Homework 3 Solutions

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1 Homework 3 Solutions

1.1 Problem 1

For a functional $I[y] = \int_0^1 (\dot{y}^2 + 12xy)dx$ with $y(0) = 0$ and $y(1) = 0$, find the function y which corresponds to the stationary value of I.

First let us define the functional

```
In [7]: import sympy as sm
        sm.init_printing()
        x = sm.symbols('x')
        y = sm.Function('y')(x)
        fun = sm.diff(y,x)**2 + 12*x*y
        fun
```

Out[7]:

$$12xy(x) + \frac{d}{dx}y(x)^2$$

Next we solve the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$$

to find the stationary values of the functional

```
In [10]: sm.dsolve(sm.diff(fun,y)-sm.diff(sm.diff(fun,sm.diff(y,x)),x))
```

Out[10]:

$$y(x) = C_1 + C_2x + x^3$$

To satisfy the boundary conditions of $y(0) = 0$ and $y(1) = 0$ we see that $y(x) = x^3 - x$

```
In [11]: y = x**3 - x
         y
```

Out[11]:

$$x^3 - x$$

```
In [12]: fun = sm.diff(y,x)**2 + 12*x*y
         I = sm.integrate(fun,(x,0,1))
         I
```

Out[12]:

$$-\frac{4}{5}$$

We can verify that this is a stationary value by choosing some other function which satisfies the boundary conditions, here we will try both $y_1 = (1-x)x$ and $y_2 = (1-x)x^2$

```
In [14]: y1 = x*(1-x)
         y2 = x**2*(1-x)
         fun1 = sm.diff(y1,x)**2 + 12*x*y1
         fun2 = sm.diff(y2,x)**2 + 12*x*y2

         I1 = sm.integrate(fun1,(x,0,1))
         I2 = sm.integrate(fun2,(x,0,1))
         I1
```

Out[14]:

$$\frac{4}{3}$$

```
In [15]: I2
```

Out[15]:

$$\frac{11}{15}$$

Both are greater than the true solution, which is some confirmation that it is, indeed, the minimum.

1.2 Problem 2

Use two different approaches to find the maximum area of a rectangle of given perimeter L.

First we will use a traditional approach of manually eliminating variables. We know that area is given by

$$A = xy$$

with the given perimeter constraint we know that $L = 2x + 2y$ which can be solved for $y = (L-2x)/2$

We can make this substition and take the derivative to find the stationary value. The second derivative will tell us whether this is a maximum or minimum value.

```
In [16]: x,y,L = sm.symbols('x y L')
         A = x*y
         A = A.subs(y,(L-2*x)/2)
         A
```

Out[16]:

$$x\left(\frac{L}{2} - x\right)$$

```
In [17]: sm.diff(A,x)
```

Out[17]:

$$\frac{L}{2} - 2x$$

```
In [19]: sm.solve(sm.diff(A,x),x)
```

Out[19]:

$$\left[\frac{L}{4}\right]$$

As we may have expected, we find $x = L/4$, which corresponds to a square. We verify that this is a maximum by taking the second derivative

```
In [20]: sm.diff(A,x,x)
```

Out[20]:

$$-2$$

Which is negative, so this is, indeed, the maximum.

Alternatively, we could solve this problem without making any substitutions and instead use the method of Lagrange multipliers. We formulate the alternative energy function

$$A^* = xy + \lambda(2x + 2y - L)$$

And we solve the system of equations for each of the partial derivatives of A^*

```
In [21]: l = sm.symbols(r'\lambda')
         vs = x*y + l*(2*x + 2*y - L)
         dvsdx = sm.diff(vs,x)
         dvsdy = sm.diff(vs,y)
         dvsdl = sm.diff(vs,l)
         sm.solve([dvsdx,dvsdy,dvsdl],[x,y,l])
```

Out[21]:

$$\left\{\lambda : -\frac{L}{8}, \quad x : \frac{L}{4}, \quad y : \frac{L}{4}\right\}$$

We see that we arrive at the same solution.

1.3 Problem 3

Find the stationary curve of the functional $I[y] = \int_{-1}^{1} \sqrt{y(1+\dot{y}^2)}dx$ with boundary conditions $y(-1) = 1$ and $y(1) = 1$

First we build the function in python.

```
In [22]: y = sm.Function('y')(x)
         fun = sm.sqrt(y*(1+sm.diff(y,x)**2))
         fun
```

Out[22]:

$$\sqrt{\left(\frac{d}{dx}y(x)^2 + 1\right)y(x)}$$

Since there is no explicit dependence on x, we can use the simplified form of the Euler-Lagrange equation.

$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} = C$$

```
In [28]: C, C1 = sm.symbols('C C1')
         el = fun-sm.diff(y,x)*sm.diff(fun,sm.diff(y,x))
         el.simplify()
```

Out[28]:

$$\frac{\sqrt{\left(\frac{d}{dx}y(x)^2 + 1\right)y(x)}}{\frac{d}{dx}y(x)^2 + 1}$$

Which is equal to some constant C. Rearranging terms we solve this differential equation

$$y = C\sqrt{y(\dot{y}^2 + 1)}$$

```
In [39]: eq = sm.dsolve(C*sm.sqrt(y*(sm.diff(y,x)**2+1))-y,y)
         eq
```

Out[39]:

$$y(x) = \frac{1}{C^2}\left(C^4 + \frac{1}{8}\left(C_1 + \sqrt{2}x\right)^2\right)$$

```
In [46]: bcs = sm.solve([eq.subs(x,-1).rhs-1,eq.subs(x,1).rhs-1],[C,C1])
         bcs
```

Out[46]:

$$\left[\left(-\frac{\sqrt{2}}{2}, \quad 0\right), \quad \left(\frac{\sqrt{2}}{2}, \quad 0\right)\right]$$

There are two solutions, in each of them $C_1 = 0$ an and $C = \pm\sqrt{2}/2$. We find

```
In [49]: eq = eq.subs([(C,bcs[1][0]),(C1,bcs[1][1])])
         eq
```

Out[49]:

$$y(x) = \frac{x^2}{2} + \frac{1}{2}$$

1.4 Problem 4

Find the natural conditions to minimize the functional

$$I[x, y, z] = \int_{t_0}^{t_1} \left[\frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} k(x^2 + y^2 + z^2) \right] dt$$

To find the natural conditions, we take the first variation of the functional

$$\delta I = \int_{t_0}^{t_1} \left[m(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}) - k(x\delta x + y\delta y + z\delta z) \right] dt$$

We perform integration by parts for to express $\delta\dot{x}$, $\delta\dot{y}$, and $\delta\dot{z}$ as δx, δy and δz. Here we show only one of the integrations by parts, since the terms are identical and the other results can be found simply by interchanging x, y and z.

Consider $u = m\dot{x}$, $dv = \delta\dot{x}dx$, integration by parts gives $\int udv = uv - \int vdu$ in this case, $\int m\dot{x}\delta\dot{x}dx = m\dot{x}\delta x|_{t_0}^{t_1} - \int m\ddot{x}\delta x dx$. combining all terms (and repeating the integration by parts for all terms as well) we find

$$\delta I = 0 = m\dot{x}\delta x|_{t_0}^{t_1} + m\dot{y}\delta y|_{t_0}^{t_1} + m\dot{z}\delta z|_{t_0}^{t_1} - \int_{t_0}^{t_1} (m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) + k(x\delta x + y\delta y + z\delta z))$$

We see that the natural boundary conditions are $\dot{x}(t_0) = 0$, $\dot{x}(t_1) = 0$, $\dot{y}(t_0) = 0$, $\dot{y}(t_1) = 0$, $\dot{z}(t_0) = 0$ and $\dot{z}(t_1) = 0$.

1.5 Problem 5

Does the following functional have stationary points. If so, under which conditions, if not, why not?

$$I[y] = \int_0^{\pi/2} \left[x \sin y + \left(\frac{x^2}{2} \cos y \right) \dot{y} \right] dx \tag{1}$$

with $y(0) = 0$ and $y(\pi/2) = \pi/2$

First we evaluate the Euler-Lagrange equation

```
In [53]: x = sm.symbols('x')
         y = sm.Function('y')(x)
         y_ = sm.diff(y,x)
         F = x*sm.sin(y) + y_*(x**2/2*sm.cos(y))
         el = sm.diff(F,y) - sm.diff(sm.diff(F,y_),x)
         el
```

Out[53]:

$$0$$

This is always true, which means any $y(x)$ will satisfy the Euler-Lagrange equation, and any $y(x)$ satisfying teh boundary conditions will be a stationary value to the problem. We can verify this by trying a few different functions, first we see that $y = x$ satisfies the boundary conditions

In [54]:
```
y1 = x
y1_ = sm.diff(y1,x)
F1 = x*sm.sin(y1) + y1_*(x**2/2*sm.cos(y1))
I1 = sm.integrate(F1,(x,0,sm.pi/2))
I1
```

Out[54]:

$$\frac{\pi^2}{8}$$

We also see that $y = \pi/2 \sin x$ satisfies the boundary conditions, let us try that as a solution

In [70]:
```
y2 = sm.pi/2*sm.sin(x)
y2_ = sm.diff(y2,x)
F2 = x*sm.sin(y2) + y2_*(x**2/2*sm.cos(y2))
I2 = sm.N(sm.integrate(F2,(x,0,sm.pi/2)))
#need to evaluate numerically
I2 -sm.N(sm.pi**2/8)
```

Out[70]:

$$0$$

1.6 Problem 6

Find the curve corresponding to the stationary value of the functional

$$I[y,z] = \int_0^1 (\dot{y}\dot{z} + \dot{y}^2 + \dot{z}^2)dx \tag{2}$$

with $y(0) = z(0) = 0$ and $y(1) = z(1) = 1$

For a functional of two functions but only first derivatives, we can use the Euler-Lagrange equations

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) = 0$$
$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{z}}\right) = 0$$

This system of equations is

```
In [73]: x = sm.symbols('x')
         y = sm.Function('y')(x)
         y_ = sm.diff(y,x)
         z = sm.Function('z')(x)
         z_ = sm.diff(z,x)

         #functional
         F = y_*z_ + y_**2 + z_**2
         #euler-lagrange
         el_y = sm.diff(F,y) - sm.diff(sm.diff(F,y_),x)
         el_z = sm.diff(F,z) - sm.diff(sm.diff(F,z_),x)
         #print system of equations
         (el_y,el_z)
```

Out[73]:

$$\left(-2\frac{d^2}{dx^2}y(x) - \frac{d^2}{dx^2}z(x), \quad -\frac{d^2}{dx^2}y(x) - 2\frac{d^2}{dx^2}z(x)\right)$$

The only way for both equations to be solved simultaneously is if $\ddot{y} = \ddot{z} = 0$. Integrating and solving for the boundary conditions gives $y = x$ and $z = x$

1.7 Problem 7

The potential energy of a circular plate with radius R under axisymmetric distributed load, $q(r)$, with $r \in [0, R]$ can be expressed in terms of deflection, $w(r)$ as

$$I[w] = \int_0^R \left(r\ddot{w}^2 + \frac{\dot{w}^2}{r} + 2\mu\dot{w}\ddot{w} - \frac{2q}{D}rw\right) dr$$

where D and μ are elastic constants. Show that $w(r)$ must satisfy the following equation of equilibrium

$$r\dddot{w} + 2\dddot{w} - \frac{\ddot{w}}{r} + \frac{\dot{w}}{r^2} = \frac{qr}{D}$$

Since we have higher order derivatives, we see that we will need to reduce those using integration by parts before we can apply the Euler-Lagrange equation. Alternatively, we have a form of the Euler-Lagrange equations where this has already been done:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

```
In [79]: r, mu, q, D = sm.symbols('r \mu q D')
         w = sm.Function('w')(r)
         w_ = sm.diff(w,r)
         w__ = sm.diff(w,r,r)
         #functional
         F = r*w__**2 + w_**2/r + 2*mu*w_*w__ - 2*q/D*r*w

         #euler-lagrange
         el = sm.diff(F,w) - sm.diff(sm.diff(F,w_),r) + sm.diff(sm.diff(F,w__),r,r)
         el.simplify()
```

Out[79]:

$$2r\frac{d^4}{dr^4}w(r) + 4\frac{d^3}{dr^3}w(r) - \frac{2}{r}\frac{d^2}{dr^2}w(r) + \frac{2}{r^2}\frac{d}{dr}w(r) - \frac{2q}{D}r$$

We see that by moving $2qr/D$ to the right side and divding by the common factor 2, we do, indeed, recover the equation of equilibrium provided in the problem.

1.8 Problem 8

Find the Euler-Lagrange equation for the following functional

$$J[u(x,y,z)] = \int_G \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + 2uf(x,y,z)\right] dxdydz$$

where $f(x,y,z)$ is a given known function.

In [81]:
```
x, y, z = sm.symbols('x y z')
u = sm.Function('u')(x,y,z)
f = sm.Function('f')(x,y,z) #known function
ux = sm.diff(u,x)
uy = sm.diff(u,y)
uz = sm.diff(u,z)

#integrand
F = ux**2 + uy**2 + uz**2 + 2*u*f
```

For a functional of multiple fundamental variables, but only one function, the Euler-Lagrange equation is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{,x}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{,y}}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F}{\partial u_{,z}}\right) = 0$$

In [83]:
```
#euler-lagrange equation
el = sm.diff(F,u) - sm.diff(sm.diff(F,ux),x) - sm.diff(sm.diff(F,uy),y) -

#solve for f
sm.solve(el,f)
```

Out[83]:

$$\left[\frac{\partial^2}{\partial x^2}u(x,y,z) + \frac{\partial^2}{\partial y^2}u(x,y,z) + \frac{\partial^2}{\partial z^2}u(x,y,z)\right]$$

We see that we must have

$$f(x,y,z) = u_{xx} + u_{yy} + u_{zz}$$

In [ ]: