AE333

Mechanics of Materials

Lecture 17 - Exam Review

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schedule

- 6 Mar - Exam 2 Review, HW 5 Due
- 9 Mar - Exam 2
- 11 Mar - Transverse Shear
- 13 Mar - Transverse Shear

outline

- exam
- topics
- axial load
- torsion
- bending

exam format

- Similar format to last exam
- Four questions
- Covers Axial Load, Torsion, and Bending
- Past exams included Transverse Shear, which will not be on this exam

topics

axial load

- Saint Venant’s Principle
- Elastic Deformation
- Superposition
- Statically Indeterminate
- Force Method
- Thermal Stress

torsion

- Torsional deformation
- Torsion formula
- Power transmission
- Angle of twist
- Statically indeterminate torsion
- Thin-walled tubes

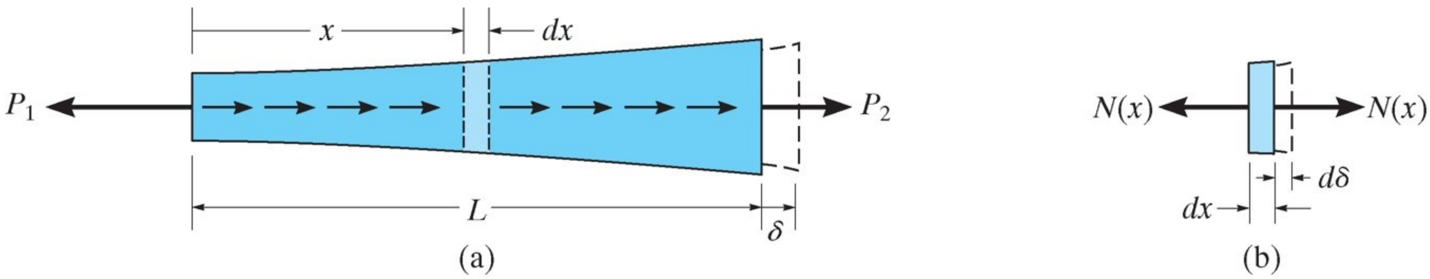
bending

- Shear and moment diagrams
- Bending deformation
- Flexure formula

axial load

axially loaded member

- We can use Hooke's Law to find the deformation of a general body under axial loading (below the elastic limit)



axially loaded member

- For some differential element, we can consider the internal forces and stresses

$$\sigma = \frac{N(x)}{A(x)} = E(x)\epsilon(x) = E(x)\left(\frac{d\delta}{dx}\right)$$

- We can solve this for $d\delta$ to find

$$d\delta = \frac{N(x)dx}{A(x)E(x)}$$

- We integrate this over the length of the bar to find the total displacement

sign convention

- In general, we consider a force or stress to be positive if it causes tension and elongation
- It is negative if it causes compression and contraction

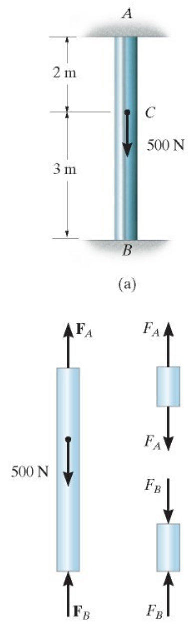
statically indeterminate

- There are many problems that are at least slightly over-constrained
- While this is common engineering practice, it creates too many variables for statics analysis
- These problems are called “statically indeterminate”

statically indeterminate

- One extra equation we can use is called “compatibility” or the “kinematic condition”
- We know that at the displacement must be equal on both sides of any arbitrary section we make in a member
- We can separate a member into two parts, then use compatibility to relate the two unknown forces

statically indeterminate



thermal stress

- A change in temperature cases a material to either expand or contract
- For most materials this is linear and can be described using the coefficient of linear expansion

$$\delta T = \alpha \Delta T L$$

thermal stress

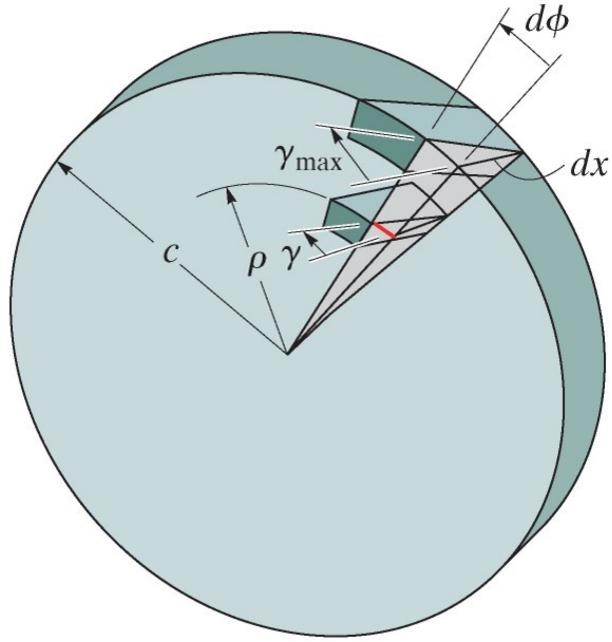
- When a body is free to expand, the deformation can be readily calculated
- If it is not free to expand, however, thermal stresses develop
- We can use the force method described previously to find the thermal stresses developed

torsion

torsion

- Torque is a moment that tends to twist a member about its axis
- For small deformation problems, we assume that the length and radius do not change significantly under torsion
- The primary deformation we are concerned with in torsion is the angle of twist, denoted with ϕ

shear



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{\max}$.

torsion formula

- For a linearly elastic material, Hooke's Law in shear will hold ($\tau = G\gamma$)
- This means that, like shear strain, shear stress will vary linearly along the radius

torsion formula

- We can find the total force on an element, dA by multiplying the shear stress by the area

$$dF = \tau dA$$

- The torque ($dT = \rho dF$) produced by this force is then

$$dT = \rho(\tau dA)$$

torsion formula

- Integrating over the whole cross-section gives

$$T = \int_A \rho(\tau dA) = \frac{\tau_{max}}{c} \int_A \rho^2 dA$$

- The integral $\int_A \rho^2 dA$ is also called the Polar Moment of Inertia, J, so we can write

$$\tau_{max} = \frac{Tc}{J}$$

polar moment of inertia

- We know that $J = \int_A \rho^2 dA$, so we can compute this for some common shapes
- For a solid circular cross-section, we have

$$J = \int_0^c \rho^2 (2\pi\rho d\rho) = \frac{\pi}{2} c^4$$

- For a circular tube we have

$$J = \int_{c_1}^{c_2} \rho^2 (2\pi\rho d\rho) = \frac{\pi}{2}(c_2^4 - c_1^4)$$

power transmission

- Shafts and tubes are often connected to belts and drives, and the torque, speed, and power are all related
- Power is the rate of work done, for rotation problems, $P = T\omega$
- We are often given the frequency f instead of the angular velocity, ω, in this case $P = 2\pi f T$

power units

- In SI Units, power is expressed in Watts 1 W = 1 N m / sec
- In Freedom Units, power is expressed in Horsepower 1 hp = 550 ft lb / sec

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shaft design

- We often know the power and frequency of a drive, and need to design a shaft such that the shear stress is acceptable
- We can easily find the torque as $T = P/2\pi f$, we can use this combined with the torsion equation

$$\tau_{max} = \frac{Tc}{J}$$

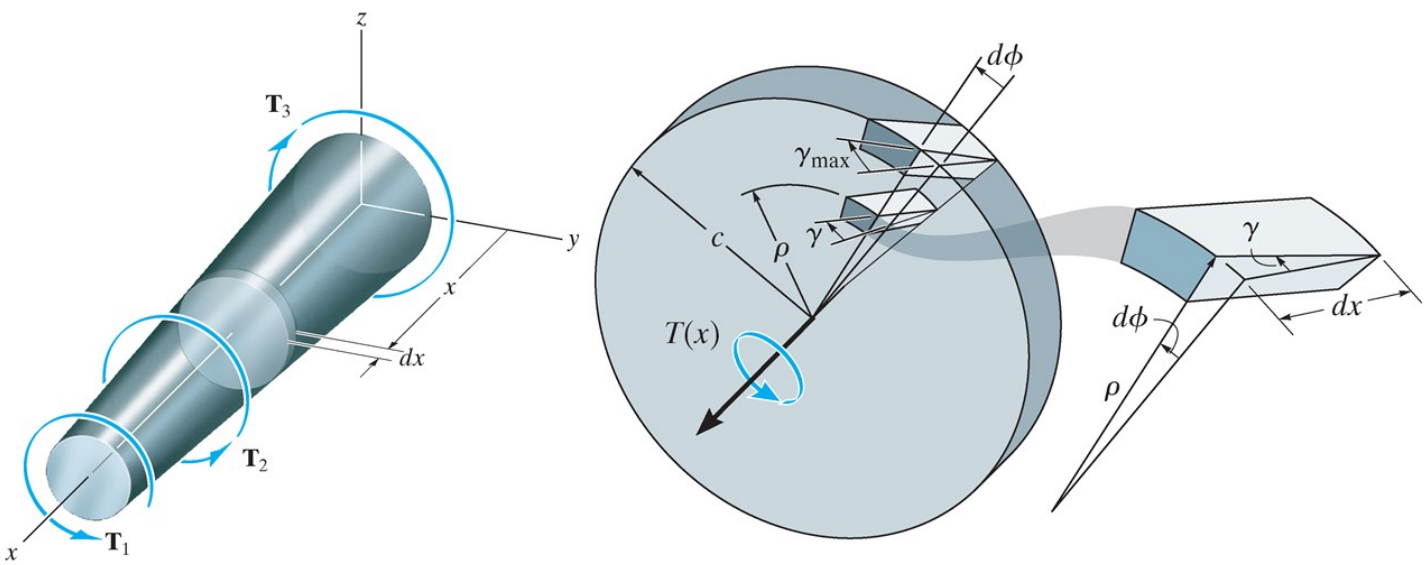
to find the appropriate shaft diameter.

- For solid shafts we can solve for c uniquely, but not for hollow shafts

angle of twist

- While in axial problems we examined the total deformation for the general case, in torsion we will examine the total angle of twist in general
- Using the method of sections, we can consider a differential disk which has some internal torque as a function of x, $T(x)$.
- On this section, the shear strain will be related to the angle of twist by the thickness of the section (dx) and the radial distance (ρ).

angle of twist



angle of twist

- γ and $d\phi$ are related by

$$d\phi = \gamma \frac{dx}{\rho}$$

- We also know that $\gamma = \tau/G$ and $\tau = T(x)\rho/J(x)$ substituting both this gives

$$d\phi = \frac{T(x)}{J(x)G(x)} dx$$

multiple torques

- If a shaft is subjected to multiple torques, or if there is a discontinuous change in shape or material we can sum the change in angle over various segments

$$\phi = \sum \frac{TL}{JG}$$

sign convention

- When we section a shaft and consider the internal torque, it is important to be consistent with our signs
- Both torque and angle of twist should follow the same convention
- The convention is to use the right hand rule with the thumb pointing normal to the cut, and the fingers curling in the positive direction

shear flow

- Thin-walled tubes of non-circular cross-sections are commonly found in aerospace and other applications
- We can analyze these using a technique called shear flow
- Because the walls of the tube are thin, we assume that the shear stress is uniformly distributed through the wall thickness

shear flow

- If we consider an arbitrary segment of a tube with varying thickness, we find from equilibrium that the product of the average shear stress and the thickness must be the same at every location on the cross-section

$$q = \tau_{avg} t$$

- q is called the shear flow

average shear stress

- We can relate the average shear stress to the torque by considering the torque produced about some point within the tubes boundary

$$T = \oint h\tau_{avg} t ds$$

- Where h is the distance from the reference point, ds is the differential arc length and t is the tube thickness.

average shear stress

- We notice that $\tau_{avg}t$ is equal to the shear flow, q, which must be constant
- We can also simplify the integral by relating a similar area integral to the arc length integral

$$dA_m = 1/2hds$$

- Thus

$$T = \oint h\tau_{avg}tds = 2q\int dA_m = 2qA_m$$

angle of twist

- The angle of twist for a thin-walled tube is given by

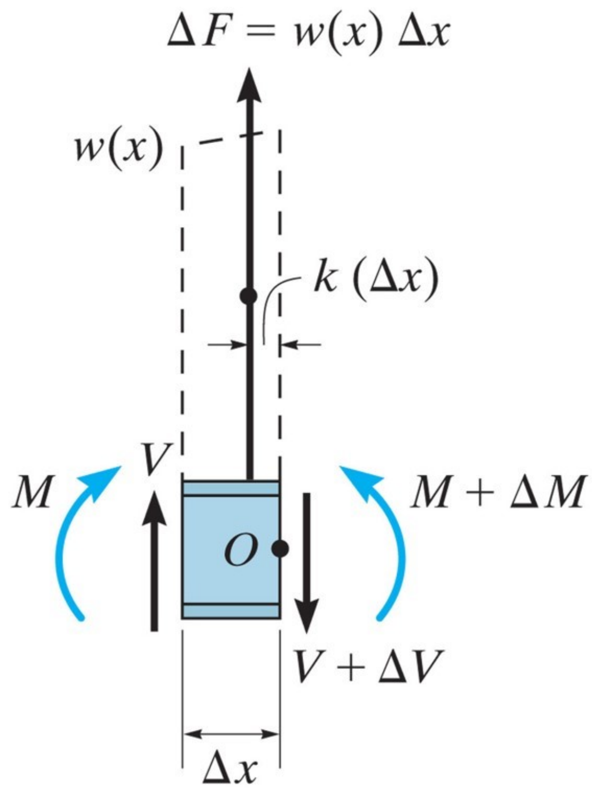
$$\phi = \frac{TL}{4A_m^2 G} \oint \frac{ds}{t}$$

bending

relation between load, shear, moment

- When there are several forces, supports, or loading conditions applied to a beam, the piecewise method can be cumbersome
- In this section we will examine the differential relationships between distributed load, shear, and bending moments

distributed load



distributed load

- Consider a beam subjected to only distributed loading
- If we section this beam in the middle (to remove both supports) we can relate V to the loading function $w(x)$
- Considering the sum of forces in y:

$$\begin{aligned} V + w(x)\Delta x - (V + \Delta V) &= 0 \\ \Delta V &= w(x)\Delta x \end{aligned}$$

distributed load

- If we divide by Δx and let $\Delta x \to 0$ we find

$$\frac{dV}{dx} = w(x)$$

- Thus the slope of the shear diagram is equal to the distributed load function

moment diagram

- If we consider the sum of moments about O on the same section we find

$$(M + \Delta M) - (w(x)\Delta x)k\Delta x - V\Delta x - M = 0$$
$$\Delta M = V\Delta x + kw(x)\Delta x^2$$

- Dividing by Δx and letting $\Delta x \to 0$ gives

$$\frac{dM}{dx} = V$$

concentrated forces

- If we consider a concentrated force (instead of a distributed load) we find

$$\Delta V = F$$

- This means that concentrated loads will cause the shear diagram to “jump” by the amount of the concentrated force (causing a discontinuity on our graph)

couple moments

- If our section includes a couple moment, we find (from the moment equation) that

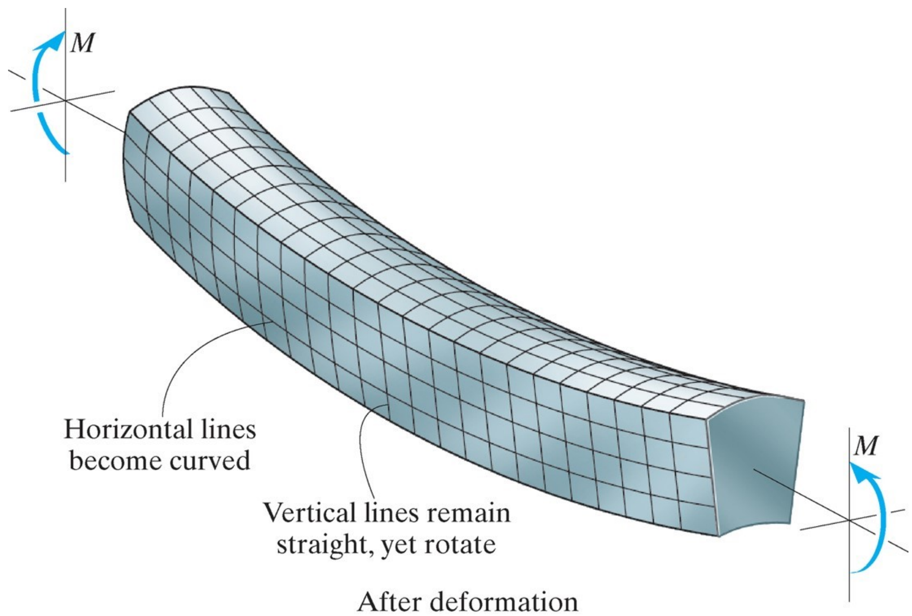
$$\Delta M = M_0$$

- Thus the moment diagram will have a jump discontinuity

bending deformation

- If we drew a grid on a specimen in bending, we would find that vertical lines tend to stay straight (but rotate)
- Horizontal lines will become curved
- If bending lifts the ends up (like a smile), then the top face will be in compression (and expand), while the bottom face will be in tension (and contract)

bending deformation



After deformation

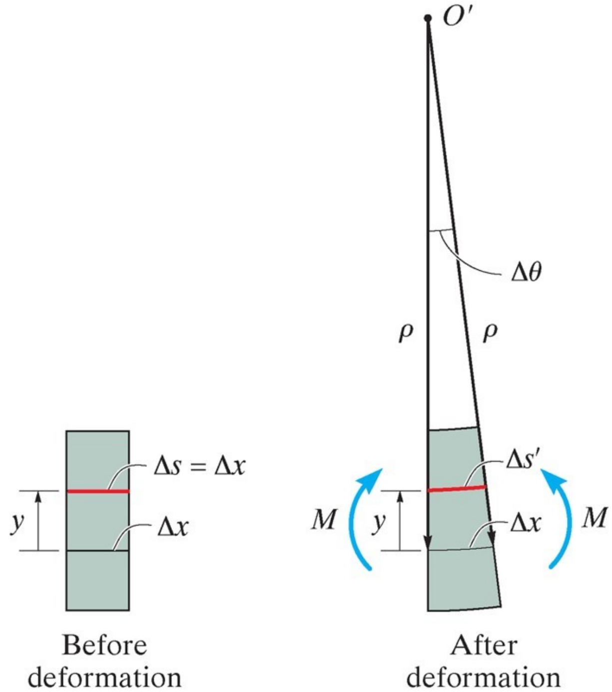
neutral axis

- Since the bottom is in tension and the top is in compression, there must be somewhere in between that is under no stress
- We call that the neutral axis, and assume it does not change in length
- We also assume that planar sections remain planar (no warping)
- Finally, Poisson’s effects are neglected (cross-sections keep the same size and shape)

strain

- We will now consider an infinitesimal beam element before and after deformation
- Δx is located on the neutral axis and thus does not change in length after deformation
- Some other line segment, Δs is located y away from the neutral axis and changes its length to $\Delta s'$ after deformation

strain



O'
$\Delta\theta$
ρ
ρ
$\Delta s = \Delta x$
y
Δx
$\Delta s'$
M
y
Δx
M
Before deformation
After deformation

strain

- We can now define strain at the line segment Δs as

$$\epsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

strain

- If we define ρ as the radius of curvature after deformation, thus $\Delta x = \Delta s = \rho\Delta\theta$
- The radius of curvature at Δs is $\rho - y$, thus we can write

$$\epsilon = \lim_{\Delta\theta\to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta}$$

- Which gives

$$\epsilon = -\frac{y}{\rho}$$

hooke's law

- If the beam is a linear material that follows Hooke's Law, the stress must be linearly proportional to the strain
- Thus the stress will vary linearly through the beam, just like the strain does

neutral axis

- We have already hypothesized that a neutral axis must exist, we will now find its location
- To be in equilibrium, the resultant force(s) produced by the stress must sum to zero, which means

$$\begin{aligned}
\sum F_x = 0 = \int_A dF &= \int_A \sigma dA \\
&= \int_A -\left(\frac{y}{c}\right)\sigma_{max} dA \\
&= -\frac{\sigma_{max}}{c}\int_A y dA
\end{aligned}$$

neutral axis

- Since σ_{max} and c are both non-zero constants, we know that

$$\int_A y dA = 0$$

- Which can only be satisfied at the horizontal centroidal axis, so the neutral axis is the centroidal axis

bending moment

- The internal bending moment must be equal to the total moment produced by the stress distribution

$$
\begin{aligned}
M &= \int_A y dF = \int_A y(\sigma dA) \\
&= \int_A y \left(\frac{y}{c} \sigma_{max} \right) dA \\
&= \frac{\sigma_{max}}{c} \int_A y^2 dA
\end{aligned}
$$

bending moment

- We recognize that $\int_A y^2 dA = I$, and we re-arrange to write

$$\sigma_{max} = \frac{Mc}{I}$$

procedure

- Find the internal moment at the point of interest, or draw a moment diagram to find the maximum point (if needed)
- Determine the moment of inertia for the beam section
- Find the neutral axis and/or the distance from the neutral axis to the point of interest
- Use the flexure formula to find the stress

moment of inertia

- We know that $I = \int_A y^2 dA$
- For common shapes, this integral has been pre-calculated (about the centroid of the shape)
- For compound shapes, we use the parallel axis theorem to combine inertias from multiple areas

parallel axis theorem

- The parallel axis theorem is used to find the moment about any axis parallel to an axis passing through the centroid
- If we consider an axis parallel to the *x*-axis, separated by some *dy* we have

$$I_X = \int_A (y + dy)^2 dA$$

- Which gives

$$I_x = \int_A y^2 dA + 2dy \int_A y dA + dy^2 \int_A dA$$

parallel axis theorem

- The first integral is the moment of inertia about the original *x*-axis, which we will call $\bar{I}_x$
- The second integral is zero since the *x*-axis passes through the centroid
- This gives the parallel axis theorem

$$I_x = \bar{I}_x + Ady^2$$

parallel axis theorem

- Similarly for the y-axis and polar moment of inertia we find

$$I_y = \bar{I}_y + Adx^2$$

$$J_O = \bar{J}_C + Ad^2$$