

## Lecture 8 - Variational Calculus

Dr. Nicholas Smith

Wichita State University, Department of Aerospace Engineering

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### schedule

- Feb 25 - Variational Calculus
- Mar 2 - Variational Calculus
- Mar 4 - Boundary Conditions (HW3 Due)
- Mar 9 - Project Descriptions

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- lagrange multipliers
- calculus of variations

## lagrange multipliers

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## differential and variational statements

- A differential statement includes a set of governing differential equations established inside a domain and a set of boundary conditions to be satisfied along the boundaries
- A variational statement is to find stationary conditions for an integral with unknown functions in the integrand
- Variational statements are advantageous in the following aspects
  - Clear physical meaning, invariant to coordinate system
  - Can provide more realistic descriptions than differential statements (concentrated loads)
  - More easily suited to solving problems numerically or approximately
  - Can be more systematic and consistent than building a set of differential equations

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## stationary problems

- If the function  $F(u_1)$  is defined on a domain, then at  $\frac{dF}{du_1} = 0$  it is considered to be stationary
- This stationary point could be a minimum, maximum, or saddle point
- We use the second derivative to determine which of these it is:  $>0$  for a minimum,  $<0$  for a maximum and  $=0$  for a saddle point
- For a function of  $n$  variables,  $F(u_n)$  the stationary points are

$$\frac{\partial F}{\partial u_i} = 0$$

for all values of  $i$  - and to determine the type of stationary point we use

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## lagrange multipliers

- Let us now consider a function of several variables, but the variables are subject to a constraint

$$f(u_1, u_2, \dots) = 0$$

- Algebraically, we could use each provided constraint equation to reduce the number of variables
- For large problems, it can be cumbersome or impossible to eliminate some variables
- The Lagrange Multiplier method is an alternative, systematic approach

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## lagrange multiplier

- For a constrained problem at a stationary point we will have

$$dF = \frac{\partial F}{\partial u_1} du_1 + \dots + \frac{\partial F}{\partial u_n} du_n = 0$$

- The relationship between  $du_i$  can be found by differentiating the constraint

$$df = \frac{\partial f}{\partial u_1} du_1 + \dots + \frac{\partial f}{\partial u_n} du_n = 0$$

- We can combine these two equations using a Lagrange Multiplier

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## lagrange multiplier

- The Lagrange Multiplier,  $\lambda$  is an arbitrary function of  $u_i$
- We can choose the Lagrange Multiplier such that

$$\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0$$

- Which now leaves

$$\frac{\partial F}{\partial u_i} + \lambda \frac{\partial f}{\partial u_i} = 0 \quad i = 1, 2, \dots, n-1$$

- We now define a new function  $F^* = F + \lambda f$

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## lagrange multiplier

- This converts a constrained problem in  $n$  variables to an unconstrained problem in  $n + 1$  variables
- Notice that while the stationary values of  $F^*$  will be the same as the stationary values to  $F$ , they will not necessarily correspond
- For example, a minimum in  $F^*$  might be a maximum in  $F$
- This provides a systematic method for solving problems with any number of variables and constraints, and is also well-posed for numeric solutions

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- Design a box with given surface area such that the volume is maximized
- The box has no cover along one of the surfaces (open-face box)
- This gives the surface area as  $A = xy + 2yz + 2xz = C$
- worked example<sup>1</sup>

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<sup>1</sup><http://nbviewer.jupyter.org/github/ndaman/multiscale/blob/master/examples/Lagrange%20Multipliers.ipynb>

## calculus of variations

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## functional

- A functional of some unknown function  $y(x)$  is defined as

$$I = I[y(x)]$$

- A functional depends on all values of  $y(x)$  over some interval
- We will often use the form

$$I[y] = \int_a^b F(x, y(x), \dot{y}(x)) dx$$

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## bernoulli

- The original problem that motivated study of variational calculus
- Bernoulli 1696
- Design a chute between two points, A and B
- such that a particle sliding without friction under its own weight
- travels from A to B in the shortest time

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## variational statement

- To solve Bernoulli's problem we denote the arc length as  $s$ , speed as

$$v = \frac{ds}{dt}$$

- And we can find the total time as

$$t = \int_A^B \frac{ds}{v}$$

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## variational statement

- The arc length  $s$  can be found from

$$ds = \sqrt{dx^2 + dy^2}$$

- Since  $y = y(x)$  we can write  $dy = \dot{y}dx$
- We can now re-write  $ds$  as

$$ds = \sqrt{1 + \dot{y}^2} dx$$

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## variational statement

- From the conservation of energy we can also say that

$$\frac{1}{2}mv^2 = mgy$$

- Such that

$$v = \sqrt{2gy}$$

- We now need to find some function  $y(x)$  which minimizes the integral

$$t = \int_0^a \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}} dx$$

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## euler lagrange

- Now we develop a method for finding  $y(x)$
- Consider the functional

$$I[y] = \int_{x_0}^{x_1} F(x, y, \dot{y}) dx$$

- Where  $y(x)$  is subject to boundary conditions

$$y(x_0) = y_0$$

$$y(x_1) = y_1$$

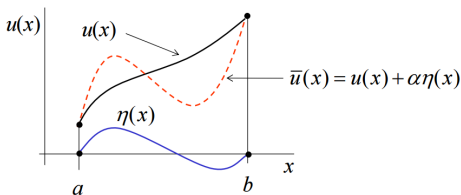
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- We assume that there is some solution,  $y(x)$  for which  $I$  is stationary
- We also assume that  $y(x)$  is continuous and differentiable in the problem domain
- Let us now choose some trial function

$$\bar{y}(x) = y(x) + \alpha\eta(x)$$

- Where  $\eta(x)$  is some arbitrary continuous function which vanishes at the boundaries

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- We can take the derivative of  $\bar{y}$  to find

$$\dot{\bar{y}} = \dot{y}(x) + \alpha \dot{\eta}(x)$$

- This now gives

$$I[\alpha] = \int_{x_0}^{x_1} F(x, \bar{y}, \dot{\bar{y}}) dx = \int_{x_0}^{x_1} F(x, y(x) + \alpha \eta(x), \dot{y}(x) + \alpha \dot{\eta}(x)) dx$$

- Once  $y(x)$  and  $\eta(x)$  are chosen,  $I$  is a function of  $\alpha$

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- We find the stationary function by letting  $\frac{dI}{d\alpha} = 0$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \frac{\partial F}{\partial \alpha} dx = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial F}{\partial \dot{\bar{y}}} \frac{\partial \dot{\bar{y}}}{\partial \alpha} \right) dx$$

- This simplifies to

$$\int_{x_0}^{x_1} \left( \frac{\partial F}{\partial \bar{y}} \eta + \frac{\partial F}{\partial \dot{\bar{y}}} \dot{\eta} \right) dx$$

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- Now we know that  $I$  will be stationary when  $\alpha = 0$  in which case  $\bar{y} = y$  therefore we can write

$$\int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial \dot{y}} \dot{\eta} \right) dx = 0$$

- And now we perform integration by parts on the second term

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## integration by parts

- Recall that

$$\int u dv = uv - \int v du$$

- We choose

$$u = \frac{\partial F}{\partial \dot{y}}$$

$$du = \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right)$$

$$v = \eta(x)$$

$$dv = \dot{\eta} dx$$

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- This gives (for the second term)

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial \dot{y}} \dot{\eta} dx = \frac{\partial F}{\partial \dot{y}} \eta \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \left( \frac{\partial F}{\partial y} \right) \eta(x)$$

- Combining with the original equation and simplifying gives

$$\int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y} \right) \right] \eta dx + \frac{\partial F}{\partial \dot{y}} \eta \Big|_{x_0}^{x_1} = 0$$

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## euler lagrange

- We already know that  $\eta|_{x_0}^{x_1} = 0$ , so we only need concern ourselves with the terms inside the integral
- Since this must be true for any arbitrary function,  $\eta$ , we can say that

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y} \right) = 0$$

- This is known as the Euler-Lagrange equation
- A solution to the Euler-Lagrange equation is called an extremal, and an extremal which satisfies the boundary conditions is called a stationary function

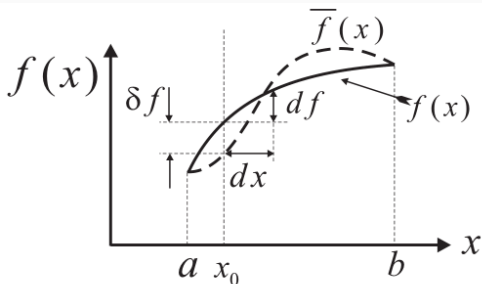
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- In variational calculus, we define the first variation as

$$\delta y = \bar{y} - y$$

- Note: while the variation follows many of the same rules as differentiation, it does not correspond to any slope, since  $\eta$  is completely arbitrary

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## variations

- Variational laws are analogous to differentiation

$$\begin{aligned}\delta(F_1 F_2) &= F_1 \delta F_2 + \delta F_1 F_2 \\ \delta \left( \frac{F_1}{F_2} \right) &= \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}\end{aligned}$$

- The variation and derivative are commutative

$$\frac{d}{dx}(\delta u) = \delta \left( \frac{du}{dx} \right)$$

- Similarly, the variation is commutative with the integral

$$\delta \int F dx = \int \delta F dx$$

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## variations

- We can also take the variation of a functional

$$\Delta F = F(x, y + \alpha \eta, \dot{y} + \alpha \dot{\eta}) - F(x, y, \dot{y})$$

- Expanding this function via a Taylor series gives

$$\Delta F = \left[ F(x, y, \dot{y}) + \left( \delta y \frac{\partial F}{\partial y} + \delta \dot{y} \frac{\partial F}{\partial \dot{y}} + \dots \right) \right] - F(x, y, \dot{y})$$

- And thus we call the variation of  $F$

$$\delta F = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial \dot{y}} \delta \dot{y} + \epsilon_1$$

Where  $\epsilon_1$  are terms of higher order than  $\sqrt{(\delta y)^2 + (\delta \dot{y})^2}$

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- We can use variational notation to find the Euler-Lagrange equation

$$I[y] = \int_{x_0}^{x_1} F(x, y, \dot{y}) dx$$

- and taking the variation

$$\delta I = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial \dot{y}} \delta \dot{y} \right] dx = 0$$

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- Using integration by parts on the second term, as before, we find

$$\delta I = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right) \right] \delta y dx = 0$$

- Since  $\delta y(x_0) = \delta y(x_1) = 0$
- Since this must be true for any arbitrary variation, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{y}} \right)$$

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- If the functional,  $F$ , does not depend on  $x$  explicitly (i.e. the only  $x$  dependence comes from  $y(x)$ ) then we can say

$$\frac{d}{dx} \left( F - \dot{y} \frac{\partial F}{\partial \dot{y}} \right) = 0$$

- or, similarly

$$F - \dot{y} \frac{\partial F}{\partial \dot{y}} = C$$

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## brachistochrone problem

- If we return now to Bernoulli's problem, we had found

$$t = \int_0^a \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}} dx$$

- Since this does not depend on  $x$  explicitly, we can use the simpler form of the Euler-Lagrange equation.

$$F = \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gy}}$$

- with

$$F - \dot{y} \frac{\partial F}{\partial \dot{y}} = C$$

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## brachistochrone problem

- Computing the partial derivative we find

$$\frac{\partial F}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{2gy}\sqrt{1+\dot{y}^2}}$$

- Which gives in the Euler-Lagrange equation

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} - \frac{\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = C$$

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## brachistochrone problem

- Simplifying gives

$$\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = C$$

- We can square both sides and lump constants together

$$y(1+\dot{y}^2) = \frac{1}{2gc^2} = c_1$$

- And solving for  $\dot{y}$ , taking only the positive solution

$$\dot{y} = \frac{\sqrt{c_1 - y}}{\sqrt{y}}$$

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- The Brachistochrone problem can be solved using parametric equations

$$x = k^2(\theta - \sin \theta)$$

$$y = k^2(1 - \cos \theta)$$

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### example

- We can also use variational calculus to prove that the shortest distance between two points is a straight line
- The distance along a curve is given by

$$L = \int_a^b ds$$

- Where  $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \dot{y}^2} dx$
- So we can find the minimum of the functional

$$I[y] = \int_a^b \sqrt{1 + \dot{y}^2} dx$$

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- Find the Euler-Lagrange equation for

$$I[y] = \int y \sqrt{1 + \dot{y}^2} dx$$

- Find the Euler-Lagrange equation for

$$I[y] = \int [\dot{y}^2 + y^2 + 2xy] dx$$

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## next class

- Boundary conditions
- Multi-variate variational calculus
- Approximate solutions
- Variational Asymptotic Method

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