Name:

Homework 2

Due 12 September 2019

1. Find the stress function or Westergaard function that solves the problem of a crack of length $2a$ in an infinite plate subjected to remote uniaxial tension.

   Solution:

   First let us specify the boundary conditions for an infinite plate under uniaxial remote tension. At $y = \pm\infty$ we have $\sigma_{yy} = \sigma_0$ and $\sigma_{xy} = 0$. At $x = \pm\infty$ we have $\sigma_{xx} = \sigma_{xy} = 0$. In the cracked region, where $y = 0$ and $|x| \leq a$ we have $\sigma_{yy} = \sigma_{xy} = 0$.

   For Mode I problems (where loads are symmetric about the x-axis), we can use the Westergaard function method with stresses

$$\begin{aligned}
\sigma_{xx} &= \mathrm{Re}\{Z_I\} - y\mathrm{Im}\{Z_I'\} + 2A \\
\sigma_{yy} &= \mathrm{Re}\{Z_I\} + y\mathrm{Im}\{Z_I'\} \\
\sigma_{xy} &= -y\mathrm{Re}\{Z_I'\}
\end{aligned}$$

   We note that $2A$ only appears for σ_{xx}, thus we can use the biaxial Westergaard function for some remote tension, and then correct it using $2A$ to satisfy all boundary conditions.

   Consider

$$\begin{aligned}
Z_I &= \frac{\sigma_0 z}{\sqrt{z^2 - a^2}} \\
Z_I' &= -\frac{\sigma_0 a^2}{(z^2 - a^2)^{3/2}}
\end{aligned}$$

   We can check boundary conditions using a polar coordinate system where

$$\begin{aligned}
z &= re^{i\theta} \\
z - a &= r_1 e^{i\theta_1} \\
z + a &= r_2 e^{i\theta_2}
\end{aligned}$$

   where r_1 and θ_1 originate from the right crack tip (at $z = x = a$) and r_2 and θ_2 originate from the left crack tip (at $z = x = -a$).

   Substituting we find

$$\begin{aligned}
Z_I &= \frac{\sigma_0 r}{\sqrt{r_1 r_2}} e^{i(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2)} \\
Z_I' &= -\frac{\sigma_0 a^2}{(r_1 r_2)^{3/2}} e^{-i\frac{3}{2}(\theta_1 + \theta_2)}
\end{aligned}$$

And taking real and imaginary parts

$$
\begin{aligned}
Re\{Z_I\} &= \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) \\
Re\{Z_I'\} &= -\frac{\sigma_0 a^2}{(r_1 r_2)^{3/2}} \cos\left(\frac{3}{2}(\theta_1 + \theta_2)\right) \\
Im\{Z_I'\} &= \frac{\sigma_0 a^2}{(r_1 r_2)^{3/2}} \sin\left(\frac{3}{2}(\theta_1 + \theta_2)\right)
\end{aligned}
$$

And we find for stresses

$$
\begin{aligned}
\sigma_{xx} &= \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) - \frac{\sigma_0 a^2 r \sin\theta}{(r_1 r_2)^{3/2}} \sin\left(\frac{3}{2}(\theta_1 + \theta_2)\right) + 2A \\
\sigma_{yy} &= \frac{\sigma_0 r}{\sqrt{r_1 r_2}} \cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) + \frac{\sigma_0 a^2 r \sin\theta}{(r_1 r_2)^{3/2}} \sin\left(\frac{3}{2}(\theta_1 + \theta_2)\right) \\
\sigma_{xy} &= \frac{\sigma_0 a^2 r \sin\theta}{(r_1 r_2)^{3/2}} \cos\left(\frac{3}{2}(\theta_1 + \theta_2)\right)
\end{aligned}
$$

Inside the crack we will have either $\theta = 0$, $\theta_1 = \pi$ and $\theta_2 = 0$ (for the right half) or $\theta = \pi$, $\theta_1 = \pi$ and $\theta_2 = 0$.

In the first case, we find $\cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) = \cos\left(0 - \frac{1}{2}\pi - 0\right) = 0$ and $\sin\theta = \sin 0 = 0$.

In the second case, we find $\cos\left(\theta - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) = \cos\left(\pi - \frac{1}{2}\pi - 0\right) = 0$ and $\sin\theta = \sin\pi = 0$.

In both cases, this leads to $\sigma_{yy} = \sigma_{xx} = 0$.

At $y = \pm\infty$ we know that $r = r_1 = r_2 = \infty$ and $\theta = \theta_1 = \theta_2$. This leads to

$$
\sigma_{yy} = \frac{\sigma_0 r}{\sqrt{r^2}} \cos\left(\theta - \frac{1}{2}\theta - \frac{1}{2}\theta\right) + \frac{\sigma_0 a^2 r \sin\theta}{r^3} \sin\left(\frac{3}{2}(\theta + \theta)\right) = \sigma_0 \sigma_{xy} \quad = \frac{\sigma_0 a^2 r \sin\theta}{r^3} \cos\left(\frac{3}{2}(\theta_1 + \theta_2)\right)
$$

Which satisfies the boundary conditions.

At $x = \pm\infty$ we also have $r = r_1 = r_2 = \infty$ and $\theta = \theta_1 = \theta_2$. This leads to

$$
\begin{aligned}
\sigma_{xx} &= \frac{\sigma_0 r}{\sqrt{r^2}} \cos\left(\theta - \frac{1}{2}\theta - \frac{1}{2}\theta\right) - \frac{\sigma_0 a^2 r \sin\theta}{r^3} \sin\left(\frac{3}{2}(\theta + \theta)\right) = \sigma_0 + 2A \\
\sigma_{xy} &= \frac{\sigma_0 a^2 r \sin\theta}{r^3} \cos\left(\frac{3}{2}(\theta_1 + \theta_2)\right) = 0
\end{aligned}
$$

this can satisfy the boundary conditions when $2A = -\sigma_0/2$.

2. Show that the Westergaard function

$$
Z_I = \sigma_0 \sin\left(\frac{\pi z}{2b}\right) / \sqrt{\sin^2\left(\frac{\pi z}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)} \tag{1}
$$

is the solution for an infinite plate containing a periodic array of cracks. Determine the stress intensity factor for this problem.

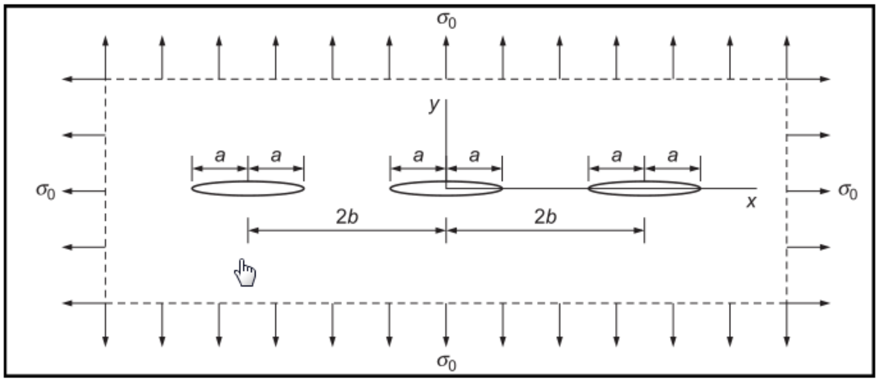

Figure 1: Illustration of Problem 2

First we will consider the boundary conditions. As $x^2+y^2 \to \infty$ we have $\sigma_{yy} = \sigma_{xx} = \sigma_0$ and $\sigma_{xy} = 0$. We can represent the region inside a crack as $|x - 2bk| \le a$ and $y = 0$. In these regions, for any integer k, we have $\sigma_{yy} = \sigma_{xy} = 0$.

We can now find

$$Z_I = \sigma_0 \sin\left(\frac{\pi z}{2b}\right) / \sqrt{\sin^2\left(\frac{\pi z}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)}$$
$$Z_I' = -\frac{\pi\sigma_0 \cos\left(\frac{\pi z}{2b}\right)}{2b} \sin^2\left(\frac{\pi a}{2b}\right) / \left(\sin^2\left(\frac{\pi z}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)\right)^{3/2}$$

We know that stresses are found using

$$\sigma_{xx} = \mathrm{Re}\{Z_I\} - y\mathrm{Im}\{Z_I'\} + 2A$$
$$\sigma_{yy} = \mathrm{Re}\{Z_I\} + y\mathrm{Im}\{Z_I'\}$$
$$\sigma_{xy} = -y\mathrm{Re}\{Z_I'\}$$

We note that along the cracks we have $y = 0$ and with $A = 0$ these equations reduce to

$$\sigma_{xx} = \mathrm{Re}\{Z_I\}$$
$$\sigma_{yy} = \mathrm{Re}\{Z_I\}$$
$$\sigma_{xy} = 0$$

So to satisfy the boundary condition of $\sigma_{yy} = 0$ we need only show that $\mathrm{Re}\{Z_I\} = 0$ when $|z - 2bk| \le a$. Since $y = 0$ we have $z = x$, and for any arbitrary crack we have $x = x - 2bk$, substituting we see that

$$Z_I = \sigma_0 \sin\left(\frac{\pi(x+2bk)}{2b}\right) / \sqrt{\sin^2\left(\frac{\pi(x+2bk)}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)}$$

And we note that

$$
\begin{aligned}
\sin\left(\frac{\pi(x+2bk)}{2b}\right) &= \sin\left(\frac{\pi x}{2b}\right)\cos\left(\frac{2bk\pi}{2b}\right) + \sin\left(\frac{2bk\pi}{2b}\right)\cos\left(\frac{\pi x}{2b}\right) \\
&= \sin\left(\frac{\pi x}{2b}\right)\cos\left(k\pi\right)
\end{aligned}
$$

which gives (note that $\cos k\pi = \pm 1$ and that $\cos^2 k\pi = 1$)

$$
Z_I = \pm\sigma_0 \sin\left(\frac{\pi x)}{2b}\right) / \sqrt{\sin^2\left(\frac{\pi x}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)}
$$

Since $|x| < a$, we know that

$$
\sin^2\left(\frac{\pi x}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right) < 0
$$

Thus this term is imaginary, and $\Re Z_I = 0$, thus the boundary conditions are satisfied along the cracks.

To satisfy the boundary conditions as $|z| \to \infty$, we note that $\sin z = \sin x \cosh y + i \cos x \sinh y$ and thus $\operatorname{Re} \sin z = \sin x \cosh y$ which leads to

$$
\lim_{|z|\to\infty} \operatorname{Re}(Z_I) = \sigma_0
$$

Similary we note that $\cos z = \sin x \cosh y - i \sin x \sinh y$ which leads to the following

$$
\begin{aligned}
\lim_{|z|\to\infty} \operatorname{Im}(yZ_I') = 0 \\
\lim_{|z|\to\infty} \operatorname{Re}(yZ_I') = 0
\end{aligned}
$$

To find the stress intensity factor we use a change of variables with $\zeta = z - a$ as $\zeta \to 0$ (which is equivalent to $z \to a$, but allows a more direct solution to the limit). We can then use the angle addition formulae noting that as $\zeta \to 0$, $\cos\frac{\pi\zeta}{2b} = 1$ and $\sin\frac{\pi\zeta}{2b} = \frac{\pi\zeta}{2b}$.

$$
\begin{aligned}
K_I &= \lim_{\zeta\to 0} \sqrt{2\pi}\sqrt{\zeta} Z_I \\
&= \lim_{\zeta\to 0} \sqrt{2\pi}\sqrt{\zeta}\sigma_0 \sin\left(\frac{\pi(\zeta-a)}{2b}\right) / \sqrt{\sin^2\left(\frac{\pi(\zeta-a)}{2b}\right) - \sin^2\left(\frac{\pi a}{2b}\right)} \\
&= \sqrt{2\pi}\lim_{\zeta\to 0} \sqrt{\zeta}\sigma_0 \left[\sin\left(\frac{\pi a)}{2b}\right) + \frac{\pi\zeta}{2b}\cos\left(\frac{\pi a)}{2b}\right)\right] / \sqrt{\frac{\pi\zeta}{4b^2}\cos^2\left(\frac{\pi a}{2b}\right) + \frac{1}{2b}\sin\left(\frac{\pi a}{2b}\right)\cos\left(\frac{\pi a}{2b}\right)} \\
&= \lim_{\zeta\to 0} \sigma_0 \sin\left(\frac{\pi a)}{2b}\right) / \sqrt{\frac{1}{2b}\sin\left(\frac{\pi a}{2b}\right)\cos\left(\frac{\pi a}{2b}\right)} \\
&= \sigma_0\sqrt{\pi a}\sqrt{\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)}
\end{aligned}
$$