Lecture 8 - Variational Calculus

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schedule

- Feb 25 - Variational Calculus
- Mar 2 - Variational Calculus
- Mar 4 - Boundary Conditions (HW3 Due)
- Mar 9 - Project Descriptions

outline

- lagrange multipliers
- calculus of variations

lagrange multipliers

differential and variational statements

- A differential statement includes a set of governing differential equations established inside a domain and a set of boundary conditions to be satisfied along the boundaries
- A variational statement is to find stationary conditions for an integral with unknown functions in the integrand
- Variational statements are advantageous in the following aspects
  - Clear physical meaning, invariant to coordinate system
  - Can provide more realistic descriptions than differential statements (concentrated loads)
  - More easily suited to solving problems numerically or approximately
  - Can be more systematic and consistent than building a set of differential equations

stationary problems

- If the function $F(u_1)$ is defined on a domain, then at $\frac{dF}{du_1} = 1$ it is considered to be stationary
- This stationary point could be a minimum, maximum, or saddle point
- We use the second derivative to determine which of these it is: >0 for a minimum, <0 for a maximum and $=0$ for a saddle point
- For a function of n variables, $F(u_n)$ the stationary points are

$$\frac{\partial F}{\partial u_i} = 0$$

for all values of i - and to determine the type of stationary point we use

lagrange multipliers

- Let us now consider a function of several variables, but the variables are subject to a constraint

$$f(u_1, u_2, \ldots) = 0$$

- Algebraically, we could use each provided constraint equation to reduce the number of variables
- For large problems, it can be cumbersome or impossible to eliminate some variables
- The Lagrange Multiplier method is an alternative, systematic approach

lagrange multiplier

- For a constrained problem at a stationary point we will have

$$dF = \frac{\partial F}{\partial u_1} du_1 + \ldots + \frac{\partial F}{\partial u_n} du_n = 0$$

- The relationship between *dui* can be found by differentiating the constraint

$$df = \frac{\partial f}{\partial u_1} du_1 + \ldots + \frac{\partial f}{\partial u_n} du_n = 0$$

- We can combine these two equations using a Lagrange Multiplier

lagrange multiplier

- The Lagrange Multiplier, λ is an arbitrary function of u_i
- We can choose the Lagrange Multiplier such that

$$\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0$$

- Which now leaves

$$\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} = 0 \qquad i = 1, 2, ..., n-1$$

- We now define a new function $F^* = F + \lambda f$

lagrange multiplier

- This converts a constrained problem in n variables to an unconstrained problem in $n+1$ variables
- Notice that while the stationary values of F^* will be the same as the stationary values to F, they will not necessarily correspond
- For example, a minimum in F^* might be a maximum in F
- This provides a systematic method for solving problems with any number of variables and constraints, and is also well-posed for numeric solutions

- Design a box with given surface area such that the volume is maximized
- The box has no cover along one of the surfaces (open-face box)
- This gives the surface area as $A = xy + 2yz + 2xz = C$
- worked example

calculus of variations

functional

- A functional of some unknown function $y(x)$ is defined as

$$I = I[y(x)]$$

- A functional depends on all values of $y(x)$ over some interval
- We will often use the form

$$I[y] = \int_a^b F(x, y(x), \dot{y}(x))dx$$

bernoulli

- The original problem that motivated study of variational calculus
- Bernoulli 1696
- Design a chute between two points, A and B
- such that a particle sliding without friction under its own weight
- travels from A to B in the shortest time

variational statement

- To solve Bernoulli's problem we denote the arc length as s, speed as

$$v = \frac{ds}{dt}$$

- And we can find the total time as

$$t = \int_A^B \frac{ds}{v}$$

variational statement

- The arc length s can be found from

$$ds = \sqrt{dx^2 + dy^2}$$

- Since $y = y(x)$ we can write $dy = \dot{y}dx$
- We can now re-write ds as

$$ds = \sqrt{1 + \dot{y}^2}dx$$

variational statement

- From the conservation of energy we can also say that

$$\frac{1}{2}mv^2 = mgy$$

- Such that

$$v = \sqrt{2gy}$$

- We now need to find some function $y(x)$ which minimizes the integral

$$t = \int_0^a \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} dx$$

euler lagrange

- Now we develop a method for finding $y(x)$
- Consider the functional

$$I[y] = \int_{x_0}^{x_1} F(x, y, \dot{y}) dx$$

- Where $y(x)$ is subject to boundary conditions

$$y(x_0) = y_0$$
$$y(x_1) = y_1$$