3. NUMERICAL METHODS AND ALGORITHMS OF ELEMENTARY ALGEBRA

This chapter discusses basic numerical methods and algorithms of computational mathematics in the field of elementary algebra. At first, numerical methods for refining the roots of nonlinear equations are presented. In the general case, such equations do not have analytical formulas for their roots or these formulas are too cumbersome and inconvenient for practical use. In particular, in the beginning of the 19^{th} century, it was proved that nonlinear equations higher than the fourth degree are unresolvable in radicals, even if radicals of arbitrary degree are used. The considered numerical methods for refining the roots of nonlinear equations are of an iterative nature and, in principle, are suitable for finding the roots of equations of any kind. The interval halving (dichotomy) method, the fixed-point iteration method, the Newton (tangent) method, and the secant (chord) method are described in detail. For refining the roots of a system of nonlinear equations, generalizations of the fixed-point iteration method and the Newton method considered for one nonlinear equation are presented.

3.1. SOLVING NONLINEAR EQUATIONS

Suppose it is required to find a solution to the nonlinear (algebraic or transcendental) equation

$$f(x) = 0,$$

i.e. all the values of x (roots) that make this equation an identity.

The numerical solution of the nonlinear equations in the form $f(x) = 0$ implies finding the values of x that satisfy this equation with a given accuracy of ε and consists of the following main steps:

1) separation (isolation, localization) of the roots of the equation;
2) refinement using a certain computational algorithm of a specific selected root with the given accuracy ε.

The purpose of the first step is to find intervals from the domain of the function $f(x)$, inside which only one root of the equation being solved is contained. Sometimes, calculators are limited to considering only some part of the domain, which is of interest for one reason or another. In the general case, the root separation step cannot be algorithmized. For certain classes of equations (the most famous of which is the class of algebraic equations), special methods for roots separation have been developed, which greatly facilitate this separation. Often, roots of nonlinear equations are separated "manually" using all possible information about the function $f(x)$. In some cases, the approximate value of the root can be determined from physical considerations if it comes to solving a nonlinear equation related to a specific applied problem. In general, methods used to implement this step can be divided into graphic and analytical.

The following theorem is useful for the analytical method of root separation. The continuous strictly monotonic function $f(x)$ has a unique zero on the interval $[a,b]$, if and only if it takes on the values of different signs at its ends. If the function has the same signs at the ends of the interval, then on this interval there are either no roots or their quantity is an even number. A sufficient sign of the monotonicity of the function $f(x)$ on the interval $[a,b]$ is the preservation of the sign of the derivative function.

It is advisable to use the graphic method of root separation, which is more illustrative, when it is possible to plot the graph of the function $y = f(x)$. The presence of the graph of the original function gives a direct idea about the quantity and location of the zeros of the function, which allows us to determine the intervals within which only one root is contained. If plotting the graph of the function $y = f(x)$ is difficult, it is often convenient to convert the original equation to an equivalent form $f_1(x) = f_2(x)$ and plot the graphs of the functions $y = f_1(x)$ and $y = f_2(x)$. The abscissas of the intersection points of these graphs will correspond to the values of the roots of the equation being solved.

One way or another, the completion of the first step should result in finding intervals, each of which contains only one root of the equation.

Example. Isolate the roots of the equation $f(x) = e^{2x} + 3x - 4 = 0$.

Let's apply the analytical method. Let's find the derivative:

$f'(x) = 2e^{2x} + 3 > 0$ for $\forall x$.

The function $f(x)$ is strictly monotonically increasing and has no more than one root. To localize it, let's apply the graphical method. Let's transform the original equation to the following equivalent form: $e^{2x} = 4 - 3x$.

Now let's plot the graphs of the functions $f_1(x) = e^{2x}$ and $f_2(x) = 4 - 3x$ (fig. 2).

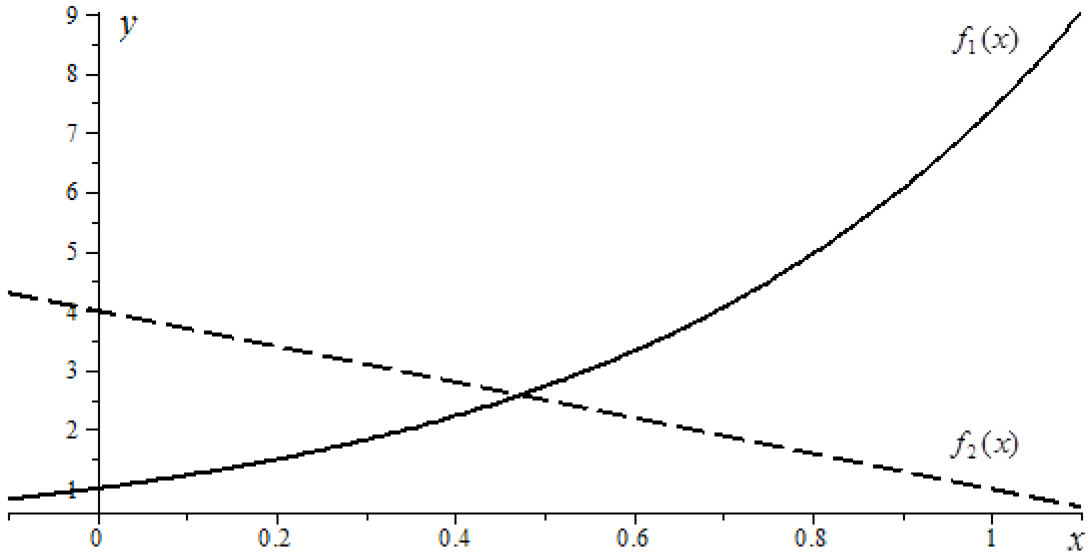

Fig. 2. The graphical method of root localization for the equation

From them, we determine that the equation to be solved has one root (the point of graphs intersection), which is within the interval $0.4 \le x^* \le 0.6$ (as $f_1(0.4) < f_2(0.4)$ and $f_1(0.6) > f_2(0.6)$). Later, we shall refine the root value on this interval with the required accuracy using the iterative methods given below.

At the second step of refinement, a certain interval containing only one root of the equation being solved is considered. When finding the root, two types of methods are used: direct and iterative. In direct methods, the root of the equation is fundamentally found for a finite, predetermined number of operations. Direct methods can be used to solve only some of the simplest algebraic and trigonometric equations. Therefore, to refine the root with the required accuracy ε, a certain iterative method is usually applied that consists of constructing the numerical sequence $x^{(k)}, k = 0,1,2,...$ converging in the limit $k \to \infty$ to the desired root x^* of the original equation. One of the points of the interval is taken as the initial approximation of the root. In this case, the exact solution – the limit of this sequence – fundamentally cannot be achieved for a finite, predetermined number of operations.

Definition. An important characteristic of iterative methods is the *order of convergence* of the process. It is said that the method has the *n*-th order of convergence if $\left|x^{(k)} - x^*\right| = C\left|x^{(k-1)} - x^*\right|^n$, where C is a constant independent of *n*. When $n=1$, it is the first-order convergence or linear convergence, when $n=2$, it is the second-order convergence or quadratic convergence, etc.

Definition. It is said that a method is *n-step* if to construct the iterative sequence $x^{(k)}, k = 0,1,2,...$, it is necessary to calculate the function $f(x)$ at *n* previous points of the sequence; for example, one-step – if it is necessary to calculate the function at one point of the sequence, two-step – if at two points, etc.

3.2. INTERVAL HALVING (DICHOTOMY) METHOD

The process of refining the root of the equation $f(x)=0$ on the interval $[a,b]$ by the interval halving method, provided that the function $f(x)$ is continuous on this interval and $f(a)\cdot f(b)<0$, is as follows.

The initial interval $[a^{(0)},b^{(0)}]$ is divided in half. If $f\left(\frac{a+b}{2}\right)=0$, then $x^* = \frac{a+b}{2}$ is the root of the equation. If $f\left(\frac{a+b}{2}\right)\neq 0$, then the approximation of the root is $x^{(0)} = \frac{a+b}{2}$, after which that half of the interval $\left[a,\frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2},b\right]$ is selected, at which ends the function $f(x)$ has opposite signs. The new interval $[a^{(1)},b^{(1)}]$ is again halved and the same consideration is held, etc.

As a result, at some stage *k*, either the exact root of the equation is found or there is a sequence of intervals embedded in each other $[a^{(0)},b^{(0)}]$, $[a^{(1)},b^{(1)}]$,

$[a^{(2)},b^{(2)}]$, …, $[a^{(k)},b^{(k)}]$, for which $f(a^{(k)})\cdot f(b^{(k)})<0,\ k=0,1,2,\ldots$, and the root approximation is the middle of the last interval

$$x^{(k)}=\frac{a^{(k)}+b^{(k)}}{2}.$$

The iterative process stops if the length of the interval $[a^{(k)},b^{(k)}]$ becomes less than 2ε, i.e.

$$\varepsilon^{(k)}=\frac{b^{(k)}-a^{(k)}}{2}=\frac{b-a}{2^{k+1}}.$$

Obviously, as k increases, the error tends to zero not slower than the geometric progression with the denominator ½. The dichotomy is simple and reliable, always converges, although slowly, and is resistant to rounding errors.

Example. Refine the value of the root of the equation $f(x)=e^{2x}+3x-4=0$ in the interval $[0.4, 0.6]$ with an accuracy of $\varepsilon=10^{-3}$ using the interval halving method.

The calculation results are presented in table 1.

The final approximation of the root is $x^*\approx x^{(7)}=0.4742$.

Table 1

k	$a^{(k)}$	$b^{(k)}$	$f(a^{(k)})$	$f(b^{(k)})$	$\frac{a^{(k)}+b^{(k)}}{2}$	$f\left(\frac{a^{(k)}+b^{(k)}}{2}\right)$	$\varepsilon^{(k)}$
0	0.4000	0.6000	-0.5745	1.1201	0.5000	0.2183	0.1000
1	0.4000	0.5000	-0.5745	0.2183	0.4500	-0.1904	0.0500
2	0.4500	0.5000	-0.1904	0.2183	0.4750	0.0107	0.0250
3	0.4500	0.4750	-0.1904	0.0107	0.4625	-0.0906	0.0125
4	0.4625	0.4750	-0.0906	0.0107	0.4688	-0.0402	0.0062
5	0.4688	0.4750	-0.0402	0.0107	0.4719	-0.0148	0.0031
6	0.4719	0.4750	-0.0148	0.0107	0.4734	-0.0024	0.0016
7	0.4734	0.4750	-0.0020	0.0107	**0.4742**	0.0042	**0.0008**

3.3. PROGRAM #06

Below is a proposed variant of the program algorithm for refining the value of the root of the equation using the interval halving method.

```
ALGORITHM "Interval halving method"
INPUT     f(), a, b, e
OUTPUT    x, k
BEGIN
    IF (f(a)*f(b)≥0) END
    x:=(a+b)/2
    k:=0
    CYCLE "Iteration" WHILE ((b-a)/2>e)
        IF (f(x)=0) BREAK
        IF (f(a)*f(x)<0) b:=x
        IF (f(x)*f(b)<0) a:=x
        x:=(a+b)/2
        k:=k+1
    PRINT x, k
END
```

3.4. FIXED-POINT ITERATION METHOD (NONLINEAR EQUATION)

When using the fixed-point iteration method, the equation $f(x)=0$ is replaced by an equivalent equation with the located linear term:

$$x=\varphi(x).$$

The solution is sought by constructing the sequence with the formula:

$$x^{(k+1)}=\varphi(x^{(k)}),\ k=0,1,2,\ldots,$$

starting from some given value $x^{(0)}$. If $\varphi(x)$ is a continuous function, and $x^{(k)},\ k=0,1,2,\ldots$ is a converging sequence, then the value $x^{*}=\lim\limits_{k\to\infty}x^{(k)}$ is a solution to the equation $f(x)=0$.

The conditions for the convergence of the method and the estimation of its error are determined by the theorem. Suppose the function $\varphi(x)$ is defined and differentiable on the interval $[a,b]$. Then, if these conditions are met

$$\varphi(x) \in [a,b], \forall x \in [a,b],$$

$$\exists q : |\varphi'(x)| \le q < 1, \forall x \in (a,b),$$

i.e. $\varphi(x)$ is a contracting mapping, then the equation $f(x) = 0$ has the only on $[a,b]$ root x^*, to which the sequence $x^{(k)}, k = 0,1,2,...$ determined by the fixed-point iteration method converges, starting with any $x^{(0)} \in [a,b]$, and the convergence is linear. Usually, $x^{(0)} = \dfrac{a+b}{2}$ is taken.

Indeed, since $x^{(k)} = \varphi(x^{(k-1)})$ and $x^* = \varphi(x^*)$ is true for the root, according to the Lagrange finite increment theorem for the differentiable on $[a,b]$ function $\varphi(x)$:

$$x^* - x^{(k)} = \varphi(x^*) - \varphi(x^{(k-1)}) = \varphi'(\xi) \cdot (x^* - x^{(k-1)}), \xi \in [x^*, x^{(k-1)}],$$

$$\left| x^* - x^{(k)} \right| \le \left| \max_{\xi \in [a,b]} \varphi'(\xi) \right| \cdot \left| x^* - x^{(k-1)} \right|.$$

Assuming $q = \left| \max_{\xi \in [a,b]} \varphi'(\xi) \right|$, we get:

$$\left| x^* - x^{(k)} \right| \le q \cdot \left| x^* - x^{(k-1)} \right| \le q^2 \cdot \left| x^* - x^{(k-2)} \right| \le ... \le q^k \cdot \left| x^* - x^{(0)} \right|,$$

which means that $\lim_{k \to \infty} \left| x^* - x^{(k)} \right| = 0$ only if $q < 1$ (sufficient condition). If though $|\varphi'(x)| > 1$, then iterations may fail to converge.

Four cases of mutual arrangement of the lines $y = x$ and $y = \varphi(x)$ near the root of the equation $x = \varphi(x)$ and the corresponding iterative processes are shown in fig. 3.

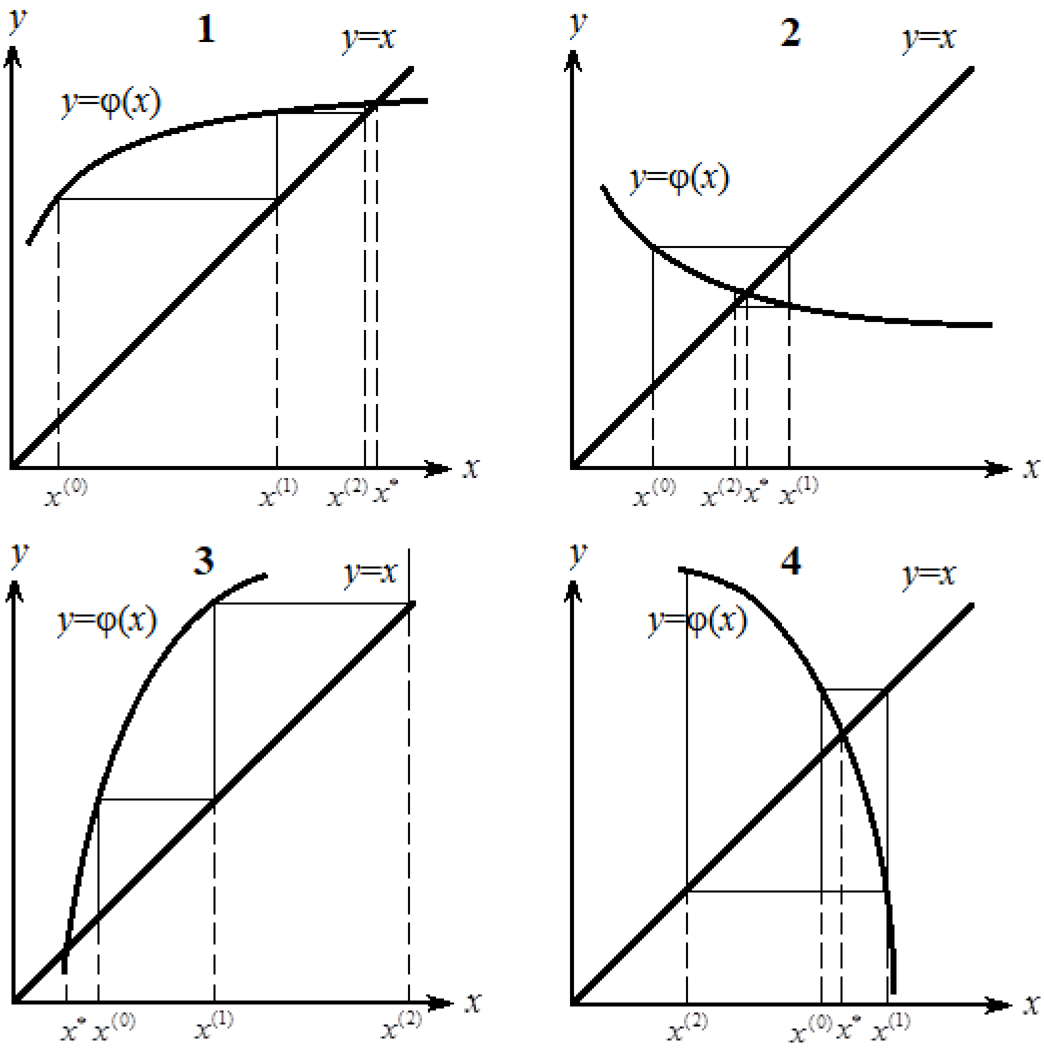

Fig. 3. Geometric interpretation of the fixed-point iteration method

Fig. 3.1 and 3.2 correspond to the case $|\varphi'(x)|<1$ – the iteration process converges. Moreover, in the first case $\varphi'(x)>0$ and the convergence is of one-sided nature, and in the second case $\varphi'(x)<0$ and the convergence is of two-sided nature. Fig. 3.3 and 3.4 correspond to the case $|\varphi'(x)|>1$ – the iteration process diverges, wherein there is one-sided and two-sided divergence respectively.

For the fixed-point iteration method, the error estimates are valid:

$$\left|x^* - x^{(k)}\right| \le \frac{q}{1-q}\left|x^{(k)} - x^{(k-1)}\right| \text{ and } \left|x^* - x^{(k)}\right| \le \frac{q^k}{1-q}\left|x^{(1)} - x^{(0)}\right|.$$

Indeed, according to the Lagrange theorem:

$$x^* - x^{(k)} = \varphi(x^*) + \varphi(x^{(k)}) - \varphi(x^{(k)}) - \varphi(x^{(k-1)}) = \varphi'(\xi_1)\cdot(x^* - x^{(k)}) + \varphi'(\xi_2)\cdot(x^{(k)} - x^{(k-1)}),$$

$$\left|x^* - x^{(k)}\right| \le q\cdot\left|x^* - x^{(k)}\right| + q\cdot\left|x^{(k)} - x^{(k-1)}\right|, \text{ i.e. } \left|x^* - x^{(k)}\right| \le \frac{q}{1-q}\cdot\left|x^{(k)} - x^{(k-1)}\right|.$$

So, in practical calculations, the rule $\varepsilon^{(k)} = \frac{q}{1-q} \cdot \left| x^{(k)} - x^{(k-1)} \right|$ is used as a condition for the termination of iterations. Herewith, it is clear that the convergence rate of the fixed-point iteration method depends on the value of *q*: the smaller *q* is, the faster the method converges. Since the original equation $f(x) = 0$ can be converted to the form $x = \varphi(x)$ in many ways, it is obvious that for the fixed-point iteration method it is advisable to use that equation $x = \varphi(x)$, for which *q* has the smallest value.

If the direct transformation of the equation $f(x) = 0$ to the form $x = \varphi(x)$ does not allow to obtain an equation, for which the conditions for convergence of the fixed-point iteration method are satisfied, then we can transform the equation $f(x) = 0$ to the following equivalent equation:

$$x = x - \lambda f(x).$$

For this equation, $\varphi(x) = x - \lambda f(x)$. Here, $\lambda > 0$ is a parameter that is selected in such a way that the inequality $|\varphi'(x)| = |1 - \lambda f'(x)| \le q < 1$, i.e. $\lambda < \frac{sign\left(f'(x)\right)}{\max\left|f'(x)\right|}$, is satisfied within the desired region.

In the general case, we can even assume $\varphi(x) = x + \psi(x) \cdot f(x)$, where $\psi(x)$ is a continuous arbitrary function of a constant sign.

Example. Refine the value of the root of the equation $f(x) = e^{2x} + 3x - 4 = 0$ in the interval $[0.4, 0.6]$ with an accuracy of $\varepsilon = 10^{-3}$ using the fixed-point iteration method.

The equation $f(x) = e^{2x} + 3x - 4 = 0$ can be rewritten as follows:

$$x = \frac{4 - e^{2x}}{3} \text{ or } x = \frac{\ln(4 - 3x)}{2}.$$

Out of these two options, the second option is acceptable, as assuming that $\varphi(x) = \frac{\ln(4-3x)}{2}$ we shall have:

$$\varphi(x) \in [0.4, 0.55], \forall x \in [0.4, 0.55];$$

$\varphi'(x) = -\dfrac{3}{2(4-3x)}$, in the interval $[0.4, 0.55]$ $|\varphi'(x)| < \left|-\dfrac{3}{2(4-3\cdot 0.55)}\right| \approx 0.64 = q$.

The convergence conditions are satisfied. Moreover, for their fulfilment, the original interval $[0.4, 0.6]$ was reduced to $[0.4, 0.55]$. This is an acceptable technique in such problems if the discarded interval does not contain the desired root (which is obvious, as $f(0.55) > 0$ or $f_1(0.55) > f_2(0.55)$).

Let's take as the initial approximation

$x^{(0)} = (0.4 + 0.55)/2 = 0.475$.

Successive approximations of $x^{(k)}$ are calculated by the formula

$x^{(k+1)} = \varphi(x^{(k)}),\ k = 0, 1, 2, \ldots$, where $\varphi(x^{(k)}) = \dfrac{\ln(4-3x^{(k)})}{2}$.

Iterations stop when the following condition is met:

$\varepsilon^{(k+1)} = \dfrac{q}{1-q}\cdot\left|x^{(k+1)} - x^{(k)}\right| \le \varepsilon$.

The calculation results are presented in table 2.

The final approximation of the root is $x^* \approx x^{(4)} = 0.4738$.

Table 2

k	$x^{(k)}$	$\varphi(x^{(k)})$	$\varepsilon^{(k)}$
0	0.4750	0.4729	–
1	0.4729	0.4741	0.0037
2	0.4741	0.4734	0.0021
3	0.4734	0.4738	0.0012
4	**0.4738**		**0.0007**

3.5. PROGRAM #07

Below is a proposed variant of the program algorithm for refining the value of the root of the equation using the fixed-point iteration method.

```
ALGORITHM “Fixed-point iteration method”
INPUT     f(), q, a, b, e
OUTPUT    r, y, x, k
BEGIN
    IF (q>=1)OR(q<=0) END
    x:=(a+b)/2
    r:=2*e
    k:=0
    CYCLE “Iteration” WHILE (r>e)
        y:=x
        x:=f(y)
        IF (x<a)OR(x>b) END
        IF (x>y) r:=x-y ELSE r:=y-x
        r:=r*q/(1-q)
        k:=k+1
    PRINT x, k
END
```

3.6. NEWTON (TANGENT) METHOD

When finding the root of the equation $f(x)=0$ using the Newton method, the iterative process is determined by the formula:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, \quad k = 0,1,2,\ldots .$$

Indeed, the expansion of the function $f(x)$ according to the Taylor formula at point $x^{(k+1)}$ will look as follows:

$$f(x^{(k+1)}) = f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)}) + O((x^{(k+1)} - x^{(k)})^2) .$$

Based on the requirement of $f(x^{(k+1)}) = 0$ when $k \to \infty$, and discarding the summands of the order higher than the first one, we get:

$$f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)}) = 0, \; x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} .$$

Since only the linear part of the expansion in a series was used, for $x^{(k+1)}$ we shall instead of the root x^* obtain only its approximate refined value, after that the process must be repeated, which allows to construct a sequence of approximations.

Geometrically, the process (fig. 4) means replacement at each iteration of the curve $y = f(x)$ by a tangent to it at point $[x^{(k)}, f(x^{(k)})]$ and finding the value $x^{(k+1)}$ as the coordinate of the intersection point of the tangent and abscissa axis; second name of the method, the tangent method, is associated with this interpretation.

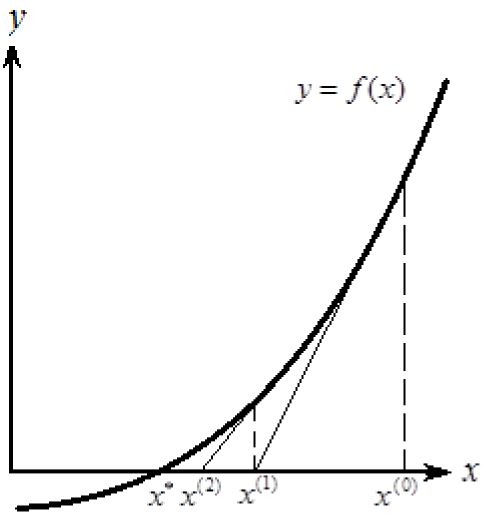

Fig. 4. Geometric interpretation of the Newton method

A sufficient condition for the convergence of the Newton method is obtained from the corresponding condition for the fixed-point iteration method: it can be seen that the Newton method is a special case of the fixed-point iteration method, in which $\varphi(x) = x - f(x)/f'(x)$.

Using the convergence condition for the fixed-point iteration method $|\varphi'(x)| < 1$, we get:

$$\varphi'(x) = 1 - \frac{\left(f'(x)\right)^2 - f(x)f''(x)}{\left(f'(x)\right)^2} = \frac{-f(x)f''(x)}{\left(f'(x)\right)^2},$$

$$\left|\frac{-f(x)f''(x)}{\left(f'(x)\right)^2}\right| < 1, \ \left|f(x)f''(x)\right| < \left(f'(x)\right)^2.$$

To start the calculation process, it is required to set the initial approximation $x^{(0)}$. The conditions for choosing the initial approximation $x^{(0)}$ are determined by the following theorem. Suppose that on the interval $[a,b]$ the function $f(x)$ has the first and second derivatives of constant sign and suppose $f(a)\cdot f(b)<0$. Then, if the point $x^{(0)}$ is chosen on $[a,b]$ so that $f(x^{(0)})\cdot f''(x^{(0)})>0$, then the sequence $x^{(k)}$ started from it, which is determined by the Newton method, converges monotonously to the root $x^{*}\in[a,b]$ of the equation $f(x)=0$. Usually, $x^{(0)}=a$ or $x^{(0)}=b$ is taken – depending on the point, at which $f(x^{(0)})\cdot f''(x^{(0)})>0$ is fulfilled.

As a condition for the termination of iterations in practical calculations, the rule $\varepsilon^{(k)}=\left|x^{(k)}-x^{(k-1)}\right|$ is used.

The Newton method has the second order of convergence near the root: at each iteration, the error changes in proportion to the square of the error at the previous iteration. Obviously, the Newton method is a one-step tool.

Example. Refine the value of the root of the equation $f(x)=e^{2x}+3x-4=0$ in the interval $[0.4, 0.6]$ with an accuracy of $\varepsilon=10^{-3}$ using the Newton method.

For the correct use of this method, it is first necessary to determine the behavior of the first and second derivatives of the function $f(x)$ on the root refinement interval and correctly choose the initial approximation $x^{(0)}$. For the function $f(x)=e^{2x}+3x-4=0$, we have:

$f'(x)=2e^{2x}+3$, $f''(x)=4e^{2x}$ are positive functions within the entire domain that satisfy the convergence condition $|f(x)f''(x)|<\left(f'(x)\right)^{2}$, the condition $f(0.4)\cdot f(0.6)<0$ is also satisfied.

As the initial approximation, we can choose the right end of the interval $[a,b]$ $x^{(0)}=0.6$, for which the inequality $f(0.6)\cdot f''(0.6)>0$ is satisfied.

Further calculations are carried out according to the formula

$$x^{(k+1)}=x^{(k)}-\frac{f(x^{(k)})}{f'(x^{(k)})},\text{ where } f(x^{(k)})=e^{2x^{(k)}}+3x^{(k)}-4,\; f'(x^{(k)})=2e^{2x^{(k)}}+3.$$

Iterations stop when the condition $\varepsilon^{(k+1)}=\left|x^{(k+1)}-x^{(k)}\right|<\varepsilon$ is met.

The calculation results are presented in table 3.

The final approximation of the root is $x^{*}\approx x^{(3)}=0.4737$.

Table 3

k	$x^{(k)}$	$f(x^{(k)})$	$f'(x^{(k)})$	$-f(x^{(k)})/f'(x^{(k)})$	$\varepsilon^{(k)}$
0	0.6000	1.1201	9.6402	-0.1162	–
1	0.4838	0.0831	8.2633	-0.0101	0.1162
2	0.4738	0.0005	8.1585	-0.0001	0.0101
3	**0.4737**				**0.0001**

3.7. PROGRAM #08

Below is a proposed variant of the program algorithm for refining the value of the root of the equation using the Newton method.

```
ALGORITHM "Newton method"
INPUT      f(), g(), h(), a, b, e
OUTPUT     r, y, x, k
BEGIN
    IF (f(a)*h(a)>0) x:=a ELSE IF (f(b)*h(b)>0) x:=b ELSE END
    r:=2*e
    k:=0
    CYCLE "Iteration" WHILE (r>e)
        y:=x
        x:=y-f(y)/g(y)
        IF (x>y) r:=x-y ELSE r:=y-x
        k:=k+1
    PRINT x, k
END
```

3.8. SECANT (CHORD) METHOD

The use of the Newton method implies calculating the value of the function and its derivative at each iteration. Sometimes, for example, if the function $f(x)$ is table, carrying out direct calculation of the derivative is difficult. Then, replacing the derivative of the function by the approximate difference relation

$f'(x^{(k)}) \approx \frac{f(x^{(k)}) - f(x^{(k-1)})}{x^{(k)} - x^{(k-1)}}$ and substituting it in the formula of the Newton method, we shall obtain the iterative formula of the secant method:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})(x^{(k)} - x^{(k-1)})}{f(x^{(k)}) - f(x^{(k-1)})},\ k = 1, 2, \ldots.$$

Geometrically, the process (fig. 5) means replacement at each iteration of the tangent curve to $y = f(x)$ at point $[x^{(k)};\ f(x^{(k)})]$ by a secant drawn through two points $[x^{(k)};\ f(x^{(k)})]$, $[x^{(k-1)};\ f(x^{(k-1)})]$, which is associated with the name of the method.

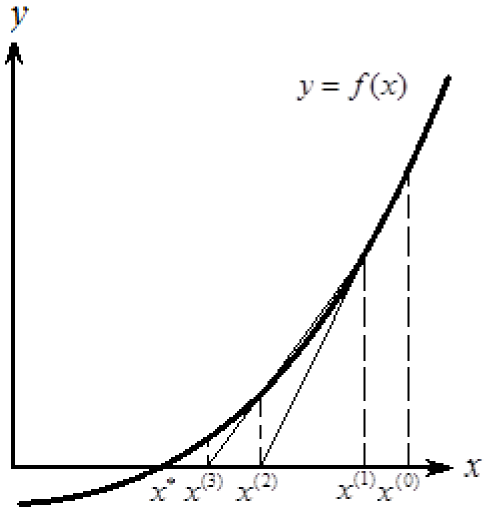

Fig. 5. Geometric interpretation of the secant method

The use of this method eliminates the need to calculate the derivative function in the process of calculation at the expense of some loss of accuracy.

The method is a two-step one: as it can be seen from its formula, the result after the $(k+1)$-th step depends on the results of the k-th and $(k-1)$-th steps. Accordingly, to perform the first iteration, it is required to set two initial points $x^{(0)}$ and $x^{(1)}$.

The convergence conditions for the secant method are similar to the convergence conditions for the Newton method. The starting point $x^{(0)}$ is chosen according to the same principle as in the Newton method. The second starting point $x^{(1)}$ is chosen in close proximity to $x^{(0)}$, preferably between the point $x^{(0)}$ and the desired root x^*. As a condition for the termination of iterations in practical calculations the rule $\varepsilon^{(k)} = \left| x^{(k)} - x^{(k-1)} \right|$ is used.

The order of convergence of the secant method is $\frac{1+\sqrt{5}}{2}\approx 1.618$.

Example. Refine the value of the root of the equation $f(x)=e^{2x}+3x-4=0$ in the interval $[0.4, 0.6]$ with an accuracy of $\varepsilon=10^{-3}$ using the secant method.

Let's set $x^{(0)}=0.6$ and $x^{(1)}=0.55$ as the starting points.

Further calculations are carried out according to the formula

$$x^{(k+1)}=x^{(k)}-\frac{f(x^{(k)})(x^{(k)}-x^{(k-1)})}{f(x^{(k)})-f(x^{(k-1)})}, \text{ where } f(x^{(k)})=e^{2x^{(k)}}+3x^{(k)}-4.$$

Iterations stop when the condition $\varepsilon^{(k+1)}=\left|x^{(k+1)}-x^{(k)}\right|<\varepsilon$ is met.

The calculation results are presented in table 4.

The final approximation of the root is $x^{*}\approx x^{(4)}=0.4737$. It can be seen that, in this case, the secant method converges somewhat slower than the Newton method.

Table 4

k	$x^{(k)}$	$f(x^{(k)})$	$\varepsilon^{(k)}$
0	0.6000	1.1201	–
1	0.5500	0.6542	–
2	0.4798	0.0501	0.0702
3	0.4740	0.0024	0.0058
4	**0.4737**		**0.0003**

3.9. PROGRAM #09

Below is a proposed variant of the program algorithm for refining the value of the root of the equation using the secant method.

```
ALGORITHM "Secant method"
INPUT     f(), c, d, e
OUTPUT    r, z, y, x, k
BEGIN
    y:=c
    x:=d
```

```
        r:=2*e
        k:=1
        CYCLE "Iteration" WHILE (r>e)
                z:=y
                y:=x
                x:=y-f(y)*(y-z)/(f(y)-f(z))
                IF (x>y) r:=x-y ELSE r:=y-x
                k:=k+1
        PRINT x, k
END
```