CHAPTER

Griffith Theory of Fracture 2

2.1 THEORETICAL STRENGTH

The theoretical strength of a solid is usually understood as the applied stress that fractures a perfect crystal of the material by breaking the atomic bonds along the fractured surfaces. The theoretical strength may be estimated using the interatomic bonding force versus the atomic separation relation. This section gives two estimates that relate the theoretical strength to the Young's modulus of the material based on the atomic bonding strength and surface energy concept.

2.1.1 An Atomistic Model

In general, failure of a solid is characterized by separation of the body. At the atomistic level the fracture strength of a "perfect" material depends on the strength of its atomic bonds. Consider two arrays of atoms in a perfect crystal as shown in Figure 2.1. Let a_0 be the equilibrium spacing between atomic planes in the absence of applied stresses. The stress σ required to separate the planes to a distance $a > a_0$ increases until the theoretical strength σ_c is reached and the bonds are

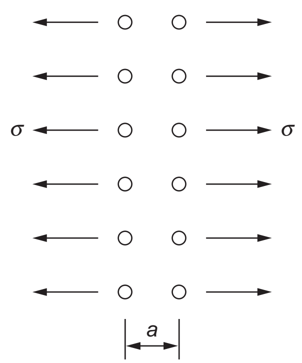

FIGURE 2.1

Atomic planes in a perfect crystal.

Fracture Mechanics. DOI: 10.1016/B978-0-12-385001-0.00002-X

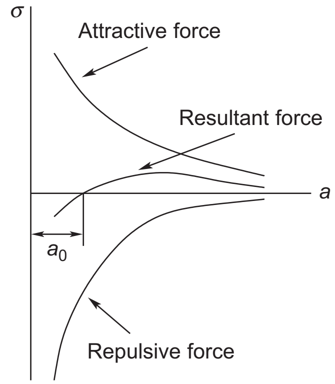

FIGURE 2.2

Cohesive force versus separation between atoms.

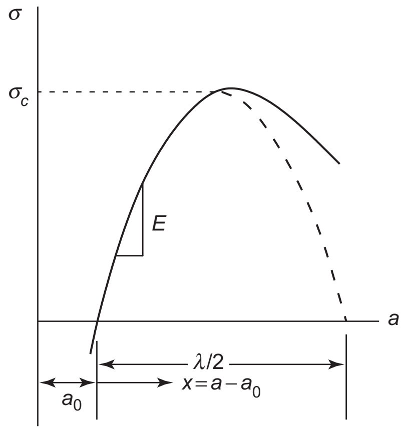

FIGURE 2.3

Cohesive force between atoms.

broken. Further displacements of the atoms can occur under a decreasing applied stress (see Figure 2.2). This stress-displacement curve can be approximated by a sine curve (Figure 2.3) having wavelength λ as

$$\sigma = \sigma_c \sin\left(\frac{2\pi x}{\lambda}\right) \tag{2.1}$$

where $x = a - a_0$ is the relative displacement between the atoms.

At small displacement x we have

$$\sin x \simeq x$$

and, thus,

$$\sigma \simeq \sigma_c \frac{2\pi x}{\lambda} \tag{2.2}$$

The modulus of elasticity is

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{x/a_0} \Rightarrow \sigma = \frac{Ex}{a_o} \tag{2.3}$$

Using Eqs. (2.2) and (2.3), we obtain

$$\frac{Ex}{a_0} = \sigma_c \frac{2\pi x}{\lambda}$$

or

$$\sigma_c = \frac{\lambda E}{2\pi a_0} \tag{2.4}$$

A reasonable value for λ is $\lambda = a_o$, which yields the bond strength

$$\sigma_c = \frac{E}{2\pi} \tag{2.5}$$

2.1.2 The Energy Consideration

Theoretical strength may also be estimated using the simple atomic model with a surface energy concept. We now define a quantity called the surface energy γ (energy per unit area) as the work done in creating new surface area by the breaking of atomic bonds. From the sine-curve approximation of the atomic force (see Figure 2.3), this is simply one-half the area under the stress-displacement curve since two new surfaces are created each time a bond is broken. Thus,

$$2\gamma = \int_0^{\lambda/2} \sigma_c \sin\left(\frac{2\pi x}{\lambda}\right) dx = \frac{\lambda \sigma_c}{\pi}$$

from which

$$\sigma_c = \frac{2\gamma\pi}{\lambda} \tag{2.6}$$

However, from Eq. (2.4),

$$\lambda = \frac{2\pi a_0 \sigma_c}{E} \tag{2.7}$$

We obtain from substitution of Eq. (2.7) in Eq. (2.6)

$$\sigma_c^2 = \frac{2\pi \gamma E}{2\pi a_0} = \frac{\gamma E}{a_0}$$

Finally,

$$\sigma_c = \sqrt{\frac{\gamma E}{a_0}} \tag{2.8}$$

For many materials, γ is on the order of $0.01Ea_0$ [2-1]. Thus, an approximate estimation of the theoretical strength is often given by

$$\sigma_c = \frac{E}{10} \tag{2.9}$$

which agrees with Eq. (2.5) in terms of order of magnitude. For most metals, the theoretical strength varies between 7 GPa (1×10^6 psi) and 21 GPa (3×10^6 psi). However, bulk materials that are commercially produced for engineering applications commonly fracture at applied stress levels 10 to 100 times below these values, and the theoretical strength is rarely obtained in engineering practice.

The main reasons for the discrepancies are

1. The existence of stress concentrators (flaws such as cracks and notches)
2. The existence of planes of weakness such as grain boundaries in polycrystalline materials.

Another type of fracture process (e.g., shear or rupture), which occurs by plastic deformation, intervenes at a lower level of applied stress.

2.2 THE GRIFFITH THEORY OF FRACTURE

Alan Griffith's work [2-2] on brittle fracture of glass was motivated by the desire to explain the discrepancy between the theoretical strength and actual stength of materials. According to the preceding theoretical strength calculation, we may conclude that glass should be very strong. However, laboratory test results often indicate otherwise. Griffith argued that what we must account for is not the strength but rather the weakness, which is normally dominant in the failure process. One clue obviously lies in the fact that actual glasses display a far more complex fracture behavior than predicted by our simple assumptions regarding the cohesive strength. In his pioneer paper [2-2], Griffith postulated that all bulk glasses contain numerous minute flaws in the form of microcracks that act as stress concentration generators. This new concept accompanied by the energy release approach that he introduced started the era of modern fracture mechanics.

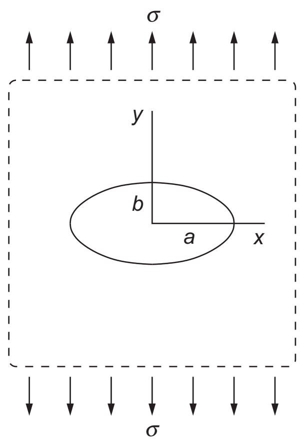

FIGURE 2.4

An elliptic hole in an infinite plate subjected to tension.

The solution of an elliptic hole in a plate of infinite extent under tension by Inglis [2-3] was the first step toward relating observed failure stress to ultimate strength. He solved the problem as shown in Figure 2.4 and found that the greatest stress occurs at the ends of the major axis:

$$\sigma_{yy} = \sigma\left(1 + \frac{2a}{b}\right) \tag{2.10}$$

If $a = b$ (a circular hole), then $\sigma_{yy} = 3\sigma$, which yields the well-known stress concentration factor near a circular hole. If $b \to 0$, then we have a "line crack," and the stress σ_{yy} increases without limit. If a stress-based failure criterion is used to predict the extension of such a "crack," one would find the unreasonable answer that any amount of applied stress would cause the crack to grow.

Griffith took an energy balance point of view and reasoned that the unstable propagation of a crack must result in a decrease in the strain energy of the system (for a body with a fixed boundary where no work is done by external forces during the crack extension), and proposed that a crack would advance when the incremental release of energy dW associated with a crack extension da in a body becomes greater than the incremental increase of surface energy dW_S as new crack surfaces are created. That is,

$$dW \geq dW_S \tag{2.11}$$

The equality indicates the critical point for crack propagation. In other words, if the supply of energy from the cracked plate is equal to or greater than the energy required to create new crack surfaces, the crack can extend.

It is easy to calculate the surface energy for a crack (having two crack tips) with length $2a$, that is,

$$W_S = 2(2a\gamma) = 4a\gamma \tag{2.12}$$

in which γ is the surface energy density and the fact that two crack surfaces for a crack has been accounted for.

Griffith used Inglis' solution to obtain the total energy released W due to the presence of a crack of length $2a$ in an infinite two-dimensional body. His method for calculating energy release was very complicated since he considered the energy change in the body as a whole. He obtained, for plane strain,

$$W = \frac{\pi a^2 \sigma^2 (1 - \nu^2)}{E} \tag{2.13}$$

and for plane stress,

$$W = \frac{\pi a^2 \sigma^2}{E} \tag{2.14}$$

Thus, the critical stress σ_{cr} under which the crack would start propagating may be obtained by substituting Eqs. (2.12) and (2.13) or Eq. (2.14) into Eq. (2.11):

$$\frac{2\pi a (1 - \nu^2)\sigma_{cr}^2}{E} da = 4\gamma da \qquad \text{(plane strain)}$$

From this equation, we have

$$\sigma_{cr}^2 = \frac{4\gamma E}{2\pi a (1 - \nu^2)}$$

or

$$\sigma_{cr} = \sqrt{\frac{2E\gamma}{\pi(1 - \nu^2)a}} \tag{2.15}$$

Similarly, for plane stress we have

$$\sigma_{cr} = \sqrt{\frac{2E\gamma}{\pi a}} \tag{2.16}$$

Comparing the previous critical stress σ_{cr}, or the fracture strength of the infinite plate with a crack of microscopic or macroscopic length $2a$, and the theoretical strength σ_c in Eq. (2.8), we note $\sigma_c \gg \sigma_{cr}$ if $a \gg a_o$. This explains qualitatively why actual strengths of materials are much smaller than their theoretical strengths.

The energy released dW for a crack extension of da can be expressed in terms of the "strain energy release rate per crack tip" G as

$$dW = 2G da \tag{2.17}$$

Thus,

$$G = \frac{1}{2}\frac{dW}{da} = \frac{\pi a\sigma^2(1-\nu^2)}{E} \quad \text{for plane strain} \tag{2.18}$$

$$= \frac{\pi a\sigma^2}{E} \quad \text{for plane stress} \tag{2.19}$$

The instability condition then reads

$$G \geq 2\gamma \tag{2.20}$$

The value of G when equal to 2γ is denoted by G_c and is called the **fracture toughness** or the **crack-resistant force** of the material. This is, in fact, the Griffith energy criterion of brittle fracture. In theory, a crack would extend in a brittle material when the load produces an energy release rate G equal to 2γ. However, such an energy release rate turns out to be much smaller than the test data since most materals are not perfectly brittle and plastic deformation occurs near the crack tip.

To take this additional crack resistant force into account, Orowan [2-4] suggested to add to 2γ the plastic work γ_p associated with the creation of the new crack surfaces. For metals, 2γ is usually much smaller than γ_p and, thus, can be neglected. On the other hand, Irwin [2-5] took G_c as a new material constant to be measured directly from fracture tests. However, Eq. (2.15) points to a correct relation between the failure stress and crack size.

2.3 A RELATION AMONG ENERGIES

The Griffith theory for fracture of perfectly brittle elastic solids is founded on the principle of energy conservation that is, energy added to and released from the body must be the same as that dissipated during crack extension. It states that, during crack extension of da, the work done dW_e by external forces, the increment of surface energy dW_S, and the increment of elastic strain energy dU must satisfy

$$dW_S + dU = dW_e \tag{2.21}$$

For a conservative force field, this condition can be expressed in the form

$$\partial(W_S + U + V)/\partial a = 0 \tag{2.22}$$

where

W_S = total crack surface energy associated with the entire crack

U = total elastic strain energy of the cracked body

V = total potential of the external forces

Note that a negative dV implies a positive work dW_e done by external forces.

Consider a single-edge-cracked elastic specimen subjected to a tensile load P or displacement δ as shown in Figure 2.5. The relationship between the applied tensile

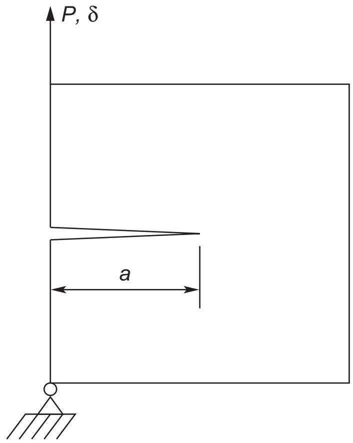

FIGURE 2.5

A single-edge-cracked specimen.

force P and the elastic extension, or displacement, δ, is

$$\delta = SP \tag{2.23}$$

where S denotes the elastic compliance of the specimen containing the crack. The strain energy stored in this specimen is

$$\begin{aligned} U &= \int_{\delta=0}^{\delta=SP} P d\delta = \int_{\delta=0}^{\delta=SP} \frac{\delta}{S} d\delta \\ &= \frac{1}{2S}[\delta^2]_0^{SP} = \frac{1}{2}SP^2 \end{aligned} \tag{2.24}$$

The compliance S is a function of the crack length. The incremental strain energy under the condition of varying a and P is

$$dU = \frac{1}{2}P^2 dS + SPdP \tag{2.25}$$

Case 2.1

Suppose that the boundary is fixed during the extension of the crack so that

$$\delta = SP = \text{ constant}$$

Consequently,

$$d\delta = SdP + PdS = 0$$

from which we obtain

$$SdP = -PdS$$

Substitution of the preceding equation into Eq. (2.25) yields

$$dU|_\delta = -\frac{1}{2}P^2 dS \tag{2.26}$$

Furthermore, $dW_e = 0$ in this case because $d\delta = 0$ and, thus, the external load does no work. Substituting Eq. (2.26) into Eq. (2.21) and using $dW_e = 0$, we have

$$dW_S = -dU|_\delta = \frac{1}{2}P^2 dS \tag{2.27}$$

Thus, a decrease in strain energy U is compensated by an increase of the same amount in the surface energy. In other words, the energy consumed during crack extension is entirely supplied by the strain energy stored in the cracked body.

Case 2.2
Suppose that the applied force is kept constant during crack extension; then

$$dP = 0$$

From Eq. (2.25) we have

$$dU|_P = \frac{1}{2}P^2 dS \tag{2.28}$$

Thus, there is a gain in strain energy during crack extension in this case. Moreover, we note that

$$dW_e = Pd\delta = P^2 dS \tag{2.29}$$

Substituting Eqs. (2.28) and (2.29) into Eq. (2.21), we again obtain Eq. (2.27), that is,

$$dW_S = \frac{1}{2}P^2 dS$$

which is half of the work done by the external force. It is interesting to note that the work done by the external force is split equally into the surface energy and an increase in strain energy.

For both boundary conditions discussed before, the energy released during crack extension is

$$dW = dW_e - dU = \frac{1}{2}P^2 dS$$

The corresponding energy release rate is

$$G = \frac{dW}{da} = \frac{1}{2} P^2 \frac{dS}{da} \tag{2.30}$$

Hence, the strain energy release rate is independent of the type of loading.

The two loading cases can be illustrated graphically as in Figures 2.6a and 2.6b, respectively. In the figures, point B indicates the beginning of crack extension and point C the termination of crack extension. The area $\overline{OBC}$ is the strain energy released, dW. It can be shown rather easily from the graphic illustration that the energies released in the two cases are equal.

Under the fixed load condition, we have

$$dW = dU = \frac{1}{2} dW_e$$

Thus, the energy release rate can be obtained with

$$G = \frac{dU}{da} \tag{2.31}$$

in which the differentiation is performed assuming that the applied load is independent of a.

Under the fixed displacement condition, we have $dW = -dU$, and hence

$$G = -\frac{dU}{da} \tag{2.32}$$

In the previous equation, the applied load P should be considered as a function of crack length a in the differentiation. The result should be the same as that given by Eq. (2.31). It is noted that the relation $dW = dU = dW_e/2$ is not true for nonlinear solids.

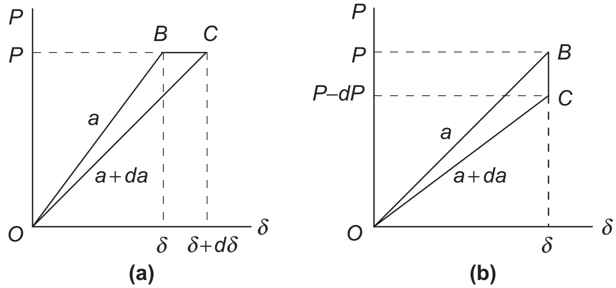

FIGURE 2.6

Energy released during crack extension: (a) constant load, (b) constant displacement.

Example 2.1

The double cantilever beam (DCB) is often used for measuring fracture toughness of materials. Consider the geometry shown in Figure 2.7 where b is the width of the beam, and the crack length a is much larger than h and, thus, the simple beam theory is suitable for modeling the deflection of the two split beams.

Noting that the unsplit portion of the DCB is not subjected to any load and that in each leg the bending moment is $M = Px$, we calculate the total strain energy stored in the two legs of the DCB as

$$U_T = 2\int_0^a \frac{P^2x^2}{2EI}dx = \frac{P^2a^3}{3EI}$$

where

$$I = \frac{bh^3}{12}$$

The total strain energy per unit width is

$$U = U_T/b$$

The strain energy release rate is obtained as

$$G = \frac{dU}{da} = \frac{P^2a^2}{bEI}$$

If the fracture toughness G_c of the material is known, then the load that could further split the beam is

$$P_{cr} = \frac{\sqrt{bEIG_c}}{a}$$

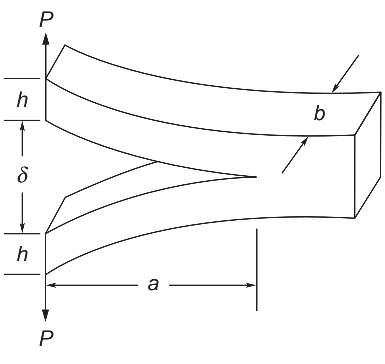

FIGURE 2.7

A double cantilever beam subjected to concentrated forces.

References

[2-1] A.H. Cottrell, Tewksbury Symposium on Fracture, University of Melbourne, 1963, p. 1.

[2-2] A.A. Griffith, The phenomena of rupture and flow in solids, Phil. Trans. Roy. Soc. (London) A221 (1920) 163–198.

[2-3] C.E. Inglis, Stresses in a plate due to the presence of cracks and sharp corners, Trans. Inst. Naval Architects 55 (1913) 219–230.

[2-4] E. Orowan, Notch brittleness and strength of metals, Trans. Inst. Engrs. Shipbuilders Scot. 89 (1945) 165–215.

[2-5] G.R. Irwin, Fracture dynamics, in: Fracture of Metals, ASM, Cleveland, OH, 1948, pp. 147–166.

PROBLEMS

2.1 Consider the cracked beam subjected to uniaxial tension shown in Figures 2.8 and 2.9. Find the strain energy release rate (per crack tip). Consider both fixed-end and constant force boundary conditions.

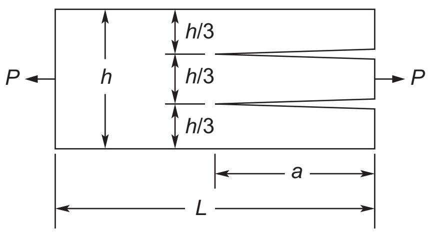

FIGURE 2.8

A cracked beam subjected to tension.

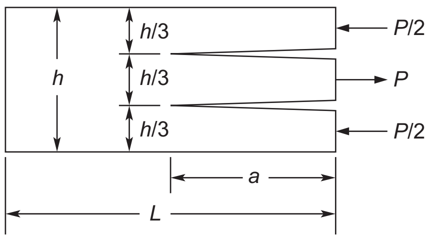

FIGURE 2.9

A cracked beam subjected to tension and compression.

2.2 Find the strain energy release rate G for the cracked beam shown in Figures 2.10 and 2.11. Use simple beam theory to model the cracked and uncracked regions. The thickness of the beam is t.

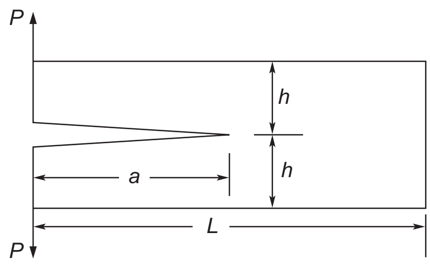

FIGURE 2.10

A cracked beam subjected to concentrated forces.

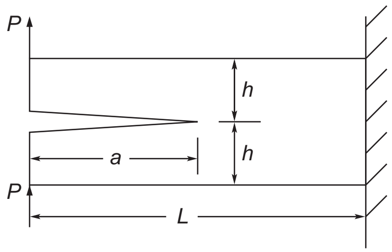

FIGURE 2.11

A cracked beam subjected to concentrated forces.

2.3 A cracked beam is subjected to a pair of forces at the center of the crack (see Figure 2.12). Find the minimum P that can split the beam. Assume $E = 70$ GPa and $G_c = 200$ N•m/m^2.

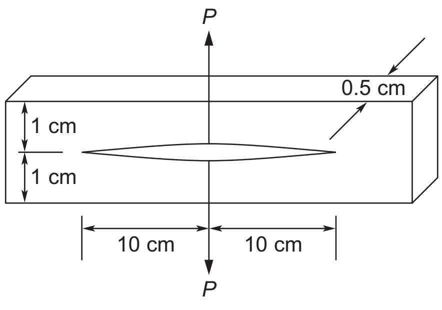

FIGURE 2.12

A center-cracked beam.

2.4 Find the strain energy release rate for the problem shown in Figure 2.13 wherein a thin elastic film of unit width is peeled from a rigid surface.

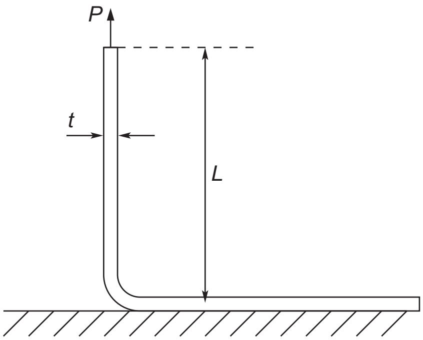

FIGURE 2.13

A thin film peeled from the rigid substrate.

2.5 Assume that the bending rigidity of the film is negligible, that L is large, and that the elastic constants of the film are known. The thin film is pulled parallel to the rigid surface as shown in Figure 2.14. Compare the strain energy release rate and the strain energy gained by the film during crack extension for both problems in Figures 2.13 and 2.14. For the problem of Figure 2.13, why is the strain energy released not the same as the strain energy gained by the film?

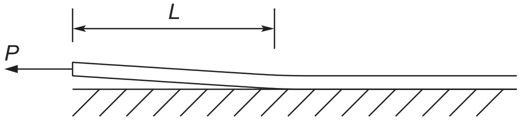

FIGURE 2.14

A thin film pulled parallel to the surface of the rigid substrate.

2.6 Show that the area $\overline{OBC}$ in Figure 2.6 does not depend on the loading condition during crack extension.