Lecture 4 - Eshelby

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schedule

- 27 Jan - 1D Micromechanics (HW1 Due)
- 1 Feb - Mean-field
- 3 Feb - Orientation Averaging (HW2 Due)
- 8 Feb - Variational Calculus

outline

- eshelby
- aspect ratio

eshelby's equivalent inclusion

eshelby

- Eshelby formulated the exact elastic solution for an elliptical inclusion in an infinite matrix
- While this is not often useful, it serves as an exact analytical model to compare numerical results with
- It is also the base for more useful mean-field theories

eshelby's thought experiment

- Eshelby solution starts with a thought experiment
- Suppose we have a homogeneous, elastic body in equilibrium
- We now cut an ellipsoidal pieces out of that body and allow it to undergo a stress-free transformation, such as thermal expansion
- This stress-free transformation is referred to as the transformation strain, ϵ^T

eshelby's thought experiment

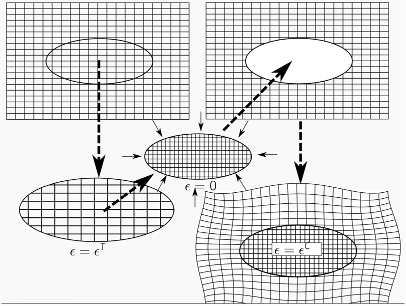


Figure 1: Eshelby's thought experiment

eshelby's thought experiment

- Now, we weld that expanded ellipsoid back into the original body
- Tractions need to be applied to force it to fit
- Once the stresses equilibrate, the ellipsoid has a constrained strain, ϵ^C

eshelby

- After equilibrium is reached the inclusion is still under a state of uniform strain
- The inclusion stress, σ_I can be found as:

$$\sigma_I = C_m(\epsilon^C - \epsilon^T)$$

Where C_m is the stiffness of the material.

eshelby

- One of Eshelby's critical findings is that

$$\epsilon^C = S\epsilon^T$$

- S is known as the Eshelby Tensor, and is a fourth-order tensor
- Function of shape and poisson's ratio
- It has been calculated exactly for ellipsoids, and numerically for other shapes

eshelby tensor

- ν represents the matrix Poisson's ratio
- s is the aspect ratio of the fibers
- $I_1 = \frac{2s}{(s^2-1)^{\frac{3}{2}}}(s(s^2-1)^{\frac{1}{2}} - \cosh^{-1} s)$
- $Q = \frac{3}{8(1-\nu)}$
- $R = \frac{1-2\nu}{8(1-\nu)}$
- $T = \frac{Q(4-3I_1)}{3(s^2-1)}$
- $I_3 = 4 - 2I_1$

eshelby tensor

S_{ijkl}	Long Fibers	Short Fibers (Ellipsoids)
$S_{1111} = S_{2222}$	$\frac{5-\nu}{8(1-\nu)}$	$Q + RI_1 + \frac{3T}{4}$
S_{3333}	0	$\frac{4Q}{3} + RI_3 + 2s^2 T$
$S_{1122} = S_{2211}$	$\frac{-1+4\nu}{8(1-\nu)}$	$\frac{Q}{3} - RI_1 + \frac{4T}{}$

inclusions

- Eshelby's initial thought experiment was for a homogeneous material
- To consider a different type of inclusion, we need to relate the transformation strain between some fictitious ellipsoid of matrix material which would be equivalent to our inclusion.
- We will refer to the inclusion stiffness as C_f, the transformation strain in the matrix as ϵ^T, and the transformation strain in the inclusion ϵ^{T*}.

inclusions

- We are trying to find a transformation equivalent to our inclusion, so we set

$$\sigma_I = C_m(\epsilon^C - \epsilon^T) = C_f(\epsilon^C - \epsilon^{T*})$$

- Now we substitute the relation $\epsilon^C = S\epsilon^T$

$$C_m(S - I)\epsilon^T = C_f(S\epsilon^T - \epsilon^{T*})$$

- We can solve this to find the transformation strain

$$\epsilon^T = \left[(C_f - C_m)S + C_m\right]^{-1} C_f\epsilon^{T*}$$

stiffness

- Since the transformation strain is arbitrary, we can choose ϵ^T such that ϵ^{T*} is 0
- Now suppose we impose some strain, ϵ^0 on the composite
- The stress in the inclusion will be

$$\sigma_I = C_m(\epsilon^0 + \epsilon^C - \epsilon^T) = C_f(\epsilon^0 + \epsilon^C)$$

- Simplifying terms gives

$$(C_f - C_m)\left(\epsilon^0 + \epsilon^C\right) = -C_m\epsilon^T$$

stiffness

- We now assume $\epsilon^0 + \epsilon^C = \bar{\epsilon}^f$ and multiply both sides by SC_m^{-1}

$$S\left(C_m\right)^{-1}\left(C_f - C_m\right)\bar{\epsilon}^f = -\epsilon^C$$

- Recall $S\epsilon^T = \epsilon^C$
- We can also write ϵ^C in terms of $\bar{\epsilon}^f$

$$S\left(C_m\right)^{-1}\left(C_f - C_m\right)\bar{\epsilon}^f = \epsilon^0 - \bar{\epsilon}^f$$

strain concentration tensor

- Finally, we can add $I\bar{\epsilon}^f$ to both sides to find

$$[I + S(C_m)^{-1}(C_f - C_m)]\bar{\epsilon}^f = \epsilon^0$$

- We define the inverse of the left-hand side the Eshelby strain-concentration tensor

$$A^E = [I + S(C_m)^{-1}(C_f - C_m)]^{-1}$$

- The stiffness can be calculated as

$$C = C_m + v_i(C_f - C_m)A^E$$

stiffness

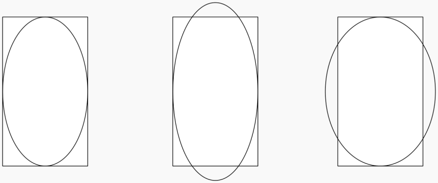
- This stiffness calculation is valid for any number of inclusions
- However, it is only appropriate for very dilute concentrations (<1% volume fraction)
- This ensures that the assumption $\epsilon^0 + \epsilon^C = \bar{\epsilon}^f$

aspect ratio

aspect ratio

- Some studies have been done to evaluate Eshelby tensors for short fibers
- Long fibers are approximated by an ellipsoid with infinitely long major axis
- This is not appropriate for short fibers
- We could logically consider three different ellipsoids to represent a short fiber

aspect ratio



(a) The first ellipsoid considered, with aspect ratio equal to fiber aspect ratio

(b) The second ellipsoid considered, with minor axis equal to fiber diameter and volume equal to fiber volume

(c) The third ellipsoid considered, with major axis equal to fiber length and volume equal to fiber volume

Figure 2: aspect ratio comparisons

aspect ratio

- Steif and Hoysan investigated the effect of aspect ratio numerically
- Found that (a) and (c) were good for short fibers
- As fibers get longer, and as stiffness ratio of fiber to matrix increases, (a) gives best results
- (a) is also the easiest to use (same aspect ratio), so that is what is done in Eshelby-based models

fiber orientation

- With Eshelby (and derivative models), fibers at different orientations are modeled as a different inclusion
- Since the Eshelby tensor, S is a fourth-order tensor, we can treat it the same way as C
- Write it as 6x6 matrix, transform using R^{σ}

example

- As an example, let us consider a "laminate" of short fiber composites
- This is a good approximate for many 3D printed composites
- We have a $\pm 45^{\circ}$ laminate, with very short carbon fibers, $s = 15$

example

- First we find the Eshelby tensor for $s = 15$
- We also need the matrix Poisson's ratio, $\nu = 0.4$
- We find the parameters here
- Then we use this slide to find S_{ijkl}
- Notice that this assumes fibers are pointed in the 3-direction

example

- Next, we rotate S_{ijkl} to find S^{45} and S^{-45}
- Notice: the eshelby tensor, S accounts for rotation, we do not rotate C_f
- So we find A^{45} and A^{-45}

$$A^{45} = \left[I + S^{45}(C_m)^{-1}(C_f - C_m)\right]^{-1}$$

$$A^{-45} = \left[I + S^{-45}(C_m)^{-1}(C_f - C_m)\right]^{-1}$$

example

- This gives our total stiffness calculation as

$$C = C_m + v^{45}(C_f - C_m)A^{45} + v^{-45}(C_f - C_m)A^{-45}$$

- If we assume the volume fraction of fibers in our part is 20%
- And that there are equally many fibers in 45 and -45 directions
- Then $v^{45} = v^{-45} = 0.1$
- Note: Since this is not a dilute concentration, we would not expect this to be very accurate

example

- Python code for this example (with some typical values for C_m and C_f) is posted here[1]

[1] https://colab.research.google.com/drive/1aUs_BfeWugyUfwQiKoPx5g-hxvrsQqNs?usp=sharing

fiber orientation

fiber orientation

- While a laminate analogy works well for some cases, in general short fibers are not aligned in laminates
- It is not practical to model each possible fiber orientation as a separate inclusion
- Advani-Tucker introduced a tensorial representation of fiber orientation

fiber in spherical coordinates

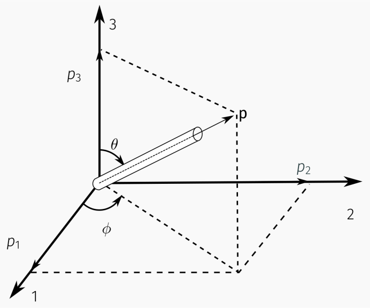


Figure 3: single fiber coordinate system

fiber direction components

Component	Definition
p_1	$\sin\theta\cos\phi$
p_2	$\sin\theta\sin\phi$
p_3	$\cos\theta$

orientation tensor

- Within a given volume, a distribution of fibers can be defined by some orientation distribution function, $\psi(\theta, \phi)$.
- Advani and Tucker introduced tensor representations of fiber orientation distribution functions

$$a_{ij} = \oint p_i p_j \psi(p) dp$$

- And

$$a_{ijkl} = \oint p_i p_j p_k p_l \psi(p) dp$$

- Note: any order tensor may be defined in this manner, the orientation distribution function must be even, due to fiber symmetry, and thus any odd-ordered tensor will be zero.

orientation tensor

- It can be noted that some symmetries must exist due to the way the tensors are defined.
- In the second order tensor we have

$$a_{ij} = a_{ji}$$

- and in the fourth order tensor

$$a_{ijkl} = a_{jikl} = a_{kijl}$$

and so on for any permutation of i, j, k and l.

orientation tensor

- The orientation tensor is also normalized such that:

$$a_{ii} = 1$$

- And any lower-order tensor can be expressed in terms of the next higher-order tensor, for example

$$a_{ijkk} = a_{ij}$$

example - 2D random

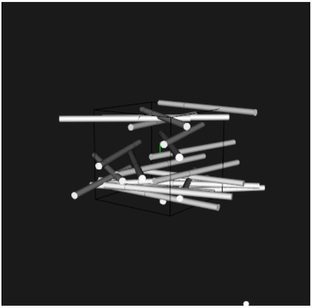


Figure 4: A visualization of a 2D random orientation distribution. This is expressed with the second-order tensor a_11 = a_22 = 0.5, with all other a_ij = 0.

example - 3D random

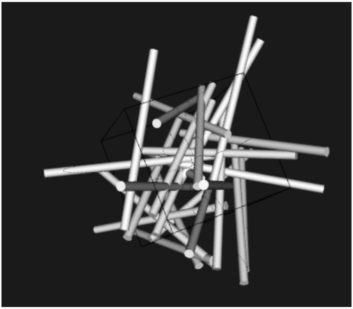


Figure 5: A visualization of a 3D random orientation distribution. This is expressed with the second-order tensor a_11 = a_22 = a_33 = 1/3, with all other a_ij = 0.

example - aligned 45

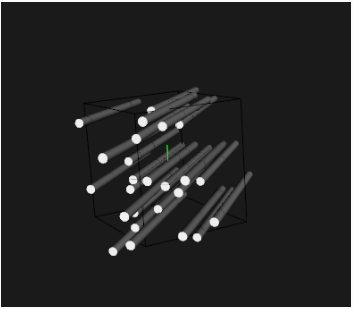


Figure 6: A visualization of a perfectly aligned, off-axis orientation distribution. This is expressed by rotating the tensor with a11 = 1 and all other aij = 0.

next class

- Orientation averaging
- Self-consistent and Mori-Tanaka methods
- Textbook pages 131-150