CHAPTER 4

The Method of Cells Micromechanics

Chapter Outline

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The purpose of this chapter is the presentation of the Method of Cells (MOC) for the micromechanical analysis of composite materials. Like the methods that were reviewed in the previous chapter, the present approach is approximate. It relies on the fundamental assumption that the two-phased composite has a periodic structure in which the reinforcing material (e.g., fibers) is arranged in a periodic manner, thus forming a periodic array. This assumption allows the analysis of a single repeating unit cell (RUC) rather than the whole composite with its many fibers. The RUC is defined as the fundamental building block such that the continuum can be constructed by repeated use of this unit cell. The analysis of the single RUC consists of the imposition of the displacement and traction continuity conditions at the interfaces within the unit cell as well as at the interfaces between neighboring unit cells, in conjunction with the equilibrium conditions. For elastic composites, in which both phases are linear elastic materials, the final micromechanics analysis should lead to relations between the average stresses and strains, from which the effective elastic moduli can be determined. The main advantage of the MOC, however, is its applicability to composites whose constituents experience nonlinear behavior (e.g., damage, inelasticity). In contrast, the generalization of the various approaches given in Chapter 3 to such composites is not always possible.

Within this chapter, the MOC will be presented for both continuous- and discontinuous-fiber-reinforced materials, and the predicted effective elastic constants, effective coefficients of thermal expansion (CTEs), effective thermal conductivities, and specific heats are derived.

The RUC in the MOC for unidirectional fibrous composites consists of four subcells, usually depicted as rectangular, one of which is occupied by the reinforcing phase (fiber). Such a construct appears to imply that the fiber is of a square cross-section. This is not actually the case, however, since the interfacial conditions are imposed on an average basis rather than pointwise. Consequently, the model involves neither rectangular fibers nor corners (see Figure 4.1). Rather, the subcell fields are associated with the subcell centroid, and this centroid has a region of influence that has been taken to be rectangular. The fact that three matrix subcells are present enables the MOC to capture some variation of the matrix fields. In contrast, mean field approaches such as the generalized self-consistent scheme (GSCS) and the Mori-Tanaka (MT) method (see Chapter 3) discern only the average matrix field variables (i.e., a single value applicable to the entire matrix).

4.1 The MOC for Continuously Fiber-Reinforced Materials (Doubly Periodic)

Here the MOC micromechanics theory is presented for the case of a doubly periodic RUC, which represents a continuously reinforced composite material. This section is followed by the triply periodic version of the MOC for discontinuous composites.

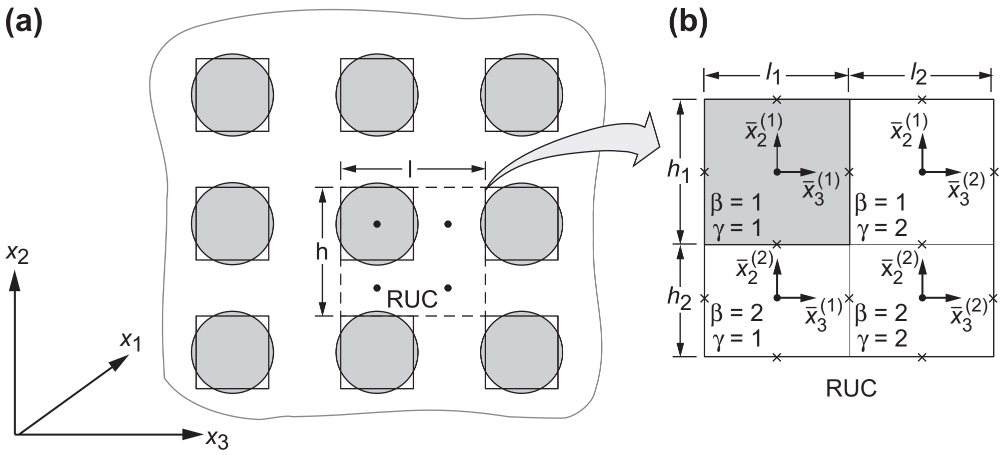

Figure 4.1:
MOC model of continuously reinforced fiber composite. (a) A composite with a doubly periodic array of fibers extending in the x_1-direction. (b) A repeating unit cell with four subcells ($\beta,\gamma = 1,2$).

4.1.1 Thermomechanical Formulation

The thermomechanical formulation develops the effective constitutive equations, effective stiffness matrix, effective CTEs, and effective inelastic strains for the composite material. In addition, it provides the localization equations, which provide the local stresses and strains in terms of the global (RUC-level) stresses and strains. Section 4.1.2 then derives the effective thermal conductivities of the composite.

4.1.1.1 Geometry and Basic Relations

The continuum model for unidirectional fiber-reinforced materials is based on the assumption that the continuous fibers extend in the x_1-direction and are arranged in a doubly periodic array in the x_2- and x_3-directions (see Figure 4.1(a)). The cross-section of the fiber is $h_1 l_1$; l_2 and h_2 represent its spacing in the matrix. As a result of this periodic arrangement, it is sufficient to analyze an RUC as shown in Figure 4.1(b). This RUC contains four subcells denoted by $\beta,\gamma = 1,2$. Let four local coordinate systems $(x_1, \bar{x}_2^{(\beta)}, \bar{x}_3^{(\gamma)})$ be introduced, all of which have origins that are located at the centroid of each subcell.

Since the average behavior of the composite is sought, as will be shown, a first-order theory in which the displacement in the subcell is expanded linearly in terms of the distances from the center of the subcell (i.e., in terms of $\bar{x}_2^{(\beta)}$ and $\bar{x}_3^{(\gamma)}$) can be used. Note that the presented framework is not limited to a first-order displacement field. Later in the book, second-order theories are used in the High-Fidelity Generalized Method of Cells (HFGMC, Chapter 6), the Higher-Order Theory for Functionally Graded Materials (HOTFGM, Chapter 11), and wave propagation problems (Chapter 12); in the latter, average properties are certainly not sufficient for the prediction of dispersion and attenuation. For the MOC, the following first-order displacement expansion in each subcell is considered:

$$u_i^{(\beta\gamma)} = w_i^{(\beta\gamma)}(\mathbf{x}) + \bar{x}_2^{(\beta)}\phi_i^{(\beta\gamma)} + \bar{x}_3^{(\gamma)}\psi_i^{(\beta\gamma)} \qquad i = 1, 2, 3 \tag{4.1}$$

where $w_i^{(\beta\gamma)}(\mathbf{x})$ are the displacement components of the center of the subcell and $\phi_i^{(\beta\gamma)}$ and $\psi_i^{(\beta\gamma)}$ characterize the linear dependence of the displacements on the local coordinates, $\bar{x}_2^{(\beta)}$ and $\bar{x}_3^{(\gamma)}$, respectively. It is noted that for full generality, $\phi_i^{(\beta\gamma)}$ and $\psi_i^{(\beta\gamma)}$ are dependent on x_1. In Eq. (4.1) and in all that follows, repeated β or γ do not imply summation. It is noted that because of the linearity of Eq. (4.1), the equations of equilibrium of the material within subcell $(\beta\gamma)$ are satisfied.

The microvariables can be related to the subcell strain components through the standard strain-displacement relations:

$$\varepsilon_{ij}^{(\beta\gamma)} = \frac{1}{2}\left[\partial_j u_i^{(\beta\gamma)} + \partial_i u_j^{(\beta\gamma)}\right] \qquad i,j = 1,\ 2,\ 3 \tag{4.2}$$

where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial\bar{x}_2^{(\beta)}$ and $\partial_3 = \partial/\partial\bar{x}_3^{(\gamma)}$

Key Point

In Figure 4.1, a continuously reinforced composite material is idealized as a doubly periodic array of fibers embedded in a matrix. The RUC and its four subvolumes, or 'subcells,' are identified whose centroids represent one fiber point and three matrix points, respectively. The rectangular graphical representation of the subcells indicates the region of influence of its centroid, not the actual fiber or matrix shape. It should be emphasized that because of the MOC enforcement of continuity, no influence of corners (i.e., stress risers) is present. Therefore the modeled fiber has no real shape and is more appropriately considered a pseudo-circle (or pseudo-ellipse if the fiber subcell is represented as rectangular).

4.1.1.2 Displacement Interfacial Conditions

Continuity of interfacial displacements between the four subcells interior to the unit cell requires that:

$$u_i^{(1\gamma)}\Big|_{\bar{x}_2^{(1)} = \frac{-h_1}{2}} = u_i^{(2\gamma)}\Big|_{\bar{x}_2^{(2)} = \frac{h_2}{2}} \tag{4.3}$$

and

$$u_i^{(\beta 1)}\bigg|_{\bar{x}_3^{(1)}=\frac{l_1}{2}} = u_i^{(\beta 2)}\bigg|_{\bar{x}_3^{(2)}=\frac{-l_2}{2}} \tag{4.4}$$

These continuity conditions are applied in an average sense; the integrals of the displacement components along the boundary are required to be continuous. Thus:

$$\int_{\frac{-l_\gamma}{2}}^{\frac{l_\gamma}{2}} \left(u_i^{(1\gamma)}\bigg|_{\bar{x}_2^{(1)}=\frac{-h_1}{2}} - u_i^{(2\gamma)}\bigg|_{\bar{x}_2^{(2)}=\frac{h_2}{2}} \right) \mathrm{d}\bar{x}_3^{(\gamma)} = 0 \tag{4.5}$$

and

$$\int_{\frac{-h_\beta}{2}}^{\frac{h_\beta}{2}} \left(u_i^{(\beta 1)}\bigg|_{\bar{x}_3^{(1)}=\frac{l_1}{2}} - u_i^{(\beta 2)}\bigg|_{\bar{x}_3^{(2)}=\frac{-l_2}{2}} \right) \mathrm{d}\bar{x}_2^{(\beta)} = 0 \tag{4.6}$$

Substituting for $u_i^{(\beta\gamma)}$ given in Eq. (4.1), inserting it into Eqs. (4.5) and (4.6), and performing the integration yields:

$$w_i^{(1\gamma)} - \frac{h_1}{2}\,\phi_i^{(1\gamma)} = w_i^{(2\gamma)} + \frac{h_2}{2}\,\phi_i^{(2\gamma)} \tag{4.7}$$

and

$$w_i^{(\beta 1)} + \frac{l_1}{2}\,\psi_i^{(\beta 1)} = w_i^{(\beta 2)} - \frac{l_2}{2}\,\psi_i^{(\beta 2)} \tag{4.8}$$

Continuity between adjacent unit cells is also required. Considering first the x_2-direction yields:

$$\left[u_i^{(1\gamma)}\bigg|_{\bar{x}_2^{(1)}=\frac{h_1}{2}} \right]_{\text{below}} = u_i^{(2\gamma)}\bigg|_{\bar{x}_2^{(2)}=\frac{-h_2}{2}} \tag{4.9}$$

and

$$u_i^{(1\gamma)}\Big|_{\bar{x}_2^{(1)}=\frac{h_1}{2}} = \left[u_i^{(2\gamma)}\Big|_{\bar{x}_2^{(2)}=\frac{-h_2}{2}}\right]_{\text{above}} \tag{4.10}$$

where 'above' and 'below' refer to the adjacent unit cells. Applying these conditions in an average sense, substituting with Eq. (4.1), and integrating yields:

$$\left[w_i^{(1\gamma)} + \frac{h_1}{2}\phi_i^{(1\gamma)}\right]_{\text{below}} = w_i^{(2\gamma)} - \frac{h_2}{2}\phi_i^{(2\gamma)} \tag{4.11}$$

and

$$w_i^{(1\gamma)} + \frac{h_1}{2}\phi_i^{(1\gamma)} = \left[w_i^{(2\gamma)} - \frac{h_2}{2}\phi_i^{(2\gamma)}\right]_{\text{above}} \tag{4.12}$$

The quantities from the unit cells above and below are represented using a Taylor series expansion of the form:

$$f(x_0+\Delta x) = f(x_0) + \frac{\partial f}{\partial x}\Big|_{x_0}\Delta x + \frac{\partial^2 f}{\partial x^2}\Big|_{x_0}(\Delta x)^2 + \ldots \tag{4.13}$$

On the left-hand side of Eq. (4.11), $w_i^{(1\gamma)}$ is a function of $\mathbf{x}$, and the term $\phi_i^{(1\gamma)}h_1/2$ is a constant. From Eq. (4.13):

$$\left[w_i^{(1\gamma)}\right]_{\text{below}} = w_i^{(1\gamma)} - (h_1+h_2)\frac{\partial w_i^{(1\gamma)}}{\partial x_2} + (h_1+h_2)^2\frac{\partial^2 w_i^{(1\gamma)}}{\partial x_2^2} + \ldots \tag{4.14}$$

Retaining only terms up to the first order in the subcell dimension h_β in Eq. (4.14), and substituting into Eq. (4.11) yields:

$$w_i^{(1\gamma)} - (h_1+h_2)\frac{\partial w_i^{(1\gamma)}}{\partial x_2} + \frac{h_1}{2}\phi_i^{(1\gamma)} = w_i^{(2\gamma)} - \frac{h_2}{2}\phi_i^{(2\gamma)} \tag{4.15}$$

Similarly, Eq. (4.12) becomes:

$$w_i^{(1\gamma)} + \frac{h_1}{2}\phi_i^{(1\gamma)} = w_i^{(2\gamma)} + (h_1+h_2)\frac{\partial w_i^{(2\gamma)}}{\partial x_2} - \frac{h_2}{2}\phi_i^{(2\gamma)} \tag{4.16}$$

Subtracting Eq. (4.16) from Eq. (4.15) yields:

$$\frac{\partial w_i^{(1\gamma)}}{\partial x_2} = \frac{\partial w_i^{(2\gamma)}}{\partial x_2} \tag{4.17}$$

and subtracting Eq. (4.7) from Eq. (4.16) yields:

$$h_1 \phi_i^{(1\gamma)} + h_2 \phi_i^{(2\gamma)} = (h_1 + h_2) \frac{\partial w_i^{(2\gamma)}}{\partial x_2} \tag{4.18}$$

A similar application of displacement continuity between adjacent unit cells in the x_3-direction yields:

$$\frac{\partial w_i^{(\beta 1)}}{\partial x_3} = \frac{\partial w_i^{(\beta 2)}}{\partial x_3} \tag{4.19}$$

and

$$l_1 \psi_1^{(\beta 1)} + l_2 \psi_1^{(\beta 2)} = (l_1 + l_2) \frac{\partial w_i^{(\beta 2)}}{\partial x_3} \tag{4.20}$$

Requiring the longitudinal strain $\varepsilon_{11}^{(\beta\gamma)}$ to be uniform in the cell and using (4.2) yields:

$$\frac{\partial w_1^{(11)}}{\partial x_1} = \frac{\partial w_1^{(12)}}{\partial x_1} = \frac{\partial w_1^{(21)}}{\partial x_1} = \frac{\partial w_1^{(22)}}{\partial x_1} \tag{4.21}$$

The conditions (4.17), (4.19), and (4.21) then require that:

$$\frac{\partial w_i^{(\beta\gamma)}}{\partial x_j} = \frac{\partial w_i}{\partial x_j} \tag{4.22}$$

From Eqs. (4.1), (4.2), and (4.22) it is clear that the subcell total strain components are independent of $\bar{x}_2^{(\beta)}$ and $\bar{x}_3^{(\gamma)}$ and are thus constant within a given subcell. Given that the CTEs are constant within a given subcell, the thermal strain components are constant within a given subcell as well. The fact that the subcell total strain and thermal strain components are constant within a given subcell requires the subcell stress components to be constant within a given subcell, provided the subcell is deforming elastically. This then requires the inelastic strain components to evolve such that they remain constant in a given subcell as well. These arguments show that the average fields and pointwise fields within the subcells are identical. That is:

$$\bar{\sigma}_{ij}^{(\beta\gamma)} = \sigma_{ij}^{(\beta\gamma)}, \quad \bar{\varepsilon}_{ij}^{(\beta\gamma)} = \varepsilon_{ij}^{(\beta\gamma)}, \quad \bar{\varepsilon}_{ij}^{T(\beta\gamma)} = \varepsilon_{ij}^{T(\beta\gamma)}, \quad \bar{\varepsilon}_{ij}^{I(\beta\gamma)} = \varepsilon_{ij}^{I(\beta\gamma)} \tag{4.23}$$

where for each subcell $\beta\gamma$, $\bar{\sigma}_{ij}^{(\beta\gamma)}$ and $\sigma_{ij}^{(\beta\gamma)}$ are the average and pointwise stresses, respectively; $\bar{\varepsilon}_{ij}^{(\beta\gamma)}$ and $\varepsilon_{ij}^{(\beta\gamma)}$ are the total average and pointwise strains, respectively; $\bar{\varepsilon}_{ij}^{T(\beta\gamma)}$ and $\varepsilon_{ij}^{T(\beta\gamma)}$ are the thermal average and pointwise strains, respectively; and $\bar{\varepsilon}_{ij}^{I(\beta\gamma)}$ and $\varepsilon_{ij}^{I(\beta\gamma)}$ are the inelastic average and pointwise strains, respectively. For a continuously reinforced composite (see Figure 4.1), the strain field is independent of x_1. Therefore, utilizing Eqs. (4.1) and (4.2) along with Eq. (4.22), the subcell strain field is given by:

$$\begin{aligned}
\varepsilon_{11}^{(\beta\gamma)} &= \frac{\partial}{\partial x_1} w_1 \\
\varepsilon_{22}^{(\beta\gamma)} &= \phi_2^{(\beta\gamma)} \\
\varepsilon_{33}^{(\beta\gamma)} &= \psi_3^{(\beta\gamma)} \\
2\varepsilon_{12}^{(\beta\gamma)} &= \phi_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1} w_2 \\
2\varepsilon_{13}^{(\beta\gamma)} &= \psi_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1} w_3 \\
2\varepsilon_{23}^{(\beta\gamma)} &= \phi_3^{(\beta\gamma)} + \psi_2^{(\beta\gamma)}
\end{aligned} \tag{4.24}$$

The volume-averaged strain components in the composite are given by:

$$\bar{\varepsilon}_{ij} = \frac{1}{V} \sum_{\beta,\gamma=1}^{2} h_\beta \; l_\gamma \;\; \varepsilon_{ij}^{(\beta\gamma)} \tag{4.25}$$

where $V = (h_1 + h_2)\,(l_1 + l_2)$. Substituting using (4.2), (4.18), (4.20), and (4.22) yields:

$$\bar{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \tag{4.26}$$

This expression relates the gradients of the subcell centroid displacements to average cell strains. It will be used to relate the subcell strains to the average cell strains.

The subcell constitutive equations are used to relate the subcell stresses and strains. For transversely isotropic constituents wherein x_2—x_3 defines the plane of isotropy, they are given by:

$$\begin{bmatrix} \sigma_{11}^{(\beta\gamma)} \\ \sigma_{22}^{(\beta\gamma)} \\ \sigma_{33}^{(\beta\gamma)} \\ \sigma_{23}^{(\beta\gamma)} \\ \sigma_{13}^{(\beta\gamma)} \\ \sigma_{12}^{(\beta\gamma)} \end{bmatrix} = \begin{bmatrix} C_{11}^{(\beta\gamma)} & C_{12}^{(\beta\gamma)} & C_{12}^{(\beta\gamma)} & 0 & 0 & 0 \\ C_{12}^{(\beta\gamma)} & C_{22}^{(\beta\gamma)} & C_{23}^{(\beta\gamma)} & 0 & 0 & 0 \\ C_{12}^{(\beta\gamma)} & C_{23}^{(\beta\gamma)} & C_{22}^{(\beta\gamma)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(\beta\gamma)} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66}^{(\beta\gamma)} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{(\beta\gamma)} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^{(\beta\gamma)} - \varepsilon_{11}^{I(\beta\gamma)} - \varepsilon_{11}^{T(\beta\gamma)} \\ \varepsilon_{22}^{(\beta\gamma)} - \varepsilon_{22}^{I(\beta\gamma)} - \varepsilon_{22}^{T(\beta\gamma)} \\ \varepsilon_{33}^{(\beta\gamma)} - \varepsilon_{33}^{I(\beta\gamma)} - \varepsilon_{33}^{T(\beta\gamma)} \\ 2\left(\varepsilon_{23}^{(\beta\gamma)} - \varepsilon_{23}^{I(\beta\gamma)} - \varepsilon_{23}^{T(\beta\gamma)}\right) \\ 2\left(\varepsilon_{13}^{(\beta\gamma)} - \varepsilon_{13}^{I(\beta\gamma)} - \varepsilon_{13}^{T(\beta\gamma)}\right) \\ 2\left(\varepsilon_{12}^{(\beta\gamma)} - \varepsilon_{12}^{I(\beta\gamma)} - \varepsilon_{12}^{T(\beta\gamma)}\right) \end{bmatrix} \tag{4.27}$$

where $C_{ij}^{(\beta\gamma)}$ are the components of the stiffness matrix $\mathbf{C}^{(\beta\gamma)}$, $C_{44}^{(\beta\gamma)} = \frac{1}{2}(C_{22}^{(\beta\gamma)} - C_{23}^{(\beta\gamma)})$, $\varepsilon_{ij}^{I(\beta\gamma)}$ are the subcell inelastic strains, and $\varepsilon_{ij}^{T(\beta\gamma)}$ are the subcell thermal strains. Note that the shear strains in this equation are tensorial quantities.

The volume-averaged stress components in the composite are given by:

$$\overline{\sigma}_{ij} = \frac{1}{V} \sum_{\beta,\gamma=1}^{2} h_\beta \, l_\gamma \, \sigma_{ij}^{(\beta\gamma)} \tag{4.28}$$

where $\sigma_{ij}^{(\beta\gamma)}$ are the average subcell stresses, since as stated in Eq. (4.23):

$$\overline{\sigma}_{ij}^{(\beta\gamma)} = \sigma_{ij}^{(\beta\gamma)} \tag{4.29}$$

Using Eq. (4.24), the subcell constitutive equation, Eq. (4.27), can be written as:

$$\begin{aligned}
\sigma_{11}^{(\beta\gamma)} &= C_{11}^{(\beta\gamma)}\overline{\varepsilon}_{11} + C_{12}^{(\beta\gamma)}\left(\phi_2^{(\beta\gamma)} + \psi_3^{(\beta\gamma)}\right) - \Gamma_1^{(\beta\gamma)}\Delta T - \left(C_{11}^{(\beta\gamma)} - C_{12}^{(\beta\gamma)}\right)\varepsilon_{11}^{I\ (\beta\gamma)} \\
\sigma_{22}^{(\beta\gamma)} &= C_{12}^{(\beta\gamma)}\overline{\varepsilon}_{11} + C_{22}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + C_{23}^{(\beta\gamma)}\psi_3^{(\beta\gamma)} - \Gamma_2^{(\beta\gamma)}\Delta T - \left(C_{11}^{(\beta\gamma)} - C_{12}^{(\beta\gamma)}\right)\varepsilon_{22}^{I\ (\beta\gamma)} \\
\sigma_{33}^{(\beta\gamma)} &= C_{12}^{(\beta\gamma)}\overline{\varepsilon}_{11} + C_{23}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + C_{22}^{(\beta\gamma)}\psi_3^{(\beta\gamma)} - \Gamma_3^{(\beta\gamma)}\Delta T - \left(C_{11}^{(\beta\gamma)} - C_{12}^{(\beta\gamma)}\right)\varepsilon_{33}^{I\ (\beta\gamma)} \\
\sigma_{23}^{(\beta\gamma)} &= C_{44}^{(\beta\gamma)}\left(\psi_2^{(\beta\gamma)} + \phi_3^{(\beta\gamma)}\right) - 2C_{44}^{(\beta\gamma)}\varepsilon_{23}^{I\ (\beta\gamma)} \\
\sigma_{13}^{(\beta\gamma)} &= C_{66}^{(\beta\gamma)}\left(\psi_1^{(\beta\gamma)} + \frac{\partial w_3}{\partial x_1}\right) - 2C_{66}^{(\beta\gamma)}\varepsilon_{13}^{I\ (\beta\gamma)} \\
\sigma_{12}^{(\beta\gamma)} &= C_{66}^{(\beta\gamma)}\left(\phi_1^{(\beta\gamma)} + \frac{\partial w_2}{\partial x_1}\right) - 2C_{66}^{(\beta\gamma)}\varepsilon_{12}^{I\ (\beta\gamma)}
\end{aligned} \tag{4.30}$$

where ΔT is the temperature change from a reference temperature and

$$\Gamma_1^{(\beta\gamma)} = C_{11}^{(\beta\gamma)}\alpha_1^{(\beta\gamma)} + 2C_{12}^{(\beta\gamma)}\alpha_2^{(\beta\gamma)} \tag{4.31}$$

$$\Gamma_2^{(\beta\gamma)} = \Gamma_3^{(\beta\gamma)} = C_{12}^{(\beta\gamma)}\alpha_1^{(\beta\gamma)} + \left(C_{22}^{(\beta\gamma)} + C_{23}^{(\beta\gamma)}\right)\ \alpha_2^{(\beta\gamma)} \tag{4.32}$$

and $\alpha_i^{(\beta\gamma)}$ are the subcell CTEs.

It should be noted that the coefficients of the inelastic terms have been specialized to an isotropic material ($C_{22} = C_{11}$, $C_{23} = C_{12}$), and the inelastic strains have been assumed to be deviatoric. This deviatoric condition requires $\varepsilon_{11}^{I\ (\beta\gamma)} + \varepsilon_{22}^{I\ (\beta\gamma)} + \varepsilon_{33}^{I\ (\beta\gamma)} = 0$, which has been employed in writing Eq. (4.30). It should also be noted that in the isotropic case, $C_{11} - C_{12} = 2\mu$, where μ is the shear modulus.

As the last step in generating the effective constitutive equations, continuity of tractions is applied along the subcell and unit cell interfaces in an average sense, yielding:

$$\sigma_{2i}^{(1\gamma)} = \sigma_{2i}^{(2\gamma)} \tag{4.33}$$

$$\sigma_{3i}^{(\beta 1)} = \sigma_{3i}^{(\beta 2)} \tag{4.34}$$

At this point, there is a sufficient number of equations to solve for the microvariables to obtain the overall composite stress-strain relations. From Eq. (4.18) with $i = 2$:

$$\phi_2^{(12)} = \frac{\left(h\,\bar{\varepsilon}_{22} - h_2\,\phi_2^{(22)}\right)}{h_1} \tag{4.35}$$

$$\phi_2^{(21)} = \frac{\left(h\,\bar{\varepsilon}_{22} - h_1\,\phi_2^{(11)}\right)}{h_2} \tag{4.36}$$

where $h = h_1 + h_2$. Similarly, from Eq. (4.20) with $i = 3$:

$$\psi_3^{(12)} = \frac{\left(l\,\bar{\varepsilon}_{33} - l_1\,\psi_3^{(11)}\right)}{l_2} \tag{4.37}$$

$$\psi_3^{(21)} = \frac{\left(l\,\bar{\varepsilon}_{33} - l_2\,\psi_3^{(22)}\right)}{l_1} \tag{4.38}$$

where $l = l_1 + l_2$. From Eq. (4.33) with $i = 2$, using Eqs. (4.30) and (4.35) to (4.38):

$$C_{22}^{(m)}\left(1+\frac{h_2}{h_1}\right)\phi_2^{(22)} + C_{23}^{(m)}\frac{l_1}{l_2}\psi_3^{(11)} + C_{23}^{(m)}\psi_3^{(22)} = C_{22}^{(m)}\frac{h}{h_1}\bar{\varepsilon}_{22} + C_{23}^{(m)}\frac{l}{l_2}\bar{\varepsilon}_{33} - 2\mu_m\left(\varepsilon_{22}^{I\ (12)} - \varepsilon_{22}^{I\ (22)}\right) \tag{4.39}$$

$$\left(C_{22}^{(f)} + \frac{h_1}{h_2}C_{22}^{(m)}\right)\phi_2^{(11)} + C_{23}^{(f)}\psi_3^{(11)} + C_{23}^{(m)}\frac{l_2}{l_1}\psi_3^{(22)} = \left(C_{12}^{(m)} - C_{12}^{(f)}\right)\bar{\varepsilon}_{11} + C_{22}^{(m)}\frac{h}{h_2}\bar{\varepsilon}_{22} + C_{23}^{(m)}\frac{l}{l_1}\bar{\varepsilon}_{33} - \left(\Gamma_2^{(m)} - \Gamma_2^{(f)}\right)\Delta T - 2\mu_m\,\varepsilon_{22}^{I\ (21)} + 2\mu_f\,\varepsilon_{22}^{I\ (11)} \tag{4.40}$$

where the (11) subcell is occupied by the fiber (f) while the remaining subcells are occupied by matrix (m) allows the material stiffness components to be written with (f) and (m) superscripts. Note also that μ_m and μ_f are the shear moduli of the matrix and fiber, respectively. From Eq. (4.34) with $i = 3$, using Eqs. (4.30) and (4.35) to (4.38):

$$C_{23}^{(f)}\phi_2^{(11)} + C_{23}^{(m)}\frac{h_2}{h_1}\phi_2^{(22)} + \left(C_{22}^{(f)} + C_{22}^{(m)}\frac{l_1}{l_2}\right)\psi_3^{(11)} = \left(C_{12}^{(m)} - C_{12}^{(f)}\right)\bar{\varepsilon}_{11} + C_{23}^{(m)}\frac{h}{h_1}\bar{\varepsilon}_{22} + C_{22}^{(m)}\frac{l}{l_2}\bar{\varepsilon}_{33} - \left(\Gamma_2^{(m)} - \Gamma_2^{(f)}\right)\Delta T - 2\mu_m\,\varepsilon_{33}^{I\ (12)} + 2\mu_f\,\varepsilon_{33}^{I\ (11)} \tag{4.41}$$

$$C_{23}^{(m)}\frac{h_1}{h_2}\phi_2^{(11)} + C_{23}^{(m)}\phi_2^{(22)} + C_{22}^{(m)}\left(1+\frac{l_2}{l_1}\right)\psi_3^{(22)} = C_{23}^{(m)}\frac{h}{h_2}\bar{\varepsilon}_{22} + C_{22}^{(m)}\frac{l}{l_1}\bar{\varepsilon}_{33} - 2\mu_m\left(\varepsilon_{33}^{I\ (21)} - \varepsilon_{33}^{I\ (22)}\right) \tag{4.42}$$

Equations (4.39) to (4.42) form a set of four simultaneous linear algebraic equations in terms of four unknown microvariables that can be written:

$$\begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ a_4 & 0 & a_5 & a_6 \\ a_7 & a_8 & a_9 & 0 \\ a_{10} & a_{11} & 0 & a_{12} \end{bmatrix} \begin{bmatrix} \phi_2^{(11)} \\ \phi_2^{(22)} \\ \psi_3^{(11)} \\ \psi_3^{(22)} \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix} \tag{4.43}$$

Equation (4.43) can then be inverted to solve for the microvariables:

$$\begin{bmatrix} \phi_2^{(11)} \\ \phi_2^{(22)} \\ \psi_3^{(11)} \\ \psi_3^{(22)} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \\ T_5 & T_6 & T_7 & T_8 \\ T_9 & T_{10} & T_{11} & T_{12} \\ T_{13} & T_{14} & T_{15} & T_{16} \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{bmatrix} \tag{4.44}$$

where the explicit expressions for $T_1,\ldots,T_{16}$ are given by:

$$\begin{aligned}
T_1 &= -(a_5a_8a_{12} + a_6a_9a_{11})/D \\
T_2 &= [a_2a_8a_{12} + a_3a_9a_{11} - a_1a_9a_{12}]/D \\
T_3 &= [a_1a_5a_{12} + a_2a_6a_{11} - a_3a_5a_{11}]/D \\
T_4 &= [a_1a_6a_9 + a_8(a_3a_5 - a_2a_6)]/D \\
T_5 &= [a_6a_9a_{10} + a_{12}(a_5a_7 - a_4a_9)]/D \\
T_6 &= -(a_2a_7a_{12} + a_3a_9a_{10})/D \\
T_7 &= [a_3a_5a_{10} + a_2(a_4a_{12} - a_6a_{10})]/D \\
T_8 &= [a_2a_6a_7 + a_3(a_4a_9 - a_5a_7)]/D \\
T_9 &= [a_4a_8a_{12} + a_6(a_7a_{11} - a_8a_{10})]/D \\
T_{10} &= [a_1a_7a_{12} + a_3(a_8a_{10} - a_7a_{11})]/D \\
T_{11} &= [a_3a_4a_{11} + a_1(a_6a_{10} - a_4a_{12})]/D \\
T_{12} &= -(a_1a_6a_7 + a_3a_4a_8)/D \\
T_{13} &= [a_4a_9a_{11} + a_5(a_8a_{10} - a_7a_{11})]/D \\
T_{14} &= [a_1a_9a_{10} + a_2(a_7a_{11} - a_8a_{10})]/D \\
T_{15} &= -(a_1a_5a_{10} + a_2a_4a_{11})/D \\
T_{16} &= [a_1(a_5a_7 - a_4a_9) + a_2a_4a_8]/D
\end{aligned}$$

and

$$\begin{aligned}
D = a_1&[a_{12}(a_5a_7 - a_4a_9) + a_6a_9a_{10}] \\
&+a_2[a_4a_8a_{12} + a_6(a_7a_{11} - a_8a_{10})\,] \\
&+a_3[a_4a_9a_{11} + a_5(a_8a_{10} - a_7a_{11})\,]
\end{aligned} \tag{4.45}$$

and the coefficients a_i ($i = 1,\ldots,12$) and J_i ($i = 1,\ldots,4$), with f and m denoting fiber and matrix quantities, are given as follows:

$$
\begin{aligned}
&a_1 = C_{22}^{(m)}\left(1+\frac{h_2}{h_1}\right), \quad a_2 = C_{23}^{(m)}\,\frac{l_1}{l_2} \\
&a_3 = C_{23}^{(m)}, \quad a_4 = C_{22}^{(m)}\frac{h_1}{h_2} + C_{22}^{(f)} \\
&a_5 = C_{23}^{(f)}, \quad a_6 = C_{23}^{(m)}\,\frac{l_2}{l_1}, \quad a_7 = C_{23}^{(f)} \\
&a_8 = C_{23}^{(m)}\,\frac{h_2}{h_1}, \quad a_9 = C_{22}^{(m)}\,\frac{l_1}{l_2} + C_{22}^{(f)} \\
&a_{10} = C_{23}^{(m)}\,\frac{h_1}{h_2}, \quad a_{11} = a_3, \quad a_{12} = C_{22}^{(m)}\left(1+\frac{l_2}{l_1}\right) \\
&J_1 = C_{22}^{(m)}\,\frac{h}{h_1}\bar{\varepsilon}_{22} + C_{23}^{(m)}\,\frac{l}{l_2}\bar{\varepsilon}_{33} - 2\mu_m\,\left(\varepsilon_{22}^{I\;(12)} - \varepsilon_{22}^{I\;(22)}\right) \\
&J_2 = \left(C_{12}^{(m)} - C_{12}^{(f)}\right)\;\bar{\varepsilon}_{11} + C_{22}^{(m)}\frac{h}{h_2}\bar{\varepsilon}_{22} + C_{23}^{(m)}\frac{l}{l_1}\bar{\varepsilon}_{33} \\
&\qquad -\left(\Gamma_2^{(m)} - \Gamma_2^{(f)}\right)\,\Delta T - 2\mu_m\,\varepsilon_{22}^{I\;(21)} + 2\mu_f\,\varepsilon_{22}^{I\;(11)} \\
&J_3 = \left(C_{12}^{(m)} - C_{12}^{(f)}\right)\;\bar{\varepsilon}_{11} + C_{23}^{(m)}\,\frac{h}{h_1}\bar{\varepsilon}_{22} + C_{22}^{(m)}\,\frac{l}{l_2}\bar{\varepsilon}_{33} \\
&\qquad -\left(\Gamma_2^{(m)} - \Gamma_2^{(f)}\right)\,\Delta T - 2\mu_m\,\varepsilon_{33}^{I\;(12)} + 2\mu_f\,\varepsilon_{33}^{I\;(11)} \\
&J_4 = C_{23}^{(m)}\,\frac{h}{h_2}\bar{\varepsilon}_{22} + C_{22}^{(m)}\,\frac{l}{l_1}\bar{\varepsilon}_{33} - 2\mu_m\,\left(\varepsilon_{33}^{I\;(21)} - \varepsilon_{33}^{I\;(22)}\right)
\end{aligned}
\tag{4.46}
$$

The remaining microvariables involved in the normal stress-strain relations can be determined using Eqs. (4.35) to (4.38). Then using Eqs. (4.28) and (4.30), the composite normal constitutive relations are obtained in the form:

$$
\begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} \left\{ \begin{bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ \bar{\varepsilon}_{33} \end{bmatrix} - \begin{bmatrix} \bar{\varepsilon}_{11}^I \\ \bar{\varepsilon}_{22}^I \\ \bar{\varepsilon}_{33}^I \end{bmatrix} - \begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \end{bmatrix} \Delta T \right\}
\tag{4.47}
$$

where b_{ij} are components of the effective stiffness matrix, $\bar{\varepsilon}_{ij}^I$ are the effective inelastic strains, and α_i^* are the effective composite CTEs. The effective CTEs that arise from the MOC are identical to those predicted by Levin's formula (Aboudi, 1991); see also Chapter 3. The components b_{ij} (multiplied by the known unit cell volume, $V = hl$) are given by:

$$
\begin{aligned}
V\, b_{11} &= \mathrm{v}_{11} C_{11}^{(f)} + C_{11}^{(m)} (\mathrm{v}_{12} + \mathrm{v}_{21} + \mathrm{v}_{22}) + \left(C_{12}^{(m)} - C_{12}^{(f)} \right) \; (Q_2 + Q_3) \\
V\, b_{12} &= \frac{h}{h_1} \left(C_{12}^{(m)} \mathrm{v}_{12} + Q_1 C_{22}^{(m)} + Q_3 C_{23}^{(m)} \right) + \frac{h}{h_2} \left(C_{12}^{(m)} \mathrm{v}_{21} + Q_2 C_{22}^{(m)} + Q_4 C_{23}^{(m)} \right) \\
V\, b_{13} &= \frac{l}{l_1} \left(C_{12}^{(m)} \mathrm{v}_{21} + Q_2 C_{23}^{(m)} + Q_4 C_{22}^{(m)} \right) + \frac{l}{l_2} \left(C_{12}^{(m)} \mathrm{v}_{12} + Q_1 C_{23}^{(m)} + Q_3 C_{22}^{(m)} \right) \\
V\, b_{22} &= \frac{h}{h_1} \left[C_{22}^{(m)} (\mathrm{v}_{12} + Q_1') + Q_3' C_{23}^{(m)} \right] + \frac{h}{h_2} \left[C_{22}^{(m)} (\mathrm{v}_{21} + Q_2') + Q_4' C_{23}^{(m)} \right] \\
V\, b_{23} &= \frac{l}{l_1} \left[C_{23}^{(m)} (\mathrm{v}_{21} + Q_2') + Q_4' C_{22}^{(m)} \right] + \frac{l}{l_2} \left[C_{23}^{(m)} (\mathrm{v}_{12} + Q_1') + Q_3' C_{22}^{(m)} \right] \\
V\, b_{33} &= \frac{l}{l_1} \left[C_{22}^{(m)} (\mathrm{v}_{21} + Q_4'') + Q_2'' C_{23}^{(m)} \right] + \frac{l}{l_2} \left[C_{22}^{(m)} (\mathrm{v}_{12} + Q_3'') + Q_1'' C_{23}^{(m)} \right]
\end{aligned}
\tag{4.48}
$$

whereas before, $C_{ij}^{(f)} = C_{ij}^{(11)}$ and $C_{ij}^{(m)} = C_{ij}^{(\beta\gamma)} (\beta + \gamma \neq 2)$ are the elastic stiffness matrix components of the fiber and matrix phases, both of which are assumed to be either transversely isotropic elastic materials with x_1 being the direction of anisotropy, or inelastic isotropic materials. In addition the effective thermal stresses are given by:

$$
\begin{aligned}
V\, \Gamma_1^* &= (\Gamma_2^{(m)} - \Gamma_2^{(f)}) \; (Q_2 + Q_3) + \mathrm{v}_{11} \Gamma_1^{(f)} + (\mathrm{v}_{12} + \mathrm{v}_{21} + \mathrm{v}_{22}) \; \Gamma_1^{(m)} \\
V\, \Gamma_2^* &= (\Gamma_2^{(m)} - \Gamma_2^{(f)}) \; (Q_2' + Q_3') + \mathrm{v}_{11} \Gamma_2^{(f)} + (\mathrm{v}_{12} + \mathrm{v}_{21} + \mathrm{v}_{22}) \; \Gamma_2^{(m)} \\
V\, \Gamma_3^* &= (\Gamma_2^{(m)} - \Gamma_2^{(f)}) \; (Q_2'' + Q_3'') + \mathrm{v}_{11} \Gamma_2^{(f)} + (\mathrm{v}_{12} + \mathrm{v}_{21} + \mathrm{v}_{22}) \; \Gamma_2^{(m)}
\end{aligned}
\tag{4.49}
$$

from which the effective CTEs are given by:

$$
\begin{bmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_1^* \\ \Gamma_2^* \\ \Gamma_3^* \end{bmatrix}
\tag{4.50}
$$

The effective inelastic strains are given by:

$$
\begin{aligned}
V\,\bar{\varepsilon}^{I}_{11} &= 2Q_1\mu_m\left(\varepsilon^{I(12)}_{22}-\varepsilon^{I(22)}_{22}\right)+2Q_2\left(\mu_m\varepsilon^{I(21)}_{22}-\mu_f\varepsilon^{I(11)}_{22}\right)+2Q_3\left(\mu_m\varepsilon^{I(12)}_{33}-\mu_f\varepsilon^{I(11)}_{33}\right)\\
&\quad+2Q_4\mu_m\left(\varepsilon^{I(21)}_{33}-\varepsilon^{I(22)}_{33}\right)+2\left[\mu_f \mathrm{v}_{11}\varepsilon^{I(11)}_{11}+\mu_m\left(\mathrm{v}_{12}\varepsilon^{I(12)}_{11}+\mathrm{v}_{21}\varepsilon^{I(21)}_{11}+\mathrm{v}_{22}\varepsilon^{I(22)}_{11}\right)\right]\\
V\,\bar{\varepsilon}^{I}_{22} &= 2Q'_1\mu_m\left(\varepsilon^{I(12)}_{22}-\varepsilon^{I(22)}_{22}\right)+2Q'_2\left(\mu_m\varepsilon^{I(21)}_{22}-\mu_f\varepsilon^{I(11)}_{22}\right)+2Q'_3\left(\mu_m\varepsilon^{I(12)}_{33}-\mu_f\varepsilon^{I(11)}_{33}\right)\\
&\quad+2Q'_4\mu_m\left(\varepsilon^{I(21)}_{33}-\varepsilon^{I(22)}_{33}\right)+2\left[\mu_f \mathrm{v}_{11}\varepsilon^{I(11)}_{22}+\mu_m\left(\mathrm{v}_{12}\varepsilon^{I(12)}_{22}+\mathrm{v}_{21}\varepsilon^{I(21)}_{22}+\mathrm{v}_{22}\varepsilon^{I(22)}_{22}\right)\right]\\
V\,\bar{\varepsilon}^{I}_{33} &= 2Q''_1\mu_m\left(\varepsilon^{I(12)}_{22}-\varepsilon^{I(22)}_{22}\right)+2Q''_2\left(\mu_m\varepsilon^{I(21)}_{22}-\mu_f\varepsilon^{I(11)}_{22}\right)+2Q''_3\left(\mu_m\varepsilon^{I(12)}_{33}-\mu_f\varepsilon^{I(11)}_{33}\right)\\
&\quad+2Q''_4\mu_m\left(\varepsilon^{I(21)}_{33}-\varepsilon^{I(22)}_{33}\right)+2\left[\mu_f \mathrm{v}_{11}\varepsilon^{I(11)}_{33}+\mu_m\left(\mathrm{v}_{12}\varepsilon^{I(12)}_{33}+\mathrm{v}_{21}\varepsilon^{I(21)}_{33}+\mathrm{v}_{22}\varepsilon^{I(22)}_{33}\right)\right]
\end{aligned}
\tag{4.51}
$$

Finally, the coefficients $Q_i,\ Q'_i,$ and $Q''_i\ \ (i=1,\ldots,\ 4)$ are given by:

$$
\begin{aligned}
Q_1 &= \mathrm{v}_{11}C^{(f)}_{12}(T_1+T_9)-\mathrm{v}_{12}C^{(m)}_{12}\left(\frac{h_2T_5}{h_1}+\frac{l_1T_9}{l_2}\right)\\
&\quad-\mathrm{v}_{21}C^{(m)}_{12}\left(\frac{h_1T_1}{h_2}+\frac{l_2T_{13}}{l_1}\right)+\mathrm{v}_{22}C^{(m)}_{12}(T_5+T_{13})
\end{aligned}
\tag{4.52}
$$

Q_2 is obtained from Q_1 by replacing T_i in the latter by T_{i+1} (e.g., T_1 is replaced by T_2, T_9 by T_{10}, etc.). Similarly, Q_3 and Q_4 are obtained from Q_1 by replacing T_i in the latter by T_{i+2} and T_{i+3} respectively:

$$
\begin{aligned}
Q'_1 &= \mathrm{v}_{11}\left(C^{(f)}_{22}T_1+C^{(f)}_{23}T_9\right)-\mathrm{v}_{12}\left(\frac{C^{(m)}_{22}h_2T_5}{h_1}+\frac{C^{(m)}_{23}l_1T_9}{l_2}\right)\\
&\quad-\mathrm{v}_{21}\left(\frac{C^{(m)}_{22}h_1T_1}{h_2}+\frac{C^{(m)}_{23}l_2T_{13}}{l_1}\right)+\mathrm{v}_{22}\left(C^{(m)}_{22}T_5+C^{(m)}_{23}T_{13}\right)
\end{aligned}
\tag{4.53}
$$

and Q'_2, Q'_3, and Q'_4 are obtained from Q'_1 by replacing T_i in the latter by T_{i+1}, T_{i+2}, and T_{i+3}, respectively:

$$
\begin{aligned}
Q''_1 &= \mathrm{v}_{11}\left(C^{(f)}_{23}T_1+C^{(f)}_{22}T_9\right)-\mathrm{v}_{12}\left(C^{(m)}_{23}T_5\frac{h_2}{h_1}+C^{(m)}_{22}T_9\frac{l_1}{l_2}\right)\\
&\quad-\mathrm{v}_{21}\left(C^{(m)}_{23}T_1\frac{h_1}{h_2}+C^{(m)}_{22}T_{13}\frac{l_2}{l_1}\right)+\mathrm{v}_{22}\left(C^{(m)}_{23}T_5+C^{(m)}_{22}T_{13}\right)
\end{aligned}
\tag{4.54}
$$

and Q_2'', Q_3'', and Q_4'' are obtained from Q_1'' by replacing T_i in the latter by T_{i+1}, T_{i+2}, and T_{i+3}, respectively.

4.1.1.3 Average Transverse Shear Stress-Strain Relation

The previous sections derived the composite normal stress-strain relations (Eq. (4.47)). Here the composite transverse shear stress-strain relation is derived.

Equation (4.18) with $i = 3$ and Eq. (4.20) with $i = 2$ give:

$$h_1 \phi_3^{(1\gamma)} + h_2 \phi_3^{(2\gamma)} = h \frac{\partial w_3}{\partial x_2} \tag{4.55}$$

$$l_1 \psi_2^{(\beta 1)} + l_2 \psi_2^{(\beta 2)} = l \frac{\partial w_2}{\partial x_3} \tag{4.56}$$

where $h = h_1 + h_2$, $l = l_1 + l_2$, and Eq. (4.22) have been employed.

Multiplication of Eq. (4.55) with $\gamma = 1$ by l_1 and Eq. (4.56) with $\beta = 1$ by h_1 and adding them together provides:

$$l_1 h_1 N^{(11)} + h_2 l_1 \phi_3^{(21)} + h_1 l_2 \psi_2^{(12)} = M_1 \tag{4.57}$$

where in general,

$$N^{(\beta\gamma)} = \phi_3^{(\beta\gamma)} + \psi_2^{(\beta\gamma)} \tag{4.58}$$

with

$$M_1 = h l_1 \frac{\partial w_3}{\partial x_2} + h_1 l \frac{\partial w_2}{\partial x_3} \tag{4.59}$$

If, however, Eq. (4.55) with $\gamma = 2$ is multiplied by l_2 and Eq. (4.56) with $\beta = 2$ is multiplied by h_2 and the results are added together, then:

$$h_2 l_2 N^{(22)} + h_1 l_2 \phi_3^{(12)} + h_2 l_1 \psi_2^{(21)} = M_2 \tag{4.60}$$

with

$$M_2 = h l_2 \frac{\partial w_3}{\partial x_2} + h_2 l \frac{\partial w_2}{\partial x_3} \tag{4.61}$$

Adding Eq. (4.57) to Eq. (4.60) and using Eq. (4.26) gives:

$$\mathrm{v}_{11}N^{(11)} + \mathrm{v}_{12}N^{(12)} + \mathrm{v}_{21}N^{(21)} + \mathrm{v}_{22}N^{(22)} = 2hl\bar{\varepsilon}_{23} \tag{4.62}$$

Equation (4.62) and the continuity conditions Eq. (4.33) with $i = 3$ and Eq. (4.34) with $i = 2$, provide altogether four independent algebraic equations in the four unknowns $N^{(\beta\gamma)}$. It should be noted in deriving these equations that:

$$\sigma_{23}^{(\beta\gamma)} = C_{44}^{(\beta\gamma)}\left(N^{(\beta\gamma)} - 2\bar{\varepsilon}_{23}^{I\,(\beta\gamma)}\right) \tag{4.63}$$

which follows from Eqs. (4.30) and (4.58). The following expressions can be readily established:

$$\begin{aligned}
N^{(11)} &= \frac{2\left[hlC_{44}^{(m)}\bar{\varepsilon}_{23} + C_{44}^{(f)}\Delta_1\varepsilon_{23}^{I\,(11)} - C_{44}^{(m)}\left(\mathrm{v}_{12}\varepsilon_{23}^{I\,(12)} + \mathrm{v}_{21}\varepsilon_{23}^{I\,(21)} + \mathrm{v}_{22}\varepsilon_{23}^{I\,(22)}\right)\right]}{\Delta_2}\\
N^{(12)} &= C_{44}^{(f)}\frac{\left(N^{(11)} - 2\varepsilon_{23}^{I\,(11)}\right)}{C_{44}^{(m)}} + 2\varepsilon_{23}^{I\,(12)}\\
N^{(21)} &= C_{44}^{(f)}\frac{\left(N^{(11)} - 2\varepsilon_{23}^{I\,(11)}\right)}{C_{44}^{(m)}} + 2\varepsilon_{23}^{I\,(21)}\\
N^{(22)} &= C_{44}^{(f)}\frac{\left(N^{(11)} - 2\varepsilon_{23}^{I\,(11)}\right)}{C_{44}^{(m)}} + 2\varepsilon_{23}^{I\,(22)}
\end{aligned} \tag{4.64}$$

where

$$\Delta_1 = (h_1 l_2 + h_2 l_1 + h_2 l_2) \tag{4.65}$$

and

$$\Delta_2 = h_1 l_2 C_{44}^{(m)} + \Delta_1 C_{44}^{(f)} \tag{4.66}$$

The average transverse shear stress $\bar{\sigma}_{23}$ is obtained from Eqs. (4.28) and (4.63) with Eq. (4.64). The resulting constitutive relation is:

$$\bar{\sigma}_{23} = 2b_{44}\left(\bar{\varepsilon}_{23} - \bar{\varepsilon}_{23}^{I}\right) \tag{4.67}$$

where the effective elastic transverse shear modulus, b_{44}, and the effective inelastic transverse shear strain, $\bar{\varepsilon}^I_{23}$, are given by:

$$b_{44} = C_{44}^{(f)} C_{44}^{(m)} \frac{hl}{\Delta_2} \tag{4.68}$$

$$\bar{\varepsilon}^I_{23} = \frac{C_{44}^{(f)} C_{44}^{(m)} \left(h_1 l_1 \varepsilon^{I}_{23}{}^{(11)} + h_1 l_2 \varepsilon^{I}_{23}{}^{(12)} + h_2 l_1 \varepsilon^{I}_{23}{}^{(21)} + l_2^2 \varepsilon^{I}_{23}{}^{(22)} \right)}{b_{44} \Delta_2} \tag{4.69}$$

4.1.1.4 Average Axial Shear Stress-Strain Relations

From Eq. (4.18) with $i = 1$ and $\gamma = 1$ and 2:

$$\phi_1^{(21)} = \frac{h \dfrac{\partial w_1}{\partial x_2} - h_1 \phi_1^{(11)}}{h_2} \tag{4.70}$$

$$\phi_1^{(12)} = \frac{h \dfrac{\partial w_1}{\partial x_2} - h_2 \phi_1^{(22)}}{h_1} \tag{4.71}$$

where Eq. (4.22) has been used. Similarly, from Eq. (4.33) with $i = 1$, using Eq. (4.30), there is for $\gamma = 1$ and $\gamma = 2$:

$$C_{66}^{(f)} \left(\phi_1^{(11)} + \frac{\partial w_2}{\partial x_1} \right) - 2C_{66}^{(f)} \varepsilon^{I}_{12}{}^{(11)} = C_{66}^{(m)} \left(\phi_1^{(21)} + \frac{\partial w_2}{\partial x_1} \right) - 2C_{66}^{(m)} \varepsilon^{I}_{12}{}^{(21)} \tag{4.72}$$

$$\phi_1^{(12)} - 2\varepsilon^{I}_{12}{}^{(12)} = \phi_1^{(22)} - 2\varepsilon^{I}_{12}{}^{(22)} \tag{4.73}$$

Equations (4.70) and (4.72) can be solved for $\phi_1^{(11)}$ and $\phi_1^{(21)}$ while Eqs. (4.71) and (4.73) can be solved for $\phi_1^{(12)}$ and $\phi_1^{(22)}$, providing:

$$\phi_1^{(11)} = \frac{h C_{66}^{(m)} \dfrac{\partial w_1}{\partial x_2} + h_2 \left(C_{66}^{(m)} - C_{66}^{(f)} \right) \dfrac{\partial w_2}{\partial x_1} - 2h_2 \left(C_{66}^{(m)} \varepsilon^{I}_{12}{}^{(21)} - C_{66}^{(f)} \varepsilon^{I}_{12}{}^{(11)} \right)}{h_2 C_{66}^{(f)} + h_1 C_{66}^{(m)}} \tag{4.74}$$

$$\phi_1^{(21)} = \frac{hC_{66}^{(f)}\dfrac{\partial w_1}{\partial x_2} - h_1\left(C_{66}^{(m)} - C_{66}^{(f)}\right)\dfrac{\partial w_2}{\partial x_1} + 2h_1\left(C_{66}^{(m)}\varepsilon_{12}^{I\ (21)} - C_{66}^{(f)}\varepsilon_{12}^{I\ (11)}\right)}{h_2C_{66}^{(f)} + h_1C_{66}^{(m)}} \tag{4.75}$$

$$\phi_1^{(12)} = \frac{\partial w_1}{\partial x_2} + 2\frac{h_2}{h}\left(\varepsilon_{12}^{I\ (12)} - \varepsilon_{12}^{I\ (22)}\right) \tag{4.76}$$

$$\phi_1^{(22)} = \frac{\partial w_1}{\partial x_2} - 2\frac{h_1}{h}\left(\varepsilon_{12}^{I\ (12)} - \varepsilon_{12}^{I\ (22)}\right) \tag{4.77}$$

A similar procedure using Eqs. (4.20) and (4.34) allows one to solve for $\psi_1^{(\beta\gamma)}$, thereby enabling determination of the remaining constitutive relations:

$$\overline{\sigma}_{13} = \frac{1}{V}\sum_{\beta,\gamma=1}^{2} h_\beta l_\gamma \sigma_{13}^{(\beta\gamma)} = 2b_{55}(\overline{\varepsilon}_{13} - \overline{\varepsilon}_{13}^I) \tag{4.78}$$

$$\overline{\sigma}_{12} = \frac{1}{V}\sum_{\beta,\gamma=1}^{2} h_\beta l_\gamma \sigma_{12}^{(\beta\gamma)} = 2b_{66}(\overline{\varepsilon}_{12} - \overline{\varepsilon}_{12}^I) \tag{4.79}$$

where b_{55} and b_{66} are the effective elastic axial shear moduli of the unidirectional composite:

$$b_{55} = \frac{C_{55}^{(m)}C_{55}^{(f)}\ [\ l\ (\mathrm{v}_{11} + \mathrm{v}_{12}) + l_2(\mathrm{v}_{21} + \mathrm{v}_{22})\ ] + C_{55}^{(m)}(\mathrm{v}_{21} + \mathrm{v}_{22})\ \ l_1}{hl\ \left(\ l_1C_{55}^{(m)} + l_2C_{55}^{(f)}\right)} \tag{4.80}$$

and

$$b_{66} = \frac{C_{66}^{(m)}C_{66}^{(f)}[\ h\ (\mathrm{v}_{11} + \mathrm{v}_{21}) + h_2\ (\mathrm{v}_{12} + \mathrm{v}_{22})\ ] + C_{66}^{(m)}(\mathrm{v}_{12} + \mathrm{v}_{22})\ h_1}{hl\left(h_1C_{66}^{(m)} + h_2C_{66}^{(f)}\right)} \tag{4.81}$$

The inelastic axial shear strains are given by:

$$V\,\overline{\varepsilon}_{13}^I = 2\left[\mathrm{v}_{11}C_{55}^{(f)}\varepsilon_{13}^{I(11)} + C_{55}^{(m)}\left(\mathrm{v}_{12}\varepsilon_{13}^{I(12)} + \mathrm{v}_{21}\varepsilon_{13}^{I(21)} + \mathrm{v}_{22}\varepsilon_{13}^{I(22)}\right)\ \right]$$

$$+\frac{2(\mathrm{v}_{22}l_1-\mathrm{v}_{21}l_2)C_{55}^{(m)}\left(\varepsilon_{13}^{I(12)}-\varepsilon_{13}^{I(22)}\right)}{l}$$
$$-\frac{2\left[\mathrm{v}_{11}l_2C_{55}^{(f)}-\mathrm{v}_{12}l_1C_{55}^{(m)}\left(C_{55}^{(f)}\varepsilon_{12}^{I(11)}-C_{55}^{(m)}\varepsilon_{12}^{I(21)}\right)\right]}{\left(l_1C_{55}^{(m)}+l_2C_{55}^{(f)}\right)} \tag{4.82}$$

$$V\,\bar{\varepsilon}_{12}^{I}=2\left[\mathrm{v}_{11}C_{66}^{(f)}\varepsilon_{12}^{I(11)}+C_{66}^{(m)}\left(\mathrm{v}_{12}\varepsilon_{12}^{I(12)}+\mathrm{v}_{21}\varepsilon_{12}^{I(21)}+\mathrm{v}_{22}\varepsilon_{12}^{I(22)}\right)\right]$$
$$+\frac{2(\mathrm{v}_{22}h_1-\mathrm{v}_{12}h_2)C_{66}^{(m)}\left(\varepsilon_{12}^{I(12)}-\varepsilon_{12}^{I(22)}\right)}{h}-\frac{2\left[\mathrm{v}_{11}h_2C_{66}^{(f)}-\mathrm{v}_{21}h_1C_{66}^{(m)}\left(C_{66}^{(f)}\varepsilon_{12}^{I(11)}-C_{66}^{(m)}\varepsilon_{12}^{I(21)}\right)\right]}{\left(h_1C_{66}^{(m)}+h_2C_{66}^{(f)}\right)} \tag{4.83}$$

Consequently, the unidirectional composite is characterized by the nine effective constants b_{11}, b_{12}, b_{13}, b_{22}, b_{23}, b_{33}, b_{44}, b_{55}, and b_{66} forming altogether the elastic stiffness matrix $\mathbf{B}=[b_{ij}]$:

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ & b_{22} & b_{23} & 0 & 0 & 0 \\ & & b_{33} & 0 & 0 & 0 \\ \text{symmetric} & & & b_{44} & 0 & 0 \\ & & & & b_{55} & 0 \\ & & & & & b_{66} \end{bmatrix} \tag{4.84}$$

4.1.1.5 Composite Constitutive Relations—Orthotropic

The composite (macroscopic) constitutive relations are of the form:

$$\bar{\boldsymbol{\sigma}}=\mathbf{B}(\bar{\boldsymbol{\varepsilon}}-\bar{\boldsymbol{\varepsilon}}^{I}-\boldsymbol{\alpha}^{*}\Delta T) \tag{4.85}$$

where

$$\begin{aligned}\bar{\boldsymbol{\sigma}}&=[\bar{\sigma}_{11},\bar{\sigma}_{22},\bar{\sigma}_{33},\bar{\sigma}_{23},\bar{\sigma}_{13},\bar{\sigma}_{12}]\\ \bar{\boldsymbol{\varepsilon}}&=[\bar{\varepsilon}_{11},\bar{\varepsilon}_{22},\bar{\varepsilon}_{33},2\bar{\varepsilon}_{23},2\bar{\varepsilon}_{13},2\bar{\varepsilon}_{12}]\\ \bar{\boldsymbol{\varepsilon}}^{I}&=[\bar{\varepsilon}_{11}^{I},\bar{\varepsilon}_{22}^{I},\bar{\varepsilon}_{33}^{I},2\bar{\varepsilon}_{23}^{I},2\bar{\varepsilon}_{13}^{I},2\bar{\varepsilon}_{12}^{I}]\end{aligned} \tag{4.86}$$

and the effective CTE vector is:

$$\boldsymbol{\alpha}^* = \left[\alpha_1^*, \alpha_2^*, \alpha_3^*, 0, 0, 0\right] \tag{4.87}$$

The composite's overall behavior is governed by these equations and is seen to be solely dependent on the material properties of the individual constituents and their geometrical dimensions. If the geometric parameters l_1 and l_2 are chosen such that $l_1/l_2 \to \infty$, the special case of a laminated medium consisting of a periodic array of two alternating elastoplastic layers of width h_1 and h_2 is obtained. In this case, the MOC elastic results correspond to the exact solution derived by Postma (1955) for elastic constituents. The derived constitutive relations provide the overall behavior of an inelastic laminated composite for this special case.

Key Point

Clearly, Eq. (4.85) provides an explicit analytical expression for the effective generalized Hooke's Law for a composite material and is in contrast to numerical methods where **B** is obtained from multiple simulations.

4.1.1.6 Local Field Equations

4.1.1.6.1 Strain Concentration Tensor

The local subcell strains can be related to the external macroscopic strains through the concentration tensor as follows:

$$\boldsymbol{\varepsilon}^{(\beta\gamma)} = \mathbf{A}^{(\beta\gamma)}\bar{\boldsymbol{\varepsilon}} + \mathbf{A}_{IT}^{(\beta\gamma)} \tag{4.88}$$

where $\mathbf{A}^{(\beta\gamma)}$ is the strain concentration tensor and $\mathbf{A}_{IT}^{(\beta\gamma)}$ are the additional terms that account for inelastic and thermal effects. In Chapter 3, the strain concentration tensors were given for the self-consistent scheme and the Mori-Tanaka (MT) method (for the Voigt and Reuss models, the concentration tensors are equal to the identity tensor). In order to determine the components of $\mathbf{A}^{(\beta\gamma)}$ and $\mathbf{A}_{IT}^{(\beta\gamma)}$ in Eq. (4.88), recall that the stresses in the subcells are given in terms of the microvariables, inelastic strains in the subcells, and temperature deviation (see Eq. (4.30)). The microvariables, on the other hand, have been determined in terms of average strain $\bar{\boldsymbol{\varepsilon}}$, inelastic strains in the subcells, and temperature deviation (see Eqs. (4.44), (4.62), (4.74), and (4.75)). Hence one can immediately establish (after some algebraic manipulation) from the subcell stress expressions Eq. (4.30), the strains in the subcells in terms of the average strains $\bar{\boldsymbol{\varepsilon}}$, the inelastic strains in the subcells, and temperature deviation, which is exactly what is expressed by Eq. (4.88).

For illustration purposes, only the elements of the subcell strain concentration matrix elements are provided below:

$$\mathbf{A}^{(\beta\gamma)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix}^{(\beta\gamma)} \tag{4.89}$$

whereas the inelastic and thermal terms are given in Brayshaw (1994). The nonzero subcell strain concentration matrix components are given below for each of the four subcells:

$$A_{11}^{(\beta\gamma)} = 1, \quad \beta, \gamma = 1,\ 2 \tag{4.90}$$

$$A_{21}^{(11)} = T_1 c_1 + T_2 c_4 + T_3 c_7 + T_4 c_{10} \tag{4.91}$$

$$A_{21}^{(22)} = T_5 c_1 + T_6 c_4 + T_7 c_7 + T_8 c_{10} \tag{4.92}$$

$$A_{21}^{(21)} = -\frac{h_1}{h_2} A_{21}^{(11)} \tag{4.93}$$

$$A_{21}^{(12)} = -\frac{h_2}{h_1} A_{21}^{(22)} \tag{4.94}$$

$$A_{22}^{(11)} = T_1 c_2 + T_2 c_5 + T_3 c_8 + T_4 c_{11} \tag{4.95}$$

$$A_{22}^{(22)} = T_5 c_2 + T_6 c_5 + T_7 c_8 + T_8 c_{11} \tag{4.96}$$

$$A_{22}^{(21)} = 1 + \frac{h_1}{h_2} - \frac{h_1}{h_2} A_{22}^{(11)} \tag{4.97}$$

$$A_{22}^{(12)} = 1 + \frac{h_2}{h_1} - \frac{h_2}{h_1} A_{22}^{(22)} \tag{4.98}$$

$$A_{23}^{(11)} = T_1 c_3 + T_2 c_6 + T_3 c_9 + T_4 c_{12} \tag{4.99}$$

$$A_{23}^{(22)} = T_5 c_3 + T_6 c_6 + T_7 c_9 + T_8 c_{12} \tag{4.100}$$

$$A_{23}^{(21)} = -\frac{h_1}{h_2} A_{23}^{(11)} \tag{4.101}$$

$$A_{23}^{(12)} = -\frac{h_2}{h_1} A_{23}^{(22)} \tag{4.102}$$

$$A_{31}^{(11)} = T_9 c_1 + T_{10} c_4 + T_{11} c_7 + T_{12} c_{10} \tag{4.103}$$

$$A_{31}^{(22)} = T_{13} c_1 + T_{14} c_4 + T_{15} c_7 + T_{16} c_{10} \tag{4.104}$$

$$A_{31}^{(12)} = -\frac{l_1}{l_2} A_{31}^{(11)} \tag{4.105}$$

$$A_{31}^{(21)} = -\frac{l_2}{l_1} A_{31}^{(22)} \tag{4.106}$$

$$A_{32}^{(11)} = T_9 c_2 + T_{10} c_5 + T_{11} c_8 + T_{12} c_{11} \tag{4.107}$$

$$A_{32}^{(22)} = T_{13} c_2 + T_{14} c_5 + T_{15} c_8 + T_{16} c_{11} \tag{4.108}$$

$$A_{32}^{(12)} = -\frac{l_1}{l_2} A_{32}^{(11)} \tag{4.109}$$

$$A_{32}^{(21)} = -\frac{l_2}{l_1} A_{32}^{(22)} \tag{4.110}$$

$$A_{33}^{(11)} = T_9 c_3 + T_{10} c_6 + T_{11} c_9 + T_{12} c_{12} \tag{4.111}$$

$$A_{33}^{(22)} = T_{13} c_3 + T_{14} c_6 + T_{15} c_9 + T_{16} c_{12} \tag{4.112}$$

$$A_{33}^{(12)} = 1 + \frac{l_1}{l_2} - \frac{l_1}{l_2} A_{33}^{(11)} \tag{4.113}$$

$$A_{33}^{(21)} = 1 + \frac{l_2}{l_1} - \frac{l_2}{l_1} A_{33}^{(22)} \tag{4.114}$$

$$\begin{aligned} A_{44}^{(11)} &= \frac{hl}{\Delta C_{44}^{(11)}} \\ A_{44}^{(12)} &= \frac{hl}{\Delta C_{44}^{(12)}} \\ A_{44}^{(21)} &= \frac{hl}{\Delta C_{44}^{(21)}} \\ A_{44}^{(22)} &= \frac{hl}{\Delta C_{44}^{(22)}} \end{aligned} \tag{4.115}$$

$$\Delta = \frac{h_1 l_1}{C_{44}^{(11)}} + \frac{h_1 l_2}{C_{44}^{(12)}} + \frac{h_2 l_1}{C_{44}^{(21)}} + \frac{h_2 l_2}{C_{44}^{(22)}} \tag{4.116}$$

$$\begin{aligned} A_{55}^{(11)} &= \frac{l}{\Delta_3} C_{66}^{(12)} \\ A_{55}^{(12)} &= \frac{l}{\Delta_3} C_{66}^{(11)} \\ A_{55}^{(21)} &= \frac{l}{\Delta_4} C_{66}^{(22)} \\ A_{55}^{(22)} &= \frac{l}{\Delta_4} C_{66}^{(21)} \end{aligned} \tag{4.117}$$

$$
\begin{aligned}
\Delta_1 &= h_1 C_{66}^{(22)} + h_2 C_{66}^{(12)} \\
\Delta_2 &= h_1 C_{66}^{(21)} + h_2 C_{66}^{(11)} \\
\Delta_3 &= l_1 C_{66}^{(12)} + l_2 C_{66}^{(11)} \\
\Delta_4 &= l_1 C_{66}^{(22)} + l_2 C_{66}^{(21)}
\end{aligned}
\tag{4.118}
$$

$$
\begin{aligned}
A_{66}^{(11)} &= \frac{l}{\Delta_2} C_{66}^{(21)} \\
A_{66}^{(12)} &= \frac{l}{\Delta_1} C_{66}^{(22)} \\
A_{66}^{(21)} &= \frac{l}{\Delta_2} C_{66}^{(11)} \\
A_{66}^{(22)} &= \frac{l}{\Delta_1} C_{66}^{(12)}
\end{aligned}
\tag{4.119}
$$

where

$$
\begin{aligned}
c_1 &= C_{12}^{(12)} - C_{12}^{(22)} \\
c_2 &= \frac{h}{h_1} C_{22}^{(12)} \\
c_3 &= \frac{l}{l_2} C_{23}^{(12)} \\
c_4 &= C_{12}^{(21)} - C_{12}^{(11)} \\
c_5 &= \frac{h}{h_2} C_{22}^{(21)} \\
c_6 &= \frac{l}{l_1} C_{23}^{(21)} \\
c_7 &= C_{13}^{(12)} - C_{13}^{(11)} \\
c_8 &= \frac{h}{h_1} C_{23}^{(12)} \\
c_9 &= \frac{l}{l_2} C_{33}^{(12)} \\
c_{10} &= C_{13}^{(21)} - C_{13}^{(22)} \\
c_{11} &= \frac{h}{h_2} C_{23}^{(21)} \\
c_{12} &= \frac{l}{l_1} C_{33}^{(21)}
\end{aligned}
\tag{4.120}
$$

and the T_1, T_2,…,T_{16} are given by Eq. (4.45).

4.1.1.6.2 Stress Concentration Tensor

Given the local subcell strains from above, one can obtain the local subcell stresses by invoking the constituent constitutive equation (see Eq. (4.27)), here rewritten using matrix notation:

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)}\left(\boldsymbol{\varepsilon}^{(\beta\gamma)} - \boldsymbol{\varepsilon}^{I(\beta\gamma)} - \boldsymbol{\alpha}^{(\beta\gamma)}\Delta T\right) \tag{4.121}$$

Substituting into the above, the expression for local strain (Eq. (4.88)) gives an expression for the local stress in terms of the far field strain, that is:

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)}\left(\mathbf{A}^{(\beta\gamma)}\boldsymbol{\varepsilon} + \mathbf{A}_{IT}^{(\beta\gamma)} - \boldsymbol{\varepsilon}^{I(\beta\gamma)} - \boldsymbol{\alpha}^{(\beta\gamma)}\Delta T\right) \tag{4.122}$$

Now, if one desires to express the local stress in terms of the far field stress, one must use the global constitutive equation (Eq. (4.85)), $\overline{\boldsymbol{\varepsilon}} = \mathbf{B}^{-1}\overline{\boldsymbol{\sigma}} + \overline{\boldsymbol{\varepsilon}}^I + \boldsymbol{\alpha}^*\Delta T$, resulting in:

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)}\left(\mathbf{A}^{(\beta\gamma)}\left[\mathbf{B}^{-1}\overline{\boldsymbol{\sigma}} + \overline{\boldsymbol{\varepsilon}}^I + \boldsymbol{\alpha}^*\Delta T\right] + \mathbf{A}_{IT}^{(\beta\gamma)} - \boldsymbol{\varepsilon}^{I(\beta\gamma)} - \boldsymbol{\alpha}^{(\beta\gamma)}\Delta T\right) \tag{4.123}$$

Simplifying, the final expression is obtained by relating the local stresses in each subcell to the applied stress along with inelastic and thermal effects:

$$\boldsymbol{\sigma}^{(\beta\gamma)} = \mathbf{B}^{(\beta\gamma)}\overline{\boldsymbol{\sigma}} + \underbrace{\mathbf{C}^{(\beta\gamma)}\left(\mathbf{A}^{(\beta\gamma)}\overline{\boldsymbol{\varepsilon}}^I - \boldsymbol{\varepsilon}^{I(\beta\gamma)} + \mathbf{A}_{IT}^{(\beta\gamma)} + \mathbf{A}^{(\beta\gamma)}\boldsymbol{\alpha}^*\Delta T - \boldsymbol{\alpha}^{(\beta\gamma)}\Delta T\right)}_{\mathbf{B}_{IT}^{(\beta\gamma)}} \tag{4.124}$$

where $\mathbf{B}^{(\beta\gamma)} = \mathbf{C}^{(\beta\gamma)}\mathbf{A}^{(\beta\gamma)}\mathbf{B}^{-1}$ is the stress concentration tensor and $\mathbf{B}_{IT}^{(\beta\gamma)}$ is the inelastic-thermal contribution for the subcell.

4.1.1.7 Composite Constitutive Relations—Transverse Isotropy

For a square unit cell and square subcells (i.e., $l_1 = h_1$ and $l_2 = h_2$), $b_{12} = b_{13}$, $b_{22} = b_{33}$, and $b_{44} = b_{55}$. This leaves six independent elastic constants, rather than five, for the transversely isotropic case. An averaging procedure can be applied within the MOC that results in the desired five independent constants. Here the discussion on establishing transversely isotropic behavior of the composite is confined to thermoelastic behavior. For a treatment on transversely isotropic inelastic behavior see Aboudi (1991) and Brayshaw (1994), who adopted the averaging of the stiffness tensor and concentration tensors, respectively. A transformation needs to be applied in order to reduce Eq. (4.85) to constitutive relations that effectively represent the unidirectional composite as a transversely isotropic material. This is achieved by rotating the (x_1, x_2, x_3) coordinates of Figure 4.1(a) around the x_1-axis by an angle ξ. The transformation law for $\mathbf{B} = [b_{ij}]$ is given in indicial notation by:

$$b'_{ijk\ell} = a_{ip}a_{jq}a_{kr}a_{\ell s}b_{pqrs} \tag{4.125}$$

where for a rotation around the x_1-axis, the transformation matrix $\mathbf{Tr} = [a_{ij}]$ is given by:

$$\mathbf{Tr} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & \sin\xi \\ 0 & -\sin\xi & \cos\xi \end{bmatrix} \tag{4.126}$$

By performing this transformation, it can be shown that:

$$\begin{aligned}
b'_{11} &= b_{11} \\
b'_{12} &= b_{12} \\
b'_{22} &= b_{22}(\cos^4\xi + \sin^4\xi) + 2(b_{23} + 2b_{44})\sin^2\xi\cos^2\xi \\
b'_{23} &= b_{23}(\cos^4\xi + \sin^4\xi) + 2(b_{22} - 2b_{44})\sin^2\xi\cos^2\xi \\
b'_{44} &= b_{44}(\cos^4\xi + \sin^4\xi) + 2(b_{22} - b_{23} - 2b_{44})\sin^2\xi\cos^2\xi \\
b'_{66} &= b_{66}
\end{aligned} \tag{4.127}$$

which form the nonzero elements of the matrix $\mathbf{B}' = [b'_{ij}]$ whose structure is the same as $\mathbf{B}$ in Eq. (4.85).

The effective elastic constants of the transversely isotropic material can be determined from:

$$\mathbf{B}^* = \frac{1}{\pi}\int_0^{\pi} \mathbf{B}'(\xi)\mathrm{d}\xi \tag{4.128}$$

where the transversely isotropic elastic stiffness matrix $\mathbf{B}^* = [b^*_{ij}]$ takes on the following form:

$$\mathbf{B}^* = \begin{bmatrix} b^*_{11} & b^*_{12} & b^*_{12} & 0 & 0 & 0 \\ & b^*_{22} & b^*_{23} & 0 & 0 & 0 \\ & & b^*_{22} & 0 & 0 & 0 \\ & & & b^*_{44} & 0 & 0 \\ & & & & b^*_{66} & 0 \\ \text{symmetric} & & & & & b^*_{66} \end{bmatrix} \tag{4.129}$$

In Eq. (4.129), b_{ij}^* are the effective stiffness components given by:

$$\begin{aligned} b_{11}^* &= b_{11} \\ b_{12}^* &= b_{12} \\ b_{22}^* &= \frac{3}{4}b_{22} + \frac{1}{4}b_{23} + \frac{1}{2}b_{44} \\ b_{23}^* &= \frac{1}{4}b_{22} + \frac{3}{4}b_{23} - \frac{1}{2}b_{44} \\ b_{44}^* &= \frac{1}{2}(b_{22}^* - b_{23}^*) \\ b_{66}^* &= b_{66} \end{aligned} \tag{4.130}$$

which form altogether five independent constants, thus representing a transversely isotropic material.

Since $\boldsymbol{\alpha}^*$ in Eq. (4.87) are not affected by the above transformation, it follows that effective transversely isotropic thermoelastic behavior of the unidirectional composite can be written in the form:

$$\overline{\boldsymbol{\sigma}} = \mathbf{B}^*(\overline{\boldsymbol{\varepsilon}} - \boldsymbol{\alpha}^*\Delta T) \tag{4.131}$$

The transversely isotropic behavior can also be achieved by considering the effective orthotropic effective compliance $\mathbf{S} = \mathbf{B}^{-1}$. If the elements of $\mathbf{S}$ are denoted by s_{ij}, it can be seen that the following transformation holds in indicial notation:

$$s'_{ijk\ell} = a_{ip}a_{jq}a_{kr}a_{\ell s}s_{pqrs} \tag{4.132}$$

where for a rotation around the x_1-axis, the transformation matrix $\mathbf{Tr} = [a_{ij}]$ is given by Eq. (4.126). The resulting relations between s'_{ij} and s_{ij} are given as in Eq. (4.127). The effective compliance matrix is determined from:

$$\mathbf{S}^* = \frac{1}{\pi}\int_0^{\pi} \mathbf{S}'(\xi)\mathrm{d}\xi \tag{4.133}$$

The elements of $\mathbf{S}^*$ can be computed from those of $\mathbf{S}$ as in Eq. (4.130). The engineering effective constants can be readily determined from:

$$\mathbf{S}^* = \begin{bmatrix} 1/E_A^* & -\nu_A^*/E_A^* & -\nu_A^*/E_A^* & 0 & 0 & 0 \\ & 1/E_T^* & -\nu_T^*/E_T^* & 0 & 0 & 0 \\ & & 1/E_T^* & 0 & 0 & 0 \\ & & & 1/G_T^* & 0 & 0 \\ \text{symmetric} & & & & 1/G_A^* & 0 \\ & & & & & 1/G_A^* \end{bmatrix} \tag{4.134}$$

Note E_A^* and E_T^* are the axial and transverse Young's moduli, respectively; ν_A^* and ν_T^* are the axial and transverse Poisson's ratios, respectively; and G_A^* and G_T^* are the axial and transverse shear moduli, respectively. The present prediction of the engineering constants is identical to those determined from Eq. (4.130) except for E_T^* and ν_T^*. For the latter two properties, the difference is very small for the commonly used constituents in composites (see Aboudi (1991)).

4.1.1.8 Strain Concentration Tensor—Transverse Isotropy

Brayshaw (1994) suggested the averaging of the concentration matrices rather than the stiffness matrix to obtain effective transversely isotropic material behavior of continuously reinforced unidirectional composites. The following presents this averaging procedure to obtain the transversely isotropic effective elastic stiffness matrix from the averaged concentration matrices.

The following averaging procedure is employed:

$$\widehat{\mathbf{A}}^{(\beta\gamma)} = \int_{\xi=-\pi/4}^{\pi/4} \mathbf{A}_\xi^{(\beta\gamma)} \mathrm{d}\xi \tag{4.135}$$

where $\mathbf{A}_\xi^{(\beta\gamma)}$ is obtained by rotating $\mathbf{A}^{(\beta\gamma)}$ by the angle ξ around the x_1- (fiber) direction (see Eq. (4.126)). The resulting expressions are:

$$\begin{aligned}
\widehat{A}_{11}^{(\beta\gamma)} &= A_{11}^{(\beta\gamma)}\\
\widehat{A}_{21}^{(\beta\gamma)} &= \left(\frac{1}{2}+\frac{1}{\pi}\right)A_{21}^{(\beta\gamma)} + \left(\frac{1}{2}-\frac{1}{\pi}\right)A_{31}^{(\beta\gamma)}\\
\widehat{A}_{31}^{(\beta\gamma)} &= \left(\frac{1}{2}-\frac{1}{\pi}\right)A_{21}^{(\beta\gamma)} + \left(\frac{1}{2}+\frac{1}{\pi}\right)A_{31}^{(\beta\gamma)}\\
\widehat{A}_{22}^{(\beta\gamma)} &= \left(\frac{3}{8}+\frac{1}{\pi}\right)A_{22}^{(\beta\gamma)} + \left(\frac{3}{8}-\frac{1}{\pi}\right)A_{33}^{(\beta\gamma)} + \frac{1}{8}\left(A_{23}^{(\beta\gamma)}+A_{32}^{(\beta\gamma)}\right) + \frac{1}{4}A_{44}^{(\beta\gamma)}\\
\widehat{A}_{32}^{(\beta\gamma)} &= \left(\frac{3}{8}+\frac{1}{\pi}\right)A_{32}^{(\beta\gamma)} + \left(\frac{3}{8}-\frac{1}{\pi}\right)A_{23}^{(\beta\gamma)} + \frac{1}{8}\left(A_{22}^{(\beta\gamma)}+A_{33}^{(\beta\gamma)}\right) - \frac{1}{4}A_{44}^{(\beta\gamma)}\\
\widehat{A}_{23}^{(\beta\gamma)} &= \left(\frac{3}{8}+\frac{1}{\pi}\right)A_{23}^{(\beta\gamma)} + \left(\frac{3}{8}-\frac{1}{\pi}\right)A_{32}^{(\beta\gamma)} + \frac{1}{8}\left(A_{22}^{(\beta\gamma)}+A_{33}^{(\beta\gamma)}\right) - \frac{1}{4}A_{44}^{(\beta\gamma)}\\
\widehat{A}_{33}^{(\beta\gamma)} &= \left(\frac{3}{8}+\frac{1}{\pi}\right)A_{33}^{(\beta\gamma)} + \left(\frac{3}{8}-\frac{1}{\pi}\right)A_{22}^{(\beta\gamma)} + \frac{1}{8}\left(A_{23}^{(\beta\gamma)}+A_{32}^{(\beta\gamma)}\right) + \frac{1}{4}A_{44}^{(\beta\gamma)}\\
\widehat{A}_{44}^{(\beta\gamma)} &= \frac{1}{4}\left(A_{22}^{(\beta\gamma)}+A_{33}^{(\beta\gamma)}\right) - \frac{1}{4}\left(A_{23}^{(\beta\gamma)}+A_{32}^{(\beta\gamma)}\right) + \frac{1}{2}A_{44}^{(\beta\gamma)}\\
\widehat{A}_{55}^{(\beta\gamma)} &= \left(\frac{1}{2}+\frac{1}{\pi}\right)A_{55}^{(\beta\gamma)} + \left(\frac{1}{2}-\frac{1}{\pi}\right)A_{66}^{(\beta\gamma)}\\
\widehat{A}_{66}^{(\beta\gamma)} &= \left(\frac{1}{2}-\frac{1}{\pi}\right)A_{55}^{(\beta\gamma)} + \left(\frac{1}{2}+\frac{1}{\pi}\right)A_{66}^{(\beta\gamma)}
\end{aligned} \tag{4.136}$$

Once $\widehat{\mathbf{A}}^{(\beta\gamma)}$ has been established, it is possible to determine the transversely isotropic effective stiffness matrix $\mathbf{B}^*$ of the composite by employing the following relation:

$$\mathbf{B}^* = \frac{1}{hl} \sum_{\beta,\gamma=1}^{2} h_\beta l_\gamma \mathbf{C}^{(\beta\gamma)} \widehat{\mathbf{A}}^{(\beta\gamma)} \tag{4.137}$$

This provides global transversely isotropic elastic behavior, identical to Eq. (4.130). Further, using the rotationally averaged strain concentration matrix, local strains that are consistent with the transversely isotropic effective stiffness matrix are obtained. As shown by Brayshaw (1994), rotational averaging can be performed on the strain concentration inelastic-thermal terms $\mathbf{A}_{IT}^{(\beta\gamma)}$ (see Eq. (4.88)) as well to enable fully consistent thermal-inelastic simulations for transversely isotropic composites with the MOC.

4.1.1.9 Strain Concentration Tensor—Isotropy

Just like the averaging of the strain concentration tensor to obtain effective transverse isotropy that was described above, it is possible to average the strain concentration tensor to establish the two independent constants that represent a composite with randomly oriented fibers or inclusions that is effectively isotropic. This can be achieved by employing Eqs. (4.199) to (4.201) that are given later in this chapter, in which the strain concentration tensor $A(\beta\gamma)$ is rotated according to Eq. (4.199) and then averaged by employing Eq. (4.201).

It should be emphasized that similar procedures can be employed to obtain effective transverse isotropy and isotropy from the strain concentration tensors established by GMC and HFGMC, which will be discussed in Chapters 5 and 6, respectively.

4.1.2 Thermal Conductivities

The Fourier law of heat conduction states that the heat flux vector is proportional to the negative vector gradient of temperature. It follows that for isotropic materials:

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{4.138}$$

where T is the temperature, q_i are the components of the heat flux vector, and k is the coefficient of heat conductivity. For anisotropic materials this equation takes the form:

$$q_i = -k_{ij} \frac{\partial T}{\partial x_j} \tag{4.139}$$

where k_{ij} is the thermal conductivity tensor. In a composite material, the effective coefficients of heat conductivity are of great physical importance and may be predicted from the knowledge of the properties of the phases themselves. This can be performed by employing the MOC. To do this, consider a unidirectional fibrous composite as shown in Figure 4.1(a). In accordance with the MOC, the deviation of the temperature from a reference temperature T_R (at which the material is stress-free when its strain is zero), $\Delta\Theta^{(\beta\gamma)}$, is expanded in the form:

$$\Delta\Theta^{(\beta\gamma)} = \Delta T + \bar{x}_2^{(\beta)} \xi_2^{(\beta\gamma)} + \bar{x}_3^{(\gamma)} \xi_3^{(\beta\gamma)} \tag{4.140}$$

where $\xi_2^{(\beta\gamma)}$ and $\xi_3^{(\beta\gamma)}$ characterize the linear dependence of the temperature deviation on the local coordinates. Compared to Eq. (4.1), the temperature deviation takes on the role analogous to the displacement vector in the mechanical problem. The heat flux vector $\bar{q}_i$ takes on the role of the stress tensor.

The continuity conditions of temperature at the interfaces on an average basis (as in the mechanical case) lead to:

$$\begin{aligned} h_1 \xi_2^{(1\gamma)} + h_2 \xi_2^{(2\gamma)} &= (h_1 + h_2) \frac{\partial T}{\partial x_2} \\ l_1 \xi_3^{(\beta 1)} + l_2 \xi_3^{(\beta 2)} &= (l_1 + l_2) \frac{\partial T}{\partial x_3} \end{aligned} \tag{4.141}$$

For the average heat flux in the subcell:

$$\begin{aligned} \bar{q}_1^{(\beta\gamma)} &= -k_1^{(\beta\gamma)} \frac{\partial T}{\partial x_1} \\ \bar{q}_2^{(\beta\gamma)} &= -k_2^{(\beta\gamma)} \frac{\partial T}{\partial x_2} \\ \bar{q}_3^{(\beta\gamma)} &= -k_3^{(\beta\gamma)} \frac{\partial T}{\partial x_3} \end{aligned} \tag{4.142}$$

where $k_i^{(\beta\gamma)}$ denote the thermal conductivity coefficients of the subcells.

The average heat flux in the composite $\bar{q}$ is determined from:

$$\bar{q}_i = \frac{1}{V} \sum_{\beta,\,\gamma=1}^{2} \mathrm{v}_{\beta\gamma} \bar{q}_i^{(\beta\gamma)} \tag{4.143}$$

The continuity conditions of the heat flux at the interfaces give:

$$\begin{aligned} \bar{q}_2^{(1\gamma)} &= \bar{q}_2^{(2\gamma)} \\ \bar{q}_3^{(\beta 1)} &= \bar{q}_3^{(\beta 2)} \end{aligned} \tag{4.144}$$

The average heat flux components are related to the temperature gradients by the effective thermal conductivities k_i^*:

$$\begin{aligned} \bar{q}_1 &= -k_1^* \frac{\partial T}{\partial x_1} \\ \bar{q}_2 &= -k_2^* \frac{\partial T}{\partial x_2} \\ \bar{q}_3 &= -k_3^* \frac{\partial T}{\partial x_3} \end{aligned} \tag{4.145}$$

By eliminating the microvariables $\xi_2^{(\beta\gamma)}$ and $\xi_3^{(\beta\gamma)}$ and using the continuity conditions (4.144), after some manipulations the effective thermal conductivities of the unidirectional composite are given by:

$$k_1^* = \frac{\left[v_{11} k_1^{(f)} + (v_{12} + v_{21} + v_{22})\, k_1^{(m)}\right]}{hl}$$
$$k_2^* = \frac{k_2^{(m)} \left\{ k_2^{(f)} [h(v_{11} + v_{21}) + h_2(v_{12} + v_{22})] + k_2^{(m)} h_1 (v_{12} + v_{22}) \right\}}{\left[hl\left(h_1 k_2^{(m)} + h_2 k_2^{(f)}\right)\right]} \tag{4.146}$$
$$k_3^* = \frac{k_3^{(m)} \left\{ k_3^{(f)} [l(v_{11} + v_{12}) + l_2(v_{21} + v_{22})] + k_3^{(m)} l_1 (v_{21} + v_{22}) \right\}}{\left[hl\left(l_1 k_3^{(m)} + l_2 k_3^{(f)}\right)\right]}$$

It can be readily seen that the effective axial coefficient of thermal conductivity is given by the rule of mixtures, and the transverse coefficients are of a form identical to that for the effective axial shear moduli (Eqs. (4.80) and (4.81)). The analogy between the effective axial shear modulus and transverse CTE of unidirectional composites was established by Hashin (1972).

4.1.3 Specific Heats

The specific heats at constant deformation and pressure are defined, respectively, by:

$$c_v = \left[\frac{\partial Q}{\partial T}\right]_{\varepsilon_{ij}} \tag{4.147}$$

$$c_p = \left[\frac{\partial Q}{\partial T}\right]_{\sigma_{ij}} \tag{4.148}$$

where ΔQ is the gain of heat of an element of an elastic solid from its surroundings, when the solid is heated in a thermodynamically reversible manner. In this situation $\mathrm{d}Q = T\,\mathrm{d}\eta$, where η is the entropy per unit mass. Consequently, at the reference temperature T_R,

$$c_v = T_R \left[\frac{\partial \eta}{\partial T}\right]_{\varepsilon_{ij}} \tag{4.149}$$

$$c_p = T_R \left[\frac{\partial \eta}{\partial T}\right]_{\sigma_{ij}} \tag{4.150}$$

The expressions for η in a transversely isotropic phase (with x_1 being the direction of anisotropy) are given in each subcell by:

$$\rho_{\beta\gamma}\eta^{(\beta\gamma)} = \Gamma_1^{(\beta\gamma)}\varepsilon_{11}^{(\beta\gamma)} + \Gamma_2^{(\beta\gamma)}\left[\varepsilon_{22}^{(\beta\gamma)} + \varepsilon_{33}^{(\beta\gamma)}\right] + \rho_{\beta\gamma}\ c_v^{(\beta\gamma)}\Delta\Theta^{(\beta\gamma)}/T_R \tag{4.151}$$

$$\rho_{\beta\gamma}\eta^{(\beta\gamma)} = \alpha_A^{(\beta\gamma)}\sigma_{11}^{(\beta\gamma)} + \alpha_T^{(\beta\gamma)}\left[\sigma_{22}^{(\beta\gamma)} + \sigma_{33}^{(\beta\gamma)}\right] + \rho_{\beta\gamma}c_p^{(\beta\gamma)}\Delta\Theta^{(\beta\gamma)}/T_R \tag{4.152}$$

where $\rho_{\beta\gamma}$ is the mass density of the material. It follows that the average entropies in the subcells are:

$$\rho_{\beta\gamma}\overline{\eta}^{(\beta\gamma)} = \Gamma_1^{(\beta\gamma)}\overline{\varepsilon}_{11} + \Gamma_2^{(\beta\gamma)}\left[\phi_2^{(\beta\gamma)} + \psi_3^{(\beta\gamma)}\right] + \rho_{\beta\gamma}c_v^{(\beta\gamma)}\Delta T/T_R \tag{4.153}$$

$$\rho_{\beta\gamma}\overline{\eta}^{(\beta\gamma)} = \alpha_A^{(\beta\gamma)}\overline{\sigma}_{11}^{(\beta\gamma)} + \alpha_T^{(\beta\gamma)}\left[\overline{\sigma}_{22}^{(\beta\gamma)} + \overline{\sigma}_{33}^{(\beta\gamma)}\right] + \rho_{\beta\gamma}c_p^{(\beta\gamma)}\Delta T/T_R \tag{4.154}$$

The effective specific heat of a unidirectional composite at constant deformation is determined from the continuity of displacements and tractions and by imposing the zero strain conditions:

$$\overline{\varepsilon}_{11} = \overline{\varepsilon}_{22} = \overline{\varepsilon}_{33} = 0 \tag{4.155}$$

The set of eight equations obtained from the traction continuity conditions, Eq. (4.33) with $i = 2$ and Eq. (4.34) with $i = 3$, and the displacement continuity conditions, Eq. (4.18) with $i = 2$ and Eq. (4.20) with $i = 3$, are solved for the eight unknown microvariables $\phi_2^{(\beta\gamma)}$ and $\psi_3^{(\beta\gamma)}$, where $\beta,\gamma = 1,2$. The solution is given by Eq. (4.44), which can be used in conjunction with the conditions Eq. (4.155), to determine the average entropies in the subcells in accordance with Eq. (4.153).

To obtain the average entropy in the composite, one merely volume averages the subcell quantities as follows:

$$\overline{\rho\eta} = \frac{1}{hl}\sum_{\beta,\gamma=1}^{2} h_\beta l_\gamma \rho_{\beta\gamma}\overline{\eta}^{(\beta\gamma)} \tag{4.156}$$

which yields the effective specific heat at constant deformation of the composite:

$$(\rho c_v)^* = \overline{\rho\eta}\ T_R/\Delta T \tag{4.157}$$

Similarly, using stress-free conditions:

$$\overline{\sigma}_{11} = \overline{\sigma}_{22} = \overline{\sigma}_{33} = 0 \tag{4.158}$$

one needs to recompute the corresponding $\overline{\rho\eta}$ to obtain the effective specific heat at constant pressure of the composite:

$$(\rho c_p)^* = \overline{\rho\eta}\, T_R/\Delta T \tag{4.159}$$

It can be easily verified that:

$$\left(\rho c_p\right)^* - (\rho c_v)^* = \left[E_A^* {\alpha_A^*}^2 + 4\kappa^* \left(\nu_A^* \alpha_A^* + \alpha_T^*\right)\right] T_R \tag{4.160}$$

where κ^* is the effective plane strain bulk modulus, $\kappa^* = 0.25E_A^*/[0.5(1 - \nu_T^*)(E_A^*/E_T^*) - {\nu_T^*}^2]$.

4.1.4 MOC with Imperfect Bonding

Interfaces are of critical importance to composite materials because the load transfer between fiber and matrix strongly depends on, and is controlled by, the degree of contact and cohesive forces at the interface. However, there is a lack of knowledge of the exact behavior and processes at the interface. Mechanical models for, and properties of, interfacial zones are necessary for incorporation of these interfacial effects into any micromechanical analysis, with the goal being to provide the overall behavior of composite materials. In most existing models, perfect bonding is assumed to exist between the constituents. Perfect bonding, however, is a demanding requirement since it is in essence the desire to 'weld' materials together, which, by their nature, may not be directly 'weldable.' Perfect bonding and complete debonding are actually idealized extremes which set bounds for the behavior of the complex situation that actually occurs at the interface, where an interphase is known to exist between the phases.

The effect of imperfect bonding between fibers and matrix in a composite material may be incorporated within the MOC by assuming that an explicit interfacial layer exists between the two constituents. Thus, the composite is considered to consist of three phases; that is, the matrix, the fibers, and an interphase, which is the zone of imperfections surrounding each fiber. The thickness and material properties of the interphase can be used to describe the quality of adhesion between the matrix and fiber. The determination of the properties of the interphase, which may be heterogeneous, can obviously be problematic.

This difficulty can be circumvented by adopting the simplified model of Jones and Whittier (1967), see Chapter 2, in which a flexible interface of infinitesimal thickness is introduced to model imperfect bonding between the fiber and matrix of fiber-reinforced composites. This is accomplished by modifying the continuity of displacements at the fiber/matrix interface (i.e., Eqs. (4.3) and (4.4)) by reinterpreting them based on the flexible interface model of Jones and Whittier such that Eq. (4.18) becomes:

$$h_1 \phi_i^{(1\gamma)} + h_2 \phi_i^{(2\gamma)} + 2R\overline{\sigma}_{2i}^{(11)} = (h_1 + h_2) \frac{\partial w_i^{(2\gamma)}}{\partial x_2} \tag{4.161}$$

and Eq. (4.20) becomes:

$$l_1 \psi_i^{(\beta 1)} + l_2 \psi_i^{(\beta 2)} + 2R\overline{\sigma}_{3i}^{(11)} = (l_1 + l_2) \frac{\partial w_i^{(\beta 2)}}{\partial x_3} \tag{4.162}$$

In these equations $R = R_n$ or R_t, the normal or tangential compliance of the interface (see Chapter 2), with $R \neq 0$ along the interface between the fiber subcell ($\beta = 1$, $\gamma = 1$) and its neighboring matrix subcells, assuming imperfect bonding in both directions. The influence of these changes on the resulting micromechanical equations is given in Aboudi (1988a) as well as in Chapter 5 when the MOC is generalized.

4.2 The Method of Cells for Discontinuously Fiber-Reinforced Composites (Triply Periodic)

In Section 4.1, the MOC was developed for composite materials with unidirectional continuous fibers. Here, the method is extended to short-fiber composites, thus allowing the consideration of inclusions of a finite length. For simplicity, the presentation given here is restricted to thermo-elastic constituents (i.e., no inelasticity). If one wishes to include inelastic effects into the triply periodic MOC, see Aboudi (1986a).

4.2.1 Thermomechanical Formulation

As in the doubly periodic MOC, the thermomechanical derivation of the triply periodic theory is first presented and followed by development of the effective thermal conductivities.

Consider a two-phase composite, which is described by a triply periodic array of identical rectangular parallelepipeds (representing the inclusions) embedded in the matrix (see Figure 4.2(a)). The volume of a parallelepiped is $d_1 h_1 l_1$, and the parameters d_2, h_2, and l_2

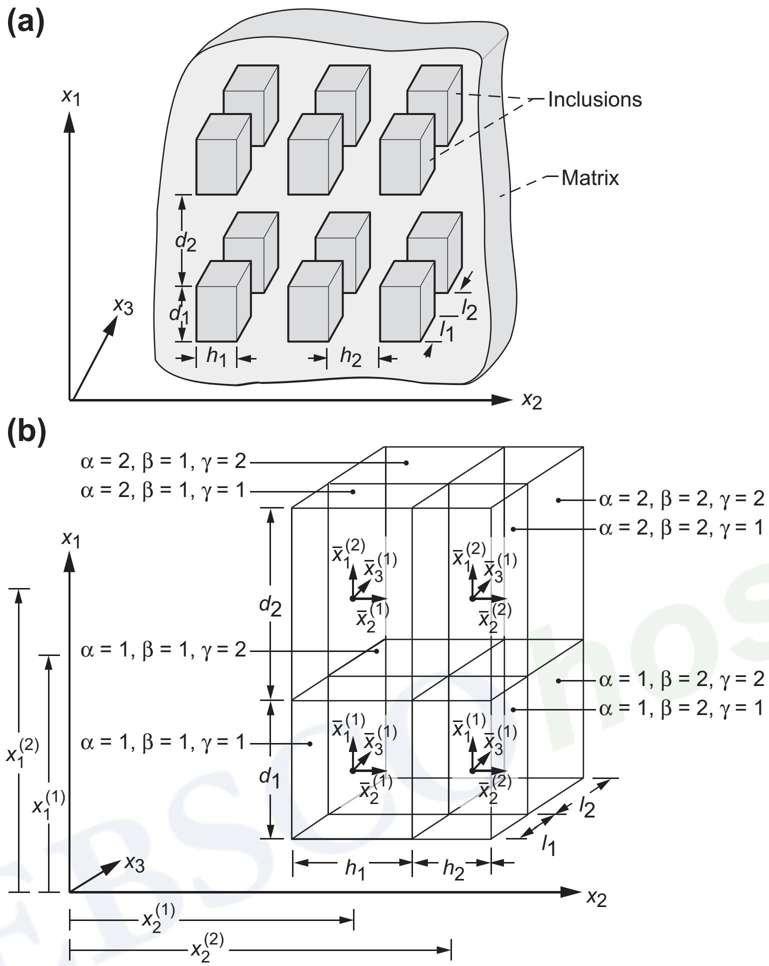

Figure 4.2:
MOC model of short-fiber composite. (a) A composite with a triply periodic array of parallelepiped inclusions. (b) A repeating unit cell with eight subcells (α, β, and γ take the values of 1 and 2).

denote the spacing of the inclusion within the matrix in the x_1-, x_2-, and x_3-directions, respectively. Because of the assumed periodic arrangement, it is sufficient to analyze a repeating unit cell (RUC) whose dimensions are $d_1 + d_2$, $h_1 + h_2$ and $l_1 + l_2$ as shown in Figure 4.2(b). The unit cell is divided into eight subcells (i.e., α, β, and γ each taking the value of 1 and 2), and eight local coordinate systems $(\bar{x}_1^{(\alpha)}, \bar{x}_2^{(\beta)}, \bar{x}_3^{(\gamma)})$ are introduced whose origins are located at the centroid of each subcell and are oriented parallel to the global Cartesian system (x_1, x_2, x_3).

The geometrical arrangement shown in Figure 4.2(a), which describes a short-fiber composite, is general. For example, a particulate composite is obtained by selecting $d_1 = h_1 = l_1$ and $d_2 = h_2 = l_2$, and a porous material results from the latter for vacuous inclusions. Furthermore, a unidirectional long-fiber composite in which the fibers are oriented in the x_1-direction (for example) would be obtained by choosing $d_1/d_2 >> 1$. The resulting distribution in this special case corresponds to longfibers embedded within the matrix at spacing h_2 and l_2. Obviously, doubly periodic layers can also be obtained as a special case by selecting $h_1/h_2 >> 1$ and $l_1/ l_2 >> 1$. Here the parameters d_1 and d_2 correspond to the widths of the two layers.

Key Point

In Figure 4.2(a), a discontinuously reinforced composite material is idealized as a triply periodic array of particles/whiskers/fibers embedded in a matrix. The RUC—and its eight subvolumes, or 'subcells'—are identified, whose centroids represent one inclusion point and seven matrix points, respectively. The rectangular graphical representation of each subcell indicates the region of influence of its centroid, not the actual inclusion or matrix shape. Thus it is emphasized that within the MOC formulation enforcement of continuity is in an average sense, no influence of rectangular corners (stress risers) is present, and therefore the modeled inclusion in reality has no associated shape and therefore is more appropriately considered as a pseudo-ellipsoid.

A higher order continuum theory can be developed for the modeling of the short-fiber composite by the expansion of the displacement vector in each subcell in terms of the distances from its center as shown in Chapters 6 and 11. Here, however, as in the case of continuous fibers, the triplyperiodic MOC is a first-order theory. Referring to Figure 4.2(b), the displacement components at any point within the subcell $(\alpha\beta\gamma)$ are restricted to a first-order expansion:

$$u_i^{(\alpha\beta\gamma)} = w_i^{(\alpha\beta\gamma)}(\mathbf{x}) + \bar{x}_1^{(\alpha)}\chi_i^{(\alpha\beta\gamma)} + \bar{x}_2^{(\beta)}\phi_i^{(\alpha\beta\gamma)} + \bar{x}_3^{(\gamma)}\psi_i^{(\alpha\beta\gamma)} \quad i = 1, 2, 3 \tag{4.163}$$

where $w_i^{(\alpha\beta\gamma)}$ are the displacement components at the centroid of the subcell; and $\chi_i^{(\alpha\beta\gamma)}$, $\phi_i^{(\alpha\beta\gamma)}$, and $\psi_i^{(\alpha\beta\gamma)}$ characterize the linear dependence of the displacements on the local coordinates of the subcell.

The components of the small strain tensor are given by:

$$\varepsilon_{ij}^{(\alpha\beta\gamma)} = \frac{1}{2}\left[\partial_j u_i^{(\alpha\beta\gamma)} + \partial_i u_j^{(\alpha\beta\gamma)}\right] \tag{4.164}$$

where $\partial_1 = \partial/\partial\overline{x}_1^{(\alpha)}$, $\partial_2 = \partial/\partial\overline{x}_2^{(\beta)}$, and $\partial_3 = \partial/\partial\overline{x}_3^{(\gamma)}$.

At the interfaces between the subcells within the unit cell of Figure 4.2(b) the following continuity relations are satisfied:

$$
\begin{aligned}
u_i^{(1\beta\gamma)}\Big|_{\overline{x}_1^{(1)}=d_1/2} &= u_i^{(2\beta\gamma)}\Big|_{\overline{x}_1^{(2)}=-d_2/2} \\
u_i^{(\alpha1\gamma)}\Big|_{\overline{x}_2^{(1)}=h_1/2} &= u_i^{(\alpha2\gamma)}\Big|_{\overline{x}_2^{(2)}=-h_2/2} \\
u_i^{(\alpha\beta1)}\Big|_{\overline{x}_3^{(1)}=l_1/2} &= u_i^{(\alpha\beta2)}\Big|_{\overline{x}_3^{(2)}=-l_2/2}
\end{aligned}
\tag{4.165}
$$

Similar relations must be imposed at the interfaces between the unit cell of Figure 4.2(b) and its neighboring unit cells (not shown in the figure). This is accomplished using a Taylor series to relate the displacements in the adjacent unit cells to those in the unit cell under consideration (see Eqs. (4.9) to (4.16)).

Imposing the displacement continuity conditions on an average basis (as was previously shown for continuous fibers) gives the following relations:

$$
\begin{aligned}
&\frac{\partial w_i^{(\alpha\beta\gamma)}}{\partial x_j} = \frac{\partial w_i}{\partial x_j} \\
&d_1\chi_i^{(1\beta\gamma)} + d_2\chi_i^{(2\beta\gamma)} = (d_1 + d_2)\ \frac{\partial}{\partial x_1} w_i \\
&h_1\phi_i^{(\alpha1\gamma)} + h_2\phi_i^{(\alpha2\gamma)} = (h_1 + h_2)\ \frac{\partial}{\partial x_2} w_i \\
&l_1\psi_i^{(\alpha\beta1)} + l_2\psi_i^{(\alpha\beta2)} = (l_1 + l_2)\ \frac{\partial}{\partial x_3} w_i
\end{aligned}
\tag{4.166}
$$

A detailed derivation of these equations can be found in Aboudi (1986a).

As in the doubly periodic MOC, it can be shown from Eqs. (4.163) and (4.164) as well as the first of Eq. (4.166) that the strain components, and thus the stress components, are constant within each subcell. Therefore, the average fields within a subcell are identical to the pointwise fields within a subcell.

The strains in the subcell can be deduced from Eq. (4.164) as follows:

$$
\begin{aligned}
\varepsilon_{11}^{(\alpha\beta\gamma)} &= \chi_1^{(\alpha\beta\gamma)} \\
\varepsilon_{22}^{(\alpha\beta\gamma)} &= \phi_2^{(\alpha\beta\gamma)} \\
\varepsilon_{33}^{(\alpha\beta\gamma)} &= \psi_3^{(\alpha\beta\gamma)} \\
\varepsilon_{23}^{(\alpha\beta\gamma)} &= \frac{\left(\psi_2^{(\alpha\beta\gamma)} + \phi_3^{(\alpha\beta\gamma)}\right)}{2} \\
\varepsilon_{13}^{(\alpha\beta\gamma)} &= \frac{\left(\psi_1^{(\alpha\beta\gamma)} + \chi_3^{(\alpha\beta\gamma)}\right)}{2} \\
\varepsilon_{12}^{(\alpha\beta\gamma)} &= \frac{\left(\phi_1^{(\alpha\beta\gamma)} + \chi_2^{(\alpha\beta\gamma)}\right)}{2}
\end{aligned}
\tag{4.167}
$$

The average strains $\bar{\varepsilon}_{ij}$ in the composite are given in terms of the average strains in the subcells of the representative cell in the form:

$$
\bar{\varepsilon}_{ij} = \frac{1}{V} \sum_{\alpha,\ \beta,\ \gamma=1}^{2} d_\alpha h_\beta l_\gamma \ \ \varepsilon_{ij}^{(\alpha\beta\gamma)} \tag{4.168}
$$

where $V = (d_1 + d_2)(h_1 + h_2)(l_1 + l_2)$ is the total volume of the representative unit cell.

It follows directly from Eqs. (4.166), (4.167), and (4.168) that:

$$
\bar{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} w_j + \frac{\partial}{\partial x_j} w_i\right) \tag{4.169}
$$

The continuity of tractions at the interfaces between the subcells of Figure 4.2(b) and between neighboring cells is expressed by:

$$
\begin{aligned}
\sigma_{1i}^{(1\beta\gamma)}\Big|_{\bar{x}_1^{(1)}=\pm d_1/2} &= \sigma_{1i}^{(2\beta\gamma)}\Big|_{\bar{x}_1^{(2)}=\mp d_2/2} \\
\sigma_{2i}^{(\alpha1\gamma)}\Big|_{\bar{x}_2^{(1)}=\pm h_1/2} &= \sigma_{2i}^{(\alpha2\gamma)}\Big|_{\bar{x}_2^{(2)}=\mp h_2/2} \\
\sigma_{3i}^{(\alpha\beta1)}\Big|_{\bar{x}_3^{(1)}=\pm l_1/2} &= \sigma_{3i}^{(\alpha\beta2)}\Big|_{\bar{x}_3^{(2)}=\mp l_2/2}
\end{aligned}
\tag{4.170}
$$

where $\sigma_{ij}^{(\alpha\beta\gamma)}$ denotes the components of the stress tensor. Imposing these conditions in an average sense leads to:

$$\begin{aligned} \sigma_{1i}^{(1\beta\gamma)} &= \sigma_{1i}^{(2\beta\gamma)} \\ \sigma_{2i}^{(\alpha 1\gamma)} &= \sigma_{2i}^{(\alpha 2\gamma)} \\ \sigma_{3i}^{(\alpha\beta 1)} &= \sigma_{3i}^{(\alpha\beta 2)} \end{aligned} \tag{4.171}$$

The average stresses in the composite are given by:

$$\overline{\sigma}_{ij} = \frac{1}{V} \sum_{\alpha,\ \beta,\ \gamma=1}^{2} d_\alpha h_\beta l_\gamma \sigma_{ij}^{(\alpha\beta\gamma)} \tag{4.172}$$

The overall behavior of the composite is determined once relations between the average stresses $\overline{\sigma}_{ij}$ and the average strains $\overline{\varepsilon}_{ij}$ are established.

The constitutive relations in each subcell are of the standard form, which can also be written presently as:

$$\boldsymbol{\sigma}^{(\alpha\beta\gamma)} = \mathbf{C}^{(\alpha\beta\gamma)} \boldsymbol{\varepsilon}^{(\alpha\beta\gamma)} - \boldsymbol{\Gamma}^{(\alpha\beta\gamma)} \Delta T \tag{4.173}$$

The following expression for the average stresses in the subcell can be established as:

$$\boldsymbol{\sigma}^{(\alpha\beta\gamma)} = \mathbf{C}^{(\alpha\beta\gamma)} \mathbf{X}^{(\alpha\beta\gamma)} - \boldsymbol{\Gamma}^{(\alpha\beta\gamma)} \Delta T \tag{4.174}$$

where

$$\mathbf{X}^{(\alpha\beta\gamma)} = \left[\varepsilon_{11}^{(\alpha\beta\gamma)},\ \varepsilon_{22}^{(\alpha\beta\gamma)},\ \varepsilon_{33}^{(\alpha\beta\gamma)},\ 2\varepsilon_{23}^{(\alpha\beta\gamma)},\ 2\varepsilon_{13}^{(\alpha\beta\gamma)},\ 2\varepsilon_{12}^{(\alpha\beta\gamma)} \right] \tag{4.175}$$

The displacement and traction continuity relations, Eqs. (4.166) and (4.171), in conjunction with the constitutive relations Eq. (4.174), provide a sufficient number of algebraic equations in the unknown microvariables. The microvariables are solved in terms of the global strain components. The subcell strains and stresses are then written in terms of the global strains using Eqs. (4.167) and (4.173). Then, using Eq. (4.172), expressions for the global stress in terms of the global strain are obtained: the effective constitutive equation for the short-fiber composite. For the most general case of parallelepiped inclusions, this matrix is of the form:

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 & 0 & 0 \\ & b_{22} & b_{23} & 0 & 0 & 0 \\ & & b_{33} & 0 & 0 & 0 \\ & & & b_{44} & 0 & 0 \\ \text{symmetric} & & & & b_{55} & 0 \\ & & & & & b_{66} \end{bmatrix} \tag{4.176}$$