2.5. GAUSSIAN ELIMINATION METHOD

There are a lot of known Gaussian elimination method schemes intended to find a solution to the system of linear algebraic equations $Ax = b$, which are adapted for manual or machine calculation of general or special type matrices, where $A \equiv \left[a_{ij}\right]$ is the square $n \times n$ matrix of coefficients with unknowns; $x \equiv \left[x_j\right]$ is the column vector of the unknowns; $b \equiv \left[b_j\right]$ is the column vector of the right-hand sides of the system.

So, the following SLAE is given:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}.$$

The Gaussian method can be interpreted as a method in which initially the matrix A is reduced by means of equivalent transformations for the SLAE (rows exchange, multiplication of a row by a number, addition of rows and replacement of one of them with the result) to the upper triangular form (forward path) and then to the unity form (backward path). The unknowns are determined during the backward path: obviously, if the SLAE matrix is a unity one, then $x_i = b_i, i = 1, \ldots, n$.

Let's write the extended matrix of the system:

$$\begin{array}{r c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & b \end{array} \\ \textit{Leading row} \rightarrow & \left[\begin{array}{c|ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array}\right] \\ & \uparrow \textit{Leading column} \end{array} \begin{array}{c} \left(-{}^{a_{21}}\!/_{a_{11}}\right);\left(-{}^{a_{31}}\!/_{a_{11}}\right);\ldots;\left(-{}^{a_{n1}}\!/_{a_{11}}\right) \\ \\ \xrightarrow[1st\ \ step]{} \end{array} .$$

At the first step of the Gaussian algorithm, we should choose the diagonal element $a_{11} \neq 0$ (if it is equal to zero, then we must replace the first row with any underlying row, where $a_{i1} \neq 0, i = 2, \ldots, n$) and declare it the *leading* element; the corresponding row and column at the intersection of which it stands will be the *leading* row and column. Let's zero out elements $a_{21}, \ldots, a_{n1}$ of the leading column.

For this purpose, we should form the numbers $(-a_{21}/a_{11});(-a_{31}/a_{11});\ldots;(-a_{n1}/a_{11})$, which we shall correlate with the leading row. Multiplying the leading row by the number $(-a_{21}/a_{11})$, adding it to the second row and putting the result in place of the second row, we shall get zero instead of the element a_{21}, and elements $a^1_{2j}=a_{2j}+a_{1j}(-a_{21}/a_{11}),\ j=2,\ldots,n$, $b^1_2=b_2+b_1(-a_{21}/a_{11})$ instead of the elements $a_{2j},\ j=2,\ldots,n$, b_2. We continue so up to the last row: multiplying the leading row by the number $(-a_{n1}/a_{11})$, adding it to the *n*-th row and putting the result in place of the *n*-th row, we shall get zero instead of the element a_{n1}, and the remaining elements of this row will be as follows: $a^1_{nj}=a_{nj}+a_{1j}(-a_{n1}/a_{11})$, $b^1_n=b_n+b_1(-a_{n1}/a_{11})$.

Keeping the leading first row of the matrix unchanged at the first step, we shall obtain the following matrix as a result of the first step of the Gaussian algorithm:

$$
\begin{array}{rcl}
 & \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & b \end{array} & \\
\begin{array}{r} \\ \textit{Leading row}\rightarrow \\ \\ \\ \\ \end{array} &
\left[\begin{array}{c|c|cccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ \hline
0 & a^1_{22} & a^1_{23} & \cdots & a^1_{2n} & b^1_2 \\ \hline
0 & a^1_{32} & a^1_{33} & \cdots & a^1_{3n} & b^1_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & a^1_{n2} & a^1_{n3} & \cdots & a^1_{nn} & b^1_n
\end{array}\right] &
\begin{array}{l} \\ \left(-a^1_{32}/a^1_{22}\right);\ldots;\left(-a^1_{n2}/a^1_{22}\right) \\ \xrightarrow[2nd\quad step]{} \\ \\ \\ \end{array} . \\
 & \uparrow\ \ \textit{Leading column} &
\end{array}
$$

At the second step of the Gaussian algorithm, the element $a^1_{22}\neq 0$ is selected as the leading element (if it is equal to zero, then the second row is mutually replaced with the underlying row, where $a_{i1}\neq 0,\ i=3,\ldots,n$). The numbers $\left(-\dfrac{a^1_{32}}{a^1_{22}}\right);\ldots;\left(-\dfrac{a^1_{n2}}{a^1_{22}}\right)$ are formed, which are placed near the leading row. Multiplying the leading row by the number $\left(-\dfrac{a^1_{32}}{a^1_{22}}\right)$ and adding the result to the third row, we shall get zero instead of the element a^1_{32}, and elements $a^2_{3j}=a^1_{3j}+a^1_{2j}\left(-\dfrac{a^1_{32}}{a^1_{22}}\right),\ j=3\ldots n$, $b^2_3=b^1_3+b^1_2\left(-\dfrac{a^1_{32}}{a^1_{22}}\right)$ instead of the elements $a^1_{3j},\ j=3,\ldots,n$, b^1_3. And so on up to the last row: multiplying the leading row by the number $\left(-\frac{a_{n2}^1}{a_{22}^1}\right)$, adding the result to the n-th row and putting the resulting sum in the place of the n-th row, we shall get zero instead of the element a_{n2}^1, and elements $a_{nj}^2 = a_{nj}^1 + a_{2j}^1\left(-\frac{a_{n2}^1}{a_{22}^1}\right)$, $j = 3,\ldots,n$, $b_n^2 = b_n^1 + b_2^1\left(-\frac{a_{n2}^1}{a_{22}^1}\right)$ instead of the elements a_{nj}^1, b_n^1.

Keeping the first and leading second rows of the matrix unchanged at the second step, we shall obtain the following matrix as a result of the second step of the Gaussian algorithm:

$$\begin{array}{rcl}
 & \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & b \end{array} & \\
\textit{Leading row} \rightarrow & \left[\begin{array}{cc|c|ccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{n1} & b_1 \\
0 & a_{22}^1 & a_{23}^1 & \cdots & a_{2n}^1 & b_2^1 \\
\hline
0 & 0 & a_{33}^2 & \cdots & a_{3n}^2 & b_3^2 \\
\hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & a_{n3}^2 & \cdots & a_{nn}^2 & b_n^2
\end{array}\right] & \xrightarrow[3rd\ \ step]{} \cdots \xrightarrow[(n-1)-th\ \ step]{} . \\
 & \uparrow\ \textit{Leading column} &
\end{array}$$

As a result, after $(n-1)$-th step of the Gaussian algorithm, we shall obtain the following extended matrix containing the upper triangular matrix of the SLAE:

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & \ldots & x_n & b \end{array} \\
\left[\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\
0 & a_{22}^1 & a_{23}^1 & \ldots & a_{2n}^1 & b_2^1 \\
0 & 0 & a_{33}^2 & \ldots & a_{3n}^2 & b_3^2 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & a_{nn}^{n-1} & b_n^{n-1}
\end{array}\right].
\end{array}$$

The forward path of the Gaussian algorithm is completed.

If the elements of any row of the matrix of the system become equal to zero as a result of transformations, and the right-hand side is not equal to zero, then the SLAE is incompatible and has no solution. If the elements of any row of the matrix of the system and the right-hand side become equal to zero as a result of transformations, then the SLAE is compatible, but has an infinite number of solutions.

The determinant of the triangular matrix is equal to the product of the diagonal elements. As a result of the forward path of the Gaussian elimination method, the system of linear equations is reduced to the upper triangular matrix. Therefore, as a result of the forward path of the Gaussian method, we can immediately calculate the determinant of the matrix A of the original SLAE as the product of the leading elements:

$$\det A = (-1)^p a_{11} a_{22}^1 a_{33}^2 \cdot \ldots \cdot a_{nn}^{n-1},$$

where p is the amount of row permutations during the forward path.

During the backward path of the Gaussian algorithm, the extended matrix of the system obtained after the forward path can be reduced to the diagonal matrix by similar equivalent transformations (if we also turn to zero all coefficients located above the main diagonal) and then (if we divide each equation by the corresponding element located on the main diagonal) to the unity matrix, which will allow us to obtain the solution to the original SLAE.

However, a more efficient technique is to write down the solution to the system sequentially through recurrence relations, when x_n is immediately determined from the last equation, x_{n-1} – from the second-to-last one and so on, until x_1 is determined from the first equation:

$$\begin{cases} a_{nn}^{n-1} x_n = b_n^{n-1} & \Rightarrow x_n \\ a_{n-1n-1}^{n-2} x_{n-1} + a_{n-1n}^{n-2} x_n = b_{n-1}^{n-2} & \Rightarrow x_{n-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{11} x_1 + \ldots + a_{1n} x_n = b_1 & \Rightarrow x_1 \end{cases}, \quad \begin{cases} x_n = \dfrac{b_n}{a_{nn}} \\ x_i = \dfrac{1}{a_{ii}} \left[b_i - \sum\limits_{k=i+1}^{n} a_{ik} x_k \right], i = n-1, \ldots, 1 \end{cases}.$$

It is important to note that the implementation of the Gaussian method requires around n^3 arithmetic operations, which makes this method *labor-intensive*. Moreover, the number of operations is determined mainly by the operations carried out when performing the forward path of the Gaussian method. The backward path of the Gaussian method (through recurrent relations) requires around n^2 operations. Therefore, if it is required to solve several systems of linear algebraic equations $Ax = b$ with the same matrix and different right-hand sides, then, in this case, it is advisable to implement the Gaussian method algorithm in the form of two subprograms: the first subprogram must implement the forward path of the algorithm and obtain the upper triangular matrix of the SLAE at the output, and the second subprogram must implement the forward path of the algorithm for the right-hand sides and, using the resulting upper triangular matrix, calculate the solution to the system for the specific right-hand side.

2.6. MATRIX INVERSION USING GAUSSIAN METHOD

The Gaussian elimination method can also be applied to invert a nondegenerate $(\det A \neq 0)$ matrix. Indeed, suppose it is required to invert the nondegenerate matrix *A*. Then, having made the notation $A^{-1} = X$, we can write the matrix equation $AX = E$, where *E* is the unity matrix, on the basis of which we can write the chain of SLAEs:

$$A \cdot \begin{pmatrix} x_{11} \\ x_{21} \\ \cdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, \qquad A \cdot \begin{pmatrix} x_{12} \\ x_{22} \\ \cdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \qquad \ldots, \qquad A \cdot \begin{pmatrix} x_{1n} \\ x_{2n} \\ \cdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix},$$

each of which can be solved using the Gaussian method.

So, for matrix inversion, it is necessary to solve *n* systems of linear equations of *n*-th order with the same matrix *A* and different right-hand sides. Since the upper triangular matrix for all these SLAEs is the same, the forward path of the Gaussian method is applied for the matrix *A* just once. The following extended matrix is constructed:

$$\begin{array}{cccc|cccc|}
x_{1n} & x_{2n} & \cdots & x_{nn} & & & & \\
\cdots & \cdots & \cdots & \cdots & & & & \\
x_{12} & x_{22} & \cdots & x_{n2} & & & & \\
x_{11} & x_{21} & \cdots & x_{n1} & b^1 & b^2 & \cdots & b^n \\
\hline
a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1
\end{array}.$$

As a result of applying $(n-1)$ step of the Gaussian method, we obtain:

$$\begin{array}{cccc|cccc}
x_{1n} & x_{2n} & \dots & x_{nn} & & & & \\
\dots & \dots & \dots & \dots & & & & \\
x_{12} & x_{22} & \dots & x_{n2} & & & & \\
x_{11} & x_{21} & \dots & x_{n1} & b^1 & b^2 & \dots & b^n \\
\hline
a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1n} \\
0 & a_{22}^1 & \dots & a_{2n}^1 & b_{21}^1 & b_{22}^1 & \dots & b_{2n}^1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & a_{nn}^{n-1} & b_{n1}^{n-1} & b_{n2}^{n-1} & \dots & b_{nn}^{n-1}
\end{array}\Bigg| \, .$$

Herewith, the first column $(x_{11} \quad x_{21} \quad ... \quad x_{n1})^T$ of the inverse matrix is determined during the backward path of the Gaussian method with the right-hand side b^1, the second column $(x_{12} \quad x_{22} \quad ... \quad x_{n2})^T$ – with the right-hand side b^2, and so on. The column $(x_{1n} \quad x_{2n} \quad ... \quad x_{nn})^T$ is determined with the right-hand side b^n.

However, as an alternative, it is possible during the backward path of the Gaussian method to reduce by equivalent transformations the whole matrix A from the upper triangular matrix directly to the unity matrix in order to immediately obtain its inverse matrix, this time without losing algorithmical efficiency.

Example. Solve the SLAE using the Gaussian method, calculate the determinant and the inverse matrix for the matrix of the SLAE:

$$\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 13 \\ 6x_1 - 2x_2 + 2x_4 = 20 \\ 3x_1 - 5x_2 + x_3 + 8x_4 = 7 \\ -x_1 + 4x_2 - 5x_3 + 9x_4 = 7 \end{cases}; \; A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 6 & -2 & 0 & 2 \\ 3 & -5 & 1 & 8 \\ -1 & 4 & -5 & 9 \end{pmatrix}, \; b = \begin{pmatrix} 13 \\ 20 \\ 7 \\ 7 \end{pmatrix}.$$

Forward path:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} \mathbf{1} & 3 & -1 & 2 & 13 \\ 6 & -2 & 0 & 2 & 20 \\ 3 & -5 & 1 & 8 & 7 \\ -1 & 4 & -5 & 9 & 7 \end{array}\right) \end{array} (-6/1); (-3/1); (1/1); \xrightarrow[1st \quad step]{}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 13 \\ 0 & \mathbf{-20} & 6 & -10 & -58 \\ 0 & -14 & 4 & 2 & -32 \\ 0 & 7 & -6 & 11 & 20 \end{array}\right) \end{array} (-14/20); (7/20); \xrightarrow[2nd \quad step]{}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 13 \\ 0 & -20 & 6 & -10 & -58 \\ 0 & 0 & \mathbf{-1/5} & 9 & 43/5 \\ 0 & 0 & -39/10 & 15/2 & -3/10 \end{array}\right) \end{array} (-39/2); \xrightarrow{3rd\quad step}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 13 \\ 0 & -20 & 6 & -10 & -58 \\ 0 & 0 & -1/5 & 9 & 43/5 \\ 0 & 0 & 0 & -168 & -168 \end{array}\right) \end{array}.$$

The determinant of the matrix: $\det A = 1\cdot(-20)\cdot(-1/5)\cdot(-168) = -672$.

Backward path:

$$\begin{cases} -168x_4 = -168 \\ (-1/5)x_3 + 9x_4 = 43/5 \\ -20x_2 + 6x_3 - 10x_4 = -58 \\ x_1 + 3x_2 - x_3 + 2x_4 = 13 \end{cases}; \quad \begin{cases} x_4 = 1 \\ (-1/5)x_3 = 43/5 - 9\cdot 1 = -2/5,\ x_3 = 2 \\ -20x_2 = -58 - 6\cdot 2 + 10\cdot 1 = -60,\ x_2 = 3 \\ x_1 = 13 - 3\cdot 3 + 2 - 2\cdot 1 = 4,\ x_1 = 4 \end{cases}.$$

Solution to the SLAE: $x = \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}^T$.

Finding the inverse matrix A^{-1}.

Forward path:

$$\begin{array}{c} \begin{array}{cccccccc} x_{14} & x_{24} & x_{34} & x_{44} & & & & \\ x_{13} & x_{23} & x_{33} & x_{43} & & & & \\ x_{12} & x_{22} & x_{32} & x_{42} & & & & \\ x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4 \end{array} \\ \left(\begin{array}{cccc|cccc} \mathbf{1} & 3 & -1 & 2 & 1 & 0 & 0 & 0 \\ 6 & -2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & -5 & 1 & 8 & 0 & 0 & 1 & 0 \\ -1 & 4 & -5 & 9 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} (-6/1);(-3/1);(1/1); \xrightarrow{1st\quad step}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_{14} & x_{24} & x_{34} & x_{44} & & & & \\ x_{13} & x_{23} & x_{33} & x_{43} & & & & \\ x_{12} & x_{22} & x_{32} & x_{42} & & & & \\ x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4 \end{array} \\ \left(\begin{array}{cccc|cccc} 1 & 3 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & \mathbf{-20} & 6 & -10 & -6 & 1 & 0 & 0 \\ 0 & -14 & 4 & 2 & -3 & 0 & 1 & 0 \\ 0 & 7 & -6 & 11 & 1 & 0 & 0 & 1 \end{array}\right) \end{array} (-14/20);(7/20); \xrightarrow{2nd\quad step}$$

$$\begin{array}{cccc|cccc}
x_{14} & x_{24} & x_{34} & x_{44} & & & & \\
x_{13} & x_{23} & x_{33} & x_{43} & & & & \\
x_{12} & x_{22} & x_{32} & x_{42} & & & & \\
x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4
\end{array}$$

$$\left(\begin{array}{cccc|cccc}
1 & 3 & -1 & 2 & 1 & 0 & 0 & 0 \\
0 & -20 & 6 & -10 & -6 & 1 & 0 & 0 \\
0 & 0 & \mathbf{-1/5} & 9 & 6/5 & -7/10 & 1 & 0 \\
0 & 0 & -39/10 & 15/2 & -11/10 & 7/20 & 0 & 1
\end{array}\right)\left|(-39/2);\right. \xrightarrow{3rd\quad step}$$

$$\begin{array}{cccc|cccc}
x_{14} & x_{24} & x_{34} & x_{44} & & & & \\
x_{13} & x_{23} & x_{33} & x_{43} & & & & \\
x_{12} & x_{22} & x_{32} & x_{42} & & & & \\
x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4
\end{array}$$

$$\left(\begin{array}{cccc|cccc}
1 & 3 & -1 & 2 & 1 & 0 & 0 & 0 \\
0 & -20 & 6 & -10 & -6 & 1 & 0 & 0 \\
0 & 0 & -1/5 & 9 & 6/5 & -7/10 & 1 & 0 \\
0 & 0 & 0 & -168 & -49/2 & 14 & -39/2 & 1
\end{array}\right).$$

Backward path:

$$\begin{cases} -168x_{41} = -49/2 \\ (-1/5)x_{31} + 9x_{41} = 6/5 \\ -20x_{21} + 6x_{31} - 10x_{41} = -6 \\ x_{11} + 3x_{21} - x_{31} + 2x_{41} = 1 \end{cases}; \quad \begin{cases} -168x_{42} = 14 \\ (-1/5)x_{32} + 9x_{42} = -7/10 \\ -20x_{22} + 6x_{32} - 10x_{42} = 1 \\ x_{12} + 3x_{22} - x_{32} + 2x_{42} = 0 \end{cases};$$

$$\begin{cases} -168x_{43} = -39/2 \\ (-1/5)x_{33} + 9x_{43} = 1 \\ -20x_{23} + 6x_{33} - 10x_{43} = 0 \\ x_{13} + 3x_{23} - x_{33} + 2x_{43} = 0 \end{cases}; \quad \begin{cases} -168x_{44} = 1 \\ (-1/5)x_{34} + 9x_{44} = 0 \\ -20x_{24} + 6x_{34} - 10x_{44} = 0 \\ x_{14} + 3x_{24} - x_{34} + 2x_{44} = 0 \end{cases}.$$

From here $A^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} 1/12 & 1/6 & -1/28 & -1/42 \\ 19/48 & -1/12 & 1/112 & -13/168 \\ 9/16 & -1/4 & 25/112 & -15/56 \\ 7/48 & -1/12 & 13/112 & -1/168 \end{pmatrix}.$

2.7. GAUSSIAN METHOD WITH PIVOT ELEMENT SELECTION

At some *i*-th step of the forward path of the Gaussian elimination method, it may turn out that the coefficient $a_{ii}^{i-1} \neq 0$, but it is small in comparison with the remaining elements of the system matrix and, in particular, small in comparison with the elements of the *i*-th column below. Division of the system coefficients by a small quantity can lead to significant rounding errors or even the overflow of variables.

To reduce these errors, proceed as follows. Select the largest in absolute value (pivot) element among the elements of the ***i***-th column $a_{ki}^{i-1}, k = i, \dots, n$ of each intermediate matrix and make the pivot element the leading element by rearranging the ***i***-th row and the row containing the pivot element. Such a modification of the Gaussian elimination method is called the Gaussian method with pivot element selection. The case of zero leading elements emergence in this goes by itself.

Example. Solve the SLAE using the Gaussian method with pivot element selection, calculate the determinant and the inverse matrix for the matrix of the SLAE:

$$\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 13 \\ 6x_1 - 2x_2 + 2x_4 = 20 \\ 3x_1 - 5x_2 + x_3 + 8x_4 = 7 \\ -x_1 + 4x_2 - 5x_3 + 9x_4 = 7 \end{cases}; \; A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 6 & -2 & 0 & 2 \\ 3 & -5 & 1 & 8 \\ -1 & 4 & -5 & 9 \end{pmatrix}, \; b = \begin{pmatrix} 13 \\ 20 \\ 7 \\ 7 \end{pmatrix}.$$

Forward path:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 13 \\ \mathbf{6} & -2 & 0 & 2 & 20 \\ 3 & -5 & 1 & 8 & 7 \\ -1 & 4 & -5 & 9 & 7 \end{array}\right) \end{array} \xrightarrow[1st \quad step]{} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 6 & -2 & 0 & 2 & 20 \\ 0 & 10/3 & -1 & 5/3 & 29/3 \\ 0 & \mathbf{-4} & 1 & 7 & -3 \\ 0 & 11/3 & -5 & 28/3 & 31/3 \end{array}\right) \end{array} \xrightarrow[2nd \quad step]{}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 6 & -2 & 0 & 2 & 20 \\ 0 & -4 & 1 & 7 & -3 \\ 0 & 0 & -1/6 & 15/2 & 43/6 \\ 0 & 0 & \mathbf{-49/12} & 63/4 & 91/12 \end{array}\right) \end{array} \xrightarrow[3rd \quad step]{} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left(\begin{array}{cccc|c} 6 & -2 & 0 & 2 & 20 \\ 0 & -4 & 1 & 7 & -3 \\ 0 & 0 & -49/12 & 63/4 & 91/12 \\ 0 & 0 & 0 & 48/7 & 48/7 \end{array}\right) \end{array}.$$

The amount of permutations performed in 3 steps: $p = 3$.

The determinant of the matrix: $\det A = (-1)^3 \cdot 6 \cdot (-4) \cdot (-49/12) \cdot (48/7) = -672$.

Backward path:

$$\begin{cases} 48/7x_4 = 48/7 \\ (-49/12)x_3 + (63/4)x_4 = 91/12 \\ -4x_2 + x_3 + 7x_4 = -3 \\ 6x_1 - 2x_2 + 2x_4 = 20 \end{cases}; \quad \begin{cases} x_4 = 1 \\ (-49/12)x_3 = 91/12 - (63/4)\cdot 1 = -98/12,\ x_3 = 2 \\ -4x_2 = -3 - 2 - 7\cdot 1 = -12,\ x_2 = 3 \\ 6x_1 = 20 + 2\cdot 3 - 2\cdot 1 = 24,\ x_1 = 4 \end{cases}.$$

Solution to the SLAE: $x = \begin{pmatrix} 4 & 3 & 2 & 1 \end{pmatrix}^T$.

Finding the inverse matrix A^{-1}.

Forward path:

$$\begin{array}{c} \begin{array}{cccccccc} x_{14} & x_{24} & x_{34} & x_{44} & & & & \\ x_{13} & x_{23} & x_{33} & x_{43} & & & & \\ x_{12} & x_{22} & x_{32} & x_{42} & & & & \\ x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4 \end{array} \\ \left(\begin{array}{cccc|cccc} 1 & 3 & -1 & 2 & 1 & 0 & 0 & 0 \\ \mathbf{6} & -2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & -5 & 1 & 8 & 0 & 0 & 1 & 0 \\ -1 & 4 & -5 & 9 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \xrightarrow[1st\ \ step]{}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_{14} & x_{24} & x_{34} & x_{44} & & & & \\ x_{13} & x_{23} & x_{33} & x_{43} & & & & \\ x_{12} & x_{22} & x_{32} & x_{42} & & & & \\ x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4 \end{array} \\ \left(\begin{array}{cccc|cccc} 6 & -2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 10/3 & -1 & 5/3 & 1 & -1/6 & 0 & 0 \\ 0 & \mathbf{-4} & 1 & 7 & 0 & -1/2 & 1 & 0 \\ 0 & 11/3 & -5 & 28/3 & 0 & 1/6 & 0 & 1 \end{array}\right) \end{array} \xrightarrow[2nd\ \ step]{}$$

$$\begin{array}{c} \begin{array}{cccccccc} x_{14} & x_{24} & x_{34} & x_{44} & & & & \\ x_{13} & x_{23} & x_{33} & x_{43} & & & & \\ x_{12} & x_{22} & x_{32} & x_{42} & & & & \\ x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4 \end{array} \\ \left(\begin{array}{cccc|cccc} 6 & -2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 7 & 0 & -1/2 & 1 & 0 \\ 0 & 0 & -1/6 & 15/2 & 1 & -7/12 & 5/6 & 0 \\ 0 & 0 & \mathbf{-49/12} & 63/4 & 0 & -7/24 & 11/12 & 1 \end{array}\right) \end{array} \xrightarrow[3rd\ \ step]{}$$

$$\begin{array}{cccccccc}
x_{14} & x_{24} & x_{34} & x_{44} & & & & \\
x_{13} & x_{23} & x_{33} & x_{43} & & & & \\
x_{12} & x_{22} & x_{32} & x_{42} & & & & \\
x_{11} & x_{21} & x_{31} & x_{41} & b^1 & b^2 & b^3 & b^4
\end{array}$$

$$\left(\begin{array}{cccc|cccc}
6 & -2 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & -4 & 1 & 7 & 0 & -1/2 & 1 & 0 \\
0 & 0 & -49/12 & 63/4 & 0 & -7/24 & 11/12 & 1 \\
0 & 0 & 0 & 48/7 & 1 & -4/7 & 39/49 & -2/49
\end{array}\right).$$

Backward path:

$$\begin{cases} 48/7x_{41} = 1 \\ (-49/12)x_{31} + (63/4)x_{41} = 0 \\ -4x_{21} + x_{31} + 7x_{41} = 0 \\ 6x_{11} - 2x_{21} + 2x_{41} = 0 \end{cases}; \quad \begin{cases} 48/7x_{42} = -4/7 \\ (-49/12)x_{32} + (63/4)x_{42} = -7/24 \\ -4x_{22} + x_{32} + 7x_{42} = -1/2 \\ 6x_{12} - 2x_{22} + 2x_{42} = 1 \end{cases};$$

$$\begin{cases} 48/7x_{43} = 39/49 \\ (-49/12)x_{33} + (63/4)x_{43} = 11/12 \\ -4x_{23} + x_{33} + 7x_{43} = 1 \\ 6x_{13} - 2x_{23} + 2x_{43} = 0 \end{cases}; \quad \begin{cases} 48/7x_{44} = -2/49 \\ (-49/12)x_{34} + (63/4)x_{44} = 1 \\ -4x_{24} + x_{34} + 7x_{44} = 0 \\ 6x_{14} - 2x_{24} + 2x_{44} = 0 \end{cases}.$$

From here $A^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix} = \begin{pmatrix} 1/12 & 1/6 & -1/28 & -1/42 \\ 19/48 & -1/12 & 1/112 & -13/168 \\ 9/16 & -1/4 & 25/112 & -15/56 \\ 7/48 & -1/12 & 13/112 & -1/168 \end{pmatrix}$.

2.8. PROGRAM #02

Below is a proposed variant of the program algorithm for solving SLAE, calculating the determinant and the inverse matrix of the SLAE's matrix using the Gaussian elimination method or the Gaussian method with pivot element selection.

```
ALGORITHM “Gaussian method with pivot element selection”
INPUT     n, a[n][n], b[n]
OUTPUT    i, j, k, m, p, f, g, h, x[n], y, z[n][n]
BEGIN
        CYCLE “Rows” FOR i FROM 1 TO n BY 1
                CYCLE “Columns” FOR j FROM 1 TO n BY 1
                        IF (i=j) z[i][j]:=1 ELSE z[i][j]:=0
```

```
p:=0
CYCLE “Forward Path” FOR k FROM 1 TO n-1 BY 1
    #For the Gaussian method with pivot element selection
    IF (a[k][k]>0) g:=a[k][k] ELSE g:=-a[k][k]
    m:=k
    CYCLE “Rows” FOR i FROM k+1 TO n BY 1
        IF (a[i][k]>0) f:=a[i][k] ELSE f:=-a[i][k]
        IF (f>g) g:=f, m:=i
    IF (g=0)
        PRINT “Solution is not unique or not available!”
        END
    IF (m≠k)
        CYCLE “Columns” FOR j FROM k TO n BY 1
            g:=a[k][j], a[k][j]:=a[m][j], a[m][j]:=g
        g:=b[k], b[k]:=b[m], b[m]:=g
        CYCLE “Columns” FOR j FROM 1 TO n BY 1
            g:=z[k][j], z[k][j]:=z[m][j], z[m][j]:=g
        p:=p+1
    #General part of the Gaussian method
    CYCLE “Rows” FOR i FROM k+1 TO n BY 1
        h:=-a[i][k]/a[k][k]
        CYCLE “Columns” FOR j FROM k TO n BY 1
            a[i][j]:=a[i][j]+h*a[k][j]
        b[i]:=b[i]+h*b[k]
        CYCLE “Columns” FOR j FROM 1 TO n BY 1
            z[i][j]:=z[i][j]+h*z[k][j]
y:=1-2*REMAINDER(p,2)
CYCLE “Backward Path” FOR k FROM n TO 1 BY -1
    y:=y*a[k][k]
    h:=0
```

```
            CYCLE “Rows” FOR i FROM k+1 TO n BY 1
                h:=h+a[k][i]*x[i]
            x[k]:=(b[k]-h)/a[k][k]
            CYCLE “Columns” FOR j FROM 1 TO n BY 1
                h:=0
                CYCLE “Rows” FOR i FROM k+1 TO n BY 1
                    h:=h+a[k][i]*z[i][j]
                z[k][j]:=(z[k][j]-h)/a[k][k]
        PRINT x, y, z
    END
```