Lecture 8 - Variational Calculus

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schedule

- 10 Feb - Variational Calculus (HW2 Due)
- 15 Feb - Variational Calculus
- 17 Feb - Boundary Conditions (HW3 Due)
- 22 Feb - Project Descriptions

outline

- lagrange multipliers
- calculus of variations

lagrange multipliers

differential and variational statements

- A differential statement includes a set of governing differential equations established inside a domain and a set of boundary conditions to be satisfied along the boundaries
- A variational statement is to find stationary conditions for an integral with unknown functions in the integrand

variational statements

- Variational statements are advantageous in the following aspects
  - Clear physical meaning, invariant to coordinate system
  - Can provide more realistic descriptions than differential statements (concentrated loads)
  - More easily suited to solving problems numerically or approximately
  - Can be more systematic and consistent than building a set of differential equations

stationary problems

- If the function $F(u_1)$ is defined on a domain, then at $\frac{dF}{du_1} = 1$ it is considered to be stationary
- This stationary point could be a minimum, maximum, or saddle point
- We use the second derivative to determine which of these it is: >0 for a minimum, <0 for a maximum and $=0$ for a saddle point

stationary points

- For a function of n variables, $F(u_n)$ the stationary points are

$$\frac{\partial F}{\partial u_i} = 0$$

for all values of i - and to determine the type of stationary point we use

$$\sum_{i,j=1,n} \frac{\partial^2 F}{\partial u_i \partial u_j}$$

lagrange multipliers

- Let us now consider a function of several variables, but the variables are subject to a constraint

$$f(u_1, u_2, ...) = 0$$

- Algebraically, we could use each provided constraint equation to reduce the number of variables
- For large problems, it can be cumbersome or impossible to eliminate some variables
- The Lagrange Multiplier method is an alternative, systematic approach

lagrange multiplier

- For a constrained problem at a stationary point we will have

$$dF = \frac{\partial F}{\partial u_1} du_1 + ... + \frac{\partial F}{\partial u_n} du_n = 0$$

- The relationship between *dui* can be found by differentiating the constraint

$$df = \frac{\partial f}{\partial u_1} du_1 + ... + \frac{\partial f}{\partial u_n} du_n = 0$$

lagrange multiplier

- We can combine these two equations using a Lagrange Multiplier

$$\frac{\partial F}{\partial u_1} du_1 + ... + \frac{\partial F}{\partial u_n} du_n + \lambda \left[\frac{\partial f}{\partial u_1} du_1 + ... + \frac{\partial f}{\partial u_n} du_n \right]$$

- We can re-group terms as

$$\sum_{i=1}^{n} \left[\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} \right] du_i = 0$$

lagrange multiplier

- The Lagrange Multiplier, λ is an arbitrary function of u_i
- We can choose the Lagrange Multiplier such that

$$\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0$$

- Which now leaves

$$\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} = 0 \qquad i = 1, 2, ..., n-1$$

- We now define a new function $F^* = F + \lambda f$

lagrange multiplier

- This converts a constrained problem in n variables to an unconstrained problem in $n+1$ variables
- Notice that while the stationary values of F^* will be the same as the stationary values to F, they will not necessarily correspond
- For example, a minimum in F^* might be a maximum in F
- This provides a systematic method for solving problems with any number of variables and constraints, and is also well-posed for numeric solutions

example

- Design a box with given surface area such that the volume is maximized
- The box has no cover along one of the surfaces (open-face box)
- This gives the surface area as $A = xy + 2yz + 2xz = C$
- worked example[1]

[1] https://colab.research.google.com/drive/1FwPoZyTFqZnGFyHpBhRj2PCqdtDACpNA?usp=sharing

calculus of variations

functional

- A functional of some unknown function $y(x)$ is defined as

$$I = I[y(x)]$$

- A functional depends on all values of $y(x)$ over some interval
- We will often use the form

$$I[y] = \int_a^b F(x, y(x), \dot{y}(x))dx$$

bernoulli

- The original problem that motivated study of variational calculus
- Bernoulli 1696
- Design a chute between two points, A and B
- such that a particle sliding without friction under its own weight
- travels from A to B in the shortest time

variational statement

- To solve Bernoulli's problem we denote the arc length as s, speed as

$$v = \frac{ds}{dt}$$

- And we can find the total time as

$$t = \int_A^B \frac{ds}{v}$$

variational statement

- The arc length s can be found from

$$ds = \sqrt{dx^2 + dy^2}$$

- Since $y = y(x)$ we can write $dy = \dot{y}dx$
- We can now re-write ds as

$$ds = \sqrt{1 + \dot{y}^2}dx$$

variational statement

- From the conservation of energy we can also say that

$$\frac{1}{2}mv^2 = mgy$$

- Such that

$$v = \sqrt{2gy}$$

- We now need to find some function $y(x)$ which minimizes the integral

$$t = \int_0^a \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}}dx$$

euler lagrange

- Now we develop a method for finding $y(x)$
- Consider the functional

$$I[y] = \int_{x_0}^{x_1} F(x, y, \dot{y}) dx$$

- Where $y(x)$ is subject to boundary conditions

$$y(x_0) = y_0$$
$$y(x_1) = y_1$$

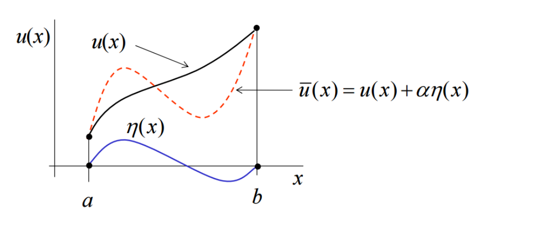
euler lagrange

- We assume that there is some solution, $y(x)$ for which I is stationary
- We also assume that $y(x)$ is continuous and differentiable in the problem domain
- Let us now choose some trial function

$$\bar{y}(x) = y(x) + \alpha\eta(x)$$

- Where $\eta(x)$ is some arbitrary continuous function which vanishes at the boundaries

euler lagrange



euler lagrange

- We can take the derivative of $\bar{y}$ to find

$$\dot{\bar{y}} = \dot{y}(x) + \alpha\dot{\eta}(x)$$

- This now gives

$$I[\alpha] = \int_{x_0}^{x_1} F(x, \bar{y}, \dot{\bar{y}})dx = \int_{x_0}^{x_1} F(x, y(x)+\alpha\eta(x), \dot{y}(x)+\alpha\dot{\eta}(x))dx$$

- Once $y(x)$ and $\eta(x)$ are chosen, I is a function of α

euler lagrange

- We find the stationary function by letting $\frac{dI}{d\alpha} = 0$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \frac{\partial F}{\partial \alpha} dx = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial F}{\partial \dot{\bar{y}}} \frac{\partial \dot{\bar{y}}}{\partial \alpha} \right) dx$$

- This simplifies to

$$\int_{x_0}^{x_1} \left(\frac{\partial F}{\partial \bar{y}} \eta + \frac{\partial F}{\partial \dot{\bar{y}}} \dot{\eta} \right) dx$$

euler lagrange

- Now we know that I will be stationary when $\alpha = 0$ in which case $\bar{y} = y$ therefore we can write

$$\int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial \dot{y}} \dot{\eta} \right) dx = 0$$

- And now we perform integration by parts on the second term

integration by parts

- Recall that

$$\int u dv = uv - \int v du$$

- We choose

$$\begin{aligned} u &= \frac{\partial F}{\partial \dot{y}} \\ du &= \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) \\ v &= \eta(x) \\ dv &= \dot{\eta} dx \end{aligned}$$

integration by parts

- This gives (for the second term)

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial \dot{y}} \dot{\eta} dx = \frac{\partial F}{\partial \dot{y}} \eta|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) \eta(x)$$

- Combining with the original equation and simplifying gives

$$\int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)\right] \eta dx + \frac{\partial F}{\partial \dot{y}} \eta|_{x_0}^{x_1} = 0$$

euler lagrange

- We already know that $\eta|_{x_0}^{x_1} = 0$, so we only need concern ourselves with the terms inside the integral
- Since this must be true for any arbitrary function, η, we can say that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$$

- This is known as the Euler-Lagrange equation
- A solution to the Euler-Lagrange equation is called an extremal, and an extremal which satisfies the boundary conditions is called a stationary function

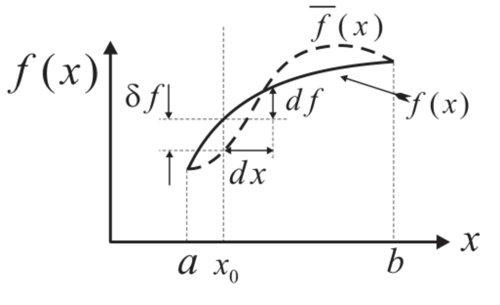
variations

- In variational calculus, we define the first variation as

$$\delta y = \bar{y} - y$$

- Note: while the variation follows many of the same rules as differentiation, it does not correspond to any slope, since η is completely arbitrary

variations



variations

- Variational laws are analogous to differentiation

$$\delta(F_1F_2) = F_1\delta F_2 + \delta F_1 F_2$$
$$\delta\left(\frac{F_1}{F_2}\right) = \frac{F_2\delta F_1 - F_1\delta F_2}{F_2^2}$$

- The variation and derivative are commutative

$$\frac{d}{dx}(\delta u) = \delta\left(\frac{du}{dx}\right)$$

- Similarly, the variation is commutative with the integral

$$\delta\int F dx = \int \delta F dx$$

variations

- We can also take the variation of a functional

$$\Delta F = F(x, y + \alpha\eta, \dot{y} + \alpha\dot{\eta}) - F(x, y, \dot{y})$$

- Expanding this function via a Taylor series gives

$$\Delta F = \left[F(x, y, \dot{y}) + \left(\delta y \frac{\partial F}{\partial y} + \delta \dot{y} \frac{\partial F}{\partial \dot{y}} + ...\right)\right] - F(x, y, \dot{y})$$

- And thus we call the variation of F

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \dot{y}}\delta \dot{y} + \epsilon_1$$

variations

- We can use variational notation to find the Euler-Lagrange equation

$$I[y] = \int_{x_0}^{x_1} F(x, y, \dot{y}) dx$$

- and taking the variation

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \dot{y}}\delta \dot{y}\right] dx = 0$$

variations

- Using integration by parts on the second term, as before, we find

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}} \right) \right] \delta y dx = 0$$

- Since $\delta y(x_0) = \delta y(x_1) = 0$
- Since this must be true for any arbitrary variation, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{y}} \right)$$

variations

- If the functional, F, does not depend on x explicitly (i.e. the only x dependence comes from $y(x)$) then we can say

$$\frac{d}{dx}\left(F - \dot{y}\frac{\partial F}{\partial \dot{y}} \right) = 0$$

- or, similarly

$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} = C$$

brachistochrone

- If we return now to Bernoulli's problem, we had found

$$t = \int_0^a \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} dx$$

- Since this does not depend on x explicitly, we can use the simpler form of the Euler-Lagrange equation.

$$F = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

- with

$$F - \dot{y}\frac{\partial F}{\partial \dot{y}} = C$$

brachistochrone

- Computing the partial derivative we find

$$\frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{2gy}\sqrt{1+\dot{y}^2}}$$

- Which gives in the Euler-Lagrange equation

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} - \frac{\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = C$$

brachistochrone

- Simplifying gives

$$\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = C$$

- We can square both sides and lump constants together

$$y(1+\dot{y}^2) = \frac{1}{2gc^2} = c_1$$

- And solving for $\dot{y}$, taking only the positive solution

$$\dot{y} = \frac{\sqrt{c_1 - y}}{\sqrt{y}}$$

brachistochrone

- The Brachistochrone problem can be solved using parametric equations

$$x = k^2(\theta - \sin\theta)$$
$$y = k^2(1 - \cos\theta)$$

example

- We can also use variational calculus to prove that the shortest distance between to points is a straight line
- The distance along a curve is given by

$$L = \int_a^b ds$$

- Where $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \dot{y}^2} dx$
- So we can find the minimum of the functional

$$I[y] = \int_a^b \sqrt{1 + \dot{y}^2} dx$$

group problems

- Find the Euler-Lagrange equation for

$$I[y] = \int y\sqrt{1 + \dot{y}^2} dx$$

- Find the Euler-Lagrange equation for

$$I[y] = \int [\dot{y}^2 + y^2 + 2xy] dx$$

next class

- Boundary conditions
- Multi-variate variational calculus
- Approximate solutions
- Variational Asymptotic Method