

Lecture 4 - Eshelby

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27 January 2022

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schedule

- 27 Jan - 1D Micromechanics (HW1 Due)
- 1 Feb - Orientation Averaging
- 3 Feb - Mean-field (HW2 Due)
- 8 Feb - Variational Calculus

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- eshelby
- aspect ratio

eshelby's equivalent inclusion

- Eshelby formulated the exact elastic solution for an elliptical inclusion in an infinite matrix
- While this is not often useful, it serves as an exact analytical model to compare numerical results with
- It is also the base for more useful mean-field theories

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eshelby's thought experiment

- Eshelby solution starts with a thought experiment
- Suppose we have a homogeneous, elastic body in equilibrium
- We now cut an ellipsoidal piece out of that body and allow it to undergo a stress-free transformation, such as thermal expansion
- This stress-free transformation is referred to as the transformation strain, ϵ^T

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eshelby's thought experiment

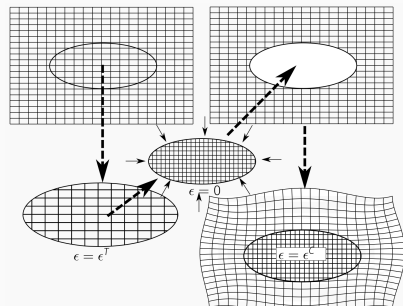


Figure 1: Eshelby's thought experiment

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eshelby's thought experiment

- Now, we weld that expanded ellipsoid back into the original body
- Traction need to be applied to force it to fit
- Once the stresses equilibrate, the ellipsoid has a constrained strain, ϵ^C

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- After equilibrium is reached the inclusion is still under a state of uniform strain
- The inclusion stress, σ_I can be found as:

$$\sigma_I = C_m(\epsilon^C - \epsilon^T)$$

Where C_m is the stiffness of the material.

- One of Eshelby's critical findings is that

$$\epsilon^C = S\epsilon^T$$

- S is known as the Eshelby Tensor, and is a fourth-order tensor
- Function of shape and poisson's ratio
- It has been calculated exactly for ellipsoids, and numerically for other shapes

- ν represents the matrix Poisson's ratio
- s is the aspect ratio of the fibers
- $l_1 = \frac{2s}{(s^2-1)^{\frac{3}{2}}} (s(s^2-1)^{\frac{1}{2}} - \cosh^{-1} s)$
- $Q = \frac{3}{8(1-\nu)}$
- $R = \frac{1-2\nu}{8(1-\nu)}$
- $T = \frac{Q(4-3l_1)}{3(s^2-1)}$
- $l_3 = 4 - 2l_1$

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eshelby tensor

S_{ijkl}	Long Fibers	Short Fibers (Ellipsoids)
$S_{1111} = S_{2222}$	$\frac{5-\nu}{8(1-\nu)}$	$Q + Rl_1 + \frac{3T}{4}$
S_{3333}	0	$\frac{4Q}{3} + Rl_3 + 2s^2 T$
$S_{1122} = S_{2211}$	$\frac{-1+4\nu}{8(1-\nu)}$	$\frac{Q}{3} - Rl_1 + \frac{4T}{3}$
$S_{1133} = S_{2233}$	$\frac{\nu}{2(1-\nu)}$	$-Rl_1 - s^2 T$
$S_{3311} = S_{3322}$	0	$-Rl_3 - T$
$S_{1212} = S_{1221}$ $= S_{2112} = S_{2121}$	$\frac{3-4\nu}{8(1-\nu)}$	$\frac{Q}{3} + Rl_1 + \frac{T}{4}$
$S_{1313} = S_{1331}$ $= S_{3113} = S_{3131}$ $= S_{3232} = S_{3223}$ $= S_{2332} = S_{2323}$	$\frac{1}{4}$	$2R - \frac{l_1 R}{2} - \frac{1+s^2}{2} T$
all other S_{ijkl}	0	0

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- Eshelby's initial thought experiment was for a homogeneous material
- To consider a different type of inclusion, we need to relate the transformation strain between some fictitious ellipsoid of matrix material which would be equivalent to our inclusion.
- We will refer to the inclusion stiffness as C_f , the transformation strain in the matrix as ϵ^T , and the transformation strain in the inclusion ϵ^{T*} .

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inclusions

- We are trying to find a transformation equivalent to our inclusion, so we set

$$\sigma_I = C_m(\epsilon^C - \epsilon^T) = C_f(\epsilon^C - \epsilon^{T*})$$

- Now we substitute the relation $\epsilon^C = S\epsilon^T$

$$C_m(S - I)\epsilon^T = C_f(S\epsilon^T - \epsilon^{T*})$$

- We can solve this to find the transformation strain

$$\epsilon^T = [(C_f - C_m)S + C_m]^{-1} C_f \epsilon^{T*}$$

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- Since the transformation strain is arbitrary, we can choose ϵ^T such that ϵ^{T*} is 0
- Now suppose we impose some strain, ϵ^0 on the composite
- The stress in the inclusion will be

$$\sigma_I = C_m(\epsilon^0 + \epsilon^C - \epsilon^T) = C_f(\epsilon^0 + \epsilon^C)$$

- Simplifying terms gives

$$(C_f - C_m)(\epsilon^0 + \epsilon^C) = -C_m\epsilon^T$$

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- We now assume $\epsilon^0 + \epsilon^C = \bar{\epsilon}^f$ and multiply both sides by $S C_m^{-1}$

$$S(C_m)^{-1}(C_f - C_m)\bar{\epsilon}^f = -\epsilon^C$$

- Recall $S\epsilon^T = \epsilon^C$
- We can also write ϵ^C in terms of $\bar{\epsilon}^f$

$$S(C_m)^{-1}(C_f - C_m)\bar{\epsilon}^f = \epsilon^0 - \bar{\epsilon}^f$$

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strain concentration tensor

- Finally, we can add $I\bar{\epsilon}^f$ to both sides to find

$$[I + S(C_m)^{-1}(C_f - C_m)]\bar{\epsilon}^f = \epsilon^0$$

- We define the inverse of the left-hand side the Eshelby strain-concentration tensor

$$A^E = [I + S(C_m)^{-1}(C_f - C_m)]^{-1}$$

- The stiffness can be calculated as

$$C = C_m + v_i(C_f - C_m)A^E$$

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stiffness

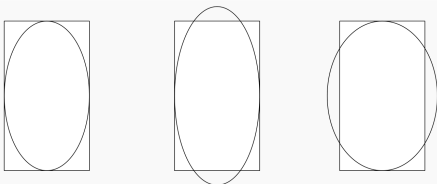
- This stiffness calculation is valid for any number of inclusions
- However, it is only appropriate for very dilute concentrations (<1% volume fraction)
- This ensures that the assumption $\epsilon^0 + \epsilon^C = \bar{\epsilon}^f$

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aspect ratio

aspect ratio

- Some studies have been done to evaluate Eshelby tensors for short fibers
- Long fibers are approximated by an ellipsoid with infinitely long major axis
- This is not appropriate for short fibers
- We could logically consider three different ellipsoids to represent a short fiber



- (a) The first ellipsoid considered, with aspect ratio equal to fiber aspect ratio
- (b) The second ellipsoid considered, with minor axis equal to fiber diameter and volume equal to fiber volume
- (c) The third ellipsoid considered, with major axis equal to fiber length and volume equal to fiber volume

Figure 2: aspect ratio comparisons

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- Steif and Hoysan investigated the effect of aspect ratio numerically
- Found that (a) and (c) were good for short fibers
- As fibers get longer, and as stiffness ratio of fiber to matrix increases, (a) gives best results
- (a) is also the easiest to use (same aspect ratio), so that is what is done in Eshelby-based models

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- With Eshelby (and derivative models), fibers at different orientations are modeled as a different inclusion
- Since the Eshelby tensor, S is a fourth-order tensor, we can treat it the same way as C
- Write it as 6x6 matrix, transform using R^σ

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example

- As an example, let us consider a “laminate” of short fiber composites
- This is a good approximate for many 3D printed composites
- We have a $\pm 45^\circ$ laminate, with very short carbon fibers, $s = 15$

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- First we find the Eshelby tensor for $s = 15$
- We also need the matrix Poisson's ratio, $\nu = 0.40$
- We find the parameters here
- Then we use this slide to find S_{ijkl}
- Notice that this assumes fibers are pointed in the 3-direction

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- Next, we rotate S_{ijkl} to find S^{45} and S^{-45}
- Notice: the eshelby tensor, S accounts for rotation, we do not rotate C_f
- So we find A^{45} and A^{-45}

$$A^{45} = \left[I + S^{45}(C_m)^{-1}(C_f - C_m) \right]^{-1}$$

$$A^{-45} = \left[I + S^{-45}(C_m)^{-1}(C_f - C_m) \right]^{-1}$$

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example

- This gives our total stiffness calculation as

$$C = C_m + v^{45}(C_f - C_m)A^{45} + v^{-45}(C_f - C_m)A^{-45}$$

- If we assume the volume fraction of fibers in our part is 20%
- And that there are equally many fibers in 45 and -45 directions
- Then $v^{45} = v^{-45} = 0.1$
- Note: Since this is not a dilute concentration, we would not expect this to be very accurate

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example

- Python code for this example (with some typical values for C_m and C_f) is posted here¹

¹https://colab.research.google.com/drive/1aUs_BfeWugyUfwQiKoPx5g-hxvrsQqNs?usp=sharing

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fiber orientation

fiber orientation

- While a laminate analogy works well for some cases, in general short fibers are not aligned in laminates
- It is not practical to model each possible fiber orientation as a separate inclusion
- Advani-Tucker introduced a tensorial representation of fiber orientation

fiber in spherical coordinates

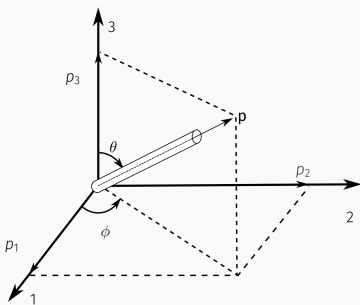


Figure 3: single fiber coordinate system

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fiber direction components

Component	Definition
p_1	$\sin \theta \cos \phi$
p_2	$\sin \theta \sin \phi$
p_3	$\cos \theta$

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orientation tensor

- Within a given volume, a distribution of fibers can be defined by some orientation distribution function, $\psi(\theta, \phi)$.
- Advani and Tucker introduced tensor representations of fiber orientation distribution functions

$$a_{ij} = \oint p_i p_j \psi(p) dp$$

- And

$$a_{ijkl} = \oint p_i p_j p_k p_l \psi(p) dp$$

- Note: any order tensor may be defined in this manner, the orientation distribution function must be even, due to fiber symmetry, and thus any odd-ordered tensor will be zero.

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orientation tensor

- It can be noted that some symmetries must exist due to the way the tensors are defined.
- In the second order tensor we have

$$a_{ij} = a_{ji}$$

- and in the fourth order tensor

$$a_{ijkl} = a_{jikl} = a_{kijl}$$

and so on for any permutation of i, j, k and l.

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- The orientation tensor is also normalized such that:

$$a_{ii} = 1$$

- And any lower-order tensor can be expressed in terms of the next higher-order tensor, for example

$$a_{ijkk} = a_{ij}$$

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example - 2D random

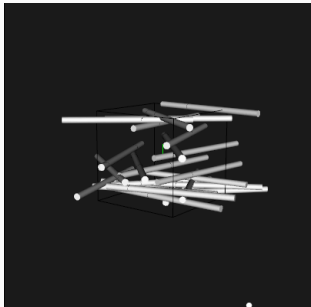


Figure 4: A visualization of a 2D random orientation distribution. This is expressed with the second-order tensor $a_{11} = a_{22} = 0.5$, with all other $a_{ij} = 0$.

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example - 3D random



Figure 5: A visualization of a 3D random orientation distribution. This is expressed with the second-order tensor $a_{11} = a_{22} = a_{33} = 1/3$, with all other $a_{ij} = 0$.

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example - aligned 45

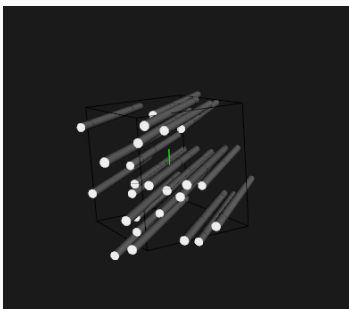


Figure 6: A visualization of a perfectly aligned, off-axis orientation distribution. This is expressed by rotating the tensor with $a_{11} = 1$ and all other $a_{ij} = 0$.

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- Orientation averaging
- Self-consistent and Mori-Tanaka methods
- Textbook pages 131-150