AE837

Advanced Mechanics of Damage Tolerance

Dr. Nicholas Smith
Wichita State University, Department of Aerospace Engineering August 22, 2019

upcoming schedule

- Aug 22 - Elasticity Review
- Aug 27 - Griffith Fracture
- Aug 29 - Griffith Fracture
- Sep 5 - Elastic Stress Field

outline

- tensor calculus
- other coordinate systems
- equilibrium equations
- spherical and cylindrical coordinates
- field equations
- boundary conditions
- stress formulation
- strain energy
- example
- airy stress functions

tensor calculus

gradient

- The gradient operator, ∇, is often used to indicate partial differentiation in matrix and vector notation
- We can represent ∇ as a vector

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle$$

- ∇ is also referred to as the *del operator*

gradient

- We can convert between vector notation and index notation for many common operations using the ∇.

$$\nabla\phi = \phi_{,i}$$
$$\nabla^2\phi = \phi_{,ii}$$
$$\nabla\hat{u} = u_{i,j}$$
$$\nabla\cdot\hat{u} = u_{i,i}$$
$$\nabla\times\hat{u} = \epsilon_{ijk}u_{k,j}$$
$$\nabla^2\hat{u} = u_{i,kk}$$

divergence theorem

- The Divergence Theorem (or Gauss Theorem) for a vector field, $\hat{u}$,

$$\iint_S \hat{u} \cdot \hat{n} dS = \iiint \nabla \cdot \hat{u} dV$$

- is also valid for tensors of any order $\iint_S a_{ij\ldots k} n_k dS = \iiint_V a_{ij\ldots k,k} dV$

stokes theorem

- Stokes theorem for a vector field, $\hat{u}$,

$$\oint \hat{u} \cdot d\hat{r} = \iint_S (\nabla \times \hat{u}) \cdot \hat{n} dS$$

- also applies for tensors of any order $\oint a_{ij\ldots k} dx_t = \iint_S \epsilon_{rst} a_{ij\ldots k,s} n_r dS$

green's theorem

- Green's theorem is merely a simplification of Stokes theorem in a planar domain.
- If we write the vector field, $\hat{u} = f\hat{e_1} + g\hat{e_2}$, we find

$$\iint_S \left(\frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) dxdy = \int_C (fdx + gdy)$$

zero-value theorem

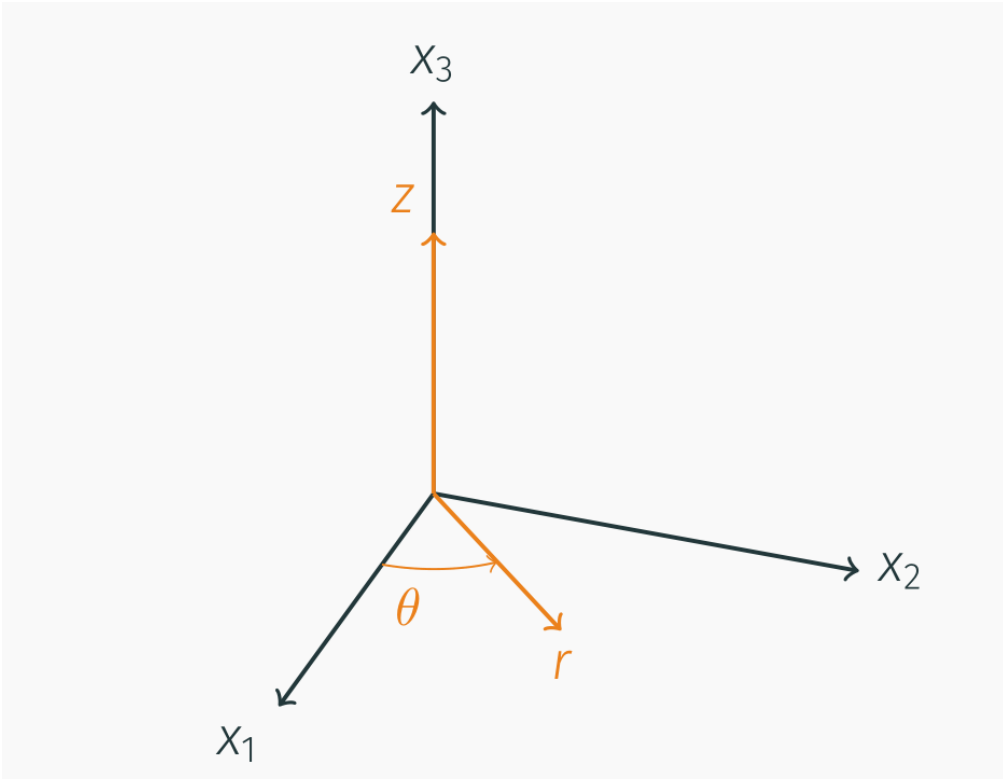
- The zero-value theorem is particularly useful in variational calculus, which we will use later in the course
- If we know that $\iiint_V f_{ij\ldots k} dV = 0$
- then $f_{ij\ldots k} = 0$

other coordinate systems

curvilinear coordinates

- We discussed coordinate transformations earlier
- However, we often desire to use other coordinate systems entirely
- Polar coordinates (in 2D) are an example of this
- In 3D, we can use cylindrical or spherical coordinates

cylindrical coordinates



cylindrical coordinates

- We can convert between Cartesian and cylindrical coordinate systems

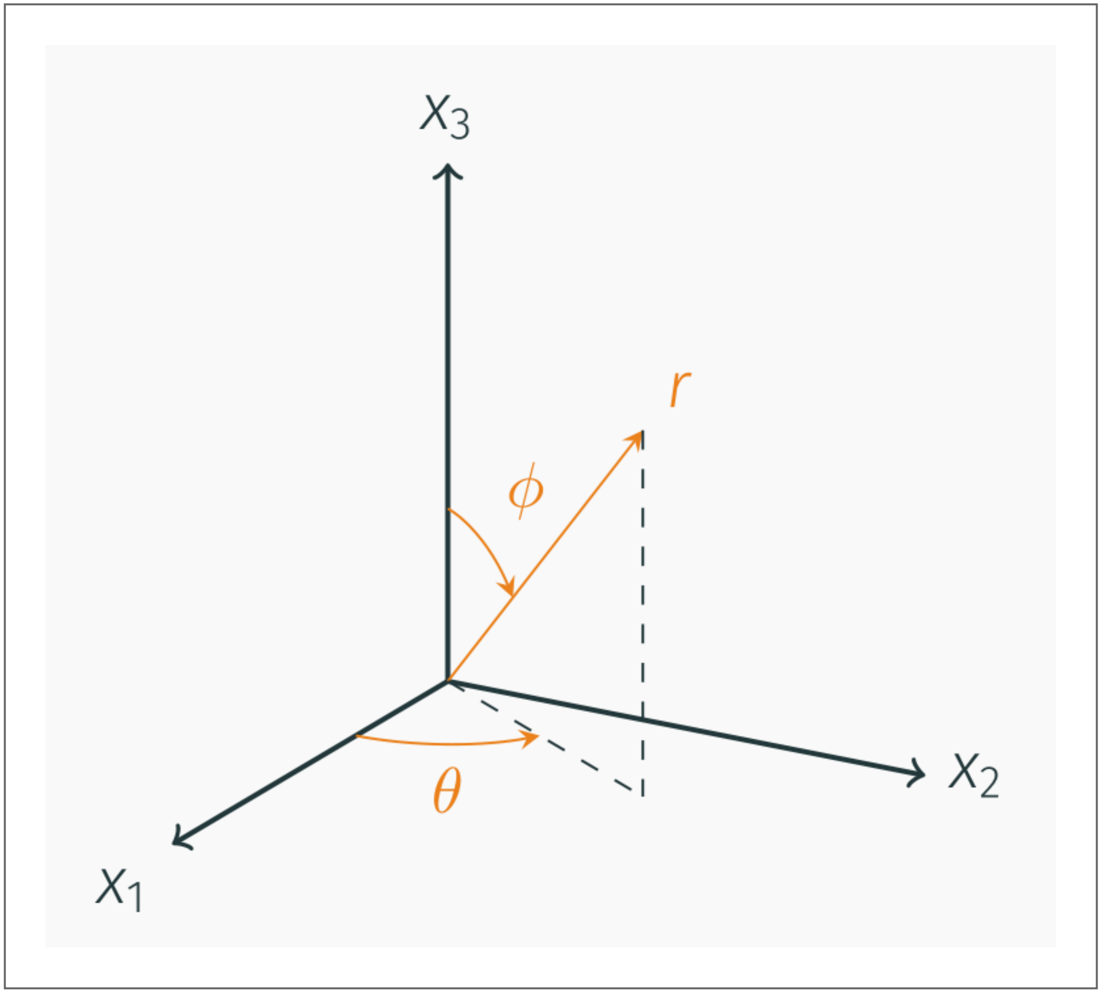
$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$
$$x_3 = z$$

cylindrical coordinates

- Or to convert from Cartesian to cylindrical

$$r = \sqrt{x_1^2 + x_2^2}$$
$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$
$$z = x_3$$

spherical coordinates



spherical coordinates

- We can convert between Cartesian and spherical coordinate systems

$$x_1 = r\cos\theta\sin\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\phi$$

spherical coordinates

- Or to convert from Cartesian to cylindrical

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\phi = \cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right)$$

$$\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

calculus in cylindrical coordinates

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z} \\
\nabla \cdot \mathbf{u} &= \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
\nabla \times \mathbf{u} &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\hat{z}
\end{aligned}
$$

calculus in spherical coordinates

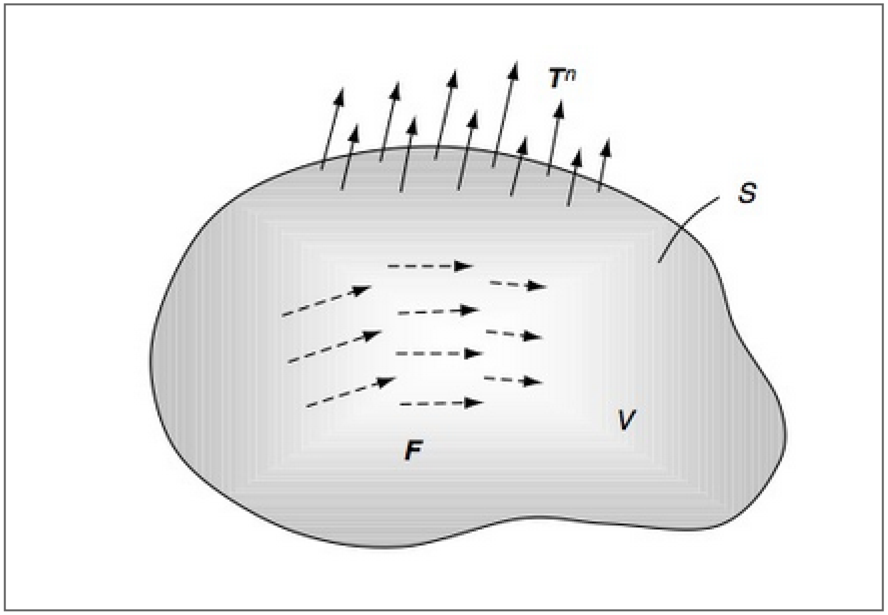
$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\hat{\theta}$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial (u_\phi \sin\phi)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial \theta}$$

$$\nabla \times \mathbf{u} = \frac{1}{r\sin\phi}\left(\frac{\partial (u_\theta \sin\phi)}{\partial \phi} - \frac{\partial u_\phi}{\partial \theta}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\phi}\frac{\partial u_r}{\partial \theta} - \frac{\partial (r u_\theta)}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial (r u_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi}\right)\hat{\theta}$$

equilibrium equations

static equilibrium



- We primarily deal with bodies in static equilibrium
- This means that all forces and moments must sum to zero
- For a closed sub-domain of volume V and surface area S with internal body forces and applied tractions, we find $\iint_S T_i^n dS + \iiint_V F_i dV = 0$

static equilibrium

- Using the Cauchy stress theorem, we can replace the traction vector with the stress tensor $\iint_S \sigma_{ji} n_j dS + \iiint_V F_i dV = 0$
- We can also apply the divergence theorem to convert the surface integral to a volume integral $\iiint_V (\sigma_{ji,j} + F_i) dV = 0$
- Since the volume is arbitrary (we could choose any volume and the conditions for equilibrium would still hold), the integrand must vanish $\sigma_{ji,j} + F_i = 0$

equilibrium equations

- Written in scalar form, the equilibrium equations are

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z &= 0
\end{aligned}
$$

angular momentum

- Similarly, the principle of angular momentum states that the moment forces must all sum to zero as well $\iint_S \epsilon_{ijk} x_j T_k{}^n dS + \iiint_V \epsilon_{ijk} x_j F_k dV = 0$
- Once again we use Cauchy's stress theorem $\iint_S \epsilon_{ijk} x_j \sigma_{lk} n_l dS + \iiint_V \epsilon_{ijk} x_j F_k dV = 0$
- And the divergence theorem $\iiint_V [(\epsilon_{ijk} x_j \sigma_{lk})_{,l} + \epsilon_{ijk} x_j F_k] dV = 0$

angular momentum

- Expanding the derivative using the chain rule gives $\iiint_V [\epsilon_{ijk} x_{j,l} \sigma_{lk} + \epsilon_{ijk} x_j \sigma_{lk,l} + \epsilon_{ijk} x_j F_k] dV = 0$
- Which can be simplified (recall that $\sigma_{ji,j} + F_i = 0$)

$$\iiint_V [\epsilon_{ijk} \delta_{jl} \sigma_{lk} + \epsilon_{ijk} x_j \sigma_{lk,l} + \epsilon_{ijk} x_j F_k] dV = 0$$

$$\iiint_V [\epsilon_{ijk} \sigma_{jk} - \epsilon_{ijk} x_j F_k + \epsilon_{ijk} x_j F_k] dV = 0$$

$$\iiint_V \epsilon_{ijk} \sigma_{jk} \, dV = 0$$

angular momentum

- Using the same argument as before (arbitrary volume) the integrand must vanish $\epsilon_{ijk}\sigma_{jk} = 0$
- Since the alternating symbol is antisymmetric in jk, σ_{jk} must be symmetric in jk for this to vanish
- And thus we have proved that the stress tensor is symmetric, thus equilibrium and angular momentum equations are satisfied when $\sigma_{ji,j} + F_i = 0$

example

- Under what circumstances is the following stress field in static equilibrium?
- $\sigma_{11} = 3x_1 + k_1 x_2^2, \sigma_{22} = 2x_1 + 4x_2, \sigma_{12} = \sigma_{21} = a + bx_1 + cx_1^2 + dx_2 + ex_2^2 + fx_1 x_2$
- We are only examining the stress field, so we neglect any internal body forces

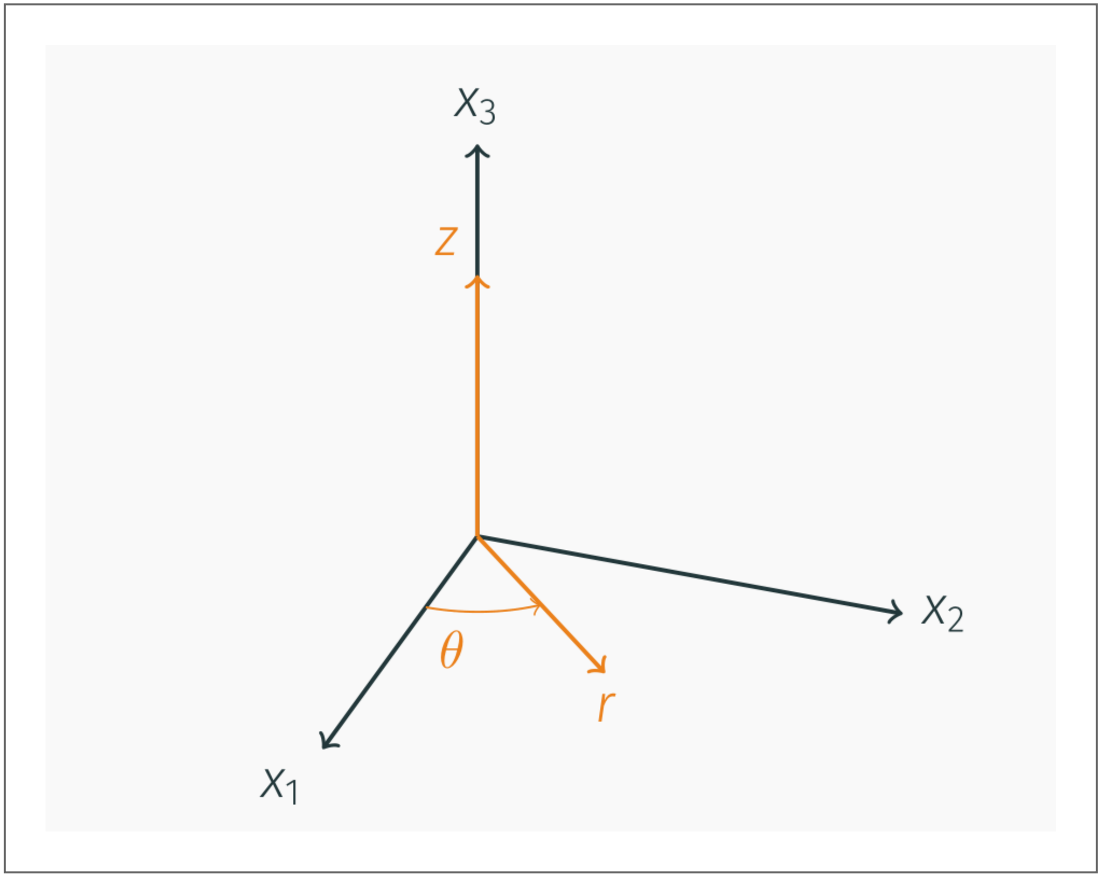
example

- The first equilibrium equation gives
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$3 + d + 2ex_2 + fx_1 = 0$

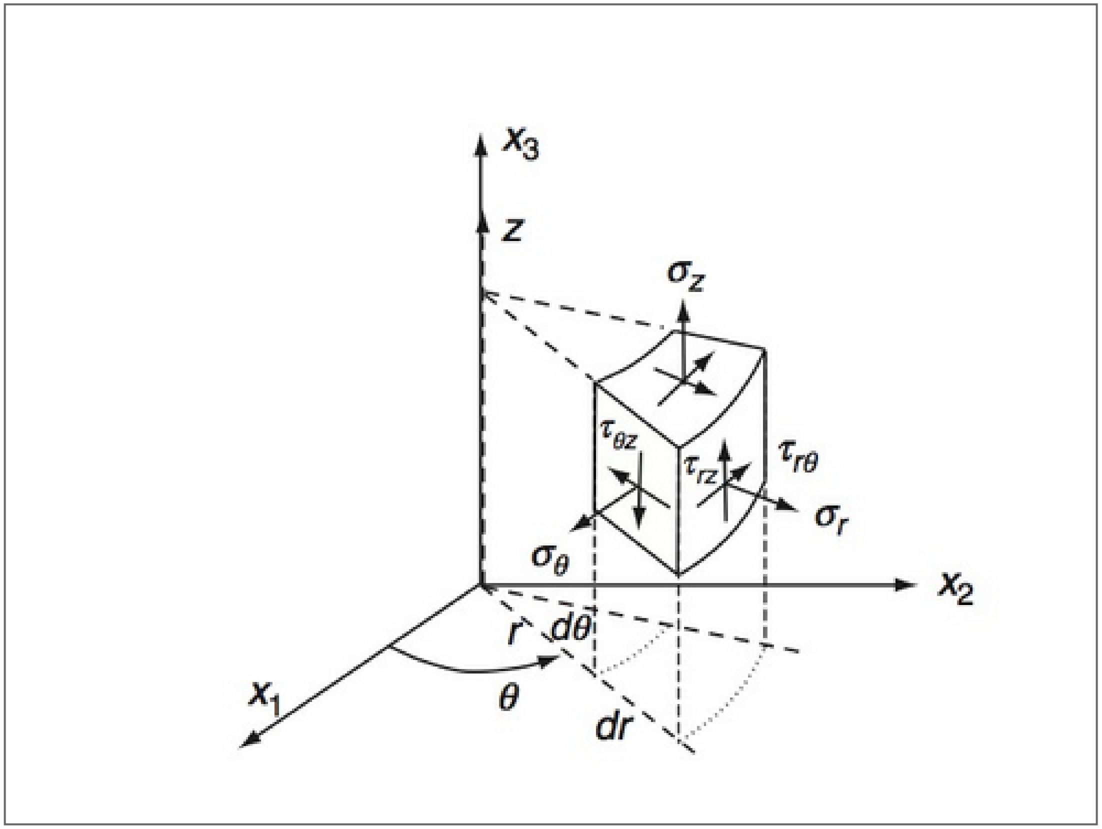
- The second equilibrium equation gives
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$
$b + 2cx_1 + fx_2 + 4 = 0$

spherical and cylindrical coordinates

cylindrical coordinates



stress



x_3
z
σ_z
$\tau_{\theta z}$
τ_{rz}
$\tau_{r\theta}$
σ_r
σ_θ
x_2
r
$d\theta$
x_1
θ
dr

stress in cylindrical coordinates

- The stress tensor in cylindrical coordinates is

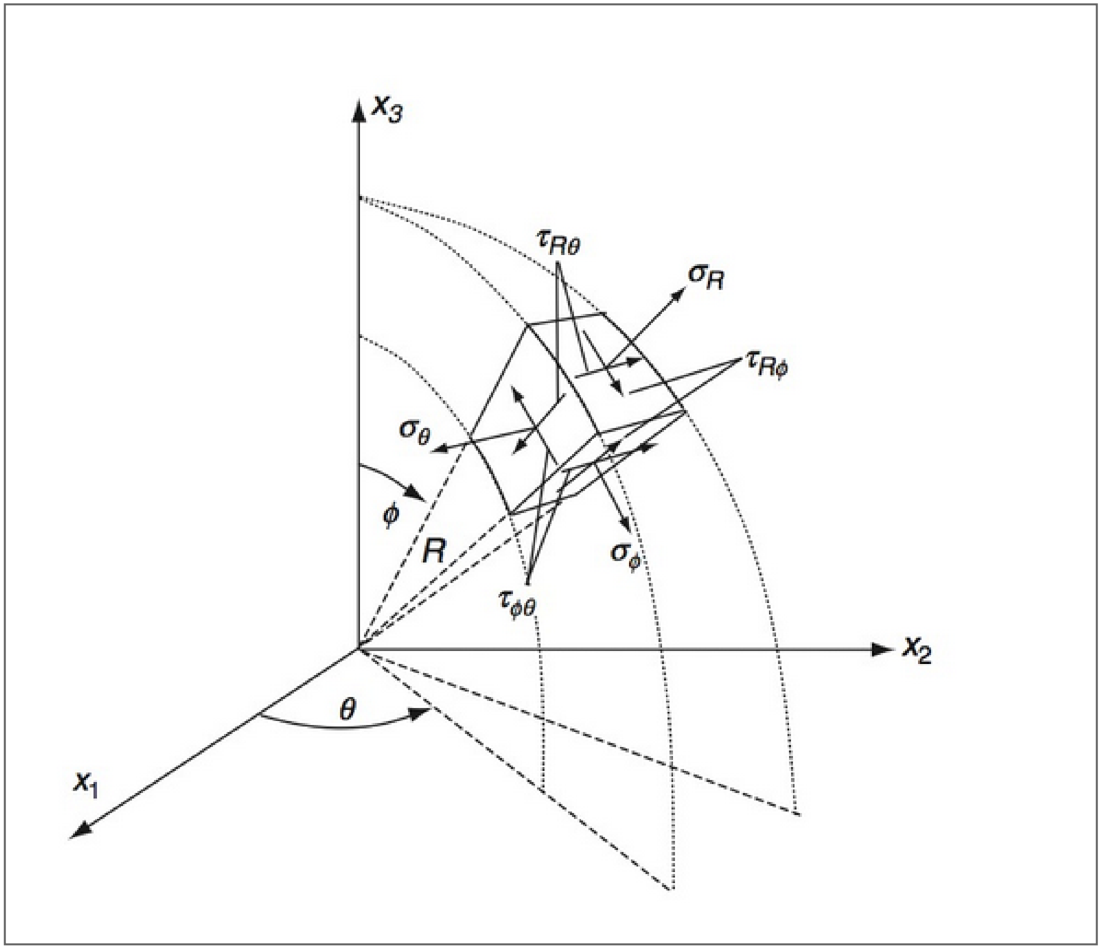
$$\sigma_{ij} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

equilibrium in cylindrical coordinates

- Using the derivative relationships developed in Chapter 1, we can express the equilibrium equations as

$$
\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}(\sigma_r - \sigma_\theta) + F_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta &= 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z &= 0
\end{aligned}
$$

spherical coordinates



spherical coordinates

- The stress tensor in spherical coordinates is

$$\sigma_{ij} = \begin{bmatrix} \sigma_r & \tau_{r\phi} & \tau_{r\theta} \\ \tau_{r\phi} & \sigma_\phi & \tau_{\phi\theta} \\ \tau_{r\theta} & \tau_{\phi\theta} & \sigma_\theta \end{bmatrix}$$

spherical equilibrium

$$
\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(2\sigma_r - \sigma_\phi - \sigma_\theta + \tau_{r\phi}\cot\phi) + F_r &= 0 \\
\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r}[(\sigma_\phi - \sigma_\theta)\cot\phi + 3\tau_{r\phi}] + F_\phi &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r}(2\tau_{\phi\theta}\cot\phi + 3\tau_{r\theta}) + F_\theta &= 0
\end{aligned}
$$

field equations

field equations

- Field equations that we have already found
- Strain-displacement

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- Equilibrium $\sigma_{ij,j} + F_i = 0$

- Constitutive (Hooke's Law)

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}$$
$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

field equations

- There are 15 unique field equations to solve for the 15 unknowns
- 3 displacements (u_i), 6 unique strain tensor terms (ϵ_{ij}), and 6 unique stress tensor terms (σ_{ij})
- These equations also depend on a knowledge of the material behavior (λ, μ) and body forces (usually gravity or zero)

compatibility equations

- If continuous, single-valued displacements are specified, differentiation will result in well-behaved strain field
- The inverse relationship, integration of a strain field to find displacement, may not always be true
- There are cases where we can integrate a strain field to find a set of discontinuous displacements

compatibility

- The compatibility equations enforce continuity of displacements to prevent this from occurring
- To enforce this condition we consider the strain-displacement relations:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- and differentiate with respect to x_k and x_l

$$\epsilon_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

- Or $2\epsilon_{ij,kl} = u_{i,jkl} + u_{j,ikl}$

compatibility

- We can eliminate the displacement terms from the equation by interchanging the indexes to generate new equations

$$2\epsilon_{ik,jl} = u_{i,jkl} + u_{k,ijl}$$
$$2\epsilon_{jl,ik} = u_{j,ikl} + u_{l,ijk}$$

- Solving for $u_{i,jkl}$ and $u_{j,ikl}$

$$u_{i,jkl} = 2\epsilon_{ik,jl} - u_{k,ijl}$$
$$u_{j,ikl} = 2\epsilon_{jl,ik} - u_{l,ijk}$$

compatibility

- Substituting these values into the equations gives $2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} - u_{k,ijl} + 2\epsilon_{jl,ik} - u_{l,ijk}$
- We now consider one more change of index equation $2\epsilon_{kl,ij} = u_{k,ijl} + u_{l,ijk}$
- and substituting this result gives $2\epsilon_{ij,kl} = 2\epsilon_{ik,jl} + 2\epsilon_{jl,ik} - 2\epsilon_{kl,ij}$
- Or, simplified $\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0$

compatibility equations

- The so-called *Saint-Venant compatibility equations* in full are a system of 81 equations, but only six are useful (although even these six are not entirely linearly independent)
- These six are found by setting $k = l$

compatibility

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$
$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$
$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}$$
$$\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$
$$\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right)$$
$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y}\right)$$

compatibility

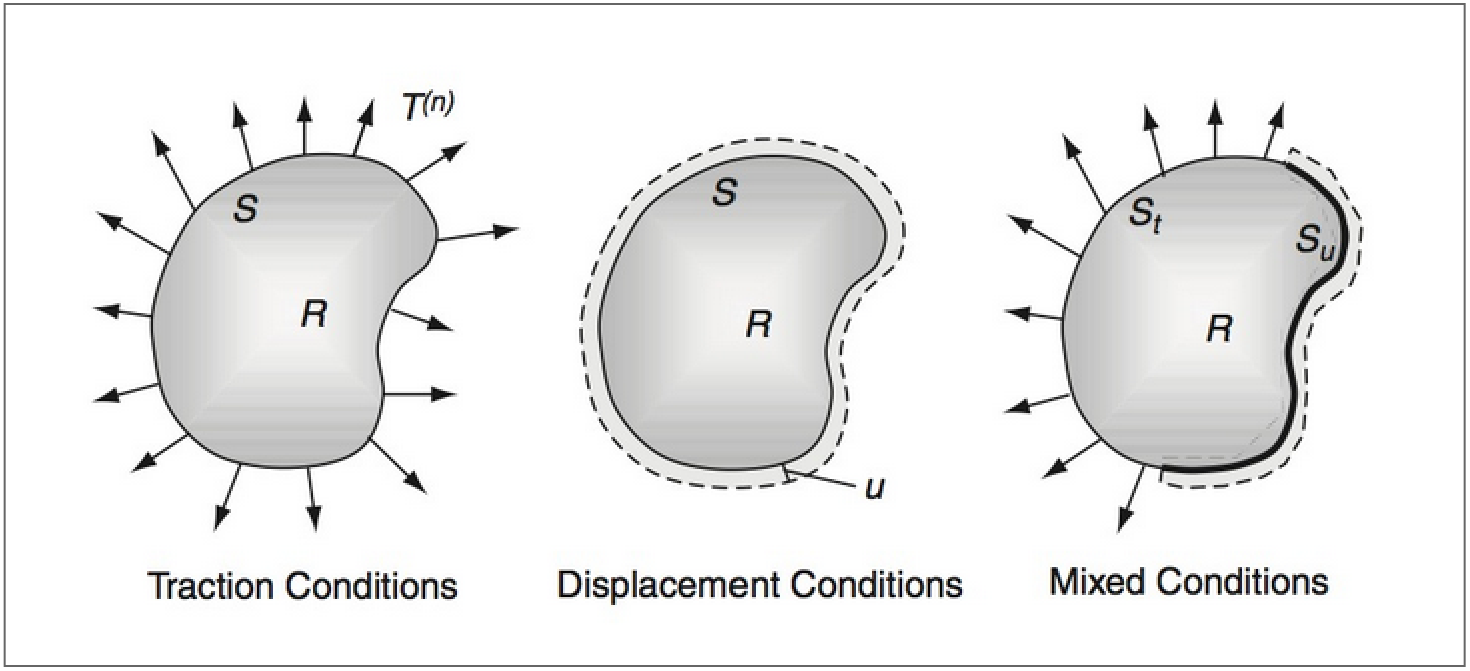
- The compatibility equations are necessary to ensure that the strain field is valid and will produce a continuous displacement field
- While these equations are important and necessary in solving elasticity problems, they are not sufficient
- 15 equations with 15 “unknowns” but each of these “unknowns” could actually be a function with many more unknowns, we need to develop framework for simplifying the problem into something we can solve

boundary conditions

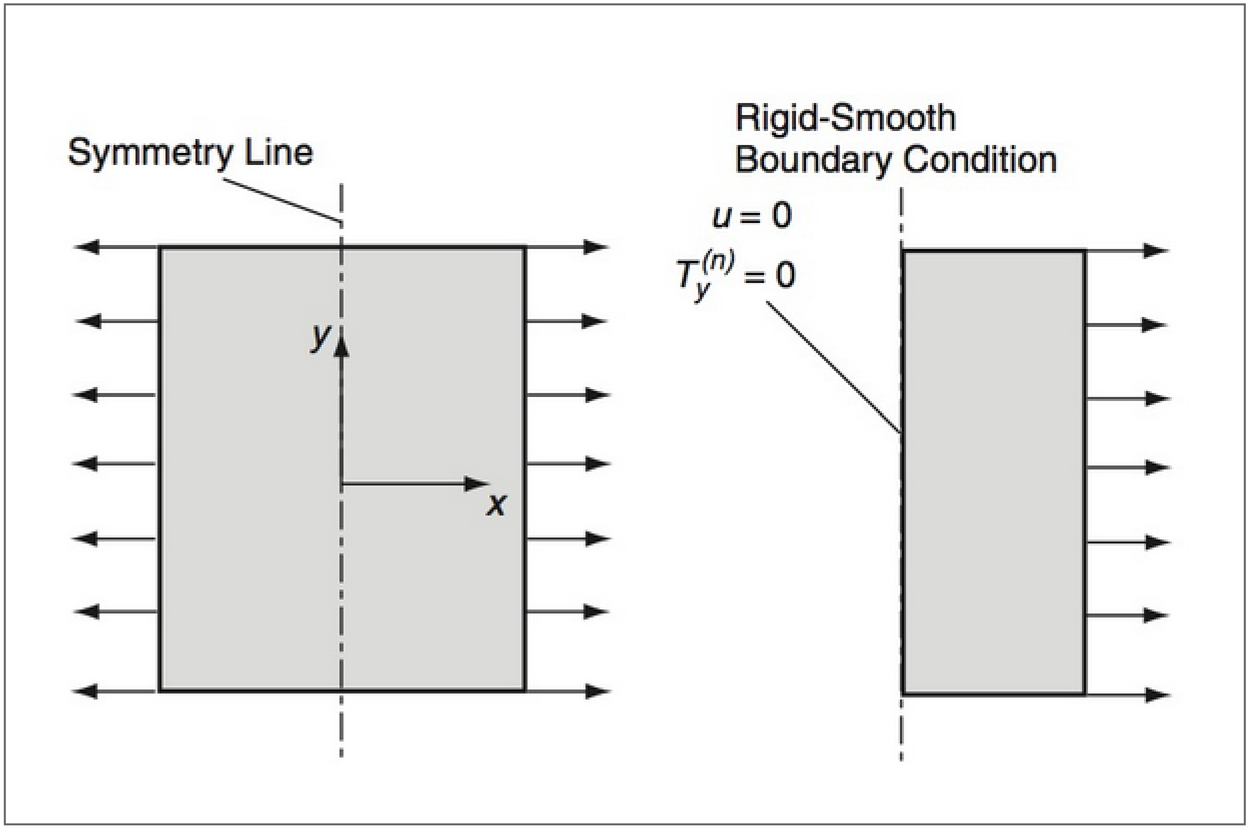
boundary conditions

- Boundary conditions commonly specify how a body is supported and/or how it is loaded
- Mathematically we treat this conditions as *displacements* or *tractions* at boundary points.
- Symmetry boundary conditions are also common, can reduce computational cost and simplify analytic solutions.

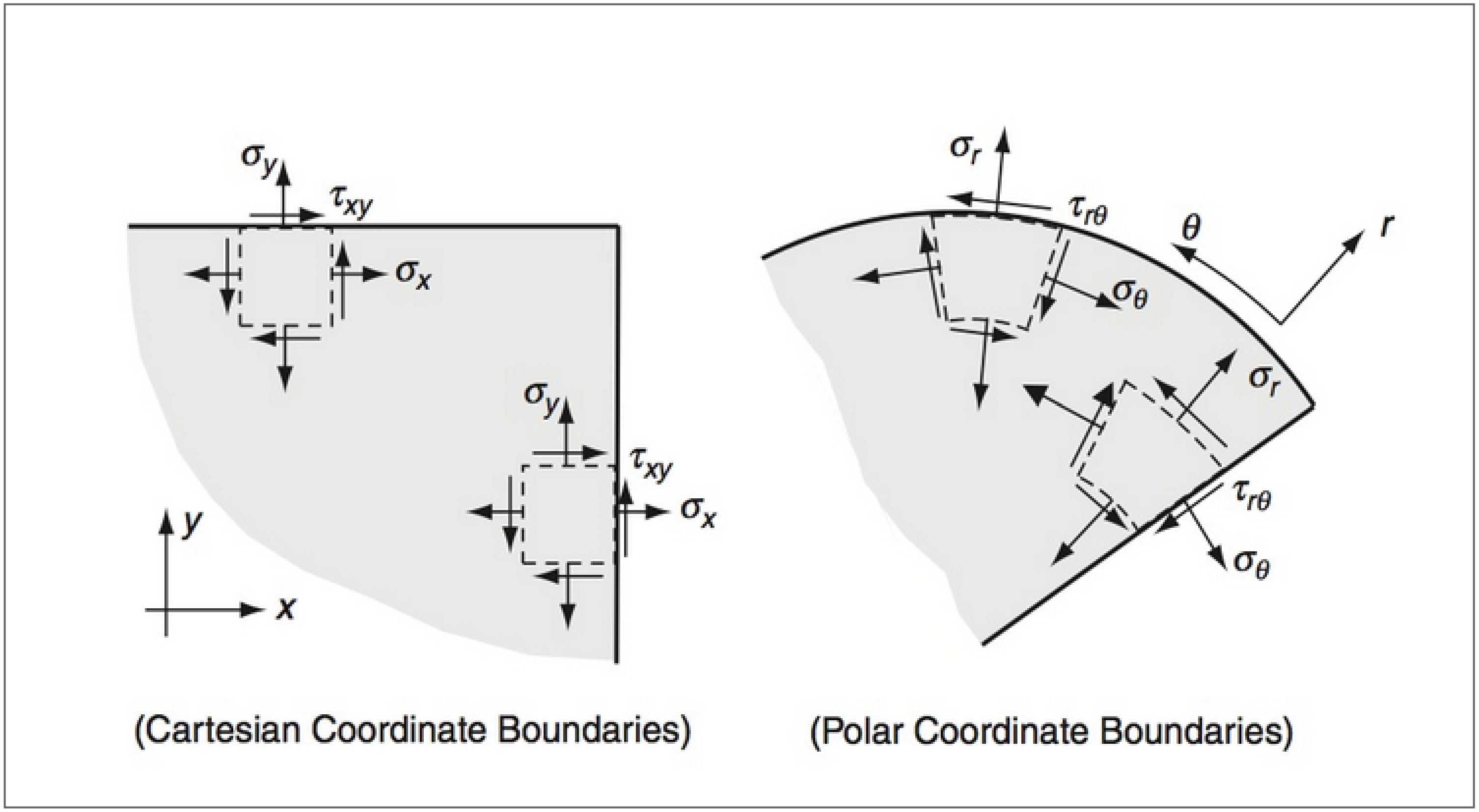
boundary conditions



symmetric boundaries

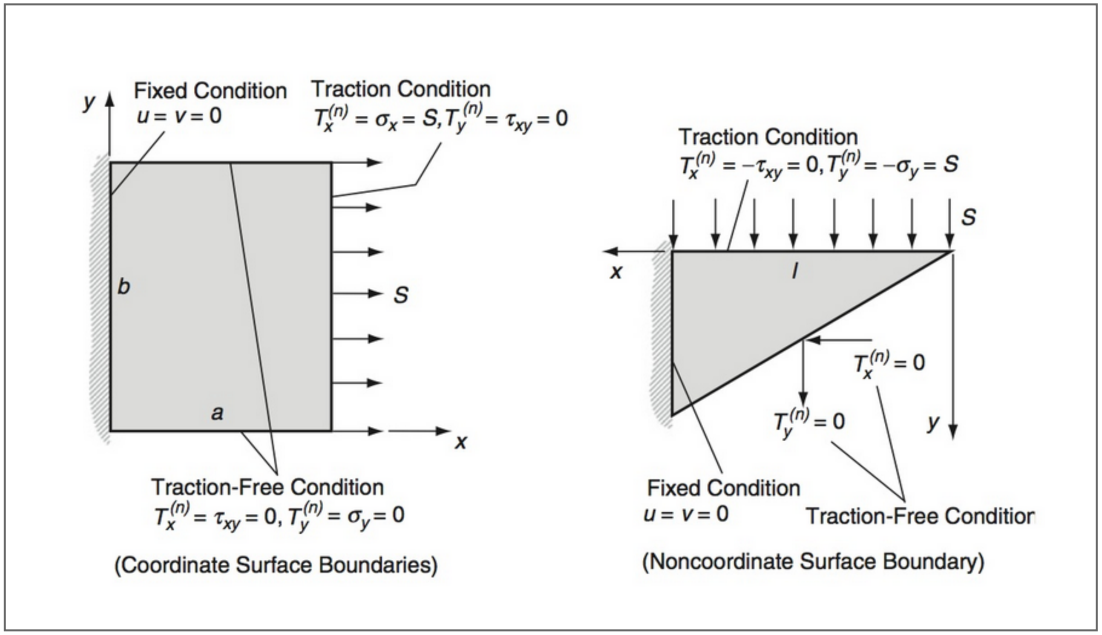


coordinate systems



boundaries

- In many systems, the boundaries are parallel to the coordinate system, but this is not always the case



boundaries

- We often translate traction boundary conditions into stress boundary conditions using Cauchy's Stress Theorem
- When the condition is on a face parallel to the coordinate system, this gives a zero-stress condition $t_j = \sigma_{ij} n_i$
- This results in $\sigma_{xy} = \sigma_{yy} = 0$

boundaries

- When the boundary is not parallel to the coordinate system, we do not necessarily have any zero-stress conditions

$$
\begin{aligned}
t_x &= \sigma_x n_x + \tau_{xy} n_y = 0 \\
t_y &= \tau_{xy} n_x + \sigma_y n_y = 0
\end{aligned}
$$

interfaces

- When we deal with multiple materials, we must prescribe conditions at the interface of these materials
- Some common *interface conditions* are
  - *Perfectly bonded interface* where displacements and tractions are continuous at the interface
  - *Slip interface* where only normal displacements and tractions are continuous at the interface, with no tangential traction and potentially discontinuous tangential displacement

stress formulation

stress formulation

- For traction problems (i.e. traction is defined on all surfaces) it is convenient to re-formulate field equations in terms of stress only
- Since displacements are eliminated, we will need to use the compatibility equations to ensure a continuous displacement field
- It is desirable for this formulation to write the compatibility equations in terms of stress

stress formulation

- We start by using Hooke's law for each of the strain terms
  $$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$
- After some tedious algebra, we find

$$kk + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1+\nu}(\sigma_{mm,kk}\delta_{ij} + \sigma_{mm,ij}\delta_{kk} - \sigma_{mm,jk}\delta_{ik} - \sigma_{mm}$$

stress formulation

- If we also include the equilibrium equations ($\sigma_{ij,j} - F_i$) in the formulation, we find

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = \frac{\nu}{1+\nu}\sigma_{mm,kk}\delta_{ij} - F_{i,j} - F_{j,i}$$

- We can further simplify the equation by consider the case when $i = j$ and nothing that

$$\sigma_{ii,kk} = -\frac{1+\nu}{1-\nu}F_{i,i}$$

stress formulation

- Which we can substitute into the equation to find

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1+\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

beltrami-michell compatibility

- The compatibility equations in terms of stress are commonly known as the *Beltrami-Michell compatibility equations*
- When there are no body forces, we can write the six expanded form equations

beltrami-michell

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x+\sigma_y+\sigma_z) = 0$$

$$(1+\nu)\nabla^2\sigma_y + \frac{\partial^2}{\partial y^2}(\sigma_x+\sigma_y+\sigma_z) = 0$$

$$(1+\nu)\nabla^2\sigma_z + \frac{\partial^2}{\partial z^2}(\sigma_x+\sigma_y+\sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x\partial y}(\sigma_x+\sigma_y+\sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y\partial z}(\sigma_x+\sigma_y+\sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{zx} + \frac{\partial^2}{\partial z\partial x}(\sigma_x+\sigma_y+\sigma_z) = 0$$

stress formulation

- When working with traction boundary problems, these compatibility equations, together with the equilibrium equations, are sufficient to solve the problem
- These partial differential equations are not easy to solve, and analytic problems approached this way are often solved only in 2D
- Solutions are also commonly based on *stress functions*, which gives a base equation form that automatically satisfies equilibrium

solution methods

- Direct method
  - Solved via direction integration
  - Limited to very simple geometries
- Inverse method
  - Choose a basic form for the solution based on our knowledge of the problem
  - Solve for coefficients
  - Usually we know the answer before we know the problem, it can be difficult to find useful problems for our solution

solution methods

- Semi-inverse method
  - Only part of the solution is assumed
  - Use direct integration to find the rest

strain energy

strain energy

- Energy stored by deformation
- In linear elasticity it is given as

$$U = \frac{1}{2} V \sigma_{ij} \epsilon_{ij}$$

rod

- Strain energy in a 1D rod in tension can be expressed as

$$U = \int_0^L \frac{P^2}{EA} dx$$

beam

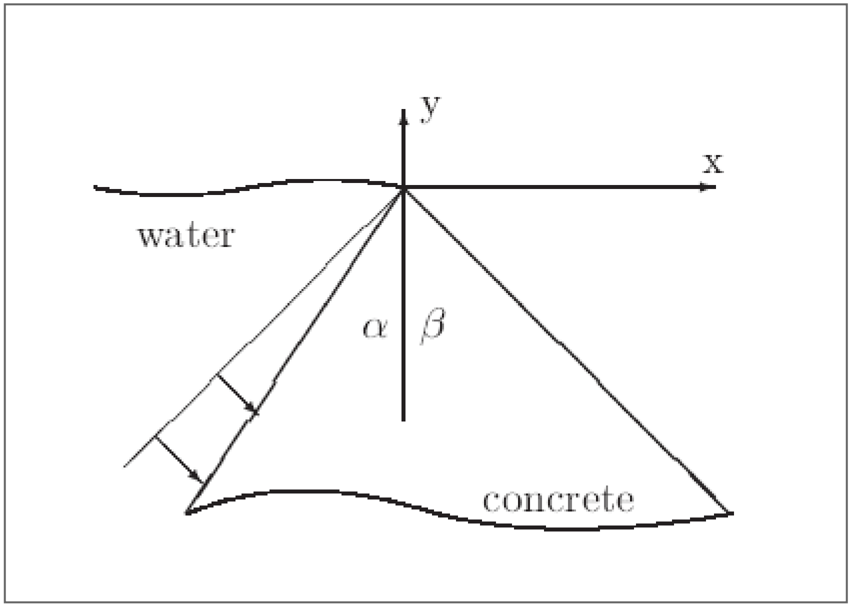
- In a beam under pure bending we find the strain energy as

$$U = \int_0^L \frac{M^2}{2EI} dx$$

example

Levy's problem

- Find the stresses in a semi-infinite wedge due to fluid pressure and its own self-weight



Levy's problem

- Since pressure varies linearly with depth, we will assume a linear state of stress

$$\begin{aligned}\sigma_x &= a_1 x + b_1 y + c_1 \\ \sigma_y &= a_2 x + b_2 y + c_2 \\ \tau_{xy} &= a_{12} x + b_{12} y + c_{12}\end{aligned}$$

- This leaves 9 coefficients to be determined

Levy's problem

- First let us consider the boundary conditions at the apex of the dam
- If we let the origin be at the apex of the dam, which must be traction free, we find $c_1 = c_2 = c_{12} = 0$

Levy's problem

- Next let us consider the equilibrium equations

$$\sigma_{x,x} + \tau_{xy,y} + \rho b_x = 0$$
$$\tau_{xy,x} + \sigma_{y,y} + \rho b_y = 0$$

- Which in this case become

$$a_1 + b_{12} + 0 = 0$$
$$a_{12} + b_2 - \rho g = 0$$

Levy's problem

- The stresses can now be written as

$$\begin{aligned}\sigma_x &= a_1 x + b_1 y \\ \sigma_y &= a_2 x + b_2 y \\ \tau_{xy} &= -b_2 x + \rho g x - a_1 y\end{aligned}$$

Levy's problem

- The compatibility equations are all satisfied, as these linear functions will all go to zero when taking second derivatives
- We now consider the boundary conditions along both faces

airy stress functions

airy stress function

- A stress function technique that can be used to solve many planar problems is known as the *Airy stress function*
- This method reduces the governing equations for a planar problem to a single unknown function
- We assume first that body forces are derivable from a *potential function*, V

airy stress function

$$F_x = -\frac{\partial V}{\partial x}$$
$$F_y = -\frac{\partial V}{\partial y}$$

- How restrictive is this assumption?
- Most body forces are linear (gravity) and can easily be represented this way

airy stress function

- Consider the following

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

- The function $\phi = \phi(x, y)$ is known as the Airy stress function
- Equilibrium is automatically satisfied

compatibility

- Substituting the Airy Stress function and potential function into the relationships, we find

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=-\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\qquad\text{plane strain}$$

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=-(1-\nu)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\qquad\text{plane stress}$$

compatibility

- If there are no body forces, or the potential function satisfies Laplace's Equation $\nabla^2 V = 0$ Then both plane stress and plane strain reduce to

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

polar coordinates

- Recall that an Airy Stress function must satisfy the Beltrami-Mitchell compatibility equations

$$\nabla^4 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0$$

polar coordinates

- One method which gives several useful solutions assumes that the Airy Stress function has the form $\phi(r, \theta)$=$f(r)e^{b\theta}$
- Substituting this into the compatibility equations (and canceling the common $e^{b\theta}$) term gives

$$f'''' + \frac{2}{r} f''' - \frac{1 - 2b^2}{r^2} f'' + \frac{1 - 2b^2}{r^3} f' + \frac{b^2(4 + b^2)}{r^4} f = 0$$

polar coordinates

- To solve this, we perform a change of variables, letting $r = e^{\xi}$, which gives
  $f'''' - 4f''' + (4 + 2b^2)f'' - 4b^2 f' + b^2(4 + b^2)f = 0$
- We know consider f to have the form $f = e^{a\xi}$ which generates the characteristic equation
  $(a^2 + b^2)(a^2 - 4a + 4 + b^2)=0$

polar coordinates

- This has solutions

$$a = \pm ib, \pm 2ib$$

OR

$$b = \pm ia, \pm i(a-2)$$

polar coordinates

- All solutions to the Beltrami-Mitchell equations in polar coordinates which are periodic in θ can be summarized as

$$
\begin{aligned}
\phi &= a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \\
&+ (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
&+ \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r\right) \cos\theta \\
&+ \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r\right) \sin\theta \\
&+ \sum_{n=2}^{\infty} (a_{n1} r^n + a_{n2} r^{2+n} + a_{n3} r^{-n} + a_{n4} r^{2-n}) \cos n\theta \\
&+ \sum_{n=2}^{\infty} (b_{n1} r^n + b_{n2} r^{2+n} + a_{n3} r^{-n} + b_{n4} r^{2-n}) \sin n\theta
\end{aligned}
$$

polar coordinates

- For axisymmetric problems, all field quantities are independent of θ
- This reduces the general solution to $\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$

polar coordinates

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\log r$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^2 \log r$	$2 \log r + 1$	0	$2 \log r + 3$
$r^2\theta$	2θ	-1	2θ
$r^3 \cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\log r\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$r\log r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$

polar coordinates

$r^4 \cos 2\theta$	0	$6r^2 \sin 2\theta$	$12r^2 \cos 2\theta$
$r^4 \sin 2\theta$	0	$-6r^2 \cos 2\theta$	$12r^2 \sin 2\theta$
$r^2 \cos 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$	$2 \cos 2\theta$
$r^2 \sin 2\theta$	$-2 \sin 2\theta$	$-2 \cos 2\theta$	$2 \sin 2\theta$
$\cos 2\theta$	$-4 \cos 2\theta / r^2$	$-2 \sin 2\theta / r^2$	0
$\sin 2\theta$	$-4 \sin 2\theta / r^2$	$2 \cos 2\theta / r^2$	0
$\cos 2\theta / r^2$	$-6 \cos 2\theta / r^4$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$
$\sin 2\theta / r^2$	$-6 \sin 2\theta / r^4$	$6 \cos 2\theta / r^4$	$6 \sin 2\theta / r^4$