University of Waterloo

MATH 135 Midterm Examination

Algebra for Honours Mathematics   Fall 2019

Instructors: S. Bauman, P. Das, B. Ferguson, M. Hamdy, F. Hu, G. Islambouli, C. Knoll, W. Kuo, Y.-R. Liu, A. Mahmoud, C. Morland, A. Mosunov, J. Nelson, M. Penney, J.P. Pretti, P. Roh, N. Rollick, D. Santos, D. Stebila, J. Wang

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ID number: ____________________________________

Date of exam:	October 5, 2019
Exam period:	1:00 PM to 2:50 PM
Duration of exam:	110 minutes
Number of exam pages:	12 (includes cover page)
Exam type:	Closed book. No calculator.

MATH 135: Lecture 9

Dr. Nike Dattani

27 September 2021

- Monday 27 September:
  - Mobius Quiz 7
- Tuesday 28 September:
  - **Look at your WA02 results thoroughly! Where did you lose marks?**
- Tuesday 28 September:
  - Complete reading from Chapter 3.6 up to 4.4 of the course notes. **Pages 55-75.**
- **Wednesday 29 September:**
  - **Complete Written Assignment 3: WA3**
- Wednesday 29 September:
  - Mobius Quiz 8
- Thursday 30 September:
  - WA03 solutions will be posted, hopefully before 12pm: Check the solutions in detail!
- Friday 1 October:
  - Mobius Quiz 9
- Sunday 3 October:
  - You'll need to know more before 0.4 (Polynomials), so use this time to review **Pages 55-75**, and do practice problems!

View By: Groups Groups: 1219-MATH.135.019.1.LEC Apply

MQ6 (Fri Sep 24) Class Statistics

Number of submitted grades: 54 / 56

Minimum: 71.67 %

Maximum: 100 %

Average: 94.51 %

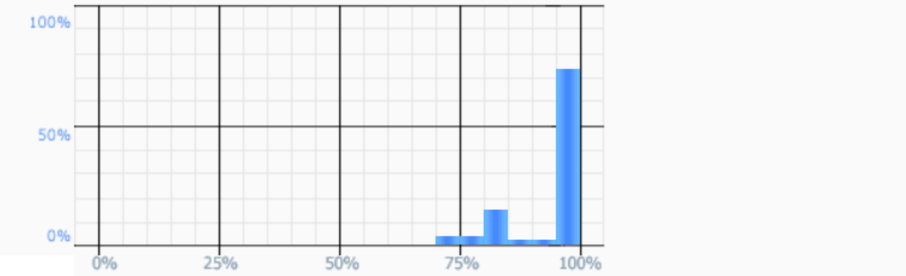
Mode: 100 %

Median: 100 %

Standard Deviation: 9.14 %

Grade Distribution

Number of Users (%)



View By: Groups Groups: 1219-MATH.135.016.1.LEC Apply

MQ6 (Fri Sep 24) Class Statistics

Number of submitted grades: 47 / 51

Minimum: 55 %

Maximum: 100 %

Average: 90.43 %

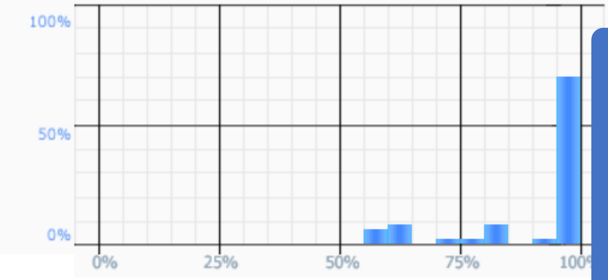
Mode: 100 %

Median: 100 %

Standard Deviation: 15.17 %

Grade Distribution

Number of Users (%)



View By: Groups Groups: MATH135_1219_LEC App

MQ6 (Fri Sep 24) Class Statistics

Number of submitted grades: 1,182 / 1,260

Minimum: 0 %

Maximum: 100

Average: 92 %

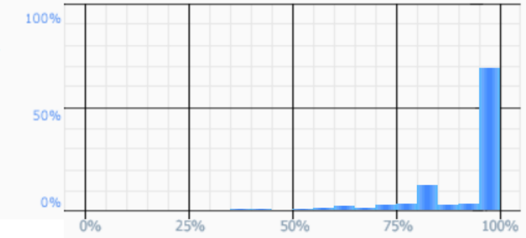
Mode: 100 %

Median: 100 %

Standard Deviation: 13.54 %

Grade Distribution

Number of Users (%)



Midterm Exam, Fall 2019

Question 2 (7 points)
Let $x, y \in \mathbb{R}$. Consider the implication S:

$$\text{If } xy > 6, \text{ then } x > 2 \text{ and } y > 3.$$

(a) State the hypothesis of S.

(b) State the conclusion of S.

(c) State the converse of S.

(d) State the contrapositive of S.

(e) State the negation of S in a form that does not contain an implication.

(f) Indicate clearly whether the given implication S is true or false for all $x, y \in \mathbb{R}$. Then prove or disprove the statement.

Circle the correct answer: True False

xy > 6 => (x > 2) ^ (y > 3)

X = 1 , y = 7 => xy > 6, but x < 2

Midterm Exam, Fall 2019

Question 4 (5 points)
For each of the following statements indicate clearly whether the statement is true or false and then prove or disprove the statement.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + 2y = 0.$

Circle the correct answer: True False

$\forall$ x $\in \mathbb{R}$, -x /2 $\in \mathbb{R}$

Let y = -x /2

x + 2y = x + 2 (-x / 2)

= x – x

= 0

(b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + 2y = 0.$

$$\neg\,(\exists\, y \in \mathbb{R}\,,\, \forall\, x \in \mathbb{R}\,,\, x + 2y = 0)$$

$$\forall\, y \in \mathbb{R}\,,\, \exists\, x \in \mathbb{R}\,,\, x + 2y \neq 0$$

$$\forall\, y \in \mathbb{R}\,,\, \exists\, x \in \mathbb{R}\,,\, x = -2y + 1$$

Now let's consider $x + 2y$, when $x = -2y + 1$:

$$\begin{aligned} x + 2y &= -2y + 1 + 2y \\ &= 1 \\ &\neq 0 \end{aligned}$$

Question 7 (5 points)
Let $a, b \in \mathbb{Z}$ with $a \geq 2$. Prove that if $a \neq 13$, then $a \nmid (3b+1)$ or $3a \nmid (7b-2)$.

Don't like the ∤ symbol:

¬((a ∤ 3b + 1) V (3a ∤ 7b - 2)) => ¬(a ≠ 13)
¬(a ∤ 3b + 1) ^ ¬(3a ∤ 7b - 2) => ¬(a ≠ 13)
(a | 3b + 1) ^ (3a | 7b - 2) => a = 13

(a | 3a) ^ (3a | 7b -2) => a | 7b -2

(a | 3b + 1) ^ (a | 7b - 2) => a |x(3b + 1) + y(7b-2), ∀x,y ∈ ℤ

x = 7, y = -3
a | 7(3b + 1) – 3(7b – 2)
a | 21b - 21b + 7 + 6
a | 13

The only integers that divide 13 are -1, 1, -13 and 13,

But a >= 2, so a = 13.

Question 7 (5 points)

Let $a, b \in \mathbb{Z}$ with $a \geq 2$. Prove that if $a \neq 13$, then $a \nmid (3b+1)$ or $3a \nmid (7b-2)$.

Try to prove the contrapositive:

$\neg((a \nmid 3b + 1) \vee (3a \nmid 7b - 2)) \Rightarrow \neg(a \neq 13)$

$\neg(a \nmid 3b + 1) \wedge \neg(3a \nmid 7b - 2) \Rightarrow \neg(a \neq 13)$

$(a \mid 3b + 1) \wedge (3a \mid 7b - 2) \Rightarrow a = 13$

$a \mid 3a$ (corollary of the definition of divisibility)

$(a \mid 3a) \wedge (3a \mid 7b - 2) \Rightarrow a \mid 7b - 2$ (by transitivity of divisbility)

$(a \mid 3b + 1) \wedge (a \mid 7b - 2) \Rightarrow a \mid x(3b + 1) + y(7b - 2), \forall x, y \in \mathbb{Z}$ (by D.I.C. Maybe also talk about $3b-1 \in \mathbb{Z}$)

Now let, $x = 7$, $y = -3$

$a \mid 7(3b + 1) - 3(7b - 2)$ (by D.I.C.)

$a \mid 21b - 21b + 7 + 6$

$a \mid 13$

The only integers that divide 13 are -1, 1, -13 and 13,

But ***a*** >= 2, so ***a*** must be 13.

Question 8 (5 points)

(a) Prove that for all $n \in \mathbb{N}$, $n^2 + n + 1$ is odd.

Case 1: n = 2k for $k \in \mathbb{Z}$ (n is even)

$(2k)^2 + (2k) + 1$

$4k^2 + 2k + 1$

$4k^2$ is even because it contains a factor of 2

$2k$ is even for the same reason

$4k^2 + 2k$ is even because an even number plus an even number is even.

$4k^2 + 2k + 1$ is odd because an even number plus an odd number is odd.

Question 8 (5 points)

(a) Prove that for all $n \in \mathbb{N}$, $n^2 + n + 1$ is odd.

Case 1: n = 2k for $k \in \mathbb{Z}$ (n is even)

$(2k)^2 + (2k) + 1$

$4k^2 + 2k + 1$

$4k^2$ is even because it contains a factor of 2

2k is even for the same reason

$4k^2 + 2k$ is even because an even number plus an even number is even.

Let $E = 4k^2 + 2k$, which we know is even.

$4k^2 + 2k + 1 = E + 1$, which is odd because an even number plus an odd number is odd.

Case 2: n = 2k + 1$k \in \mathbb{Z}$ (n is odd)

(2k + 1)^2 = $4k^2 + 4k + 1$

Question 8 (5 points)

(a) Prove that for all $n \in \mathbb{N}$, $n^2 + n + 1$ is odd.

Case 1: $n = 2k$ for $k \in \mathbb{Z}$ (n is even)

$(2k)^2 + (2k) + 1$

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Let $E = 4k^2 + 2k$, which we know is even.

$4k^2 + 2k + 1 = E + 1$, which is odd because an even number plus an odd number is odd.

Case 2: $n = 2k + 1 k \in \mathbb{Z}$ (n is odd)

$(2k + 1)$^2 $= 4k^2 + 4k + 1$ (odd because of Case 1?)

Question 8 (5 points)

(a) Prove that for all $n \in \mathbb{N}$, $n^2 + n + 1$ is odd.

Case 1: $n = 2k$ for $k \in \mathbb{Z}$ (n is even)

$(2k)^2 + (2k) + 1$

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$4k^2$ is even because it contains a factor of 2

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$4k^2 + 2k$ is even because an even number plus an even number is even.

Let $E = 4k^2 + 2k$, which we know is even.

$4k^2 + 2k + 1 = E + 1$, which is odd because an even number plus an odd number is odd. (*)

Case 2: $n = 2k + 1$ $k \in \mathbb{Z}$ (n is odd)

$(2k + 1)^2 = 4k^2 + 4k + 1$

$(2k + 1)^2 + (2k + 1) + 1$

$4k^2 + 4k + 1 + 2k + 1 + 1$

$4k^2 + 6k + 3$

Let $E = 4k^2 + 6k$, which is even because it contains a factor of 2.

$4k^2 + 6k + 3 = E + 3$, which is odd by the same reasoning as in (*).

Since all integers are either even or odd, we have proven that the statement is true for all possible cases.

(b) Let $d, n \in \mathbb{N}$. Prove that if $d \mid (n^2 + n + 1)$ and $d \mid (n^2 + n + 3)$, then $d = 1$.

$(d \mid n^2 + n + 1) \wedge (d \mid n^2 + n + 3) \Rightarrow d \mid (n^2 + n + 1)(1) + (n^2 + n + 3)(-1)$ (by D.I.C. with x=1, y= -1)

$\therefore\ d \mid -2$

$\therefore\ d \mid 2$ (corollary to definition of divisibility)

$\therefore\ d = -2, -1, +1, \text{ or } +2$

(b) Let $d, n \in \mathbb{N}$. Prove that if $d \mid (n^2 + n + 1)$ and $d \mid (n^2 + n + 3)$, then $d = 1$.

$(d \mid n^2 + n + 1) \wedge (d \mid n^2 + n + 3) \Rightarrow d \mid (n^2 + n + 1)(1) + (n^2 + n + 3)(-1)$ (by D.I.C. with x=1, y= -1)

$\therefore\ d \mid -2$

$\therefore\ d \mid 2$ (corollary to definition of divisibility)

$\therefore$ d is either -2, -1, +1, or + 2

(b) Let $d, n \in \mathbb{N}$. Prove that if $d \mid (n^2 + n + 1)$ and $d \mid (n^2 + n + 3)$, then $d = 1$.

$(d \mid n^2 + n + 1) \wedge (d \mid n^2 + n + 3) \Rightarrow d \mid (n^2 + n + 1)(1) + (n^2 + n + 3)(-1)$ (by D.I.C. with x=1, y= -1)

$\therefore d \mid -2$

$\therefore d \mid 2$ (corollary to definition of divisibility)

$\therefore$ d is either -2, -1, +1, or + 2

But $n^2 + n + 1$ is odd, so $2 \nmid n^2 + n + 1$ and $-2 \nmid n^2 + n + 1$

If d = 2 or -2, then we would have a contradiction.

So d is either -1 or +1

But d is in $\mathbb{N}$

So d must be +1.

Thank you for coming to class!