Lecture 3 - Coordinate Transformation

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schedule

- Feb 9 - Coordinate Transformation
- Feb 11 - 1D Micromechanics (HW1 Due)
- Feb 16 - Mean-field
- Feb 18 - Orientation Averaging (HW2 Due)

outline

- transformation
- engineering notation

transformation

general coordinate transformation

- Coordinate transformation can become much more complicated in three dimensions, and with higher-order tensors
- It is convenient to define a general form of the coordinate transformation in index notation
- We define Q_{ij} as the cosine of the angle between the x_i' axis and the x_j axis.
- This is also referred to as the "direction cosine"

$$Q_{ij} = \cos(x_i', x_j)$$

mental and emotional health warning

- Different textbooks flip the definition of Q_{ij} (Elasticity and Continuum texts have opposite definitions, for example)
- The result gives the transpose
- Always use equations (next slide) from the same source as your Q_{ij} definition

general coordinate transformation

- We can transform any-order tensor using Q_{ij}
- Vectors (first-order tensors): $v'_i = Q_{ij} v_j$
- Matrices (second-order tensors): $\sigma'_{ij} = Q_{im} Q_{jn} \sigma_{mn}$
- Fourth-order tensors: $C'_{ijkl} = Q_{im} Q_{jn} Q_{ko} Q_{lp} C_{mnop}$

transformation

- We can use this form on our 2D transformation example

$$
\begin{aligned}
Q_{ij} &= \cos(x'_i, x_j) \\
&= \begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \cos(90 - \theta) \\ \cos(90 + \theta) & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}
\end{aligned}
$$

general coordinate transformation

- We can similarly use Q_{ij} to find tensors in the original coordinate system
- Vectors (first-order tensors): $v_j = Q_{ij} v'_i$
- Matrices (second-order tensors): $\sigma_{mn} = Q_{im} Q_{jn} \sigma'_{ij}$
- Fourth-order tensors: $C_{mnop} = Q_{im} Q_{jn} Q_{ko} Q_{lp} C'_{ijkl}$

general coordinate transformation

- We can derive some interesting properties of the transformation tensor, Q_{ij}
- We know that $v'_i = Q_{ij} v_j$ and that $v_j = Q_{ij} v'_i$
- If we substitute (changing the appropriate indexes) we find:
- $v_j = Q_{ij} Q_{ik} v_k$
- We can now use the Kronecker Delta to substitute $v_j = \delta_{jk} v_k$
- $\delta_{jk} v_k = Q_{ij} Q_{ik} v_k$

engineering notation

engineering notation

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{1322} & C_{1222} \\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & C_{1333} & C_{1233} \\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & C_{1323} & C_{1223} \\ C_{1113} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1213} \\ C_{1112} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{bmatrix}
$$

orthotropic symmetry

$$
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0\\
C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0\\
C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0\\
0 & 0 & 0 & C_{2323} & 0 & 0\\
0 & 0 & 0 & 0 & C_{1313} & 0\\
0 & 0 & 0 & 0 & 0 & C_{1212}
\end{bmatrix}
\begin{bmatrix}E_{11}\\ E_{22}\\ E_{33}\\ 2E_{23}\\ 2E_{13}\\ 2E_{12}\end{bmatrix}
$$

transversely isotropic symmetry

$$
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0\\
C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0\\
C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0\\
0 & 0 & 0 & C_{1313} & 0 & 0\\
0 & 0 & 0 & 0 & C_{1313} & 0\\
0 & 0 & 0 & 0 & 0 & 1/2(C_{1111} - C_{2222})
\end{bmatrix}
\begin{bmatrix}E_{11}\\ E_{22}\\ E_{33}\\ 2E_{23}\\ 2E_{13}\\ 2E_{12}\end{bmatrix}
$$

isotropic symmetry

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

transformation

- We know that
- $$\sigma_{mn} = Q_{im} Q_{jn} \sigma'_{ij}$$
- We can expand this to write in terms of engineering stress
- We will expand only two terms, as they show the general pattern for all 6

stress transformation

$$\begin{aligned}\sigma_1' = \sigma_{11}' = {} & Q_{11}Q_{11}\sigma_{11} + Q_{11}Q_{12}\sigma_{12} + Q_{11}Q_{13}\sigma_{13} \\ & + Q_{12}Q_{11}\sigma_{21} + Q_{12}Q_{12}\sigma_{22} + Q_{12}Q_{13}\sigma_{23} \\ & + Q_{13}Q_{11}\sigma_{31} + Q_{13}Q_{12}\sigma_{32} + Q_{13}Q_{13}\sigma_{33}\end{aligned}$$

$$\begin{aligned}\sigma_1' = {} & Q_{11}^2\sigma_1 + Q_{12}^2\sigma_2 + Q_{13}^2\sigma_3 \\ & + 2Q_{11}Q_{12}\sigma_6 + 2Q_{11}Q_{13}\sigma_5 + 2Q_{12}Q_{13}\sigma_4\end{aligned}$$

stress transformation

$$\begin{aligned}\sigma_4' = \sigma_{23}' = {} & Q_{21}Q_{31}\sigma_{11} + Q_{21}Q_{32}\sigma_{12} + Q_{21}Q_{33}\sigma_{13} \\ & + Q_{22}Q_{31}\sigma_{21} + Q_{22}Q_{32}\sigma_{22} + Q_{22}Q_{33}\sigma_{23} \\ & + Q_{23}Q_{31}\sigma_{31} + Q_{23}Q_{32}\sigma_{32} + Q_{23}Q_{33}\sigma_{33}\end{aligned}$$

$$\begin{aligned}\sigma_4' = {} & Q_{21}Q_{31}\sigma_1 + Q_{22}Q_{32}\sigma_{22} + Q_{23}Q_{33}\sigma_3 \\ & + (Q_{21}Q_{32} + Q_{22}Q_{31})\sigma_6 + (Q_{21}Q_{33} + Q_{23}Q_{31})\sigma_5 \\ & + (Q_{22}Q_{33} + Q_{23}Q_{32})\sigma_4\end{aligned}$$

stress transformation

- We often write $\sigma' = R_\sigma \sigma$ for engineering notation

$$R_\sigma = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & 2Q_{12}Q_{13} & 2Q_{11}Q_{13} & 2Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & 2Q_{22}Q_{23} & 2Q_{21}Q_{23} & 2Q_{21}Q_{22} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & 2Q_{32}Q_{33} & 2Q_{31}Q_{33} & 2Q_{31}Q_{32} \\ Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} & Q_{23}Q_{32}+Q_{22}Q_{33} & Q_{23}Q_{31}+Q_{21}Q_{33} & Q_{22}Q_{31}+Q_{21}Q_{32} \\ Q_{11}Q_{31} & Q_{12}Q_{32} & Q_{13}Q_{33} & Q_{13}Q_{32}+Q_{12}Q_{33} & Q_{13}Q_{31}+Q_{11}Q_{33} & Q_{12}Q_{31}+Q_{11}Q_{32} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} & Q_{13}Q_{22}+Q_{12}Q_{23} & Q_{13}Q_{21}+Q_{11}Q_{23} & Q_{12}Q_{21}+Q_{11}Q_{22} \end{bmatrix}$$

strain transformation

- We can follow the exact same procedure to transform strain
- The values are almost the same, notice the highlighted terms

$$R_\epsilon = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & Q_{12}Q_{13} & Q_{11}Q_{13} & Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & Q_{22}Q_{23} & Q_{21}Q_{23} & Q_{21}Q_{22} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & Q_{32}Q_{33} & Q_{31}Q_{33} & Q_{31}Q_{32} \\ Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} & Q_{23}Q_{32}+Q_{22}Q_{33} & Q_{23}Q_{31}+Q_{21}Q_{33} & Q_{22}Q_{31}+Q_{21}Q_{32} \\ Q_{11}Q_{31} & Q_{12}Q_{32} & Q_{13}Q_{33} & Q_{13}Q_{32}+Q_{12}Q_{33} & Q_{13}Q_{31}+Q_{11}Q_{33} & Q_{12}Q_{31}+Q_{11}Q_{32} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} & Q_{13}Q_{22}+Q_{12}Q_{23} & Q_{13}Q_{21}+Q_{11}Q_{23} & Q_{12}Q_{21}+Q_{11}Q_{22} \end{bmatrix}$$

stiffness transformation

- We can now formulate the transformation of the stiffness matrix. We know that

$$\sigma' = R\sigma_\sigma = C'E'$$

- And since $\sigma = CE$, we can say

$$R_\sigma CE = C'E'$$

- Now we know that $E' = R_E E$, so we substitute that to find

$$R_\sigma CE = C'R_E E$$

stiffness transformation

- We can right multiply both sides by E-1 to cancel E
- Then we can right multiply both sides by RE-1 to get C' by itself

$$C' = R_\sigma C(R_E)^{-1}$$

- Note that $R_E^{-1} = R_\sigma^T$

conventions

- There are two things that can be very confusing when transforming engineering stiffness
- First, while I have used the most standard ordering of stress/strain terms, not everyone uses the same order
- Second, the equations used here are for engineering strain (which is the most common)
- However, tensorial strain may also be used, in which case $R_\sigma = R_E$, but that adds other complications

one dimensional micromechanics

one dimensional micromechanics

- Some simple one-dimensional micromechanics models are useful as bounding cases
- The first micromechancis models were developed by Voigt and Reuss
- These provide a type of bound to possible solutions
- Some improvements were made using the method of cells

equivalent solid

- The goal of all micromechanics models is to use the known properties of constituents to find the large-scale behavior
- We can find this by averaging the stress and strain tensors over the volume of some RVE

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ij}(x,y,z)dV$$
$$\bar{\epsilon}_{ij} = \frac{1}{V}\int_V \epsilon_{ij}(x,y,z)dV$$

equivalent solid

- If we have only two phases (fiber and matrix), and we use engineering notation, this average can be expressed as

$$\bar{\sigma}_i = \frac{1}{V}\left(\int_{V^f} \sigma_i^f(x,y,z)dV + \int_{V^m} \sigma_i^m(x,y,z)dV\right)$$
$$\bar{\epsilon}_i = \frac{1}{V}\left(\int_{V^f} \epsilon_i^f(x,y,z)dV + \int_{V^m} \epsilon_i^m(x,y,z)dV\right)$$

equivalent solid

- We also know that in the fiber and matrix, respectively, Hooke's Law still holds

$$\sigma_i = C_{ij}\epsilon_j$$

- And this must be true for the average as well

$$\bar{\sigma}_i = C_{ij}\bar{\epsilon}_j$$

voigt

- Voigt considered a two-phase composite with a uniform strain imposed on both phases
- The uniform strain assumption means that

$$\epsilon_i^f = \epsilon_i^m = \epsilon_i$$

- While a macroscopically homogeneous strain does not necessarily impose a locally homogeneous strain field, Voigt assumed that

$$\epsilon_i = \bar{\epsilon}_i$$

voigt

- This assumption results in

$$\bar{\sigma}_i = \frac{1}{V}\left(\int_{V^f} C_{ij}^f \bar{\epsilon}_j dV + \int_{V^m} C_{ij}^m \bar{\epsilon}_j dV\right)$$
$$\bar{\sigma}_i = \left(\frac{V_f}{V} C_{ij}^f + \frac{V_m}{V} C_{ij}^m\right)\bar{\epsilon}_j$$

- This gives the well-known rule of mixtures for Cij

$$C_{ij}^c = \frac{V_f}{V} C_{ij}^f + \frac{V_m}{V} C_{ij}^m$$

reuss

- If we instead assume a uniform stress imposed on both phases such that

$$\sigma_i^f = \sigma_i^m = \sigma_i = \bar{\sigma}_i$$

- We would find the identical relationship, but with compliance instead of stiffness

$$\bar{\epsilon}_i = \frac{1}{V}\left(\int_{V^f} S_{ij}^f \bar{\sigma}_j dV + \int_{V^m} S_{ij}^m \bar{\sigma}_j dV\right)$$
$$\bar{\epsilon}_i = \left(\frac{V_f}{V} S_{ij}^f + \frac{V_m}{V} S_{ij}^m\right)\bar{\sigma}_j$$

bounds

- In general, the Voigt assumption (homogeneous strain, rule of mixtures for stiffness) gives an upper bound for stiffness
- On the other hand, the Reuss assmption (homogeneous stress, rule of mixtures for compliance) gives a lower bound for stiffness
- In uni-directional composites, the Voigt model is good enough for $E1$ and $v12$ predictions, but not for $E2$ or $G12$

subregions

- Hopkins and Chamis considered a refined model to subdivide the RVE into sub-regions
- This gives reasonable predictions for $E2$ and $G12$

discontinuous composites

discontinuous fibers

- The previous models all assumed that the constituent (fiber) was infinitely long
- There are many cases where we want to consider discontinuous fibers
- Weaker than continuous composites, but easier to mass-produce, more shapes can be made
- We will consider a simple model for aligned composites (shear lag)

shear lag

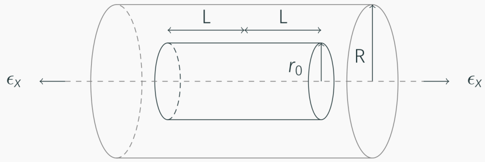

Figure 1: The RVE used for the shear-lag model.

Figure 1: shear lag diagram

shear lag

- Balancing forces on a differential element we find

$$\sum F_x = (\sigma_f + d\sigma_f)\frac{\pi d^2}{4} - \sigma_f \frac{\pi d^2}{4} - \tau_i(\pi d)dx = 0$$
$$\frac{d\sigma_f}{dx} = \frac{4\tau_i}{d}$$

shear lag

- To integrate, we need to make some assumptions
- It is commonly assumed that the normal stress on the end of the fibers is 0
- Various assumptions are made about the shear stress, τ, Kelly-Tyson assumed it is constant (rigid plastic)
- Cox assumed τ is a linear function of x

shear stress

- We can also find the shear stress by comparing adjacent annuli of matrix material around the fiber
- This assumes that fiber and matrix are perfectly bonded (continuous displacement at boundary)
- The force balance due to shear in adjacent annula means that

$$\pi dt = \pi d_0 \tau_i$$

- The shear stress far away from the fiber, $\tau = G_m \gamma$, and if $\gamma = \frac{du}{dr}$ then we can say

$$\frac{r_0}{r}\tau_i = G_m \frac{du}{dr}$$

shear stress

- We integrate to find that

$$\tau_i = \frac{G_m(u_R - u_f)}{r_0 ln(r)}$$

- Which we can substitute into our original force-balance equation to find

$$\frac{d\sigma_f}{dx} = \frac{4G_m(u_R - u_f)}{dr_0 ln(r)}$$

- But d=2r0, so we can simplify to

$$\frac{d\sigma_f}{dx} = \frac{2G_m(u_R - u_f)}{r_0^2 ln(r)}$$

shear lag

- Finally, we differentiate with respect to x to replace the displacements with strains
- We assume that duR/dx is far enough away from the fiber such that the strain is equal to far-field strain
- The solution to the differential equation is

$$\sigma_f = E_f \epsilon_1 + B \sinh(nx/r) + D \cos(nx/r)$$

stress in fibers

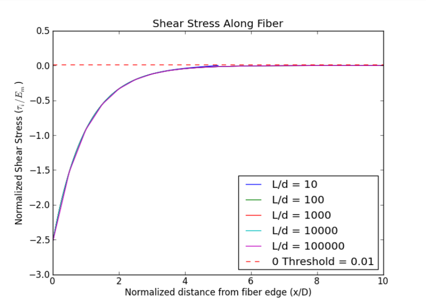

Figure 2: Stress near the edge of fibers in shear lag model

normalizing

- An interesting finding was that when we normalized distance (x) by fiber diameter
- The shear stress was the same for any fiber length
- This means that most/all shear stress transfer occurs near the ends
- If fibers are not long enough, full stress profile does not develop, fibers contribute very little to stiffness

next class

- Eshelby's equivalent inclusion
- Textbook pages 94-99 and 364 - 370 (I feel these are pretty confusing though)