Lecture 9 - Variational Calculus

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schedule

- 15 Feb - Variational Calculus
- 17 Feb - Boundary Conditions (HW3 Due)
- 22 Feb - Project Descriptions
- 24 Feb - SwiftComp (HW 4 Due)

outline

- boundary conditions
- multiple variables

homework

- My Python functions are not a substitute for understanding the math
- You can program in any language, but it is also possible to do Mori-Tanaka in Excel
- In my code I switched between tensor and matrix notation to avoid re-writing equations
- Alternatively, we could re-write tensor equations entirely

tensor equations

$a^q_{ijkl} = a_{ij} a_{kl}$

$$a^l_4 = -\frac{1}{35}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) +$$
$$\frac{1}{7}(a_{ij}\delta_{kl} + a_{ik}\delta_{jl} + a_{il}\delta_{jk} + a_{kl}\delta_{ij} + a_{jl}\delta_{ik} + a_{jk}\delta_{il})$$

- NOTE: Many of you copied my linear closure approximation, which used constants for 2D orientation
- In 2D replace $-\frac{1}{35}$ and $\frac{1}{7}$ with $-\frac{1}{24}$ and $\frac{1}{6}$, respectively

boundary conditions

boundaries

- Not all problems of functionals have well-defined boundary conditions
- The Euler-Lagrange equation will be the same
- Consider the example

$$I[y] = \int_{x_0}^{x_1} [p(x)(\dot{y})^2 + q(x)y^2 + f(x)y]dx + h_1 y^2(x_1) + h_0 y^2(x_0)$$

boundaries

- For the functional to be stationary we have

$$I[y] = 2\int_{x_0}^{x_1} [-\dot{(p\dot{y})} + qy + f]\delta y dx +$$
$$2p\dot{y}\delta y|_{x_0}^{x_1} + 2h_1 y(x_1)\delta y(x_1) + 2h_0 y(x_0)\delta y(x_0) = 0$$

boundaries

- Satisfying the Euler-Lagrange equation will ensure the first line is equal to zero
- The second line forms the natural boundary conditions

$$
\begin{aligned}
p(x_1)\dot{y}(x_1) + h_1 y(x_1) &= 0 \\
-p(x_0)\dot{y}(x_0) + h_0 y(x_0) &= 0
\end{aligned}
$$

natural and geometric boundaries

- In general, if a functional contains the derivative of an unknown function to the m^{th} order:
- Boundary conditions expressed in terms of the unknown function to the $(m-1)^{\text{th}}$ order are geometric boundary conditions
- Boundary conditions expressed in terms of the unknown function higher than the $(m-1)^{\text{th}}$ order are natural boundary conditions
- When there are geometric boundaries, the variation will be zero at the boundaries
- Otherwise the coefficients must equal zero

example

- Find the governing differential equation and boundary conditions for a bar of stiffness EA, length L
- Subjected to a tensile load, $p(x)$
- There is a spring of stiffness k attached to $x{=}L$
- The bar is fixed at $x{=}0$

subsidiary conditions

- We have discussed problems with or without prescribed boundary conditions
- We may also have additional constraints (also known as subsidiary conditions)
- The can be formulated using the same method as the Lagrange Multiplier

subsidiary conditions

- Consider a functional

$$I = \int_{x_0}^{x_1} F(y, \dot{y}, x)dx$$

- With boundary conditions, $y(x_0) = y_0$ and $y(x_1) = y_1$
- And the subsidiary condition

$$\int_{x_0}^{x_1} G(y, \dot{y}, x)dx = C$$

subsidiary conditions

- The stationary conditions for this functional can be obtained using

$$\delta I^* = 0$$

- Where

$$I^* = \int_{x_0}^{x_1} F(y, \dot{y}, x)dx + \lambda \left(\int_{x_0}^{x_1} G(y, \dot{y}, x)dx - C \right)$$

subsidiary conditions

- Carrying out the variation we find

$$\delta I^* = \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} + \lambda \left[\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial \dot{y}} \right] \right\} \delta y dx + \\ \delta\lambda \left(\int_{x_0}^{x_1} G(y, \dot{y}, x) dx - C \right) = 0$$

- Which gives the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} + \lambda \left[\frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial \dot{y}} \right]$$

subsidiary conditions

- If the subsidiary condition is given in terms of differential equations instead of an integral

$$G(x, y, \dot{y}) = 0$$

- Then we must write the functional as

$$J[y, \lambda] = \int_{x_0}^{x_1} F(y, \dot{y}, x) dx + \int_{x_0}^{x_1} \lambda G(y, \dot{y}, x) dx$$

- Since λ will be a function of x

subsidiary conditions

- The only difference in the Euler-Lagrange solution is that λ will be inside the derivative

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial \dot{y}} + \lambda\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\lambda\frac{\partial G}{\partial \dot{y}}\right)$$

example

- A uniform power line with length C and density ρ is hanging between two points, (x_0, y_0) and (x_1, y_2)
- With gravity acting in the y direction, find the shape of the power line in equilibrium

multiple variables

higher derivatives

- While our development has only used one derivative of y, it can easily be extended

$$I[y] = \int_{x_0}^{x} 1 F(x, y, \dot{y}, \ddot{y}, \dddot{y}, ..., y^{(n)}) dx$$

- The first variation is

$$\delta I[y] = \int_{x_0}^{x} 1 \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \dot{y}}\delta \dot{y} + ... + \frac{\partial F}{\partial y^{(n)}}\delta y^{(n)} \right] dx$$

- Carrying out successive integration by parts we find

$$\delta I[y] = \int_{x_0}^{x} 1 \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) \right] \delta y dx$$

higher derivatives

- The Euler-Lagrange equation is merely in the terms inside the integral
- Boundary terms from integration vanish when $y, \dot{y}, ..., y^{(n)}$ are prescribed at the boundaries

multiple functions

- A functional could also consist of several functions, for example

$$I[y, z] = \int_{x_0}^{x} 1F(x, y, z, \dot{y}, \dot{z})dx$$

- Where both y and z are functions of x
- In this case the Euler-Lagrange equation is two equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0 \qquad \frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0$$

multiple variables

- We could also have multiple fundamental variables in the functional, for example

$$I[u] = \int \int_G F(x, y, u, u_{,x}, u_{,y}) dxdy$$

- The Euler-Lagrange equation is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{,y}}\right) = 0$$

multiple variables

- If u is prescribed along the boundary, then $\delta u = 0$ along the boundary, otherwise

$$\frac{\partial F}{\partial u_{,x}} n_x + \frac{\partial F}{\partial u_{,y}} n_y = 0$$

along the boundary

example

- Minimize the mechanical potential energy of a beam with deflection y under applied force, $f(x)$

$$I[y] = \int_0^L \left[\frac{1}{2} EI(\ddot{y})^2 - fy\right] dx$$

example

- Minimize the functional

$$I[y, z] = \int_{x_0}^{x_1} (y^2 - z^2) dx$$

- Under the constraint

$$\dot{y} - y + z = 0$$

next class

- Converting between differential and variational statements
- Approximate solutions
- Variational asymptotic method