Lecture 2 - Tensor review, Anisotropic Elasticity

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering

20 January 2022

schedule

- 20 Jan - Tensor review, Anisotropic Elasticity
- 25 Jan - Coordinate Transformation
- 27 Jan - 1D Micromechanics (HW1 Due)
- 1 Feb - Orientation Averaging

outline

- index notation
- anisotropic elasticity

index notation

index notation

- Consider the following
- $s = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$
- Which we could also write as

$$s = \sum_{i=1}^{n} a_i x_i$$

- Using index notation, and Einstein's summation convention, we can also write this as

$$s = a_i x_i$$

dummy index

- In index notation, a repeated index implies summation
- This index is also referred to as a dummy index
- It is called a "dummy index" because the expression would have the same meaning with any index in its place
- i.e. i, j, k, etc. would all have the same meaning when repeated

dummy index

- Note, no index may be repeated more than once, thus the expression

$$s = \sum_{i=1}^{n} a_i b_i x_i$$

could not be directly written in index notation

free index

- Any index which is not repeated in an index notation expression is referred to as a free index
- The number of free indexes in an expression indicate the tensor order of that expression
- No free indexes = scalar expression (0-order tensor)
- One free index = vector expression (1st-order tensor)
- Two free indexes = matrix expression (2nd-order tensor)

index notation

- Free index is not repeated (on any term)
- Free index takes all values (1,2,3)
- e.g. $u_i = \langle u_1, u_2, u_3 \rangle$
- Free indexes must match across terms in an expression or equation

Dummy index is repeated on at least one term Dummy index indicates summation over all values e.g. $\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ Index can not be used more than twice in the same term ($A_{ij}B_{jk}C_{kl}$ is good, $A_{ij}B_{ij}C_{ij}$ is not)

dummy index

- The dummy index can be triggered by any repeated index in a *term*.
- Summation or not?
  - $a_i + b_{ij}c_j$
  - $a_{ij}b_{ij}$
  - $a_{ij} + b_{ij}c_j$

matrix multiplication

- How can we write matrix multiplication in index notation?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

special symbols

kronecker delta

- For convenience we define two symbols in index notation
- *Kronecker delta* is a general tensor form of the Identity Matrix

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Is also used for higher order tensors

kronecker delta

- $\delta_{ij} = \delta_{ji}$
- $\delta_{ii} = 3$
- $\delta_{ij} a_j = a_i$
- $\delta_{ij} b_{ij} = b_{ii}$

alternating symbol

- *alternating symbol* or *permutation symbol*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an odd permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

alternating symbol

- This symbol is not used as frequently as the *Kronecker delta*
- For our uses in this course, it is enough to know that 123, 231, and 312 are even permutations
- 321, 132, 213 are odd permutations
- all other indexes are zero
- $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta mk$

tensor algebra

substitution

- When solving tensor equations, we often need to manipulate expressions
- We need to make sure the correct indexes are used when substituting, for example

$$a_i = U_{im} b_m \tag{1}$$

$$b_i = V_{im} c_m \tag{2}$$

- To substitute (2) into (1), we first need to change indexes

substitution

- We need to change the free index, i, to m in (2)
- Since m is already used as the dummy index, we need to change that too

$$b_m = V_{mj} c_j \tag{3}$$

- We can now make the substitution

$$a_i = U_{im} V_{mj} c_j \tag{4}$$

multiplication

- We need to be careful with indexes when multiplying expressions
- $p = a_m b_m$ and $q = c_m d_m$
- We can express, pq, but remember the dummy index cannot be repeated more than once
- $pq \neq a_m b_m c_m d_m$
- Instead we must change the dummy index in one of the expressions first
- $pq = a_m b_m c_n d_n$

factoring

- In the following expression, we would like to factor out n, but it has different indexes
- $\sigma_{ij} n_j - \lambda n_i = 0$
- Recall $\delta_{ij} a_j = a_i$ we can rewrite $n_i = \delta_{ij} nj$
- $\sigma_{ij} n_j - \lambda \delta_{ij} nj = 0$
- $(\sigma_{ij} - \lambda \delta_{ij}) nj = 0$

contraction

- σ_{ii} is the contraction of σ_{ij}
- This can often be a useful tool in solving tensor equations
- $\sigma_{ij} = \lambda \Delta \delta_{ij} + 2\mu E_{ij}$
- $\sigma_{ii} = 3\lambda \Delta + 2\mu E_{ii}$

tensor calculus

partial derivative

- We indicate (partial) derivatives using a comma
- In three dimensions, we take the partial derivative with respect to each variable (x, y, z or x_1, x_2, and x_3)
- For example a scalar property, such as density, can have a different value at any point in space
- $\rho = \rho(x_1, x_2, x_3)$
- $\rho_{,i} = \frac{\partial}{\partial x_i}\rho = \left\langle \frac{\partial \rho}{\partial x_1}, \frac{\partial \rho}{\partial x_2}, \frac{\partial \rho}{\partial x_3} \right\rangle$

- Similarly, if we take the partial derivative of a vector, it produces a matrix

$$u_{i,j} = \frac{\partial}{\partial x_j} u_i = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

dyadic notation

dyadic notation

- Dyadic notation is sometimes called tensor product notation
- Dyadic product: $C_{ij} = a_i b_j$ is written as $C = a \otimes b$
- Double dot product: $A_{ij} B_{ji} = c$ is written as $A : B = c$

transformation

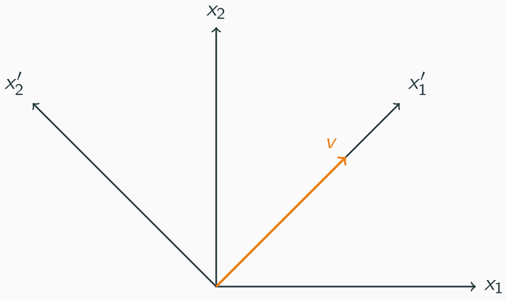
linear transformation

- Let us consider some transformation, $\mathbf{T}$, which transforms any vector into another vector
- If we transform $\mathbf{T}\ a = c$ and $\mathbf{T}\ b = d$
- We call $\mathbf{T}$ a linear transformation (and a tensor) if

$$\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta}+\mathbf{Tb}$$
$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{Ta}$$

- Where α is any arbitrary scalar and a, b are arbitrary vectors

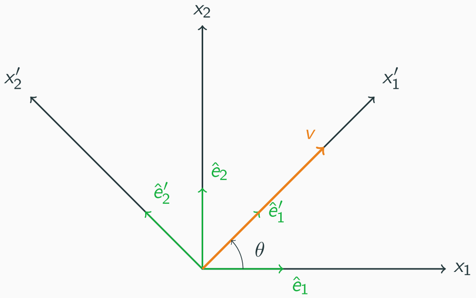
coordinate transformation in two dimensions



coordinate transformation in two dimensions

- The vector, v, remains fixed, but we transform our coordinate system
- In the new coordinate system, the x_2' portion of v is zero.
- To transform the coordinate system, we first define some unit vectors.
- $\hat{e}_1$ is a unit vector in the direction of x_1, while $\hat{e}_1'$ is a unit vector in the direction of x_1'

coordinate transformation in two dimensions



coordinate transformation in two dimensions

- For this example, let us assume $v = \langle 2, 2 \rangle$ and $\theta = 45^\circ$
- We can write the transformed unit vectors, $\hat{e}_1'$ and $\hat{e}_2'$ in terms of $\hat{e}_1$, $\hat{e}_2$ and the angle of rotation, θ.

$$\hat{e}_1' = \langle \hat{e}_1 \cos\theta, \hat{e}_2 \sin\theta \rangle$$
$$\hat{e}_2' = \langle -\hat{e}_1 \sin\theta, \hat{e}_2 \cos\theta \rangle$$

coordinate transformation in two dimensions

- We can write the vector, v, in terms of the unit vectors describing our axis system
- $v = v_1 \hat{e}_1 + v_2 \hat{e}_2$
- (note: $\hat{e}_1 = \langle 1, 0 \rangle$ and $\hat{e}_2 = \langle 0, 1 \rangle$)
- $v = \langle 2, 2 \rangle = 2\langle 1, 0 \rangle + 2\langle 0, 1 \rangle$

coordinate transformation in two dimensions

- When expressed in the transformed coordinate system, we refer to v'
- $v' = \langle v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta$
- $v' = \langle 2\sqrt{2}, 0\rangle$
- We can recover the original vector from the transformed coordinates:
- $v = v_1' \hat{e}_1' + v_2' \hat{e}_2'$
- (note: $\hat{e}_1' = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$ and $\hat{e}_2' = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle\}$
- $v = 2\sqrt{2}\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle, 0\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = \langle 2, 2\rangle$

general coordinate transformation

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- It is convenient to define a general form of the coordinate transformation in index notation
- We define Q_{ij} as the cosine of the angle between the x_i' axis and the x_j axis.
- This is also referred to as the "direction cosine"

$$Q_{ij} = \cos(x_i', x_j)$$

mental and emotional health warning

- Different textbooks flip the definition of Q_{ij} (Elasticity and Continuum texts have opposite definitions, for example)
- The result gives the transpose
- Always use equations (next slide) from the same source as your Q_{ij} definition

general coordinate transformation

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v'_i = Q_{ij} v_j$
- Matrices (second-order tensors): $\sigma'_{ij} = Q_{im} Q_{jn} \sigma_{mn}$
- Fourth-order tensors: $C'_{ijkl} = Q_{im} Q_{jn} Q_{ko} Q_{lp} C_{mnop}$

general coordinate transformation

- We can use this form on our 2D transformation example

$$
\begin{aligned}
Q_{ij} &= \cos(x'_i, x_j) \\
&= \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

general coordinate transformation

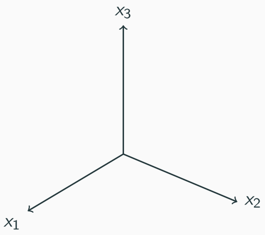
- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_j = Q_{ij} v'_i$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im} Q_{jn} \sigma'_{ij}$
- Fourth-order tensors: $C_{mnop} = Q_{im} Q_{jn} Q_{ko} Q_{lp} C'_{ijkl}$

general coordinate transformation

- We can derive some interesting properties of the transformation tensor, Q_{ij}
- We know that $v_i' = Q_{ij} v_j$ and that $v_j = Q_{ij} v_i'$
- If we substitute (changing the appropriate indexes) we find:
- $v_j = Q_{ij} Q_{ik} v_k$
- We can now use the Kronecker Delta to substitute $v_j = \delta_{jk} v_k$
- $\delta_{jk} v_k = Q_{ij} Q_{ik} v_k$

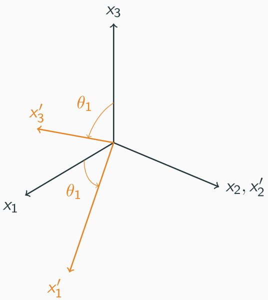
examples

example

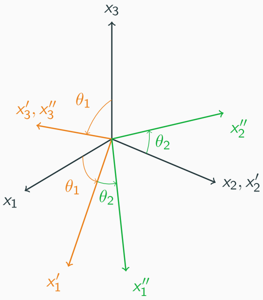


- Find Q^1_{ij} for rotation of 60° about x_2
- Find Q^2_{ij} for rotation of 30° about x'_3
- Find e''_i after both rotations

example



example



example

- $Q^1_{ij} = \cos(x'_i, x_j)$
- $Q^2_{ij} = \cos(x''_i, x'_j)$

$$Q^1_{ij} = \begin{bmatrix} \cos 60 & \cos 90 & \cos 150 \\ \cos 90 & \cos 0 & \cos 90 \\ \cos 30 & \cos 90 & \cos 60 \end{bmatrix}$$

$$Q^2_{ij} = \begin{bmatrix} \cos 30 & \cos 60 & \cos 90 \\ \cos 120 & \cos 30 & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix}$$

example

- We now use Q_{ij} to find $\hat{e}'_i$ and $\hat{e}''_i$
- First, we need to write $\hat{e}_i$ in a manner more consistent with index notation
- We will indicate axis direction with a superscript, e.g. $\hat{e}_1 = e^1_i$
- $e'_i = Q^1_{ij} e_j$
- $e''_i = Q^2_{ij} e'_j$
- How do we find e''_i in terms of e_i?

anisotropic elasticity

stiffness

- In 3D, Hooke's Law for linearly elastic materials is

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

- For isotropic materials, C_{ijkl} can be expressed in terms of two constants
- In general (anisotropic materials) more constants are needed and we use the full tensor

engineering notation

- Fourth-order tensors are cumbersome to write, we often use engineering notation
- σ and ϵ are written as vectors and C_{ijkl} is written as a matrix.
- NOTE: Although σ, ϵ and C_{ijkl} are tensors, their counterparts in engineering notation are NOT formal tensors
- This means that the usual transformation laws do not apply

engineering notation

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{1322} & C_{1222}\\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & C_{1333} & C_{1233}\\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & C_{1323} & C_{1223}\\ C_{1113} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1213}\\ C_{1112} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}\end{bmatrix} \begin{bmatrix}E_{11}\\ E_{22}\\ E_{33}\\ E_{23}\\ 2E_{13}\\ 2E_{12}\end{bmatrix}$$

compliance

$$\begin{bmatrix}E_{11}\\ E_{22}\\ E_{33}\\ E_{23}\\ 2E_{13}\\ 2E_{12}\end{bmatrix} = \begin{bmatrix}S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1113} & S_{1112}\\ S_{1122} & S_{2222} & S_{2233} & S_{2223} & S_{1322} & S_{1222}\\ S_{1133} & S_{2233} & S_{3333} & S_{2333} & S_{1333} & S_{1233}\\ S_{1123} & S_{2223} & S_{2333} & S_{2323} & S_{1323} & S_{1223}\\ S_{1113} & S_{1322} & S_{1333} & S_{1323} & S_{1313} & S_{1213}\\ S_{1112} & S_{1222} & S_{1233} & S_{1223} & S_{1213} & S_{1212}\end{bmatrix} \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}$$

physical interpretation

- If we now consider the case of uniaxial tension, we see that

$$
\begin{aligned}
E_{11} &= S_{1111}\sigma_{11} \\
E_{22} &= S_{1122}\sigma_{11} \\
E_{33} &= S_{1133}\sigma_{11} \\
2E_{23} &= S_{1123}\sigma_{11} \\
2E_{13} &= S_{1113}\sigma_{11} \\
2E_{12} &= S_{1112}\sigma_{11}
\end{aligned}
$$

- $S1111$ is familiar, acting like $1/EY$

poisson's ratio

- For isotropic materials we defined Poisson's ratio as $\nu = -E_{22}/E_{11}$
- For anisotropic materials, we can have a different Poisson's ratio acting in different directions
- We define $\nu_{ij} = -E_{jj}/E_{ii}$ (with no summation), the ratio of the transverse strain in the j direction when stress is applied in the i direction
- For this example we can find ν_{12} and ν_{13} as

$$
\begin{aligned}
\nu_{12} = -E_{22}/E_{11} = -S_{1122}/S_{1111} \\
\nu_{13} = -E_{33}/E_{11} = -S_{1133}/S_{1111}
\end{aligned}
$$

poisson's ratio

- Note that we cannot, in general, say that $\nu_{12} = \nu_{21}$
- However, due to the symmetry of the stiffness/compliance tensors, we know that

$$\nu_{21} E_x = \nu_{12} E_y$$
$$\nu_{31} E_x = \nu_{13} E_z$$
$$\nu_{32} E_y = \nu_{23} E_z$$

- Where E_x refer's to the Young's Modulus in the x-direction, etc.

shear coupling coefficients

- An unfamiliar effect is that shear strains can be introduced from a normal stress
- We define shear coupling coefficients as $\eta_{1112} = \eta_{16} = -2E_{12}/E_{11}$ due to σ_{11}
- These coupling terms can also effect shear strain in a different plane from the applied shear stress
- Like the Poisson's ratio, these are not entirely independent

$$\eta_{61} E_x = \eta_{16} G_6$$

- Where G_6 is the shear modulus in the 12 plane

shear coupling coefficients

- Shear coupling coefficients are sometimes placed in two groups
- Coefficients of mutual influence relate shear stress to normal strain and normal stress to shear strain
- Chentsov coefficients relate shear stress in one plane to shear strain in another plane
- In general we can say

$$\eta_{nm} E_m = \eta_{mn} G_n \qquad (\mathrm{m} = 1,2,3)\ (\mathrm{n} = 4,5,6)$$

and

$$\eta_{nm} G_m = \eta_{mn} G_n \qquad (\mathrm{m,n} = 4,5,6) \qquad m \neq n$$

orthotropic symmetry

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}$$

transversely isotropic symmetry

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix}=\begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0\\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0\\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{1313} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{1313} & 0\\ 0 & 0 & 0 & 0 & 0 & 1/2(C_{1111}-C_{2222})\end{bmatrix}\begin{bmatrix}E_{11}\\E_{22}\\E_{33}\\E_{23}\\2E_{13}\\2E_{12}\end{bmatrix}$$

isotropic symmetry

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu)\end{bmatrix}$$

next class

- Next class we will develop transformation laws for engineering stress/strain and stiffness