AE837

Advanced Mechanics of Damage Tolerance

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upcoming schedule

- Aug 27 - Griffith Fracture
- Aug 29 - Griffith Fracture
- Sep 3 - Elastic Stress Field, Homework 1 Due
- Sep 5 - Elastic Stress Field

outline

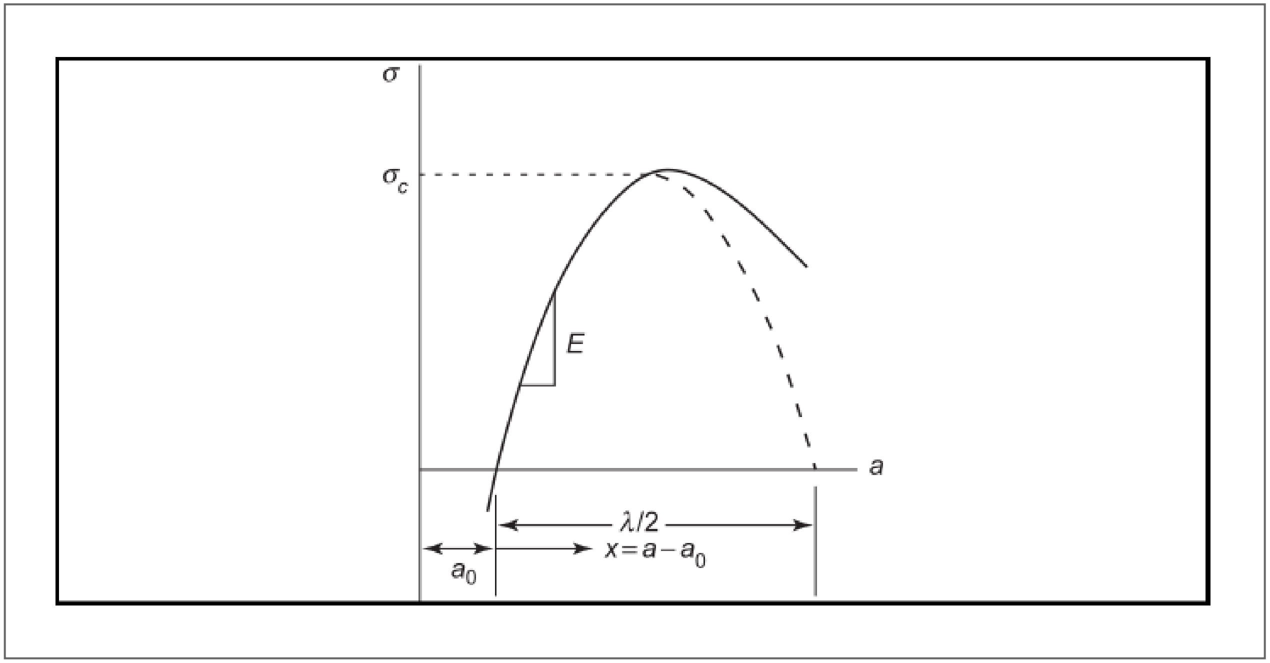
- theoretical strength
- griffith fracture
- energy release rate
- relation among energies

theoretical strength

theoretical strength

- We can (theoretically) predict the strength of a material from the strength of its atomic bonds
- As the distance between atoms increases (strain) so does the attractive force between them (stress) until some critical distance when they are no longer attracted to one another (failure)

cohesive stress



theoretical strength

- Using a sinusoidal approximation of strength we have

$$\sigma = \sigma_c \sin\left(\frac{2\pi x}{\lambda}\right)$$

- For small strains, $\sin(x) \approx x$
- We can also write in terms of the modulus of elasticity, $E = \frac{\sigma}{\epsilon} = \frac{\sigma}{x/a_0}$

theoretical strength

- This means we can write $\sigma = \frac{Ex}{a_0}$, substituting into the original equation we find
$$\frac{Ex}{a_0} = \sigma_c \frac{2\pi x}{\lambda}$$
- solving for σ_c, we find the theoretical strength as
$$\sigma_c = \frac{\lambda E}{2\pi a_0}$$
- In many materials, $\lambda \approx a_0$, in which case $\sigma_c = \frac{E}{2\pi}$

energy consideration

- Surface energy is the work done to create a new surface when an atomic bond breaks
- We define γ as the surface energy (units of energy/area)
- We can calcualte this surface energy in terms of the sinusoidal approximation of traction-separation

$$2\gamma = \int_0^{\lambda/2} \sigma_c \sin\left(\frac{2\pi x}{\lambda}\right) dx = \frac{\lambda \sigma_c}{\pi}$$

energy consideration

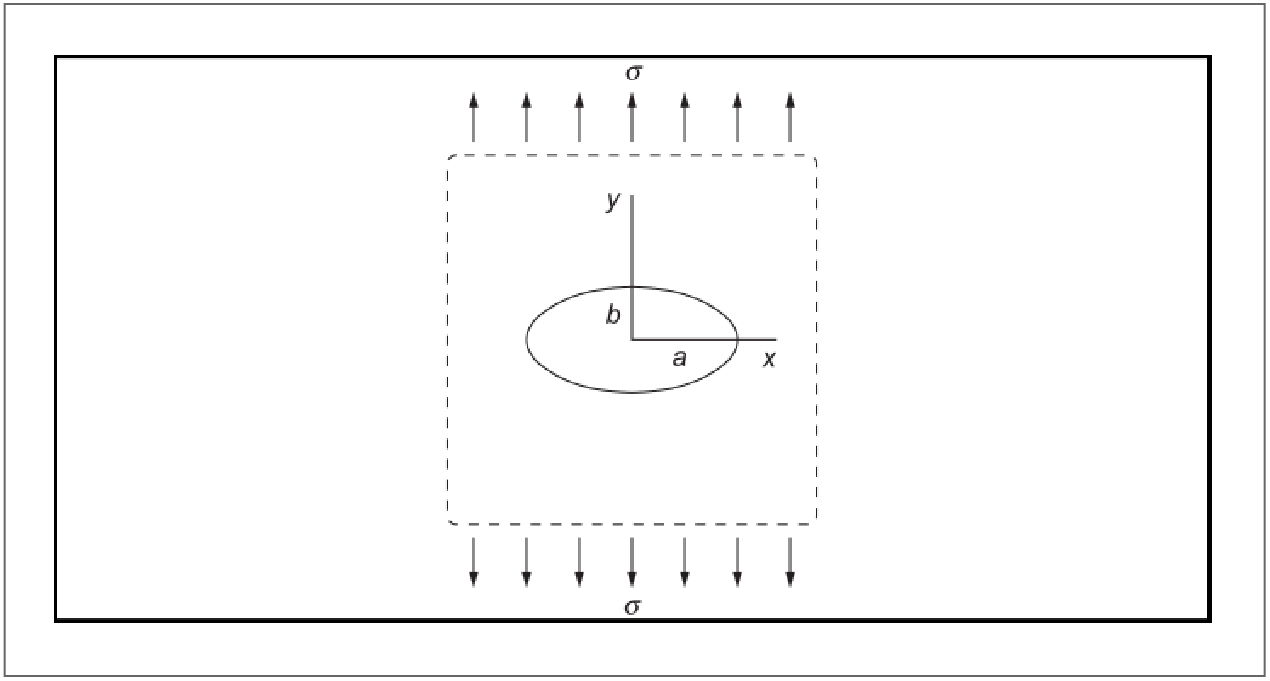
- re-arranging a previous result, we know that
  $$\lambda = \frac{2\pi a_0 \sigma_c}{E}$$
- from which we find
  $$\sigma_c^2 = \frac{2\pi\gamma E}{2\pi a_0} = \frac{\gamma E}{a_0}$$
- For many materials, the surface energy is approximately $\gamma = 0.01 E a_0$, which gives an approximate theoretical strength of
  $$\sigma_c = \frac{E}{10}$$

griffith fracture

real failure

- In practice, materials fail at loads 10 to 100 times lower than the theoretical “atomic” strength
- Griffith studied glass, which by the atomic strength models should be a very strong material, but in practice is often weaker than much less stiff materials
- He hypothesized that there were many micro-cracks which coalesced to cause failure

elliptic hole



elliptic hole

- Inglis solved the problem of an eliptic hole under remote tension, finding that at the edge of the major axis the stress is given by
  $\sigma_{yy} = \sigma \left(1 + \frac{2a}{b}\right)$
- For a crack as $b \to 0$, we would get the unreasonable prediction that any stress, no matter how small, would produce failure as the stress at the edge of the ellipse would be infinite
- Griffith took an energy balance approach

surface energy

- Griffith proposed that a crack would extend when the incremental release of energy, dW associated with a crack extension of da in a body is greater than the energy required to create the new surfaces, dW_s
- For a center crack, there are two crack tips and a total of 4 surfaces, thus
  $W_s = 2(2a\gamma) = 4a\gamma$

surface energy

- Griffith then used the Inglis solution to calculate the total energy released by a crack extension of da

$$W = \frac{\pi a^2 \sigma^2 (1-\nu^2)}{E} \qquad \text{plane strain}$$

$$W = \frac{\pi a^2 \sigma^2}{E} \qquad \text{plane stress}$$

surface energy

- We can now substitute to find under what conditions $dW \geq dW_s$

$$\frac{2\pi a^2 \sigma_{cr}^2 (1-\nu^2)}{E} da = 4\gamma da \qquad \text{plane strain}$$

$$\frac{2\pi a^2 \sigma_{cr}^2}{E} da = 4\gamma da \qquad \text{plane stress}$$

surface energy

- Which gives the critical stress as

$$\sigma_{cr} = \sqrt{\frac{2E\gamma}{\pi(1-\nu^2)a}} \qquad \text{plane strain}$$

$$\sigma_{cr} = \sqrt{\frac{2E\gamma}{\pi a}} \qquad \text{plane stress}$$

comparison

- We can compare this critical stress prediction with previous methods
- We had found

$$\sigma_c = \sqrt{\frac{\gamma E}{a_0}}$$

- Thus when there is inherent damage in a materal larger than the twice the characteristic distance between atoms (a_0), the fracture strength will be lower than the theoretical strength

energy release rate

energy release rate

- We can now define the energy release rate
- More precisely, “strain energy release rate per crack tip”
  $dW = 2Gda$
- Which after substitution and integration gives

$$G = \frac{1}{2}\frac{dW}{da} = \frac{\pi a \sigma^2 (1-\nu^2)}{E} \qquad \text{plane strain}$$

$$= \frac{\pi a \sigma^2}{E} \qquad \text{plane stress}$$

fracture toughness

- When $G \geq 2\gamma$ the strain energy from the applied load is greater than the energy required to propagate a crack, and thus a crack will propagate
- The critical strain energy release rate is called G_c, and is also referred to as the fracture toughness

fracture toughness

- Note: K_c (the critical stress intensity factor) is also referred to as the "fracture toughness," and although they give equivalent predictions, they have different units
- The G_c predicted from the atomic surface energy will be somewhat conservative, since almost all materials will absorb at least some of the strain energy in plastic deformation

relation among energies

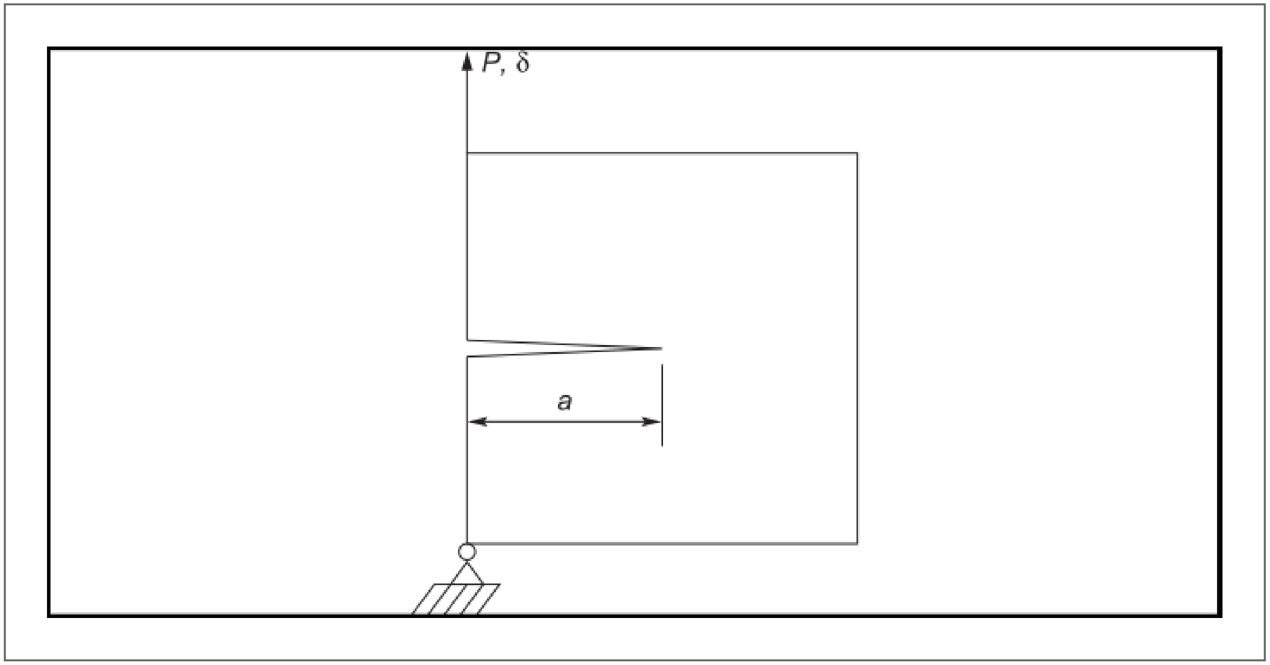
energy balance

- Griffith's theory is founded on the principle of energy balance
- During crack extension, the external work done, dW_e must equal the increment of surface energy, dW_s, and the increment of elastic strain energy, dU
  $dW_s + dU = dW_e$

energy balance

- For a conservative field we can write this as
  $$\frac{\partial}{\partial a}(W_s + U + V) = 0$$
- Where a negative potential, V, implies positive external work done dW_e

example



example

- The displacement at the top edge will be proportional to the applied load by some elastic compliance of the specimen
  $\delta = SP$
- Note that this compliance, S, will be a function of the crack length, a
- The strain energy can be expressed as
  $$U = \int_{\delta=0}^{\delta=SP} P d\delta = \int_{\delta=0}^{\delta=SP} \frac{\delta}{S} d\delta$$

example

- After integrating
  $$U = \frac{1}{2S}(\delta^2)|_0^{SP} = \frac{1}{2}SP^2$$

example

- To find the incremental strain energy increase (where both P and S should be treated as variable), we find
  $dU = \frac{1}{2}P^2 dS + SPdP$
- We will now consider two loading cases, one with fixed displacement and the other with fixed loading force

fixed displacement

- Under constant displacement we have
  $\delta = SP = \text{constant}$
- This means that the derivative of displacement will be zero, hence
  $d\delta = SdP + PdS = 0$
- and
  $SdP = -PdS$

fixed displacement

- Substituting into the previous equation gives
  $$dU = -\frac{1}{2}P^2 dS$$
- Since $d\delta = 0$, the external work, $dW_e = 0$ and we find
  $$dW_s = -dU = \frac{1}{2}P^2 dS$$

fixed load

- If instead of fixing displacement we fix the applied load we have $dP = 0$
- Which gives $$dU = \frac{1}{2}P^2 dS$$
- The strain energy increases (while under fixed displacement it decreased)

fixed load

- Further, we can find the external work done as $dW_e = Pd\delta = P^2 dS$
- And thus, from energy balance, we find the surface energy $dW_s = \frac{1}{2} P^2 dS$
- In this case the external work is equally split between strain energy and surface energy

comparison

- In both cases, the energy released is $dW = dW_e - dU$

$$dW = 0 - (-\frac{1}{2}P^2 dS) \qquad \text{fixed displacement}$$
$$dW = P^2 dS - \frac{1}{2}P^2 dS \qquad \text{fixed load}$$

- And we see that the energy released is independent of the load type

example - double cantilever beam