```
> #################################### P1
#######################################
restart;
> Digits:=15;
```

$$Digits := 15$$

```
> eq:=diff(y(t),t)=-y(t);
```

$$eq := \frac{d}{dt}\,\mathrm{y}(t) = -\mathrm{y}(t)$$

```
> sol:=dsolve({eq,y(0)=1},type=numeric):
> sol(1);
```

$$[t = 1., \mathrm{y}(t) = 0.367879356307219\ ]$$

```
> f:=rhs(eq);
```

$$f := -\mathrm{y}(t)$$

```
> f:=subs(y(t)=Y,f);
```

$$f := -Y$$

```
> F:=unapply(f,Y);
```

$$F := Y \rightarrow -Y$$

```
> F(0);F(1);
```

$$0$$

$$-1$$

```
> YY[0]:=1.0;
```

$$YY_0 := 1.0$$

```
> h:=0.1;
```

$$h := 0.1$$

```
> YY[1]:=YY[0]+h*F(YY[0]);
```

$$YY_1 := 0.90$$

```
> for i from 2 to 10 do YY[i]:=YY[i-1]+h*F(YY[i-1]);od;
```

$$YY_2 := 0.810$$

$$YY_3 := 0.7290$$

$$YY_4 := 0.65610$$

$$YY_5 := 0.590490$$

$$YY_6 := 0.5314410$$

$$YY_7 := 0.47829690$$

$$YY_8 := 0.430467210$$

$$YY_9 := 0.3874204890$$

$$YY_{10} := 0.34867844010$$

```
> err:=abs(subs(sol(1),y(t))-YY[10]);
```

$$err := 0.0192009162072187$$

```
> Eulerforward:=proc(f,y0,tf,N)
local F,h,YY,i;
F:=unapply(f,Y);
h:=tf/N;
YY[0]:=y0;
for i from 1 to N do
YY[i]:=YY[i-1]+h*F(YY[i-1]);
od;
#[seq([i*h,YY[i]],i=0..N)];#for printing all the values
YY[N];
end proc;
```

Eulerforward := **proc**(*f*, *y0*, *tf*, *N*)
local *F*, *h*, *YY*, *i*;
F := unapply(*f*, *Y*);
h := *tf*/*N*;
YY[0] := *y0*;
for *i* **to** *N* **do** $YY[i] := YY[i-1] + h \times F(YY[i-1])$ **end do**;
YY[*N*]
end proc

```
> f;
```

$$-Y$$

```
> sol(1);
```

$$[t = 1., y(t) = 0.367879356307219]$$

```
> sol3:=Eulerforward(f,1.0,1.0,2);
```

$$sol3 := 0.250000000000000$$

```
> for i from 1 to 10 do YY[i]:=Eulerforward(f,1.0,1.0,2^i);od;
```

$$YY_1 := 0.250000000000000$$

$$YY_2 := 0.316406250000000$$

$$YY_3 := 0.343608915805816$$

$$YY_4 := 0.356074130451793$$

$$YY_5 := 0.362055289256317$$

$$YY_6 := 0.364986524243911$$

$$YY_7 := 0.366437715922039$$

$$YY_8 := 0.367159754891533$$

$$YY_9 := 0.367519891254532$$

$$YY_{10} := 0.367699739411277$$

```
> for i from 1 to 9 do
Y1[i]:=abs(YY[i+1]-YY[i]);
od;
```

$$Y1_1 := 0.066406250000000$$

$$Y1_2 := 0.027202665805816$$

$$Y1_3 := 0.012465214645977$$

$$Y1_4 := 0.005981158804524$$

$$Y1_5 := 0.002931234987594$$

$$Y1_6 := 0.001451191678128$$

$$Y1_7 := 0.000722038969494$$

$$Y1_8 := 0.000360136362999$$

$$Y1_9 := 0.000179848156745$$

```
> for i from 1 to 8 do Y1[i]/Y1[i+1];od;
```

2.44116699716254

2.18228619228754

2.08408020140656

2.04049106599721

2.01988133736766

2.00985229252236

2.00490437422451

2.00244678353654

```
> #from last homework:
AutoEuler:=proc(f,y0,tf,N)
local F,h,yy,i;
F2:=unapply(f2,seq(Y[i],i=1..nops(y0)));
h:=tf/N;
if nops(y0)>1 then
#print("yes");
```

```
yy[0]:=y0;
for j from 1 to N do
yy[j]:=yy[j-1]+h*F2(seq(yy[j-1][i],i=1..nops(y0)));
#Y1[j][1]:=abs(yy[j][1]-yy[j-1][1]);
#Y1[j][2]:=abs(yy[j][2]-yy[j-1][2]);
#Y11[j][1]:=Y1[j][1]/Y1[j-1][1];
#Y11[j][2]:=Y1[j][2]/Y1[j-1][2];
#print(Y11[j][1]);
#print(Y11[j][2]);
od;
#print(yy[N]);
end if;
if nops(y0)=1 then yy[0]:=y0; print("no");
for i from 1 to N do
yy[i]:=yy[i-1]+h*F(yy[i-1]);
od;
#return yy[N];
end if;
#h;
yy[N];
end proc;
```

```
Warning, `F2` is implicitly declared local to procedure `AutoEuler`

Warning, `j` is implicitly declared local to procedure `AutoEuler`
```

AutoEuler := **proc**(*f*, *y0*, *tf*, *N*)
local *F*, *h*, *yy*, *i*, *F2*, *j*;
$F2 := \mathrm{unapply}(f2, \mathrm{seq}(Y[i], i = 1 .. \mathrm{nops}(y0)));$
$h := tf/N;$
if $1 < \mathrm{nops}(y0)$ **then**
$yy[0] := y0;$
for j **to** N **do** $yy[j] := yy[j-1] + h \times F2(\mathrm{seq}(yy[j-1][i], i = 1 .. \mathrm{nops}(y0)))$
end do
end if;
if $\mathrm{nops}(y0) = 1$ **then**
$yy[0] := y0;$
print("no");
for i **to** N **do** $yy[i] := yy[i-1] + h \times F(yy[i-1])$ **end do**
end if;
$yy[N]$
end proc

```
> eq11:=diff(y[1](t),t)=-y[1](t)^2;
eq2:=diff(y[2](t),t)=y[1](t)^2-y[2](t);
f2:=subs([y[1](t)=Y[1],y[2](t)=Y[2]],[rhs(eq11),rhs(eq2)]);
```

$$eq11 := \frac{d}{dt} y_1(t) = -y_1(t)^2$$

$$eq2 := \frac{d}{dt} y_2(t) = y_1(t)^2 - y_2(t)$$

$$f2 := [-Y_1^{\,2}, Y_1^{\,2} - Y_2]$$

```
> sol2:=dsolve({eq11,eq2,y[1](0)=1,y[2](0)=0},type=numeric):
> sol2(1);
```

$$[t = 1., y_1(t) = 0.500000005414531\ , y_2(t) = 0.281885861747178\ ]$$

```
> eff:=[1.0,0.0];
```

$$eff := [1.0, 0.]$$

```
>
for i from 1 to 10 do YY[i]:=AutoEuler(f2,eff,1.0,2^i);od;
```

$$YY_1 := [0.375000000000000\ , 0.375000000000000\ ]$$

$$YY_2 := [0.449836998246610\ , 0.320899696089327\ ]$$

$$YY_3 := [0.476811488179000\ , 0.300104275573270\ ]$$

$$YY_4 := [0.488805718500791\ , 0.290703144682500\ ]$$

$$YY_5 := [0.494495866636012\ , 0.286224758749415\ ]$$

$$YY_6 := [0.497270407411737\ , 0.284038252521431\ ]$$

$$YY_7 := [0.498640730115085\ , 0.282957833229589\ ]$$

$$YY_8 := [0.499321735441957\ , 0.282420790858355\ ]$$

$$YY_9 := [0.499661208932730\ , 0.282153056406220\ ]$$

$$YY_{10} := [0.499830689597354\ , 0.282019385231668\ ]$$

```
> for i from 1 to 9 do
Y1[i][1]:=abs(YY[i+1][1]-YY[i][1]);
Y1[i][2]:=abs(YY[i+1][2]-YY[i][2]);
od;
for i from 1 to 9 do
[Y1[i][1],Y1[i][2]];
od;
```

$$Y1_{1_1} := 0.074836998246610$$

$$Y1_{1_2} := 0.054100303910673$$

$$Y1_{2_1} := 0.026974489932390$$

$$Y1_{2_2} := 0.020795420516057$$

$$Y1_{3_1} := 0.011994230321791$$

$$Y1_{3_2} := 0.009401130890770$$

$$Y1_{4_1} := 0.005690148135221$$

$$Y1_{4_2} := 0.004478385933085$$

$$Y1_{5_1} := 0.002774540775725$$

$$Y1_{5_2} := 0.002186506227984$$

$$Y1_{6_1} := 0.001370322703348$$

$$Y1_{6_2} := 0.001080419291842$$

$$Y1_{7_1} := 0.000681005326872$$

$$Y1_{7_2} := 0.000537042371234$$

$$Y1_{8_1} := 0.000339473490773$$

$$Y1_{8_2} := 0.000267734452135$$

$$Y1_{9_1} := 0.000169480664624$$

$$Y1_{9_2} := 0.000133671174552$$

$$[0.074836998246610, 0.054100303910673]$$

$$[0.026974489932390, 0.020795420516057]$$

$$[0.011994230321791, 0.009401130890770]$$

$$[0.005690148135221, 0.004478385933085]$$

$$[0.002774540775725, 0.002186506227984]$$

$$[0.001370322703348, 0.001080419291842]$$

$$[0.000681005326872, 0.000537042371234]$$

$$[0.000339473490773, 0.000267734452135]$$

$$[0.000169480664624, 0.000133671174552]$$

```
> for i from 1 to 8 do
[Y1[i][1]/Y1[i+1][1],Y1[i][2]/Y1[i+1][2]];
od;
```

$$[2.77436194101110, 2.60154892606764]$$

$$[2.24895547348153, 2.21201265652772]$$

$$[2.10789421237540, 2.09922303062744]$$

$$[2.05084321881488, 2.04819262610295]$$

$$[2.02473531887503, 2.02375711401473]$$

$$[2.01220555739583, 2.01179525064185]$$

$$[2.00606334627577, 2.00587696858381]$$

$$[2.00302194663997, 2.00293334020827]$$

```
> ################################## P2
#####################################
> restart;
NewtonRhapson:=proc(E,V,T,G,jac)
local f,F,JAC,xold,tol,xnew;
with(ArrayTools):
with(LinearAlgebra):
f:=E;
F:=unapply(f,x);
#jac:=VectorCalculus:-Jacobian(f,[seq(x[i],i=1..nops(V))]);
#print(jac);
Jac:=unapply(jac,x);
xold:=G;
tol:=1;
for i from 1 to 50 do
dx:=LinearAlgebra:-LinearSolve(-Jac(xold),F(xold));
#print(Jac(xold));
xnew:=xold+dx;
tol:=VectorCalculus[Norm](dx);
xold:=xnew;
if tol < T then break end if
end do;
end proc;
```

```
Warning, `Jac` is implicitly declared local to procedure `NewtonRhapson`

Warning, `i` is implicitly declared local to procedure `NewtonRhapson`

Warning, `dx` is implicitly declared local to procedure `NewtonRhapson`
```

```
NewtonRhapson := proc(E, V, T, G, jac)
local f, F, JAC, xold, tol, xnew, Jac, i, dx;
    with(ArrayTools);
    with(LinearAlgebra);
    f := E;
    F := unapply(f, x);
    Jac := unapply(jac, x);
    xold := G;
    tol := 1;
    for i to 50 do
        dx := LinearAlgebra:-LinearSolve(-Jac(xold), F(xold));
        xnew := xold + dx;
        tol := VectorCalculus[Norm](dx);
        xold := xnew;
        if tol < T then break end if
    end do
end proc
```

```
> Digits:=15;
```

$$Digits := 15$$

```
> eq1:=Y1=y(t-h);
```

$$eq1 := Y1 = \mathrm{y}(t-h)$$

```
> eq1:=Y1=series(y(t-h),h);
```

$$eq1 := Y1 = \mathrm{y}(t) - \mathrm{D}(y)(t)\,h + \frac{1}{2}(\mathrm{D}^{(2)})(y)(t)\,h^2 - \frac{1}{6}(\mathrm{D}^{(3)})(y)(t)\,h^3 + \frac{1}{24}(\mathrm{D}^{(4)})(y)(t)\,h^4 - \frac{1}{120}(\mathrm{D}^{(5)})(y)(t)\,h^5 + \mathrm{O}(h^6)$$

```
> eq1:=Y1=series(y(t-h),h,3);
```

$$eq1 := Y1 = \mathrm{y}(t) - \mathrm{D}(y)(t)\,h + \frac{1}{2}(\mathrm{D}^{(2)})(y)(t)\,h^2 + \mathrm{O}(h^3)$$

```
> eq1:=y0=convert(series(y(t-h),h,2),polynom);
```

$$eq1 := y0 = \mathrm{y}(t) - \mathrm{D}(y)(t)\,h$$

```
> eq1:=convert(eq1,diff);
```

$$eq1 := y0 = \mathrm{y}(t) - \left(\frac{d}{dt}\mathrm{y}(t)\right)h$$

```
> eq2:=subs(diff(y(t),t)=F1,y(t)=Y1,eq1);
```

$$eq2 := y0 = -F1\,h + Y1$$

```
> eq3:=Y1=solve(eq2,Y1);
```

$$eq3 := Y1 = F1\,h + y0$$

```
> eq:=diff(y(t),t)=-y(t);
```

$$eq := \frac{d}{dt}\mathrm{y}(t) = -\mathrm{y}(t)$$

```
> sol:=dsolve({eq,y(0)=1},type=numeric):
> sol(1);
```

$$[t = 1., \mathrm{y}(t) = 0.367879356307219]$$

```
> f:=rhs(eq);
```

$$f := -\mathrm{y}(t)$$

```
> f:=subs(y(t)=Y,f);
```

$$f := -Y$$

```
> F:=unapply(h*f+y0-Y,y0,h);
```

$$F := (y0, h) \rightarrow -Y\,h - Y + y0$$

```
> F(1,0.1);
with(VectorCalculus):
jac:=Jacobian([f],[Y]);
Jac:=unapply(jac,Y);
#T:=.001;
xold:=1;
step:=0.1;
for i from 1 to 10 do
dx:=LinearAlgebra:-LinearSolve([-Jac(xold)],[F(xold,step)]);
#xnew:=xold+dx;
#tol:=VectorCalculus[Norm](dx);
#xold:=xnew;
#if tol < T then break end if
end do;

>
>
```

$$-1.1\,Y + 1$$

$$jac := [\,-1]$$

$$Jac := Y \rightarrow \mathrm{Matrix}(1,\ 1,\ [[...]],\ \mathrm{datatype} = \mathrm{anything})$$

$$xold := 1$$

$$step := 0.1$$

```
Error, (in LinearAlgebra:-LinearSolve) invalid input: LinearAlgebra:-
LinearSolve expects its 2nd argument, B, to be of type {Matrix,
Vector[column]} but received [-1.1*Y+1]
```

```
> fsolve(%,Y);
> YY[0]:=1.0;
```

$$YY_0 := 1.0$$

```
> h:=0.1;YY[1]:=fsolve(F(1,0.1),Y=1);
```

$$h := 0.1$$

$$YY_1 := 0.909090909090909$$

```
> F(YY[1],YY[0],0.1);
```

$$-2.0\,Y + 0.909090909090909$$

```
> for i from 2 to 10 do YY[i]:=fsolve(F(YY[i-1],0.1),Y=YY[i-
1]);od;
>
```

$$YY_2 := 0.826446280991735$$

$$YY_3 := 0.751314800901577$$

$$YY_4 := 0.683013455365070$$

$$YY_5 := 0.620921323059155$$

$$YY_6 := 0.564473930053777$$

$$YY_7 := 0.513158118230706$$

$$YY_8 := 0.466507380209733$$

$$YY_9 := 0.424097618372485$$

$$YY_{10} := 0.385543289429532$$

```
> err:=abs(subs(sol(1),y(t))-YY[10]);
```

$$err := 0.0176639331223133$$

```
> EulerBDF:=proc(f,y00,tf,N)
local F,YY,h,y0,i;
F:=unapply(h*f+y0-Y,y0,h);
h:=tf/N;
YY[0]:=y00;
YY[1]:=fsolve(F(YY[0],h),Y=YY[0]);
#print(YY[1]);
for i from 2 to N do
YY[i]:=fsolve(F(YY[i-1],h),Y=YY[i-1]);
od;
#[seq([i*h,YY[i]],i=0..N)];#for printing all the values
YY[N];
end proc;
```

```
EulerBDF := proc(f, y00, tf, N)
local F, YY, h, y0, i;
    F := unapply( VectorCalculus:-`+`(
        VectorCalculus:-`+`( VectorCalculus:-`*`( h, f ), y0 ), VectorCalculus:-`-`( Y ) )
        , y0, h );
    h := VectorCalculus:-`*`( tf, 1/N );
    YY[ 0 ] := y00;
    YY[ 1 ] := fsolve( F( YY[ 0 ], h ), Y = YY[ 0 ] );
    for i from 2 to N do YY[ i ] := fsolve( F( YY[ VectorCalculus:-`+`( i, -1 ) ], h ),
        Y = YY[ VectorCalculus:-`+`( i, -1 ) ] )
    end do;
    YY[ N ]
end proc
```

```
> EulerBDF2:=proc(f,y00,tf,N)
local F,YY,h,y0,i;
F:=unapply(h*f+y0-Y,y0,h);
h:=tf/N;
YY[0]:=y00;
YY[1]:=NewtonRhapson(F(YY[0],h),Y=YY[0]);
#print(YY[1]);
for i from 2 to N do
YY[i]:=fsolve(F(YY[i-1],h),Y=YY[i-1]);
od;
#[seq([i*h,YY[i]],i=0..N)];#for printing all the values
YY[N];
end proc;
```

```
EulerBDF2 := proc(f, y00, tf, N)
local F, YY, h, y0, i;
    F := unapply( VectorCalculus:-`+`(
        VectorCalculus:-`+`( VectorCalculus:-`*`( h, f ), y0 ), VectorCalculus:-`-`( Y ) )
        , y0, h );
    h := VectorCalculus:-`*`( tf, 1/N );
    YY[ 0 ] := y00;
    YY[ 1 ] := NewtonRhapson( F( YY[ 0 ], h ), Y = YY[ 0 ] );
    for i from 2 to N do YY[ i ] := fsolve( F( YY[ VectorCalculus:-`+`( i, -1 ) ], h ),
        Y = YY[ VectorCalculus:-`+`( i, -1 ) ] )
    end do;
    YY[ N ]
end proc
```

```
> sol(1);
```

$$[\, t = 1.,\ y(t) = 0.367879356307219 \,]$$

```
> y00:=1;N:=10;tf:=1.;
```

$$y00 := 1$$

$$N := 10$$

$$tf := 1.$$

```
> EulerBDF(f,y00,tf,N);
```

$$0.385543289429532$$

```
> for i from 1 to 10 do
N:=2^i:
YY[i]:=EulerBDF(f,1.0,1.0,N);od;
>
```

$$N := 2$$

$$YY_1 := 0.444444444444445$$

$$N := 4$$

$$YY_2 := 0.409600000000000$$

$$N := 8$$

$$YY_3 := 0.389744343128946$$

$$N := 16$$

$$YY_4 := 0.379085331917936$$

$$N := 32$$

$$YY_5 := 0.373553861490063$$

$$N := 64$$

$$YY_6 := 0.370734932900974$$

$$N := 128$$

$$YY_7 := 0.369311810602494$$

$$N := 256$$

$$YY_8 := 0.368596788526832$$

$$N := 512$$

$$YY_9 := 0.368238406359098$$

$$N := 1024$$

$$YY_{10} := 0.368058996749418$$

```
> for i from 1 to 9 do Y1[i]:=abs(YY[i+1]-YY[i]);od;
```

$$Y1_1 := 0.034844444444445$$

$$Y1_2 := 0.019855656871054$$

$$Y1_3 := 0.010659011211010$$

$$Y1_4 := 0.005531470427873$$

$$Y1_5 := 0.002818928589089$$

$$Y1_6 := 0.001423122298480$$

$$Y1_7 := 0.000715022075662$$

$$Y1_8 := 0.000358382167734$$

$$Y1_9 := 0.000179409609680$$

```
> for i from 1 to 8 do Y1[i]/Y1[i+1];od;
```

1.75488752000152

1.86280476471819

1.92697608167611

1.96225986329814

1.98080557946413

1.99031938582093

1.99513854214060

1.99756394528264

```
> restart:
N:=10;
y00:=[1,1];
eq11:=diff(y[1](t),t)=-y[1](t)^2;
eq2:=diff(y[2](t),t)=y[1](t)^2-y[2](t);
f2:=subs([y[1](t)=Y[1],y[2](t)=Y[2]],[rhs(eq11),rhs(eq2)]);
eq3:=[seq(h*f2[i]+y0[i]-Y[i],i=1..nops(f2))];
#F:=unapply(eq3,seq(Y[i],i=1..nops(f2)),seq(y0[i],i=1..nops(f2))
,h);
F:=unapply(eq3,seq(y0[i],i=1..nops(f2)),h);
seq(Y[i],i=1..nops(f2)),seq(y0[i],i=1..nops(f2));
F(seq(y00[i],i=1..nops(f2)),0.1);
printlevel:=1;
for j from 1 to N do
YY[0]:=y00;
YY[1]:=fsolve(F(YY[0][1],YY[0][2],1/(2^j)),{Y[1]=YY[0][1],Y[2]=Y
Y[0][2]});
for i from 2 to 2^j do
YY[i]:=fsolve(F(rhs(YY[i-1][1]),rhs(YY[i-1][2]),1/(2^j)),{YY[i-
1][1],YY[i-1][2]});end do;
j;
YYY[j]:=YY[2^j];
end do;
```

>
>

$$N := 10$$

$$y00 := [1, 1]$$

$$eq11 := \frac{d}{dt} y_1(t) = -y_1(t)^2$$

$$eq2 := \frac{d}{dt} y_2(t) = y_1(t)^2 - y_2(t)$$

$$f2 := [-Y_1^2, Y_1^2 - Y_2]$$

$$eq3 := [-h\, Y_1^2 - Y_1 + y0_1, h\,(Y_1^2 - Y_2) - Y_2 + y0_2]$$

$$F := (y0\_1, y0\_2, h) \to [-h\, Y_1^2 + y0\_1 - Y_1, h\,(Y_1^2 - Y_2) - Y_2 + y0\_2]$$

$$Y_1, Y_2, y0_1, y0_2$$

$$[-0.1\, Y_1^2 + 1 - Y_1, 0.1\, Y_1^2 - 1.1\, Y_2 + 1]$$

$$printlevel := 1$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{Y_1 = 0.7320508076, Y_2 = 0.8452994616\}$$

$$1$$

$$YYY_1 := \{Y_1 = 0.5697457167, Y_2 = 0.6717363683\}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{Y_1 = 0.8284271247, Y_2 = 0.9372583002\}$$

$$2$$

$$YYY_2 := \{Y_1 = 0.5385376831, Y_2 = 0.6611292569\}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{Y_1 = 0.8989794856, Y_2 = 0.9786849017\}$$

$$3$$

$$YYY_3 := \{Y_1 = 0.5203762705, Y_2 = 0.6555528553\}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{Y_1 = 0.9442719100, Y_2 = 0.9936264376\}$$

$$4$$

$$YYY_4 := \{ Y_1 = 0.5104965568, Y_2 = 0.6526879102 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9705627485, Y_2 = 0.9982421833 \}$$

$$5$$

$$YYY_5 := \{ Y_1 = 0.5053300278, Y_2 = 0.6512342251 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9848450049, Y_2 = 0.9995372259 \}$$

$$6$$

$$YYY_6 := \{ Y_1 = 0.5026860865, Y_2 = 0.6505017231 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9923072371, Y_2 = 0.9998811911 \}$$

$$7$$

$$YYY_7 := \{ Y_1 = 0.5013483943, Y_2 = 0.6501340044 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9961239728, Y_2 = 0.9999698948 \}$$

$$8$$

$$YYY_8 := \{ Y_1 = 0.5006755458, Y_2 = 0.6499497706 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9980544673, Y_2 = 0.9999924225 \}$$

$$9$$

$$YYY_9 := \{ Y_1 = 0.5003381118, Y_2 = 0.6498575604 \}$$

$$YY_0 := [1, 1]$$

$$YY_1 := \{ Y_1 = 0.9990253402, Y_2 = 0.9999980997 \}$$

$$10$$

$$YYY_{10} := \{ Y_1 = 0.5001691411, Y_2 = 0.6498116960 \}$$

```
> sol2:=dsolve({eq11,eq2,y[1](0)=1,y[2](0)=0},type=numeric):
```

```
> sol2(1);
```

$$[t = 1., y_1(t) = 0.500000005414531, y_2(t) = 0.281885861747178]$$

```
> printlevel:=1;
for i from 1 to N-1 do
Z1[i]:={seq(abs(rhs(YYY[i+1][j])-
rhs(YYY[i][j])),j=1..nops(f2))};od;
Z1[1][1]/Z1[2][1];
```

$$printlevel := 1$$

$$Z1_1 := \{0.0106071114, 0.0312080336\}$$

$$Z1_2 := \{0.0055764016, 0.0181614126\}$$

$$Z1_3 := \{0.0028649451, 0.0098797137\}$$

$$Z1_4 := \{0.0014536851, 0.0051665290\}$$

$$Z1_5 := \{0.0007325020, 0.0026439413\}$$

$$Z1_6 := \{0.0003677187, 0.0013376922\}$$

$$Z1_7 := \{0.0001842338, 0.0006728485\}$$

$$Z1_8 := \{0.0000922102, 0.0003374340\}$$

$$Z1_9 := \{0.0000458644, 0.0001689707\}$$

$$1.902142665$$

```
> for i from 1 to N-2 do
ZZ1[i]:={seq(Z1[i][j]/Z1[i+1][j],j=1..nops(f2))};od;
```

$$ZZ1_1 := \{1.718370387, 1.902142665\}$$

$$ZZ1_2 := \{1.838252924, 1.946425291\}$$

$$ZZ1_3 := \{1.912253604, 1.970815481\}$$

$$ZZ1_4 := \{1.954101250, 1.984547619\}$$

$$ZZ1_5 := \{1.976494518, 1.992017268\}$$

$$ZZ1_6 := \{1.988103117, 1.995935056\}$$

$$ZZ1_7 := \{1.994015126, 1.997976363\}$$

$$ZZ1_8 := \{1.996997113, 2.010496158\}$$

```
> ###################################### P3
######################################
> restart;
```

```
> Digits:=15;
```

$$Digits := 15$$

```
> eq:=diff(y(t),t)=-100*y(t);
```

$$eq := \frac{d}{dt}\,y(t) = -100\,y(t)$$

```
> sol:=dsolve({eq,y(0)=1},type=numeric):
> sol(1);
```

$$[t = 1.,\, y(t) = -0.311605349102452\;\; 10^{-8}]$$

```
> sol:=dsolve({eq,y(0)=1},type=numeric,stiff=true):
sol(1);
```

$$[t = 1.,\, y(t) = 0.276232095700885\;\; 10^{-9}]$$

```
> f:=rhs(eq);
```

$$f := -100\,y(t)$$

```
> f:=subs(y(t)=Y,f);
```

$$f := -100\,Y$$

```
> F:=unapply(f,Y);
```

$$F := Y \rightarrow -100\,Y$$

```
> YY[0]:=1.0;
```

$$YY_0 := 1.0$$

```
> h:=0.1;
```

$$h := 0.1$$

```
> YY[1]:=YY[0]+h*F(YY[0]);
```

$$YY_1 := -9.00$$

```
> eq1:=Y=YY[0]+h*F(Y);
```

$$eq1 := Y = 1.0 - 10.0\,Y$$

```
> fsolve(eq1,Y);sol(0.1);
```

$$0.0909090909090909$$

$$[t = 0.1,\, y(t) = 0.0000454375942981399\;\;]$$

```
> restart;
> Y[1]:=Y[0]+h*lambda*Y[0];
```

$$Y_1 := h\,\lambda\,Y_0 + Y_0$$

```
> eq1:=theta=1+z;
```

$$eq1 := \theta = 1 + z$$

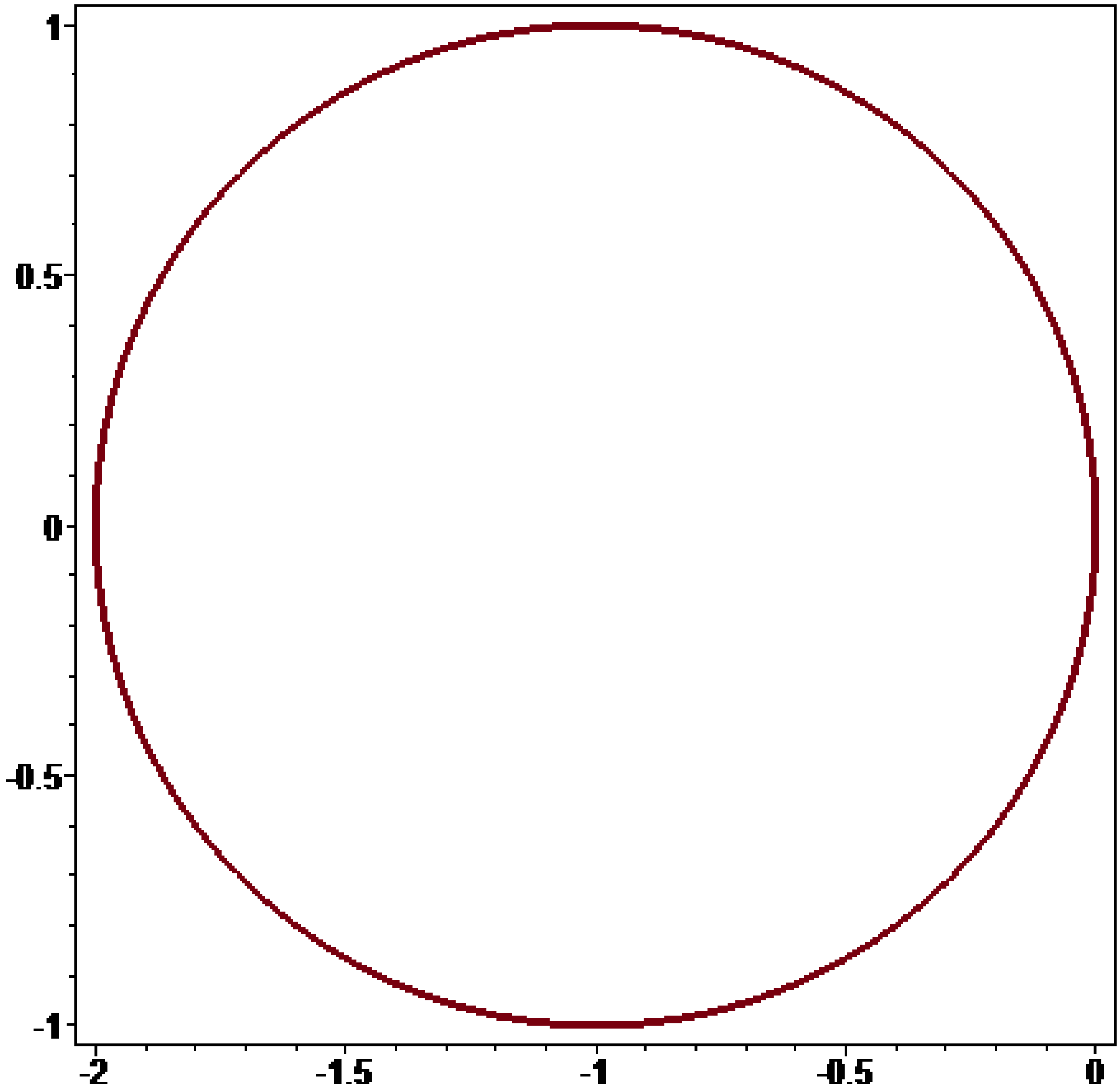
```
> z1:=solve(eq1,z);
```

$$z1 := -1 + \theta$$

```
> z1:=subs(theta=exp(I*phi),z1);
```

$$z1 := -1 + \mathbf{e}^{(\phi\, I)}$$

```
> with(plots):
p1:=complexplot(z1,phi=0..2*Pi,thickness=3,axes=boxed):
display({p1});
```




```
> eq2:=ypred=1+h*lambda*1;
```

$$eq2 := ypred = h\,\lambda + 1$$

```
> eq3:=Y1=1+h/2*(lambda*1+lambda*ypred);
```

$$eq3 := Y1 = 1 + \frac{h\,(\lambda\, ypred + \lambda)}{2}$$

```
> cons:=solve({eq2,eq3},{ypred,Y1});
```

$$cons := \{Y1 = \frac{1}{2} h^2 \lambda^2 + h \lambda + 1, ypred = h \lambda + 1\}$$

```
> eqz:=Y1=subs(cons,Y1);
```

$$eqz := Y1 = \frac{1}{2} h^2 \lambda^2 + h \lambda + 1$$

```
> eqz:=subs(Y0=1,eqz);
```

$$eqz := Y1 = \frac{1}{2} h^2 \lambda^2 + h \lambda + 1$$

```
> series(exp(lambda*h),h);
```

$$1 + \lambda h + \frac{\lambda^2}{2} h^2 + \frac{\lambda^3}{6} h^3 + \frac{\lambda^4}{24} h^4 + \frac{\lambda^5}{120} h^5 + O(h^6)$$

```
> eqz:=subs(h=z/lambda,Y1=theta,eqz);
```

$$eqz := \theta = \frac{1}{2} z^2 + z + 1$$

```
> solve(eqz,z);
```

$$-1 + \sqrt{-1 + 2\theta}, -1 - \sqrt{-1 + 2\theta}$$

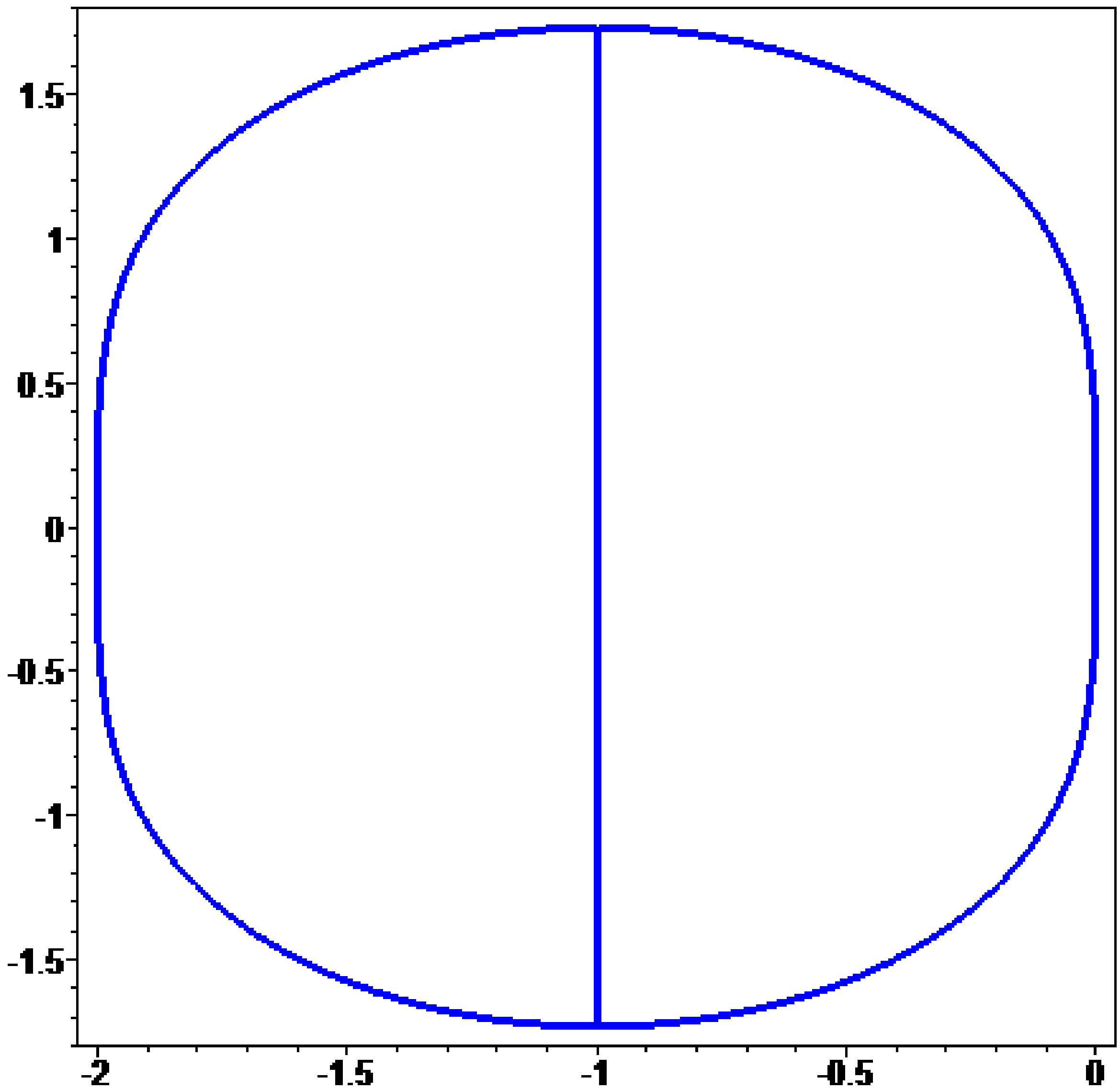
```
> L:=[solve(eqz,z)];
```

$$L := [-1 + \sqrt{-1 + 2\theta}, -1 - \sqrt{-1 + 2\theta}]$$

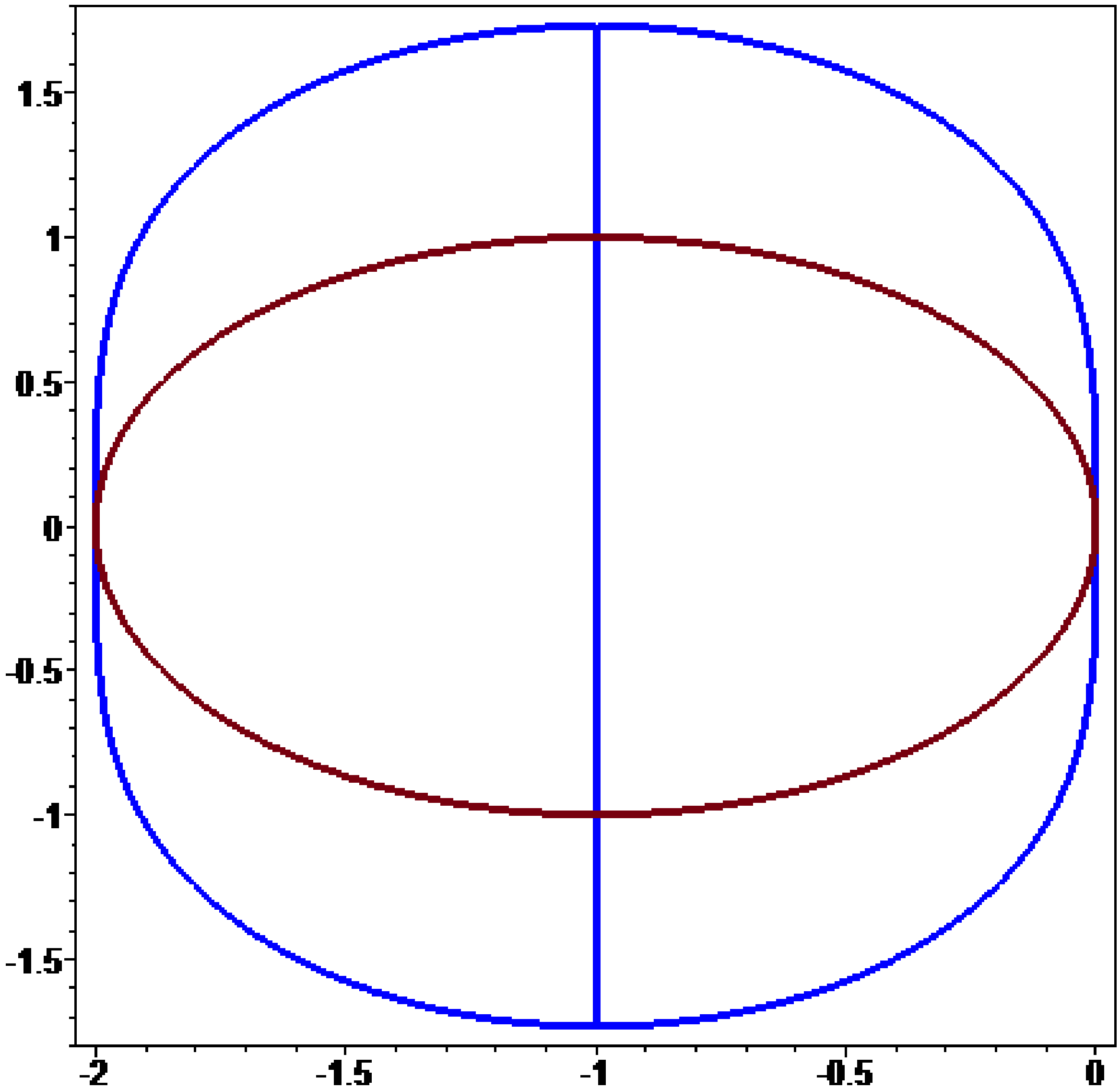
```
> L:=subs(theta=exp(I*phi),L);
```

$$L := [-1 + \sqrt{-1 + 2\,\mathbf{e}^{(\phi I)}}, -1 - \sqrt{-1 + 2\,\mathbf{e}^{(\phi I)}}]$$

```
>
p2:=complexplot(L[1],phi=0..2*Pi,thickness=3,axes=boxed,color=bl
ue):
p3:=complexplot(L[2],phi=0..2*Pi,thickness=3,axes=boxed,color=bl
ue):
display({p2,p3});
```




```
> display({p1,p2,p3});
```




```
> eq4:=Y1=1+h*lambda*Y1;
```

$$eq4 := Y1 = Y1\, h\, \lambda + 1$$

```
> cons:=solve({eq4},{Y1});
```

$$cons := \{ Y1 = -\frac{1}{h\,\lambda - 1} \}$$

```
> eqz:=Y1=subs(cons,Y1);
```

$$eqz := Y1 = -\frac{1}{h\,\lambda - 1}$$

```
> eqz:=subs(lambda=z/h,Y1=theta,eqz);
```

$$eqz := \theta = -\frac{1}{z - 1}$$

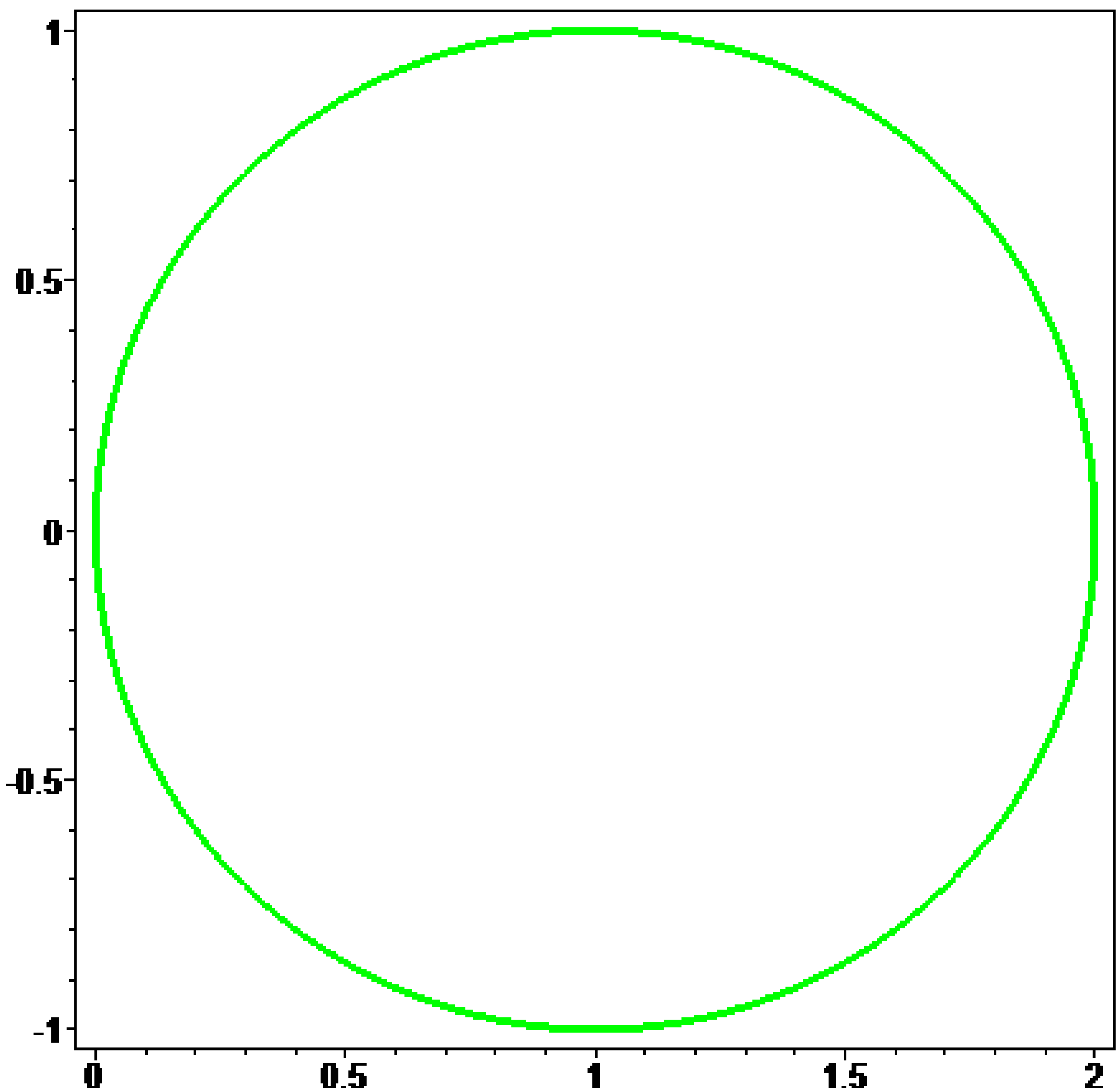
```
> z2:=solve(eqz,z);
```

$$z2 := \frac{-1+\theta}{\theta}$$

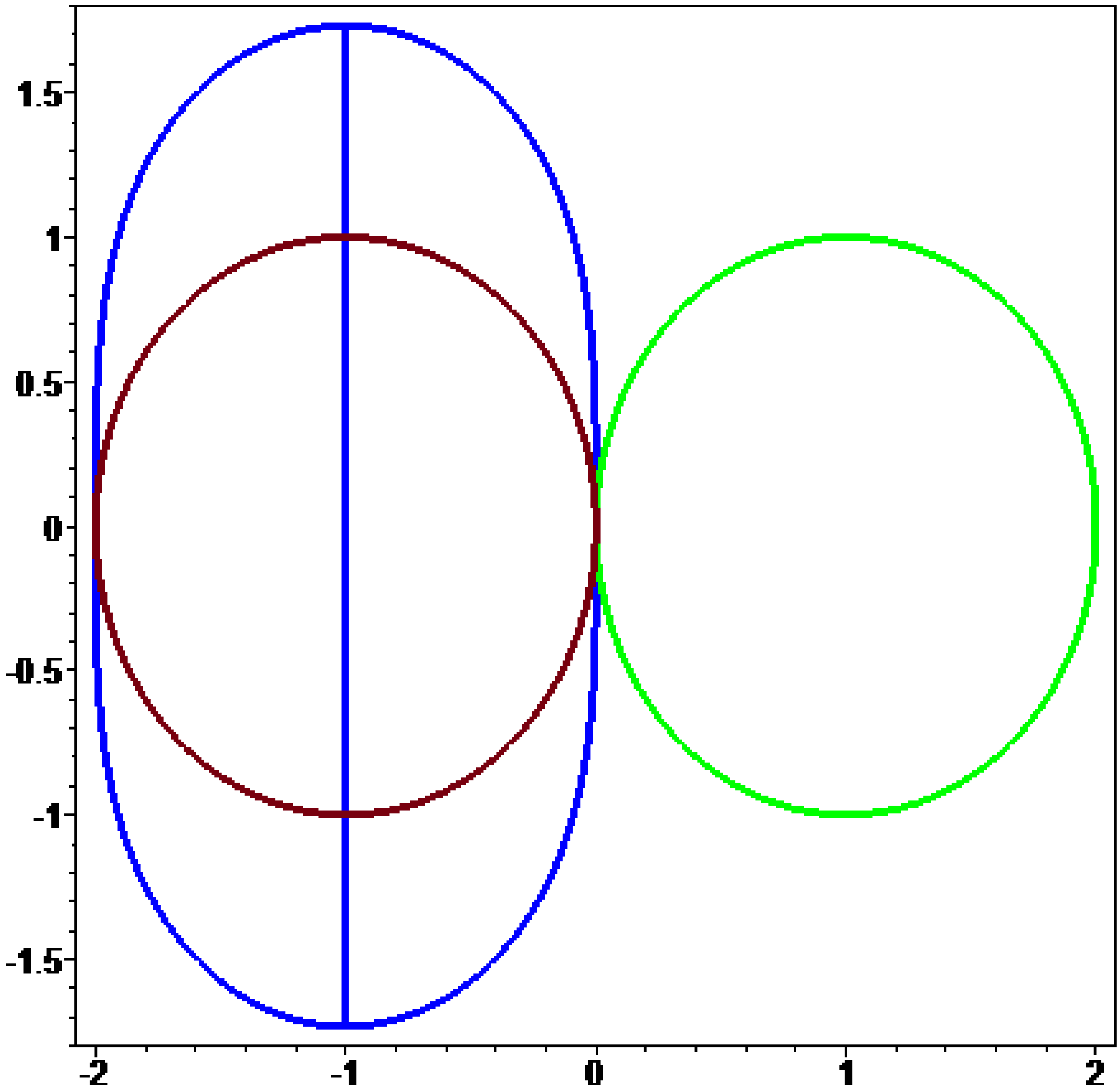
```
> z2:=subs(theta=exp(I*phi),z2);
```

$$z2 := \frac{-1+\mathbf{e}^{(\phi\, I)}}{\mathbf{e}^{(\phi\, I)}}$$

```
>
p4:=complexplot(z2,phi=0..2*Pi,thickness=3,axes=boxed,color=gree
n):
display({p4});
```




```
> display({p1,p2,p3,p4});
```




```
> yy:=y0+b*t+c*t^2;
```

$$yy := c\,t^2 + b\,t + y0$$

```
> Eq1:=subs(t=-h,yy)=yb;
```

$$Eq1 := c\,h^2 - b\,h + y0 = yb$$

```
> Eq2:=subs(t=h,diff(yy,t))=F1;
```

$$Eq2 := 2\,c\,h + b = F1$$

```
> cons:=solve({Eq1,Eq2},{b,c});
```

$$cons := \{b = \frac{F1\,h + 2\,y0 - 2\,yb}{3\,h}, c = \frac{F1\,h - y0 + yb}{3\,h^2}\}$$

```
> subs(cons,t=h,yy);
```

$$\frac{2\,F1\,h}{3}+\frac{4\,y0}{3}-\frac{yb}{3}$$

```
> eq5:=Y1=2/3*h*lambda*Y1+4/3*1-1/3*yb;
```

$$eq5 := Y1 = \frac{2\,h\,\lambda\,Y1}{3}+\frac{4}{3}-\frac{yb}{3}$$

```
> cons:=solve({eq5},{Y1});
```

$$cons := \{Y1 = \frac{-4+yb}{2\,h\,\lambda-3}\}$$

```
> eqz:=Y1=subs(cons,Y1);
```

$$eqz := Y1 = \frac{-4+yb}{2\,h\,\lambda-3}$$

```
> eqz:=subs(lambda=z/h,yb=1/theta,Y1=theta,eqz);
```

$$eqz := \theta = \frac{-4+\frac{1}{\theta}}{2\,z-3}$$

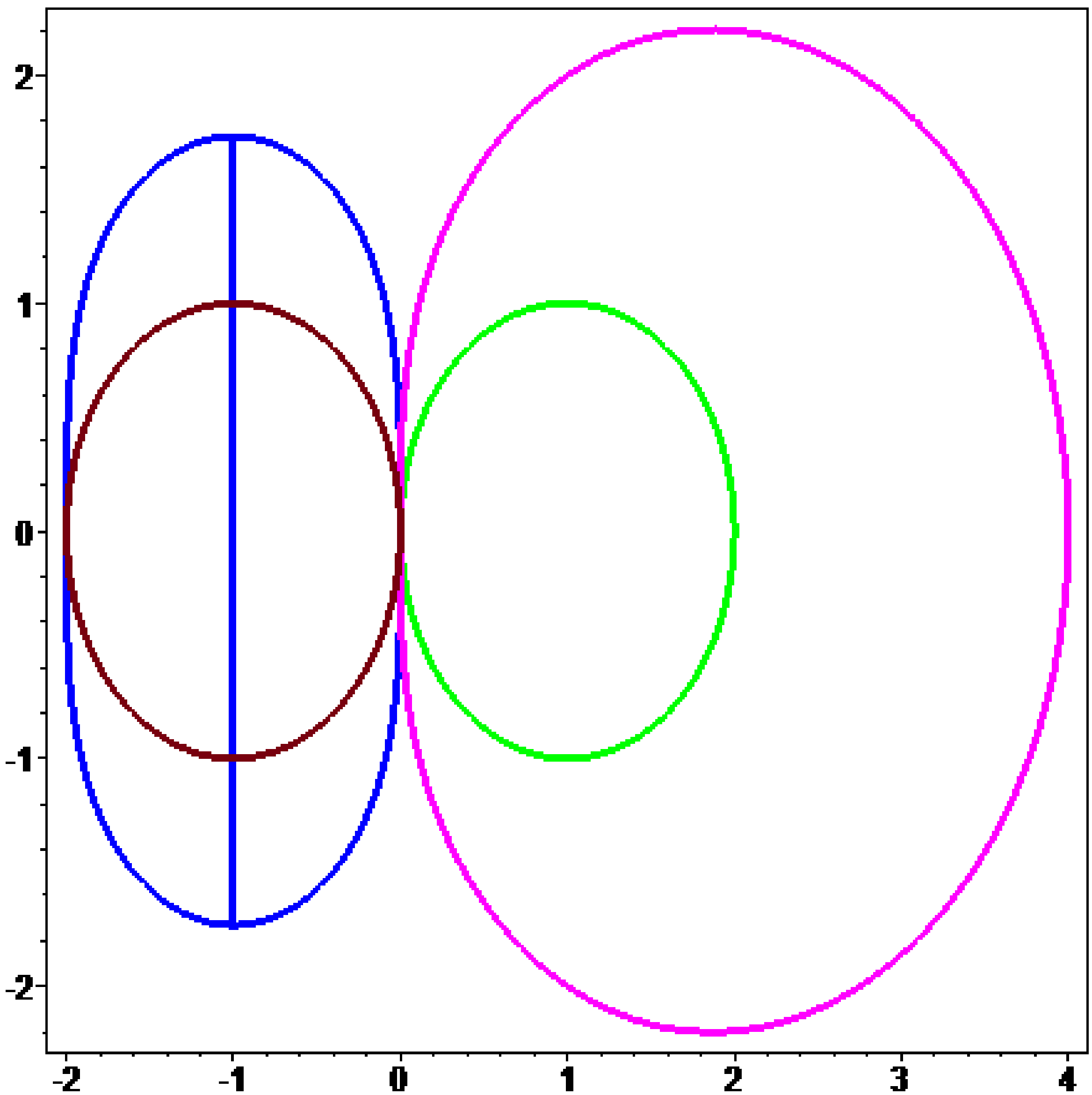
```
> z3:=solve(eqz,z);
```

$$z3 := \frac{3\,\theta^2-4\,\theta+1}{2\,\theta^2}$$

```
> z3:=subs(theta=exp(I*phi),z3);
```

$$z3 := \frac{1}{2}\,\frac{3\,(\mathbf{e}^{(\phi\,I)})^2-4\,\mathbf{e}^{(\phi\,I)}+1}{(\mathbf{e}^{(\phi\,I)})^2}$$

```
>
p5:=complexplot(z3,phi=0..2*Pi,thickness=3,axes=boxed,color=mage
nta):
display({p1,p2,p3,p4,p5});
```




```
> yy:=y0+b*t+c*t^2;
```

$$yy := c\,t^2 + b\,t + y0$$

```
> Eq1:=subs(t=alpha*h,diff(yy,t))=Fint;
```

$$Eq1 := 2\,\alpha\,c\,h + b = Fint$$

```
> Eq1:=subs(t=h,diff(yy,t))=F1;
```

$$Eq1 := 2\,c\,h + b = F1$$

```
> cons:=solve({Eq1,Eq2},{b,c});
```

$$cons := \{b = -2\,c\,h + F1, c = c\}$$

```
> subs(cons,t=h,yy);
```

$$c\,h^2 + (-2\,c\,h + F1)\,h + y0$$

```
> eqY:=Y1=subs(cons,t=h,yy);
```

$$eqY := Y1 = c\,h^2 + (-2\,c\,h + F1)\,h + y0$$

```
> eqYint:=Yint=subs(cons,t=alpha*h,yy);
```

$$eqYint := Yint = c\,\alpha^2\,h^2 + (-2\,c\,h + F1)\,\alpha\,h + y0$$

```
> ############################ P4 ##############################
```

Part a)

```
In[65]:= Clear[Fm]
         eq1 = {Fn == b, Fm == +b + 2 * c * (-h)}

Out[66]= {Fn == b, Fm == b - 2 c h}

In[67]:= sol1 = Solve[eq1, {b, c}]
```

$$\text{Out[67]= } \left\{\left\{b \to Fn,\ c \to -\frac{Fm - Fn}{2\,h}\right\}\right\}$$

```
In[68]:= ans = FullSimplify[yn + h * b + c * h ^ 2 /. sol1]
```

$$\text{Out[68]= } \left\{-\frac{1}{2}\,(Fm - 3\,Fn)\,h + yn\right\}$$

Part b)

```
In[24]:= Clear[yn, ym, yl, yk]
         eq1 = {yl == yn + b * (-h) + c * (h ^ 2) - d * (h ^ 3), Fn == b + c * 2 * h + 3 * d * h ^ 2,
           yk == yn + b * (-2 * h) + 4 * c * h ^ 2 - 8 * d * h ^ 3}
```

$$\text{Out[25]= } \{yl == -b\,h + c\,h^2 - d\,h^3 + yn,\ Fn == b + 2\,c\,h + 3\,d\,h^2,\ yk == -2\,b\,h + 4\,c\,h^2 - 8\,d\,h^3 + yn\}$$

```
In[26]:= sol1 = FullSimplify[Solve[eq1, {b, c, d}]]
```

$$\text{Out[26]= } \left\{\left\{b \to \frac{4\,Fn\,h + 5\,yk - 28\,yl + 23\,yn}{22\,h},\ c \to \frac{3\,Fn\,h + yk + yl - 2\,yn}{11\,h^2},\ d \to \frac{2\,Fn\,h - 3\,yk + 8\,yl - 5\,yn}{22\,h^3}\right\}\right\}$$

```
In[27]:= ans = FullSimplify[yn + b * h + c * h ^ 2 + d * h ^ 3 /. sol1]
```

$$\text{Out[27]= } \left\{\frac{1}{11}\,(6\,Fn\,h + 2\,yk - 9\,yl + 18\,yn)\right\}$$

```
> ############################### P5 ################################
> restart;
  eq:=diff(y(t),t)=5*y(t)*((10-y(t))/90-1/3);
```

$$eq := \frac{d}{dt}y(t) = 5\,y(t)\left(-\frac{2}{9}-\frac{1}{90}y(t)\right)$$

```
> sol:=dsolve({eq,y(0)=10},type=numeric):
> sol(1);
>
```

$$[t = 1.,\ y(t) = 2.46511962536120]$$

```
> plot(sol,t);
Error, (in plot) expected a range but received t

> eq2:={diff(y[1](t),t)=a*(1-y[1](t))-y[1](t)*y[2](t),diff(y[2](t),t)=-e*y[1](t)*y[2](t),y[
  1](0)=0,y[2](0)=1};
> eqz:=subs([e=0.1,a=1],eq2);
```

$$eq2 := \{\frac{d}{dt}y_1(t) = a\,(1-y_1(t)) - y_1(t)\,y_2(t),\ \frac{d}{dt}y_2(t) = -e\,y_1(t)\,y_2(t),\ y_1(0) = 0,\ y_2(0) = 1\}$$

$$eqz := \{\frac{d}{dt}y_1(t) = 1 - y_1(t) - y_1(t)\,y_2(t),\ \frac{d}{dt}y_2(t) = -0.1\,y_1(t)\,y_2(t),\ y_1(0) = 0,\ y_2(0) = 1\}$$

```
> sol2:=dsolve(eqz,type=numeric);
```

sol2 := **proc**(*x_rkf45*) ... **end proc**

```
> sol2(1);
```

$$[t = 1.,\ y_1(t) = 0.434971341170385,\ y_2(t) = 0.971949039067227]$$

```
> eq3:={diff(C[1](t),t)=-k[1]*C[1](t)^2-k[2]*C[1](t)^2,diff(C[2](t),t)=k[1]*C[1](t)^2-k[3]*
  C[2](t),C[1](t)+C[2](t)+C[3](t)=1,C[1](0)=1,C[2](0)=0,C[3](0)=0};
```

$eq3 :=$

$$\{C_1(t) + C_2(t) + C_3(t) = 1,\ \frac{d}{dt}C_1(t) = -k_1\,C_1(t)2 - k_2\,C_1(t)2,\ \frac{d}{dt}C_2(t) = k_1\,C_1(t)2 - k_3\,C_2(t),\ C_1(0) = 1,\ C_2(0) = 0,\ C_3(0) = 0\}$$

```
> eqy:=subs([k[1]=1,k[2]=0.1,k[3]=0.5],eq3);
```

$$eqy := \{C_1(t) + C_2(t) + C_3(t) = 1,\ \frac{d}{dt}C_1(t) = -1.1\,C_1(t)2,\ \frac{d}{dt}C_2(t) = C_1(t)2 - 0.5\,C_2(t),\ C_1(0) = 1,\ C_2(0) = 0,\ C_3(0) = 0\}$$

```
> sol3:=dsolve(eqy,type=numeric);
```

sol3 := **proc**(*x_rkf45_dae*) ... **end proc**

```
> sol3(1);
```

$$[t = 1.,\ C_1(t) = 0.476190476176131,\ C_2(t) = 0.352484120299863,\ C_3(t) = 0.171325403524006]$$

```
> |
```

(maple kept crashing on trying to save, so a screenshot is the best I can do for problem 5)