 2\mathrm{Re}\{\psi'\} + 2y\mathrm{Im}\{\psi''\} - A \\
\sigma_{xy} &= -2y\mathrm{Re}\{\psi''\}
\end{aligned}$$

mode I

- We now define
  $$\psi' = \frac{1}{2}(Z_I + A)$$
- where $\hat{Z}_I' \equiv Z_I$ and thus

$$\psi = \frac{1}{2}(\hat{Z}_I + Az)$$
$$\psi'' = \frac{1}{2}Z_I'$$

mode I

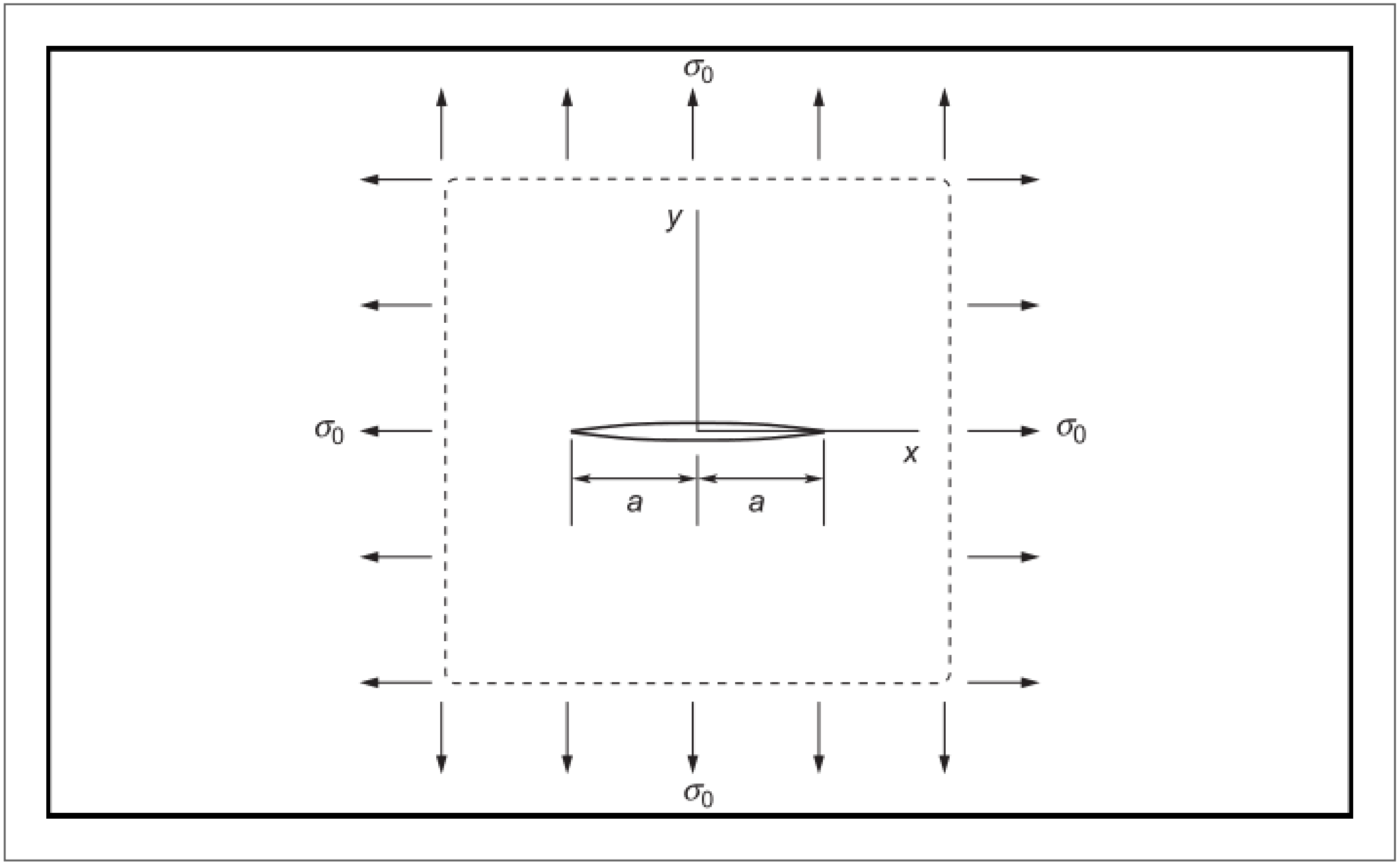
- This gives the following result for the stress field

$$
\begin{aligned}
\sigma_{xx} &= \mathrm{Re}\{Z_I\} - y\mathrm{Im}\{Z_I'\} + 2A \\
\sigma_{yy} &= \mathrm{Re}\{Z_I\} + y\mathrm{Im}\{Z_I'\} \\
\sigma_{xy} &= -y\mathrm{Re}\{Z_I'\}
\end{aligned}
$$

- and Z_I is the so-called Westergaard function

solutions using westergaard

mode I crack



mode I crack