Assignment 4:

Q4. (5 points) Alice and Bob play a game starting with a pile of n sticks. Each player on their turn can remove 1, 2 or 3 sticks from the pile. The last player to remove a stick wins. Alice goes first. Prove by induction that Alice has a winning strategy if and only if $4 \nmid n$.

- A winning strategy is a rule that a player can follow which guarantees they will win no matter what decisions their opponent makes. You may assume, without proof, that either Alice or Bob has a winning strategy.
- You may also assume without proof that any integer n can be written as $n = 4q + r$ where $q \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3\}$.

MATH 135: Lecture 10

Dr. Nike Dattani

29 September 2021

- Tuesday 28 September:
  - **Look at your WA02 results thoroughly! Where did you lose marks?**
- **Wednesday 29 September:**
  - **Complete Written Assignment 3: WA3**
- Wednesday 29 September:
  - Mobius Quiz 8
- Thursday 30 September:
  - Look at WA4 **!!!**
- Thursday 30 September:
  - WA03 solutions will be posted, hopefully before 12pm: Check the solutions in detail!
- Thursday 30 September:
  - Complete reading from Chapter 3.6 up to 5 of the course notes. **Pages 55-81.**
- Friday 1 October:
  - Mobius Quiz 9
- Sunday 3 October:
  - You'll need to know more before 0.4 (Polynomials), so use this time to review **Pages 55-81**, and do practice problems!
- Monday 4 October:
  - Mobius Quiz 10

View By: Groups Groups: 1219-MATH.135.016.1.LEC Apply

WA2 Class Statistics

Number of submitted grades: 50 / 50

Minimum:		0 %
Maximum:		100 %
Average:		79.64 %

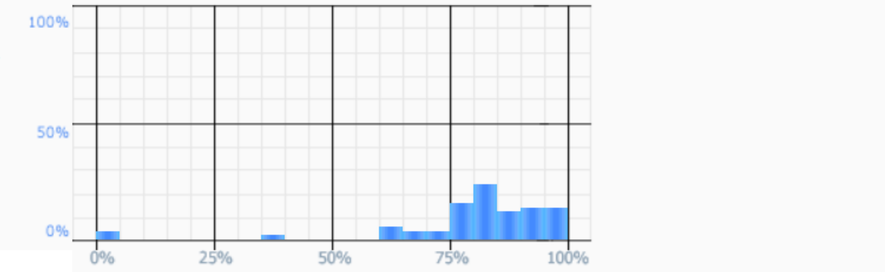
Mode: 84.09 %

Median: 84.09 %

Standard Deviation: 19.92 %

Grade Distribution

Number of Users (%)



View By: Groups Groups: 1219-MATH.135.019.1.LEC Apply

WA2 Class Statistics

Number of submitted grades: 56 / 56

Minimum:		0 %
Maximum:		100 %
Average:		83.89 %

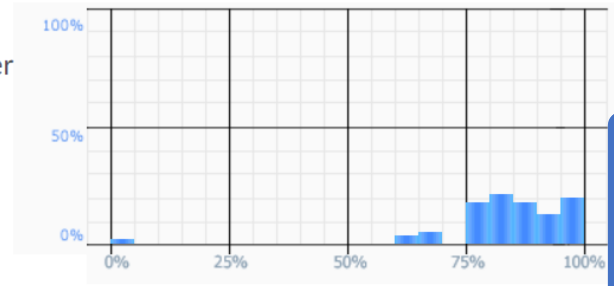
Mode: 84.09 %

Median: 84.66 %

Standard Deviation: 14.42 %

Grade Distribution

Number of Users (%)



View By: Groups Groups: MATH135_1219_LEC Apply

WA2 Class Statistics

Number of submitted grades: 1,255 / 1,256

Minimum:		0 %
Maximum:		100 %
Average:		81.53 %

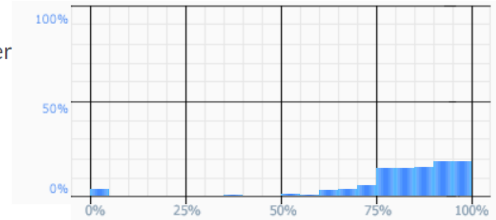
Mode: 90.91 %

Median: 86.36 %

Standard Deviation: 19.23 %

Grade Distribution

Number of Users (%)



University of Waterloo

MATH 135 Midterm Examination

Algebra for Honours Mathematics   Fall 2019

Instructors: S. Bauman, P. Das, B. Ferguson, M. Hamdy, F. Hu, G. Islambouli, C. Knoll, W. Kuo, Y.-R. Liu, A. Mahmoud, C. Morland, A. Mosunov, J. Nelson, M. Penney, J.P. Pretti, P. Roh, N. Rollick, D. Santos, D. Stebila, J. Wang

Username: ____________________ @uwaterloo.ca

ID number: ______________________________

Date of exam:	October 5, 2019
Exam period:	1:00 PM to 2:50 PM
Duration of exam:	110 minutes
Number of exam pages:	12 (includes cover page)
Exam type:	Closed book. No calculator.

Midterm Exam, Fall 2019

Question 5 (5 points)
Let $a_1, a_2, a_3, \cdots$, be a sequence of positive integers defined by

$$a_1 = 1, \quad a_2 = 5, \quad a_m = a_{m-1} + 2a_{m-2}, \quad \text{if } m \geq 3.$$

Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

Midterm Exam, Fall 2019

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Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

Who here has done induction before? Why or why not?

Midterm Exam, Fall 2019

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Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

Who here has done induction before? Why or why not?

(take note of who hasn't done induction before)

Midterm Exam, Fall 2019

Question 5 (5 points)
Let $a_1, a_2, a_3, \cdots$, be a sequence of positive integers defined by

$$a_1 = 1, \quad a_2 = 5, \quad a_m = a_{m-1} + 2a_{m-2}, \quad \text{if } m \geq 3.$$

Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

Base Cases
If $n = 1$,

$$\text{LHS} = a_1 = 1 \quad \text{and} \quad \text{RHS} = 2^1 + (-1)^1 = 1.$$

Since LHS=RHS, $P(1)$ is true.
If $n = 2$,

$$\text{LHS} = a_2 = 5 \quad \text{and} \quad \text{RHS} = 2^2 + (-1)^2 = 5.$$

Since LHS=RHS, $P(2)$ is true.

What's the difference between simple induction and strong induction?

Ask Question

Asked 7 years, 11 months ago Active 8 days ago Viewed 158k times

92

I just started to learn induction in my first year course. I'm having a difficult time grasping the concept. I believe I understand the basics but could someone summarize simple induction and strong induction and explain what the differences are? The video I'm watching explains that if $P(k)$ is true then $P(k+1)$ is true for simple induction, and for strong induction if $P(i)$ is true for all i less than equal to k then $P(k+1)$ is true. I don't really know what that means.

41

induction

Share Cite

edited Feb 9 '16 at 17:20 Mike Pierce 17.3k 11 58 110

asked Oct 7 '13 at 7:13 Jake Park 1,167 1 9 15

For further reading regarding mathematical induction read this. – user170039 Aug 12 '16 at 4:37

2 Answers

Active Oldest Votes

84

With simple induction you use "if $p(k)$ is true then $p(k+1)$ is true" while in strong induction you use "if $p(i)$ is true for all i less than or equal to k then $p(k+1)$ is true", where $p(k)$ is some statement depending on the positive integer k.

They are NOT "identical" but they are equivalent.

It is easy to see that if strong induction is true then simple induction is true: if you know that statement $p(i)$ is true for all i less than or equal to k, then you know that it is true, in particular, for $i=k$ and can use simple induction.

It is harder to prove, but still true, that if strong induction is true, then simple induction is true. That is what we mean by "equivalent".

Here we have a question. It is not why we still have "weak" induction - it's why we still have "strong" induction when this is not actually any stronger.

My opinion is that the reason this distinction remains is that it serves a pedagogical purpose. The first proofs by induction that we teach are usually things like $\forall n \left[\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right]$. The proofs of these naturally suggest "weak" induction, which students learn as a pattern to mimic.

Later, we teach more difficult proofs where that pattern no longer works. To give a name to the difference, we call the new pattern "strong induction" so that we can distinguish between the methods when presenting a proof in lecture. Then we can tell a student "try using strong induction", which is more helpful than just "try using induction".

Linked

1 In mathematical induction, does assuming $P(n)$ also assume all $P(k), 0 \leq k < n$?

1 Establish equivalence between the principle of Weak Induction and Strong Induction.

35 What exactly is the difference between weak and strong induction?

2 $(3+\sqrt{5})^n + (3-\sqrt{5})^n \equiv 0\ [2^n]$

2 Solving Complex Number Equation with Galois theory

1 Show that $\sqrt{2}$ is an irrational number with strong mathematical induction

3 Am I correct that this proof doesn't need strong induction?

3 Simple Induction vs Strong Induction proof.

0 Alternative methods to avoid induction

2 Vacuous truth and (simple and complete) induction

See more linked questions

Related

6 Is "Strong Induction" not actually stronger than normal induction?

1 Infinite descent method and strong induction

35 What exactly is the difference between weak and strong induction?

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Prove that for all $n \in \mathbb{N}$,

$$a_n = 2^n + (-1)^n.$$

<u>Inductive Step</u> Let $k \in \mathbb{N}$ with $k \geq 2$ arbitrary. Suppose that $P(1), \cdots, P(k)$ are true, i.e.,

$$a_r = 2^r + (-1)^r \quad \text{for } 1 \leq r \leq k.$$

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$$a_{k+1} = a_k + 2a_{k-1} \quad \text{(by definition since } k+1 \geq 3)$$

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$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} \quad \text{(by definition since } k+1 \geq 3) \\ &= (2^k + (-1)^k) + 2(2^{k-1} + (-1)^{k-1}) \quad \text{(by inductive hypothesis)} \end{aligned}$$

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&= 2^k + (-1)^k + 2^k + 2 \cdot (-1)^{k-1}\\
&= 2^k(1+1) + (-1)^{k-1}(-1+2)\\
&= 2^{k+1} + (-1)^{k-1}
\end{aligned}$$

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Time is precious. Check in WolframAlpha.

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&= 2^k(1+1) + (-1)^{k-1}(-1+2)\\
&= 2^{k+1} + (-1)^{k-1}\\
&= 2^{k+1} + (-1)^{k+1} \quad (\text{since } (-1)^{k-1} = (-1)^{k+1}).
\end{aligned}$$

Thus $P(k+1)$ is true. By POSI, $P(n)$ is true for all $n \in \mathbb{N}$.

University of Waterloo

MATH 135 Midterm Examination

Algebra for Honours Mathematics   Winter 2017

Instructors: L. Haykazyan, J. Pretti, R. Kim, M. Satriano, L. Narins, D. Sharma, J. Stephenson, M. Pei, R. Moosa

Username: ________________ @uwaterloo.ca

Last name: ______________________________

First name: ______________________________

ID number: ______________________________

Date of exam:	February 6, 2017
Exam period:	7:00 PM to 8:50 PM
Duration of exam:	110 minutes
Number of exam pages:	12 (includes cover page)
Exam type:	Closed book

9. Let the sequence $\{a_n\}$ be defined by $a_1 = 2$, $a_2 = 1$, and $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 3$.
Prove that $a_n \leq 3^n$ for all integers $n \geq 1$. **[4 marks]**

9. Let the sequence $\{a_n\}$ be defined by $a_1 = 2$, $a_2 = 1$, and $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 3$.
Prove that $a_n \leq 3^n$ for all integers $n \geq 1$. **[4 marks]**

Let $P(n)$ be the statement to be proven.

Base Case: When $n = 1$, $a_1 = 2$ and $3^n = 3$ so since $2 \leq 3$, $P(1)$ is true. When $n = 2$, $a_1 = 1$ and $3^n = 9$ so since $1 \leq 9$, $P(2)$ is true.

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Inductive Hypothesis: We assume that the statement $P(i)$ is true for all integers i satisfying $1 \leq i \leq k$ and some integer $k \geq 2$.

Inductive Conclusion: When $n = k + 1$,

$$a_n = 2a_k + 2a_{k-1} \quad \text{by definition}$$

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$$\begin{aligned} a_n &= 2a_k + 2a_{k-1} \quad \text{by definition} \\ &\leq 2 \cdot 3^k + 2 \cdot 3^{k-1} \quad \text{by the inductive hypothesis} \end{aligned}$$

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\end{aligned}$$

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Inductive Conclusion: When $n = k + 1$,

$$\begin{aligned} a_n &= 2a_k + 2a_{k-1} \quad \text{by definition} \\ &\leq 2 \cdot 3^k + 2 \cdot 3^{k-1} \quad \text{by the inductive hypothesis} \\ &= 2 \cdot 3 \cdot 3^{k-1} + 2 \cdot 3^{k-1} \\ &= (6 + 2) \cdot 3^{k-1} \\ &= 8 \cdot 3^{k-1} \\ &\leq 3^2 \cdot 3^{k-1} \\ &= 3^{k+1} \end{aligned}$$

Therefore, by the *Principle of Srong Induction (POSI)*, $P(n)$ is true for all natural numbers n.

10. Use induction to prove that $3 \mid (5^n + 2^{n+1})$ for every natural number n. **[4 marks]**

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Let $P(n)$ be the statement to be proven.

Base Case: When $n = 1$, $5^n + 2^{n+1} = 9$ and $3 \mid 9$ so $P(1)$ is true.

10. Use induction to prove that $3 \mid (5^n + 2^{n+1})$ for every natural number n. **[4 marks]**

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$$5^n + 2^{n+1} = 5^{k+1} + 2^{k+2}$$

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Now $3 \mid 3 \cdot 5^k$ because 5^k is an integer, and $3 \mid (5^k + 2^{k+1})$ by the inductive hypothesis.

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Now $3 \mid 3 \cdot 5^k$ because 5^k is an integer, and $3 \mid (5^k + 2^{k+1})$ by the inductive hypothesis. Thus by *Divisibility of Integer Combinations (DIC)*, $3 \mid (5^n + 2^{n+1})x + 3 \cdot 5^k y$ where $x = 2$ and $y = 1$. Therefore $3 \mid (5^n + 2^{n+1})$ and $P(n)$ is true for all $n \in \mathbb{N}$ by the *Principle of Mathematical Induction (POMI)*.

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Marking Scheme:

- 1 mark(s): correctly verifying $P(1)$
- 1 mark(s): giving a correct inductive hypothesis that is quantified correctly
  - "Let $k \geq 1$. Assume ..." is okay
  - "Fix $k \geq 1$. Assume ..." is okay
  - "Suppose $k \geq 1$. Assume ..." is okay
  - "There exists $k \geq 1$ ..." is okay
  - "for all $k \geq 1$ is not okay (deduct a mark)
  - not quantifying k is not okay (deduct a mark)
- 1 mark(s): finding a way to relate $5^{k+1} + 2^{k+2}$ to $5^k + 2^{k+1}$ (must be clear that inductive hypothesis is being used but unlike Q9, don't require explicit citation)
- 1 mark(s): completing the details of the inductive conclusion

MATH 135 Sample Midterm F19 (based on F18 and W19 Midterms)

Question 5 (5 points)

The *Fibonacci Sequence* is defined as follows: $a_1 = 1$, $a_2 = 1$ and $a_m = a_{m-1} + a_{m-2}$ for all integers $m \geq 3$. Prove that for every $n \in \mathbb{N}$, $3 \mid a_{4n}$.

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The *Fibonacci Sequence* is defined as follows: $a_1 = 1$, $a_2 = 1$ and $a_m = a_{m-1} + a_{m-2}$ for all integers $m \geq 3$. Prove that for every $n \in \mathbb{N}$, $3 \mid a_{4n}$.

Solution Let $P(n)$ denote the statement that $3|a_{4n}$.
<u>Base case</u> Consider $n = 1$ and $a_{4n} = a_4$. Since $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 2$ and $a_4 = 3$. Since $3 \mid 3$, we have3 $\mid a_4$ and hence $P(1)$ is true.

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Inductive Step: Let $k \in \mathbb{N}$ be arbitrary. Suppose that $P(k)$ is true, i.e., $3 \mid a_{4k}$.

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