Lecture 10 - Variational Asymptotic Method

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schedule

- Mar 4 - Boundary Conditions (HW3 Due)
- Mar 9 - Project Descriptions
- Mar 11 - SwiftComp
- Mar 16 - Work Day

outline

- converting to variational statements
- ritz method
- variational asymptotic method

converting to variational statements

differential to variational

- In general, a differential statement can be expressed as

$$\begin{aligned} L(u) + f = 0 \qquad & \text{in} \qquad \Omega \\ B(u) + g = 0 \qquad & \text{on} \qquad \Gamma \end{aligned}$$

- Where L is a differential operator, B can be either differential or algebraic
- Ω is the domain and Γ is the boundary

differential to variational

- The equivalent variational statement is

$$\Pi(u) = \int_\Omega \delta u[L(u) + f] d\Omega - \int_\Gamma \delta u[B(u) + g] d\Gamma = 0$$

- We can then perform integration by parts on $L(u)$ to form the variational statement with

$$\delta\Pi = 0$$

example

- 2D steady-state heat transfer

ritz method

ritz method

- Only a small set of Euler-Lagrange equations have exact solutions
- The Ritz method is one way to find approximate (and exact) solutions
- In the Ritz method we approximate some continuously differentiable function with a linear combination of functions
- We can choose the form of these functions based on our problem, polynomials and trig functions are common

$$y_n = \sum a_k w_k$$

ritz method

- The general method for using the Ritz method with variational statements can be summarized as
  1. Select a set of trial functions
  2. Form a linear combination of trial functions to approximate $y \approx y_n$
  3. Substitute y_n into the functional, $I[y] = I[a_1, a_2, ..., a_n]$
  4. Obtain a system of equations by carrying out the partial derivatives $\frac{\partial I}{\partial a_n}$
  5. Solve this system for the unknown coefficients to find y

ritz method

- We can increase the accuracy by including more terms
- If our set contains the exact solution, the solution will be exact
- The Ritz method is a direct method solving stationary problems of functionals and an indirect method for solving Euler-Lagrange equations

kantarovich method

- A slightly different approach to the Ritz method is used by Kantarovich

$$I[y] = \int_t \int_x F(x, t, y) dxdt$$

- Where boundary conditions are $y(x1,\ t)$=$y1(t)$, $y(x2,\ t)$=$y2(t)$ and $y(x,\ t1)$=$y3(x)$
- Where boundary conditions are $y(x_1, t) = y_1(t), y(x_2, t) = y_2(t)$ and $y(x, t_1) = y_3(x)$
- The trial function will then have the form

$$y(x, t) = g^P(x) + \sum f_j(t) g(j) H(x)$$

- This gives a functional which can be solved for *fi(t)*

examples

- Solve the differential equation

$$\frac{d^2u}{dx^2} + u + x = 0$$

for $0 \leq x \leq 1$ with $u(0){=}u(1){=}0$

examples

- A 2D domain defined by $x \in [0, \pi]$ and $y \in [0, 1]$ solve the following PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Where $u(0, y) = u(\pi, y) = u(0, x) = 0$ and $u(x, 1) = \sin x$

examples

- worked solutions here[1]

[1] https://nbviewer.jupyter.org/github/ndaman/multiscale/blob/master/examples/Ritz.ipynb

variational asymptotic method

asymptotic analysis

- Asymptotic analysis is a mathematical method to describe limiting behavior
- It is used to numerically approximate solutions
- Also used in probability theory (large-sample behavior of random variables)
- Computer science (algorithm performance)

example

- Compute $\sin 39^\circ$ without using trig functions on calculator
- We know $\sin 30^\circ = 0$ and $\cos 30^\circ = \sqrt{3}/2$
- Expand $\sin \theta$ about some known θ_0 using Taylor Series

$$\sin \theta = \sin \theta_0 + (\theta - \theta_0) \cos \theta_0 - \frac{1}{2}(\theta - \theta_0)^2 \sin \theta_0 + \ldots$$

- With only three terms of a Taylor series, we have a very close approximation
- This only works when $\theta - \theta_0$ is less than 1 in radians

o notation

- Suppose $f(x)$ and $g(x)$ are continuous functions with defined limits as $x \to x_0$
- $f(x) = O(g(x))$ as $x \to x_0$ if $|f(x)| \leq K|g(x)|$ in the neighborhood of x_0 where K is a constant. We say that $f(x)$ is asymptotically bounded by $g(x)$ or that $f(x)$ is of the order of $g(x)$
- $f(x) = o(g(x))$ as $x \to x_0$ if $|f(x)| \leq \epsilon|g(x)|$ in the neighborhood of x_0 for all positive values of ϵ. We say that $f(x)$ is asymptotically smaller than $g(x)$
- $f(x)\ g(x)$ as $x \to x_0$ if $f(x) = g(x) + o(g(x))$ in the neighborhood of x_0. We say that $f(x)$ is asymptotically equal to $g(x)$

characteristic length

- If we define the maximum difference of a function between too points as

$$\bar{f} = \max|f(x_1) - f(x_2)|$$

- Then for some l the following will be true

$$|\frac{df}{dx}| \leq \frac{\bar{f}}{l}$$

- The largest l which satisifes this equation is termed the characteristic length
- For estimating higher order derivatives we us

variational asymptotic method

- Let us consider a functional $I[u, \epsilon]$ which depends on some elements, u, as well as some small parameter, ϵ
- For a beam, we could say that u represents the 3D displacement field, while ϵ is the aspect ratio of the cross section with respect to the length
- Let us call the stationary value of this functional $\bar{u}$
- $\bar{u}$ will be a function of ϵ, and will approach its asymptotic limit as $\epsilon \to 0$
- This is often referred to as the zeroth order approximation

varational asymptotic method

- We start with a zeroth-order approximation and let $I_0[u] = I[u, 0]$ and find the stationary values
- The following cases could be encountered
  1. $I_0[u]$ has isolated stationary points
  2. $I_0[u]$ has non-isolated stationary points
  3. $I_0[u]$ does not have stationary points
  4. $I_0[u]$ is meaningless (undefined)

case one

- If I_0 has isolated stationary points, we can use them as a first approximate for stationary points of I
- We now write $u = \bar{u} + u'$ and we can arrange terms to find $I_1[u', \epsilon]$
- The stationary points of I_1 can then be found, this process is repeated to the desired order

example

- Approximate the stationary values of

$$f(u, \epsilon) = u^2 + u^3 + 2\epsilon u + \epsilon u^2 + \epsilon^2 u$$

case two

- Consider the following

$$f(x, y, \epsilon) = f_0(x) + \epsilon g(x, y)$$

- If we drop the small term, $\epsilon g(x, y)$, we find stationary lines in the y-direction

example

- Approximate the stationary values of

$$f(x, y, \epsilon) = \cos(x - y) + \epsilon \left(\frac{1}{x} + y \right)$$

example

- Approximate the stationary values of

$$f(x, y, \epsilon) = x^2 - 2x + 4\epsilon(x-1)y + \epsilon^2 y^2 + 2\epsilon^2 y$$

cases three and four

- It is not uncommon to have a problem where I_0 has no stationary points

$$f(u, \epsilon) = u + \epsilon u^2 + \sin \epsilon u$$

- The only way to approach such problems is to make a substitution
- For the above problem, if we let $v = \epsilon u$ and $g = \epsilon f$ we find

$$g(v, \epsilon) = v + v^2 + \epsilon \sin v$$

next class

- Project description
- SwiftComp